O'zbekiston Respublikasi qishloq va suv xo'jaligi vazirligi

Samarqand qishloq xo'jalik instituti

Oliy matematika
(Aniq integral)
Uslubiy qo'llanma

Samarqand -2009

O’zbekiston Respublikasi qishloq va suv xo’jaligi vazirligi

Samarqand qishloq xo’jalik instituti

J.S. Sultonov

Oliy matematika

(Aniq integral)

Nazariy va amaliy mashg’ulotlar bo’yicha uslubiy qo’llanma

Samarqand -2009

Samarqand qishloq xo'jalik instituti markaziy attestasiya va uslubiy kengashi tomonidan tasdiqlangan va chop etishga tavsiya qilingan (Bayonnoma №9, 28 may, 2009)

Taqrizchilar: SamDU «Algebra va geometriya» kafedrasi dosent f.m.f.n. V.Ergashev, «Ehtimollar nazariyasi va o'qitish metodikasi» kafedrasi dosenti, ped.f.n. Q. Ostonov

Uslubiy qo'llanma «Oliy matematika» kursining «Aniq integral» bo'limini o'z ichiga oladi. Unda nazariy materiallar, yechib ko'rsatilgan mashqlar hamda mustaqil yechish uchun mashqlar majmuasi berilgan.

Qo'llanma oliy o'quv yurtlarining nomatematikaviy yo'nalishdagi fakultet talabalari uchun uslubiy qo'llanma sifatida tavsiya qilinadi. Undan yuqori sinf o'quvchilari, kollej va lisey talabalari ham foydalanishlari mumkin.

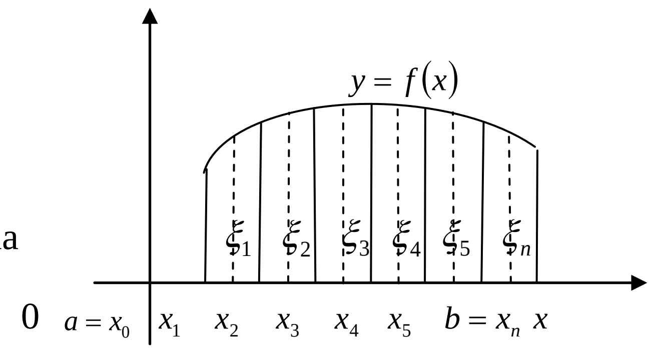
J.S. Sultonov. Oliy matematika / Aniq integral /.- Samarqand, 2009

Mundarija

1- BOB. ANIQ INTEGRAL VA UNI HISOBLASh

1§. Aniq integral tushunchasi

Ixtiyoriy $y = f(x)$ funksiya biror $[a, b]$ oraliqda berilgan bo'lib, u uzluksiz bo'lsin. $[a, b]$ oraliqda n ta $x_0, x_1, x_2, x_3, \ldots, x_n$ ketma- ket kuqtalar olamiz. U holda, bu nuqtalar $[a, b]$ oraliqni n ta qismga ajratadi. Bunda $a = x_0$ va $b = x_n$ deb olamiz. Hosil bo'lgan elementar kesmalarni quyidagicha ifodalaymiz:



$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$

$[x_0, x_1]$ kesmada ξ_1, $[x_1, x_2]$da ξ_2, $[x_2, x_3]$ da ξ_3 va hokazo, $[x_{n-1}, x_n]$da ξ_n nuqta olamiz. U holda, quyidagi yig'indi o'rinli bo'ladi:

$$S_n = f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \ldots + f(\xi_n)(x_n - x_{n-1}) \qquad (1)$$

yoki

$$\sum_{k=1}^{k=n} f(\xi_k)(x_k - x_{k-1}). \qquad (2)$$

$x_1 - x_0 = \Delta x_1, \quad x_2 - x_1 = \Delta x_2, \ldots, x_n - x_{n-1} = \Delta x_n$ belgilashlar kiritamiz. U holda (1) va (2) ni quyidagicha yozish mumkin:

$$S_n = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_n)\Delta x_n \qquad \text{yoki}$$

$$S_n = \sum_{k=n}^{n} f(\xi_k)\Delta x_k = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_n)\Delta x_n. \qquad (3)$$

(3) ga $f(x)$funksiyaning $[a, b]$ oraliqdagi ***integral yig'indisi*** deyiladi.

Ta'rif: $f(x)$ funksiyaning $[a, b]$kesmadag ***aniq integrali*** deb integral yig'indining elementar kesmalardan eng kattasining uzunligi $\lambda = 0$ bo'lgandagi limitiga aytiladi va quyidagi ko'rinishda ifodalanadi:

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k. \qquad (4)$$

Bunda a- integralning quyi, b- yuqori chegarasidir. Integralning o'qilishi: «Integral a dan b gacha, ef iks de iks».

Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda integral yig'indi S_n chekli limitga ega bo'ladi, ya'ni qarralayotgan $f(x)$ funksiya $[a,b]$da integrallanuvchi bo'lib, integral yig'indining limiti $[a,b]$ oraliqning bo'linish usuliga va har bir elementar kesmadagi ξ nuqtaning olinishiga bog'liq bo'lmaydi.

Misol. $\int_0^1 xdx$ integralni ta'rif asosida hisoblang.

Yechilishi: Berilishiga ko'ra $f(x)=x,\ \ a=0$ va $b=1.$ $[0,1]$ oraliqni quyidagi $x_0=0,\ \ x_1=\frac{1}{n},\ \ x_2=\frac{2}{n},\ldots,x_{n-1}=\frac{n-1}{n},\ \ x_n=1$ nuqtalar yordamida n ta teng elementar kesmalarga ajratamiz va berilgan funksiyaning ularga mos qiymatlarini topamiz:

$$f(x_0)=0,\ f(x_1)=\left(\frac{1}{n}\right)^2,\ f(x_2)=\left(\frac{2}{n}\right)^2,\ldots,f(x_{n-1})=\left(\frac{n-1}{n}\right)^2,\ \ f(x_n)=\left(\frac{n}{n}\right)^2=1.$$

U holda, integral yig'indining qo'shiluvchilari $f(x_k)\cdot\Delta x_k=\left(\frac{k}{n}\right)^2\cdot\frac{1}{n}=\frac{k^2}{n^3}.$

Integral yig'indi quyidagicha bo'ladi:

$$S_n=\sum_{k=0}^{n-1}f(x_k)\Delta x_k=\frac{1}{n}\sum_{k=0}^{n-1}x_k^2=\frac{1}{n}\left(x_0^2+x_1^2+\ldots+x_{n-1}^2\right)=\frac{1^2+2^2+\ldots+(n-1)^2}{n^3}=$$

$$=\frac{(n-1)\cdot n\cdot(2n-1)}{6n^3}=\frac{1}{n^3}\cdot\frac{2n^3+3n^2+n}{6}=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}.$$

U holda, $S=\lim S_n=\lim\limits_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{1}{3}.$ Demak, $\int_0^1 x^2dx=\frac{1}{3}$ kv. birl.

2§. Aniq integralning xossalari

1- xossa. Har qanday M o'zgarmas son uchun quyidagi tenglik o'rinli:

$$\int_a^b M dx = M(b-a). \qquad (1)$$

Isboti: $f(x)=M$ funksiyaning $[a,b]$ dagi integral yig'indisini qaraydik:

$$\sum_{i=1}^{n} f(\xi_1)\Delta x_i = \sum_{i=1}^{n} M \cdot \Delta x = M\sum_{i=1}^{n} \Delta x_i = M(b-a)$$

Demak, (1) tenglik o'rinli ekan.

2- xossa. O'zgarmas sonni integral belgisi oldiga chiqarish mumkin:

$$\int_a^b Mf(x)dx = M\int_a^b f(x)dx. \qquad (2)$$

Isboti: $Mf(x)$ funusiyaning $[a,b]$ dagi integral yig'indisi uchun quyidagi o'rinlidir:

$$\sum_{i=1}^{n} Mf(\xi_i)\cdot \Delta x_i = M\sum_{i=1}^{n} f(\xi_i)\cdot \Delta x_i .$$

Shuning uchun $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} Mf(\xi_i)\cdot \Delta x_i = M\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(\xi_i)\cdot \Delta x_i = M\int\limits_a^b f(x)dx.$

Demak, $Mf(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lib, (2) formula o'rinli ekan.

3-xossa. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, ularning algebraik yig'indisi ham shu oraliqda integrallanuvchi bo'ladi, ya'ni:

$$\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx. \qquad (3)$$

Isboti: $\int\limits_a^b [f(x)\pm g(x)]dx = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}[f(\xi_i)\pm g(\xi_i)]\Delta x_i = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i \pm$

$$\pm \lim_{n\to\infty}\sum_{i=1}^{n} g(\xi_i)\Delta x_i = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

4-xossa. Agar aniq integralning chegaralari o'zaro almashtirilsa, uning ishorasi qarama –qarshiga o'zgaradi:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx. \qquad (4)$$

Isboti talabalarga havola qilinadi.

5-xossa. Chegaralari o'zaro teng, ya'ni $a = b$ bo'lgan aniq integral nolga teng:

$$\int_b^a f(x)dx = 0. \qquad (5)$$

Isboti talabalarga havola qilinadi.

6-xossa. Agar $f(x)$ fnksiya $[a, b]$ da musbat bo'lib, $a < b$ bo'lsa, quyidagi tengsizlik o'rinli bo'ladi:

$$\int_a^b f(x)dx \geq 0. \qquad (6)$$

Isboti: $[a, b]$ oraliq ixtiyoriy $[x_{i-1}, x_i]$ elementar kesmalarga ajratilganda va ξ_i nuqta $[x_{i-1}, x_i]$ da ixtiyoriy tanlanganda $f(\xi_i) \geq 0$ va $\Delta x_i > 0$ bo'ladi. U holda,

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i \geq 0.$$

Bundan, $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx \geq 0.$

7-xossa. Agar $[a, b]$ oraliqda $a < b$ bo'lganda $f(x) \geq \varphi(x)$ bo'lsa,

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx. \qquad (7)$$

o'rinli bo'ladi.

Isboti: Shartga asosan $f(x) \geq \varphi(x)$. U holda, uni $[a, b]$ da integrallaymiz:

$$\int_a^b [f(x) - \varphi(x)]dx \geq 0.$$

3-xossaga asosan $\int\limits_a^b f(x)dx - \int\limits_a^b \varphi(x)dx \geq 0.$

8-xossa. Agar $[a, b]$ oraliqda $a < b$ bo'lib, m va M lar $f(x)$ funksiyaning shu oraliqdagi eng kichik va eng katta qiymatlari bo'lsa, quyidagi o'rinli bo'ladi:

$$m(b-a) \leq \int\limits_a^b f(x)dx \leq M(b-a). \qquad (8)$$

Isboti: Shartga asosan

$$m \leq f(x) \leq M.$$

Tengsizlikni $[a, b]$ oraliqda integrallaymiz:

$$\int\limits_a^b mdx \leq \int\limits_a^b f(x)dx \leq \int\limits_a^b Mdx.$$

U holda, 2- xossaga va 1- xossaga asosan

$$m(b-a) \leq \int\limits_a^b f(x)dx \leq M(b-a).$$

9-xossa. (O'rta qiymat haqidagi teorema). Agar $f(x)$ funksiya $[a, b]$ oraliqda uzluksiz bo'lsa, bu oraliqda shunday c nuqta mavjud bo'ladiki, uning uchun

$$\int\limits_a^b f(x)dx = f(c)(b-a). \qquad (9)$$

tenglik o'rinli bo'ladi.

Isboti: $[a, b]$ oraliqda m va M lar $f(x)$ funksiyaning eng kichik hamda eng katta qiymatlari bo'lsin. U holda, 8 –xossaga asosan

$$m(b-a) \leq \int\limits_a^b f(x)dx \leq M(b-a).$$

Bundan, $m \le \frac{1}{b-a}\int\limits_a^b f(x)dx \le M$. $f(x)$ funksiya uzluksiz bo'lganligi sababli u $[a,b]$ oraliqdagi barcha qiymatlarni qabul qiladi. U holda, c $[a,b]$ dagi x_1 va x_2 nuqtalar orasida yotadi, ya'ni $x_1 < c < x_2$. Bundan

$$a < c < b.$$

Demak, $\frac{1}{b-a}\int\limits_a^b f(x)dx = f(c)$ yoki $\int\limits_a^b f(x)dx = (b-a)f(c).$

3§. N'yuton-Leybnis formulasi

Faraz qilaylik, $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsin. U holda, funksiya shu oraliqda boshlang'ich funksiyaga ega bo'ladi. Boshlang'ich funksiyalaridan biri $Q(x)$ bo'lib, u quyidagidan iborat bo'lsin:

$$Q(x) = \int\limits_a^x f(t)dt, \text{ bunda } a \le x \le b. \qquad (1)$$

Shu oraliqda $f(x)$ funksiyaning boshqa boshlang'ichi $F(x)$ ham mavjud bo'lsin. U holda, bu boshlang'ich funksiyalar bir – biridan biror c o'zgarmas songa farq qilishi ma'lum, ya'ni

$$\int\limits_a^x f(t)dt = F(x) + c. \qquad (2)$$

Agar $x = a$ bo'lsa, (1) tenglik hamda 5- xossaga asosan quyidagiga ega bo'lamiz:

$$\int\limits_a^a f(t)dt = F(a) + c. \qquad (3)$$

$$F(a) + c = 0, \quad c = -F(a). \qquad (4)$$

(4) ni (2) ga qo'ysak

$$\int\limits_a^x f(t)dt = F(x) - F(a). \qquad (5)$$

hosil bo’ladi. $x=b$ deb olsak

$$\int_a^b f(x)dx = F(b)-F(a). \qquad (6)$$

(6) formulaga ***Nyuton –Leybnis formulasi*** deyiladi.

$F(b)-F(a)$ ayirmani quyidagi ko’rinishlarda yozish qabul qilingan.

$$F(x)\Big|_a^b \text{ yoki } [F(x)]_a^b$$

U holda, (6) formula bunday ifodalanadi:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a). \qquad (7)$$

Yuqori chegarasi o’zgaruvchidan iborat bo’lgan (5) aniq integralni hisoblashning Nyuton –Leybnis usuli quyidagi ko’rinishda bo’ladi:

$$\int_a^x f(t)dt = F(t)\Big|_a^x = F(x)-F(a). \qquad (8)$$

Shuningdek, quyi chegarasi o’zgaruvchidan iborat bo’lgan aniq integral ifodasi esa quyidagicha bo’ladi:

$$\int_x^b f(t)dt = F(t)\Big|_x^b = F(b)-F(x). \qquad (9)$$

$\int_a^b f(x)dx$ aniq integralni hisoblashda quyidagi bosqich ishlari ketma – ket bajariladi:

1. Quyidagi aniqmas integral topiladi:

$$\int f(x)dx = F(x)+c.$$

2. $F(x)+c$ ning $x=b$ dagi qiymati topiladi, ya’ni $F(b)+c.$
3. $F(x)+c$ ning $x=a$ dagi qiymati hisoblanadi, ya’ni $F(a)+c.$
4. $[F(b)+c]-[F(a)+c]=F(b)-F(a)$ ayirama topiladi.

1-misol. $\int_1^3 x^2dx$ integralni hisoblang.

Yechilishi: Bunda $a=1$ va $b=3$.

1. Aniqmas integral $\int x^2dx$ ni hisoblaymiz:

$$\int x^2dx = \frac{x^3}{3}+c.$$

2. $F(b)$ni topamiz:

$$F(b)=F(3)=\frac{3^3}{3}=9.$$

3. $F(a)$ni topamiz:

$$F(a)=F(1)=\frac{1}{3}.$$

4. $F(b)+F(a)=9-\frac{1}{3}=\frac{26}{3}=8\frac{2}{3}.$

Demak, $\int_1^3 x^2dx=\left.\frac{x^3}{3}\right|_1^3=\frac{3^3}{3}-\frac{1^3}{3}=9-\frac{1}{3}=8\frac{2}{3}.$

2-misol. Hisoblang: $\int_a^b \cos xdx$.

Yechilishi: Integralni hisoblashni yuqoridagi bosqichlar asosida, ya'ni (7) formulani qo'llash orqali bajaramiz:

$$\int_a^b \cos xdx = -\sin x\Big|_a^b = -(\sin b-\sin a)=\sin a-\sin b.$$

3-misol. Integralni hisoblang: $\int_2^3 \left(x^3+2x\right)dx.$

Yechilishi: Aniq integralning 3- xossasiga asosan berilgan integralni ikki qismga ajratamiz va Nyuton –Leybnis formulasidan foydalanib, hisoblaymiz:

$$\int_2^3 \left(x^3+2x\right)dx=\int_2^3 x^3dx+\int_2^3 2xdx=\left.\frac{x^4}{4}\right|_2^3+\left.\frac{2x^2}{2}\right|_2^3=\frac{81}{4}-4+9-4=\frac{85}{4}=21\frac{1}{4}.$$

Mustaqil yechish uchun mashqlar.

№1. $\int_1^2 xdx$

№2. $\int_0^3 x^2dx$

№3. $\int_{-1}^0 (x-1)dx$

№4. $\int_1^2 (4x^3-3x^2+2x)dx$

№5. $\int_0^\pi \sin xdx$

№6. $\int_{\frac{\pi}{2}}^\pi \cos xdx$

№7. $\int_1^3 (x^3+2x)dx.$

№8. $\int_0^{\frac{\pi}{4}} \left(\frac{3}{\cos^2 x}+2\sin x\right)dx.$

№9. $\int_{-1}^1 (2x^3+3)dx.$

№10. $\int_{-3}^{-1} 7(x^2+1)dx.$

№11. $\int_0^8 x^2dx$

№12. $\int_0^\pi (2e^x+5\sin x)dx.$

O'rta qiymat haqidagi teorema

Teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u holda, shu kesmada shunday c nuqta mavjud bo'ladiki, uning uchun

$$\int_a^b f(x)dx = f(c)(b-a) \tag{1}$$

tenglik o'rinli bo'ladi.

Isboti: Faraz qilaylik, $a<b$ bo'lsin. U holda, funksiyaning berilgan $[a,b]$ kesmadagi eng katta qiymati M va eng kichik qiymati m bo'lsin, ya'ni

$$m \le f(x) \le M\,. \tag{2}$$

$[a,b]$ da (2) tengsizlikni integrallaymiz:

$$\int_a^b mdx \le \int_a^b f(x)dx \le \int_a^b Mdx.$$

Bundan, $$m(b-a) \le \int_a^b f(x)dx \le M(b-a). \quad (3)$$

(3)ni $b-a>0$ ga hadma – had bo'lamiz:

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M\ . \quad (4)$$

Berilgan $f(x)$ funksiya $[a, b]$ da uzluksiz bo'lganligi uchun qo'yi va yuqori chegara oralig'idagi (ya'ni $[m, M]$) istalgan qiymatni qabul qiladi. U holda, $[a, b]$ da shunday c nuqta mavjud bo'ladiki, $f(c)=\frac{1}{b-a}\int_a^b f(x)dx$ bo'lishini ta'minlaydi. Bu esa (1) formuladan iborat. Teorema isbot bo'ldi.

4§. Aniq integralni o'zgaruvchini almashtirish (o'rniga qo'yish) usuli bilan hisoblash

Aniqmas integralni o'zgaruvchini almashtirish usulida yechishdan ma'lumki, agar integrallash qoidalari, xossalari yoki formulalar yordamida integrallash qiyinlik tug'dirsa integral ostidagi funksiyaga yangi o'zgaruvchi kiritish lozim. Aniq integralni hisoblashda ham shu usul qo'llaniladi.

$\int_a^b f(x)dx$ ni o'zgaruvchini almashtirish usulida hisoblash talab qilinsin. Yangi t o'zgaruvchini kiritaylik. U holda, $x=\varphi(t)$. $\varphi(t)$ funksiya $[a, b]$ kesmada uzluksiz va differensiallanuvchi bo'lsin. Agarda t o'zgaruvchi $[\alpha, \beta]$ kesmada o'zgarganda x o'zgaruvchi $[a, b]$ da o'zgarsa, ya'ni $\varphi(\alpha)=a,\ \varphi(\beta)=b,$ hamda $f(\varphi(t))$ murakkab funksiya $[\alpha, \beta]$ kesmada uzluksiz va aniqlangan bo'lsa, quyidagi formula o'rinli bo'ladi:

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt. \quad (1)$$

(1) formulaga o'zgaruvchini almashtirish usulida integral formulasi deyiladi.

$F(x)$ funksiya $f(x)$ ning boshlang'ichi bo'lsin. U holda, $F(\varphi(t))$ funksiya $f(\varphi(t))\cdot\varphi'(t)$ ning boshlang'ichi bo'ladi. Shuning uchun

$$\int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha)) = F(b) - F(a) = \int_{a}^{b} f(x)dx.$$

Demak, (1) formula hosil bo'ldi.

Yuqoridagilarni umumlashtirib, o'zgaruvchini almashtirish usulida integrallashni quyidagi ketma – ketlikda bajarish tavsiya qilinadi:

1. Imkoni bo'lsa, integral ostida berilgan ifodani soddalashtirish.
2. Yangi o'zgaruvchini kiritish ($x = \varphi(t)$).
3. Integralning yangi chegaralarini aniqlash.
4. Hosil bo'lgan integralni hisoblash.

1- misol. $\int_{2}^{3}(2x-1)^3 dx$ ni hisoblang.

Yechilishi: $2x-1=t$ almashtirishni bajaramiz. Uning ikkala tomonini differensiallaymiz:

$$d(2x-1) = du, \quad 2dx = du. \quad \text{Bundan} \quad dx = \frac{1}{2}du.$$

Integralning yangi chegaralarini topamiz. Buning uchun $2x-1=u$ dagi x ning o'rniga avval integralning yuqori chegarasi 3 ni, keyin esa quyi chegarasi 2 ni qo'yib hisoblaymiz:

$$u_{ю} = 2\cdot 3 - 1 = 5, \quad u_{к} = 2\cdot 2 - 1 = 3.$$

Demak, yangi chegaralar $u_{ю} = 5$ va $u_{к} = 3$ ekan. U holda,

$$\int_{2}^{3}(2x-1)^3 dx = \frac{1}{2}\int_{2}^{3} u^3 du = \frac{1}{2}\cdot\frac{u^4}{4}\Big|_{2}^{3} = \frac{1}{8}\left(5^4 - 3^4\right) = 68.$$

2- misol. Integralni hisoblang: $\int_{0}^{2}(2x+1)^4 dx$.

Yechilishi: Integral ostidagi $2x+1$ ni t o'zgaruvchi bilan almashtiramiz. U holda $2x+1=t$ ning differensiali quyidagicha bo'ladi: $2dx = dt.$ Bundan,

$dx = \frac{1}{2}dt.$ Endi yangi chegaralarni topamiz. Buning uchun $2x+1=t$ dagi x ning o'rniga yuqori chegara 2 ni, keyin esa quyi chegara 0 ni quyib hisoblaymiz:

$$x=2 \text{ da } t_{ю}=5, \quad x=0 \text{ da } t_{к}=1.$$

Demak, yangi integralning yuqori chegarasi $t_{ю}=5$, quyi chegarsi $t_{к}=1$ ga teng ekan. Yuqorida aytilganlarning analitik ifodasini keltiramiz:

$$\int_0^2 (2x+1)^4 dx = \left\{\begin{array}{l} 2x+1=t, \\ dx=\frac{1}{2}dt \end{array}\right| \left.\begin{array}{l} 2x+1=t, \ 2\cdot 2+1=t, \ t_{ю}=5; \\ 2x+1=t, \ 2\cdot 0+1=t, \ t_{к}=1. \end{array}\right\} =$$

$$= \int_1^5 t^4 \cdot \frac{1}{2}dt = \frac{1}{2}\cdot\frac{t^5}{5}\Bigg|_1^5 = \frac{t^5}{10}\Bigg|_1^5 = \frac{1}{10}\left(5^5-1^5\right) = \frac{1}{10}\cdot 3124 = 312{,}4.$$

3-misol. Integralni hisoblang: $\int_0^a \sqrt{a^2-x^2}\,dx, \quad a>0.$

Yechilishi: Bunda integral ostidagi ifodadagi x ni $a\sin t$ bilan almashtiramiz:

$$x = a\sin t.$$

U holda, $dx = a\sin tdt$ hosil bo'ladi. Endi integralning yangi chegaralarini topamiz:

$$x=a \text{ da } \quad a=a\sin t, \quad \sin t=1, \quad t_{ю}=\frac{\pi}{2};$$

$$x=0 \text{ da } \quad 0=a\sin t, \quad \sin t=0, \quad t_{к}=0.$$

Demak,

$$\int_0^a \sqrt{a^2-x^2}\,dx = \left\{\begin{array}{l} x=a\sin t, \\ dx=a\cos tdt \end{array}\right| \left.\begin{array}{l} t_{ю}=\frac{\pi}{2}, \\ t_{к}=0. \end{array}\right\} = \int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2 t}\cdot a\cos tdt =$$

$$= \int_a^{\frac{\pi}{2}} a\cdot\cos t\cdot a\cos tdt = a^2\int_0^{\frac{\pi}{2}}\cos^2 tdt = a^2\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt = \frac{a^2}{2}\int_0^{\frac{\pi}{2}}dt + \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\cos 2tdt =$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{a^2}{2}\left(\frac{\pi}{2} - 0 + \frac{1}{2}\cdot\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{1}{2}\cdot\sin 0\right) =$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} + \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 0\right) = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{a^2\pi}{4}.$$

4-misol. Integralni hisoblang: $\int_0^{\frac{1}{2}} \frac{x+1}{1+4x^2}\,dx.$

Yechilishi: Integral ostidagi ifodani ikkita kasrga ajratib, integrallar yig'indisiga keltiramiz va ularni alohida – alohida hisoblaymiz, ya'ni:

$$\int_0^{\frac{1}{2}} \frac{x+1}{1+4x^2}\,dx = \int_0^{\frac{1}{2}} \frac{x}{1+4x^2}\,dx + \int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx\,.$$

Birinchi integralda $1+4x^2 = t,$ ikkinchisida esa $2x = u$ almashtirishni bajaramiz. U holda, birinchi integralning yangi chegaralari: $t_ю = 2,$ $t_к = 1,$ ikkinchisiniki esa $u_ю = 1,$ $u_к = 0$ bo'ladi. Integralni hisoblashning analitik ifodasi quyidagicha:

$$\int_0^{\frac{1}{2}} \frac{x+1}{1+4x^2}\,dx = \int_0^{\frac{1}{2}} \frac{x}{1+4x^2}\,dx + \int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx =$$

$$= \left\{ \begin{pmatrix} 1+4x^2 = dt & \Big| t_ю = 2 \\ xdx = \frac{1}{8}dt & \Big| t_к = 1 \end{pmatrix}, \begin{pmatrix} 2x = u & \Big| u_ю = 2 \\ dx = \frac{1}{2}du & \Big| u_к = 1 \end{pmatrix} \right\} =$$

$$= \int_1^2 \frac{1}{8t}\,dt + \int_0^1 \frac{1}{2\left(1+u^2\right)}\,du = \frac{1}{8}\int_1^2 \frac{1}{t}\,dt + \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{8}\ln t\Big|_1^2 + \frac{1}{2}\,arctgu\Big|_0^1 =$$

$$= \frac{1}{8}\left(\ln 2 - \ln 1\right) + \frac{1}{2}\left(arctg1 - arctg0\right) = \frac{1}{8}\ln 2 + \frac{1}{2}\left(\frac{\sqrt{2}}{2} - 0\right) \approx 0{,}48.$$

Mustaqil yechish uchun mashqlar

Quyidagi integrallarni o'zgaruvchini almashtirish usuli yordamida hisoblang:

№13. $\int_4^5 (4-x)^3 dx$.

№14. $\int_0^1 \frac{1}{(3x+1)} dx$.

№15. $\int_1^2 \frac{1}{\sqrt{5x-1}} dx$.

№16. $\int_0^1 (2x^3+1)^4 x^2 dx$.

№17. $\int_0^2 9\sqrt{x^3+1}x^2 dx$.

№18. $\int_0^7 \frac{1}{\sqrt[3]{(8-x)^2}} dx$.

№19. $\int_0^3 x\sqrt{1+x}dx$.

№20. $\int_5^{13} \sqrt{2x-1}dx$.

№21. $\int_{-a}^{a} \sqrt{a^2-x^2}dx$.

№22. $\int_0^1 \sqrt{1-x^2}dx$.

№23. $\int_{\frac{\pi}{2}}^{\pi} \frac{2\sin x}{(1-\cos x)^2} dx$.

№24. $\int_0^3 2x^2\sqrt{9-x^2}dx$.

№25. $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \frac{\sin^2 4x+\cos^2 2x}{\sin^2 4x\cdot\cos^2 2x} dx$.

№26. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 x\cdot\cos xdx$.

№27. $\int_0^1 \frac{7e^x dx}{e^x+3} dx$.

№28. $\int_0^{\frac{\pi}{2}} 4e^{\sin x}\cdot\cos xdx$.

5§. Aniq integralni bo’laklab integrallash

$u(x)$ va $v(x)$ funksiyalar berilgan bo’lib, ular $[a, b]$ kesmada uzluksiz hosillalar ($u'(x)$ va $v'(x)$) ga ega bo’lsin. U holda, bizga ma’lum bo’lgan

$$(u(x)v(x))' = u'(x)v(x)+u(x)v'(x) \tag{1}$$

tenglikni $[a, b]$ kesmada integrallaymiz:

$$\int_a^b [u(x)v(x)]' dx = \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx. \tag{2}$$

(2)ni quyidagi ko’rinishda ham ifodalash mumkin:

$$\int_a^b (uv)' dx = \int_a^b u' v dx + \int_a^b u v' dx. \tag{3}$$

(3)ning chap tomonini quyidagicha yozish mumkin:

$$\int_a^b (uv)' dx = uv\Big|_a^b . \tag{4}$$

U holda (3):

$$uv\Big|_a^b = \int_a^b u' v dx + \int_a^b u v' dx \tag{5}$$

yoki $$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du. \tag{6}$$

hosil bo'ladi. Bu formulaga ***aniq integralni bo'laklab integrallash formulasi*** deyiladi.

1-misol. $\int_a^{\frac{\pi}{2}} x \cos x dx$ integralni hisoblang.

Yechilishi: Berilgan aniq integralni bo'laklab integrallash usulida yechish uchun

$$x = u \text{ va } \cos x dx = dv$$

deb belgilaymiz. U holda,

$$dx = du \quad \text{va } v = \sin x$$

bo'ladi. Bularni (6)ga qo'yamiz:

$$\int_0^{\frac{\pi}{2}} x \cos x dx = \left\{ \begin{array}{l|l} x = u & \cos x dx = dv, \\ dx = du & v = \int \cos x dx = \sin x \end{array} \right\} = x \cdot \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx =$$

$$= \frac{\pi}{2} \cdot \sin \frac{\pi}{2} - 0 \cdot \sin 0 + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \cdot 1 + \cos \frac{\pi}{2} - \cos 0 = \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1.$$

2-misol. Hisoblang: $\int_0^1 x e^x dx.$

Yechilishi: (6) formulani qo'llaymiz:

$$\int_0^1 xe^x dx = \left\{\begin{matrix} x = u \\ dx = du \end{matrix}\right. \left| \begin{matrix} e^x dx = dv, \\ v = \int e^x dx = e^x \end{matrix} \right\} = \left[xe^x\right]_0^1 - \int_0^1 e^x dx = 1\cdot e^1 - 0\cdot e^0 - \int_0^1 e^x dx =$$

$$= e - e^t\Big|_0^1 = e - e - e^0 = 1.$$

3-misol. Hisoblang: $\int_0^{\frac{\pi}{6}} \cos xdx.$

Yechilishi:

$$\int_0^{\frac{\pi}{6}} x\cos 3xdx = \left\{\begin{matrix} x = u \\ dx = du \end{matrix}\right. \left| \begin{matrix} \cos 3xdx = dv, \\ v = \frac{1}{3}\sin 3x \end{matrix} \right\} = \left[\frac{x}{3}\cdot\sin 3x\right]_0^{\frac{\pi}{6}} - \frac{1}{3}\int_0^{\frac{\pi}{6}} \sin 3xdx = \frac{1}{9}\left(\frac{\pi}{2} - 1\right).$$

4-misol. Hisoblang: $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x}dx.$

Yechilishi: Belgilashlar kiritamiz, so'ngra (6) formulani qo'llaymiz:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x}dx = \left\{\begin{matrix} x = u \\ dx = du \end{matrix}\right. \left| \begin{matrix} \frac{dx}{\sin^2 x} = dv, \\ v = \int \frac{dx}{\sin^2 x} = -ctgx \end{matrix} \right\} = -x\cdot ctgx\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} ctgxdx =$$

$$= -xctgx\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} ctgxdx = \frac{\pi\sqrt{3}}{6} + \ln\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\pi\sqrt{3}}{6} + \ln 1 - \ln\frac{1}{2} =$$

$$= \frac{\pi\sqrt{3}}{6} - \ln\frac{1}{2} = \frac{\pi\sqrt{3}}{6} - \ln 1 + \ln 2 = \frac{\pi\sqrt{3}}{6} + \ln 2.$$

Mustaqil yechish uchun mashqlar

№29. $\int_{-1}^{2} \left(x^2 - 1\right)^3 xdx$.

№33. $\int_1^3 \ln xdx$.

№30. $\int_{1}^{\sqrt{3}} \frac{128x}{\left(x^2+1\right)^5} dx$.

№31. $\int_{\frac{1}{2}}^{2} (2x-1)^3 dx$.

№32. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin 2x dx$.

№34. $\int_{0}^{\pi} x \sin x dx$.

№35. $\int_{1}^{e} \frac{dx}{x}$.

№36. $\int_{0}^{2} x e^x dx$.

2- BOB. ANIQ INTEGRALNING TADBIQLARI

1§. Yassi figuralarning yuzini hisoblash

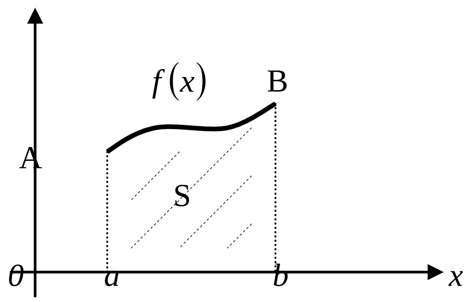
Yassi figuralarning yuzini hisoblashda aniq integralni qo’llashning bir necha hollari mavjud. Bunda chegara funksiyalarining joylashuv vaziyatlari muhim ahamiyatga ega. Ba’zi hollarini ko’rib o’tamiz.

1)Agar $y = f(x)$ funksiya OX o’qining yuqori (manfiy bo’lmagan) qismida joylashgan hamda uzluksiz bo’lib, $x = a$ va $x = b$ to’g’ri chiziq kesmalari bilan chegaralangan bo’lsa, hosil bo’lgan egri chiziqli trapesiya yuzi

y

$$S = \int_a^b y dx \quad \text{yoki} \quad S = \int_a^b f(x) dx \qquad (1)$$

formula yordamida topiladi.



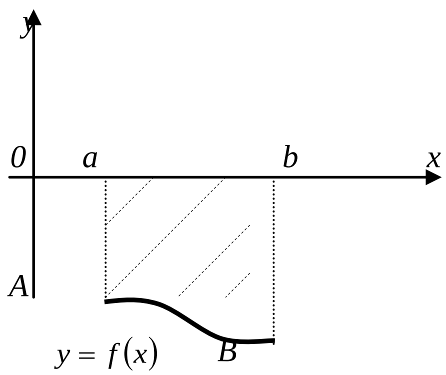
Misol: $y = x^2 + 1, \quad y = 0, \ x = -1, \quad x = 2$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechilishi: Shartga asosan figura $y = x^2 + 1$ egri chiziq, absissalar o’qi ($y = 0$) hamda $x = -1$ **va** $x = 2$ to’g’ri chiziqlar bilan chegaralangan. U holda, (1) formuladan foydalanib, quyidagi integralni hisoblaymiz:

$$S = \int_{-1}^{2} (x^2 + 1) dx = \left(\frac{x^3}{3} + x \right) \Bigg|_{-1}^{2} = \frac{2^3}{3} - \frac{(-1)^3}{3} + 2 - (-1) = 6.$$

Demak, berilgan egri chiziqli trapesiyasimon figuraning yuzi 6 ga teng ekan.

2) Agar $y = f(x)$ funksiya OX o’qining pastki qismida joylashgan hamda uzluksiz bo’lib, $x = a$ va $x = b$ to’g’ri chiziq kesmalari bilan chegaralangan bo’lsa, hosil bo’lgan egri chiziqli trapesiyasimon figuraning yuzi quyidagi formula yordamida topiladi:



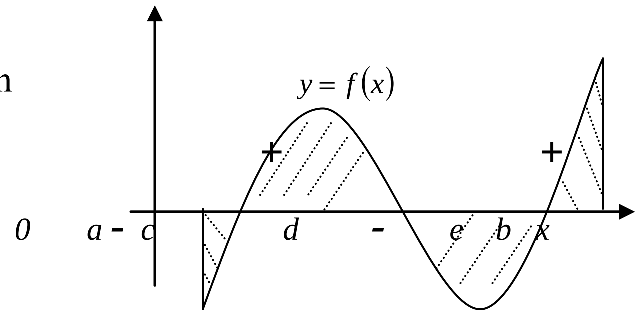
$$S = -\int_a^b y dx \quad \text{yoki} \quad S = -\int_a^b f(x) dx. \qquad (1)$$

Misol: $y=-x^2-2,\quad y=0,\ x=-1,\quad x=1$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechilishi: Berilgan masalani yechish uchun (2) formuladan foydalanib, chegaralari -1 va 1 dan iborat bo'lgan quyidagi aniq integralni hisoblaymiz:

$$S=-\int_{-1}^{1}\left(-x^2-2\right)dx=\int_{-1}^{3}\left(x^2+2\right)dx=\left(\frac{x^3}{3}+2x\right)\Bigg|_{-1}^{1}=3\frac{2}{3}.$$

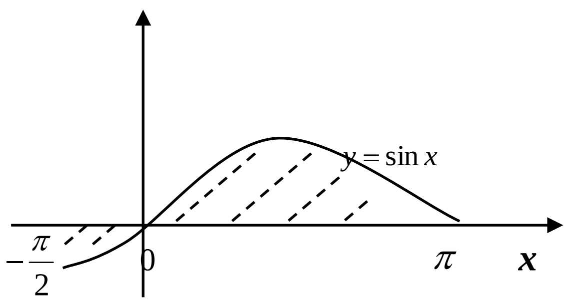
3) $y=f(x)$ uzluksiz funksiya grafigi $[a,b]$ kesmada OX o'qini chekli sondagi nuqtalarda kesib o'tsin. U holda, $[a,b]$ kesma funksiyaning ishorasi almashinishiga asoslanib, bir xil ishorali qismlari alohida –alohida kesmachalarga ajratiladi, ya'ni $[a;c]$, $[c;d]$, $[d;e]$ va $[e,b]$. U holda izlangan S yuza hosil bo'lgan yuzachalarning algebraik yig'indisidan iborat bo'ladi. Bunda qism funksiyalarning ishoralari e'tiborda bo'ladi. Izlanayotgan S yuza quyidagi integrallarning algebraik yiqindilari yordamida topiladi:



$$S=-\int_{a}^{c}f(x)dx+\int_{c}^{d}f(x)dx-\int_{d}^{e}f(x)dx+\int_{e}^{b}f(x)dx. \qquad (3)$$

Misol: $y=\sin x,\ y=0,\ x=-\dfrac{\pi}{2}$ va $x=\pi$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechilishi: Berilganlarga hamda chizmalarga asosan barcha $x\in\left[-\dfrac{\pi}{2};0\right]$lar uchun $\sin x\le 0$ va barcha $x\in[0;\pi]$lar uchun $\sin x\ge 0$dir. U holda, (3) formulaga asosa:



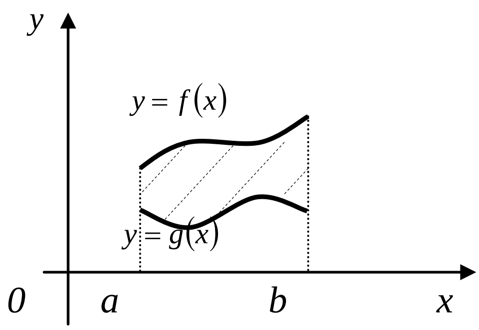
$$S=-\int_{-\frac{\pi}{2}}^{0}\sin xdx+\int_{0}^{\pi}\sin xdx=\cos x\Big|_{-\frac{\pi}{2}}^{0}-\cos x\Big|_{0}^{\pi}=$$

$$=\cos 0-\cos\left(-\frac{\pi}{2}\right)-\cos\pi+\cos 0=1-0-(-1)+1=3.$$

4) Agar figura $[a,b]$ kesmada ikkita uzluksiz $f(x)$ va $g(x)$ funksiyalar, $x=a$ hamda $x=b$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, uning yuzi quyidagi formula yordamida hisoblanadi:

$$S=\int_a^b\left(f(x)-g(x)\right)dx. \qquad (4)$$

Bunda $f(x)\geq g(x)$ va $a\leq x\leq b$ dir.



Misol: $y=x+2$ va $y=x^2$ chiziqlar bilan chegaralangan figuraning yuzini toping.

Yechilishi: Integrallash chegaralarini, ya'ni a va b ni berilgan chiziq tenglamalarini o'zaro tenglashtirib, topamiz:

$$x+2=x^2,\qquad x^2-x-2=0.$$

Bundan, $x_1=-2$, $x_2=1$ yani $a=-2$, $b=1$. U holda, (4) formulaga asosan:

$$S=\int_{-2}^{1}\left((x+2)-x^2\right)dx=\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-2}^{1}=\frac{1}{2}-\frac{1}{2}+2-(-4)-\frac{1}{3}-\left(-\frac{8}{3}\right)=8\frac{1}{3}.$$

Demak, izlanayotgan figuraning yuzasi $S=8\frac{1}{3}$ dan iborat ekan.

Quyida ba'zi egri chiziqli figuralarning yuzalarini topish formulalarni qaraymiz.

Ellipsning yuzi

Ma'lumki, ellipsning tenglamasi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (5)$$

dan iborat. Ellipsni 4 ta chorakka ajratib, uning bir bo'lagi, ya'ni $\frac{1}{4}S$ ni topish yetarlidir. (5) ga asosan

$$y = \frac{b}{a}\sqrt{a^2 - x^2}\ . \qquad (6)$$

(1) formulaga asosan $\quad \frac{1}{4}S = \int_a^b ydx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}dx.$

Quyidagi almashtirishlar olamiz: $\quad x = a\sin t,\quad dx = a\sin tdt.$

U holda, integralning yangi chegaralarini aniqlaymiz: $0 = a\sin t$ va $a = a\sin t$ lardan $\alpha = 0$ va $\beta = \frac{\pi}{2}\cdot$ Bulardan ,

$$\frac{1}{4}S = \frac{b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\cdot a\cos tdt = ab\int_0^{\frac{\pi}{2}} \cos^2 tdt = \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)dt =$$

$$= \frac{ab}{2}\left(1 + \frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}ab.$$

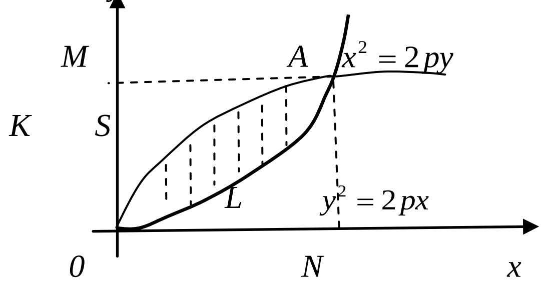
Demak, $\frac{1}{4}S = \frac{\pi}{4}ab.$ Bundan, $\qquad S = \pi ab. \qquad (7)$

Ellips yuzini topishning umumiy fomulasi quyidagi ko'rinishda bo'ladi:

$$S = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}dx = \pi ab. \qquad (8)$$

Ikkita parbolaning kesishmasidan hosil bo'lgan figuraning yuzi

$y^2 = 2px$ va $x^2 = 2py$ parabolalar berilan. S yuza $OKAN$ va $OLAN$ yuzalar ayirmasiga teng. Berilgan paralar $O(0;\,0)$ va $A(2p;\,2p)$ nuqtalarda kesishishadi. Shuning uchun



$$S = \int_0^{2p} \sqrt{2px}dx - \int_0^{2p} \frac{x^2}{2p}dx = \frac{4}{3}p^2 = \frac{(2p)^2}{3}. \qquad (9)$$

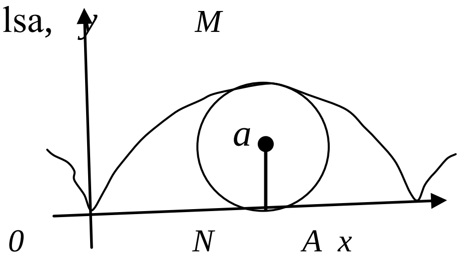
Demak, izlangan $OKALO$ yuza $OMAN$ kvadratining uchdan bir qismidan iborat.

Sikloidaning yuzi

$x = a(t - \sin t)$ va $y = a(1 - \cos t)$ berilgan bo'lsa,

$$S_{OMANO} = \int_0^{2\pi} y dx = a^2 \int_0^{2\pi} (1 - \cos t)^2 dt = 3\pi a^2. \quad (10)$$

Demak, sikloidaning yuzi $S = 3\pi a^2$ iborat ekan .



Qutb koordinatalarida yuzani topish

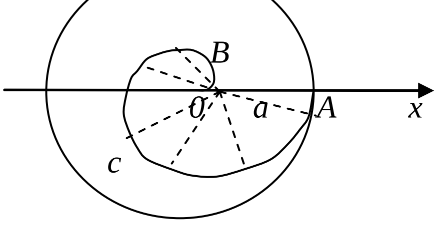
AOB sektor AB yoy, OA va OB nurlar bilan chegaralangan bo'lsin. Bunday sektorning yuzi qutb koordinatalarida quyidagi formula yordamida topiladi:

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2 d\varphi. \quad (11)$$

Bunda r-qutb radiusi, φ -qutb radiusining OX o'q bilan tashkil qilgan burchagi, ya'ni qutb burchagi.

Arximed spirali birinchi o'ramining OX o'qi bilan chegaralangan qismining yuzi

Arximed spiralining birinchi o'rami O nuqtadan (qutb markazidan) boshlanib, A nuqtada tugagan bo'lsin.



U holda, qutb burchaklari $\varphi_1 = 0$ va $\varphi_2 = 2\pi$ bo'ladi. Qutb radiusi

$$r = \frac{a}{2\pi}\varphi \quad (12)$$

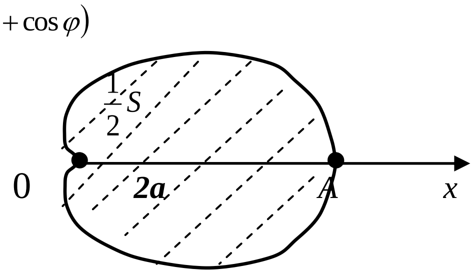
dan iboratdir. Bunda a - spiral qadami, ya’ni $OA = a$.

(11) formulaga asosan BCA egri chiziq va OA spiral qadami bilan chegaralangan figuraning yuzi quyidagi formula yoramida hisoblanadi:

$$S = \frac{1}{2}\int_0^{2\pi} r^2 d\varphi = \frac{a^2}{8\pi^2}\int_0^{2\pi} \varphi^2 d\varphi = \frac{1}{3}\pi a^2. \qquad (13)$$

Kardioidaning yuzi

$\rho = a(1+\cos\varphi)$- kardioda bilan chegaralangan yuzani hisoblash talaba qilinsin. Ma’lumki, kardoida OX qutb o’qiga nisbatan simmetrik. Shuning uchun uning yuqori qismi yuzasini topib, natija ikkilantirilsa, yetarli bo’ladi. U holda,



$$\frac{1}{2}S = \frac{1}{2}\int_0^{\pi} \rho^2 d\varphi. \qquad (14)$$

Bundan,

$$S = \int_0^{\pi} \rho^2 d\varphi = a^2\int_0^{\pi} (1+\cos\varphi)^2 d\varphi = a^2\left(\int_0^{\pi} d\varphi + 2\int_0^{\pi}\cos\varphi d\varphi + \int_0^{\pi}\cos^2\varphi d\varphi\right). \qquad (15)$$

$$\int_0^{\pi} d\varphi = \pi, \qquad \int_0^{\pi}\cos\varphi d\varphi = \sin\varphi\Big|_0^{\pi} = 0 \quad \text{hamda}$$

$$\int_0^{\pi}\cos^2\varphi d\varphi = \frac{1}{2}\int_0^{\pi}(1+\cos 2\varphi)d\varphi = \frac{1}{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\Bigg|_0^{\pi} = \frac{\pi}{2} \quad \text{ekanligini}$$

hisobga olsak, (15) quyidagidan iborat bo’ladi:

$$S = \int_0^{\pi} \rho^2 d\varphi = a^2\left(\pi + 0 + \frac{\pi}{2}\right) = \frac{3\pi}{2}a^2. \qquad (16)$$

Demak, kardioidaning yuzi $S = \frac{3\pi}{2}a^2$ ekan.

Mustaqil yechish uchun mashqlar

№37. $y=x^2$ va $y^2=x$ parabolalar bilan chegaralangan figuraning yuzini toping.

№38. $xy=a^2$ giperbola va $y=0$, $x=b$, $x=2b$ $(b>0)$ to'g'ri chiziqlar bilan chegaralangan figuraning yuzini toping.

№39. OX o'q hamda $y=2x-x^2$ parabola bilan chegaralangan figura yuzini toping.

№40. $y=\dfrac{a^3}{a^2+x^2}$ $(a>0)$ va absissalar o'qi bilan chegaralangan figura yuzni hisoblang.

№41. OX o'q va $y=2(x-1)(3-x)$ parabola bilan chegaralangan figura yuzini toping.

№42. $y=\sqrt{2(x-1)}$ parabola va $x=3$ to'g'ri chiziq bilan chegaralangan yuzani toping.

№43. $y=2^x$ egri chiziq va $y=2,$ $x=0$ to'g'ri chiziqlar bilan chegaralangan yuzani hisoblang.

№44. $y=1-x^2$ va $y=x^2-7$ parabolalar bilan chegaralangan yuzani toping.

№45. $x=a\cos^3 t$ va $y=a\sin^3 t$ astroidalar bilan chegaralangan yuzani toping (bunda $0\le t\le 2\pi$).

№46. $\rho^2=a^2\cos 2\varphi$ lemniskatani yasang va bu egri chiziq bilan chegaralangan barcha sohalar yuzini toping.

2§. Yoy uzunligini hisoblash

$y=f(x)$ egri chiziq $[a,b]$ kesmada berilgan bo'lib, yassi va uzluksiz bo'lsin. U holda, funksiya shu kesmada uzluksiz hosilaga ega bo'ladi. Egri chiziqni n ta bo'lakka ajratamiz va bo'linish nuqtalarini kesmalar yordamida ketma- ket tutashtiramiz. Natijada, hosil bo'lgan qism yoychalarning har biriga bitta kesmacha mos keladi. Agar egri chiziqni bo'lishni davom ettirsak, qism yoychalarning uzunligiga ularga mos keluvchi kesmalarning uzunligi yaqinlashadi. Funksiya grafigining bo'linish nuqtalaridan OX o'qiga proyeksiyalar tushiramiz. Undagi har ikki nuqta orasidagi masofalarni $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ lar bilan belgilaymiz. Ixtiyoriy M_{i-1} va M_i nuqtalar ordinatalari farqini Δy_i bilan belgilaymiz. U holda, Pifagor teoremasiga asosan $M_{i-1}\,M_i$ kesmaning uzunligi quyidagicha bo'ladi.

$$|M_{i-1}M_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}. \qquad (1)$$

Hosilaning ta'rifiga asosan: $\frac{\Delta y_i}{\Delta x_i} = y'$, u holda

$$|M_{i-1}M_i| = \sqrt{1 + y'^2} \cdot \Delta x_i. \qquad (2)$$

Kesmalar hosil qilgan siniq chiziqning uzunligi

$$\sum_{i=1}^{n} \sqrt{1 + y'^2} \cdot \Delta x_i \qquad (3)$$

dan iborat bo'ladi. Egri chiziqning uzunligi l ni topish uchun (3) ning $\Delta x_i \to 0$ dagi limitini olish lozim, ya'ni:

$$l = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + y'^2} \cdot \Delta x_i\,. \qquad (4)$$

(4) – integral yig'indidan iborat. Uni integral ko'rinishida ifodalash mumkin:

$$l = \int_a^b \sqrt{1 + y'^2}\,dx \quad \text{yoki} \quad l = \int_a^b \sqrt{1 + f'^2(x)}\,dx. \qquad (5)$$

(5) formula yassi egri chiziq, ya'ni ***yoyning uzunligini*** topish formulasidir.

To'g'ri burchakli koordinatalar sistemasida ***yoy differensiali*** quyidagi formula ko'rinishida ifodalanadi:

$$dl = \sqrt{1 + y'^2}\,dx \quad \text{yoki} \quad dl = \sqrt{(dx)^2 + (dy)^2}. \qquad (6)$$

1-misol. $O(0;0)$ va $N(2;3)$ nuqtalar bilan chegaralangan $y=x^2+2$ parabola yoyining uzunligini toping.

Yechilishi: Parabola tenglamasidan hosila olamiz:

$$y'=\left(x^2+2\right)'=2x, \text{ ya'ni } y'=2x.$$

(5)- formulani qo'llaymiz:

$$l=\int_0^2\sqrt{1+(2x)'}dx=\int_0^2\sqrt{1+4x^2}dx=2\int_0^2\sqrt{\frac{1}{4}+x^2}dx=$$

$$=2\left(\frac{x}{2}\sqrt{\frac{1}{4}+x^2}+\frac{1}{8}\ln\left|x+\sqrt{\frac{1}{4}+x^2}\right|\right)\Bigg|_0^2=2\left(\sqrt{\frac{17}{4}}+\frac{1}{8}\ln\left(2+\sqrt{\frac{17}{4}}\right)-\frac{1}{8}\ln\frac{1}{2}\right)=$$

$$=\sqrt{17}+\frac{1}{4}\ln\left(4+\sqrt{17}\right)\approx 4{,}65.$$

Demak, izlanayotgan ON yoyning uzunligi 4,65 uzunlik o'lchov birligiga teng ekan.

2-misol. $O(0;0)$ va $N\left(\sqrt{3};\frac{3}{2}\right)$ nuqtalar orasidagi $y=\frac{x^2}{2}$ parabola yoyining uzunligini toping.

Yechilishi: Berilgan parabolaning $y=\frac{x^2}{2}$ tenglamasini differensiallab, $\frac{dy}{dx}=x$ ni, so'ngra, (5) formulaga asosan ON yoyning uzunligini topamiz:

$$l=\int_0^{\sqrt{3}}\sqrt{1+x^2}dx=\left(\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)\right)\Bigg|_0^{\sqrt{3}}\approx 2{,}4.$$

Demak, yoy uzunligi 2,4 uzunlik o'lchov birligidan iborat ekan.

3-misol. $0\le t\le T$ da $x=a\cos t,\quad y=a\sin t$ aylananing uzunligini toping.

Yechilishi: x' va y' larni topamiz:

$$dx=-a\sin t dt \quad \text{va} \quad dy=a\cos t dt.$$

U holda, $dt=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a^2\sqrt{\sin^2 t+\cos^2 t}dt=adt.$

Bundan, $l=\int_0^T a dt = aT$.

Mustaqil yechish uchun mashqlar.

№46. $O(0;0)$ va $N(3;2\sqrt{3})$ nuqtalar bilan chegaralangan $9y^2=4x^3$ egri chiziqli yoyning uzunligini toping.

№47. $M(2;-1)$ va $N(5;-8)$ nuqtalar bilan chegaralangan $y^2=(x-1)^3$ egri chiziqli yoyning uzunligini toping.

№48. $x^2+y^2=r^2$ aylananing uzunligini toping.

№49. $y=x^2$ parabola yoyining $O(0;0)$ va $N\left(\frac{\sqrt{3}}{2};\frac{3}{4}\right)$ nuqtalar orasidagi uzunligini toping.

№50. $y=4-x^2$ parabola yoyining OX o'qi bilan kesishish nuqtalari orasidagi qismining uzunligini hisoblang.

№51. $x^2-2y-3=0$ egri chiziqning OX o'qi bilan kesishish nuqtalari orasidagi yoy uzunligini toping.

№52. $y^2=x^3$ egri chiziq yoyining $3x-4=0$ to'g'ri chiziq bilan kesishgan qismi uzunligini toping.

№53. $x_1=0$ va $x_2=8$ bo'lganda $y=\frac{2}{3}x^{\frac{3}{2}}$ yarimkubik parabola yoyining uzunligini toping.

№54. $x=a(t-\sin t)$, $y=a(1-\cos t)$, $(0\le t\le 2\pi)$ sikloidaning bitta arki yoyining uzunligini toping.

№55. $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ zanjir yoyining $M(0;a)$ va $N\left(a;\frac{a(e^2+1)}{2e}\right)$ nuqtalar orasidagi uzunligini toping.

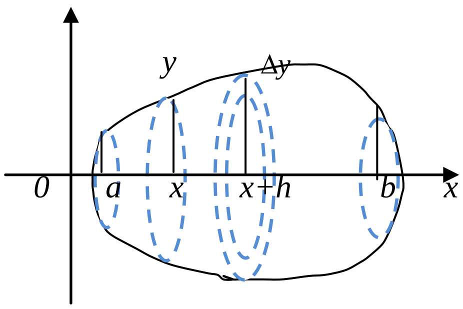
№56. $0\le t\le 2\pi$ bo'lganda $x=a\cos^2 t$, $y=a\sin^2 t$ egri chiziq yoyining uzunligini toping.

№57. $x^2 + y^2 = R^2$ aylana yoyining koordinatalar tekisligidagi birinchi chorakda yotgan qismi uzunligini toping.

3§. Aylanish jismini hajmi

$y = f(x)$ formula bilan berilgan AB egri chiziqning $[a, b]$ kesmada OX o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini topish talab qilinsin.

y



Aylanish jismini OX ga perpendikulyar tekislikdar bilan n ta bo'laklarga ajratamiz. Perpendikulyar tekisliklarning biri 0 nuqtadan a masofada, ikkinchi tekislik x masofada, keyingisi esa $x+h$ masofada bo'lsin. Bunda, h - orttirma bo'lib, $h = dx$ dir. U holda, jismning birinchi ikki tekislik bilan kesilgan qismining hajmi $v(x)$, undan keyingi qismining hajmi esa $v(x) + \Delta v(x)$dan iborat bo'ladi.

Birinchi silindrsimon jismning balandligi $h = dx$, asos radiusi $y = f(x)$; ikinchisining balandligi ham $h = dx$, asos radiusi $y + \Delta y$. U holda, birinchi jism hajmi $\pi y^2 dx$, ikkinchisiniki esa $\pi(y + \Delta y)^2 dx$ bo'ladi. Ikki silindr orasidagi Δv orttirma hajm $2\pi hy \cdot \Delta y$dan iborat bo'ladi. Ammo Δv hajm $\Delta y \to 0$ va $h \to 0$da cheksiz kichik miqdor bo'lib, 0ga intiladi. Shuning uchun hajmning differensiali kichik silindrsimon jismning hajmi $\pi y^2 dx$ bo'ladi. Buni integrallaymiz:

$$v = \int_a^b \pi y^2 dx = \pi \int_a^b (f(x))^2 dx. \tag{1}$$

(1) tenglik ***aylanish jismining hajmini*** topish formulasidan iborat.

1-misol. Asos radiusi $MN = r$ va balandligi $ON = h$ bo'lgan aylanish paraboloidi segmentining hajmini toping.

Yechilishi: Ma'lumki, parabola tenglamasi $y^2 = 2px$ bo'lib, parabolaning ixtiyoriy $N(h; r)$ nuqtadan o'tishini e'tiborga olsak.

$$r^2 = 2ph. \tag{2}$$

Parabola tenglamasi va (2) dan $y = \frac{r}{\sqrt{h}}\sqrt{x}.$ (3)

bo'ladi. Bundan, $y^2 = \frac{r^2}{h}x.$ (4)

U holda, (1) formulaga asosan ***paraboloid segmentining hajmi*** quyidagicha bo'ladi:

$$v = \pi\int_a^b \frac{r^2 x}{h}dx = \frac{1}{2}\pi r^2 h. \tag{5}$$

2-misol. $y = x^2$ parabola, OX o'q va $x = 1$ to'g'ri chiziq bilan chegaralangan egri chiziqli trapesiyaning OX o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping.

Yechilishi: (1) formuladan foydalanamiz. Bunda, $f(x) = x^2$, $a = 0$ va $b = 1$ larni formulaga qo'yib, integralni hisoblaymiz:

$$v = \pi\int_0^1 (f(x))^2 dx = \pi\int_0^1 (x^2)^2 dx = \pi\int_0^1 x^4 dx = \pi\cdot\frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}.$$

Demak, jismning hajmi $\frac{\pi}{5}$ dan iborat ekan.

4§. Ko'ndalang kesim yuzi ma'lum bo'lgan jismning hajmi

$x = a$ va $x = b$ kesmalardan o'tgan hamda OX o'qqa perpendikulyar bo'lgan tekisliklar bilan chegaralangan V jismning hajmini topish talab qilinsin. U holda jismni n ta o'zaro parallel bo'lgan tekisliklar bilan OX o'qiga parallel holda bo'laklarga ajratamiz. Ixtiyoriy $x = x_{i-1}$ va $x = x_i$ tekisliklar bilan chegaralangan jismning hajmi Δv_i, asos yuzi $S(x_i)$, balandligi Δx_i bo'lsin. U holda,

$$\Delta v_i = S(x_i)\cdot \Delta x_i \qquad (6)$$

o'rinli bo'ladi. Jismning umumiy hajmi quyidagicha bo'ladi:

$$v = \lim_{\varepsilon\to 0}\sum_{i=1}^{n} S(x)\Delta x_i\ . \qquad (7)$$

Bunda, ε balandlik Δx_i larning eng kattasi. (7) tenglik integral yig'indidan iborat. Shuning uchun (7) ni quyidagicha ifodalash mumkin:

$$v = \int_a^b S(x)dx. \qquad (8)$$

(8) - ***ko'ndalang kesim yuzi ma'lum bo'lgan jismning hajmini*** topish formulasi.

Misol. $y^2 = 4x,\ \ y = 0,\ \ x = 0,\ \ x = 4$ chiziqlar bilan chegaralangan hamda OX o'q atrofida aylanishdan hosil bo'lgan jismning hajmini toping.

Yechilishi: Hosil bo'ladigan jism aylanish paraboloididan iborat bo'ladi. Uning hajmini (8) formula yordamida topamiz. Bunda $a = 0$, $b = 4$ va $S(x) = 4x$ dir.

$$v = \pi\int_0^4 4xdx = \pi\cdot 4\cdot\frac{x^2}{2}\Big|_0^4 = 2\pi x^2\Big|_0^4 = 32\pi.$$

Ba'zi jismlarning hajmini topish formulalari:

Piramidaning hajmi:

$$v = \frac{S}{H^2}\int_0^H x^2dx = \frac{1}{3}SH.$$

Paraboloid segmentiining hajmi:

$$v = \pi\int_0^h \frac{r^2x}{h}dx = \frac{1}{2}\pi r^2 h$$

(ya'ni silindr hajmining yarimiga teng- Arximed tadqiqoti)

Elliptik asosli konus:

$$v = \frac{1}{3} Sa \qquad (a \text{ - kata yarim o'q})$$

Ellipsoid:

$$v = \pi \int_a^b \frac{b^2}{a^2}\left(a^2 - x^2\right)dx = \frac{4}{3}\pi ab^3.$$

Sharning hajmi:

$$v = \pi \int_{-r}^{r} y^2 dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right)dx = \frac{4}{3}\pi r^3.$$

Mustaqil yechish uchun mashqlar

№56. Birinchi chorakda yotgan, koordinata o'qlari hamda $y = 4 - x^2$ parabola yoyi bilan chegaralangan figuraning OX o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping.

№57. Birinchi chorakda yotgan hamda $x^2 + y^2 = 1,$ $y = x$ va $y = 0$ chiziqlar bilan chegaralangan tekislikning OX o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping.

№58. $y = \frac{3}{4}x^2$ parabola va $y = 1 - x$ to'g'ri chiziqning kesishishidan hosil bo'lgan figuraning OX o'qi arofida aylanishidan hosil bo'lgan jismning hajmini toping.

№59. OY o'q, $y = \sin x$ - sinusoida yoyi hamda $y = \cos x$ - kosinusoida yoyi bilan chegaralangan figuraning OX o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping.

№60. $y = \cos x$ $\left(0 \le x \le \frac{\pi}{2}\right)$ kosinusoida va koordinatalar markazidan o'tib, OX o'qiga perpendikulyar bo'lgan tekislik bilan chegaralangan figuraning OX o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping.

№61. Quyidagi chiziqlar bilan chegaralangan hamda OX o'qi atrofida aylanishidan hosil bo'lgan jismlar hajmini toping:

a) $y^2 = 4(x-2), \quad y=0, \quad x=3, \quad x=6;$ v) $y=\sin x, \quad y=0, \quad x=0, \quad x=\pi;$

b) $y=x^2-4, \quad y=0;$ g) $y=4-x^2, \quad x-y+2=0.$

№62. Quyidagi chiziqlar bilan chegaralangan va OY o'qi atrofida aylanishidan hosil bo'lgan jismlar hajmini toping:

a) $y=x^2, \quad y=1, \quad y=4,$ va $x=0;$ v) $y^2=9x, \quad y=3x;$

b) $y=x^2+1, \quad y=5;$ g) $y^2=2x, \quad 2x+2y-3=0.$

3- BOB. ANIQ INTEGRALLARNI TAQRIBIY HISOBLASh

1§. To'g'ri to'rtburchaklar formulasi

To'g'ri burchakli koordinatalar sistemasida $y=f(x)$ uzluksiz funksiya berilgan bo'lsin. Integrallash oralig'i $[a,b]$ ni $x_0, x_1, x_2, x_{3,} \dots, x_n$ nuqtalar yordamida n ta teng bo'laklarga ajratamiz. U holda, bir bo'lakning uzunligi

$$h=\frac{x_n-x_0}{n} \quad \text{yoki} \quad h=\frac{b-a}{n} \tag{1}$$

dan iborat bo'ladi. $y=f(x)$, $y=0$, $x=a$, $x=b$ chiziqlar bilan chegaralangan egri chiziqli figura ham n ta bo'lakka ajralib, ordinatalari $y_0, y_1, y_2, \ldots, y_n$ bo'ladi. Bo'laklar kami va ortig'i bilan to'g'ri to'rtburchaklarga to'ldiriladi.

Kami bilan to'ldirilgan to'g'ri to'rtburchaklarning yuzalari yig'indisi:

$$S=\int_a^b ydx \approx \frac{b-a}{n}\left(y_0+y_1+y_2+\ldots+y_{n-1}\right). \tag{2}$$

Ortig'i bilan to'ldirilgan yuzalar yig'indisi esa

$$S=\int_a^b ydx \approx \frac{b-a}{n}\left(y_1+y_2+\ldots,+y_n\right). \tag{3}$$

dan iborat bo'ladi.

Agar yuzalar yana ikkiga bo'linsa, (2) va (3) larning xatosi kamaya boradi. $n \to \infty$ va $h \to 0$ bo'lsa, ular aniq integralning yanada haqiqiyroq qiymatini beradi. (3) ni quyidagicha ifodalasa ham bo'ladi:

$$S=\int_a^b ydx \approx \frac{b-a}{n}\left(y_{\frac{1}{2}}+y_{\frac{3}{2}}+\ldots,+y_{\frac{2n-1}{2}}\right). \tag{3'}$$

2§. Trapesiyalar formulasi

Trapesiyalar formulasini keltirib chiqarishda $[a,b]$ kesmani n ta teng bo'laklarga ajratamiz. U holda, bo'linish nuqtalari (ya'ni absissalar) $x_0=a, x_1, x_2, \ldots, x_n=b$ bo'ladi. Kesmalar o'zaro teng bo'lganligi uchun ularni umumiy holda h bilan belgilasak,

$$x_0=a,\ \ x_1=x_0+h,\ \ x_2=x_1+h, \ldots, x_n=x_{n-1}+h=b \tag{4}$$

bo'ladi. Ordinata chiziqlari mos ravishda

$$y_0, y_1, y_2, \ldots, y_n \tag{5}$$

lardan iborat bo'lib, ular ketma- ket tutashtirilsa, siniq chiziqlar hosil bo'ladi. Absissa, ordinatalar va funksiya chiziqlari bilan chegaralangan egri chiziqli trapesiyalar hosil bo'lib, ularning yuzalari

$$S_1=\frac{y_0+y_1}{2}h,\ S_2=\frac{y_1+y_2}{2}h,\ldots,S_n=\frac{y_{n-1}+y_n}{2}h \qquad (6)$$

lardan iborat. U holda, egri chiziqli trapesiyasimon S figuraning yuzi quyidagicha bo'ladi:

$$S=\int_a^b ydx=S_1+S_2+\ldots+S_n=\frac{y_0+y_1}{2}h+\frac{y_1+y_2}{2}h+\ldots+\frac{y_{n-1}+y_n}{2}h=$$

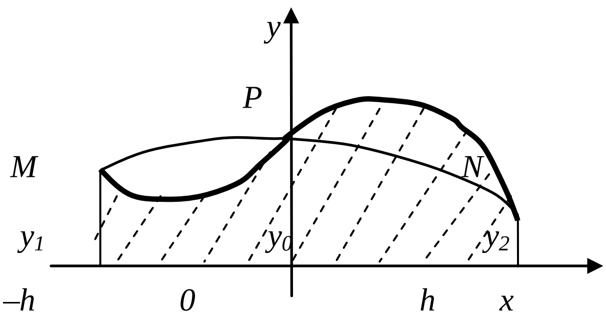
$$=\frac{h}{2}\left(\frac{y_0+y_n}{2}+y_1+y_2+\ldots+y_{n-1}\right). \qquad (7)$$

Bunda $h=\frac{b-a}{n}$ dan ibort.

Bu formulaga ***trapesiyalar formulasi*** deyiladi. (6) formulaning xatosi to'g'ri to'rtburchaklar formulasiga nisbatan kichikroq. Agar $n\to\infty$ bo'lsa, $h\to 0$ bo'lib, hato yanada kamayadi. Natijada, bu formula integralning izlangan qiymatini yetarlicha aniqlikda beradi.

3§. Sipson formulasi (Parabolik formula)

Yuqoridan $y=f(x)$ egri chiziqli funksiya, yon tomonlari $x=-h$ va $x=h$ kesmalar, pastdan $y=0$ absissalar o'qi bilan chegaralangan figuraning yuzini topish talab qilinsin.



Agar h cheksiz kichik bo'lsa, $y=f(x)$ funksiyani taqriban $M(-h;\,y_1)$, $P(0;\,y_0)$ va $N(h;\,y_2)$ nuqtalardan o'tuvchi

$$Y=\alpha x^2+\beta x+\lambda \qquad (7)$$

parabola yoyi bilan almashtirish mumkin bo'ladi. U holda, bu yoy bilan chegaralangan figuraning yuzi quyidagicha bo'ladi:

$$S=\int_{-h}^{h}\left(\alpha x^2+\beta x+\lambda\right)dx=\left(\frac{ax^3}{3}+\frac{\beta x^2}{2}+\lambda x\right)\Bigg|_{-h}^{h}=\frac{2\alpha}{3}h^3+2\lambda h. \qquad (8)$$

MPN yoy o’tgan nuqtalarning $x=-h$, $x=0$ va $x=h$ absissalarini (7) ga ketma – ket qo’yamiz:

$$y_1=\alpha h^2-\beta h+\lambda,\quad y_0=\gamma,\; y_2=\alpha h^2+\beta h+\lambda.$$

Bulardan, $\gamma=y_0$ va $\qquad \alpha=\dfrac{y_1-2y_0+y_2}{2h^2}.$ (9)

(9) qiymatlarini (8)ga qo’ysak, ***parabolik yoy formulasi*** kelib chiqadi:

$$\int_{-h}^{h} ydx=\frac{1}{3}h(y_1-2y_0+y_2)+2y_0h=\frac{h}{3}(y_1+4y_0+y_2). \qquad (10)$$

Bu formulaning egri chiziqli figura yuzini topishdagi xatosi ***«to’g’ri burchakli to’rtburchaklar»*** va ***«trapesiyalar formulasi»*** ga nisbatan kichikroqdir.

Koordinatalar sistemasini parallel ko’chirishni hisobga olsak, (10) formulani quyidagicha ham ifodalash mumkin:

$$\int_a^b ydx=\frac{h}{3}\left(y(a)+4y\left(\frac{a+b}{2}\right)+y(b)\right). \qquad (11)$$

Agar $[a,b]$ kesma n bo’lakka ajratilsa, undagi egri chiziqli yuzalar soni ham n ta bo’ladi. U holda, ***parabolik yoki Simpson formulasi***ning umumiy ko’rinishi quyidagicha bo’ladi:

$$S=\int_a^b ydx=\frac{b-a}{3n}\left(\frac{y_0+y_n}{2}+(y_1+y_2+\ldots+y_{n-1})+2\left(y_{\frac{1}{2}}+y_{\frac{3}{2}}+\ldots+y_{n-\frac{1}{2}}\right)\right) \qquad (12)$$

yoki

$$S=\int_a^b ydx=\frac{h}{6n}\left((y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})+4\left(y_{\frac{1}{2}}+y_{\frac{3}{2}}+\ldots+y_{n-\frac{1}{2}}\right)\right). \qquad (13)$$

Misol. Quyidagi integralning $n=10$ bo’lganda taqribiy qiymatini toping: $\displaystyle\int_0^1\frac{dx}{1+x^2}.\quad\left(\approx\frac{\pi}{4}\approx 0{,}785398\ldots\right).$

Yechilishi. Berilgan integralning taqribiy qiymatini «*to’g’ri to’rtburchaklar formulasi*», «*trapesya formulasi*» va «*Sipson formulasi*» yordamida hisoblab ko'rib, natijalarini taqqoslab ko'ramiz. (Uning qiymati $\frac{\pi}{4} \approx 0{,}785398\ldots$ dan iborat).

1) To'g'ri to'rtburchaklar formulasi yordamida taqribiy qiymatini topish uchun ($3'$) formuladan foydalanamiz. Buning uchun $[a, b]$ kesmani, ya'ni $[0, 1]$ni 10 bo'lakka ajratamiz va quyidagilarni hosil qilamiz (bunda $k = 0{,}10$ $\quad ва \quad$ $\frac{b-a}{n} = \frac{1}{10}$):

$x_{\frac{1}{2}} = 0{,}55$ $\qquad$ $y_{\frac{1}{2}} = 0{,}9975$

$x_{\frac{3}{2}} = 0{,}15$ $\qquad$ $y_{\frac{3}{2}} = 0{,}9780$

$x_{\frac{5}{2}} = 0{,}25$ $\qquad$ $y_{\frac{5}{2}} = 0{,}9412$

$x_{\frac{7}{2}} = 0{,}35$ $\qquad$ $y_{\frac{7}{2}} = 0{,}8909$

$x_{\frac{9}{2}} = 0{,}45$ $\qquad$ $y_{\frac{9}{2}} = 0{,}8316$

$x_{\frac{11}{2}} = 0{,}55$ $\qquad$ $y_{\frac{11}{2}} = 0{,}7678$

$x_{\frac{13}{2}} = 0{,}63$ $\qquad$ $y_{\frac{13}{2}} = 0{,}7029$

$x_{\frac{15}{2}} = 0{,}75$ $\qquad$ $y_{\frac{15}{2}} = 0{,}6400$

$x_{\frac{17}{2}} = 0{,}85$ $\qquad$ $y_{\frac{17}{2}} = 0{,}5806$

$x_{\frac{19}{2}} = 0{,}95$ $\qquad$ $y_{\frac{19}{2}} = 0{,}5256.$

Yig'indisi:7,8561

U holda, ($3'$) dan:

$$\int_0^1 y dx = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{10} \cdot 7{,}8561 = 0{,}78561.$$

Demak, integral qiymati 0,78361 bo’lib , uning xatosi taxminan 0,0002 ga teng.

2) trapesiyalar formulasi yordamida yuqoridagi integralni hisoblayiz. $n=10$ deb, to’rtta raqamgacha hisoblaymiz. Trapesya formulasiga asosan quyidagilarni aniqlaymiz:

$x_0 = 0,0 \qquad y_0 = 1,0000$

$x_{10} = 1,0 \qquad \underline{y_{10} = 0,5000}$

Yig’indisi:1,5000

$x_1 = 0,1$	$y_1 = 0,9901$
$x_2 = 0,2$	$y_2 = 0,9615$
$x_3 = 0,3$	$y_1 = 0,9174$
$x_4 = 0,4$	$y_4 = 0,8621$
$x_5 = 0,5$	$y_5 = 0,8000$
$x_6 = 0,6$	$y_6 = 0,7353$
$x_7 = 0,7$	$y_7 = 0,6711$
$x_8 = 0,8$	$y_8 = 0,6098$
$x_9 = 0,9$	$\underline{y_9 = 0,5525}$

Yig’indisi:7,0996

U holda, (6) formuladan:

$$\int_0^1 ydx = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{10}\cdot\left(\frac{1,5000}{2} + 7,0998\right) = 0,78498.$$

Demak, trapesiyalar formulasi qo’llanib, integral hisoblaganda, u 0,78998 dan iborat bo’ldi. Topilgan taqribiy qiymat haqiqiy qiymatdan 0,0004 ga farq qiladi.

3) Simpson formulasi yordamida berilgan integralning taqribiy qiymatini topamiz. Buning uchun $n = 2$ deb, beshta raqamgacha hisoblaymiz. Bunda $\frac{b-a}{3n} = \frac{1}{6}$ dan iborat bo'ladi. U holda,

$$x_0 = 0 \qquad \frac{1}{2}y_0 = 0,50000$$

$$x_{\frac{1}{2}} = 0,25 \qquad 2y_{\frac{1}{2}} = 1,88235$$

$$x_1 = 0,50 \qquad y_1 = 0,80000$$

$$x_{\frac{3}{2}} = 0,75 \qquad 2y_{\frac{3}{2}} = 1,28000$$

$$x_2 = 1,00 \qquad \frac{1}{2}y_2 = 0,25000$$

Yig'indi:4,71235

(12) formulaga asosan:

$$\int_0^1 ydx = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{6} \cdot 4,71235 = 0,78539$$

ni hosil qilamiz. Topilgan taqribiy qiymat haqiqiy qiymatdan 0,00001 ga farq qiladi.

Demak, Simpson formulasi yordamida berilgan integralni topish yuqoridagi ($3'$) va (6) formulalar bilan topishga nisbatan ancha aniqroq ekanligi ko'rindi.

Mustaqil yechish uchun mashqlar.

№63. Quyidagi itegralning taqribiy qiymatini toping. Bunda $n = 10$.

№64. $\int_0^1 \frac{dx}{1+x}$ $(\approx \ln 2 \approx 0,69315)$ integralning taqribiy qiymatini $n = 10$ da to'g'ri to'rtburchaklar formulasi va trapesiyalar formulasi bo'yicha toping.

№65. Trapesiyalar formulasi bo'yicha $\int_0^1 e^{-x^2} dx$ integralni 0,001 gacha aniqlik bilan hisoblang.

№66. $\int_0^3 \frac{dx}{x}$ integralni trapesiyalar formulasi bo'yicha integrallash oralig'ini 4 bo'lakka bo'lib, hisoblang.

№67. $\int_0^1 e^{x^2} dx$ integralni 0,01 gacha aniqlik bilan hisoblang.

№68. $\int_0^1 \sqrt{x^2+1} \cdot \sin x dx$ integralni 0,01 gacha aniqlik bilan hisoblang.

№69. Trapesiya formulasi yordamida $n=5$ bo'lganda $\int_0^6 \frac{dx}{\sqrt{8+x^2}}$ integralni taqribiy hisoblang.

№70. $n=8$ deb, $\int_2^{10} \frac{dx}{\lg x}$ integralni taqribiy hisoblang.

№71. Trapesiya formulasi bo'yicha $n=8$ da $\int_0^4 \sqrt{2+x^2} dx$ integralni hisoblang.

№72. $h=\frac{\pi}{2}$ da $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ integralni Simpson formulasi yordamida taqribiy hisoblang.

№73. $\int_0^1 \frac{dx}{1+x^2}$ aniq integralni $n=2$ deb olib, Simpson formulasi yordamida hisoblang.

№74. $\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$ integralni $2n = 10$ da Simpson formulasi yordamida hisoblang.

4- BOB. FIZIKAVIY VA TEXNIKAVIY MASALALARNI YeChIShDA ANIQ INTEGRALNI QO'LLASh

1§. Moddiy nuqtaning statik momenti va og'irlik markazi

Faraz qilaylik, XOY koordinatalar tekisligida quyidagi moddiy nuqtalar sistemasi berilgan bo'lsin:

$$A_1(x_1, y_1),\ A_2(x_2, y_2), \ldots, A_n(x_n, y_n). \qquad (1)$$

Bu nuqtalarning massalari mos ravishda

$$m_1, m_2, \ldots, m_n \qquad (2)$$

dan ibort bo'lsin. U holda, moddiy nuqtalarning OX o'qqa nisbatan statik momenti (M_x) quyidagidan iborat bo'ladi:

$$M_x = m_1 y_1 + m_2 y_2 + \ldots + m_n y_n, \qquad (3)$$

yoki
$$M_x = \sum_{i=1}^{n} m_i y_i = m y_c. \qquad (3')$$

OY o'qqa nisbatan statik momenti esa

$$M_y = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n \qquad (4)$$

yoki
$$M_y = \sum_{i=1}^{n} m_i x_i = m x_c. \qquad (4')$$

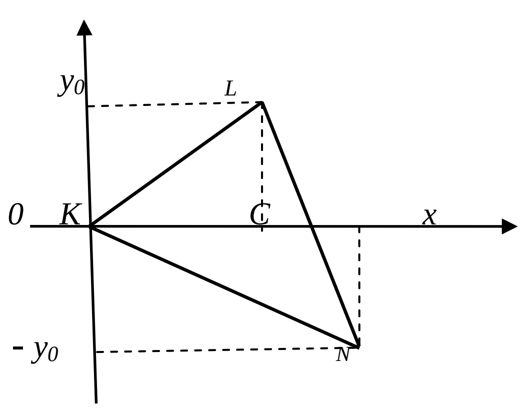
Ta'rif: $m = m_1 + m_2 + \ldots + m_n$ bo'lganda $(x_c; y_c) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$ koordinatali nuqtaga moddiy nuqtalar sistemasining ***og'irlik markazi*** deyiladi.

Og'irlik markazining koordinatalari $(3')$ va $(4')$ larga asosan quyidagi formulalar orqali topiladi:

$$x_c = \frac{M_y}{m} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}, \qquad (5)$$

$$y_c = \frac{M_x}{m} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}. \qquad (6)$$

Misol. $m_K = m_L = m_N = 1$ birlik massalari qo'yilgan K, L va N nuqtalarning og'irlik markazi uchburchak medianalarining kesishish nuqtasi bo'lishini ko'rsating.



Yechilishi. K, L, N nuqtalarni koordinatalar sistemasida tasvirlashda K nuqtani koordinatalar markaziga, L va N nuqtalarning o'rtasi, ya'ni C nuqtani OX o'qida yotadigan qilib joylashtiramiz. U holda, K nuqtaning koordinatalari $(0;0)$, L nuqtaning ordinatasi y_0, N nuqtaning ordinatasi $(-y_0)$dan iborat bo'ladi. Bundan ko'rinadiki, C og'irlik markazining y_c ordinatasi quyidagidan iborat bo'ladi:

$$y_c = \frac{0 \cdot 1 + y_0 \cdot 1 - y_0 \cdot 1}{2} = \frac{0}{2} = 0.$$

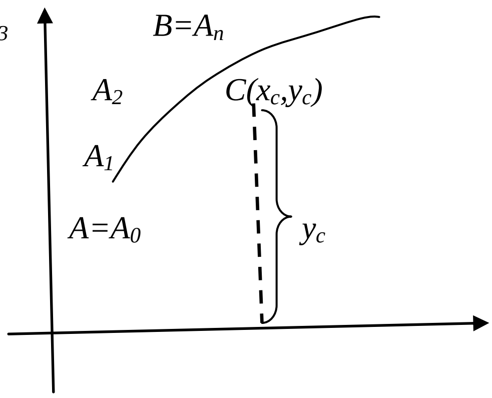
Demak, C nuqta OX o'qida yotishi ma'lum bo'ldi.

Agar massalar uchta nuqtaga emas, balki figuraning hamma joyiga tekis qo'yilgan bo'lsa, bunday figuralarning statik momentini topishda yig'indining o'rniga integraldan foydalaniladi.

2§. Tekis egri chiziqning statik momenti va og'irlik markazi

$[a, b]$ oraliqda uzluksiz hosilaga ega bo'lgan AB egri chiziqli $f(x)$ funksiya berilgan bo'lib, u L uzunlikka ega bo'lsin. Bunda AB egri chiziqni bir jinsli, ya'ni chiziqli zichligi ρ o'zgarmas, soddalik uchun $\rho = 1$ deb qaraymiz.

AB egri chiziqning OX va OY o'qlarga nisbatan statik momentlarini hamda egri chiziqning og'irlik markazini topamiz. Buning uchun AB egri chiziqni



$A = A_0(x_0, y_0), A_1(x_1, y_1), \ldots, A_n(x_n, y_n) = B$

nuqtalar orqali n ta bo’lakka x

ajratamiz. Bu nuqtalarga l parametrning 0 x_c

$$l_0 = 0 < l_1 < l_2 < \ldots < l_n = L$$

qiymatlari mos kelsin. l parametr A nuqtadan boshlangan yoydan iborat. $A_i A_{i+1}$ yoyning uzunligini

$$\Delta l_i = l_{i+1} - l_i,$$

shu yoyning massasini esa Δm_i bilan belgilaymiz. U holda, $\rho = 1$ bo’lganda massa quyidagicha bo’ladi:

$$\Delta m_i = \rho \cdot \Delta l_i = \Delta l_i.$$

Qism yoyning uzunligini moddiy nuqta deb qarasak, u holda, uning og’irlik markazi unga mos keladigan qism yoy uzunligiga teng bo’ladi, ya’ni $\Delta m_i = \Delta l_i$. Moddiy nuqtaning OX o’qdan y_i, OY o’qdan esa x_i masofalarda yotganligini e’tiborga olsak, quyidagi tenglamalar o’rinli bo’ladi:

$$\begin{cases} \Delta M x_i = y_i \cdot \Delta m_i = y_i \cdot \Delta l_i \\ \Delta M y_i = x_i \cdot \Delta m_i = x_i \cdot \Delta l_i \end{cases} \quad \text{va}$$

$\Delta l_i \approx \sqrt{1 + \left(f'(x_i)\right)^2} \cdot \Delta x_i$ bo’lganligi sababli,

$$\begin{cases} \Delta M x_i \approx f(x_i)\sqrt{1 + \left(f'(x_i)\right)^2} \cdot \Delta x_i \\ \Delta M y_i \approx x_i \sqrt{1 + \left(f'(x_i)\right)^2} \cdot \Delta x_i. \end{cases} \quad \text{va}$$

U holda, AB egri chiziq bo’laklari yig’indisining $\lambda = \max\limits_{1 \le i \le n}\{\Delta l_i\} \to 0$ dagi limitlari quyidagilardan iborat bo’ladi:

$$M_x = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(x_i)\Delta l_i = \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx, \qquad (7)$$

$$M_y = \lim_{\lambda \to 0} \sum_{i=1}^{n} x_i \cdot \Delta l_i = \int_a^b x\sqrt{1 + \left(f'(x)\right)^2}\,dx, \qquad (8)$$

$$m = l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx. \qquad (9)$$

(5)-(9) formulalardan foydalanib, tekis egri chiziq og'irlik markazining formulalarini hosil qilamiz:

$$x_c = \frac{\int_a^b x\sqrt{1+\left(f'(x)\right)^2}\,dx}{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx}, \tag{10}$$

$$y_c = \frac{\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx}{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx}. \tag{11}$$

Agar $f(x)$ni $y = f(x)$ va $L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$ ekanligini e'tiborga olsak, (10) va (11) ni soddalik uchun quyidagi ko'rinishlarda yozish ham mumkin:

$$x_c = \frac{\int_a^b x\sqrt{1+(y')^2}\,dx}{L}, \tag{12}$$

$$y_c = \frac{\int_a^b y\sqrt{1+(y')^2}\,dx}{L}. \tag{13}$$

(13) formuladan:

$$L \cdot y_c = \int_a^b y\sqrt{1+(y')^2}\,dx \tag{14}$$

hosil bo'ladi. (14)ning ikkala tomonini 2π ga ko'paytirib, quyidagi formulaga ega bo'lamiz:

$$2\pi y_c \cdot L = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx, \tag{15}$$

(15)ning chap tomoni y_c radiusli aylana uzunligini, o'ng tomoni esa AB yassi egri chiziqning OX o'qi atrofida aylanishidan hosil bo'lgan sirt yuzasini ifodalaydi, ya'ni $2\pi y_c l = S$.

Demak, $L=\int_a^b y\sqrt{1+(y')^2}dx$ ***yoy uzunligining***, $S=2\pi\int_a^b y\sqrt{1+(y')^2}dx$ esa ***aylanma jism sirtining yuzini*** topish formulasidir. Yuqridagilarni xulosalash uchun quyidagi Gulden teoremasini keltiramiz:

Teorema. Tekis egri chiziqni shu chiziq bilan kesishmaydigan biror o'q atrofida aylantirishdan hosil qilingan sirtning yuzi bu chiziq uzunligini C og'irlik markazi tomonidan chizilgan aylana uzunligiga ko'paytirilganiga tengdir.

Misol. $y=\sqrt{R^2-x^2}$ (bunda $-R\le x\le R$) yarim aylananing koordinata o'qlariga nisbatan statik momentini va og'irlik markazi koordinatalarini toping.

Yechilishi. Berilgan funksiya, ya'ni yarim aylana formulasidan hosila olamiz.

$$y'=\left(\sqrt{R^2-x^2}\right)'=\frac{-2x}{2\sqrt{R^2-x^2}}=-\frac{x}{\sqrt{R^2-x^2}}.$$

U holda, $\sqrt{1+(y')^2}=\frac{R}{\sqrt{R^2-x^2}}$. (7) va (8) formulalarni qo'llab M_x va M_y ni topamiz: $M_x=\int_{-R}^{R}\sqrt{R^2-x^2}\cdot\frac{R}{\sqrt{R^2-x^2}}dx=R\int_{-R}^{R}dx=2R^2$ hamda

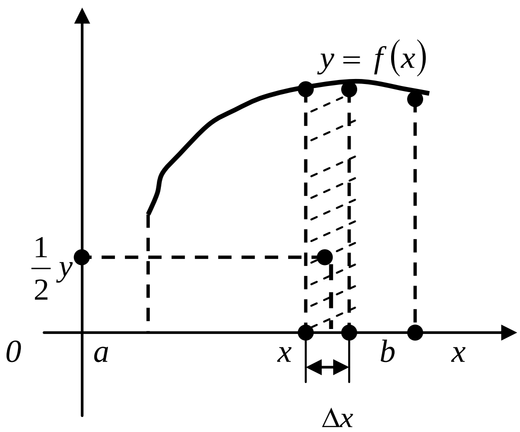
$$M_y=\int_{-R}^{R}x\cdot\frac{R}{\sqrt{R^2-x^2}}dx=-R\sqrt{R^2-x^2}\Big|_{-R}^{R}=0.$$

Yarim aylananing uzunligi $\frac{L}{2}=\pi R$ ga tengligi ma'lum. (12) va (13) formulalarni qo'llab, og'irlik markazining koordinatalarini topamiz:

$$x_c=\frac{0}{2R}=0 \quad \text{va} \quad y_c=\frac{2R^2}{\pi R}=\frac{2R^2}{\pi}. \qquad \text{Demak, } C\left(0;\ \frac{2R}{\pi}\right).$$

3§. Tekis figuralarning statik momenti va og'irlik markazi

$[a, b]$ oraliqda $x = a$ va $x = b$ to'g'ri chizig'lar hamda OX o'q bilan chegaralangan, manfiymas egri chiziqli trapesiya $y = f(x)$ funksiya orqali berilgan. Egri chiziq massasi $\rho = 1$ zichlikda uzluksiz taqsimlangan bo'lsin. U holda, egri chiziqli trapesiyaning massasi uning yuzasini son qiymatiga teng. Egri chiziqli trapesiyaning statik momentlari M_x va M_y ni topish uchun uni $x_i = a + \frac{b-a}{n} i$ ($i = 0,1,2,\ldots,n$) nuqtadan o'tuvchi va ordinata o'qiga parallel bo'lgan to'g'ri chiziqlar yordamida n ta bo'lakka ajratamiz. Ajratilgan bo'laklardan birining massasi



$$\Delta S_i \approx y_i \cdot \Delta x_i$$

dan iborat. Bo'lakning og'irlik markazi $\left(x_i, \frac{y_i}{2}\right)$ ni hisobga olib, quyidagilarni hosil qilamiz:

$$\Delta M_{xi} = y_i \Delta x_i \cdot \frac{y_i}{2} = \frac{y_i^2}{2} \Delta x_i, \qquad \Delta M_{y_i} = y_i \Delta x_i \cdot x_i = x_i y_i \cdot \Delta x_i.$$

Ularning barchasining yig'indisini $\lambda = \max\limits_{1 \le i \le n} \{\Delta x_i\} \to 0$ dagi limitini topamiz:

$$M_x = \frac{1}{2} \lim_{\lambda \to 0} \sum_{i=1}^{n} y^2 \Delta x_i = \frac{1}{2} \int_a^b y^2 dx, \tag{16}$$

$$M_y = \lim_{\lambda \to 0} \sum_{i=1}^{n} x \cdot y \Delta x_i = \int_a^b xydx \tag{17}$$

$$m = \lim_{\lambda \to 0} \sum_{i=1}^{n} y \cdot \Delta x_i = \int_a^b ydx = S. \tag{18}$$

(18) formula berilgan egri chiziqli trapesiyaning massasidan iborat.

(10) va (11) yoki (12) va (13) dan og'irlik markazining koordinatalarini aniqlaymiz:

$$x_c = \frac{M_y}{m} = \frac{\int_a^b xydx}{\int_a^b ydx}, \qquad (19)$$

$$y_c = \frac{M_x}{m} = \frac{\int_a^b y^2 ydx}{\int_a^b ydx}. \qquad (20)$$

(20) tenglamaning ikkala tomonini $2\pi m$ ga ko'paytiramiz:

$$2\pi y_c \cdot m = \pi \int_a^b y^2 dx. \qquad (21)$$

Hosil bo'lgan tenglamaning o'ng tomoni aylanish jismining hajmi formulasidir. Agar $m = S$ ekanligini e'tiborga olsak, (21)ni quyidagicha ifodalash mumkin bo'ladi:

$$V = S \cdot 2\pi y_c. \qquad (22)$$

Tekis figurani u bilan kesishmagan o'q atrofida aylanishidan hosil bo'lgan jismning hajmi- shu figura yuzining shu shakl og'irlik markazi chizgan aylana uzunligi bilan ko'paytimasiga teng. Bu Guldenning ikkinchi teoremasi edi.

2-misol. Plastinka $x = a(t - \sin t),\ y = a(1 - \cos t)$ (bunda $t \in [0;\ 2\pi]$) sikloidaning bitta arkasi va asosi bilan plastinkaning OX o'qqa nisbatan statik momentini toping.

Yechilishi: Plastinkaning OX o'qqa nisbatan statik momentini topish uchun (16) formuladan foydalanamiz:

$$M_x = \frac{1}{2}\int_0^{2\pi} y^2 dx = \frac{1}{2}\int_0^{2\pi} a^3(1 - \cos t)^3 dt = \frac{5a^3}{2}\pi.$$

3-misol. $x^2 + y^2 \le r^2$ $(y > 0)$ yarim aylana og'irlik markazining koordinatalarini toping.

Yechilishi: Berilgan yarim aylananing og'irlik markazi ordinatasini topish uchun Guldenning ikkinchi teoremasi, (29) formuladan foydalanamiz, ya'ni:

$$V = S\cdot 2\pi y_c \quad \text{dan} \quad y_c = \frac{v}{2\pi S}.$$

Aylanish jismi shar bo'lganligi sababli uning hajmi $V = \frac{4}{3}\pi r^3$, yarim aylana yuzasi $S = \frac{\pi r^2}{2}$. U holda, $y_c = \dfrac{\frac{4}{3}\pi r^3}{2\pi \frac{\pi r^2}{2}} = \dfrac{4r}{3\pi}$.

Figura OY o'qiga nisbatan simmetrik bo'lganligi uchun $x_c = 0$. Shuning uchun og'irlik markasi $C\left(0; \dfrac{4r}{3\pi}\right)$dan iborat bo'ladi.

Mustaqil yechish uchun mashqlar

№75. $y = x,\quad y = \dfrac{1}{x^2},\quad y = 0$ va $x = 3$ chiziqlar bilan chegaralangan figuraning yuzini toping.

№76. $y = \sqrt{x},\quad y = 2$ va $x = 0$ chiziqlar bilan chegaralangan figuraning yuzini toping.

№77. $y = f_1(x) = x$ va $z = f_2(x) = 2 - x^2$ figuralar bilan chegaralangan figuraning yuzini toping.

№78. $y = x^{\frac{3}{2}}$ yarimkubik parabolaning $x = 0$ dan $x = 5$ gacha bo'lgan yoyi uzunligini toping.

№79. $y^2 = 2x$ (bunda $y > 0,\quad 0 \le x \le 2$) parabolaning OX va OY o'qlarga nisbatan statik momentilarini toping.

№80. $y = \cos x$ kosinusoida yoyining (bunda $-\dfrac{\pi}{2} \le x \dfrac{\pi}{2}$) OX o'qqa nisbatan statik momentini toping.

№81. $y = 2\sqrt{x}$, OX o'q va $x = 1$ chiziqlar bilan chegaralangan figuraning og'irlik markazini toping.

№82. $y^2 = 20x$ va $x^2 = 20y$ chiziqlar bilan chegaralangan figuraning og'irlik markazini toping.

№83. $y = 4 - x^2$ va $y = 0$ chiziqlar bilan chegaralangan koordinatalarini toping.

№84. Guldenning birinchi teoremasidan foydalanib, a radiusli yarim aylananing og'irlik markazini toping.

№85. Gulden teoremasidan foydalanib, H balandlik, va asos radiusi r bo'lgan konusning yon sirti va hajmini toping.

4§. Kuch ishini hisoblash

O'zgarmas F kuch OX o'qi bo'ylab yo'naltirilgan hamda uning P nuqtasi OX o'qi bo'ylab (a, b) kesmada joylashgan bo'lsin. U holda, shu kesmada kuchning bajargan ishi

$$A = F|ab| \tag{1}$$

formula yordamida hisoblanadi. Agar F kuch o'zining kattaligini o'zgartirsa, kuch ishini (1) formula yordamida hisoblashning imkoni bo'lmay qoladi. Shuningdek, kuch bajargan ishni hisoblashda ko'pincha quyidagi ***Guk qonunidan*** foydalaniladi:

$$F = kx, \tag{2}$$

bunda F -kuch, x - prujinaning F kuch ta'sirida absolyut uzayishi, k - proporsionallik koeffisiyenti. Biz quyidagi o'zgaruvchan F kuch ishini hisoblashga doir masalani qaraymiz.

Masala. Birorta F kuch OX o'q bo'yicha yo'naltirilgan bo'lsin. $F(x)$ kuchning (a, b) kesmadagi ishini hisoblang.

Yechilishi: Berilgan F kuch OX o'qi bo'yicha yo'nalgan bo'lsa, uning kattaligi x ga bog'liq bo'ladi, ya'ni $F = F(x)$. Aniq integral integrallar yig'indisining limiti ekanligini e'tiborga olib, (a, b) kesmani

$$a = x_0 < x_1 < x_2 < \ldots < x_{k-1} < x_k < \ldots = b$$

nuqtalar yordamida kichik qismlarga ajratamiz. Bunda $F(x)$ kuch x_{k-1} dagi qiymatini (x_{k-1}, x_k) kesmada ham saqlaydi deb qaraymiz, ya'ni $F(x_{k-1})$. U holda, (1) formula yordamida $F(x)$ kuch ishini hisoblab,

$$A = F(x_{k-1})(x_k - x_{k-1}) \tag{3}$$

ekanligini topamiz. Shuningdek, (a, b) dagi barcha kichik kesmalarda $F(x)$ kuchning bajargan ishlarini topib, quyidagi jadvalni tuzamiz:

Kesmalar tartibi	***Kesma uzunligi***	***Kuch kattaligi***	***Kuchning kesmadagi ishi***
1	$x_1 - a$	$F(a)$	$F(a)\,(x_1 - a)$
2	$x_2 - x_1$	$F(x_1)$	$F(x_1)\,(x_2 - x_1)$
3	$x_3 - x_2$	$F(x_2)$	$F(x_2)\,(x_3 - x_2)$
...	...	...	...
k	$x_k - x_{k-1}$	$F(x_{k-1})$	$F(x_{k-1})\,(x_k - x_{k-1})$
$k+1$	$x_{k+1} - x_k$	$F(x_k)$	$F(x_k)\,(x_{k+1} - x_k)$
...	...	...	...
n	$b - x_{n-1}$	$F(x_{n-1})$	$F(x_{n-1})\,(b - x_{n-1})$

Alohida – alohida kesmalarda bajarilgan ishlarning yig'indisi quyidagicha bo'ladi:

$$A_n = F(a)(x_1 - a) + F(x_1)(x_2 - x_1) + F(x_2)(x_3 - x_2) + \ldots tF(x_{n-1})(b - x_{n-1}). \tag{4}$$

Ushbu yig'indi – integral yig'indidan iboratdir. Ma'lumki, integral yig'indining limiti aniq integraldir, ya'ni:

$$\int_a^b F(x)dx.$$

Demak, $F(x)$ kuchning (a, b) kesmada bajargan ishi aniq integraldan iborat ekan:

$$A = \int_a^b F(x)dx. \qquad (5)$$

1-misol. $(0, 1)$ kesmada $F(x) = x + 1$ tenglama bilan berilgan F kuchning bajargan ishini toping.

Yechilishi: Kesmaning chegaralari $a = 0$ va $b = 1$dir. U holda, (5) formuladan foydalanib, quyidagi integralni hisoblaymiz:

$$A = \int_0^1 (x+1)dx = \left[\frac{x^2}{2} + x\right]\Bigg|_0^1 = \frac{1}{2} + 1 = \frac{3}{1}.$$

Demak, F kuchning berilgan kesmada bajargan ishi $A = \frac{3}{2}$.

2-misol. Kesmaning chegara nuqtalari $a = -2$ va $b = 2$ bo'lganda $F = x^2 + x$ tenglama yordamida berilgan F kuchining bajargan ishini hisoblang.

Yechilishi: (5) formuladan foydalanib, A ishni topamiz:

$$A = \int_{-2}^{2} (x^2 + x)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_{-2}^{2} = -\frac{8}{3} + \frac{4}{2} + \frac{8}{3} + \frac{4}{2} = 4.$$

Demak, bajarilgan ish $A = 4$ ekan.

3-misol: Prujinani 0,04m qisish uchun 24 Joul ish bajarilishi ma'lum bo'lsa, uni 0,2 m qisish uchun qanday ish bajarish lozim?

Yechilishi: Berilganlarga ko'ra, prujina qisilgan kattalik 0,04m, bajarilgan ishi esa 24 Joul ekanligi ma'lum. U holda, (5) formulaga asosan:

$$24 = \int_0^{0,04} kxdx = k\,\frac{x^2}{2}\Bigg|_0^{0,04} = \frac{0,0016}{2}k = 0,0008k.$$

U holda, $0,0008k = 24$, $k = \dfrac{24}{0,0008} = 30000(H/м)$.

Endi prujinani 0,2 m qisish uchun qancha ish bajarish lozimligini topamiz:

$$A=\int_0^{0,2} 30000x\,dx = 30000\frac{x^2}{2}\Big|_0^{0,2} = 15000x^2\Big|_0^{0,2} = 600(Ж).$$

Mustaqil yechish uchun mashqlar

№86. $a=0$ va $b=2$ lar bilan chegaralangan kesmada $F=x$ tenglama bilan berilgan F kuchning bajargan ishini toping.

№87. F kuch $F=x+1$ tenglama yordamida berilgan bo'lsin, kuchning $[1;\,3]$ kesmada bajargan ishini toping.

№88. $F=x^2-x+1$ tenglama bilan berilgan kuchning $a=-2$ va $b=-1$ lar bilan chegaralangan kesmada bajargan ishini toping.

№89. $0\le x\le 2$ kesmada OX o'qi bo'ylab, harakatlanayotgan hamda $F(x)=\frac{1}{x+1}$ tenglama bilan berilgan kuchning bajargan ishini toping.

№90. Prujinani 0,01m.ga cho'zish uchun $1H$ kuch talab qilinsa, uni 0,05m. cho'zish uchun qancha ish bajarish kerak bo'ladi.

№91. Agar prujinani 1sm.ga cho'zish uchun $100H$ kuch talab qilinsa, uni 5sm cho'zish uchun qancha ish bajarish talab qilinadi?

№92. Ressor 2Joul nagruzka ostida 1sm egilishi ma'lum bo'lsa, uni 3sm deformasiyalash uchun qancha ish bajarish lozim.

№93. Vintsimon prujinani 0,2sm siqish uchun 0,32kg ish bajarilsa, uni 6sm siqish uchun sarf bo'ladigan ishni hisoblang.

№94. $P=2\cdot 10^4 H$ og'irlikdagi raketaning yerdan $h=1500$km. balandlikka ko'tarilishi uchun qancha ish bajarilishi lozim.

Javoblar

№1. 1,5. №2. 9. №3. 0,5. №4. 11. №5. 2. №6. -1. №7. 28. №8. $5-\sqrt{2}$. №9. 6. №10. 168. №11. $9\frac{1}{3}$. №12. $2e^{\pi}+8$. №13. -0,25. №14. $\frac{7}{64}$. №15. 0,4. №16. $8\frac{1}{15}$. №17.52. №18. 3. №19. $7\frac{11}{15}$. №20. $32\frac{2}{3}$. №21. $\frac{a^2\pi}{3}$. №22. $\frac{\pi}{4}$.

№23. 1. №24. $10\frac{1}{8}\pi.$ № 25. $\approx 0{,}36$. №26. $\frac{3}{16}$. №27. $7\ln\frac{e+3}{4}$. №28. $4(e-1)$.

№29. $10\frac{1}{8}$. №30. $\frac{15}{16}$. №31. $10\frac{1}{8}$. №32. 0,25. №33. $3\ln 3-2$. №34. π. №35.1.

№36. e^2+1. №37. $\frac{1}{3}$. №38. $a^2\ln 2$. №39. $\frac{4}{3}$. №40. πa^2. №41. $2\frac{2}{3}$. №42. $5\frac{1}{3}$. №42. $2-\frac{1}{\ln 2}\approx 0{,}56$. №44. $21\frac{1}{3}$. №45. $\frac{3}{8}\pi a^2$. №46. $4\frac{2}{3}$. №47. 7,634.

№48. $2\pi r$. №49. 1,196. №50. $2\sqrt{17}-\frac{1}{2}\ln\left(4+\sqrt{17}\right)$. №51. $2\sqrt{3}+\ln\left(2+\sqrt{3}\right)$.

№52. $4\frac{4}{27}$. №53. $17\frac{1}{2}$. №54. $8a$. №53. $\frac{a}{2}\left(e-\frac{1}{e}\right)$. №54. $6a$. №55. $\frac{\pi R}{2}$.

№56. $17\frac{1}{15}\pi$. №57. $\frac{\pi}{2}\left(2-\sqrt{2}\right)$. №58. $\frac{11}{405}\pi$. №59. $\frac{\pi}{2}$. №60. $\frac{\pi^2}{4}$. №61. a) 30π; b) $34\frac{2}{15}\pi$; v) $\frac{\pi^2}{2}$; g) $21\frac{3}{5}\pi$. №62. a) $7{,}5\pi$; b) 8π; v) $0{,}4\pi$; g) $18\frac{2}{15}\pi$. №63. 0,75682. №65. 0,7462. №66. 1,12. №67. 1,46. №68. 0,56.

№69. 0,82. №70. 11,9826. №71. 14,84. №72. 2,07. №73. 0,78539. №74. 1,37.

№75. $\frac{7}{6}$. №76. $\frac{8}{3}$. №77. $\frac{9}{2}$. №78. $12\frac{11}{27}$. №79. $M_x=\frac{3\sqrt{5}-1}{3}$; $M_y=\frac{9\sqrt{5}}{8}+\frac{1}{16}\ln\left(2+\sqrt{5}\right)$. №80. $M_x=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$. №81. $C\left(\frac{3}{5};\frac{3}{4}\right)$.

№82. $C(9;9)$. №83. $C\left(0;\frac{8}{5}\right)$. №4. $C\left(0;\frac{4a}{3\pi}\right)$. №85. πRl; $\frac{1}{3}\pi R^2H$. №86. 2.

№87. 6. №88. $3\frac{5}{6}$. №89. $\ln 3$. №90. 0,125Joul. №91. 12,5 Joul. №92. 900Joul.

№93. 0,288 Joul. №94. $2{,}43\cdot 10^{10}$ *Жоул.* (Ko'rsatma: yerning tortish kuchi $f(x)=\frac{\lambda}{x^2}$, λ - o'zgarmas son. Yer sirtidagi jismning ρ og'irligi, x esa yerning radiusi R ga teng. U holda, $\rho = \frac{\lambda}{R^2}$, $\lambda = \rho R^2$. Bundan, $f(x) = \frac{\rho R^2}{x^2}$, $x = R$ va $x = R + h$. Demak, $A = \rho R^2 \int_R^{R+h} \frac{dx}{x^2}$)

Adabiyotlar

1. Abdalimov V., Solixov Sh. Oliy matematika qisqa kursi.- Toshkent: O'qituvchi, 1981.
2. Bogomolov N.V. Matematikadan amaliy mashg'ulotlar. – Toshkent: O'qituvchi, 1984.
3. Vыgodskiy M.Ya. Spravochnik po vыsshey matematike.-Moskva: Nauka, 1977.

4. Glagolev N.S. va boshqalar. Matematika, III qism.-Toshkent: O’qituvchi, 1947.
5. Kachenovskiy M.I. va boshqalar. Algebra va analiz asoslari. 2-qism. – Toshkent: O’qituvchi, 1982.
6. Kudryavsev V.A., Demidovich V.R. Kratkiy kurs vыsshey matematiki. - Moskva: Nauka, 1985.
7. Loboskaya N.L. Osnovы vыsshey matematiki. – Minsk, 1978.
8. Minorskiy V.P. Sbornik zadach po vыsshey matematike. – Moskva: Nauka, 1977.
9. Piskunov N.S. Differensial va integral hisobi. I – II tomlar. -Toshkent: O’qituvchi, 1974.
10. Saxarnikov N.A. Vыsshaya matematika. – Leningrad, 1973.
11. Fixtengols G.M. Matematik analiz asoslari. 1- tom. – Toshkent: O’qituvchi, 1970.
12. Cherskasov A.N. Vvedeniye v vыsshuyu matematiku. – Moskva: Nauka, 1964.