O`ZBEKISTON RESPUBLIKASI OLIY VA
O`RTA MAXSUS TA`LIM VAZIRLIGI
ANDIJON DAVLAT UNIVERSITETI

Miladjonov V.G'., Mullajonov R.V., Turg'unova K.X.,
Abdugapporova Sh.N., Mullajonova J.V.

MATRITSALAR NAZARIYASINING TANLANGAN BOBLARI

(O'quv qo'llanma)

Andijon 2012 y

UDK 512.83

Mazkur o’quv qo’llanma universitetning yuqori kurs talabalari va margistrlari uchun mo’lgallangan bo’lib, unda matritsalar nazariyasining matritsalar algebrasi, kompleks simmetrik, kososimmetrik va ortoganal matritsalar, manfiymas elementli matritsalar, xos qiymatlarni regulyarligi va lokalligining har-xil kriteriyalari, matritsali tenglamalar, kvadratik formalar va ularning tadbiqlari, yirik masshtabli sistemalar turg’unligining umumiy masalasi kabi boblar bayon etilgan.

Mualliflar: Miladjonov V.G’ (ADU, Matematika kafedrasi mudiri), Mullajonov R.V (ADU, Matematika kafedrasi katta o’qituvchisi), Turg’unova K.X(ADU, Matematika kafedrasi katta o’qituvchisi), Abdugapporova Sh.N(ADU, Matematika kafedrasi katta o’qituvchisi) , Mullajonov J.V.(ADU, matematika yo’nalishi talabasi)

Taqrizchilar: ADU Matematika kafedrasi katta o’qituvchisi, f.-m.f.n. Arziqulov F., And MI dotsenti, Ergashev S

«Matematika» kafedrasining umumiy majlisida muhokama etildi va ma`qullandi (Bayonnoma № 2 4- sentyabr 2012 y).

Universitet ilmiy kengashida muxokama qilindi va chop etishga tavsiya etildi. (Bayonnoma № 3 29 oktyabr 2012 y).

SO'Z BOSHI

Ma'lumki xozirgi kunda matritsalar matematika, mexanika, nazariy fizika, nazariy elektrotexnika va boshqa ko'plab soxalarda keng qo'llanilmoqda. Ammo matritsalar nazariyasini to'la yoritib beruvchi o'zbek tilida yozilgan adabiyotlar mavjud emas. Ushbu o'quv qo'llanma universitetning yuqori kurs talabalari, margistrlari va ilmiy izlanishlar olib borayotgan barcha mutaxasislar uchun mo'lgallangan bo'lib, unda matritsalar nazariyasining matritsalar algebrasi, kompleks simmetrik, kososimmetrik va ortoganal matritsalar, manfiymas elementli matritsalar, xos qiymatlarni regulyarligi va lokalligining har-xil kriteriyalari, matritsali tenglamalar, kvadratik formalar va ularning tadbiqlari, yirik masshtabli sistemalar turg'unligining umumiy masalasi kabi boblar bayon etilgan. Har-bir bobning oxirida shu bobni mustaxkamlash uchun mashqlar keltirilgan.

Ushbu qo'llanmani o'rganish uchun o'quvchi universitet dasturi xajmida, algebra va sonlar nazariyasi, matematik taxlil, kompleks o'zgaruvchili funktsiyalar nazariyasi, differentsial tenglamalar kabi fanlarni to'la o'zlashtirgan bo'lishi kerak.

Qo'lanma sakkiz bobdan iborat.

Birinchi bob, matritsalar algebrasiga bag'ishlangan bo'lib, unda matritsalar va ular ustida amallar,umumlashgan transponirlangan matritsalar, simmetrik matritsalar, λ- matritsalar. Elementar bo'luvchilar, Jordon kataklari, asosiy teoremalar bayon qilingan.

Ikkinchi bobda, kompleks simmetrik, kososimmetrik va ortogonal matritsalar qarab chiqilgan bo'lib, unda kompleks ortaganal va unitar matritsyalar uchun ba`zi formulalar, kompleks matritsalarni qutub yoyilmasi, ko`mpleks simmetrik matritsalarning normal ko`rinishi, kompleks kososimmetkir matritsaning normal ko`rinishi, kompleks ortogonal matritsaning normal ko`rinishi keltirilgan.

Uchinchi bobda, matritsalarning singulyarlik dastasi o'rganilib, unda masalani qo`yilishi, matritsalarning regulyar dastasi, singulyar dastalar, keltirish xaqida teorema, matritsalar singulyar dastasining kanonik formasi, dastaning minimal indeksi,kvadratik formalarining singulyar dastasi, differentsial tenglamalarga tadbiqlari ko'rib, chiqilgan.

To'rtinchi bob, manfiymas elementli matritsalarni o'rganishga bag'ishlangan bo'lib, unda umumiy xossa, yoyilmaydigan manfiymas matritsaning spektral xossasi, yoyiluvchi matritsa, yoyiluvchi matritsaning normal formasi, primitiv va imirimitiv matritsalar, to'la manfiymas matritsalar to'la bayon qilingan.

Beshinchi bob, xos qiymatlarni regulyarligi va lokalligining har-xil kriteriyalarini o'rganishga bag'ishlangan bo'lib, unda Adamarning regulyarlik kriteryasi va uning umumlashgani, matritsa normasi, Adamar kriteriyasini blok matritsalarga kengaytirish, Fidlerning regulyarlik kriteryasi, Gershgoran doirasi va boshqa lokallashtirish sohalari qarab chiqilgan.

Oltinchi bobda, matritsali tenglamalar o'rganilib, unda $AX = XB$ tenglama, $A = B$ bo'lgan hususiy hol. O'rin almashinuvchi matritsalar, $Ax - xB = C$ tenglama, $f(x) = 0$ skalyar tenglama, matritsali ko'phadli tenglamalar, xosmas matritsadan m-darajali ildiz chiqarish, xos matritsadan m-darajali ildiz chiqarish, matritsa logarifmi bayon etilgan.

Ettinchi bob, kvadratik formalar va ularning tadbiqlarini o'rganishga bag'ishlangan bo'lib, unda kvadratik formalarda o'zgaruvchilarni almashtirish, inertsiya qonuni, Lagranj metodi, Yakobi formulasi, kvadratik formalarning ishoralari, kvadratik formalarni bosh o'qlarga keltirish, kvadratik formalar dastasi, formalar regulyar dastasi harakteristik sonlarining ekstremal xossasi, kvadratik formalar ustida amallar, n-o'zgaruvchili kvadratik formalarni ikki o'zgaruvchili kvadratik formalar yig'indisi shaklida yozish, erkinlik darajasi n bo'lgan sistemalarning kichik tebranishlari, chiziqli yirik masshtabli sistemalar turg'unligi masalasiga bog'liq bo'lgan ba'zi teoremalar, dempfirlanishi va bikirligi oshkor xolatda vaqtga bog'liq bo'lib, chiziqsiz bo'lgan sistema asimptotik turg'unligining yetarli shartlari ko'rib chiqilgan.

Sakkizinchi bobda, matritsalar nazariyasini tadbiqi sifatida yirik masshtabli sistemalar turg’unligining masalasiga bag’ishlangan bo’lib, unda masalaning qo’yilishi, yirik masshtabli sistemalarning dekompozitsiyasi, Lyapunov matritsa funktsiyasi usuli bayon etilgan.

Qo’llanmani I, VI,VIII- bobolari V.G’. Miladjonov, II,III- boblari K.X.Turg’unova, IV,V-boblari R.V.Mullajonov, VII-bobi esa Sh.N. Abdugapporova va J.V.Mullajonovalar tomonidan yozilgan bo’lib, u V.G’. Miladjonov va K.X.Turg’unovalarning taxriri ostida chop etishga tayorlandi.

Mualliflar fizika-matematika fanlari nomzodi F. Arziqulov va dotsent S. Ergashevlarga qo’llanmani yozishdagi qimmatli maslaxatlari uchun chuqur minnatdorchilik bildiradi

I-BOB

MATRITSALAR ALGEBRASI

Matritsa tushunchasi chiziqli algebraning asosiy tushunchalaridan biri bo’lib, uning talaba tomonidan chuqur o’zlashtirilishi muhim ahamiyatga ega. Chunki, bu tushunchaning tadbiqlari zamonaviy ishlab chiqarishdagi muhim iqtisodiy, texnikaviy masalalarni yechishda keng qo’llaniladi.

§1. Matritsalar va ular ustida amallar.

Ushbu paragraf yordamchi xarakterda bo‘lib, unda matritsalar va kvadratik formalar xaqidagi umumiy tushunchalar esga tushiriladi. Ma’lumki bu tushunchalar "Oliy algebra" kursida to‘la ko‘rib chiqilgan, shuning uchun isbotlanadigan jumlalarning isbotlariga to‘xtalib o‘tmaymiz

Ta’rif 1.1: n ta satr va m ta ustundan iborat bo‘lib, to‘g‘ri turtburchak shaklida joylashgan, n·m ta elementdan tuzilgan ixtiyoriy jadval nxm tipdagi matritsa deyiladi. Matritsani tashkil qiluvchi narsalar uning elementlari deyiladi. nxm tipdagi A matritsa quyidagicha yoziladi:

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \, . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}$$

yoki qisqacha ko‘rinishda

$$A=[a_{ki}], \ k=1,2,...,n, \ i=1,2,...,m$$

Agar matritsaning ustunlar soni bitta (m=1) bo‘lsa, u xolda ustun matritsani xosil qilamiz.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Shuningdek, satrlari soni bitta (n=1) bo‘lsa,

$$y=[y_1, y_2, ..., y_m]$$

satr matritsani xosil qilamiz.

Agar matritsaning satrlari soni bilan ustunlar soni o‘zaro teng bo‘lsa, u xolda matritsa kvadrat matritsa deyilib, uning satrlar (yoki ustunlar) soni matritsaning tartibi deyiladi.

Ta’rif 1.2. Matritsaning k ta satri va k ta ustunidan tuzilgan determinant bu matritsaning k-tartibli minori deyiladi.

Masalan, birinchi tartibli minorlar shu matritsa elementlarining o‘zlari bo‘lib, ularning soni n·m ta bo‘ladi, quyidagi 2x3 tipdagi

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

matritsa uchun uchta xar xil ikkinchi tartibli

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

minorlarni tuzish mumkin. n-tartibli A kvadratik matritsaning n-tartibni minori shu matritsaning determinantiga teng bo‘lib, detA yoki ¦A¦ ko‘rinishda belgilanadi.

Ta’rif 1.3. Satrlar soni va ustunlar soni o‘zaro teng bo‘lib, mos elementlari ham o‘zaro teng bo‘lgan matritsalar o‘zaro teng deyiladi. Shuning uchun ikkita matritsaning o‘zaro tengligi A=B, n·m ta skalyarlarning o‘zaro tengligi $a_{ki}=b_{ki}$, k=1,2,...,n, i=1,2,...,m bilan teng kuchlidir.

Ta’rif 1.4. Matritsani songa ko‘paytirish deb, shu matritsaning hamma elementlarini shu songa ko‘paytirishdan xosil bo‘lgan matritsaga aytiladi, ya’ni

$$\lambda A = \lambda[a_{ki}] = [\lambda a_{ki}] \quad k=1,2,...,n, \ i=1,2,...,m.$$

Hamma elementlari nolga teng bo‘lgan matritsa nol matritsa deyiladi.

Ta’rif 1.5. Bir xil tipdagi ikkita matritsaning yig‘indisi deb shunday matritsaga aytiladiki, bu matritsaning elementlari, qo‘shiluvchi matritsalar mos elementlarining yig‘indisidan iborat bo‘lib, yig‘indi matritsaning tipi qo‘shiluvchi matritsalar tipi bilan bir xil bo‘ladi.

Bu aytilganlardan quyidagilar kelib chiqadi:

$$A+(B+C)=(A+B)+C,$$

$$A+B=B+A,$$

$$A+0=A,$$

$$(\alpha+\beta)A=\alpha A+\beta A,$$

$$\alpha(A+B)=\alpha A+\alpha B,$$

bu yerda A, B, C - matritsalar, α,β -skalyar.

Ta'rif 1. 6. A matritsaning ustunlari soni B matritsaning satrlari soniga teng bo'lgan shartda A va B matritsalarning ko'paytmasi deb, shunday C matritsaga aytiladiki, uning elementlari

$$c_{ki} = \sum_{j=1}^{m} a_{kj} b_{ji}$$

qoida bo'yicha aniqlangan bo'ladi. Agar A matritsa $n\times m$ tipda B matritsa $m\times s$ tipda bo'lsa, C=AB matritsa $n\times s$ tipdagi matritsa bo'ladi.

Bu ta'rifdan quyidagilar kelib chiqadi;

$$AB\neq BA$$

$$(A+B)C = AC+ BC.$$

Ikkita kvadratik matritsa ko'paytmasining determinanti shu matritsalar determinantlari ko'paytmasiga teng, ya'ni

$$\det(AB) = \det A \det B.$$

Ta'rif 1.7. Kvadratik matritsa bosh dioganalida turgan elementlari yig'indisi, shu matritsaning izi deyiladi va Sp belgi bilan belgilanadi. Demak,

$$SpA = a_{11}+a_{22} +...+ a_{nn}$$

Diogonalidagi barcha elementlari birga teng bo'lib, qolgan barcha elementlari nollardan iborat bo'lgan matritsa birlik matritsa deyiladi va E bilan belgilanadi.

Bevosita xisoblash bilan

$$AE = EA=A$$

ekanligini ko'rsatish mumkin.

Quyidagi ko'rinishdagi kvadrat matritsa

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

diogonal matritsa deyilib, diagA=(a_{11},a_{22},...,a_{nn}) ko‘rinishda yoziladi.

Ta’rif 1.8. Agar kvadratik matritsaning determinanti noldan farqli bo‘lsa, u xolda bu matritsa maxsusmas aks xolda maxsus deyiladi.

Agar A·A'=E tenglik bajarilsa A' matritsa A matritsaga teskari matritsa deyilib, A'=A^{-1} bo’ladi. Ixtiyoriy maxsusmas matritsani teskari matritsaga ega ekanligini isbotlash mumkin.

Ta’rif 1.9. Agar A matritsaning satrlarini ustun, ustunlarini satr qilib yozsak, xosil bo‘lgan matritsa A matritsaning transponirlangan matritsasi deyilib, A^T ko‘rinishda belgilanadi. Demak,

$$A=[a_{ki}] \text{ bo‘lsa, } A^T = [a_{ik}], \quad i=1,2,...,m, \ k=1,2,...,n.$$

Transponirlangan va teskari matritsalarning ta’riflaridan bevosita quyidagi tengliklar kelib chiqadi.

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$\det A^T = \det A.$$

Ta’rif 1.10. A=[a_{ki}], i,k=1,2,...,n kvadratik matritsaning elementlari bosh dioganalga nisbatan simmetrik joylashgan bo‘lsa, ya’ni $a_{ki} = a_{ik}$ bo‘lsa, u simmetrik matritsa deyiladi. Simmetrik matritsa uchun A^T =A tenglik o‘rinli.

Ta’rif 1.11. A kvadratik matritsaning elementlari

$$a_{ki}=-a_{ik}, \quad i,k=1,2,...,n,$$

tenglikni qanoatlantirib, bosh dioganaldagi elementlari nolga teng, ya’ni a_{ii}=0, i=1,2,...,n bo‘lsa, u kososimmetrik matritsa deyiladi. Kososimmetrik matritsalar uchun

$$A^T=-A$$

tenglik o‘rinli.

Oliy algebradan ma’lumki, toq tartibli kososimmmetrik matritsalarning determinantlari aynan nolga teng, juft tartibli kososimmetrik matritsalarning determinantlari esa uning elementlari butun ratsional funksiyasi kvadratini ifodalaydi. Demak, xaqiqiy elementli kososimmetrik matritsalarning determinantlari manfiymas bo‘ladi.

Ixtiyoriy kvadratik matritsani simmetrik va kososimmetrik matritsalar yig‘indisi ko‘rinishida tasvirlash mumkin. Xaqiqatan,

$$\Lambda=[\alpha_{ki}]$$

Ixtiyoriy kvadratik matritsa bo‘lsin. Undan

$$A=\frac{1}{2}\left(\Lambda+\Lambda^{T}\right), \quad B=\frac{1}{2}\left(\Lambda-\Lambda^{T}\right)$$

matritsalarni tuzamiz. Aniqki, A matritsa simmetrik, B matritsa kososimmetrik bo‘lib,

$$\Lambda = A + B$$

bo‘ladi.

Ta’rif 1.12. Agar Λ kvadrat matritsa uchun

$$\Lambda\cdot\Lambda^{T}=E$$

tenglik o‘rinli bo‘lsa, u ortogonal matritsa deyiladi.

Ortogonal matritsaning bu ta’rifidan quyidagi natijalar kelib chiqadi:

1) $\Lambda^{T}=\Lambda^{-1}$

2) Ortogonal matritsaning determinanti ±1 ga teng ya’ni

$$\Delta=\det\Lambda=\pm1 ;$$

3) Ixtiyoriy satr (yoki ustun) elementlari kvadratlari yig‘indisi birga teng, ya’ni

$$\sum_{i}\alpha^{2}_{ki}=\sum_{k}\alpha^{2}_{ik}=1;$$

4) Qandaydir satr (ustun) elementlarini boshqa satr (ustun) mos elementlariga ko‘paytmasining yig‘indisi nolga teng, ya’ni

$$\sum_{i}\alpha_{ki}\alpha_{mi}=\sum_{i}\alpha_{ik}\cdot\alpha_{im}=0 \qquad k\neq m$$

Agar matritsa elementlari skalyar parametrga masalan, t vaqtga bog‘liq bo‘lsa, u holda matritsani bu parametr bo‘yicha hosilasi deb, elementlari berilgan matritsa mos elementlaridan shu parametr bo‘yicha olingan hosilalardan iborat bo‘lgan matritsaga aytiladi. Demak, agar $X=[x_{ki}]$, bo‘lsa,

$$\dot{X} = [\dot{x}_{ki}] \quad \text{yoki} \quad \frac{dX}{dt} = \left[\frac{dx_{ki}}{dt}\right].$$

Biz qarab chiqqan matritsalarning elementlari sonlardangina iborat edi. Umuman olganda matritsalarning elementlari ixtiyoriy ob’ektlar bo‘lishi mumkin, xususan shunday matritsalarni qarash mumkinki, ularning elementlari o‘zlari matritsalardan iborat bo’ladi.

Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & c_1 & c_2 \\ a_{21} & a_{22} & a_{23} & d_{11} & d_{12} \\ b_1 & b_2 & b_3 & d_{21} & d_{22} \end{vmatrix}$$

matritsani qisqacha quyidagicha yozish mumkin

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix},$$

bu yerda $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$, $C = [c_1, c_2]$,

$$B = [b_1, b_2, b_3], \qquad A = \begin{vmatrix} d_1 & d_{12} \\ d_{21} & d_{22} \end{vmatrix}.$$

Matritsalar yordamida quyidagi o‘zgarmas koeffitsientli chiziqli differensial tenglamalar sistemasini sodda va ixcham ko‘rinishda yozish mumkin. Xaqiqatan,

$$\begin{aligned} \dot{x}_1 &= a_n x_1 + ... + a_{1n} x_n \\ &.............................. \\ \dot{x}_n &= a_{n1} x_1 + ... + a_{nn} x_n \end{aligned} \qquad (1.1)$$

differensial tenglamalar sistemasini matritsa ko’rinishida yozish uchun, quyidagi 2 ta matritsalarni kiritamiz.

1. (1.1) tenglamalar o‘ng tomonlaridagi koeffitsientlardan tuzilgan matritsani

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{bmatrix}$$

2. Ustun matritsa yoki vektorni

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

bu matritsalarni ko‘paytirib, quyidagi ustun matritsani tuzamiz.

$$AX = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n \\ \dots & & \dots & & \dots & & \dots \\ a_{n1}x_1 & + & a_{n1}x_2 & + & \dots & + & a_{nn}x_n \end{bmatrix}$$

Nixoyat ikki matritsaning tenglik shartidan foydalanib isbotlash mumkinki (1.1) sistema quyidagi matritsali tenglamaga teng kuchli bo’ladi.

$$\dot{X} = A \cdot X$$

Bundan murakkab bo‘lgan differensial tenglamalar sistemasini ham matritsa ko‘rinishida yozish mumkin.

Xususiy xolda quyidagi

$$\sum_{j=1}^{s}\left(a_{kj}\ddot{x}_j + b_{kj}\dot{x}_j + c_{kj}x_j\right) = X_k\,, \qquad k=1,2,...s$$

ikkinchi tartibli tenglamalar sistemasini matritsa ko‘rinishidagi yozuvi

$$A\ddot{x} + B\dot{x} + Cx = X$$

bo‘lib, bu yerda A=[a_{kj}], B=[b_{kj}], C=[c_{kj}], k,j=1,2,...,s -kvadratik matritsalar, x va X lar elementlari mos ravishda x_i va X_i, i=1,2,...,n lardan iborat bo‘lgan ustun matritsalardir.

A-kvadrat matritsa va x-ustun matritsalarni o‘zaro ko‘paytirib, Ax - ustun matritsa (vektori) ni xosil qilamiz. Ma’lumki, ustun-matritsa bu vektordir, shuning uchun Ax va x ustun-matritsa (vektor) larni o‘zaro skalyar ko‘paytirib, xadlarni qayta gruppalab chiqsak,

$$x^T Ax = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki}x_i \cdot x_k \qquad (1.2)$$

xosil bo‘ladi.

Agar A matritsa simmetrik, ya’ni $a_{ki} = a_{ik}$ bo‘lsa,

$$x^T Ax = a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \ldots + 2a_{n-1}x_{n-1}x_n = \sum_{k}^{n}\sum_{i}^{m} a_{ki}x_kx_i,$$

oddiy kvadratik forma xosil bo‘ladi.

Agar $x^T Ax$ kvadratik forma musbat aniqlangan bo‘lsa, u xolda soddalik uchun A matritsa musbat- aniqlangan deyiladi.

Agar A matritsa kososimmetrik, ya’ni $a_{kk}=0$, $a_{ki}=-a_{ik}$ bo‘lsa, u xolda $A\vec{x}\cdot\overline{x}=0$ bo’ladi.

Bizga n ta satr va m ta ustundan iborat bo‘lgan nxm tipdagi A matritsa berilgan bo‘lsin.

$$A=[a_{ki}], \quad k=1, 2,..., n, \quad i=1, 2,..., m.$$

Ta’rif 1.13 . A matritsaning normasi deb

$$\|A\| = \sup\left|\sum_{k=1}^{n}\sum_{i=1}^{m} a_{ki}\xi_k\eta_i\right|$$

songa aytiladi. Bu yerda ξ_k va η_i lar mos ravishda

$$\sum_{k=1}^{n}|\xi_k|^2 = 1 \qquad \text{va} \qquad \sum_{i=1}^{m}|\eta_i|^2 = 1$$

tengliklarni qanoatlantiruvchi sonlar.

Amalda, ko‘p xollarda quyidagi ko‘rinishdagi normalar ham ishlatiladi.

$$\|A\|_I = \sup\sum_{k=1}^{n}|a_{ki}|, \qquad \|A\|_{II} = \sup\sum_{i=1}^{m}|a_{ki}|$$

$$\|A\|_P = (\sum_{k=1}^{n}\sum_{i=1}^{m}|a_{ki}|^p)^{\frac{1}{P}}, \qquad p=1,2,...$$

$$\|A\|_1 = \sum_{k=1}^{n}\sum_{i=1}^{m}|a_{ki}|, \qquad \|A\|_2 = (\sum_{k=1}^{n}\sum_{i=1}^{m}|a_{ki}|^2)^{\frac{1}{2}}, \quad \text{(Evklidcha norma)}$$

Agar A matritsa kvadrat matritsadan iborat bo‘lsa, uning normasini quyidagicha xam aniqlash mumkin.

$$\|A\| = \lambda_M^{1/2}\left(A \cdot A^T\right) ,$$

bu yerda λ_M (AA^T) - AA^T matritsaning maksimal xos qiymati.

Agar A-simmetrik kvadrat matritsadan iborat bo‘lsa, uning normasi shu matritsa maksimal xos qiymatiga teng bo‘ladi, ya’ni

$$\|A\| = \lambda_M(A)$$

Matritsalarning normalari uchun quyidagi munosabatlar o‘rinli:

$$\|A + B\| \le \|A\| + \|B\| \ , \quad \|\alpha * A\| = \alpha * \|A\| \ ,$$

$$\|A \cdot B\| \le \|A\| \cdot \|B\|$$

Xususiy xolda $\|A \cdot x\| \le \|A\| \cdot \|x\|$.

§ 2. Umumlashgan transponirlangan matritsalar

Ta’rif 1.14. Matritsani transponirlash deb, biror aniq qonun yoki qoida bo‘yicha uning barcha elementlarini o‘rinlarini almashtirishga aytiladi.

Bizga $m \times n,\ (m \le n)$ o‘lchovli $A = (a_{ij}),\ (i = 1.2....m,\ j = 1,2,...n,) -$ to‘g‘ri to‘rtburchakli matritsa berilgan bo‘lsin. Matritsaning barcha elementlarini o‘rinlarini almashtiruvchi trivial (sodda) qoidalarni qarab chiqaylik:

1. Matritsa satrlarini (ustunlarini) uning ustunlari (satrlari) bilan to‘g‘ridan to‘g‘ri (to‘g‘ri tartibda) almashtirish,
2. matritsa satrlarini (ustunlarini) uning ustunlari (satrlari) bilan teskari tartibda almashtirish,
3. matritsa i- chi satrini (i=1,2,…m) mos ravishda m+1-i-chi satri bilan almashtirish,
4. matritsa j-ustunini (j=1,2,…,n) mos ravishda n+1-j- ustuni bilan almashtirish,
5. matritsa i- satrini (i=1,2,…m) mos ravishda m+1-i-chi satri bilan, j-ustunini (j=1,2,…,n) mos ravishda n+1-j- ustuni bilan almashtirish.

Avval matritsa bilan bog‘liq bo‘lgan ba’zi tushunchalarni aniqlab olamiz. Ma’lumki xar bir to‘g‘ri to‘rtburchakli matritsaga shu matritsa elementlari ichida yotuvchi to‘g‘ri to‘trburchak mos keladi.

a) A matritsaning bosh (bosh bo‘lmagan) diagonali deb, shu matritsaning a_{ii} , i=1,2,…,m ($a_{i,m+1-i}$,i=1,2,…,m) elementlari joylashgan nuqtalardan o‘tuvchi to‘g‘ri chiziq kesmasiga aytiladi.

b) A matirsaning vertikal (gorizontal) o‘qi deb, shu matritsaga mos to‘g‘ri burchakli to‘rtburchakning vertikal (gorizontal) simmetriya o‘qlariga aytiladi,

v) A matritsaning markazi deb, unga mos to‘g‘ri to‘rtburchakning simmetriya markaziga aytiladi.

To‘g‘ri to‘rtburchakli A matritsaning bosh va bosh bo‘lmagan diagonallari unga mos to‘g‘ri to‘rtburchakning diagonallari bilan ustma - ust tushmaydi. Shuning uchun bunday matritsalar transponirlanganda ularning o‘lchovi nxm ga almashadi. Agar nqm bo‘lsa, ya’ni A kvadrat matritsadan iborat bo‘lsa, u xolda bu matritsaning bosh (bosh bo‘lmagan) diagonali unga mos kvadratning chap (o‘ng) diagonali bilan ustma - ust tushadi. Demak , geometrik nuqtai nazardan matritsani transponirlash nuqta yoki to‘g‘ri chiziqqa nisbatan amalga oshiriladi. Agar nuqta yoki to‘g‘ri chiziq kesmasi shu matritsaga mos to‘g‘ri to‘rtburchak (kvadrat) ning simmetriya markazi yoki simmetriya o‘qi bilan ustma - ust tushsa, u xolda transponirlangan matritsaning o‘lchovi o‘zgarmaydi, aks xolda transponirlangan matritsaning o‘lchovi o‘zgaradi. Agar A matritsa biror nuqta yoki to‘g‘ri chiziqqa nisbatan transponirlansa, u xolda bu matritsaning shu nuqta yoki to‘g‘ri chiziqda yotgan elementlari (agar bo‘lsa) o‘zgarmay qoladi.

Agar A matritsaga biror to‘g‘ri to‘rtburchak (kvadrat) mos kelib, A matritsa shu to‘g‘ri to‘rtburchak (kvadrat)da yotuvchi nuqta yoki to‘g‘ri chiziq kesmasiga nisbatan transponirlangan bo‘lsa, u xolda transponirlangan matritsaga shu to‘g‘ri to‘rtburchak (kvadrat) ni transponirlash o‘tkazilgan nuqta yoki to‘g‘ri chiziq kesmasi atrofida 180^o ga burilgani mos keladi.

Matritsalarni transponirlashning mexanik ma’nosini ochish uchun matritsa bilan yirik masshtabli mexanik sistemalar (YMMS) o‘rtasida quyidagicha moslik o‘rnatamiz.

$A=(a_{ij})$ – to‘g‘ri burchakli mxn (aniqlik uchun $m \le n$ deb olamiz) o‘lchovli matritsa bo‘lib, R^n da aniqlangan (YMMS) m ta erkin qism sistemalardan tashkil topgan bo‘lsin. A matritsaning bosh diagonalida yotuvchi elementlariga YMMS ning erkin qism sistemalarini shunday mos qo‘yamiz, unda a_{ii} , $i=1,2,\ldots,m$ elementga mos keluvchi erkin qism sistema $a_{m+1-i,m+1-i}$ elementga mos keluvchi erkin qism sistema bilan muvozanatlashsin, A matritsaning $a_{ij},\ i,j=1,2,..,m\ i<j\ (i>j)$ elementlariga mos a_{ii} va $a_{jj},\ i,j=1,2,..,m$ erkin qism sistemalar orasidagi bog‘lanishlar (teskari bog‘lanishlar) ni, ya’ni $a_{ii}\ (a_{jj})$ elementga mos keluvchi erkin qism sistemani $a_{jj}\ (a_{ii})$ elementga mos keluvchi erkin qism sistemaga ta’sirini ifodalovchi funksiyalarni mos qo‘yamiz. Bu bog‘lanishlar YMMS ning ichki bog‘lanishlari deyladi. A matritsaning qolgan elementlariga, ya’ni $a_{ij},\ i=1,2,...m,\ j=m+1,m+2...n,\ \ i<j\ (i>j)$ elementlariga erkin qism sistemalar bilan berilgan YMMS bilan birga xarakterlanuvchi tashqi sistemalar orasidagi bog‘lanishlar (teskari bog‘lanishlar) ni mos qo‘yamiz. Bu bog‘lanishlar tashqi bog‘lanishlar deyiladi. Agar $m=n$ bo‘lsa, tashqi bog‘lanishlar qaralmaydi, ya’ni barcha bog‘lanishlar ichki bog‘lanishlar bo‘ladi. Bunday o‘rnatilgan moslikda matritsaning mos bo‘lmagan diagonalidagi elementlarga o‘zaro muvozanatlashuvchi erkin qism sistemalar orasidagi bog‘lanishlar va teskari bog‘lanishlar mos keladi. Agar m – juft bo‘lsa, u xolda xar bir erkin qism sistemaga mos muvozanatlashtiruvchi erkin qism sistema mavjud bo‘ladi. Agar m – toq bo‘lsa, u xolda $a_{\frac{m+1}{2},\frac{m+1}{2}}$ elementga mos erkin qism sistemaga muvozanatlashuvchi qism sistema mavjud bo‘lmaydi. Shuning uchun bu erkin qism sistema etalon qism sistema deyilib, aloxida qaraladi. (Masalan, yirik masshtabli energetik sistemalarda sistemani tashkil etuvchi mashinalar soni toq bo‘lib, bitta mashina etalon mashina sifatida qaraladi). Bunday moslikdan ko‘rinadiki, transponirlash YMMS lar ichki strukturasi o‘zgarishini aniqlaydi.

Endi A matritsaning barcha elementlarining o‘rinlarini almashtiruvchi, yuqorida keltirilgan, trivial (sodda) qoidalarga mos keluvchi, matritsani transponirlashning ta’riflarini keltiramiz.

Ta’rif 1.15.

1. A matritsaning satrlarini (ustunlarini) ustunlari (satrlari) bilan to‘g‘ri tartibda almashtirib, xosil qilingan $A^T = (a_{ji}),\ j = 1,2,...,n,\ i = 1,2,...,m,$ matritsa,

2. A matritsaning satrlarini (ustunlarini) ustunlari (satrlari) bilan teskari tartibda almashtirib xosil qilingan $A^{\perp} = (a_{n+1-j,m+1-j}),\ j = 1,2,...n,\ i = 1,2,...,m$ matritsa,

3. A matritsaning i- satrini m+1-i satri bilan almashtirib xosil qilingan $A = (a_{m+1-i,j}),\ i = 1,2,..,m,\ j = 1,2,...,n,$ matritsa,

4. A matritsaning j- ustunini n+1-j- ustuni bilan almashtirib, xosil qilingan $A^{!} = (a_{i,n+1-j}),\ i = 1,2,...m,\ j = 1,2,...,n$ matritsa,

5. A matritsaning i- satrini m+1-i satri bilan, j- ustunini n+1-j- ustuni bilan almashtirib xosil qilingan $A^0 = (a_{m+1-j,n+1-j}),\ i = 1,2,...,m,\ j = 1,2,...,n,$ matritsa,

A matritsani

1) bosh diagonali bo‘yicha,

2) bosh bo‘lmagan diagonali bo‘yicha,

3) gorizontal o‘qi bo‘yicha,

4) vertikal o‘qi bo‘yicha,

5) markazi bo‘yicha transponirlangan matritsasi deyiladi.

Bu ta’rifning geometrik ma’nosi A matritsaga mos keluvchi to‘g‘ri to‘rtburchakni

1) bosh diagonalidan o‘tuvchi to‘g‘ri chiziq atrofida,

2) bosh bo‘lmagan diagonalidan o‘tuvchi to‘g‘ri chiziq atrofida,

3) gorizontal o‘qi atrofida,

4) vertikal o‘qi atrofida,

5) A matritsa markazi atrofida 180^o ga burishni ifodalaydi.

Yuqorida matritsa bilan YMMS o‘rtasida o‘rnatilgan moslikka asosan shuni ayta olamizki, ta’rif 1.15 da keltirilgan transponirlangan matritsa mos ravishda qaralayotgan YMMS ichki strukturasini

1) erkin qism sistemalarni o‘zgartirmay, erkin qism sistemalar o‘rtasidagi bog‘lanishlarni ularga mos teskari bog‘lanishlar bilan o‘zaro almashtirib,

2) o‘zaro muvozanatlashuvchi erkin qism sistemalar o‘rtasidagi bog‘lanishlar va teskari bog‘lanishlar o‘zgarmay, muvozanatlanuvchi erkin qism sistemalarni o‘zaro va qolgan bog‘lanishlarni (teskari bog‘lanishlarni) mos ravishda o‘zaro almashtirib,

3) erkin qism sistemalarni o‘zaro muvozanatlashuvchi erkin qism sistemalar orasidagi bog‘lanishlar va teskari bog‘lanishlar bilan teskari tartibda almashtirib,

4) erkin qism sistemalarni o‘zaro muvozanatlashuvchi erkin qism sistemalar orasidagi bog‘lanishlar va teskari bog‘lanishlar bilan to‘g‘ri tartibda almashtirib,

5) etalon qism sistema (agar bor bo‘lsa) dan tashqari muvozanatlashuvchi qism sistemalarni o‘zaro va ularga mos barcha bog‘lanishlarni mos teskari bog‘lanishlar bilan almashtirib, o‘zgartirilishini ifodalaydi.

Eslatma 1. Agar n (m) – toq bo‘lsa, u xolda vertikal (gorizontal) o‘q bo‘yicha transponirlashda etalon qism sistema va unga mos vertikal (gorizontal) bog‘lanishlar va teskari bog‘lanishlar o‘zgartirilmaydi. Agar n (m) – juft bo‘lsa, bunday qism sistema mavjud emas.

2. Agar n va m – toq bo‘lsa, u xolda markaz bo‘yicha transponirlashda faqat etalon qism sistema o‘zgartirilmaydi, bu qism sistemaga mos bog‘lanishlar va teskari bog‘lanishlar teskari tartibda o‘zaro almashadi. Agar n va m – juft bo‘lsa, bunday qism sistema mavjud bo‘lmaydi.

Misollar : 1. $X = (x_1, x_2, ..., x_n)$ bo‘lsin. U xolda

$$X^T = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \qquad X^{\perp} = \begin{pmatrix} x_n \\ x_{n-1} \\ . \\ x_1 \end{pmatrix}, \quad X^{!} = \left(x_n, x_{n-1}, ..., x_2, x_1\right)$$

$$X^{-}=(x_1,x_2,...,x_n)=X, \qquad X^{0}=(x_n,x_{n-1},...,x_1)$$

2. $$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

bo‘lsin, u xolda

$$A^{T}=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{44} \end{pmatrix}, \quad A^{\perp}=\begin{pmatrix} a_{34} & a_{24} & a_{14} \\ a_{33} & a_{23} & a_{13} \\ a_{32} & a_{22} & a_{12} \\ a_{31} & a_{21} & a_{11} \end{pmatrix},$$

$$A^{!}=\begin{pmatrix} a_{14} & a_{13} & a_{12} & a_{11} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{34} & a_{33} & a_{32} & a_{31} \end{pmatrix}, \quad A^{-}=\begin{pmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}, \quad A^{0}=\begin{pmatrix} a_{34} & a_{33} & a_{32} & a_{31} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{14} & a_{13} & a_{12} & a_{11} \end{pmatrix},$$

Bevosita tekshirib quyidagi xossalarni o‘rinli ekanligiga ishonch xosil qilish mumkin.

1. Agar A va V $m\times n$ o‘lchovli, to‘g‘ri to‘rtburchakli matritsalar bo‘lsa, u xolda

$$(\mathrm{A}+\mathrm{B})^{\mathrm{T}}=\mathrm{A}^{\mathrm{T}}+\mathrm{B}^{\mathrm{T}}, \quad (\mathrm{A}+\mathrm{B})^{\perp}=\mathrm{A}^{\perp}+\mathrm{B}^{\perp}, \quad (\mathrm{A}+\mathrm{B})^{!}=\mathrm{A}^{!}+\mathrm{B}^{!},$$
$$(\mathrm{A}+\mathrm{B})^{-}=\mathrm{A}^{-}+\mathrm{B}^{-}, \quad (\mathrm{A}+\mathrm{B})^{0}=\mathrm{A}^{0}+\mathrm{B}^{0}$$

2. Agar $\mathrm{A}-m\times n,$ o‘lchovli, to‘g‘ri to‘rtburchakli matritsa bo‘lib, $\alpha\neq 0$ xaqiqiy son bo‘lsa, u xolda

$$(\alpha A)^{T}=\alpha A^{T}, \ (\alpha A)^{\perp}=\alpha A^{\perp}, \ (\alpha A)^{!}=\alpha A^{!}, \ (\alpha A)^{-}=\alpha A^{-}, \ (\alpha A)^{0}=\alpha A^{0}.$$

3. Agar $\mathrm{A}-m\times n,$ o‘lchovli, to‘g‘ri to‘rtburchakli matritsa bo‘lsa, u xolda $(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}}=\mathrm{A}, \ (\mathrm{A}^{\perp})^{\perp}=\mathrm{A}, \ (\mathrm{A}^{!})^{!}=\mathrm{A}, \ (\mathrm{A}^{-})^{-}=\mathrm{A}, \ (\mathrm{A}^{0})^{0}=\mathrm{A}.$

4. Agar A $m\times n$, V $n\times m$ o‘lchovli to‘g‘ri burchakli matritsalar bo‘lsa, u xolda $(\mathrm{AB})^{\mathrm{T}}=\mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}, \ (\mathrm{AB})^{\perp}=\mathrm{B}^{\perp}\mathrm{A}^{\perp}, \ (\mathrm{AB})^{0}=\mathrm{B}^{0}\mathrm{A}^{0}$

5. Agar A n - tartibli kvadrat matritsa bo‘lsa, u xolda

$$(A^{T})^{\perp}=(A^{\perp})^{T}=A^{0}, \qquad (A^{!})^{-}=(A^{-})^{!}=A^{0}, \ (A^{0})^{!}=(A^{!})^{0}=A^{-},$$

$$(A^0)^T = (A^T)^0 = A^{\perp}, \qquad (A^0)^{\perp} = (A^{\perp})^0 = A^T, \ (A^0)^{-} = (A^{-})^0 = A^{!}$$

6. Agar A –n tartibli kvadrat matritsa bo‘lsa, u xolda

$$\mathrm{A} = (\mathrm{A}^0)^{-1}(\mathrm{A}^{\mathrm{T}}\mathrm{A}^{\perp})^{\mathrm{T}} = (\mathrm{A}^0)^{-1}(\mathrm{A}^{\perp}\mathrm{A}^{\mathrm{T}})^{\perp} = (\mathrm{A}^{\perp}\mathrm{A}^{\mathrm{T}})^{\mathrm{T}}(\mathrm{A}^0)^{-1} = (\mathrm{A}^{\mathrm{T}}\mathrm{A}^{\perp})^{\perp}(\mathrm{A}^0)^{-1}$$

7. Agar A maxsusmas kvadrat matritsa bo‘lsa, u xolda

$$(A^{-1})^T = (A^T)^{-1}, \ (A^{-1})^{\perp} = (A^{\perp})^{-1}, \ (A^{-1})^{!} = (A^{!})^{-1}, \ (A^{-1})^{-} = (A^{-})^{-1}, \ (A^{-1})^0 = (A^0)^{-1}$$

8. Agar A n –tartibli kvadrat matritsa bo‘lsa, u xolda

$$\left|A^T\right| = \left|A^{\perp}\right| = \left|A^0\right| = |A|, \quad \left|A^{!}\right| = \left|A^{-}\right| = (-1)^{\alpha}|A|,$$

bu yerda α – A matritsadan $A^{!}$ yoki A^{-} matritsalarni xosil qilish uchun A matritsaning satr yoki ustunlarini almashtirishlar soni.

Bu tengliklarning to‘g‘riligi ta’rif 1.15 va determinantning xossalaridan kelib chiqadi.

9. Agar A n –tartibli kvadrat matritsa, E n-tartibli birlik matritsa va λ sonli parametr bo‘lsa, u xolda $\left|A^T - \lambda E\right| = \left|A^{\perp} - \lambda E\right| = \left|A^0 - \lambda E\right| = |A - \lambda E|$

Bu tengliklarning to‘g‘riligi 1.,2.,7. xossalar va $E = E^T = E^{\perp} = E^0$ ekanligidan kelib chiqadi.

10. Agar Sp(A) – A matritsaning izi bo‘lsa, u xolda

$$Sp(A^T) = Sp(A^{\perp}) = Sp(A^0) = Sp(A),$$

11. Agar $rang(A)$ -A matritsaning rangi bo‘lsa, u xolda

$$rang(A^T) = rang(A^{\perp}) = rang(A^{!}) = rang(A^{-1}) = rang(A^0) = rang(A)$$

12. A kvadrat matritsa bo‘lib, $\Delta_i, \Delta_i^T, \Delta_i^{\perp}, \Delta_i^0$ $(\overline{\Delta}_i, \overline{\Delta}_i^T, \overline{\Delta}_i^{\perp}, \overline{\Delta}_i^0)$ $i = 1,2,...,n$ lar mos ravishda $A, A^T, A^{\perp}, A^0$ matritsalarning bosh minorlari (ularning mos to‘ldiruvchi minorlari) bo‘lsin. U xolda quyidagi tengliklar o‘rinli:

$$\begin{aligned} &\Delta_i = \overline{\Delta}_{n-i}^0, \ i = 1,2,...,n-1, \ \Delta_n = \Delta_n^0 = |A| = \left|A^0\right| \\ &\Delta_i^0 = \overline{\Delta}_{n-i}, \ i = 1,2,...,n-1, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\Delta_i = \Delta_i^T, \ i = 1,2,...,n, \ \Delta_n = \Delta_n^T = |A| = \left|A^T\right| \\ &\Delta_i = \overline{\Delta}_{n-i}^{\perp}, \ i = 1,2,...,n-1, \ \Delta_n = \Delta_n^{\perp} = |A| = \left|A^{\perp}\right| \end{aligned} \tag{1.4}$$

Bu tengliklarning to‘g‘riligi ta’rif 1.15 , 7. xossa va determinantning xossalaridan kelib chiqadi.

§ 3. Simmetrik matritsalar

A- n- tartibli kvadrat matritsa bo‘lsin, ya’ni $A=(a_{ij}),\ i,\ j=1,2,...,n$

Ta’rif 1.16. A matritsa simmetrik deyiladi, agarda uning xar bir elementi uchun shunday element mavjud bo‘lib, bu elementlar juftliklari biror nuqta yoki to‘g‘ri chiziqqa nisbatan o‘zaro simmetrik bo‘lsa. Bu nuqta yoki to‘g‘ri chiziqda yotuvchi elementlar o‘z o‘ziga simmetrik deyiladi.

Simmetrik kvadrat matritsaning barcha ko‘rinishlarini aniqlash uchun quyidagicha belgilashlar kiritamiz. A- n- tartibli kvadrat matritsaga qandaydir kvadrat mos keladi.

1. Kvadratning chap (o‘ng) diagonalini A matritsaning bosh (bosh bo‘lmagan) diagonali deb ataymiz,

2. Kvadratning vertikal (gorizontal) simmetriya o‘qini A matritsaning vertikal (gorizontal) o‘qi deb aytamiz.

3. Kvadratning simmetriya markazini A matritsaning markazi deb aytamiz.

Ta’rif 1.17. A- n- tartibli kvadrat matritsa

1) bosh diagonalga nisbatan simmetrik matritsa deyiladi, agarda $A^T=A,$ ya’ni $a_{ij}=a_{ji},\ i,\ j=1,2,...,n$ bo‘lsa,

2) bosh bo‘lmagan diagonalga nisbatan simmetrik matritsa deyiladi, agarda $A^{\perp}=A,$ ya’ni $a_{ij}=a_{n+1-j,\,n+1-i},\ i,\ j=1,2,...,n$ bo‘lsa,

3) vertikal o‘qqa nisbatan simmetrik matritsa deyiladi, agarda $A^{!}=A,$ ya’ni $a_{ij}=a_{i,n+1-j},\ i,\ j=1,2,...,n$ bo‘lsa,

4) gorizontal o‘qqa nisbatan simmetrik matritsa deyiladi, agarda $A^{-}=A,$ ya’ni $a_{ij}=a_{n+1-i,\,j},\ i,\ j=1,2,...,n$ bo‘lsa,

5) matritsa markaziga nisbatan simmetrik matritsa deyiladi, agarda $A^{0}=A,$ ya’ni $a_{ij}=a_{n+1-i,n+1-j},\ i,\ j=1,2,...,n$ bo‘lsa,

Shuni aytib o‘tamizki, bosh va bosh bo‘lmagan diagonallarda A matritsaning elementlari mavjud, vertikal va gorizontal o‘qlarda esa n- juft bo‘lganda A matritsaning elementlari mavjud bo‘lmaydi, n- toq bo‘lganda mavjud bo‘ladi, matritsa markazida n- juft bo‘lganda matritsa elementi mavjud emas, n- toq bo‘lganda $a_{\frac{n+1}{2},\frac{n+1}{2}}$ element matritsa markazida yotadi.

E birlik matritsa bosh va bosh bo‘lmagan diagonallar, xamda matritsa markaziga nisbatan simmetrik bo‘ladi.

Ta’rif 1.18. R^n fazodagi n ta erkin qism sistemalardan tashkil topgan YMMS

1) erkin qism sistemalarga nisbatan simmetrik deyiladi, agarda uning mos bog‘lanishlari va teskari bog‘lanishlari bir xil bo‘lsa,

2) o‘zaro muvozanatlashuvchi erkin qism sistemalar o‘rtasidagi bog‘lanishlar va teskari bog‘lanishlarga nisbatan simmetrik deyiladi, agarda muvozanatlashuvchi erkin qism sistemalar juftliklari o‘zaro va o‘zaro muvozanatlashuvchi erkin qism sistemalar o‘rtasidagi bog‘lanishlardan boshqa bog‘lanishlar o‘zlariga mos teskari bog‘lanishlar bilan bir xil bo‘lsa.

3) YMMS markaziga nisbatan simmetrik deyiladi, agarda muvozanatlashuvchi erkin qism sistemalar juftliklari o‘zaro va barcha bog‘lanishlar o‘zlariga mos teskari bog‘lanishlar bilan bir xil bo‘lsa.

Ta’rif 1.17 dan simmetrik matritsalarning quyidagi xossalari kelib chiqadi.

1. Bosh va bosh bo‘lmagan diagonallariga nisbatan simmetrik bo‘lgan matritsalar shu matritsa markaziga nisbatan xam simmetrik bo‘ladi.

2. Vertikal va gorizantal o‘qlarga nisbatan simmetrik bo‘lgan matritsalar shu matritsa markaziga nisbatan xam simmetrik bo‘ladi.

3. Vertikal (gorizontal) o‘qga nisbatan simmetrik bo‘lgan matritsalar maxsus matritsalar bo‘ladi.

4. Ixtiyoriy A kvadrat matritsa uchun quyidagilar mos ravishda bosh diagonalga, bosh bo‘lmagan diagonalga, vertikal o‘qga, gorizontal o‘qga va matritsa markaziga nisbatan simmetrik matritsalar bo‘ladi.

$$S_1 = \frac{1}{2}(A + A^T),\ S_2 = \frac{1}{2}(A + A^{\perp}),\quad S_3 = \frac{1}{2}(A + A^{|}),\ S_4 = \frac{1}{2}(A + A^{-}),\ S_5 = \frac{1}{2}(A + A^{0}),$$

5. Agar A kvadrat matritsa bosh (bosh bo‘lmagan) diagonalga, vertikal (gorizontal) o‘qga, matritsa markaziga nisbatan simmetrik matritsa bo‘lsa, u xolda

$$A^i\,(i=1,2,...),\quad \alpha A,\quad T^* AT$$

lar xam mos ravishda bosh (bosh bo‘lmagan) diagonalga, vertikal (gorizontal) o‘qga, matritsa markaziga nisbatan simmetrik matritsa bo‘ladi. Bu yerda T - A matritsa bilan bir xil tartibli bo‘lgan maxsusmas kvadrat matritsa, α - xaqiqiy son, *- mos transponirlash belgisini bildiradi.

6. Agar A maxsusmas kvadrat matritsa bosh (bosh bo‘lmagan) diagonalga, matritsa markaziga nisbatan simmetrik bo‘lsa, u xolda A^{-1} xam mos ravishda bosh (bosh bo‘lmagan) diagonalga, matritsa markaziga nisbatan simmetrik bo‘ladi.

7. Agar A va B kvadratik matritsalar o‘z markazlariga nisbatan simmetrik matritsalar bo‘lsa, u xolda AB va BA matritsalar o‘z markazlariga nisbatan simetrik matritsalar bo‘ladi.

8. Agar A n- tartibli kvadrat matritsa o‘z markaziga nisbatan simmetrik bo‘lib, $\Delta_i,\ i=1,2,...,n$ bu matritsaning bosh minorlari, $\overline{\Delta}_i$ - shu minorlarga mos to‘ldiruvchi minorlar bo‘lsa, u xolda

$$\Delta_i=\overline{\Delta}_{n-i},\ i=1,2,...,n-1 \tag{1.5}$$

9. Agar A n- tartibli kvadrat matritsa o‘z markaziga nisbatan simmetrik bo‘lsa, u xolda

$$\overline{\Delta}_i>0,\ i=1,2,...,n-1,\ \Delta_n=|A|>0 \tag{1.6}$$

shartlar A matritsaning musbat aniqlangan bo‘lishi uchun zarur va yetarli shartlar bo‘ladi. A matritsaning manfiy aniqlangan bo‘lishi uchun (1.6) shartlar quyidagi ko‘rinishda bo‘ladi.

$$(-1)^i\,\overline{\Delta}_i>0,\ i=1,2,...,n-1,\ (-1)^n\,\Delta_n=|A|>0 \tag{1.7}$$

10. Agar A n- tartibli kvadrat matritsa 1) bosh diagonalga, 2) bosh bo‘lmagan diagonalga, 3) vertikal o‘qga, 4) gorizontal o‘qga, 5) matritsa markaziga nisbatan simmetrik matritsa bo‘lsa, u xolda bu matritsani mos

ravishda quyidagicha blok matritsalar ko‘rinishida yozish mumkin:

1) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ B_1^T & A_2 \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & B_1 \\ a_1^T & a_{k+1,k+1} & a_2^T \\ B_1^T & a_2 & A_2 \end{pmatrix}$,

2) n=2k da $A=\begin{pmatrix} A_1 & C_1 \\ C_2 & A_1^{\perp} \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & C_1 \\ a_2^T & a_{k+1,k+1} & a_1^T \\ C_1^T & a_2 & A_1^{\perp} \end{pmatrix}$,

3) n=2k da $A=\begin{pmatrix} A_1 & A_1' \\ A_2 & A_2' \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & A_1' \\ (a_1^T)' & a_{k+1,k+1} & (a_2^T)' \\ A_2 & a_2 & A_2' \end{pmatrix}$,

4) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ A_1^- & B_1^- \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & B_1 \\ a_1^T & a_{k+1,k+1} & a_2^T \\ A_1^- & a_2^- & B_1^- \end{pmatrix}$,

5) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ B_1^0 & A_1^0 \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_2 & B_1 \\ a_1^T & a_{k+1,k+1} & (a_1^T)^0 \\ B_1^0 & a_2^0 & A_1^0 \end{pmatrix}$,

bu yerda barcha blok matritsalar k-tartibli

$$a_1=(a_{k+1,1},a_{k+1,2},...,a_{k+1,k})^T,\quad a_2=(a_{k+1,k+2},a_{k+1,k+3},...,a_{k+1,n})^T$$

Ta’rif 1.19. $A=(a_{ij})$ n- tartibli kvadrat matritsa

1) bosh diagonalga nisbatan kososimmetrik (antisimmetrik) deyiladi, agarda $A^T=-A,$ ya’ni $a_{ij}=-a_{ji},\quad i,j=1,2,...,n,$ bo‘lsa:

2) bosh bo‘lmagan diagonalga nisbatan kososimmetrik deyiladi, agarda $A^{\perp}=-A,$ ya’ni $a_{ij}=-a_{n+1-j,n+1-i},\quad i,j=1,2,...,n,$ bo‘lsa:

3) vertikal o‘qqa nisbatan kososimmetrik deyiladi, agarda $A'=-A,$ ya’ni $a_{ij}=-a_{i,n+1-j},\quad i,j=1,2,...,n,$ bo‘lsa :

4) gorizontal o‘qqa nisbatan kososimmetrik deyiladi, agarda $A^-=-A,$ ya’ni $a_{ij}=-a_{n+1-i,j},\quad i,j=1,2,...,n,$ bo‘lsa:

5) matritsa markaziga nisbatan kososimmetrik deyiladi, agarda $A^0=A,$ ya’ni $a_{ij}=-a_{n+1-i,n+1-j},\quad i,j=1,2,...,n,$ bo‘lsa.

Bu ta’rifdan quyidagilar kelib chiqadi:

1. Xar qanday A kvadrat matritsa uchun quyidagilar mos ravishda bosh diagonalga, bosh bo‘lmagan diagonalga, vertikal o‘qga, gorizontal o‘qqa, matritsa markaziga nisbatan kososimmetrik matritsalar bo‘ladi

$$\overline{S}_1=\frac{1}{2}(A-A^T),\ \overline{S}_2=\frac{1}{2}(A-A^{\perp}),\ \overline{S}_3=\frac{1}{2}(A-A'),\ \overline{S}_4=\frac{1}{2}(A-A^{-}),\ \overline{S}_5=\frac{1}{2}(A-A^{0})$$

2. Agar A kvadrat matritsa bo‘lsa, u xolda

$$A=S_1+\overline{S}_i,\quad i=1,2,3,4,5$$

A matritsani mos ravishda bosh diagonalga, bosh bo‘lmagan diagonalga, vertikal o‘qga, gorizontal o‘qqa, matritsa markaziga nisbatan simmetrik va kososimmetrik bo‘lgan matritsalar yig‘indisiga yoyilmasi bo‘ladi.

3. Agar A n- tartibli kvadrat matritsa 1) bosh diagonalga, 2) bosh bo‘liagan diagonalga, 3) vertika o‘qga, 4) gorizontal o‘qqa, 5) matritsa markaziga nisbatan kososimmetrik bo‘lsa, u xolda bu matritsani mos ravishda quyidagicha blok matritsalarga ajratib yozish mumkin

1) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ -B_1^T & A_2 \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & B_1 \\ -a_1^T & -a_{k+1,k+1} & a_2^T \\ -B_1^T & -a_2 & A_2 \end{pmatrix}$,

2) n=2k da $A=\begin{pmatrix} A_1 & C_1 \\ C_1^T & -A_1^{\perp} \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_1 & C_1 \\ a_2^T & -a_{k+1,k+1} & -a_1^T \\ C_1^T & -a_2 & -A_1^{\perp} \end{pmatrix}$,

3) n=2k da $A=\begin{pmatrix} A_1 & -A_1' \\ A_2 & -A_2' \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & -a_1 & -A_1' \\ (a_1^T)' & -a_{k+1,k+1} & -(a_2^T)' \\ A_2 & -a_2 & -A_2' \end{pmatrix}$,

4) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ -A_1^{-} & -B_1^{-} \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & -a_1 & B_1 \\ -a_1^T & -a_{k+1,k+1} & -a_2^T \\ -A_1^{-} & -a_2^{-} & -B_1^{-} \end{pmatrix}$,

5) n=2k da $A=\begin{pmatrix} A_1 & B_1 \\ -B_1^{0} & -A_1^{0} \end{pmatrix}$, n=2k+1 da $A=\begin{pmatrix} A_1 & a_2 & B_1 \\ a_1^T & -a_{k+1,k+1} & -(a_1^T)^0 \\ -B_1^{0} & -a_2^{0} & -A_1^{0} \end{pmatrix}$,

bu yerda barcha blok matritsalar k-tartibli

$$a_1=(a_{k+1,1},a_{k+1,2},...,a_{k+1,k})^T,\quad a_2=(a_{k+1,k+2},a_{k+1,k+3},...,a_{k+1,n})^T$$

Ta'rif 1.20. $A=(a_{ij})$ n – tartibli kvadrat matritsa

1) bosh diagonalga nisbatan ortoganal deyiladi, agarda $A^T = A^{-1}$ bo'lsa,
2) bosh bo'lmagan diagonalga nisbatan ortoganal deyiladi, agarda $A^{\perp} = A^{-1}$ bo'lsa,
3) vertikal o'qqa nisbatan ortoganal deyiladi, agarda $A' = A^{-1}$ bo'lsa,
4) gorizontal o'qqa nisbatan ortoganal deyiladi, agarda $A^{-} = A^{-1}$ bo'lsa,
5) matritsa markaziga nisbatan ortoganal deyiladi, agarda $A^0 = A^{-1}$ bo'lsa.

Bu ta'rifdan kelib chiqadiki, agarda A va B kvadrat matritsalar bosh (bosh bo'lmagan) diagonalga, vertikal (gorizontal) o'qqa, matritsa markaziga nisbatan ortoganal bo'lsa u xolda A^{-1} va AB matritsalar xam mos ravida bosh (bosh bo'lmagan) diagonalga, vertikal (gorizontal) o'qqa, matritsa markaziga nisbatan ortoganal bo'ladi.

§ 4. λ- matritsalar. Elementar bo'luvchilar.

Ushbu Ma'ruza yordamchi xarakterda bo'lib, chiziqli avtonom sistemalarning turg'unlik shartlarini aniqlash uchun kerak bo'ladigan yordamchi tushunchalarni o'z ichiga oladi.

Elementlari qandaydir λ parametrning f_{ij} (λ) ko'rinishdagi ko'pxadlaridan iborat bo'lgan

$$F(\lambda)=\begin{bmatrix} f_{11}(\lambda) & \dots & f_{1n}(\lambda) \\ \dots & \dots & \dots \\ f_{n1}(\lambda) & \dots & f_{nn}(\lambda) \end{bmatrix}$$

kvadratik matritsani qaraylik. Bunday matritsalar λ- matritsalar deyiladi.

ν_k (λ) (k=1,2,3,...,n) orkali F(λ) matritsaning barcha k-tartibli minorlarining eng katta umumiy bo'luvchisini belgilab, bosh xad oldidagi koeffitsientni birga teng qilib tanlaymiz. Osongina ko'rsatish mumkinki $\nu_k(\lambda)$ ko'pxadning bu aniqlanishidan quyidagi xulosani chiqarish mumkin: agar qandaydir k- tartibli minor o'zgarmas songa teng bo'lsa, u xolda $\nu_k = \nu_{k-1} =...= \nu_1 =1$ bo'ladi. Chunki bu minor ν_k ga bo'linishi, ν_k esa

$$\nu_{k-1},\ \nu_{k-2},\ldots,\ \nu_1$$

larga bo‘linishi kerak.

$$\frac{\nu_k(\lambda)}{\nu_{k-1}(\lambda)}=E_k(\lambda) \quad k=1,2,3...n\ , \ \nu_0=1$$

(1.8)

isbot bilan aniqlanuvchi ko‘pxad F(λ) matritsaning invariant ko‘paytuvchisi deyiladi. Ravshanki,

$$\nu_k(\lambda)=E_1(\lambda)\cdot E_2(\lambda)\ ...\ E_k(\lambda)$$

bo‘lib, $\nu_n(\lambda)$ o‘zgarmas ko‘paytuvchi aniqligida F(λ) ning determinantiga teng, ya’ni

$$\nu_n(\lambda)=\delta\det F(\lambda)=E_1(\lambda)\,E_2(\lambda)\ldots E_n(\lambda).$$

$E_k(\lambda)$ invariant ko‘paytuvchini ko‘paytuvchilarga ajratamiz.

$$E_k(\lambda)=(\lambda-\lambda_l)^{i}{}_{k1}\cdot(\lambda-\lambda_2)^{i}{}_{k2}\cdot\ldots\cdot(\lambda-\lambda_p)^{i}{}_{kp}$$

bu yerda

$$\lambda_l,\lambda_2,...,\lambda_p \text{ lar } \det F(\lambda)=0$$

tenglamaning xar xil ildizlari.

Aniqki $I_{kr}>0$, k=1,2,...,n; r=1,2,...,p.

Bundan tashqari, agar $k<k^I$ bo‘lsa, $l_{kj}<l_{kj}$ bo‘ladi. Chunki $E_k(\lambda)$ (1.8) ko‘pxad E_k ko‘pxadga bo‘linadi. $E_k(\lambda)$ ning ko‘paytuvchilari tarkibiga kiruvchi o‘zgarmas sondan farqli bo‘lgan $(\lambda-\lambda_r)^{Ikr}$ ikkixad λ matritsaning elementar bo‘luvchilari deyiladi. Ularning umumiy sonini m bilan belgilab, ularni o‘zlarini

$$(\lambda-\lambda_1)^{i_1},\ (\lambda-\lambda_2)^{i_2},\ (\lambda-\lambda_m)^{i_m}$$

lar orqali belgilaymiz. Chunki λ_l sonlarning ichida o‘zaro tenglari bo‘lib , $(\lambda-\lambda_i)^{Ii}$ binom xar xil E_k invariant ko‘paytuvchilar tarkibiga kirishi mumkin.

Misol: $$F(\lambda)=\begin{bmatrix}(\lambda+1)^3 & (\lambda+1)^2\\ \lambda+1 & \lambda+1\end{bmatrix}$$

matritsa uchun quyidagi to‘rtta birinchi tartibli $(\lambda+1)^3$, $(\lambda+1)^2$, $\lambda+1$, $\lambda+1$ minorlarni tuzish mumkin bo‘lib, ularning eng katta bo‘luvchisi

$$\nu_1=\lambda+1$$

bo‘ladi.

Berilgan misoldagi matritsa uchun bitta ikkinchi tartibli minor bo‘lib,

$$\lambda(\lambda+1)^3 = \begin{vmatrix} (\lambda+1)^3 & (\lambda+1)^2 \\ \lambda+1 & \lambda+1 \end{vmatrix}$$

uning eng katta umumiy bo‘luvchisi

$$\nu_2 = \lambda(\lambda+1)^3$$

bo‘ladi. (1.8) formuladan foydalanib invariant ko‘paytuvchilarni topamiz.

$$E_1 = \nu_1 = \lambda+1, \; E_2 = \frac{\nu_2}{\nu_1} = \lambda(\lambda+1)^2$$

Misolda qaralayotgan matritsa uchun elementar bo‘luvchilar $\lambda+1$, λ, $(\lambda+1)^2$ bo‘ladi. Bu yerda ildizlar $\lambda_1=-1$, $\lambda_2=0$, $\lambda_3=\lambda_4=-1$

Bu ildizlar

$$\det F(\lambda)=0,$$

tenglamaning xam ildizlari bo‘ladi. Ammo $\lambda=-1$ tenglamaning uch karrali ildizi bo‘lib, bir elementar bo‘luvchi uchun oddiy, boshqasi uchun ikki karralidir.

$F(\lambda)$ matritsaning normal diogonal ko‘rinishi deb

$$\begin{bmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E_n \end{bmatrix}$$

matritsaga aytiladi. Bu yerda $E_1, E_2, ..., E_n$- $F(\lambda)$ matritsaning invariant ko‘paytuvchilari. Masalan, yuqorida qaralgan misoldagi matritsaning normal diogonal ko‘rinishi,

$$\begin{bmatrix} \lambda+1 & 0 \\ 0 & \lambda(\lambda+1)^2 \end{bmatrix}$$

matritsadan iborat bo‘ladi.

λ- matritsalarni elementar almashtirishlar deb quyidagi operatsiyalarga aytiladi:

a) ikkita satr yoki ikkita ustunini o‘zaro almashtirish;

b) qandaydir satri (ustuni) ning barcha elementlarini bitta noldan farqli o‘zgarmas ko‘paytivchilarga ko‘paytirish;

v) qandaydir satri (ustuni) ning barcha elementlarini ko‘paytirilgan eementlarini boshqa satr (ustun) ning mos elementlariga ko‘shish, +uyidagilarni isbotlash mumkin:

a) Elementar almashtirishlar λ-matritsa elementar bo‘luvchilarni o‘zgartirmaydi;

b) ixtiyoriy λ-matritsani chekli sondagi almashtirishlar bilan normal dioganal ko‘rinishiga keltirish mumkin.

Bu jumlalarning to‘g‘riligining isbotini keltirmay, yuqoridagi misoldagi matritsani normal shaklga keltiramiz:

$$\begin{bmatrix} (\lambda+1)^3 & (\lambda+1)^2 \\ \lambda+1 & \lambda+1 \end{bmatrix} \to \begin{bmatrix} \lambda+1 & \lambda+1 \\ (\lambda+1)^3 & (\lambda+1)^2 \end{bmatrix} \to \begin{bmatrix} \lambda+1 & \lambda+1 \\ (\lambda+1)^2 & (\lambda+1)^3 \end{bmatrix} \to$$
$$\to \begin{bmatrix} \lambda+1 & 0 \\ (\lambda+1)^2 & \lambda(\lambda+1)^2 \end{bmatrix} \to \begin{bmatrix} \lambda+1 & 0 \\ 0 & \lambda(\lambda+1)^2 \end{bmatrix}$$

Bu yerda avval birinchi satrni ikinchisi bilan , birinchi ustunni щam ikkinchisi bilan almashtirdik. Keyin birinchi ustundan ikkinchisini ayirdik. Nixoyat oxirida birinchi satrni λ+l ga ko‘paytirib ikkinchi satrdan ayirdik

§ 5. Jordon kataklari.

Umuman aytganda, ko‘p xollarda elementar almashtirishlar elementar bo‘luvchilarni topishda ishlatiladi.

Quyidagi l_1 tartibli matritsani qaraylik.

$$J_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & \lambda_1 \end{bmatrix}$$

Bunday ko‘rinishdagi matritsalar Jordon kataklari yoki elementar yashiklar deyiladi.

Bundan foydalanib J_1-λE-λ-matritsani tuzamiz;

$$J_1-\lambda E=\begin{bmatrix} \lambda_1-\lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_1-\lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda_1-\lambda & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda_1-\lambda & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & \lambda_1-\lambda \end{bmatrix}$$

Bu matritsaning birinchi satr va oxirgi ustunini o‘chirib qolgan elementlardan l_1-1 tartibli minor tuzamiz.

$$\begin{bmatrix} 1 & \lambda_1-1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda_1-\lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda_1-\lambda \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Bu minor 1 ga teng bo‘lgani uchun $\nu_1 = \nu_2 = \nu_{i-1} = 1$ bo‘ladi. Ikkinchi tomondan yagona l_1-tartibli minor quyidagiga teng:

$$\det(J_1 - \lambda E)=(\lambda_1 - \lambda)^{i_1}$$

Demak,

$$\nu_{l_1} =(\lambda- \lambda_1)^{i_1}$$

Bu yerda λ va λ_1 larning o‘rinlari almashtirildi, chunki ν_{l1} ning bosh xadi oldidagi koeffitsienti 1 ga teng bo‘lishi kerak

(1.8) formuladan foydalanib invariant ko‘paytuvchilarni topamiz:

$$E_1 =1,\quad E_2 =1,\ \dots,\ E_{i_1} =1,\ E_{i_1} =(\lambda- \lambda_1)^{i_1}.$$

Bundan ko‘rinadiki, J_1- λE matritsa faqat bitta $(\lambda- \lambda_1)^{i_1}$ ga teng elementlar bo‘luvchiga ega.

Endi elementlari a_{kj} o‘zgarmas sonlardan iborat bo‘lgan A ixtiyoriy kvadrat matritsani karaymiz. A-λE λ - matritsani tuzamiz (u A matritsaning xarakteristikasi deyiladi)

$$A-\lambda E=\begin{vmatrix} a_{11}-\lambda & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn}-\lambda \end{vmatrix}$$

Bu matritsaning elementar bo‘luvchilarini topamiz

$$(\lambda-\lambda_1)^{i_1},\ (\lambda-\lambda_2)^{i_2},\ \dots,\ (\lambda-\lambda_m)^{i_m}.$$

Bu elementar bo‘luvchilarning xar biri λ_k (k=1,2,3,...,m) ildiziga o‘zining mos J_k Jordon katagi mos keladi. Berilgan A matritsa uchun Jordonning normal ko‘rinishi deb, diogonaldagi elementlari Jordon kataklaridan, qolgan elementlari nollardan iborat bo‘lgan

$$J=\begin{vmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_m \end{vmatrix}$$

ko‘rinishdagi matritsaga aytiladi.

Ravshanki, J-λE matritsaning elementar bo‘luvchilari xarakteristik matritsa elementar bo‘luvchilari bilan ustma ust tushadi.

Bundan tashkari,

$$|A-\lambda E| = 0$$

xarakteristik tenglamaning ildizlari elementar bo‘luvchilarning ildizlari bilan ustma-ust tushadi.

Misol 1.

$$A=\begin{vmatrix} -2 & -1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -4 & 1 & -1 & -1 \\ 5 & 1 & 2 & 2 \end{vmatrix}$$

Bu matritsani Jordonning normal ko‘rinishiga keltirish uchun avval A-λE xarakteristik matritsaning elementar bo‘luvchilarini topamiz.

$$A-\lambda E=\begin{vmatrix} -2-\lambda & -1 & -1 & -1 \\ 1 & -1-\lambda & 0 & 0 \\ -4 & 1 & -1-\lambda & -1 \\ 5 & 1 & 2 & 2-\lambda \end{vmatrix}$$

Buning uchun elementar almashtirishlardan foydalanamiz. Birinchi satrni -1 ga ko‘paytiramiz. Keyin oxirgi ustunni -(2+ λ) ga ko‘paytiramiz va birinchi ustunga qo‘shamiz; bundan keyingi oxirgi

ustunni ikkinchi va uchinchi ustunlargday ayiramiz:

$$A-\lambda E=\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -1-\lambda & 0 & 0 \\ -2+\lambda & 2 & -\lambda & -1 \\ (\lambda+1)^2 & -(1-\lambda) & \lambda & 2-\lambda \end{vmatrix}$$

Birinchi satr uchinsiga qo‘shamiz, keyingi birinchi satrni 2-λ ga ko‘paytirib to‘rtinchi satrdan ayiramiz; bundan keyin oxirgi ustunni birinchi ustun o‘rniga keltiramiz:

$$A-\lambda E=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 \\ 0 & -2+\lambda & 2 & -\lambda \\ 0 & (1+\lambda) & -1-\lambda & \lambda \end{vmatrix}$$

Ikkinchi ustunni 1+λ ga ko‘paytirib, uchinchi ustunga qo‘shamiz

$$A-\lambda E=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2+\lambda & -\lambda(1-\lambda) & -\lambda \\ 0 & (1+\lambda)^2 & \lambda(2+\lambda+\lambda^2) & \lambda \end{bmatrix}$$

Endi ikkinchi satrni avval -2-λ ga ko‘paytirib, uchinchi satrga ko‘shamiz; keyin ikkinchi satrni $-(l+\lambda)^2$ ga ko‘paytirib to‘rtinchi ustunga ko‘shamiz. Bundan keyin to‘rtinchi ustunni -(1-λ) ga ko‘paytirib uchinchi ustunga qo‘shamiz:

$$A-\lambda E=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda(1+\lambda)^2 & \lambda \end{bmatrix}$$

Uchinchi satrni to‘rtinchi satrga qo‘shamiz, keyin bu satrni -1 ga ko‘paytirib, to‘rtinchi ustunni uchinchi ustun bilan almashtiramiz:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda(1+\lambda)^2 \end{bmatrix}$$

Nihoyat A-λE xarakteristik matritsaning normal diogonal ko‘rinishi hosil bo‘ldi. Bundan quyidagilarni topamiz:

$$E_1=1,\ E_2=1,\ E_3=\lambda,\ E_4=\lambda(1+\lambda)^2$$

Demak, A-λE matritsa ildizlari $\lambda_1=0$, $\lambda_2=0$,

$$\lambda_3=\lambda_4=-1,$$

bo‘lgan uchta

$$\lambda,\ \lambda\ ,\ (\lambda+l)^2$$

elementar bo‘luvchiga ega.

Har bir elementar bo‘luvchiga o‘zining Jordon katagi mos keladi:

$$\lambda_1=0 \text{ da } l_1=1;\ \ \lambda_2=0 \text{ da } l_2=0\ ;\ \ \lambda_3=-1 \text{ da } l_3=2$$

bo‘lgani uchun

$$J_1=[0] \quad J_2=[0]$$

$$J_3=\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

Endi qaralayotgan matritsa uchun Jordonning normal ko‘rinishini quyidagicha yozamiz:

$$J=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$A=\begin{bmatrix} -2 & -1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -5 & 0 & -2 & -3 \\ 6 & 2 & 3 & 3 \end{bmatrix}$$

Avval xarakteristik matritsani tuzamiz.

$$A-\lambda E=\begin{bmatrix}-2-\lambda & -1 & -1 & -1\\ 1 & -1-\lambda & 0 & 0\\ -5 & 0 & -2-\lambda & -2\\ 6 & 2 & 3 & 3-\lambda\end{bmatrix}$$

Elementar almashtirishlar yordamida bu λ matritsani quyidagi ko‘rinishdagi normal diogonal ko‘rinishga keltiramiz:

$$A-\lambda E=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \lambda_2(\lambda+1)^2\end{bmatrix}$$

Bundan invariant ko‘paytuvchilarni topamiz:

$$E_1=1,\ E_2=1,\ E_3=1,\ E_4=\lambda^2(\lambda+1)^2$$

Demak, A-λE matritsa ildizlari

$$\lambda_1=\lambda_2=0\ ,\ \lambda_3=\lambda_4=-1$$

bo‘lgan faqat ikkita λ^2, $(\lambda+1)^2$ elementar bo‘luvchilarga ega bulib, xar bir elementar bo‘luvchiga bittadan

$$J_1=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}\qquad J_2=\begin{vmatrix}-1 & 0\\ 1 & -1\end{vmatrix}$$

Jordon kletkasa mos keladi.

Endi qaralayotgan matritsa uchun jordonning normal formasini yoza olamiz.

$$J=\begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$$

Bu misolardan shuni ko‘ramizki, xar ikkala misolning xarakteristik tenglamalari bir xil ildizga ega, ammo Jordonning normal formasi xar xil. Buning sababi shuki birinchi misolning xarakteristik matritsasi uchta elementar bo‘luvchiga, ikkinchi misoldagi matritsa esa faqat ikkita elementar bo‘luvchiga ega.

§ 6. Asosiy teoremalar.

Endi bizning keyingi izlanishlarimizda kerak bo‘ladigan ikkita chiziqli algebraning teoremalarini isbotsiz keltiramiz

Teorema1.1. Agar Λ matritsa maxsusmas bo‘lsa, u xolda $A-\lambda E$ va $\Lambda A\Lambda^{-1}-\lambda E$ matritsalarning elementar bo‘luvchilari bir xil bo‘ladi. Aksincha, agar $A-\lambda E$ va $B-\lambda E$ matritsalarning elementar bo‘luvchilari bir xil bo‘lsa, u xolda xar doim $B=\Lambda A\Lambda^{-1}$ tenglikni qanoatlantiruvchi Λ maxsusmas matritsa topiladi.

Ayrim mualliflar bu teoremani algebraning asosiy teoremasi deb ataydilar.

Teorema1.2. Agar A va C lar s - tartibli simmetrik, kvadratik matritsalar bo‘lib, A aniq ishorali bo‘lsa, u xolda

1) $\det(A\lambda+C)=0$ xarakteristik tenglamaning barcha ildizlari haqiqiy;

2) har doim shunday Λ maxsusmas matritsa topiladiki, unda

$$\Lambda^T A\Lambda=E\ ,\ \Lambda^T C\Lambda=S_0$$

bo‘ladi.

Bu yerda E birlik matritsa.

$$C_0=\begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_s \end{bmatrix}$$

bo‘lib, $c_1, c_2,..., c_s$ lar xarakteristik tenglamaning ildizlari.

Teoremaning ikkinchi qismi quyidagi tasdiqqa teng kuchlidir: Agar quyidagi ikkita

$$T=\frac{1}{1}A\overline{x}\cdot\overline{x}=\frac{1}{2}\sum_{k=1}^{s}\sum_{i=1}^{s}a_{ki}x_k x_i\ ,$$

$$\Pi=\frac{1}{1}C\overline{x}\cdot\overline{x}=\frac{1}{2}\sum_{k=1}^{s}\sum_{i=1}^{s}c_{ki}x_k x_i\ ,$$

kvadratik formalar berilgan bo‘lib, ularning birinchisi musbat aniqlangan bo‘lsa, u xolda shunday Λ maxsusmas matritsali $x=\Lambda z$ chiziqli almashtirish topiladiki, unda

$$T=\frac{1}{2}\overline{z}\cdot\overline{z}=\frac{1}{2}\left(z_1^2+z_2^2+\ldots+z_s^2\right)$$

$$\Pi = \frac{1}{2} c_0 \overline{z} \cdot \overline{z} = \frac{1}{2}\left(c_1 z_1^2 + c_2 z_2^2 + \ldots + c_s z_s^2\right)$$

Teorema1.2 ning ikkinchi qismidagi ikkinchi tengligidan

$$\det C_0 = \det \Lambda^t \det C \det \Lambda$$

tenglikni hosil qilib, $\det S_0 = \det C$ ekanligini e'tiborga olsak va $\det \Lambda = \Delta$ deb olsak,

$$\det C_0 = \Delta^2 \det C$$

hosil bo‘ladi.

Shuning uchun C_0 diogonal matritsa bo‘lgani uchun $\det C_0 = c_1\ c_2\ \ldots\ c_s$, bo‘lib, $s_1\ s_2\ \ldots\ s_s = \det C$ ko‘rinishni oladi .

Bundan tashkari ortogonal almashtirishda ixtiyoriy B kvadrat matritsaning izi $\Lambda^T B \Lambda$ matritsaning iziga teng, ya'ni

$$C_p\ B = C_p \Lambda^T\ B\ \Lambda$$

ekanligini isbotlash mumkin.

Mashqlar:

1. Quyidagi matritsalar ustida algebraik amallarni bajaring.

a) $A=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$

c) $A = \begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$

d) $A = \begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$

e) $A = \begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 8 & 30 & -14 \\ -5 & -19 & 9 \\ -6 & -23 & 11 \end{pmatrix}$

f) $A = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ $\qquad B=\begin{pmatrix} 3 & 7 & -3 \\ -2 & -5 & 2 \\ -4 & -10 & 3 \end{pmatrix}$

g) $A = \begin{pmatrix} 9 & 22 & -6 \\ -1 & -4 & 1 \\ 8 & 16 & -5 \end{pmatrix}$ $\qquad$ $B=\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$

2. Quyidagi matritsalarning normasini xisoblang.

1. $\begin{pmatrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{pmatrix}$ $\qquad$ 2. $\begin{pmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ $\qquad$ 3. $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & 6 \end{pmatrix}$ $\qquad$ 5. $\begin{pmatrix} -1 & 1 & 2 \\ -1 & 4 & 0 \\ 1 & -1 & 3 \end{pmatrix}$ $\qquad$ 6. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

3. Umumlashgan transponirlangan matritsalarning 1-12 xossalarni isbotlang.
4. Umumlashgan transponirlangan matritsalarning boshqa xossalarini aniqlang.
5. Umumlashgan simmetrik matritsalarning 1-10 xossalarni isbotlang.
6. Umumlashgan simmetrik matritsalarning boshqa xossalarini aniqlang.
7. Quyidagi λ - matritsalarni avval elementar almashtirishlar yo'li bilan, so'ngra invariant ko'paytuvchilardan foydalanib kanonik ko'rinishga keltiring.

1. $A(\lambda) = \begin{Vmatrix} \lambda^4+\lambda^2+\lambda-1 & \lambda^3+\lambda^2+\lambda+2 \\ 2\lambda^3-\lambda & 2\lambda^2+2\lambda \end{Vmatrix}$

2. $A(\lambda) = \begin{Vmatrix} \lambda^2+\lambda+1 & \lambda^3-\lambda+2 \\ 2\lambda & \lambda^2-3\lambda-1 \end{Vmatrix}$

3. $A(\lambda) = \begin{Vmatrix} \lambda^2+1 & 1 \\ \lambda & \lambda^2+\lambda \end{Vmatrix}$

4. $A(\lambda) = \begin{Vmatrix} \lambda & \lambda+1 \\ \lambda^2-\lambda & \lambda^2-1 \end{Vmatrix}$

5. $A(\lambda) = \begin{Vmatrix} 0 & 1 & \lambda \\ \lambda & \lambda & 1 \\ \lambda^2-\lambda & \lambda^2-1 & \lambda^2-1 \end{Vmatrix}$

6. $A(\lambda) = \begin{Vmatrix} \lambda^2 & \lambda+1 \\ \lambda-1 & \lambda^2 \end{Vmatrix}$

7. $A(\lambda) = \begin{Vmatrix} \lambda & \lambda^2 & 0 \\ \lambda^2 & \lambda^2 & 0 \\ 0 & 0 & 2\lambda \end{Vmatrix}$

8. Quyidagi matritsalarni Jordonning normal formasiga keltiring.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix} \qquad \begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix} \qquad \begin{pmatrix} 8 & 30 & -14 \\ -5 & -19 & 9 \\ -6 & -23 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 7 & -3 \\ -2 & -5 & 2 \\ -4 & -10 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 22 & -6 \\ -1 & -4 & 1 \\ 8 & 16 & -5 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$$

II-BOB

KOMPLEKS SIMMETRIK, KOSOSIMMETRIK VA ORTOGONAL MATRITSALAR.

Ushbu bob kompleks simetrik, kososimmetrik va ortoganal matritsyalarni o`rganishga bag`ishlangan bo`lib, unda bu matritsiyalar qanday elementlar bo`luvchilarga ega bo`lishi mumkinligi va ularning norma formalari qarab chiqiladi. Bu formalar oddiy xolatdagiga qaraganda sezilarli darajada murakkab strukturaga ega.

§1. Kompleks ortaganal va unitar matritsyalar uchun ba`zi formulalar.

Lemma 2.1. L agar G matritaya bir vaqitning o`zida Ermit matritsyasi bo`lib, ortogonal bo`lsa, ya`ni $G^T = \overline{G} = G^{-1}$ bo`lsa, u holda u quydagi ko`rinishda tasvirlanadi:

$$G = Ie^{iK}, \tag{2.1}$$

bu yerda I-xaqiqiy simmetrik involyutiv $\left(I^2 = E\right)$ klatritsa, $K - I$ bilan o`rin almashinuvchi bo`lgan kososimmetrik matritsya:

$$I = \overline{I} = I^T, \quad I^2 = E, \quad K = \overline{K} = -K^T \tag{2.2}$$

Agar $G-$ yuqoridagilarga qo`shimcha ravishda musbat aniqlangan ermit matritsasi bo`lsa, (1) formulasida $I = E$ bo`lib,

$$G = e^{iK}, \tag{2.3}$$

bo`ladi.

Isboti.

$$G = S + iT \tag{2.4}$$

bo`lsin, bu yerda S va T-xaqiqiy matritsalar. U holda

$$\overline{G} = S - iT \quad \text{va} \quad G^T = S^T + iT^T \tag{2.5}$$

Shuning uchun $\overline{G} = G^T$ tenglikdan $S = S^T$, $T = -T^T$, ya`ni S-simmetrik, T- kososimmetrik ekanligi kelib chiqadi. Bundan tashqari (2.4) va (2.5) ga asosan $G\overline{G}=E$ tenglikdan

$$S^2+T^2=E\ ,\quad ST=TS \tag{2.6}$$

ni hosil qilamiz. Buning ikkinchisidan S va T ning o`zaro kommutativligi kelib chiqadi.

Malumki, o`zaro kommutativ matritsalar bir hil xaqiqiy ortogonal almashtirish bilan kvazidioganal kanonik formaga keltiriladi. Shuning uchun

$$S=O(s_1,s_1,s_2,s_2,...s_q,s_q,s_{2q+1},...s_n)O^{-1},\left(O=\overline{O}=O^{-1}\right), \tag{2.7}$$

$$T=O\left\{\begin{Vmatrix}0 & t_1\\ -t_1 & 0\end{Vmatrix},\begin{Vmatrix}0 & t_2\\ -t_2 & 0\end{Vmatrix},...,\begin{Vmatrix}0 & t_q\\ -t_q & 0\end{Vmatrix},O,O,...,O\right\}O^{-1},$$

bu yerda s_i, t_i -xaqiqiy sonlar. Bundan,

$$G=S+iT=O\left\{\begin{Vmatrix}s_1 & it_1\\ -it_1 & s_1\end{Vmatrix},\begin{Vmatrix}s_2 & it_2\\ -it_2 & s_2\end{Vmatrix},\ ...\ ,\begin{Vmatrix}s_q & it_q\\ -it_q & s_q\end{Vmatrix},\ s_{2q+1},...s_n\right\}O^{-1} \tag{2.8}$$

Ikkinchi tomondan (2.7) ifodalarni (2.6) ga qo`yib quyidagilarni topamiz;

$$s_1^2-t_1^2=1,\ s_2^2-t_2^2=1,...,s_q^2-t_q^2=1,s_{2q+1}=\pm1,...,s_n=\pm1 \tag{2.9}$$

Endi tekshirib ko`rish mumkinki $\begin{Vmatrix}s & it\\ -it & s\end{Vmatrix}$ tipdagi matritsalarni $s^2-t^2=1$ shartdan har doim quyidagi ko`rinishda tasvirlash mumkin:

$$\begin{Vmatrix}s & it\\ -it & s\end{Vmatrix}=\varepsilon\, e^{i\begin{Vmatrix}0 & \varphi\\ -\varphi & 0\end{Vmatrix}},$$

bu yerda $|s|=ch\varphi,\ \varepsilon t=sh\varphi,\ \varepsilon=signS.$. Shuning uchun (2.8) va (2.9) ga asosan quydagiga ega bo`lamiz:

$$G=O\left\{\pm e^{i\begin{Vmatrix}0 & \varphi_1\\ -\varphi_1 & 0\end{Vmatrix}},\pm e^{i\begin{Vmatrix}0 & \varphi_2\\ -\varphi_2 & 0\end{Vmatrix}},...,\pm e^{i\begin{Vmatrix}0 & \varphi_q\\ -\varphi_q & 0\end{Vmatrix}},\pm1,\ ...\ ,\pm1\right\}O^{-1} \tag{2.10}$$

yani

$$G=Ie^{iK},$$

bu yerda

$$I=O(\pm1,\pm1,...,\pm1,)O^{-1},$$

$$T=O\left\{\begin{Vmatrix}0 & \varphi_1\\ -\varphi_1 & 0\end{Vmatrix},...,\begin{Vmatrix}0 & \varphi_q\\ -\varphi_q & 0\end{Vmatrix},O,O,...,O\right\}O^{-1} \tag{2.11}$$

va

$$IK = KI$$

Agar G-musbat aniqlangan ermat matritsasi bo`lsa, uning barch xarakteristik sonlari musbat bo`ladi. Ammo (2.10)ga asosan Gning xarakteristik sonlari

$$\pm e^{\varphi_1}, \pm e^{-\varphi_1}, \pm e^{\varphi_q}, \pm e^{-\varphi_q}, \pm 1, \ldots, \pm 1$$

bo`ladi.

Shuning uchun G-musbat aniqlangan bo`lganda (2.10) va (2.11) dagi $\pm$ ishoralar $+$ ishora bilan almashtirilib,

$$I = O(1, 1, \ldots, 1,)O^{-1} = E$$

bo`ladi.

Teorema 2.1. O-kompleks orthogonal matritsa xar doim quydagi ko`rinishda tasvirlanadi;

$$O = \mathrm{R}e^{iK} \tag{2.12}$$

bu yerda R-xaqiqiy orthogonal matritsa, K-xaqiqiy kososimmetrik matritsa yani

$$R = \overline{R} = R^{T-1}, \; K = \overline{K} = -K^T \tag{2.13}$$

Isbot. Faraz qilaylik (2.12) formula o`rinli bo`lsin. U holda

$$O^* = \overline{O}^T = e^{iK} R^T \quad \text{va} \quad O^* O = e^{iK} R^T \mathrm{R}e^{iK} = e^{2iK}$$

bo`lib,

$$O^* O = e^{2iK} \tag{2.14}$$

tenglikdan K matritsani aniqlashimiz mumkin. Kni aniqlaganimizdan keyin (2.12) tenglikdan Rni topamiz

$$R = O e^{-iK} \tag{2.15}$$

U holda $R^* R = e^{-iK} O^* O e^{iK} = E$, yani R-unitar matritsa bo`ladi. Ikkinchi tomondan (2.15) dan kelib chiqadiki, ikkita orthogonal matritsaning ko`paytmasidan iborat bo`lgan $R$ matritsa o`zi orthogonal, yani $R^T K = E$ bo`ladi.

Demak, R bir vaqitning o`zida ham orthogonal, ham unitary bo`ladi, bundan uni xaqiqiy ortogonalligi kelib chiqadi. (2.15) ni (2.12) ko`rinishda yozish mumkin.

Lemma 2.2. Agar D matritsa bir vaqitda simmetrik va unitar, yani

$$D = D^T = D^{-1}$$

bo`lsa, u xolda u xar doim quydagi ko`rinishda tasvirlanadi.

$$D = e^{iS}, \tag{2.16}$$

bu yerda S-xaqiqiy simmetrik matritsa, yani $S = \overline{S} = S^T$

Isboti.

$$D = U + iV \qquad \left(U = \overline{U}, V = \overline{V}\right) \tag{2.17}$$

bo`lsin. U holda

$$\overline{D} = U - iV \qquad D^T = U^T + iV^T$$

bo`lsin $D = D^T$ dan $U = U^T$, $V = V^T$ kelib chiqadi, yani U va V lar xaqiqiy simmetrik matritsalar.

$$D\overline{D} = E$$

tenglikdan

$$U^2 + V^2 = E, \quad UV = VU \tag{2.18}$$

kelib chiqadi.

U va V matritsalar o`zaro kamutativ bo`lgani uchun ular bir xil ortogonal almashtirish bilan kanonik ko`rinishga keladi. Shuning uchun quydagilarni xosil qilamiz:

$$U = O(s_1, s_2, \ldots, s_n)O^{-1}, \; V = O(t_1, t_2, \ldots, t_n)O^{-1}, \tag{2.19}$$

bu yerda $O = \overline{O} = O^{-1}$, s_k, t_t, $k = \overline{1,n}$-xaqiqiy sonlar (2.18) ning birinch tengligidan $s_k^2 + t_k^2 = 1$, $k = \overline{1,n}$ kelib chiqadi. Shuning uchun shunday φ_k, $k = \overline{1,n}$ xaqiqiy sonlar mavjud bo`lib, $s_k = \cos\varphi_k$, $t_k = \sin\varphi_k$, $k = \overline{1,n}$ bo`ladi. Bu ifodalarni (2.19) ga qo`yib, (2.17) ga asosan quydagini hosil qilamiz.

$$D = O\left(e^{i\varphi_1}, \ldots, e^{i\varphi_n}\right)O^{-1} = e^{iS}, \tag{2.20}$$

bu yerdan $S = O(\varphi_1, \varphi_2, \ldots, \varphi_n)O^{-1}$ (2.20) dan $S = \overline{S} = S^T$ ekanligi kelib chiqadi.

Teorema 2.2. U-unitar matritsani xar doim quydagi ko`rinishda tasvirlash mumkin:

$$U = \mathrm{Re}^{iS}, \tag{2.21}$$

bu yerda R-xaqiqiy ortogonal matritsa, S-xaqiqiy simmetrik matritsa, yani

$$R = \overline{R} = R^{-1}, \quad S = \overline{S} = S^T \tag{2.22}$$

Isboti. (2.21) formuladan

$$U^T = e^{iS} R^T \tag{2.23}$$

kelib chiqadi. (2.21) va (2.23) ni xadlab ko`paytirib, (2.22) ga asosan
ni xosil qilamiz

$$U^T U = e^{2iS} \tag{2.24}$$

tenglikdan lemma 2 ga asosan S ni aniqlash mumkin. Shundan so`ng R matritsani

$$R = Ue^{-iS} \tag{2.25}$$

ko`rinishda aniqlaymiz. U holda $R^T = e^{-iS} U^T$ bo`lib, (2.24), (2.25) va (2.26) dan

$$R^T R = e^{-iS} U^T U e^{-iS} = E$$

kelib chiqadi.

Ikkinchi tomondan, (2.25) ga asosan, R ikkita unitary matritsalar ko`paymasidan iborat, demak, $R$-unitar matritsa bo`lib, u bir vaqitning o`zida ham ortogon, ham unitar matritsa bo`ladi. Bundan R ning xaqiqiy matritsa ekanligi kelib chiqadi. (2.25) formulani (2.21) ko`rinishda yozish mumkin.

§2. Kompleks matritsalarni qutub yoyilmasi.

Teorema 2.3. Agar $A = \| a_{ik} \|_{i,k=1}^{n}$-kompleks elementli xosmas matritsa bo`lsa, y holda quydagi yoyilma o`rinli:

$$A = SO, \tag{2.26}$$

va

$$A = O_1 S_1 \qquad (2.27)$$

bu yerda S va S_1 simmetrik kompleks matritsa, O va O_1 esa ortogonal kompleks matritsa bo`ladi,

$$S = \sqrt{AA^T} = f\left(AA^T\right),\ S_1 = \sqrt{A^T A} = f\left(A^T A\right),$$

$f(\lambda)$ va $f_1(\lambda)$- λ ga nisbatan qandaydir ko`phadlar.

(2.26) yoyilmadagi, shuningdek (2.27) yoyilmadagi S va O , mos ravishda O_1 va S_1 matritsalar faqat va faqat A va A^T o`rin almashunuvchi bo`lgandagina o`rin almashinuvchi bo`ladi.

Isboti: (2.26) yoyilmani hosil qilish yetarli, shuningdek bu yoyilma A^T matritsaga qo`yilib, va xosil qilingan formuladan A matritsani aniqlab, (2.27) yoyilmaga kelamiz.

Agar (2.26) o`rinli bo`lsa, u holda

$$A = SO,\quad A^T = O^{-1}S$$

bo`lib,

$$AA^T = S^2 \qquad (2.28)$$

bo`ladi.

Aksinch, AA^T -xosmas matritsa, $\left|AA^T\right| = |A|^2 \neq 0$, u holda $\sqrt{\lambda}$ funksiyasi bu matritsaning spektrida aniqlangan bo`ladi. Demak, shunday $f(\lambda)$ interpolyatsion kopxad mavjudki,

$$\sqrt{AA^T} = f\left(AA^T\right) \qquad (2.29)$$

bo`ladi. (2.29) simmetrik matritsani $S = \sqrt{AA^T}$ orqali belgilaymiz. U holda (2.28) o`rinli bo`ladi, $|S| \neq 0$ bo`ladi. (2.26) tenglikdan O ni aniqlab,

$$O = S^{-1}A,$$

osongina tekshirib ko`ramizki, bu matritsa ortogonal. Shunday qilib, (2.26) da $S$ va $O$ o`zaro o`rin almashinuvchi bo`lsa, u holda $A = SO$ va $A^T = O^{-1}S$ matritsalar ham o`rin almashinuvchi bo`lib, $AA^T = S^2$, $A^T A = O^{-1}S^2 O$ bo`ladi. Aksincha, agar $AA^T = A^T A$ bo`lsa,

$$S^2 = O^{-1}S^2 O,$$

ya`ni $O$ matritsa $S^2 = AA^T$ matritsa bilan o`rin almashinuvchi. Ammo bu holda O matritsa $S = f\left(AA^T\right)$ matritsa bilan o`rin almashinuvchi bo`ladi.

Teorema 2.4. Agar ikkita kompleks simmetrik, yoki kososimmetrik, yoki ortogonal matritsalar

$$B = T^{-1}AT \tag{2.30}$$

bo`lsa, u holda bu matritsalar ortogonal-o`xshash, yani shunday O ortogonal matritsa mavjudki, unda

$$B = O^{-1}AO \tag{2.31}$$

bo`ladi.

Isboti. Teorema shartidan kelib chiqsa, $q(\lambda)$ ko`phad mavjud bo`lib,

$$A^T = q(A), \; B^T = q(B) \tag{2.32}$$

bo`ladi. Bu ko`phad matritsalar simmetrik bo`lgan xolda $\lambda$ ga teng, kososimmetrik bo`lgan xolda esa $-\lambda$ ga teng. Agar A va B ortogonal matritsalar bo`lsa u holda $A$ va $B$ matritsalarning umumiy spektrida $\frac{1}{\lambda}$ uchun $g(\lambda)$ interpolyatsion ko`phad bo`ladi.

(2.32) tengliklardan foydalansak, (2.30) dan

$$g(B) = T^{-1}q(A)T$$

kelib chiqadi, yoki (2.32) ga asosan

$$B^T = T^{-1}A^T T$$

bo`ladi. Bundan $B = T^T A T^{T^{-1}}$. Bu tenglikni (2.30) ga qo`yib

$$TT^T A = ATT^T \tag{2.33}$$

ni topamiz.

T matritsaga teorema 2.3 ni qo`yamiz.

$$T = SO, \quad \left(S = S^T = f\left(TT^T\right), \; O^T = O^{-1}\right)$$

(2.33) ga asosan TT^T matritsa A matritsa bilan o`rin almashinuvchi, u holda $S = f\left(TT^T\right)$ matritsa ham $A$ matritsa bilan o`rin almashinuvchi bo`ladi, quydagiga ega bo`lamiz:

$$B = O^{-1}S^{-1}ASO = O^{-1}AO.$$

§3. Ko`mpleks simmetrik matritsalarning normal ko`rinishi.

Teorema2. 5. Avvaldan berilgan ixtiyoriy elementar bo`luvchilarga ega bo`lgan kompleks simmetrik matritsa mavjud.

Isboti. O`ng diogonalidan pastdagi elementlari birga teng, qolgan elementlari nolga teng bo`lgan $n-$tartibli H matritsani qaraymiz. matritsaga o`hshash bo`lgan S simmetrik matritsa mavjudligini isbotlaymiz: H

$$S = THT^{-1} \tag{2.34}$$

T -almashtiruvchi matritsani

$$S = THT^{-1} = S^T = T^{T^{-1}} H^T T^T$$

shartdan kelib chiqib izlaymiz. Bu shartni quydagich yozish mumkin:

$$VH = H^T V, \tag{2.35}$$

bu yerda V -simmetrik matritsa bo`lib, T matritsa bilan

$$T^T T = -2iV \tag{2.36}$$

tenglik orqali bog`langan.

H va $F = H^T$ matritsalarning xossalariga ko`ra, (2.35) matritsani tenglamaning ixtiyoriy $V$ yechim quydagi ko`rinishga ega:

$$V = \begin{Vmatrix} 0 & 0 & . & . & . & 0 & a_0 \\ 0 & 0 & . & . & . & a_0 & a_1 \\ . & . & . & . & . & . & . \\ a_0 & a_1 & . & . & . & & a_{n-1} \end{Vmatrix}, \tag{2.37}$$

bu yerda $a_0, a_1, . . . , a_{n-1}$-ixtiyoriy kompleks sonlar,

Bizga bitta T almashtiruvchi matritsani izlash yetarli, shuning uchun bu formulaga $a_0 = 1,\ a_1 = \ldots = a_{n-1} = 0$ deb olib, V matritsani quydagicha aniqlaymiz:

$$V = \begin{Vmatrix} 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 & 0 \end{Vmatrix}, \tag{2.38}$$

Bundan tashqari T – almashtiruvchi matritsani simmetrik matritsa ko`rinishda izlaymiz:

$$T = T^T \tag{2.39}$$

U holda (36) tenglama T uchun quydagich yozamiz:

$$T^2 = -2iV \tag{2.40}$$

Endi T noma`lum matritsani $V$ ning ko`phad ko`rinishda izlaymiz. $V^2 = E$ bo`lgani uchun bunday ko`phad sifatida $T = \alpha E + \beta V$ birinchi darajali ko`phadni olish mumkin. (2.40) tenglamadan $V^2 = E$ ekanligini xisobga olib, $\alpha^2 + \beta^2 = 0,\ \ 2\alpha\beta = -2i$ ekanligini topamiz. Bu munosabatlardan $\alpha = 1,\ \beta = -i$ ni aniqlaymiz. U holda

$$T = E - iV \tag{2.41}$$

bo`ladi. T - xosmas simmetrik matritsa. Shu bilan birga (2.40) dan

$$T^{-1} = \frac{1}{2} iV^{-1}T = \frac{1}{2} iVT,$$

yani

$$T^{-1} = \frac{1}{2}(E + iV) \tag{2.42}$$

Shunday qilib, H matritsaning S-simmetrik ko`rinishi quydagicha aniqlanadi:

$$S = THT^{-1} = \frac{1}{2}(E - iV)(E + iV),\ \ V = \begin{Vmatrix} 0\ 0 & . & . & . & 0 & 1 \\ 0\ 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . \\ 1\ 0 & . & . & ., & 0 & 0 \end{Vmatrix} \tag{2.43}$$

S matritsa (2.35) tenglamani qanoatlantiradi va $V^2 = E$ bo`lgani uchun (2.43) tenglikni quydagi ko`rinishda ham yozish mumkin:

$$2S = (H + H^T) + i(HV - VH) = H + H^T + i(H - H^T)V =$$
$$= \begin{Vmatrix} 0\ 1 & . & . & . & 0 & 0 \\ 1\ 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0\ 0 & . & . & . & 0 & 1 \\ 0\ 0 & . & . & . & 1 & 0 \end{Vmatrix} + i \begin{Vmatrix} 0 & . & . & . & 1 & 0 \\ 0 & . & . & . & 0 & -1 \\ . & . & . & . & . & . \\ 1 & . & . & . & 0 & 0 \\ 0 & -1 & . & .0 & & 0 \end{Vmatrix} \tag{2.44}$$

(2.44) formula H matritsani S simmetrik ko`rinishini aniqlaydi.

Agar $H-n-$tartibli matritsa bo`lsa, uni $H=H^{(n)}$ deb belgilaymiz. U holda mos T,V,S matritsalarni $T^{(n)},V^{(n)},S^{(n)}$ deb belgilaymiz.

Quydagi ixtiyoriy elementar bo`luvchilar berilgan bo`lsin:

$$(\lambda-\lambda_1)^{P_1},(\lambda-\lambda_2)^{P_2},\ldots,(\lambda-\lambda_i)^{P_i}, \tag{2.45}$$

mos mordon matritsani tuzamiz:

$$J=\left\{\lambda_1E^{(P_1)}+H^{(P_1)},\lambda_2E^{(P_2)}+H^{(P_2)},\ldots,\lambda_iE^{(P_i)}+H^{(P_i)},\right\}$$

Xar bir $H^{(P_j)}$ matritsa uchun mos $S^{(P_j)}$ simmetrik formani kiritamiz.

$$S^{(P_j)}=T^{(P_j)}H^{(P_j)}\left[T^{(P_j)}\right]^{-1},\ j=1,2,\ldots,i$$

dan

$$\lambda_jE^{(P_j)}+S^{(P_j)}=T^{(P_j)}\left[\lambda_jE^{(P_j)}+H^{(P_j)}\right]\left[T^{(P_j)}\right]^{-1}$$

Shuning uchun

$$\overline{S}=\left\{\lambda_1E^{(P_1)}+S^{(P_1)},\lambda_2E^{(P_2)}+S^{(P_2)},\ldots,\lambda_iE^{(P_i)}+S^{(P_i)}\right\} \tag{2.46}$$

$$T=\left\{T^{(P_1)},T^{(P_2)},\ldots,T^{(P_i)}\right\} \tag{2.47}$$

deb olib,

$$\overline{S}=TJT^{-1}$$

ga ega bo`lamiz.

$\overline{S}-J$ ordon matritsasining simmetrik ko`rinishi , $\overline{S}$ matritsa $J$ matritsaga o`hshash va (2.46) elementar bo`luvchilarga ega.

Natija2. 1. Ixtiyoriy $A=\|a_{ik}\|_{i,k=1}^{n}$ – kvadrat kompleks matritsa simmetrik matritsaga o`xshash.

Natija 2.2. Ixtiyoriy $S=\|s_{ik}\|_{i,k=1}^{n}$ – kompleks simmetrik matritsa $\overline{S}$ normal ko`rinishga ega bo`lgan simmetrik matritsa ortogonal –o`xshash, yani shunday $O$-ortogonal matritsa mavjudki unda quydagi tenglik o`rinli.

$$S=O\overline{S}O^{-1}. \tag{2.48}$$

Kompleks simmetrik matritsaning normal ko`rinishi quydagacha kvazidogonal ko`rinishga ega:

$$\overline{S}=\left\{\lambda_1E^{(P_1)}+S^{(P_1)},\ \lambda_2E^{(P_2)}+S^{(P_2)},\ldots,\lambda_iE^{(P_i)}+S^{(P_i)}\right\} \tag{2.49}$$

bu yerda $S^{(p)}$ kataklar quydagicha aniqlanadi:

$$2S^{(P)} = \left[E^{(P)} - iV^{(P)}\right]H^{(P)}\left[E^{(P)} + iV^{(P)}\right] = \left[H^{(P)} + H^{(P)^T} + i\left(H^{(P)} - H^{(P)^T}\right)V^{(P)}\right]$$

$$= \begin{Vmatrix} 0 & 1 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & . & . & . & 1 & 0 \end{Vmatrix} + i \begin{Vmatrix} 0 & 0 & . & . & . & 1 & 0 \\ 0 & 0 & . & . & . & 0 & -1 \\ . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & . & . & . & 0 & 0 \end{Vmatrix} \qquad (2.50)$$

§4. Kompleks kososimmetkir matritsaning normal ko\`rinishi.

Teorema 2.6. Kososimmetrik matritsa xar doim juft rang bo\`lsin. U holda $K$ matritsa satirlari orasida $r$ ta chiziqli bog\`liq bo\`lganlari $i_1, i_2, \ldots, i_r$ mavjud bo\`lib, qolgan satrlar bu satrlarning chiziqli kombinatsiyasidan iborat bo\`ladi. Shuningdek $K$ matritsaning mos satirlaridan xosil qilingan ustunlari, agaroxirgi elementlarni $-1$ ga ko\`paytirsak, u holda K matritsaning ixtiyoriy ustuni $i_1, i_2, \ldots, i_r$ nomerli ustunlarning chiziqli kombinatsiyasidan iborat bo\`ladi. Shuning uchun $r$-tartibli ixtiyoriy minor quydagi ko\`rinishda tasvirlanishi mumkin.

$$LK\begin{pmatrix} i_1, i_2, i_3, \ldots, i_r \\ i_1, i_2, i_{3,} \ldots, i_r \end{pmatrix},$$

bu yerda $L-$son.

Bundan

$$K\begin{pmatrix} i_1, i_2, \ldots, i_r \\ i_1, i_2, \ldots, i_r \end{pmatrix} \neq 0$$

ekanligi kelib chiqadi. Ammo kososimmetrik toq tartibli aniqlavchi doimo nolga teng. Demak r-juft son.

Teorema 2.7. 1.Agar λ K matritsaning xarakteristik son bo\`lib,

$$(\lambda - \lambda_0)^{l_1}, (\lambda - \lambda_0)^{l_2}, \ldots, (\lambda - \lambda_0)^{l_t}$$

lar unga mos elementar bo\`luvchilar bo\`lsa, u holda $-\lambda_0$ ham K matritsaning xarakteristik soni bo\`lib,

$$(\lambda + \lambda_0)^{l_1}, (\lambda + \lambda_0)^{l_2}, \ldots, (\lambda + \lambda_0)^{l_t}$$

lar unga mos elementar bo`luvchilar bo`ladi.

2. Agar nol soni $K-$ kososimmetrik matritsaning xarakteristik son bo`lib, u holda $K$ matritsa elementar bo`luvchilari sistemasida nol xarakteristik songa mos juft darajali elementar bo`luvchilar juft son marta takrorlanadi.

Isboti. 1. K^T va K matritsalar bir xil elementar bo`luvchilariga ega. Ammo $K^T = -K,$ $K$ ning elementar bo`luvchilari $\lambda_1, \lambda_2, \ldots$ larni $-\lambda_1, -\lambda_2, \ldots$ larga almashtirib xosil qilinadi.

2. K matritsaning nol xarakteristik soniga λ ko`rinishdagi elementar bo`luvchilari b_1 ta, λ^2 ko`rinishdagilari $b_2$ ta, va xakozo bo`lsin. Umuman b_r barcha λ^p ko`rinishdagi elementar bo`luvchilarni belgilaymiz. $b_2, b_4, \ldots,$ larni juft son ekanini isbotlaymiz.

K matritsaning d defekti nol xarakteristik sonlarga mos keluvchi, chiziqli bog`lanmagan xos vektorlar soniga teng, ya`ni $\lambda, \lambda^2, \ldots$ ko`rinishdagi elementan bo`luvchilar soniga teng. Shuning uchun

$$d = \delta_1 + \delta_2 + \ldots \qquad (2.51)$$

Teorma 2.6 ga asosan K matritsa rangi juft son bo`lib, $d = n - r,$ u holda $d$ son n soni qanday juftlikka ega bo`lsa, xuddi shu juftlikka ega. Shunday tasdiqni $K^3, K^5, \ldots$ matritsalarning $d_3, d_5, \ldots$ defektlariga nisbatan ham aytish mumkin, chunki kososimmetrik matritsaning toq darajalari yana kososimmetrik matritsa bo`ladi. Shuning uchun $d_1 = d,\ d_3,\ d_5, \ldots$ lar bir xil juftlikka ega.

Ikkinchi tomondan K matritsani m darajaga ko`targanda bu matritsaning xar bir $\lambda^p$ elementar bo`luvchisi p<m da p ta birinchi darajali elementar bo`luvchilarga yoyiladi, $p > m$ da esa m ta elementar bo`luvchilarga yoyiladi. Shuning uchun $K, K^3, \ldots$ matritsaning λ ning darajalari bo`lgan elementar bo`luvchilari soni quyidagi formulalar bilan aniqlanadi:

$$
\begin{aligned}
&d_3 = \delta_1 + 2\delta_2 + 3(\delta_3 + b\delta_4 + \ldots), \\
&d_5 = \delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + 5(\delta_5 + \delta_6 + \ldots), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{2.52}
$$

(2.51) ni (2.52) bilan birga qarab, barch $d_1 = d,\, d_3,\, d_5, \ldots$ sonlar bir xil juftlikka egaligidan, $\delta_2, \delta_4, \ldots,$ lar juft sonlar deb xulosa qilamiz.

Teorema 2.8. Teorema 2.7. dagi 1. va 2. cheklashlarni qanoatlantiruvchi ixtiyoriy berilgan elementar bo\`luvchilarga ega bo\`lgan kososimmetrik matritsa mavjud.

Isboti. Avval ikkita $(\lambda + \lambda_0)^p$ va elementar bo\`luvchilarga ega bo\`lgan 2p tartibli.

$$
J_{\lambda_0}^{(pp)} = \{\lambda_0 E + H, -\lambda_0 E - H\},\; E = E^{(p)}, H = H^{(p)}, \tag{2.53}
$$

Kvazidiogonal matritsa uchun kososimmetrik matritsani topamiz.

Buning uchun shunday T almashtirish matritsasini izlaymizki, unda

$$
T\, J_{\lambda_0}^{(pp)} T^{-1}
$$

matritsa kososimmetrik, ya\`ni

$$
T\, J_{\lambda_0}^{(pp)} T^{-1} + T^{T^{-1}} \left[J_{\lambda_0}^{(pp)}\right]^T T^T = 0
$$

yoki

$$
W\, J_{\lambda_0}^{(pp)} + \left[J_{\lambda_0}^{(pp)}\right]^T W = 0 \tag{2.54}
$$

tenglik o\`rinli bo\`lib, $W-$kososimmetrik matritsa T matritsa bilan

$$
T^T T = 2iW \tag{2.55}
$$

tenglik orqali bog\`langan.

W matritsani xar biri $p-$tartibli bo\`lgan to\`rtta kvadratik blokka ajratamiz:

$$
W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}
$$

U holda (2.54) ni quydagicha tasvirlash mumkin

$$
\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}\begin{pmatrix} \lambda_0 E + H & 0 \\ 0 & -\lambda_0 E - H \end{pmatrix} + \begin{pmatrix} \lambda_0 E + H^T & 0 \\ 0 & -\lambda_0 E - H^T \end{pmatrix}\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = 0 \tag{2.56}
$$

(2.56) matritsani tenglamaning chap tomonidagi blok matritsalar ustidagi amallarni bajarib, quydagi to`rtta matritsani tenglamalar sistemasini xosil qilamiz:

$$\begin{aligned} &H^T W_{11} + W_{11}(2\lambda_0 E + H) = 0, \\ &H^T W_{12} - W_{12} H = 0, \\ &H^T W_{21} - W_{21} H = 0, \\ &H^T W_{22} + W_{22}(2\lambda_0 E + H) = 0 \end{aligned} \tag{2.57}$$

Malumki, agar A va B matritsalar umumiy xarakterstik sonlarga ega bo`lmasa $AX - XB = 0$ tenglama faqat $X = 0$ yechimga ega. Shunung uchun (2.57) ning 1- va 4- tenglamalaridan $W_{11} = W_{22} = 0$ kelib chiqadi. (2.57) ning 2- va 3- tenglamalari ustida teorema 2.5 ning isbotidagidek muloxaza yuritib,

$$W_{12} = V = \begin{Vmatrix} 0 & . & . & . & 0 & 1 \\ 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . \\ 1 & . & . & . & 0 & 0 \end{Vmatrix} \tag{2.58}$$

ni aniqlaymiz. W ning simmetrik matritsa ekanligidan

$$W_{21} = W_{12} = V$$

ekanligi kelib chiqadi.

Shunday qilib,

$$W = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} = V^{(2p)} \tag{2.59}$$

Ammo §3 da ko`rsatilganidek (2.55) tenglama qanoatlantiradi, agarda

$$T = E^{(2p)} - iV^{(2p)}, \tag{2.60}$$

bo`lsa. Bundan

$$T^{-1} = \frac{1}{2}\left(E^{(2p)} + iV^{(2p)}\right) \tag{2.61}$$

Demak, izlanayotgan kososimmetrik matritsa quydagi formula bilan topiladi:

$$K_{\lambda_0}^{(pp)} = \frac{1}{2}\left[E^{(2p)} - iV^{(2p)}\right] J_{\lambda_0}^{(pp)} \left[E^{(2p)} + iV^{(2p)}\right] = \\ = \frac{1}{2}\left[J_{\lambda_0}^{(pp)} - J_{\lambda_0}^{(pp)^T} + i\left(J_{\lambda_0}^{(pp)} V^{(2p)} - V^{(2p)} J_{\lambda_0}^{(pp)}\right)\right] \tag{2.62}$$

$J_{\lambda_0}^{(pp)}$ va $V^{(2p)}$ larning o`rniga (2.53) va (2.59) dagi mos blok matritsalarni qo`yib, quydagini topamiz:

$$2K_{\lambda_0}^{(pp)} = \begin{pmatrix} H - H^T & 0 \\ 0 & H - H^T \end{pmatrix} + i\begin{pmatrix} H + H^T + 2\lambda_0 E & 0 \\ 0 & -H - H^T - 2\lambda_0 E \end{pmatrix}\begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} = \tag{2.63}$$

$$= \begin{pmatrix} H - H^T & i(2\lambda_0 V + HV + VH) \\ -i(2\lambda_0 V + HV + VH) & H^T - H \end{pmatrix}$$

ya`ni

$$K_{\lambda_0}^{(pp)} = \frac{1}{2}\begin{Vmatrix} 0 & 1 & \dots & 0 & 0 & . & 0 & 0 & \dots & i & 2\lambda_0 \\ -1 & 0 & \dots & 0 & 0 & . & 0 & 0 & \dots & 2\lambda_0 & i \\ \dots & \dots & \dots & \dots & \dots & . & \dots & . & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & . & i & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 & . & 2\lambda_0 & i & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & . & \dots & \dots & \dots & \dots & \dots \\ 0 & . & \dots & -i & -2\lambda_0 & . & 0 & -1 & \dots & . & 0 \\ 0 & . & \dots & -2\lambda_0 & -i & . & 1 & 0 & \dots & . & 0 \\ \dots & \dots & \dots & \dots & \dots & . & \dots & \dots & \dots & \dots & \dots \\ -i & . & \dots & 0 & 0 & . & 0 & 0 & \dots & . & -1 \\ -2\lambda_0 & -i & \dots & 0 & 0 & . & 0 & 0 & \dots & 1 & 0 \end{Vmatrix}. \tag{2.64}$$

Endi λ^q bitta elementar bo`luvchiga ega bo`lgan $q-$tartibli $K^{(q)}$ kososimmetrik matritsani quramiz, bu yerda $q-$toq son. Izlanayotgan kososimmetrik matritsa quydagi matritsaga o`xshash bo`ladi.

$$J^{(q)} = \begin{Vmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{Vmatrix} \tag{2.65}$$

$$K^{(q)} = TJ^{(q)}T^{-1} \tag{2.66}$$

deb olib, kososimmetrik shartidan quydagini topamiz:

$$W_1 J^{(q)} + J^{(q)} W_1 = 0, \tag{2.67}$$

bu yerda

$$T^T T = 2iW_1 \tag{2.68}$$

Bevosita tekshirib ko`rib, ishonch xosil qilish mumkin,

$$W_1 = V^{(q)} = \begin{Vmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & 0 & 0 \end{Vmatrix}$$

matritsa (2.67) tenglamani qanoatlantiradi. W_1 ni bunday tanlab, (2.68) dan quydagini topamiz:

$$T = E^{(q)} - iV^{(q)},\ T^{-1} = \frac{1}{2}\left[E^{(q)} + iV^{(q)}\right] \tag{2.69}$$

$$2K^{(q)}\left[E^{(q)} - iV^{(q)}\right]J^{(q)}\left[E^{(q)} + iV^{(q)}\right] = J^{(q)} - J^{(q)T} + i\left(J^{(q)} + J^{(q)T}\right)V^{(q)} \tag{2.70}$$

mos xisoblashlarni bajarib, quydagini topamiz:

$$2K^{(q)} = \begin{Vmatrix} 0 & 1 & . & . & . & 0 \\ -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & \\ 0 & . & . & . & 0 & -1 \\ 0 & . & . & . & -1 & 0 \end{Vmatrix} + i\begin{Vmatrix} 0 & . & . & . & 1 & 0 \\ 0 & . & . & . & 0 & 1 \\ . & . & . & . & . & \\ -1 & 0 & . & . & . & 0 \\ 0 & -1 & . & . & . & 0 \end{Vmatrix} \tag{2.71}$$

Teorema 2.7 dagi shartlarni qanoatlantiruvchi ixtiyoriy elementar bo`luvchilar.

$$\begin{gathered} \left(\lambda - \lambda_j\right)^{p_j},\ \left(\lambda + \lambda_j\right)^{p_j},\ j = 1, 2, \dots, u \\ \lambda^{q_k}, k = 1, 2, \dots, v,\ q_1, q_2, \dots, q_v - toq\ sonlar \end{gathered} \tag{2.72}$$

berilgan bo`lsin.

U holda kvazidiogonal kososimmetrik matritsa.

$$\tilde{K} = \left\{K_{\lambda_1}^{(p_1 p_1)}, \dots, K_{\lambda_i}^{(p_i p_i)}, K^{(q_1)}, \dots, K^{(q_v)}\right\} \tag{2.73}$$

bo`ladi.

Natija2.3. Ixtiyoriy kompleks kososimmetrik K matritsa (2.64), (2.71), (2.73) formulalar bilan aniqlangan $\tilde{K}$ normal formaga ega bo`lgan kososimmetrik matritsa ortogonal-o`xshashdir, ya`ni shunday kompleks ortogonal O matritsa mavjudki, unda

$$K = O\tilde{K}O^{-1} \tag{2.74}$$

Eslatma. Agar K – xaqiqiy kososimmetrik matritsa bo`lsa, u holda quydagi chiziqli elementar bo`luvchilarga ega:

$$\lambda + i\varphi_1,\ \lambda - i\varphi_1, \ldots, \lambda + i\varphi_u, \lambda - i\varphi_u,\ \underbrace{\lambda, \ldots, \lambda}_{v\ ta},$$

φ_j – xaqiqiy sonlar. Bu holda (2.73) da $p_j = 1,\ q_k = 1$ deb olib, xaqiqiy kososimmetrik matritsa normal ko`rinishni hosil qiladi:

$$\tilde{K} = \left\{ \begin{Vmatrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{Vmatrix}, \ldots, \begin{Vmatrix} 0 & \varphi_i \\ -\varphi_i & 0 \end{Vmatrix}, 0, \ldots, 0 \right\}$$

§5. Kompleks ortogonal matritsaning normal ko`rinishi.

Teorema 2.9. 1.Agar $\lambda_0\ \left(\lambda_0^2 \neq 1\right)$ – O ortogonal matritsaning xarakteristik son bo`lib, bu xarakteristik songa mos elementar bo`luvchilar

$$(\lambda - \lambda_0)^{j_1},\ (\lambda - \lambda_0)^{j_2}, \ldots, (\lambda - \lambda_0)^{j_t}$$

lar bo`lsa u holda $\frac{1}{\lambda_0}$ ham $O$ matritsaning xarakteristik soni bo`lib, bu xarakteristik songa mos elementar bo`luvchilar

$$\left(\lambda - \lambda_0^{-1}\right)^{j_1},\ \left(\lambda - \lambda_0^{-1}\right)^{j_2}, \ldots, \left(\lambda - \lambda_0^{-1}\right)^{j_t}$$

bo`ladi.

2. Agar $\lambda_0 = \pm 1$ O ortogonal matritsaning xarakteristik soni bo`lsa, u holda bu $\lambda_0$ xarakteristik songa mos juft darajali elementar bo`luvchilar juft son marta takrorlanadi.

Isboti. 1. Ixtiyoriy O xosmas matritsadan O^{-1} matritsaga o`tganda $(\lambda-\lambda_0)i$ elementar bo`luvchi $\left(\lambda-\lambda_0^{-1}\right)i$ elementar bo`luvchi bilan almashadi. Ikkinchi tomondan $O$ va $O^T$ matritsalar xar doim bir xil elementar bo`luvchilarga ega. Shuning uchun $O^T=O^{-1}$ ortogonallik shartiga ko`ra teorema 9 ning birinchi q`ismi isbotlanadi.

2. Faraz q`ilaylik 1 soni matritsaning xarakteristik soni bolib , -1 xarakteristik soni bo`lmasin, ya`ni $|E-O|=0,\ E+O\neq 0.$ U holda $K$ matritsani quyidagi tenglik bilan aniq`laymiz:

$$K=(E-O)(E+O)^{-1}$$

Bevosita tekshirib ishonch hosil q`ilish mumkinki, $K^T=-K$ ya`ni $K-$ kososimmetrik boladi. (2.74) dan.

$$O=(E-K)(E+K)^{-1}$$

ni aniq`lab, $f(\lambda)=\dfrac{1-\lambda}{1+\lambda}$ deb olsak $f`(\lambda)=-\dfrac{2}{(1+\lambda)^2}$ bo`ladi. Demak, $K$ matritsadan $O=f(K)$ matritsaga o`tganda elementar bo`luvchilar yoyilmaydi. Shuning uchun $O$ matritsa elementar bo`luvchilar sistemasidagi $(\lambda-1)^{2p}$ ko`rinishdagi elementar bo`luvchilar juft son marta takrorlanadi, shuningdek K matritsaning λ^{2p} ko`rinishdagi elementar bo`luvchilari uchun xam bu o`rinli. $-\varepsilon$ xarakteristik son bo`lib, 1 xarakteristik son bo`lmagan xol $O$ matritsani $-O$ matritsa bilan almashtirish yo`li bilan xal qilinadi.

$\lambda_0=\pm 1$ ning xar ikkalasi ham xarakteristik son bo`gan holni qarab chiqamiz. $\varphi(\lambda)$ bilan $O$ matritsaning minimal ko`phadini belgilaymiz.

Teoremaning birinchi q`ismiga asosan $\varphi(\lambda)$ni quyidagi korinishda yozishimiz mumkin.

$$\varphi(\lambda)=(\lambda-1)^m(\lambda+1)^{m_2}\prod_{j=1}^{n}\left(\lambda-\lambda_j\right)^{p_j}\left(\lambda-\lambda_j^{-1}\right)^{p_j},\ \lambda_j^2+1,\ j=1,2,...,n.$$

Darajasi m dan ($m-\varphi(\lambda)$ ning darajasi) kichik bo`lgan $g(1)=1$ bo`lib, O matritsa spektiridagi barcha qolgan $m-1$ta qiymati no`lga teng bo`lgan $g(\lambda)$ ko`phadni qaraymiz va

$$P = g(O) \tag{2.75}$$

deb olamiz

Omatritsaning spektorida $[g(\lambda)]^2$ va $g\left(\frac{1}{\lambda}\right)$ funksiyalar $g(\lambda)$ funksiya qabul qilingan qiymatlarni qabul qiladilar.Shuning uchun

$$P^2 = P,\ P^T = g\left(O^T\right) = g\left(O^{-1}\right) = P \tag{2.76}$$

ya`ni $P-$simmetrik tasvirlovchi matritsa.

$h(\lambda)$ ko`pxad va Q matritsalarni quydagicha aniqlaymiz.

$$h(\lambda) = (\lambda-1)g(\lambda), \tag{2.77}$$

$$Q = h(O) = (O-E)P \tag{2.78}$$

$[h(\lambda)]^m$ daraja O matritsa spektorida nolga aylanib, $\varphi(\lambda)$ ga qoldiqsiz bo`linadi, Shuning uchun

$$Q^{m_1} = O,$$

ya`ni $Q-m,$ nilpoteng indeksli nilpoteng matritsa, (2.78) dan

$$Q^T = \left(O^T - E\right)P \tag{2.79}$$

endi

$$R = Q\left(O^T + 2E\right) \tag{2.80}$$

matritsani qaraymiz.

(2.76), (2.78) va (2.79)dan quydagi kelib chiqadi:

$$R = QQ^T + 2Q = \left(O - O^T\right)P$$

Bundan ko`rinadiki, $R-$kososimmetrik matritsa. Ikkinchi tomondan (2.80) dan

$$R^k = Q^k\left(O^T + 2E\right)^k \quad k = 1,2,\ldots, \tag{2.81}$$

Ammo Q^T matritsa Q matritsa kabi nilpoteng matritsa, shuning uchun

$$\left|Q^T + 2E\right| \neq O$$

Shuning uchun (2.81) dan kelibchiqadiki, ixtiyoriy K uchun R^K va Q^K lar bir xil ranga ega.

Ammo k toq bo`lganda $R^k$ kososimmetrik matritsa bo`lib, juft ranga ega. Demak,

$$Q, Q^3, Q^5, \ldots$$

matritsalar juft ranga ega bo`ladi.

Shuning uchun §4 da K matritsa uchun aytilgan muloxazalarni Q mateitsa uchun ham takrorlab, shunday xulosaga kelishimiz mumkin, Q matritsaning elementar bo`luvchilar orasidagi $\lambda^{2p}$ ko`rinishdagi bo`luvchilar juft son marta takrorlanadi. Ammo $Q$ matritsaning xar bir $\lambda^{2p}$ elementar bo`luvchisiga O matritsaning $(\lambda-1)^{2p}$ elementar bo`luvchisi mos keladi va aksincha. Bundan kelib chiqadiki, $O$ matritsaning elementar bo`luvchilarining $(\lambda-1)^{2p}$ ko`rinishdagi juft son marta takrorlanadi.

$(\lambda+1)^{2p}$ elementar bo`luvchilar uchun shunday tasdiqni isbotlanganlarni $-O$ matritsaga qo`llab xosil qilish mumkin.

Teorema 2.10. Ixtiyoriy quydagi ko`rinishdagi darajalar sistemasi qandaydir $O$ kompleks ortoganal matritsaning elementar bo`luvchilari sistemasi bo`ladi:

$$\begin{aligned}
&(\lambda-\lambda_j)^{p_j},\ (\lambda-\lambda_j^{-1})^{p_j},\ \lambda_j \neq 0,\ j=1,2,\ldots,i\\
&(\lambda-1)^{q_1},\ (\lambda-1)^{q_2},\ldots,\ (\lambda-1)^{q_v},\\
&(\lambda+1)^{t_1},\ (\lambda+1)^{t_2},\ldots,\ (\lambda+1)^{t_v},\\
&q_1, q_2,\ldots,q_v, t_1, t_2,\ldots, t_w, - \ toq\ sonlar
\end{aligned} \tag{2.82}$$

Isboti.

$$\lambda_j = e^{\mu_j},\ j=1,2,\ldots,i$$

tenglik yordamida μ_j sonlarni kiritamiz.

Elementar bo`luvchilari mos ravishda

$$(\lambda-\mu_j)^{P_j}, (\lambda+\mu_j)^{P_j}, \quad j=1,2,\ldots,i,\ \lambda^{q_1},\ldots,\lambda^{q_v},\ \lambda^{t_1},\ldots,\lambda^{t_w}$$

bo`lgan

$$K_{\mu_j}^{(p_j p_j)}, \quad j=1,2,\ldots,i, \; K^{(q_1)},\ldots,K^{(q_v)}, \;\; K^{(t_1)},\ldots,K^{(t_w)}$$

kanonik kososimmetrik matritsalarni qaraymiz.

Agar K – kososimmetrik matritsa bo`lsa,

$$O=e^K$$

ortogonal bo`lada, ya`ni

$$O^T = e^{K^T} = e^{-K} = O^{-1}.$$

K matritsaning xar biri $(\lambda-\mu)^p$ elementar bo`luvchisiga $O$ matritsaning $(\lambda-\mu)^p$ elementar bo`luvchisi mos kelishini etiborga olsak, quydagi kvazidiogonal matritsa ortogonal bo`lib, (2.82) elementar bo`luvchilarga ega bo`ladi.

$$\overline{O}=\left\{e^{K_{p_1}^{(p_1p_1)}},\ldots,e^{K_{p_i}^{(p_ip_i)}},e^{K^{(q_1)}},\ldots,e^{K^{(q_v)}},-e^{K^{(t_1)}},\ldots,-e^{K^{(t_w)}}\right\} \qquad (2.83)$$

Natija2.4. Ixtiyoriy O ortogonal matritsa $\overline{O}$ normal ko`rinishga ega bo`lgan ortogonal matritsaga ortogonal-o`xshash bo`ladi, ya`ni shunday O_1 ortogonal matritsa mavjudki, unda

$$O=O_1\overline{O}O_1^{-1} \qquad (2.84)$$

bo`ladi.

Mashqlar:

1. Agar C-kvadrat yoki to’g’ri to’rtburchakli matritsa bo’lib, $SpCC^*$ bo’lsa, u holda C=0 bo’lishini isbotlang.
2. Qanday shartlar bojarilganda
$$A=\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$
matritsa normal bo’ladi. Bu yerda B va D kataklar kvadrat matritsalar.
3. Xaqiqiy xos qiymatga ega bo’lmagan, ikkinchi tartibli, normal xaqiqiy matritsa ko’rinishini toping.
4. Xaqiqiy ortoganal simmetrik matritsaning kanonik ko’rinishi va geometrik ma’nosi qanday bo’ladi.
5. Ortoganal matritsa antisimmetrik bo’lishi mumkinmi ?

6. Har-bir ustuni uchun qo'shma kompleks ustunga ega bo'lgan unitary matritsa mavjudmi ?
7. Quyidagi ko'rinishdagi

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \cdot u^T \\ \sin\alpha \cdot v & R \end{pmatrix}, \qquad u^T u = v\, v^T = 1$$

barcha ortoganal matritsalar berilgan u ni v ustunlarda

$$R = P - (1 + \cos\alpha) v \cdot u^T$$

da xosil qilinishini ko'rsating. Bu yerda P - u ni v ga o'tkazuvchi ortoganal matritsa.
8. Unitar simmetrik matritsa qanday bo'ladi?
9. Agar B- unitary matritsa bo'lsa, $B^T B$ -ko'rinishdagi matritsani unitary simmetrik matritsa ekanligini isbotlang.

III-BOB

MATRITSALARNING SINGULYAR DASTASI.

§1. Masalani qo`yilishi.

Bu bob quyidagi masalaga bag`ishlangan.

Elementlari K sonlar maydonidan olingan bir xil $m \times n$ o`lchovli to`rtta A, B, A_1, B_1 matritsalar berilgan. Qanday shartlar bajarilganda, mos ravishda m va n o`lchovli kvadrat xosmas $P$ va $Q$ matritsalar mavjud bo`lib, bir vaqitni o`zida

$$PAQ = A_1 \ , \quad PBQ = B_1 \tag{3.1}$$

tengliklar bajariladi.

$$A + \lambda B \text{ va } A_1 + \lambda B_1$$

matritsalar dastasini qarab chiqamiz. Bunday xolda (3.1) tengliklarni quydagi bitta tenglik bilan almashtirish mumkin bo`ladi.

$$P(A + \lambda B)Q = A_1 + \lambda B_1 \tag{3.2}$$

**Ta`rif 3.1.** Mos ravishda $m$ va $n$ o`lchovlari P va Q o`zgarmas matritsalar yordamida tuzilgan (3.2) tenglik bilan bog`langan bir xil $m \times n$ o`lchovli to`g`ri to`rtburchakli $A + \lambda B$ va $A_1 + \lambda B_1$ matritsalar dastalari qa`tiy ekvivalent deyiladi.

$A + \lambda B$ va $A_1 + \lambda B_1$ dastalarining ekvivalentlik kriteriyasi $\lambda -$matritsalar ekvivalentligining umumiy kriteryasidan kelib chiqib, shu dastalar invariant ko`pxadlar yoki elementar bo`luvchilarning ustma-ust tushishidan iborat bo`ladi.

Ushbu bobda ikkta matritsalar dastasi qa`tiy ekvivalentligi kriteryasi o`rnatiladi va xar bir dasta uchun unga qat`iy ekvivalent bo`lgan kvadratik forma aniqlanadi.

Quydagi masala geometrik ma`noga ega.

R^n fazoni R^m fazoga o`tkazuvchi $\overline{A}+\lambda\overline{B}$ chiziqli operatorlar dastasini qaraymiz. Fazolarning aniq bir bazasida bu chiziqli operatorlar dastasiga to`g`ri to`rtburchakli $A+\lambda B$ matritsalar dastasi mos keladi. Fazolardagi bazislarini o`zgarishi bilan $A+\lambda B$ dasta qat'iy ekvivalent $P(A+\lambda B)Q$ dasta bilan almashadi, bu yerda $P$ va $Q$ lar mos ravishda $m$ va $n$ o`lchovli xosmas kvadrat matritsalar. Demak, qat'iy ekvivalentlik kriteryasi $m\times n$ o`lchovli $A+\lambda B$ matritsalar dastasi sinfining xarakteristikasini berib, bu dastalar sinfi R^n fazoni R^m fazoga akslantiruvchi $\overline{A}+\lambda\overline{B}$ operatorlar dastasini (shu fazolarda tanlangan xar-xil bazislarda) ifodalaydi.

Dastaning kononik formasini xosil qilish uchun R^n va R^m fazolarda shunday bazisni topish kerakki, unda $\overline{A}+\lambda\overline{B}$ operatorlar dastasi mumkin qadar sodda matritsalar bilan ifodalansin.

Barcha $m\times n$ o`lchovli matritsalar dastalari ikkita asosiy tiplarga ajratiladi: regulyar va singular dastalar.

**Ta`rif 3.2.** $A$ va $B$ matritsalar bir xil $n-$tartibli kvadrat matritsalar bo`lib, $|A+\lambda B|$ aniqlovchi aynan nolga tengmas bo`lsa, $A+\lambda B$-regulyar dasta deyiladi. Boshqa barcha xollardagi dastalar singulyar dastalar deyiladi.

Regulyar dastalarni qat'iy ekvivalent kriteriyasi va ularning kanonik formasi 1867-yilda K. Vetshtrass tomonidan o`rganilgan. Xuddi shu masalalar singulyar dastalar uchun 1890-yilda L. Kroneker tomonidan o`rganilgan.

§2. Matritsalarning regulyar dastasi.

$A+\lambda B$ va $A_1+\lambda B_1$ dastalar bir xil o`lchovli matritsalardan tashkil topgan bo`lib, $|B|\neq 0,\ |B_1|\neq 0$ bo`lgan xususiy xolni qaraymiz. Bu holda dastalarning ekvivalentligi va qat'iy ekvivalentligi tushunchalari ustma-ust tushadi. Shuning uchun $\lambda-$matritsalar ekvivalentligi umumiy kriteryasini dastalar uchun qo`llab, quydagi teoremani xosil qilamiz:

Teorema 3.1. Ikkita bir-xil tartibli $A+\lambda B$ va $A_1+\lambda B_1$ dastalar $|B|\neq 0,$ va $|B_1|\neq 0$ shartda qat'iy ekvivalent bo`ladi. Faqat va faqat ular $K$ maydonda bir xil elementar bo`luvchilarga ega bo`lsa.

§1 da keltirilgan ta`rif 3.2 ga ko`ra $A+\lambda B$ regular dastalarda $|B|=0$, xattoki $|A|=|B|=0$ xolatlar ham bo`lishi mumkin.

Bu keltirilgan umumlashgan ta`rifda teorema 3.1 o`z kuchini saqlaydimi yoki yo`q ekanligini bilish uchun quyidagi misolni qaraymiz;

$$A+\lambda B=\begin{Vmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 3 & 2 & 6 \end{Vmatrix}+\lambda\begin{Vmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{Vmatrix},\ A_1+\lambda B_1=\begin{Vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{Vmatrix}+\lambda\begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} \tag{3.3}$$

Bu yerda har ikkala dasta bitta $\lambda+1$ elementar bo`luvchiga ega. Shu bilan birga bu dastalar qat'iy ekvivalent emas, chunki $r(B)=2,\ r(B_2)=1,$ (3.2) tenglikdan esa $r(B)=r(B_2)$ kelib chiqadi. Shu bilan birga (3.3) dastalar teorema 3.1 ga ko`ra regulyar bo`ladi , chunki.

$$|A+\lambda B|=|A_1+\lambda B_1|=\lambda+1$$

Bu misoldan ko`rinadiki , regulyar dastalarning umumlashgan ta`rifida teorema 3.1 to`gri emas.

Teorema 3.1 ni saqlab qolish uchun dastalarning cheksiz elementar bo`luvchilari tushunchasini kiritishimizga to`gri keladi. $A+\lambda B$ dastani bir jinsli λ,μ parametrlar yordamida $\mu A+\lambda B$ ko`rinishda olamiz. U holda $\Delta(\lambda,\mu)=|\mu A+\lambda B|$ aniqlovchi $\lambda,\mu$ larning bir jinsli funksiyasi bo`ladi..

$A+\lambda B$ matritsaning barcha $k-$tartibli minorlari EKUBi $D_k(\lambda,\mu)$ $(k=1,2,\ .\ .\ .\ ,n)$ ni aniqlab, quyidagi invariant kophadlarni hosil qilamiz:

$$i_1(\lambda,\mu)=\frac{D_n(\lambda,\mu)}{D_{n-1}(\lambda,\mu)},\quad i_2(\lambda,\mu)=\frac{D_{n-1}(\lambda,\mu)}{D_{n-2}(\lambda,\mu)},\ .\ .\ .\ ,$$

shu bilan birga $D_k(\lambda,\mu)$ va $i_j(\lambda,\mu)$ lar λ va μ larga nisbatan bir jinsli ko`phadlardir. Invaryant ko`phadlarni K maydonda keltirilmaydigan bir jinsli ko`phadlarga yoyib, $\mu A+\lambda B$ dastani $K$ maydondagi $l_\alpha(\lambda,\mu)$ $(\alpha=1,2\ .\ .\ .\ ,)$ elementar bo`luvchilarini hosil qilamiz.

$l_\alpha(\lambda,\mu)$ da $\mu=1$ deb olib, $A+\lambda B$ dastani $l_\alpha(\lambda)$ elementar bo`luvchisiga kelamiz. Aksincha , $A+\lambda B$ dastaning har bir $q-$darajali $l_\alpha(\lambda)$ elementar bo`luvchisidan

$$l_\alpha(\lambda,\mu)=\mu^q l_\alpha\left(\frac{\lambda}{\mu}\right)$$

formula yordamida $l_\alpha(\lambda,\mu)$ elementar bo`luvchini hosil qilamiz .Shunday qilib, $\mu A+\lambda B$ dastaning $\mu^q$ ko`rinishdagidan boshqa barcha elementar bo`luvchilarini hosil qilishimiz mumkin.

μ^q korinishdagi elementar bo`luvchilar faqat va faqat $|B|=0$ dagina mavjud bo`lib , ular $A+\lambda B$ dasta uchun "cheksiz" elementar bo`luvchilar degan nom bilan ataladi.

$A+\lambda B$ va $A_1+\lambda B_1$ dastalarning qat'iy ekvivalentligidan $\mu A+\lambda B$ va $\mu A_1+\lambda B_1$ dastalarning ham qa`tiy ekvivalentligi kelib chiqadi, shuning uchun $A+\lambda B$ va $A_1+\lambda B_1$ qa`tiy ekvivalent dastalarda nafaqat "chekli" , balki "cheksiz" elementar bo`luvchilar ham ustma-ust tushadi.

Endi bizga barcha elementar bo`luvchilari ustma-ust tushgan $A+\lambda B$ va $A_1+\lambda B_1$ regulyar dastalar berilgan bo`lsin. Bir jinsli parametrlarni kiritib, $\mu A+\lambda B$ va $\mu A_1+\lambda B_1$ larni xosil qilamiz. Bu parametrlarni quyidagicha almashtiramiz:

$$\begin{aligned}\lambda &= \alpha_1\tilde{\lambda}+\alpha_2\tilde{\mu}\\ \mu &= \beta_1\tilde{\lambda}+\beta_2\tilde{\mu}\end{aligned} \qquad (\alpha_1\beta_2-\alpha_2\beta_1\neq 0)$$

Yangi parametirlarda dastalar $\tilde{\mu}\tilde{A}+\tilde{\lambda}\tilde{B},\ \tilde{\mu}\tilde{A}_1+\tilde{\lambda}\tilde{B}_1$ ko`rinishda yozib, bu yerda

$$\tilde{B}=\beta_1 A+\alpha_1 B,\ \ \tilde{B}_1=\beta_1 A_1+\alpha_1 B_1\,.$$

$\mu A+\lambda B$ va $\mu A_1+\lambda B_1$ dastalarining reguiyarligidan kelib chiqadiki, α_1 va β_1 larni $|\tilde{B}|\neq 0,\ \ |\tilde{B}_1|\neq 0$ shatlarni qanoatlantiradigan qilib tanlash mumkin. Shuning uchun teorema 3.1 ga asosan $\tilde{\mu}\tilde{A}+\tilde{\lambda}\tilde{B}$ va $\tilde{\mu}\tilde{A}_1+\tilde{\lambda}\tilde{B}_1$ dastalar, demak $\mu A+\lambda B$ va $\mu A_1+\lambda B_1$ yoki $A+\lambda B$ va $A_1+\lambda B_1$ dastalar ham qa`tiy ekvivalentdir. Shunday qilib, biz teorema 3.1 ning quyidagi ummumlashganiga keldik.

Teorema 3.2. $A+\lambda B$ va $A_1+\lambda B_1$ dastalar qa`tiy ekvivalent bo`lishi uchun bir xil va faqat bir xil ("chekli" va "cheksiz") elementar bo`luvchilarga ega bo`lishi zarur va yetarlidir.

Yuqorida ko`rilgan misolda (3.3) dastalar bir xil "chekli" elementar bo`luvchilarga ega ammo "cheksiz" elementar bo`luvchilari xar-xil, ya`ni birinchi dasta bitta μ^2 elementar bo`luvchiga, ikkinchisi esa ikkta $\mu,\ \mu$ elementar bo`luvchilarga ega. Shuning uchun bu dastalar qa`tiy ekvivalent emas.

Endi $A+\lambda B$ ixtiyoriy regulyar dasta bo`lsin. U holda shunday c soni mavjudki, unda $|A+cB|\neq 0$ bo`ladi. Berilgan dastani $A_1+(\lambda-c)B,$ bu yerda $A_1=A+cB,\ |A_1|\neq 0,$ ko`rinishda tasvirlaymiz. Bu dastani chapdan $A_1^{-1}$ ga ko`paytirib quyidagi ko`rinishdagi dastalarni xosil qilamiz.

$$E+(\lambda-c)A_1^{-1}B=E+(\lambda-c)\{J_0,J_1\}=\{E-cJ_0+\lambda J_0,\ \ E-cJ_1+\lambda J_1\} \qquad (3.4)$$

bu yerda $\{J_0,J_1\}-A_1^{-1}B$ matritsaning kvazidiogonal normal formasi, J_0 – Jordon nil`potent matritsasi va $|J_1|\neq 0.$

(3.4) ning o`ng tomonidagi birinchi dioganal biokni $(E-c\,J_0)^{-1}$ ga ko'paytiramiz. Bu yerda $\lambda$ oldidagi koeffisiyent nil`potent (qandaydir darajasi nolga teng) matritsa. Shuning uchun o`xshash almashtirish bilan bu dastani quyidagi ko`rinishga keltirish mumkin.

$$E+\hat{\lambda}\hat{J}_0=\{N^{(i_1)},N^{(i_2)},\ .\ .\ .\ ,N^{(i_s)}\}, \qquad \left(N^{(i)}=E^{(i)}+\lambda H^{(i)}\right) \qquad (3.5)$$

Teorema3.3. Ixtiyiriy $A+\lambda B$ dasta quyidagicha kvazidiogonal kanonik ko`rinishga keltirilishi mumkin.

$$\{N^{(i_1)},N^{(i_2)},\ .\ .\ .\ ,N^{(i_s)},\ \ J+\lambda E\}, \qquad \left(N^{(i)}=E^{(i)}+\lambda H^{(i)}\right) \qquad (3.6)$$

bu yerda birinchi s ta dioganal blok $A+\lambda B$ dastaning $\mu^{i_1},\lambda^{i_2},\ .\ .\ .\ ,\mu^{i_s}$ cheksiz elementar bo`luvchilarga mos kelib, oxirgi $J+\lambda E$ dioganal blok berilgan dastaning chekli elementar bo`luvchilari bilan bir qiymatli aniqlanadi.

§3. Singulyar dastalar. Keltirish xaqida teorema.

$m\times n$ o`lchovli $A+\lambda B-$matritsalarning singulyar dastasini qaraylik. $r$ bilan dastaning rangini, ya`ni aynan nolga teng bo`lmagan minorlarning eng yuqori tartibini belgilaymiz. Dastaning singulyarligidan kelib chiqadiki, xar doim $r<n$ yoki $r<m$ bo`ladi. $r<n$ bo`lsin, u holda $A+\lambda B-\lambda-$matritsaning ustunlari chiziqli bog`langan bo`ladi,

ya`ni

$$(A+\lambda B)X=0, \tag{3.7}$$

bu yerda $x-$izlanayotgan ustun, tenglama nolmas yechimga ega. Bu tenglamaning xar bir nolmas yechimi $A+\lambda B-\lambda-$matritsaning ustunlari orasidagi qandaydir chiziqli bog`lanishni ifodalaydi. Biz (3.7) tenglamani faqat $\lambda$ ning ko`pxadlari bo`ladigan. $x(\lambda)$ yechimlarni qarash bilan chegaralanamiz. Bunday yechimlar ichidan eng kichik ε darajalisini olamiz.

$$x(\lambda)=x_0-\lambda x_1+\lambda^2x_2-...+(-1)^{\varepsilon}\lambda^{\varepsilon}x_{\varepsilon} \quad (x_{\varepsilon}\neq 0) \tag{3.8}$$

Bu yechimni (3.7) tenglamaga qo`yib, λ darajaning oldidagi koeffitsentlarni nolga tenglab, quydagilarni xosil qilamiz:

$$Ax_0,\ Bx_0-Ax_1=0,\ Bx_1-Ax_2=0,\ ...,Bx_{\varepsilon-1}-Ax_{\varepsilon}=0,\ Bx_{\varepsilon}=0, \tag{3.9}$$

Bu tengliklar sistemasini $x_0,-x_1,x_2,...,(-1)^{\varepsilon}x_{\varepsilon}$ ustun elementlariga nisbatan chiziqli birjinsli tenglamalar sistemasi sifatida qarab, shunday xulosaga kelamizki, bu sistema koyffitsiyentlaridan tuzilgan quyidagi matritsa.

$$M_{\varepsilon}=M_{\varepsilon}[A+\lambda B]=\underbrace{\begin{Vmatrix} A & O & ... & O \\ B & A & ... & O \\ O & B & ... & O \\ . & . & ..... & . \\ O & O & ... & A \\ O & O & ... & B \end{Vmatrix}}_{\varepsilon+1} \tag{3.10}$$

$\rho_{\varepsilon}<(\varepsilon+1)n$ rangga ega. Shu bilan birga ε soning minimallik xossasiga ko`ra quyidagi matritsalarning

$$M_0 = \left\| \begin{matrix} A \\ B \end{matrix} \right\|, \ M_1 = \left\| \begin{matrix} A & O \\ B & A \\ O & B \end{matrix} \right\|, \ . \ . \ . \ , M_{\varepsilon-1} = \underbrace{\left\| \begin{matrix} A & O & . & . & . & O \\ B & A & . & . & . & O \\ . & . & . & . & . & . \\ O & O & . & . & . & A \\ O & O & . & . & . & B \end{matrix} \right\|}_{\varepsilon} \qquad (3.10`)$$

$\rho_1, \ . \ . \ . \ , \rho_{\varepsilon-1}$ ranglar uchun quyidagi tengliklar o`rinli.

$$\rho_0 = n, \ \rho_1 = 2n, . \ . \ ., \rho_{\varepsilon-1} = \varepsilon \, n.$$

Shunday qilib, ε son $\rho_k \leq (k+1)n$ munosabatni qanoatlantiruvchi K indeksning eng kichik qiymati.

Teorema 3.4. Agar (3.7) tenglama $\varepsilon > 0$ minimal darajali yechimga ega bo`lsa, u holda berilgan $A + \lambda B$ dasta quyidagi dastaga qa`tiy ekvivalent boladi.

$$\left\| \begin{matrix} L_\varepsilon & 0 \\ 0 & \hat{A} + \lambda \hat{B} \end{matrix} \right\| \qquad (3.11)$$

bu yerda

$$L_\varepsilon = \underbrace{\left\| \begin{matrix} \lambda & 1 & 0 & . & . & . & 0 & 0 \\ 0 & \lambda & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \lambda & 1 \end{matrix} \right\|}_{\varepsilon+1} \Bigg\} \varepsilon \qquad (3.12)$$

$A + \lambda B -$matritsalarning shunday dastasiki, unga mos (3.7)ga o`hshash tenglama ε dan kichik darajali yechimga ega emas.

Isboti. Teoremaning isbotini quyidagi uch bosqichda amalga oshiramiz.

1. Berilgan $A + \lambda B$ dastani quyidagi

$$\left\| \begin{matrix} L_\varepsilon & D + \lambda F \\ 0 & A + \lambda B \end{matrix} \right\| \qquad (3.13)$$

dastaga qa`tiy ekvivalentligini ko`rsatamiz bu yerda $D, F, \hat{A}, \hat{B}, -$ mos o`lchovli to`g`ri to`rtburchakli o`zgarmas matritsalar.

2. $(\hat{A} + \lambda \hat{B})x = 0$ tenglama ε dan kichik darajali yechimga ega emasligini ko`rsatamiz.

3. (3.13) dastani (3.11) kvazidiogonal ko`rinishga keltirish mumkin ekanligini ko`rsatamiz.

1. Isbotning birinchi qismini geometrik shakilda amalga oshiramiz. Buning uchun $A+\lambda B$-matritsalar dastasi o`rniga $R^n$ fazoni $R^m$ fazoga akslantiruvchi $\overline{A}+\lambda\overline{B}$ operatorlar dastasini qaraymiz va bu fazolarning tanlangan bazislarida $\overline{A}+\lambda\overline{B}$ operator (3.13) formaga egaligini ko`rsatamiz.

(3.7) tenglama o`rniga quyidagi vektor tenglamani

$$(A+\lambda B)\overline{x}=0 \tag{3.14}$$

va vektor yechimni

$$\overline{x_0}(\lambda)=\overline{x_1}-\lambda\overline{x_1}+\lambda^2\overline{x_2}-\ldots+(-1)^{\varepsilon}\lambda^{\varepsilon}\overline{x_{\varepsilon}} \tag{3.15}$$

olamiz. Bu holda (3.9) tengliklar quyidagi vektor tengliklar bilan almashadi.

$$\overline{Ax_0}=0,\ \overline{Ax_1}=\overline{Bx_0},\ \overline{Ax_2}=\overline{Bx_1},\ \ldots,\overline{Ax_{\varepsilon}}=\overline{Bx_{\varepsilon-1}},\ \overline{Bx_{\varepsilon}}=0 \tag{3.16}$$

Quyidagi vektorlarni chiziqli bog`liqmasligini isbotlaymiz:

$$\overline{Ax_1},\ \overline{Ax_2},\ \ldots,\overline{Ax_{\varepsilon}} \tag{3.17}$$

Bundan,

$$\overline{x_0},\ \overline{x_1},\ \ldots,\overline{x_{\varepsilon}} \tag{3.18}$$

vektorlarni chiziqli bog`liqmasligi kelib chiqadi.

Xaqiqatan,

$$\overline{Ax_0}=0$$

$a_0\overline{x_0}+a_1\overline{x_1}+\ldots+a_{\varepsilon}\overline{x_{\varepsilon}}=0$ tenglikdan $a_0\overline{Ax_0}+a_1\overline{Ax_1}+\ldots+a_{\varepsilon}\overline{Ax_{\varepsilon}}=0$ tenglikni xosil qilamiz. (3.17) vektorlarni chiziqlik bog`liq emasligidan $a_1=a_1=\ldots=a_{\varepsilon}=0$ kelib chiqadi. Ammo $\overline{x_0}\neq 0$, chunki, aks xolda $\frac{1}{\lambda}\overline{x}(\lambda)$ (3.14) tenglamani $\varepsilon-1$ darajali yechimi bo`lib qoladi, bu bo`lishi mumkin emas(ε ni minimal darajali ekanligiga zid). Shuning uchun $a_0=0$.

Agar mos ravishda R^m va R^n da yangi bazislar uchun, (13.17) va (3.18) vektorlarni birinchi bazis vektorlar deb qabul qilsak, u holda

(3.16) ga ko`ra yangi bazisda $\overline{A}$ va $\overline{B}$ operatorlarga quyidagi matritsalar mos keladi.

$$\tilde{A}=\left\|\begin{matrix} \overbrace{0 & 1 & 0 & \ldots & 0}^{\varepsilon+1} & * & \ldots & * \\ 0 & 0 & 1 & \ldots & 0 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & * & \ldots & * \\ 0 & 0 & 0 & \ldots & 0 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & * & \ldots & * \end{matrix}\right\| \qquad \tilde{B}=\left\|\begin{matrix} \overbrace{1 & 0 & \ldots & 0 & 0}^{\varepsilon+1} & * & \ldots & * \\ 0 & 1 & \ldots & 0 & 0 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 & * & \ldots & * \\ 0 & 0 & \ldots & 0 & 1 & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & * & \ldots & * \end{matrix}\right\|$$

u holda $\tilde{A}+\lambda\tilde{B}$ $\lambda-$matritsa (3.13) ko`rinishga ega bo`ladi.

Barcha avvalgi muxokamalar asoslangan bo`ladi, agarda biz (3.17) vektorlarni chiziqli bog'liq emasligini ko'rsata olsak. Teskarisini faraz qilamiz, ya'ni $\overline{Ax}_h$ $(h\geq 1)-$(3.17) qatordagi ozidan avvalgi vektorlar orqali chiziqli ifodalangan birinchi vektor bo'lsin,

$$\overline{Ax}_h=\alpha_1\overline{Ax}_{h-1}+\alpha_2\overline{Ax}_{h-2}+.\ .\ .+\alpha_{h-1}\overline{Ax}_1$$

(3.16) ga ko'ra bu tenglikni quyidagicha yozishimiz mumkin:

$$\overline{Bx}_{h-1}=\alpha_1\overline{Bx}_{h-2}+\alpha_2\overline{Bx}_{h-3}+.\ .\ .+\lambda_{h-1}\overline{Bx}_0,$$

ya'ni

$$\overline{Bx}^*_{h-1}=0$$

bu yerda

$$\overline{x}^*_{h-1}=\overline{x}_{h-1}-\alpha_1\overline{x}_{h-2}-\alpha_2\overline{x}_{h-3}-.\ .\ .-\alpha_{h-1}\overline{x}_0,$$

yana (3.16) ga ko'ra

$$\overline{Ax}^*_{h-1}=B\left(\overline{x}_{h-2}-\alpha_1\overline{x}_{h-2}-.\ .\ .-\alpha_{h-2}\overline{x}_0\right)=\overline{Bx}^*_{h-2},$$

bu yerda

$$\overline{x}^*_{h-2}=\overline{x}_{h-2}-\alpha_1\overline{x}_{h-2}-.\ .\ .-\alpha_{h-2}\overline{x}_0,$$

Bu jarayonni davom ettirib, quydagi vektorlarni xosil qilamiz:

$$\overline{x}^*_{h-3}=\overline{x}_{h-3}-\alpha_1\overline{x}_{h-3}-.\ .\ .-\alpha_{h-3}\overline{x}_0,.\ .\ .,\overline{x}^*_1=\overline{x}_1-\alpha_1\overline{x}_0,\ \ \overline{x}^*_0=\overline{x}_0$$

Natijada quydagi tengliklar hosil bo`ladi:

$$\overline{Bx}^*_{h-1}=0,\ \ \overline{Ax}^*_{h-1}=\overline{Bx}^*_{h-2},\ .\ .\ .\ ,\overline{Ax}^*_1,=\overline{Bx}^*_1,\ \overline{Ax}^*_0=0 \qquad (3.19)$$

(3.19) dan kelib chiqadiki,

$$\overline{x}^*(\lambda)=\overline{x}^*_0-\lambda\overline{x}^*_1+.\ .\ +(-1)^{h-1}\overline{x}^*_{h-1} \qquad \left(\overline{x}^*_0=\overline{x}_0\neq 0\right)$$

(3.14) tenglamani $h-1<\varepsilon$ darajadan ortmaydigan nolmas yechimi bo`lib, qarama-qarshilikka kelamiz.

Shunday qilib (3.17) vektorlar chiziqli bo`liq emas.

2. Endi $(\hat{A}+\lambda\hat{B})\hat{x}=0$ tenglamani ε dan kichik darajali yechimga ega emasligini isbotlaymiz. Avval etiborimizni $\overline{L}_\varepsilon\overline{y}=0$ tenglama (3.7) tenglama kabi eng kichik darajali nolmas yechimga ega ekanligiga qaratamiz. Bunga $\overline{L}_\varepsilon\overline{y}=0$ tenglamani

$$\lambda y_1+y_2=0,\quad \lambda y_2+y_3=0,\ .\ .\ .\ ,\lambda y_\varepsilon+y_{\varepsilon+1}=0,$$

$$\overline{y}=(y_1,y_2,\ .\ .\ .\ ,y_{\varepsilon+1})^T,\ \ y_k=(-1)^{k-1}y_1\lambda^{k-1},\quad k=1,2,.\ .\ .\ ,\varepsilon+1$$

oddiy tenglamalar sistemasi bilan almashtirib ishonch xosil qilishimiz mumkin.

Ikkinchi tomondan, agar dasta (3.13) ,,uchburchak’’ ko`rinishga ega bo`lsa, u holda bu dastaga mos keluvchi $M_k(k=1,2,\ .\ .\ .\ \varepsilon)$ matritsalar ham satr va ustunlarini kerakli almashtirishlardan so`ng quydagi uchburchak ko`rinishga keltirilishi mumkin:

$$\begin{Vmatrix} M_k[L_\varepsilon] & M_k[D+\lambda F] \\ O & M_k[\hat{A}+\lambda\hat{B}] \end{Vmatrix} \qquad (3.20)$$

$k=\varepsilon-1$ da bu matritsaning barcha ustunlari, jumladan $M_{\varepsilon-1}[L_\varepsilon]$ matritsaning ustunlari chiziqli bog`liq emas. Ammo $M_{\varepsilon-1}[L_\varepsilon]-\varepsilon(\varepsilon+1)$ tartibli kvadrat matritsa. Shuning uchun $M_{\varepsilon-1}[\hat{A}+\lambda\hat{B}]$ matritsaning ham barcha ustunlari chiziq`li bog`liq emas bo`lib, $(\hat{A}+\lambda\hat{B})\hat{x}=0$ tenglama ε dan kichik darajali yechimga ega bo`lmaydi.

3. (3.13) dastani unga qat’iy ekvivalent bo’lgan quyidagi dasta bilan almashtiramiz:

$$\begin{Vmatrix} E_1 & Y \\ O & E_2 \end{Vmatrix} \begin{Vmatrix} L_\varepsilon & D+\lambda F \\ O & \hat{A}+\lambda\hat{B} \end{Vmatrix} \begin{Vmatrix} E_3 & -X \\ O & E_4 \end{Vmatrix} = \begin{Vmatrix} L_\varepsilon & D+\lambda F+Y\left(\hat{A}+\lambda\hat{B}\right)-L_\varepsilon X \\ O & \hat{A}+\lambda\hat{B} \end{Vmatrix} \qquad (3.21)$$

bu yerda E_1, E_2, E_3, E_4 mos ravishda $\varepsilon, m-\varepsilon, \varepsilon+1, n-\varepsilon-1$ tartibli birlik kvadrat matritsalar, $X, Y-$mos o\`lchovli, ixtiyori to\`g\`ri to\`rtburchakli matritsalar. Teorema to\`la isbotlangan bo\`ladi, agarda X va Y matritsalarni

$$L_\varepsilon X = D+\lambda F+Y\left(\hat{A}+\lambda\hat{B}\right) \qquad (3.22)$$

matritsali tenglikni qanoatlantiradigan qilib tanlash mumkin ekanligini ko\`rsata olsak.

D, F, X matritsalar elementlari uchun, shundek Y matritsa satirlar va $\hat{A}, \hat{B}$ matritsalar ustunlari uchun quyidagicha belgilashlar kiritamiz:

$$D=\| d_{ik} \|, \quad F=\| f_{ik} \|, \quad X=\| x_{jk} \|, \quad i=\overline{1,\varepsilon}, \quad k=\overline{1,n-\varepsilon-1}, \quad j=\overline{1,\varepsilon+1},$$

$$Y=\left(y_1, y_2, \ldots, y_\varepsilon\right)^T, \quad \hat{A}=\left(a_1, a_2, \ldots, a_{n-\varepsilon-1}\right), \quad \hat{B}\left(b_1, b_2, \ldots, b_{n-\varepsilon-1}\right)$$

U holda (3.22) matritsali tenglamani quyidagi skalyar tenglamalar sistemasi bilan almashtirish mumkin:

$$\begin{cases} x_{2k}-\lambda x_{1k}=d_{1k}+\lambda f_{1k}+y_1 a_k+\lambda y_1 v_k \\ x_{3k}-\lambda x_{2k}=d_{2k}+\lambda f_{2k}+y_2 a_k+\lambda y_2 v_k \\ x_{4k}-\lambda x_{3k}=d_{3k}+\lambda f_{3k}+y_3 a_k+\lambda y_3 v_k \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{\varepsilon+1,k}+\lambda x_{\varepsilon k}=d_{\varepsilon k}+\lambda f_{\varepsilon k}+y_\varepsilon a_k+\lambda y_\varepsilon v_k \end{cases} \qquad (3.23)$$

$$k=1,2,\ldots n-\varepsilon-1$$

Bu tengliklarning chap tomonida λ ga nisbatan chiziqli ikkixadlar turibdi. Bu birinchi $\varepsilon-1$ ta ikkixadning ozod xadi keyingi ikkixaddagi λ oldidagi koyfitsientga teng. U holda tengliklarning o\`ng tomoni ham shu shartni qanoatlantirishi kerak. Shuning uchun quyidagi tengliklar o’rinli bo’lishi kerak:

$$\begin{aligned} & y_1 a_k - y_2 v_k = f_{2k} - d_{1k}, \\ & y_2 a_k - y_3 v_k = f_{3k} - d_{2k}, \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & y_{\varepsilon-1} a_k - y_\varepsilon v_k = f_{\varepsilon k} - d_{\varepsilon-1,k} \\ & k=1,2,\ldots,n-\varepsilon-1 \end{aligned} \qquad (3.24)$$

Agar (3.24) tengliklar o'rinli bo'lsa , u holda (3.23) dan X matritsaning elementlarini aniqlash mumkin bo`ladi.

Endi (3.24) Y matritsaning elementlariga nisbatan tenglamalar sistemasi, ixtiyoriy d_{ik} va f_{ik} $\left(i = 1, 2, \ldots, \varepsilon \quad k = 1, 2, \ldots, n - \varepsilon - 1\right)$ da har doim yechimga ega ekanligini ko`rsatish qoldi. Xaqiqatan, $y_1, -y_2, y_3 - y_4, \ldots$ noma'lum elementlar oldidagi koeffitsentlardan tuzilgan matritsa transponirlangandan so'ng , quyidagi korinishda yozilishi mumkin.

$$\begin{Vmatrix} \overbrace{\begin{matrix} \hat{A} & O & . & . & . & O \\ \hat{B} & \hat{A} & . & . & . & O \\ O & \hat{B} & . & . & . & O \\ . & . & . & . & . & \\ O & O & . & . & . & \hat{A} \\ O & O & . & . & . & \hat{B} \end{matrix}}^{\varepsilon - 1} \end{Vmatrix}$$

Ammo bu matritsa $\hat{A} + \lambda\hat{B}$ – to`gri to`rtburchak matritsalar dastasi uchun $M_{\varepsilon-2}$ matritsadan iborat bo`lib, uning rangi $(\varepsilon - 1)(n - \varepsilon - 1)$ ga teng, chunki isbotlanganiga ko`ra $\left(\hat{A} + \lambda\hat{B}\right)x = 0$ tenglama ε dan kichik darajali yechimga ega emas. Shunday qilib, (3.24) tenglamalar sistemasining rangi tenglamalar soniga teng, bunday sistema ixtiyoriy ozod xadlarda birgalashgan bo`ladi.

Teorema to`la isbotlandi.

§4. Matritsalar singulyar dastasining kanonik formasi.

Matritsaning $m \times n$ o`lchovli singulyar dastasi $A + \lambda B$ berilgan bo`lsin. Avval bu dastani ustunlari va satirlari orasida o'zgarmas koyffisentli chiziqli bog'langanlari yo'q deb faraz qilamiz.

Dastaning rangi $r < n$ bo`lsin, ya`ni $A + \lambda B$ dastaning ustunlari chiziqli bog'langan bo'lsin. Bu holda

$$(A + \lambda B)x = 0$$

tenglama ε_1 minimal darajali nolmas yechimga ega bo'ladi. U holda teorema 3.4 ga asosan berilgan dastani quyidagi ko`rinishga keltirish mumkin:

$$\begin{pmatrix} L_{\varepsilon_1} & O \\ O & A_1+\lambda B_1 \end{pmatrix},$$

bu yerda $(A_1+\lambda B_1)x^{(1)}=0$ tenglama ε_1 da kichik darajali yechimga ega emas.

Agar tenglama ε_2 minimal darajali nolmas yechimga ega bo`lsa, u holda $A_1+\lambda B_1$ dastaga teorema 3.4 ni qo'llab berilgan dastani

$$\begin{pmatrix} L_{\varepsilon_1} & O & O \\ O & L_{\varepsilon_2} & O \\ O & O & A_2+\lambda B_2 \end{pmatrix}$$

ko`rinishga keltiramiz.

Bu jarayonni davom ettirib, berilgan dastani quyidagicha kvazidioganal ko`rinishga keltiramiz:

$$\left\| \begin{matrix} L_{\varepsilon_1} & O & . & . & . & O \\ O & L_{\varepsilon_2} & . & . & . & O \\ . & . & . & . & . & . \\ O & . & . & . & L_{\varepsilon_p} & O \\ o & . & . & . & O & A_p+\lambda B_p \end{matrix} \right\| \qquad (3.25)$$

bu yerda $0<\varepsilon_1\le\varepsilon_2\le . \ . \ . \le\varepsilon_p$ $\left(A_p+\lambda B_p\right)x^{(p)}=0$ tenglama esa nolmas yechimga ega emas, ya`ni $A_p+\lambda B_p$ matritsaning ustunlari chiziqli bog`lanmagan.

Agar $A_p+\lambda B_p$ dastaning satrlari chiziqli bog`langan bo`lsa, u holda transponirlangan $A_p^T+\lambda B_p^T$ dasta (3.25) ko`rinishga keltirilishi mumkin bo`lib, $\varepsilon_1,\varepsilon_2,. \ . \ .,\varepsilon_p$ sonlar o`rniga $0<r_1\le r_2\le . \ . \ . \le_p$ sonlar olinadi. Ammo bu holda berilgan $A+\lambda B$ dasta quyidagicha kvazidioganal ko`rinishga almashtiriladi.

$$\left\| \begin{array}{ccccccccccc} L_{\varepsilon_1} & & & & & & & & & & \\ & L_{\varepsilon_2} & & & & & & & & & \\ & & \cdot & & & & & & & & \\ & & & \cdot & & & & & & & \\ & & & & \cdot & & & & & & \\ & & & & & L_{\varepsilon_p} & & & & & \\ & & & & & & L_{\eta_1} & & & & \\ & & & & & & & L_{\eta_2} & & & \\ & & & & & & & & \cdot & & \\ & & & & & & & & & \cdot & \\ & & & & & & & & & & \cdot \\ & & & & & & & & & & & L_{\eta_q} \\ & & & & & & & & & & & & A_0 + \lambda B_0 \end{array} \right\| \qquad (3.26)$$

bu yerda $A_0 + \lambda B_0$ dastaning satrlari ham, ustunlari ham chiziqli bog`lanmagan, ya`ni $A_0 + \lambda B_0$ regulyar dasta bo`ladi.

Endi umumiy holni qaraymiz, ya`ni berilgan dastaning satrlari va ustunlari o`zgarmas koyffitsientli chiziqli bog`lanish bilan bog`langan bo`lishi mumkin.

$$(A + \lambda B)x = 0 \qquad va \qquad (A^T + \lambda B^T)y = 0$$

Tenglamalar o`zgarmas bog`lanmagan yechimlari maksimal sonini mos ravishda g va h bilan belgilaymiz. Bu tenglamalarning birinchisi o`rniga teorema 3.4 ning isbotidagidek $(\overline{A} + \lambda\overline{B})\overline{x} = 0$ vektor tenglamani qaraymiz. Bu yerda $\overline{A}\ va\ \overline{B}$ lar $R^n$ fazoni $R^m$ fazoga akslantiruvchi operatorlar. Bu tenglamaning chiziqli bog`lanmagan o`zgarmas yechimlarini $e_1,\ e_2, \dots$, e_g orqali belgilab, ularni R^n fazoning birinchi bazis vektorlari deb qabul qilamiz. U holda mos $\tilde{A} + \lambda\tilde{B}$ matritsadagi birinchi g ta ustunda nollar turadi.

$$\tilde{A} + \lambda\tilde{B} = \left(\overbrace{\ 0\ }^{g\ ta} \ , \tilde{A} + \lambda\tilde{B} \right). \qquad (3.27)$$

Xuddi shundek $\widetilde{A}_1 + \lambda\tilde{B}_1$ dasta ham birinchi h ta satrni nolli qilish mumkin. U holda berilgan dasta quyidagi ko`rinishni oladi:

$$\left\| \begin{matrix} h\,ta\{ \overbrace{0}^{g\,ta} & 0 \\ 0 & A^0 + \lambda B^0 \end{matrix} \right\|, \qquad (3.28)$$

bu yerda $A^0 + \lambda B^0$ dastaning satr va ustunlari o`zgarmas koyffitsientli chiziqli bog`lanish bilan bog`lanmagan. $A^0 + \lambda B^0$ dastaga (3.26) ko`rinishdagi tasvirlashni qo`llash mumkin. Sunday qilib, eng umumiy holda A+$\lambda B$ dasta har doim quyidagi kanonik kvazidioganal ko`rinishga keltirilishi mumkin.

$$\left\{ h\,ta \left\{ \overbrace{0}^{g\,ta} \right. ,\ L_{\varepsilon_{g+1}}, \dots ,\ L_{\varepsilon_p}\ , L^T_{\eta_{h+1}}\ , \ \dots\ ,\ L^T_{\eta_g},\ A_0 + \lambda B_0 \right\} \qquad (3.29)$$

(3.29) dagi $A_0 + \lambda B_0$ regulya dastani uning (3.6) kanonik ko`rinish bilan almashtirib, quydagi kvazidiogonal matritsani xosil qilamiz:

$$\left\{ h_{ta} \left\{ \overbrace{O}^{g\,ta},\ L_{\varepsilon_{g+1}}, \dots, L_{\varepsilon_p},\ L^T_{\eta_{h+1}}, \dots, L^T_{\eta_g},\ N^{(u_1)}, \dots,\ N^{(u_s)},\ J + \lambda E \right. \right\} \qquad (3.30)$$

bu yerda J matritsa Jordon yoki oddiy normal formada, $N^{(u)} = E^{(u)} + \lambda H^{(u)}$.

(3.30) matritsa $A + \lambda B$ dastaning eng umumiy xoldagi kanonik fo`rmasini ifodalaydi.

§5. Dastaning minimal indeksi.

Dastalarning qat'iy ekvivalentlik kriteryasi.

To`g`ri to`rtburchakli matritsalarning $A + \lambda B$ singulyar dastasi berilgan bo`lsin. U holda

$$(A + \lambda B)x = 0 \qquad (3.31)$$

tenglamaning yechimi bo`lgan k ta ko`pxadli ustun $x_1(\lambda), x_2(\lambda), \dots$, $x_k(\lambda)$ chiziqli bog`liq bo`ladi, agarda bu ustunlardan tashkil topgan

$X = [x_1(\lambda), x_2(\lambda), \dots \ , \ x_k(\lambda)]$ ko`pxadli matritsaning rangi k dan kichik bo`lsa. Bu holda k ta bir vaqtda nolga teng bo`lmagan $p_1(\lambda), p_2(\lambda), \dots$ , $p_k(\lambda)$ ko`pxadlar mavjud bo`lib, ular uchun

$$p_1(\lambda)x_1(\lambda) + p_2(\lambda)x_2(\lambda) + \ \dots + p_k(\lambda)x_k(\lambda) \equiv 0$$

tenglik o`rinli bo`ladi. Agar X matritsaning rangi k ga teng bo`lsa, u holda bunday bog`lanishlar mavjud emas va $x_1(\lambda), x_2(\lambda), \dots \ , \ x_k(\lambda)$ yechim chiziqli bog`lanmagan bo`ladi.

(3.31) tenglamaning barcha yechimlari ichidan eng kichik ε_1 darajali $X_1(\lambda)$ yechimni olamiz. Bu tenglamaning boshqa barcha yechimlari $X_1(\lambda)$ bilan chiziqli bog`lanmaganlik shartidan foydalanib, $\varepsilon_2(\varepsilon_1 \le \varepsilon_2)$darajali $\tilde{o}_2(\lambda)$ yechimni tanlaymiz. Bu jarayonni davom ettirib, boshqa yechimlarni topamiz. Chiziqli bog`lanmagan yechimlar soni n dan katta emasligidan bu jarayon chekli bo`ladi. Biz (3.31) tenglamaning fundamental yechimlar qatorini xosil qilamiz:

$$X_1(\lambda), X_2\ (\lambda), \dots, X_p(\boldsymbol{\lambda}) \qquad (3.32)$$

bularning darajalari mos ravishda

$$\varepsilon_1 \le \varepsilon_2 \le \varepsilon_3 \le \cdots \varepsilon_p \qquad (3.33)$$

bo`ladi.

Umumiy xolda fundamental yechimlar qatori $A + \lambda B$ dastaning berilishi bilan bir qiymatli aniqlanmaydi. Ammo ikkita har xil fundamental yechimlar qatori har doim bitta $\varepsilon_1, \varepsilon_2, \varepsilon_3 \dots, \varepsilon_p$ darajalar qatoriga ega bo'ladi. Xaqiqatan, (3.32) bilan birga $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_p$ darajali $\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \dots$ fundamental yechimlar qatorini qaraymiz. (3.33) darajalar ichida

$$\varepsilon_1 = \dots = \varepsilon_{n_1} < \varepsilon_{n_g+1} = \dots = \varepsilon_{n_2} < \dots$$

va shunga o`xshash $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots$ qatorda

$$\tilde{\varepsilon}_1 = \dots = \tilde{\varepsilon}_{\tilde{n}_1} < \tilde{\varepsilon}_{\tilde{n}_1+1} = \dots = \tilde{\varepsilon}_{\tilde{n}_2} < \dots$$

bo`lsin.

Ko`rinib turibdiki, $\varepsilon_1 = \tilde{\varepsilon}_1$. Ihtiyoriy $\tilde{x}_i(\lambda)$ $(i = 1,2,\ldots,\tilde{n}_1)$ ustun $x_1(\lambda), x_2(\lambda),\ldots,x_{n_1}(\lambda)$ ustunlar chiziqli kambinatsiyasidan iborat, chunki aks xolda (3.32) qatordagi $x_{n_1+1}(\lambda)$ yechimni kichikroq darajali $\tilde{x}_1(\lambda)$ yechim bilan almashtirilishi mumkin bo`ladi. Aksincha, xar bir $x_i(\lambda)$ $(i = 1,2,\ldots,n_1)$ ustun $\tilde{x}_1(\lambda), \tilde{x}_2(\lambda),\ldots,\tilde{x}_{\tilde{n}_1}(\lambda)$ ustunlar chiziqli kambinatsiyasidan iborat bo`ladi. Shuning uchun $n_1 = \tilde{n}_1$ va $\varepsilon_{n_1+1} = \tilde{\varepsilon}_{\tilde{n}_1+1}$. Xuddi shunday muloxaza yuritib, $n_2 = \tilde{n}_2$ va $\varepsilon_{n_2+1} = \tilde{\varepsilon}_{\tilde{n}_2+1}$ ni xosil qilamiz va xokozo.

(3.32) fundamental qatordagi xar bir $x_k(\lambda)$ yechim $A + \lambda B$ matritsa ustunlari orasida ε_k darajali chiziqli bog`lanishni beradi $(k = 1,2,\ldots,p)$. Shuning uchun $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p$ sonlar $A + \lambda B$ dasta ustunlari uchun minimal indekslar deyiladi. Xuddi shunday $A + \lambda B$ dasta satrlari uchun $|\eta_1,\eta_2,\ldots,\eta_q|$ minimal indekslar kiritiladi. Bunda $(A + \lambda B)x = 0$ tenglama $\left(A^T + \lambda B^T\right)y = 0$ tenglama bilan almashitirilib, $|\eta_1,\eta_2,\ldots,\eta_q|$ sonlar transponirlangan $A^T + \lambda B^T$ dasta ustunlari uchun minimal indeks sifatida aniqlanadi.

Qat'iy ekvivalent dastalar bitta va faqat bitta minimal indeksga ega. Haqiqatan, 2 ta $A + \lambda B$ va $P(A + \lambda B)Q$ (P va Q – xosmas kvadrat matritsalar) o`xshash dastalar berilgan bo`lsin. Birinchi dasta uchun (3.31) tenglamani o`ngdan P matritsaga ko`paytirib, quyidagicha yozamiz:

$$P\,(A + \lambda B)QQ^{-1}x = 0$$

Bundan ko`rinadiki, (3.31) tenglamaning barcha yechimlari chapdan $Q^{-1}$ga ko`paytirilgandan so`ng

$$P\,(A + \lambda B)Qx = 0$$

tenglama yechimlarining to`la sistemasini beradi.

Shuning uchun $A + \lambda B$ va $p(A + \lambda B)Q$ dastalar ustunlar uchun bir xil minimal indekslarga ega.

Satrlar uchun minimal indekslarni ustma-ust tushishi transponirlangan dastalarga o`tish bilan ko`rsatiladi.

Kanonik kvazidiogonal matritsalar

$$\left\{ h_{ta} \left\{ \overbrace{O}^{g\ ta},\ L_{\varepsilon_{g+1}}, \ldots, L_{\varepsilon_p}, L^T_{\eta_{h+1}}, \ldots, L^T_{\eta_q}, A_0 + \lambda B_0 \right\} \right. \qquad (3.34)$$

uchun minimal indekslarni xisoblaymiz. $A_0 + \lambda B_0 -$(3.6) normal formaga ega bo`lgan regulyar dasta.

Kvazidional matritsa ustunlari (satrlari) uchun minimal indekslarning to`la sistemasi, mos alohida diagonal bloklar minimal indekslar sistemalarini birlashtirish bilan hosil qilinadi. $L_\varepsilon$ matritsa ustuni uchun faqat bitta $\varepsilon$ indekisga ega, satri uchun esa bitta $\eta$ indeksga ega bo`lib, bu matritsa ustunlari chiziqli bog`liq emas. $A_0 + \lambda B_0$ – regulyar dasta umuman minimal indekslarga ega emas. Shuning uchun (3.34) matritsa ustunlari uchun

$$\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_g = 0,\ \ \varepsilon_{g+1} \ldots, \varepsilon_p$$

satrlari uchun

$$\eta_1 = \eta_2 = \ldots = \eta_h = 0,\ \ \eta_{h+1} \ldots, \eta_q$$

minimal indekslarga ega.

L_ε matritsa elementar bo`luvchilarga ega emas, chunki maksimal tartibli minorlari 1ga yoki $\lambda^\varepsilon$ teng bo`ladi. Bu tasdiq L^T_ε matritsa uchun ham o`rinli. Shuningdek kvazidiogonal matritsa elementar bo`luvchilari uchun alohida olingan diagonal bloklri elementar bo`luvchilarni birlashtirib hosil qilinadi, shuning uchun (3.34) $\lambda -$matritsa elementar bo`luvchilaari uchun $A_0 + \lambda B_0$ regulyar yadrosi elementar bo`luvchilari bilan ustma-ust tushadi.

(3.34) dastaning kanonik formasi minimal indeslar $\varepsilon_1, \ldots, \varepsilon_p, \eta_1, \eta_2, \ldots, \eta_q$ va bu dastalar yoki unga qat’iy ekvivalent bo`lgan $A + \lambda B$ dasta elementar bo`luvchilari bilan to`la aniqlanadi. Shuningdek, bir xil kanonik formaga ega bo`lgan ikkita dasta o`zaro qat’iy ekvivalent bo`ladi. U holda biz quyidagi teoremani isbotladik:

Teorema 3.5. (Kroneker teoremasi) Ixtiyoriy bir xil mxn o`lchovli to`g`ri to`rtburchakli matritsalarning $A + \lambda B$ va $A_1 + \lambda B_1$ dastalari qat’iy ekvivalent bo`lishi uchun bu dastalar bir xil minimal indekslarga va bir xil elementar (chekli yoki cheksiz) bo`luvchilarga ega bo`lishi zarur va yetarli.

Misol tariqasida $\varepsilon_1 = 0,\ \varepsilon_2 = 1,\ \varepsilon_3 = 2,\ \eta_1 = \eta_2 = 0\ \ \eta_3 = 2$ minimal indekslarga va $\lambda^2,\ (\lambda+2)^2,\ \mu^3$ elementar bo`luvchilarga ega bo`lgan $A + \lambda B$ dastaning kanonik formasini yozamiz:

$$\left\| \begin{matrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & & & & & & \\ & [\lambda\ 1] & & & & & \\ & & \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix} & & & & \\ & & & \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \\ 0 & 1 \end{bmatrix} & & & \\ & & & & \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} & & \\ & & & & & \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} & \\ & & & & & & \begin{bmatrix} \lambda+2 & 1 \\ 0 & \lambda+2 \end{bmatrix} \end{matrix} \right\|$$

§6. Kvadratik formalarining singulyar dastasi.

Quyidagi ikkita kompleks kvadratik formalar berilgan bo`lsin:

$$A(x,x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k, \qquad B(x,x) = \sum_{i,k=1}^{n} b_{ik} x_i x_k \tag{3.36}$$

bular $A(x,x) + \lambda B(x,x)$ kvadratik formalar dastasini tashkil qiladi. Bu formalar dastasiga $A + \lambda B\ \left(A^T = A,\ B^T = B\right)$ simmetrik matritsalar dastasi mos keladi. Agar $A(x,x) + \lambda B(x,x)$ kvadratik formalar dastasida o`zgaruvchilarni $x = Tz\ \left(|T| \neq 0\right)$ chiziqli almashtirishni qo`llasak, u holda $\tilde{A}(z,z) + \lambda \tilde{B}(z,z)$ almashtirilgan formalar dastasiga quyidagi matritsalar dastasi mos keladi.

$$\tilde{A} + \lambda \tilde{B} = T^T (A + \lambda B) T \tag{3.37}$$

bu yerda $T - n -$tartibli, o`zgarmas, xosmas kvadrat matritsa.

Ikkita (3.37) ayniyat bilan bog`langan $A+\lambda B$ va $\tilde{A}+\lambda\tilde{B}$ matritsalar dastalari o`zaro kongruyent deyiladi.

Ma`lumki, kongruyentlik matritsalar dastalarining qat`iy ekvivalentligini maxsus xususiy holi bo`ladi. Agar matritsalar simmetrik ( yoki kososimmetrik) bo`lsa, kongruyentlik tushunchasi qat'iy ekvivalentlik tushunchasi bilan ustma-ust tushadi.

Teorema 3.6. Ikkita qat'iy ekvivalent kompleks simmetrik (yoki kososimmetrik) matritsalar dastasi o`zaro kongruyent bo`ladi.

Isboti. . Ikkita $\Lambda = A+\lambda B$ va $\tilde{\Lambda} = \tilde{A}+\lambda\tilde{B}$ qat'iy ekvivalent simmetrik (yoki kososimmetrik) matritsalar dastasi berilgan bo`lsin;

$$\tilde{\Lambda} = P\Lambda Q \left(\Lambda^T = \pm\Lambda,\ \tilde{\Lambda}^T = \pm\tilde{\Lambda};\ |P| \neq 0, |Q| \neq 0\right) \tag{3.38}$$

Transponirlangan matritsalarga o`tib , quyidagini xosil qilamiz:

$$\tilde{\Lambda} = Q^T \Lambda P^T. \tag{3.39}$$

(3.38) va (3.39) dan quyidagini topamiz:

$$\Lambda Q P^{T^{-1}} = P^{-1} Q^T \Lambda \tag{3.40}$$

$$U = QP^{T^{-1}} \tag{3.41}$$

deb olib, (2.40) ni quyidagicha yozamiz:

$$\Lambda U = U^T \Lambda \tag{3.42}$$

(3.42) dan quyidagilar kelib chiqadi:

$$\Lambda U^k = U^{T^k} \Lambda, \quad (k = 0,1,\dots)$$

va umumiy xolda

$$\Lambda S = S^T \Lambda \tag{3.43}$$

bu yerda

$$S = f(U) \tag{3.44}$$

$f(\lambda) - \lambda$ ga nisbatan ixtiyoriy ko`phad. Faraz qilaylik bu ko`phad shunday tanlanganki, unda $|S| \neq 0.$ U holda (3.43)dan quyidagini topamiz:

$$\Lambda = S^T \Lambda S^{-1} \tag{3.45}$$

Λ uchun olingan ifodani (3.38) ga qo`yib, quyidagiga ega bo`lamiz:

$$\tilde{\Lambda} = PS^T \Lambda S^{-1} Q \qquad (3.46)$$

Bu munosabat konkruent almashtirish bo`lishi uchun quyidagi tenglik bajarilishi kerak:

$$(PS)^T = S^{-1}Q$$

buni quyidagicha yozish mumkin:

$$S^2 = QP^{T^{-1}} = U$$

Ammo $f(\lambda)$ sifatida U matritsa spektorida $\sqrt{\lambda}$ interpolyatsion ko`phadni olsak, $S = f(U)$ bu tenglamani qanoatlantiradi. Buni qilish mumkin, chunki ko`p qiymatli $\sqrt{\lambda}$ U matritsa spektrida bir qiymatli tarmoqqa ega, shuningdek $|U| \neq 0$

Bundan keyin (3.46) tenglik kongruentlik sharti bo`ladi:

$$\tilde{\Lambda} = T^T \Lambda T \quad \left(T = SQ = \sqrt{QP^{T^{-1}} Q} \right) \qquad (3.47)$$

Bu isbotlangan teorema va teorema 3.5. dan quyidagi natija kelib chiqadi:

Natija3.1: Ikkita $A(x,x) + \lambda B(x,x)$ va $\tilde{A}(x,x) + \lambda \tilde{B}(x,x)$ kvadrat formalar dastasi $x = Tz$ $(|T| \neq 0)$ almashtirish bilan bir-biriga o`tkaziladi, shunda va faqat shunda, qachonki $A + \lambda B$ va $\tilde{A} + \lambda \tilde{B}$ simmetrik matritsalar dastalari bir xil elementar bo`luvchilarga va bir xil minimal indekslarga ega bo`lsa.

Eslatma. Simmetrik matritsalar dastasi uchun satrlar va ustunlar bir xil minimal indekslarga ega, ya`ni

$$p = q, \quad \varepsilon_1 = \eta_1, \quad \varepsilon_2 = \eta_2, ..., \varepsilon_p = \eta_p \qquad (3.48)$$

Quyidagicha savol qo`yamiz : Ikkita

$$A(x,x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k, \qquad B(x,x) = \sum_{i,k=1}^{n} b_{ik} x_i x_k$$

ixtiyoriy kompleks kvadratik formalar berilgan. Qanday shartlar bajarilganda $x = Tz$ $(|T| \neq 0)$ o`zgaruvchilarni xosmas almashtirish bilan bu formalarni bir vaqtning o`zida.

$$\sum_{i=1}^{n} a_i z_i^2 \quad \text{va} \quad \sum_{i=1}^{n} b_i z_1^2 \tag{3.49}$$

Kvadratlar yig`indisiga keltirish mumkin?

Shunga o`xshash savolni ikkita $A(x,x)$ va $B(x,x)$ Ermit formalar uchun ham qo`yish mumkin, ammo bu holda (3.49) ni o`rniga quyidagini yozish kerak bo`ladi.

$$\sum_{i=1}^{n} a_i\, z_i\, \bar{z}_i \quad \text{va} \quad \sum_{i=1}^{n} b_i\, z_i\, \bar{z}_i \tag{3.50}$$

bu yerda a_i va b_i $(i = 1,2,\ldots,n)$ – haqiqiy sonlar .

Faraz qilaylik, $A(x,x)$ va $B(x,x)$ kvadratik formalar ko`rsatilgan xossalarga ega bo`lsin. U holda $A\lambda B$ matritsalar dastasi quyidagi dioganal matritsalar dastasi kongurent bo`ladi:

$$\{a_1 + \lambda b_1, a_2 + \lambda b_2, \ldots, a_n + \lambda b_n\} \tag{3.51}$$

$a_i + \lambda b_i$ dioganal ko`phadlarning $r(r \le n)$ tasi aynan no`lga teng emas bo`lsin. Umumiylikni buzmasdan quyidagicha deb olamiz:

$$a_1 = b_1 = 0,\ldots, a_{n-r} = b_{n-r} = 0,\ a_i + \lambda b_i \ (i = n-r+1,\ldots,n).$$

$$A_0 + \lambda B_0 = \{a_{n-r+1} + \lambda b_{n-r+1},\ldots, a_n + \lambda b_n\}$$

deb olib, (3.51) ni quyidagicha yozamiz:

$$\left\{ \overbrace{O}^{n-r\ ta},\ A_0 + \lambda B_0 \right\} \tag{3.52}$$

(3.52)ni (3.34) bilan solishtirib, ko`ramizki, bu holda barcha minimal indekslar nolga teng. Bundan tashqari barcha elementlar bo`luvchilar birinchi darajaga ega.

Biz quyidagi teoremaga keldik;

Teorema 3.7. Ikkita $A(x,x)$ va $B(x,x)$ kvadratik formalar bir vaqtda o`zgaruvchilarni almashtirish bilan kvadlar yigindisiga keltirilishi mumkin, faqat va faqat shu holda , qachonki, $A + \lambda B$ matritsalar dastasida barcha elementar bo`luvchilar birinchi darajali bo`lib , barcha minimal indekslar nolga teng bo`lsa.

Umumiy holda , ikkita $A(x,x)$ va $B(x,x)$ kvadratik formalarni qandaydir kanonik ko`rinishga keltirish uchun $A+\lambda B$ matritsalar dastasini unga qat'iy ekvivalent bo`lgan simmetrik matritsalar kanonik dastasi bilan almashtirish kerak .

$A+\lambda B-$ simmetrik matritsalar dastasi

$$\varepsilon_1=0,\varepsilon_2=0,\ldots,\varepsilon_g=0,\varepsilon_{g+1}\neq 0,\ldots,\varepsilon_p\neq 0$$

minimal indekslarga va $\mu^{u_1},\mu^{u_2},\ldots,\mu^{u_s}$ – cheksiz va $(\lambda+\lambda_1)^{c_1},(\lambda+\lambda_1)^{c_2},\ldots,(\lambda+\lambda_t)^{c_t}$ chekli elementar bo`luvchilarga ega bo`lsin .U holda (3.30) kanonik formada $g=h,\ p=q,\ \varepsilon_{g+1}=\eta_{g+1}$, ... , $\varepsilon_p=\eta_p$ bo`ladi . (3.30) da har ikkita $L_\varepsilon$ va $L_\varepsilon^T$ ko`rinishdagi dioganal bloklarni bitta $\begin{pmatrix} O & L_\varepsilon^T \\ L_\varepsilon & O\end{pmatrix}$ dioganal blok bilan, $N^{(u)}=E^{(u)}+\lambda H^{(u)}$ ko`rinishdagi har bir blokni

$$\tilde{N}^{(u)}=V^{(u)}N^{(u)}=\begin{Vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & \lambda \\ .. & .. & \ldots & .. & .. \\ 1 & \lambda & \ldots & 0 & 0 \end{Vmatrix} \quad \left(V^{(u)}=\begin{Vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ .. & .. & \ldots & .. & .. \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{Vmatrix} \right) \qquad (3.53)$$

qat'iy ekvivalentlik simmetrik blok bilan almashtiramiz.

Bundan tashqari, (2.30) dagi

$$J+\lambda E=\left\{(\lambda+\lambda_1)E^{(c_1)}+H^{(c_1)},\ldots,(\lambda+\lambda_t)E^{(c_t)}+H^{(c_t)}\right\} \qquad (3.54)$$

(J -Jordon matritsasi) regulyar dioganal blok o`rniga, unga qat'iy ekvivalentlik bo`lgan

$$\left\{Z_{\lambda_1}^{(c_1)},\ldots,Z_{\lambda_t}^{(c_t)}\right\} \qquad (3.55)$$

dastani olamiz. Bu yerda

$$Z_{\lambda_i}^{(c_i)}=V^{(c_i)}\left[(\lambda+\lambda_i)E^{(c_i)}+H^{(c_i)}\right]=\begin{Vmatrix} 0 \ \ 0 \ \ldots \ 0 & \lambda+\lambda_i \\ 0 \ \ 0 \ \ldots \ \lambda+\lambda_i & 1 \\ . \ . \ . \ . \ . \ . \ . \ . \ . \ . & \\ \lambda+\lambda_i \ \ 1 \ldots 0 & 0 \end{Vmatrix}, \quad (i=1,2,\ldots,t) \qquad (3.56)$$

$A+\lambda B$ dasta quyidagi simmetrik dastaga qat'iy ekvivalent.

$$\tilde{A}+\lambda\tilde{B}=\left\{O,\left\|\begin{matrix} O & L^T_{\varepsilon_{g+1}} \\ L_{\varepsilon_{g+1}} & O\end{matrix}\right\|,\ldots,\left\|\begin{matrix} O & L^T_{\varepsilon_p} \\ L_{\varepsilon_p} & O\end{matrix}\right\|, N^{(u_1)},\ldots,N^{(u_S)}, Z^{(c_1)}_{\lambda_1},\ldots,Z^{(c_t)}_{\lambda_t}\right\} \quad (3.57)$$

Ikkita $A(x,x)$ va $B(x,x)$ kompleks koyffitsientli kvadratik formalar $x=Tz$ $(|T|\neq 0)$ o\`zgaruvchilarni almashtirish bilan (3.57) tenglik bilan aniqlangan $\tilde{A}(z,z)$ va $\tilde{B}(z,z)$ kanonik ko'rinishga bir vaqtda keltirilish mumkin.

§7. Differensial tenglamalarga tadbiqlar.

Olingan natijalarni quyidagi o'zgarmas koeffitsientli, n ta noma'lum funktsiyali, birinchi tartibli m ta chiziqli differensial tenglamalar sistemasini integrallashga tadbiqini qaraymiz.

$$\sum_{k=1}^{n} a_{ik}x_k+\sum_{k=1}^{n} b_{ik}\frac{dx_k}{dt}=f_i(t),\quad (i=1,2,\ldots,m) \qquad (3.58)$$

yoki matritsa yozuvida

$$Ax+B\frac{dx}{dt}=f(t) \qquad (3.59)$$

bu yerda $A=\|a_{ik}\|,\ B=\|b_{ik}\|,\ i=\overline{1,m},\ k=\overline{1,n},$

$$x=(x_1,x_2,\ldots,x_n)^T,\qquad f=(f_1,f_2,\ldots,f_n)^T$$

$x_1,x_2,\ldots,x_n$ noma\`lum funktsiyalar bilan o'zgarmas koyffitsentli chiziqli xosmas matritsalar

$$x=Qz\quad \left(z=(z_1,z\ ,\ldots,z_n)^T,\ |Q|\neq O\right) \qquad (3.60)$$

Orqali bog'langan yangi $z_1,z_1,\ldots,z_n$ funksiyalarni kiritamiz.

(3.59) tenglamada x ning o\`rniga Qz ni qo'yib, (3.59) ni chapdan P ga ko'paytirib quyidagini xosil qilamiz.

$$\tilde{A}z+\tilde{B}\frac{dz}{dt}=\tilde{f}(t), \qquad (3.61)$$

bu yerda

$$\tilde{A}=PAQ,\qquad \tilde{B}=PBQ,\qquad \tilde{f}=Pf=(\tilde{f}_1,\tilde{f}_2,\ldots,\tilde{f}_n) \qquad (3.62)$$

Shu bilan birga $A+\lambda B$ va $\tilde{A}+\lambda\tilde{B}$ matritsalar dastalari bir biri bilan qat'iy ekvivalent:

$$\tilde{A}+\lambda\tilde{B}=P(A+\lambda B)Q \tag{3.63}$$

P va Q matritsalarni shunday tanlaymizki, unda $\tilde{A}+\lambda\tilde{B}$ dasta quyidagicha kanonik kvazidioganal formaga ega bo'lsin:

$$\tilde{A}+\lambda\tilde{B}=\left\{O, L_{\varepsilon_{d+1}},\ldots,L_{\varepsilon_p}, L^T_{\eta_{h+1}}, \ldots, L^T_{\eta_q}, N^{(u_1)},\ldots, N^{(u_s)}, J+\lambda E\right\} \tag{3.64}$$

(3.64) ning dioganal bloklariga mos differensial tenglamalar sistemasi $v=p-d+q-h+s+2$ ta aloxida sistemalarga ajraladi.

$$O\cdot z=\tilde{f}, \tag{3.65}$$

$$L_{E_{g+i}}\left(\frac{d}{dt}\right)\overset{1+i}{z}=\overset{1+i}{\tilde{f}}, \qquad (i=1,2,\ldots,p-g) \tag{3.66}$$

$$L^T_{E_{h+j}}\left(\frac{d}{dt}\right)\overset{p-g+1+j}{z}=\overset{p-g+1+j}{\tilde{f}}, \qquad (j=1,2,\ldots,q-h) \tag{3.67}$$

$$N^{(i_k)}\left(\frac{d}{dt}\right)\overset{p-g+q-h+1+k}{z}=\overset{p-g+q-h+1+k}{\tilde{f}}, \qquad (k=1,2,\ldots,s) \tag{3.68}$$

$$\left(J+\frac{d}{dt}\right)\overset{v}{z}=\overset{v}{\tilde{f}} \tag{3.69}$$

bu yerda

$$z=\left\|\begin{matrix}\overset{1}{z}\\ \overset{2}{z}\\ .\\ .\\ .\\ \overset{v}{z}\end{matrix}\right\|, \qquad \tilde{f}=\left\|\begin{matrix}\overset{1}{\tilde{f}}\\ \overset{2}{\tilde{f}}\\ .\\ .\\ .\\ \overset{v}{\tilde{f}}\end{matrix}\right\| \tag{3.70}$$

$$\overset{1}{z}=(z_1, z_1,\ldots,z_g), \quad \overset{1}{\tilde{f}}=(\tilde{f}_1, \tilde{f}_1,\ldots,\tilde{f}_h), \quad \overset{2}{z}=(z_{g+1},\ldots), \quad \overset{2}{\tilde{f}}=(\tilde{f}_{h+1},\ldots) \tag{3.71}$$

va xakozo

$$\Lambda\left(\frac{d}{dt}\right)=A+B\frac{d}{dt}, \qquad \text{agar} \qquad \Lambda(\lambda)=A+\lambda B \ \text{ bo`lsa.} \tag{3.72}$$

Shunday qilib, (3.59) sistemani integrallash, umumiy xolda (3.65)-(3.69) xususiy sistemalarni integrallashga keltiriladi. Bu sistemalarda $A+\lambda B$ matritsalar dastasi mos ravishda $O,\ L_{\varepsilon}, \mathrm{L}_{\eta}^{\mathrm{T}},\ N^{(u)},\ J+\lambda E$ ko’rinishlarga ega.

1. (3.65) sistemada qarama-qarshilik bo’lmasligi uchun

$$\overset{1}{\tilde{f}}=0$$

ya`ni

$$\tilde{f}_1=0,\ldots,\tilde{f}_h=0 \tag{3.73}$$

bo’lishi zarur va yetarli. Bu xolda $\overset{1}{z}$ ustunni tashkil etuvchi $z_1, z_2, \ldots, z_g$ noma’lum funktsiyalar sifatida t ning ixtiyoriy funktsiyasini olish mumkin.

2. (3.66) sistema quyidagi ko’rinishdagi sistemani ifodalaydi:

$$L_{\varepsilon}\left(\frac{d}{dt}\right)z=\tilde{f} \tag{3.74}$$

yoki yoyilgan yozuvda

$$\frac{dz_1}{dt}+z_2=\tilde{f}_1(t),\ \frac{dz_2}{dt}+z_3=\tilde{f}_2(t),\ldots,\frac{dz_E}{dt}+z_{E+1}=\tilde{f}_E(t) \tag{3.75}$$

Bunday sistemalar har diom birgalashgan bo’ladi. Agar $z_{\varepsilon+1}(t)$ sifatida t ning ixtiyoriy funktsiyani olsak, u xolda (3.75) dan ketma-ket kvadraturalarda barcha qolgan $z_{\varepsilon}, z_{\varepsilon-1}, \ldots, z_1$ noma’lum funktsiyalarni aniqlaymiz:

3. (3.67) sistema quyidagi ko’rinishdagi sistemani ifodalaydi:

$$\mathrm{L}_{\eta}^{\mathrm{T}}\left(\frac{d}{dt}\right)z=\tilde{f} \tag{3.76}$$

yoki yoyilgan yozuvda

$$\frac{dz_1}{dt}=\tilde{f}_1(t),\ \frac{dz_2}{dt}+z_1=\tilde{f}_2(t),\ \ldots,\ \frac{dz_{\eta}}{dt}+z_{\eta-1}=\tilde{f}_{\eta}(t),\ \tilde{z}_{\eta}=\tilde{f}_{\eta+1}(t) \tag{3.77}$$

(3.77) ning birinchisidan boshqa barcha tenglamalaridan bir qiymatli ravishda $z_{\eta}, z_{\eta-1},\ldots, z_1$ larni aniqlaymiz:

$$z_{\eta}=\tilde{f}_{\eta+1}(t),\ z_{\eta-1}=\tilde{f}_{\eta}-\frac{d\tilde{f}_{\eta+1}}{dt},\ldots,z_1=\tilde{f}_2-\frac{d\tilde{f}_3}{dt}+\ldots+(-1)^{\eta-1}\frac{d^{\eta-1}f_{\eta+1}}{dt^{\eta-1}} \tag{3.78}$$

z_1 uchun hosil qilingan ifodani birinchi tenglamaga qo`yib , birgalashganlik shartini hosil qilamiz:

$$\tilde{f}_1 - \frac{d\tilde{f}_2}{dt} + \frac{d^2\tilde{f}_3}{dt^2} - \ldots + (-1)^{\eta}\frac{d^{\eta}\tilde{f}_{\eta+1}}{dt^{\eta}} = 0 \qquad (3.79)$$

4. (3.68) sistema quyidagi ko`rinishdagi sistemani ifodalaymiz:

$$N^{(u)}\left(\frac{d}{dt}\right)z = \tilde{f} \qquad (3.80)$$

yoki yoyilgan yozuvda

$$\frac{dz_2}{dt} + z_1 = \tilde{f}_1, \frac{dz_3}{dt} + z_2 = \tilde{f}_2, \ldots, \frac{dz_u}{dt} + z_{u-1} = \tilde{f}_{u-1}, z_u = \tilde{f}_u. \qquad (3.81)$$

Bundan yechimlarni ketma-ket bir qiymatli aniqlaymiz:

$$z_u = \tilde{f}_u, z_{u+1} = \tilde{f}_{u-1} - \frac{d\tilde{f}_u}{dt}, \ldots, z_1 = \tilde{f}_1 - \frac{d\tilde{f}_2}{dt} + \frac{d^2\tilde{f}_2}{dt^2} - \ldots + (-1)^{u-1}\frac{d^{u-1}\tilde{f}_u}{dt^{u-1}} \qquad (3.82)$$

5. (3.69) sistema quydagi sistemani ifodalaydi:

$$J_z + \frac{dz}{dt} = \tilde{f}$$

Bunday sistemaning umumiy yechimi quydagicha bo`ladi:

$$z = e^{-J t_0} + \int_0^t e^{-J(t-\tau)} f(\tau) d\tau \qquad (3.84)$$

bu yerda z_0- ixtiyoriy elementli ustun bo`lib , noma`lum funktsiyani $t = 0$ dagi boshlangich qiymati bo`ladi.

(3.61) sistemadan (3.59) sistemaga teskari o`tish (3.60) va (3.62) fo`rmulalar bilan amalga ohiriladi Bunda har bir $x_1, x_2, \ldots, x_n$ funksiyalar $z_1, z_2, \ldots, z_n$ funksiyalarning chiziqli kombinatsiyasidan iborat bo`lib , har bir $\tilde{f}_1(t), \ldots, \tilde{f}_m(t)$ funkiyalar $f_1(t), \ldots, f_m(t)$ funksiyalar orqali (o`zgarmas koeffitsientlar bilan)chiziqli ifodalanadi.

O`tkazilgan taxlil ko`rsatadiki , (3.58) sistema birgalashgan bo`lishi uchun , umumiy holda , tenglamalarning o`ng tomonlari o`rtasida ba`zi aniq chiziqli chekli va diferensial bog’lanishlar bajarilishi shart .

Agar bu shartlar bajarilsa , u holda sistemaning umumiy yechimi chiziqli ihtiyoriy o`zgarmaslar kabi ihtiyoriy funksiyalarni o`zida saqlaydi .

Birlashganlik sharti xarakteri va yechimlari xarakteri (xususiy holda ihtiyoriy o`zgarmaslar va ihtiyoriy funksiyalar soni ) $A+\lambda B$ dastaning minimal indekslari va elementar bo`luvchilari bilan aniqlanadi ,chunki (3.65) –(3.69) differensial tenglamalar sistemalarining kanonik formasi bu indekskar va bo`luvchilarga bog`liq.

Mashqlar:

1. Quyidagi matritsalar bilan berilgan $A+\lambda B$ dastalarni regulyar yoki singulyar ekanligini aniqlang.

a) $A=\begin{pmatrix}1 & 2 & 0\\ 0 & 2 & 0\\ -2 & -2 & -1\end{pmatrix}$ B= $\begin{pmatrix}-2 & 8 & 6\\ -4 & 10 & 6\\ 4 & -8 & -4\end{pmatrix}$

b) $A=\begin{pmatrix}3 & 0 & 8\\ 3 & -1 & 6\\ -2 & 0 & -5\end{pmatrix}$ B= $\begin{pmatrix}13 & 16 & 16\\ -5 & -7 & -6\\ -6 & -8 & -7\end{pmatrix}$

c) $A=\begin{pmatrix}-4 & 2 & 10\\ -4 & 3 & 7\\ -3 & 1 & 7\end{pmatrix}$ B= $\begin{pmatrix}7 & -12 & -2\\ 3 & -4 & 0\\ -2 & 0 & -2\end{pmatrix}$

d) $A=\begin{pmatrix}4 & 6 & 0\\ -3 & -5 & 0\\ -3 & -6 & 1\end{pmatrix}$ B= $\begin{pmatrix}0 & 3 & 3\\ -1 & 8 & 6\\ 2 & -14 & -10\end{pmatrix}$

e) $A=\begin{pmatrix}9 & 22 & -6\\ -1 & -4 & 1\\ 8 & 16 & -5\end{pmatrix}$ B= $\begin{pmatrix}8 & 30 & -14\\ -5 & -19 & 9\\ -6 & -23 & 11\end{pmatrix}$

f) $A=\begin{pmatrix}4 & 5 & -2\\ -2 & -2 & 1\\ -1 & -1 & 1\end{pmatrix}$ B= $\begin{pmatrix}-1 & 1 & 1\\ -5 & 21 & 17\\ 6 & -26 & -21\end{pmatrix}$

g) $A=\begin{pmatrix}1 & -1 & 2\\ 3 & -3 & 6\\ 2 & -2 & 4\end{pmatrix}$ B= $\begin{pmatrix}3 & 7 & -3\\ -2 & -5 & 2\\ -4 & -10 & 3\end{pmatrix}$

2. Yuqorida xosil qilingan dastalarni kanonik ko’rinishga keltiring va ularni minimal indiksini aniqlang.

IV BOB MANFIYMAS ELEMENTLI MATRITSALAR

Ushbu bobda manfiymas elementli haqiqiy matritsalarning xossalari o'rganiladi. Bunday matritsalar extimollar nazariyasidagi Markov zanjirlarini o'rganishda va sistemalar kichik tebranishlar nazariyasida keng qo'llaniladi.

§1. Umumiy xossa

Ta'rif 4.1: haqiqiy elementli to'g'ri to'rtburchakli

$$A = \|a_{ik}\|, i = 1,2,..., m; k = 1,2,..., n$$

matritsa manfiymas ($A \geq 0$) yoki musbat $A > 0$ deyiladi, agarda uning barcha elementlari manfiymas $(a_{ik} \geq 0)$ yoki musbat $(a_{ik} > 0)$ bo'lsa.

Ta'rif 4.2: $A = \|a_{ik}\|_{i,k=1}^{n}$ kvadrat matritsa yoyiluvchi deyiladi, agarda barcha 1,2,…,n indekslarni qandaydir ikkita qo'shimcha sistemaga $i_1, i_2,...i_m$ va $k_1, k_2,...k_y \, (m + y = n)$ bo'linishida, umumiy indekslardan boshqa holda $a_{i_\alpha k_\beta} = 0(\alpha = 1,2,.., m; \beta = 1,2,...y)$ bo'lsa.

Aks holda A matritsa yoyilmaydigan matritsa deyiladi. A kvadrat matritsa qatorlarini o'rinalmashtirish deganda satrlarni o'rin almashtirish bilan birga A matritsa ustunlarini ham huddi shunday o'rin almashtirishni tushunamiz.

Yoyiluvchi va yoyilmaydigan matritsalar ta'rifini quyidagicha ifodalash ham mumkin.

Ta'rif 4.$2'$: $A = \|a_{ik}\|_{i,k=1}^{n}$ matritsa yoyiluvchi deyiladi, agarda uni qatorlarining o'rinlarini almashtirib, quyidagi ko'rinishga keltirish mumkin bo'lsa;

$$\tilde{A} = \begin{Vmatrix} B & 0 \\ 0 & D \end{Vmatrix},$$

bu yerda B va D kvadrat matritsalar. Aks holda A matritsa yoyilmaydigan deyiladi.

A – n o’lchovli kvadrat matritsa $e_1, e_2, ..., e_n$ bazisli n-o’lchovli R^n fazodagi $\overline{A}$ chiziqli operatorga mos kelsin. Matritsada qatorlarni o’rin almashtirish bazis vektorlarni qayta nomerlashga mos keladi, ya’ni $e_1, e_2, ..., e_n$ ba’zisdan yangi $e_1' = e_{j_1}, e_2' = e_{j_2}, ..., e_n' = e_{j_n}$ bazisga o’tish mos keladi, bu yerda $(j_1, j_2, ... j_n)$ indekslarni qandaydir o’rin almashtirish, bunda A matritsa unga o’xshash bo’lgan $\tilde{A} = T^{-1}AT$ matritsaga o’tadi. (T-almashtiruvchi matritsaning har bir satr va ustunining bitta elementi birga teng bo’lib, qolgan elementlari nollardan iborat). R^n fazoning v- o’lchovli qism fazosi deganda $e_{k_1}, e_{k2}, ..., e_{k_v}$ bazisli ixtiyoriy qism fazoni tushunamiz. $(1 \le k_1 \le k_2 \le ... \le k_v)$

T a ’ r i f 4. $2''$**:** $A = \| a_{ik} \|_{i,k=1}^{n}$ matritsa yoyiluvchi deyiladi, faqat va faqata shu holdaki, agar bu matritsaga mos $\overline{A}$ operator v<n o’lchovli invariant koordinatali qism fazoga ega bo’lsa.

Lemma 4.1. agar $A \ge 0$ matritsa yoyilmaydigan bo’lib, n – o’lchovli bo’lsa, u holda

$$(E + A)^{n-1} > 0 \tag{4.1}$$

I s b o t i. Lemmani isbotlash uchun, ixtiyoriy y>0 (vector ustun) uchun

$$(E + A)^{n-1} y > 0$$

ekanligini ko’rsatish yetarli. Bu tengsizlik isbotlanadi, agarda biz $y \ge 0$ va $g \ne 0$ shartda $z = (E + A)y$ har doim y ga nisbatan kichik nomli koordinataga ega ekanligini ko’rsatsak, teskarisini faraz qilamiz. U holda g va z vektorlar bir xil nolli koordinataga ega bo’ladi. Umumiylikni buzmasdan,

$$y = \left\| \begin{matrix} u \\ 0 \end{matrix} \right\|, \quad z = \left\| \begin{matrix} v \\ 0 \end{matrix} \right\| \quad \text{(u>0, v>0)}$$

bu yerda u va v ustunlar bir xil o’lchovli, deb olamiz.

$$A = \left\| \begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix} \right\|$$

deb olib, quyidagiga ega bo’lamiz:

$$\left\| \begin{matrix} u \\ 0 \end{matrix} \right\| + \left\| \begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix} \right\| \left\| \begin{matrix} u \\ 0 \end{matrix} \right\| = \left\| \begin{matrix} v \\ 0 \end{matrix} \right\|,$$

bundan

$$A_{21}u = 0$$

bo'lib, u>0 bo'lgani uchun $A_{21} = 0$ kelib chiqadi. Bu tenglik A matritsaning yoyiluvchi emasligiga ziddir.

A matritsaning quyidagi darajasini qaraymiz:

$$A^q = \| a_{ik} \|_{i,k=1}^{n} \quad \text{(q=1,2…)}$$

u holda yuqoridagi lemmadan quyidagi natija kelib chiqadi.

Natija: Agar A>0 yoyilmaydigan matritsa bo'lsa, u holda i,k indekslar juftligi uchun shunday q butun musbat son mavjudki, unda

$$a_{ik}^{q} > 0 \tag{4.2}$$

bo'ladi. Shu bilan birga q sonini har doim quyidagicha oraliqda tanlash mumkin

$$\left.\begin{matrix} q \le m-1, & agar & i \ne k & bo'lsa \\ q \le m, & agar & i = k, & bo'lsa \end{matrix}\right\}, \tag{4.3}$$

bu yerda m- A matritsaning $\psi(\lambda)$ ko'phadning darajasi.

§2. Yoyilmaydigan manfiymas matritsaning spektral xossasi

Teorema 4.1. (Perron teoremasi). $A = \| a_{ik} \|_{i,k=1}^{n}$ musbat matritsa har doim r haqiqiy va musbat harakteristik songa ega bo'lib, u harakteristik tenglamaning oddiy ildizi bo'ladi va moduli bo'yicha barcha harakteristik ildizlardan ortiq. Bu maksimal r harakteristik songa A matritsaning z_i >0, (i=1,2, …,n) koordinatali $z = (z_1, z_2, ...z_n)$ xos vaktori mos keladi.

Musbat matritsa yoyilmaydigan manfiymas matritsaning xussiy ko'rinishi bo'ladi. Frobenius yoyilmaydigan manfiymas matritsaning spektral xossasini o'rganib, Perron teoremasini umumlashtirdi.

Teorema4.2. (Frobenius teoremasi). Yoyilmaydigan manfiymas $A = \| a_{ik} \|_{i,k=1}^{n}$ matritsa har doim mos harakteristik tenglamani oddiy ildizi bo'lgan r musbat harakteristik songa ega. Boshqa barcha harakteristik ildizlarning moduli r dan ortmaydi. R maksimal harakteristik songa musbat koordinatali z xos vektor mos keladi.

Agar shu bilan birga A matritsa moduli r ga teng bo'lgan h ta $\lambda_0 = r, \lambda_1, ..., \lambda_{h-1}$ harakteristik sonlarga ega bo'lsa, u holda bu sonlarning hammasi har xil bo'lib,

$$\lambda^h - r^h = 0 \tag{4.4}$$

tenglamaning ildizlari bo'ladi va λ-kompleks tekkislikdagi nuqtalar sistemasi sifatida qaralayotgan, $A = \| a_{ik} \|_{i,k=1}^{n}$ matritsaning $\lambda_0, \lambda_1, ..., \lambda_{h-1}$ barcha harakteristik sonlarning umumiy to'dasi bu tekkislikni $\frac{2\pi}{h}$ burchakka burganda o'zi o'ziga o'tadi, h>1 da qatorlarni almashtirish bilan A matritsani quyidagi siklik ko'rinishga keltirish mumkin:

$$A = \begin{Vmatrix} 0 & A_{12} & 0 & ... & 0 \\ 0 & 0 & A_{22} & ... & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & ... & A_{h-1,h} \\ A_{h1} & 0 & 0 & ... & 0 \end{Vmatrix} \tag{4.5}$$

bu yerda dioganal bo'ylab kvadrat matritsalar joylashgan.

Perron teoremasi Frobenius teoremasining xususiy holi bo'lgani uchun, biz Frobenius teoremasini isbotlaymiz. Avval nisbatan ba'zi belgilashlarni kelishib olamiz, faqat va faqat

$$c_{ik} \le d_{ik} \, (i = 1,2,...m; \; k = 1,2,..,n) \tag{4.6}$$

holdagina quyidagi tengsizlikni yozamiz

$$C \le D \quad \text{yoki} \quad D \ge C$$

bu yerda C va D lar bir xil $m \times n$ o'lchovli, to'g'ri to'rtburchakli matritsalar bo'lib,

$$C = \| c_{ik} \|, \quad D = \| d_{ik} \|, \quad \text{(i=1,2,…m; k=1,2,..,n).}$$

Agar (4.6) tengsizliklarda tenglik belgisini tashlab yuborsak, u holda quyidagini yozamiz:

$$C < D \quad \text{yoki} \quad D > C$$

Xususiy holda, $C \geq 0\ (C > 0)$ C matritsaning barcha elementlari manfiymas (mos ravishda musbat) ekanligini bildiradi.

Bundan tashqari, C^{+} bilan modC, ya'ni C matritsa barcha elementlarini ularning modullari bilan almashtirib hosil qilingan matritsani belgilaymiz.

Frobenius teoremasining isboti. Fiksirlangan haqiqiy $x = (x_1, x_2, ..., x_n) \geq 0 (x \neq 0)$ vektor uchun

$$r_x = \min_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} \quad \left((Ax)_i = \sum_{k=1}^{n} a_{ik} x_k, i = 1,2,...n \right)$$

deb olamiz, bunda minimumni aniqlashda $x_i = 0$ uchun i indeksning qiymatlari yo'qotiladi. Ma'lumki, $r_x \geq 0$ va r_x

$$gx \leq Ax$$

tengsizlikni qanoatlantiruvchi g haqiqiy sonlarning eng kattasi bo'ladi.

Biz r_x funksiya qandaydir z>0 vektorda o'zining eng katta r qiymatiga erishishini isbotlaymiz:

$$r = r_z = \max_{(x \geq 0)} r_x = \max_{(x \geq 0)} \min_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} \tag{4.7}$$

r_x ning aniqlanishidan kelib chiqadiki, $x \geq 0$ $(x \neq 0)$ vektorni $\lambda > 0$ songa ko'paytirganda r_x o'zgarmaydi. Shuning uchun r_x funksiya maksimumini izlashda

$$x \geq 0, \qquad (x, x) = \sum_{i=1}^{n} x_i^2 = 1$$

shartni qanoatlantiruvchi x vektorlardan tuzilgan M yopiq to'plam bilan chegaralanish mumkin.

Agar r_x funksiya M to'plamda uzluksiz bo'lsa, u holda maksimumning mavjudligi ta'minlanadi. Ammo r_x funksiya ixtiyoriy x>0 nuqtada uzluksiz bo'lib, koordinatalaridan biri nolga aylanadigan M to'plamning chegaraviy nuqtalarida uziladigan bo'lishi mumkin. Shuning uchun M to'plam o'rniga quyidagi ko'rinishdagi y vektorlardan tuzilgan N to'plamni kiritamiz:

$$y = (E + A)^{n-1} x (x \in M)$$

N to'plam M to'plam kabi chegaralangan va yopiq bo'lib, lemma4.1 ga asosan musbat vektorlardan tuzilgan.
bundan tashqari,

$$r_x x \le Ax$$

tengsizlikning ikkala tomonini $(E + A)^{n-1} > 0$ ga ko'paytirib, quyidagini hosil qilamiz:

$$r_x y \le Ay \quad \left[(y = (E + A)^{n-1} x)\right]$$

bundan, r_y ni aniqlanishiga ko'ra quyidagini topamiz:

$$r_x \le r_y$$

shuning uchun r_x ning maksimumini izlashda M to'plamni faqat musbat vektorlardan tuzilgan N to'plam bilan almashtirishimiz mumkin. N chegaralangan, yopiq toplamda r_x funksiya uzluksiz bo'lib, qandaydir z>0 vektorda o'zining eng katta qiymatiga erishadi.

$$r_z = r \tag{4.8}$$

Shartni qanoatlantiruvchi ixtiyoriy z>0 vektorni ekstremal deb ataymiz.
Endi quyidagilarni isbotlaymiz:

1) (4.7) tenglik bilan aniqlangan r son musbat va A matritsaning harakteristik soni bo'ladi;

2) ixtiyoriy z ekstremal vektor bo'ladi, ya'ni

$$r>0,\ z>0, \quad Az=rz. \tag{4.9}$$

Haqiqatan, agar $u = (\underbrace{1,1,\dots,1}_{n})$ bo'lsa, u holda $r_u = \min\limits_{1 \le i \le n} \sum\limits_{k=1}^{n} a_{ik}$. Ammo bu holda $r_u > 0$, chunki yoyilmaydigan matritsaning birorta ham satri faqat nollardan iborat bo'lishi mumkin emas. Demak, r>0, chunki $r > r_x$.

Agar

$$x = (E + A)^{n-1} z \qquad (4.10)$$

bo’lsin. U holda lemma4.1 asosan x>0 faraz qilaylik, $Az - rz \neq 0$. U holda (4.1), (4.8) va (4.10) dan quyidagini ketma-ketni hosil qilamiz:

$$Az - rz \geq 0, \quad (E + A)^{n-1}(Az - rz) > 0, \qquad Ax - rx > 0$$

oxirgi tengsizlik r sonini aniqlanishiga zid, chunki bu tengsizlikdan yetarli kichik $\varepsilon > 0$ uchun $Ax - (r + \varepsilon)x > 0$ ya’ni $r_x \geq r + \varepsilon > r$ kelib chiqadi. Demak, Az=rz. Ammo bu holda

$$0 < x = (E + A)^{n-1} z = (1 + r)^{n-1} z$$

bo’lib, bundan $z > 0$ kelib chiqadi.

Endi barcha harakteristik sonlar modullari r dan ortmasligini ko’rsatamiz.

$$Ay = \alpha y (y \neq 0) \qquad (4.11)$$

bo’lsin. (4.11) ning ikkala tomonidan modulga o’tib,

$$|\alpha| y^+ < Ay^+ \qquad (4.12)$$

ni hosil qilamiz. Bundan,

$$|\alpha| < r_{y^+} < r$$

faraz qilaylik, r harakteristik son qandaydir y vektorga mos kelsin:

$$Ay = ry (y \neq 0)$$

u holda (4.11) va (4.12) da $\alpha = r$ deb olib, xulosa qilamizki, y^+-ekstremal vektor bo’lib, $y^+ > 0$, ya’ni $y = (y_1, y_2, ..., y_n)$, $y_i \neq 0$ (i=1,2,…,n). bundan kelib chiqadiki, r harakteristik songa faqat bitta xos yo’nalish mos keladi, chunki, ikkita chiziqli bog’liqmas z va z_1 vektorlar bo’lgan holda biz shunday c va d sonlarni tanlashimiz mumkin bo’ladiki, unda $y = cz + dz_1$ xos vektor nolli koordinataga ega bo’ladi, isbotlanganga ko’ra bu mumkin emas.

$\lambda E - A$ harakteristik matritsa uchun quyidagi matritsani kiritamiz:

$$B(\lambda) = \| B_{ik}(\lambda) \|_{i,k=1}^{n} = \Delta(\lambda)(\lambda E - A)^{-1},$$

bu yerda $\Delta(\lambda)$-A matritsaning harakteristik ko’phadi, $B_{ik}(\lambda)$ esa $\Delta(\lambda)$ aniqlovchidagi $\lambda\delta_{ki} - a_{ki}$ elementning algebraic to’ldiruvchisi. R harakteristik songa faqat bitta $z = (z_1, z_2, ..., z_n)$, $z_1 > 0, z_2 > 0, ..., z_n > 0$ xos vektor mos kelishidan kelib chiqadiki, $B(r)\neq 0$ va *B(r)* matritsaning ixtiyoriy nolmas ustunida barcha elementlar noldan farqli va bir xil ishorali bo'ladi. U holda bu hol B(r) matritsa satrlari uchun ham o'rinli, chunki A matritsa uchun yuritilgan mulohazalarni A^T-transponarlangan matritsa uchun ham yuritish mumkin. Bundan kelib chiqadiki, barcha $B_{ik}(r)\ (i,k=1,2,...,n)$ noldan farqli va bitta σ ishorali. Shuning uchun

$$\sigma\cdot\Delta^T(r)=\sigma\cdot\sum_{i=1}^{n}B_{ii}(r)>0,$$

ya'ni $\Delta^T(r)\neq 0$ va r son $\Delta(\lambda)$ harakteristik ko'phadning oddiy ildizi.

Shunday qilib, r son $\Delta(\lambda)=\lambda^n+...$ ko'phadning maksimal ildizi, u holda $\Delta(\lambda)$ ko'phad $\lambda=r$ da o'sadi. Shuning uchun $\Delta^T(r)>0$ va $\tau=1$ ya'ni

$$B_{ik}(r)>0 \quad (i,k=1,2,\dots,n) \tag{4.13}$$

teoremaning ikkinchi qismini isbotlashda quyidagi lemmadan foydalanamiz:

Lemma4.2. Agar $A=\| a_{ik}\|_{i,k=1}^{n}$ va $C=\| c_{ik}\|_{i,k=1}^{n}$ ikkita bir xil n-o'lchovli matritsalar bo'lib, A- yoyilmaydigan matritsa va

$$C^+\leq A \tag{4.14}$$

bo'lsa, u holda C matritsaning γ harakteristik soni bilan A matritsa r maksimal harakteristik soni o'rtasida

$$|\gamma|\leq r \tag{4.15}$$

tengsizlik o'rinli. (4.15) munosabatda tenglik belgisi faqat va faqat

$$C=e^{i\varphi}DAD^{-1} \tag{4.16}$$

dagina o'rinli bo'lishu mumkin, bu yerda

$$e^{i\varphi}=\frac{\gamma}{r},$$

D-elementlari moduli bo'yicha birga teng bo'lgan dioganal matritsa (D^+=E).

Lemmaning isboti. C matritsaning γ harakteristik soniga mos keluvchi xos vektorni y bilan belgilaymiz:

$$C\cdot y=\gamma\cdot y \qquad (\gamma\neq 0)$$

(4.14) va (4.17) dan

$$|\gamma|y^+\leq C^+y^+\leq Ay^+ \tag{4.18}$$

shuning uchun

$$|\gamma| \le r_{y^+} \le r$$

Endi $|\gamma| = r$ holni taxlil qilamiz. Bu holda (4.18)dan kelib chiqadiki, y^+ A matritsa uchun ekstremal vektor, demak $y^+ > 0$ va y^+ A matritsaning r harakteristik soniga mos xos vektor. Shuning uchun (4.18) quyidagi ko'rinishni oladi:

$$Ay^+ = C^+ y^+ = ry^+, y^+ > 0 \tag{4.19}$$

Bundan, (4.14) ga asosan

$$C^+ = A \tag{4.20}$$

$y = (y_1, y_2, ..., y_n)$, $y_j = |y_j| e^{i\Psi_j} (j = 1, 2, ..., n)$ bo'lsin.

$$D = \{e^{i\Psi 1}, e^{i\Psi_2}, ..., e^{i\Psi_n}\}$$

dioganal matritsa olamiz.

U holda

$$y = Dy^+$$

Buni (4.17) ga qo'yib, $\gamma = re^{i\varphi}$ deb olib, quyidagini topamiz:

$$Fy^+ = ry^+ \tag{4.21}$$

Bu yerda

$$F = e^{-i\varphi} D^{-1} CD \tag{4.22}$$

(4.19) va (4.21) dan

$$Fy^+ = C^+ y^+ = Ay^+ \tag{4.23}$$

ammo (4.22) va (4.20) ga asosan

$$F^+ = C^+ = A$$

shuning uchun (4.23) dan ,

$$Fy^+ = F^+ y^+$$

$y^+ > 0$ bo'lgani uchun bu tenglik $F = F^+$ dagina, ya'ni

$$e^{-i\varphi} D^{-1} CD = A$$

da o'rinli bo'ladi. Bundan

$$C = e^{i\varphi} DAD^{-1}$$

lemma isbotlandi.

Teoremaning isbotiga qaytamiz. Isbotlangan lemmani r maksimal moduli h ta har xil harakteristik sonlarga ega bo'lgan oyilmaydigan A matritsaga qo'llaymiz:

$$\lambda_0 = re^{i\varphi_0}, \lambda_1 = re^{i\varphi_1}, ..., \lambda_{h-1} = re^{i\varphi_{h-1}} (\Delta = \varphi_0 < \varphi_1 < \varphi_2 < .... < \varphi_{h-1} < 2\pi)$$

U holda C=A va $\gamma = \lambda_k$ deb olib, ixtiyoriy k=0,1,…,h-1 uchun quyidagiga ega bo'lamiz:

$$A = e^{i\varphi_k} D_k A D_k^{-1} \qquad (4.24)$$

bu yerda D_k-dioganal matritsa bo'lib, $D_k^+ = E$.

z A matritsaning *r* maksimal harakteristik songa mos musbat xos vector bo'lsin:

$$Az = rz(z > 0) \qquad (4.25)$$

u holda

$$\overset{k}{y} = D_k z \quad \overset{k}{y}{}^+ = z > 0 \qquad (4.26)$$

deb olib, (4.25) dan

$$A\overset{k}{y} = \lambda_k \overset{k}{y} \quad (\lambda_k = re^{i\varphi_k}, k = 0,1,..,h-1) \qquad (4.27)$$

Oxirgi tenglikdan ko'rinadiki, $\overset{0}{y}, \overset{1}{y}, ..., \overset{h-1}{y}$ vektorlar A matritsaning $\lambda_0, \lambda_1, ..., \lambda_{h-1}$ harakteristik sonlar uchun xos vektorlar bo'ladi.

(4.24) dan kelib chiqadiki nafaqat $\lambda_0 = r$ balki, $\lambda_0, \lambda_1, ..., \lambda_{h-1}$ lar A matritsaning oddiy harakteristik sonlari bo'ladi. Shuning uchun $\overset{k}{y}$ xos vektorlar va $D_k (k = 0,1,2,...,h-1)$ matritsalar o'zgarmas skalyar ko'paytuvchi aniqligida aniqlanadi. $D_0, D_1, ..., D_{h-1}$ matritsalarni bir qiymatli aniqlash uchun bu matritsaning birinchi dioganal elementlarini birga teng qilib tanlaymiz. U holda

$$D_0 = E \text{ va } \dot{y} = z > 0 \ .$$

(4.24) dan quyidagi kelib chiqadi:

$$A = e^{i(\varphi_j \pm \varphi_k)} D_j D_k^{\pm 1} A D_k^{\mp 1} D_j^{-1} (j, k = 0,1,..., h-1)$$

bundan, yuqoridagi kabi xulosa qilamizki,

$$D_j D_k^{\pm 1} z$$

vektor A matritsani $re^{i(\varphi_j \pm \varphi_k)}$ harakteristik soniga mos xos vektori bo'ladi. Shuning uchun $e^{i(\varphi_j \pm \varphi_k)}$ son $e^{i\varphi_l}$ sonlarning biri bilan, $D_j D_k^{\pm 1}$ matritsa esa D_j matritsaning biri bilan ustma-ust tushadi, ya'ni qandaydir e_1, e_2 larda

$$e^{i(\varphi_j + \varphi_k)} = e^{i\varphi_{e1}}, e^{i(\varphi_j - \varphi_k)} = e^{i\varphi_2},$$

$$D_j D_k = D_{e_1,}, \quad D_j D_k^{-1} = D_{e_2}$$

Shunday qilib, $e^{i\varphi_0}, e^{i\varphi_1}, ..., e^{i\varphi_{h-1}}$ songa mos va dioganal $D_0, D_1, ..., D_{h-1}$ matritsalar o'zaro izomorf multiplikativ abel gruppalarini tashkil etadi.
Har qanday h ta har xil elementli chekli gruppada ixtiyoriy elementning h-darajasi gruppaning birlik elementiga teng. Shuning uchun $e^{i\varphi_0}, e^{i\varphi_1}, ..., e^{i\varphi_{h-1}}$ lar birning h-darajali ildizlari bo'ladi. Shuningdek , birning hta har xil ildizlari mavjud va $\varphi_0 = 0 < \varphi_1 < \varphi_2, ..., \varphi_{h-1} < 2\pi$ u holda

$$\varphi_k = \frac{2\pi}{k}(k = 0,1,..,h-1)$$

va

$$e^{i\varphi_k} = \varepsilon^k (\varepsilon = e^{i\varphi_1} = e^{\frac{2\pi}{k}i}, k = 0,1..., h-1) \tag{4.28}$$

$$\lambda_k = r\varepsilon^k, (k = 0,1,2..., h-1) \tag{4.29}$$

$\lambda_0, \lambda_1, ..., \lambda_{h-1}$ sonlar (4.4) tenglamaning to'la yechimlar sistemasini tashkil etadi. (4.28) mos ravishda quyidagicha ega bo'lamiz:

$$D_k = D^k, (D = D_1, k = 0,1,2,...h-1) \tag{4.30}$$

Endi (4.24) tenglikdan k=1 da

$$A = e^{i\frac{2\pi}{h}} DAD^{-1} \tag{4.31}$$

hosil bo'ladi. Bundan kelib chiqadiki, A matritsa $e^{i\frac{2\pi}{h}}$ ga ko'paytirilganda o'xshash matritsaga o'tadi, demak, A matritsada harakteristik sonlarning to'la sistemasi $e^{i\frac{2\pi}{h}}$ ga ko’paytirilganda o’zi-o’ziga o’tadi.

$$D^h = E$$

ekanligidan ko’rinadiki, d ning dioganalidagi barcha elementlari birning h-darajali ildizlari bo’ladi. A dagi (mos ravishda D dagi) qatorlarni o’rin almashtirish bilan D matritsa quyidagi kvazidioganal ko’rinishga kelishi mumkin:

$$D = \{\tau_0 E_0, \tau_1 E_1, ..., \tau_{s-1} E_{s-1}\} \tag{4.32}$$

bu yerda $E_0, E_1, ..., E_{s-1}$ -birlik matritsalar va

$$\tau_p = e^{i\psi_p}, \psi_p = n_p \frac{2\pi}{h}$$

(n_p -butun son, p=0,1,…,s-1, 0=n_0<n_1<…<n_{s-1}<h) ma’lumki, $s \le h$.

A matritsani quyidasicha blok ko’rinishida yozib,

$$A = \begin{Vmatrix} A_{11} & A_{12} & ... & A_{1s} \\ A_{21} & A_{22} & ... & A_{2s} \\ .. & .. & .. & .. \\ A_{s1} & A_{s2} & ... & A_{ss} \end{Vmatrix} \tag{4.33}$$

$$\varepsilon A_{pq} = \frac{\tau_{q-1}}{\tau_{p-1}} A_{pq} (p, q = 1,2,, ..., s) \quad \varepsilon = e^{\frac{2\pi}{h} i} \tag{4.34}$$

Bundan ixtiyoriy p va q da $\frac{\tau_{q-1}}{\tau_{p-1}} = \varepsilon$ yoki $A_{pq} = 0$.

p=1 deb olamiz. Barcha $A_{12}, A_{13}, ..., A_{1n}$ matritsalar bir vaqtda nolga aylanishi mumkin emas, u holda $\frac{\tau_1}{\tau_0}, \frac{\tau_2}{\tau_0}, ..., \frac{\tau_{s-1}}{\tau_0} (\tau_0 = 1)$ yechimlardan biri ε ga teng bo’lishi kerak. Bu faqat $n_1 = 1$ dagina mumkin. U holda $\frac{\tau_1}{\tau_0} = \varepsilon$ va $A_{11} = A_{13} = ... = A_{1s} = 0$ huddi shunday (4.34)da p=2 deb olib, $n_2 = 2$ va $A_{21} = A_{22} = ... = A_{2s} = 0$ va hakozo. Natijada quyidagini hosil qilamiz:

$$A = \left\| \begin{matrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & . & A_{s-1,s} \\ A_{s1} & A_{s2} & A_{s3} & \dots & A_{ss} \end{matrix} \right\|$$

shunday qilib, $n_1 = 1, n_2 = 2, \dots, n_{s-1} = s - 1$. Ammo p=s da (4.34) tenglikning o'ng tomonida quyidagi ko'paytuvchi turadi:

$$\frac{\tau_{q-1}}{\tau_{p-1}} = e^{(q-s)\frac{2\pi}{h}i} \quad (q = 1,2,..,s)$$

bu sonlardan biri $\varepsilon = \frac{2\pi}{h} i$ ga teng bo'lishi kerak. Bu faqat s=h va q=1 dagina mumkin, demak, $A_{s2} = \dots = A_{ss} = 0$ shunday qilib,

$$D = \{E_0, \varepsilon E_1, \varepsilon^2 E_2, \dots, \varepsilon^{h-1} E_{h-1}\}$$

va A matritsa (4.5) ko'rinishga ega.

Frobenus teoremasi to'la isbotlandi.

§3. Yoyiluvchi matritsa.

Avvalgi paragrifda aytilgan yoyilmaydigan manfiymas matritsalarning spektral xossasi yoyiluvchi matritsalarga o'tganda o'z kuchini yo'qotadi. Ammo, ixtiyoriy $A \geq 0$ manfiymas matritsa har doim yoyilmaydigan va hattoki musbat A_m matritsalar ketma-ketligi sifatida ifodalanishi mumkin

$$A = \lim_{m \to \infty} A_m \, ((A_m > 0, m = 1,2,\dots), \qquad (4.35)$$

u holda ba'zi yoyilmaydigan matritsalar spectral xossalari kuchsizlanlantirilgan formada yoyiluvchi matritsalar uchun ham o'rinli.

$A = \|a_{ik}\|_{i,k=1}^{n}$ manfiymas matritsa uchun quyidagi teoremani isbotlaymiz:

Teorema 4. 3. A manfiymas matritsa har doim r manfiymas harakteristik songa egaki, unda A matritsaning barcha harakteristik ildizlarining moduli r dan ortmaydi. Bu r maksimal harakteristik songa $y \geq 0$ manfiymas xos vektor mos keladi:

$$Ay = ry(y \geq 0, y \neq 0)$$

Isboti. A matritsa uchun (4.35) o'rinli bo'lsin. A_m matritsaning maksimal harakteristik sonini $r^{(m)}$ bilan unga mos normalangan musbat xos vektorni $y^{(m)}$ bilan belgilaymiz:

$$A_m\, y^{(m)} = r^{(m)}\, y^{(m)} \qquad \left[y^{(m)}g^{m} = 1,\ y^{(m)} > 0;\ m = 1,2,,...\right] \tag{4.36}$$

u holda (4.35) dan kelib chiqadiki, quyidagi limit mavjud:

$$\lim r^{(m)} = r$$

bu yerda r - A matritsaning harakteristik soni. $r^{(m)} > 0$ va $r^{(m)} > \left|\lambda_0^{(m)}\right|$, bu yerda $\lambda_0^{(m)}$ - A_m matritsaning ixtiyoriy harakteristik soni (m=1,2…) ekanligidan limitga o'tib, quyidagini hosil qilamiz:

$$r \geq 0, \qquad r \geq |\lambda_0| \tag{4.37}$$

bu yerda λ_0 - A matritsaning ixtiyoriy harakteristik soni. Bu limitik o'tishdan (4.35) bilan birga quyidagi hosil bo'ladi:

$$B(r) \geq 0 \tag{4.38}$$

normalangan xos vektorlar $y^{(m)}(m = 1,2...)$ ketma-ketligidan qandaydir normalangan y vektorga yaqinlashuvchi $y^{(m_p)}(p = 1)$ qism ketma-ketlik ajratish mumkin. (4.36) tenglikdan limitga o'tib, quyidagini hosil qilamiz:

$$Ay = ry(y \geq 0, y \neq 0)$$

teorema isbotlandi.

Manfiymas elementli matritsalar uchun muhim bo'lgan qator tasdiqlarni qarab chiqamiz:

1.Agar $A = \| a_{ik} \|_{i,k=1}^{n}$-r maksimal harakteristik sonli manfiymas matritsa bo'lsa, u holda

$$(\lambda E - A)^{-1} > 0,\ \frac{d}{d\lambda}(\lambda E - A)^{-1} \leq 0, \qquad \lambda > r \tag{4.39}$$

haqiqatan, $\lambda > r > 0$ da

$$(\lambda E - A)^{-1} = \sum_{j=0}^{\infty} \frac{A^j}{\lambda^{j+1}} > 0 \tag{4.40}$$

yoyilma o’rinli, shuning uchun

$$\frac{d}{d\lambda}(\lambda E - A)^{-1} = -\sum_{j=0}^{\infty}\frac{(j+1)A^j}{\lambda^{j+2}} \le 0 \qquad (4.41)$$

2.Agar A – r maksimal harakteristik sonli manfiymas matritsa bo’lib, $B(\lambda)$ va $C(\lambda)$ mos ravishda uning yopishgan va keltirilgan yopishgan matritsalari bo’lsa, u holda

$$\lambda \ge r \text{ da } B(\lambda) \ge 0, \quad C(\lambda) \ge 0 \qquad (4.42)$$

bo’lib,

$$B(\lambda) = (\lambda E - A)^{-1}\Delta(\lambda), \quad C(\lambda) = (\lambda E - A)^{-1}\psi(\lambda)$$

va

$$\lambda > r \text{ da } \Delta(\lambda) > 0, \psi(\lambda) > 0 \qquad (4.43)$$

ekanligidan, (4.39) dan (4.42) kelib chiqadi.

3.Agar A yoyilmaydigan, r maksimal harakteristik sonli matritsa bo’lsa, u holda

$$\lambda > r \text{ da } (\lambda E - A)^{-1} > 0, \ \frac{d}{d\lambda}(\lambda E - A)^{-1} < 0, \qquad (4.44)$$

$$\lambda \ge r \text{ da } B(\lambda) > 0, \ C(\lambda) > 0 \qquad (4.45)$$

4.r'-A manfiymas matritsaning taribi n dan kichik bo’lgan bosh minorning harakteristik soni bo’lib, r-A matritsaning maksimal harakteristik soni bo’lsa,

$$r' \le r \qquad (4.46)$$

bo’ladi. Agar (n-1)-tartibli minor uchun $r' < r$ bo’lsa, u holda

$$\Delta(\lambda) = |\lambda E - A|$$

harakteristik aniqlovchi uchun

$$r' < \lambda < r \text{ da } \Delta(\lambda) < 0 \qquad (4.47)$$

tengsizlik o’rinli.

Agar A –yoyilmaydigan matritsa bo’lsa, u holda (4.46) da tenglik belgisi bo’lmaydi, ya’ni $r' < r$ bo’ladi.

Agar A –yoyilmaydigan matritsa bo’lsa, u holda xech bo’lmaganda bitta bosh minor uchun (4.46) da tenglik belgisi o’rinli, ya’ni $r' = r$ bo’ladi.

5.Agar $A \ge 0$ yoyiluvchi matritsa bo’lib,

$$\Delta\,(r) = \begin{vmatrix} r-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & r-a_{22} & \cdots & -a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ -a_{n1} & -a_{n2} & \ldots & r-a_{nn} \end{vmatrix}$$

harakteristik aniqlovchisining qandaydir bosh minori nolga aylansa, u holda shu minorni o’rab turuvchi minor xususiy holda (n-1)-tartibli bosh minorlardan biri

$$B_{11}(\lambda), B_{22}(\lambda),\ldots, B_{nn}(\lambda)$$

nolga aylanadi.

6. $A \geq 0$ matritsa faqat va faqat

$$B_{ii}(r) \geq 0 (i = 1,2\ldots,n)$$

munosabatlardan birida tenglik belgisi o’rinli bo’lsa, yoyiluvchi bo’ladi.

7.Agar r-A>0 matritsaning maksimal harakteristik soni bo’lsa, u holda ixtiyoriy $\lambda > r$ da $A_\lambda = \lambda E - A$ harakteristik matritsaning barcha bosh minorlari musbat, ya’ni

$$A_\lambda \begin{pmatrix} i_1, i_2 \ldots i_p \\ i_1, i_2 \ldots . i_p \end{pmatrix} \geq 0, (\lambda \geq r, 1 \leq i_1 < i_2 < \ldots . < i_p \leq n;\ p = 1,2,..,n) \qquad (4.48)$$

bo’ladi.

Lemma 4.3. (Kotelyanskiy lemmesi).

Agar $G = \| g_{ik} \|_{i,k=1}^{n}$ haqiqiy matritsada dioganalda yotmagan barcha elementar manfiy yoki nolga teng,

$$g_{ik} \leq 0 \qquad (i \neq k, i.k = 1,2,\ldots.,n) \qquad (4.49)$$

bosh minorlar ketma-ketligi ega musbat

$$g_{11} = G\begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0,\ G\begin{pmatrix} 12 \\ 12 \end{pmatrix} > 0,\ \ldots,\ G\begin{pmatrix} 12\ldots.n \\ 12\ldots.n \end{pmatrix} > 0 \qquad (4.50)$$

bo’lsa, u holda G matritsaning bosh minorlari musbat,

$$G\begin{pmatrix} i_1, i_2 \ldots i_p \\ i_1, i_2 \ldots . i_p \end{pmatrix} \geq 0 \quad (1 \leq i_1 < i_2 < \ldots . < i_p \leq n;\ \ p = 1,2,..,n)$$

bo’ladi.

Teorema 4.4. λ haqiqiy son $A = \| a_{ik} \|_{i,k=1}^{n} > 0$ matritsaning r maksimal harakteristik sonidan katta, $r < \lambda$ bo'lishi uchun λ ning bu qiymatida $A_k = \lambda E - A$ ketma-ketligi musbat, $\lambda - a_{11} > 0$,

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} > 0,$$

$$,\dots,\begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{vmatrix} \qquad (4.51)$$

bo'lishi zarur va yetarli.

Teorema 4.5. Dioganalda yotmagan elementlari manfiymas bo'lgan $C = \| c_{ik} \|_{i,k=1}^{n} > 0$ haqiqiy matritsaning barcha harakteristik sonlari manfiy haqiqiy qismga ega bo'lishi uchun

$$C_{11} < 0, \ \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} > 0, \dots, (-1)^{n} \begin{vmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{vmatrix} \qquad (4.52)$$

tengsizlik bajarailishi zarur va yetarli.

Teorema 4.6. A-manfiymas matritsaning ixtiyoriy lementi o'sganda uning maksimal harakteristik soni kamaymaydi. U qat'iy o'sadi, agarda A yoyilmaydigan matritsa bo'lsa.

Teorema 4.$6'$**.** Agar mos ravishda r va r_1 maksimal harakteristik sonli A va A_1 manfiymas matritsalar berilgan bo'lsa, u holda $A \le A_1$ tengsizlikdan $r \le r_1$ tengsizlik kelib chiqadi. Agar A yoyilmaydigan matritsa bo'lsa, $r < r_1$ bo'ladi.

Isboti. A –yoyilmaydigan matritsa bo'lsin. U holda A_1 ham yoyilmaydigan matritsa bo'ladi. A_1 matritsaning r_1 harakteristik soni uchun xos vektorini x bilan belgilaymiz:

$$A_1 x = r_1 x \ \ (x > 0)$$

bundan,

$$(r_1 - r)x = Ax - rx + (A_1 - A)x \tag{4.53}$$

ammo $(A_1 - A)x \geq 0$. Shuning uchun agar x A matritsaning r harakteristik soni uchun xos vektor bo'lmasa, u holda qandaydir $i(1 \leq i \leq n)$ indeksda

$$(r_1 - r)x_i = (A_1 - A)x_i > 0$$

ya'ni, yana $r < r_1$ bo'ladi.

Yoyiluvchi matritsa bo'lgan holda $A_\varepsilon = A + \varepsilon B$ va $A_{1\varepsilon} = A_1 + \varepsilon B, B > 0, \varepsilon > 0$ matritsalarni kiritamiz, u holda $A_\varepsilon \leq A_{1\varepsilon} va A_\varepsilon \geq 0$ bo'ladi. Shuning uchun $r_\varepsilon < r_{1\varepsilon}$ mos ravishda $A_\varepsilon va A_{1\varepsilon}$ matritsalarning maksimal harakteristik sonlari. $\varepsilon \to 0$ da limitga o'tsak, $A_\varepsilon va A_{1\varepsilon}$ lar mos ravishda $A \ va \ A_1$ da, $r_\varepsilon < r_{1\varepsilon}$ tengsizlik $r < r_1$ tengsizlikka o'tadi.

Mashqlar: Manfiymas elementli matritsalar uchun muhim bo'lgan 1-7 tasdiqlarni isbotlang.

§4. Yoyiluvchi matritsaning normal formasi.

Ixtiyoriy $A = \| a_{ik} \|_{i,k=1}^{n}$ yoyiluvchi matritsani qaraymiz.. uning qatorlarini almashtirib, quyidagi ko'rinishga keltirish mumkin;

$$A = \left\| \begin{matrix} B & O \\ C & D \end{matrix} \right\| \tag{4.54}$$

bu yerda B,D-kvadrat matritsalar.

Agar B va D matritsalar yoyiluvchi bo'lsa, uni (4.54)ga o'xshash ko'rinishda tasvirlash mumkin, shundan so'ng A matritsa quyidagi ko'rinishni oladi:

$$A = \left\| \begin{matrix} K & O & O \\ H & L & O \\ F & G & M \end{matrix} \right\|$$

Agar K,L,M matritsalarning qandaydir biri yoyiluvchi bo'lsa, yuqoridagi jarayonni yana davom ettirish mumkin. Qatorlarni almashtirish natijasida biz A matritsada uchburchak bloklar (formasi) ko'rinishini beramiz:

$$A = \begin{Vmatrix} A_{11} & O & \dots. & O \\ A_{21} & A_{22} & \dots. & O \\ . & . & . & . \\ A_{S1} & A_{s2} & \dots. & A_{ss} \end{Vmatrix} \qquad (4.55)$$

bu yerda dioganaldagi bloklar yoyilmaydigan kvadrat matritsalardir.

Dioganaldagi $A_{ii}(1 \le i \le s)$ blok matritsalar ajralgan deyiladi, agarda

$$A_{ik} = 0(k = 1,2, \dots, i-1, i+1, \dots, s)$$

bo'lsa. (4.55) matritsada blokli qatorlarni o'rinlarini almashtirib, barcha ajralgan bloklarni bosh dioganal bo'ylab birinchi o'ringa qo'yamiz. Shundan so'ng A matritsa quyidagi ko'rinishni oladi:

$$A = \begin{Vmatrix} A_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_g & 0 & \dots & 0 \\ A_{g+1,1} & A_{g+1,2} & \dots & A_{g+1,g} & A_{g+1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots. & \dots & \dots \\ A_{s1} & A_{s2} & \dots & A_{sg} & A_{s,g+1} & \dots & A_s \end{Vmatrix} \qquad (4.56)$$

u yerda $A_1, A_2, \dots, A_s$ yoyilmaydigan matritsalarhar bir qatordagi,

$$A_{f1}, A_{f2}, \dots, A_{f,f-1}(f = g+1, .., s),$$

matritsaning hech bo'lmaganda bittasi noldan farqli.

(4.56) matritsani yoyiluvchi A matritsaning normal formasi deb ataymiz.

Teorema 4.7. $\mathrm{A} \ge 0$ matritsaning r maksimal harakteristik soniga faqat va faqat shu holda musbat xos vektor mos keladi, qachonki A matritsaning (4.64) normal formasida:

1.$A_1, A_2, \dots, A_g$ matritsaning har biri o'zining r harakteristik soniga ega:

2. $g < s\ da\ A_{g+1,}A_{g+2}, \dots A_s$ matritsalarning birortasi ham bunday xossaga ega emas.

Isboti. R-maksimal harakateristik soniga z>0 musbat xos vektor mos kelsin. Bloklarga bo'lish bilan mos ravishda (4.56) da z ustunni $z^k(k = 1,2, \dots, s)$ qismlarga ajratamiz. U holda

$$Az = rz \ \ (z > 0) \tag{4.57}$$

tenglik quyidagi ikkita tengliklar sistemasiga almashadi:

$$\mathrm{A_i z^i = rz^i} \quad (\mathrm{i} = 1,2,\dots,\mathrm{g}) \tag{4.57$^|$}$$

$$\sum_{\mathrm{h=1}}^{\mathrm{j-1}} \mathrm{A_{jh}\, z^h + A_j z^j = rz^j}, \ (\mathrm{j = g+1,..,s}) \tag{4.57$^{||}$}$$

(4.57$^|$) da kelib chiqadiki, r har bir $A_1, A_2, \dots, A_s$ matritsaning harakteristik soni bo'ladi. (4.57$^{||}$) dan quyidagini topamiz:

$$A_j z^j \le rz^j, A_j z^j \ne rz^j, (j = g+1,\dots,s) \tag{4.58}$$

r_j bilan $A_j\,(j = g+1,..,s)$ matritsaning maksimal harakteristik sonini belgilaymiz. U holda (4.58)dan quyidagini topamiz:

$$r_j \le max \frac{(A_j z^j)i}{z_i^j} \le r, (j = g+1,..,s)$$

ikkinchi tomondan $r_j = r$ tenglik (4.58) ning ikkinchi munosabatiga ziddir. shuning uchun

$$r_j < r \quad (j = g+1,\dots,s) \tag{4.59}$$

endi aksincha, $A_i \ \ (i = 1,2,\dots,g)$ matritsalarning r gat eng maksimal xos qiymati berilgan bo'lsin, $A_j\,(j = g+1,..,s)$ matritsalar uchun esa (4.59) tengsizliklar o'rinli. U holda izlanayotgan (4.57) tenglikni $(4.57^{'})$ va $(4.57^{''})$ tengliklar sistemasi bilan alamashtirib, $(4.57^{'})$ dan $A_i \ \ (i = 1,2,\dots,g)$ matritsaning z^i musbat xos ustunlarni aniqlashimiz mumkin. Shundan so'ng,(4.57$^{||}$) dan $z^j \ \ (j = g+1,..,n)$ ustunlarni topamiz:

$$z^i = \left(rE_j - A_j\right)^{-1} - \sum_{k=1}^{j-1} A_{jk}\, z^h \quad (j = g+1,..,s) \tag{4.60}$$

bu yerda E_j shu tartibli birlik matritsa bo'lib, A_j kabi $(j = g+1,..,s)$

$$r_j < r(j - g + 1,\dots,s)$$

bo'lgani uchun

$$\left(rE_j - A_j\right)^{-1} > 0 \ \ (j = g+1,\dots s) \tag{4.61}$$

(4.60) formulalar bilan aniqlangan $z^{g+1}, \dots, z^s$ ustun musbat ekanligini induktiv usul bilan isbotlaymiz. Buning uchun ixtiyoriy $j \ (g+1 \le j \le n)$ da $z^1, z^2, \dots, z^{j-1}$ ustunning musbatligidan $z^j > 0$ kelib chiqishini ko'rsatamiz. Haqiqatan, bu holda

$$\sum_{k=1}^{j-1} A_{jk} z^k \geq 0 \quad \sum_{k=1}^{j-1} A_{jk} z^k \neq 0$$

bo'lib, buni (4.61) bilan birgalikda qarasak (4.60) ga asosan kelib chiqadi.

Shunday qilib, $z = \left\| \begin{matrix} z^i \\ . \\ z^s \end{matrix} \right\|$ musbat ustun A matritsaning r harakteristik soni uchun xos vektor bo'ladi.

Teorema $4.7'$. A matritsaning r maksimal harakteristik soniga A matritsani va A^T transponirlangan matritsani musbat xos vektori javob beradi, agarda A matritsani qatorlarini almashtirish bilan quyidagicha kvazidioganal ko'rinishda tasvirlash mumkin bo'lsa,

$$A = \{A_1, A_2, \dots, A_s,\} \tag{4.62}$$

bu yerda $A_1, A_2, \dots, A_s$ -har biri o'zining r maksimal harakteristik soni oddiy bo'lib, unga A va A^T matritsalarning musbat xos vektorlari mos kelsa, u holda A –yoyilmaydigan matritsa bo'ladi.

Aksincha, har qanday yoyilmaydigan matritsa natijada ko'rsatilgan xossaga ega bo'ladi, u holda bu xossa yoyilmaydigan manfiymas matritsaning spektral harakteristikasini ifodalaydi.

5§ Primitiv va imirimitiv matritsalar.

Ta'rif 4.3. Agar $A \geq 0$ yoyilmaydigan matritsa hammasi bo'lib, h ta $\lambda_1, \lambda_2, \dots, \lambda_k$ r maksimal modulli

$$(|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k| = r)$$

harakteristik sonlarga ega bo'lsa, u holda hq1 da A-primitiv matritsa, h>1 da esa imprimitiv matritsa deyiladi. h son A matritsaning impritivlik indeksi deyiladi.

Agar A matritsaning harakteristik tenglamasi (ko'pxadi)

$$\Delta\ \lambda = \lambda^{\mathrm{n}} + a_1 \lambda^{\mathrm{n1}} + \cdots + a_{t_{\mathrm{a}_1}} \lambda^{\mathrm{nt}} = 0$$
$$(n > n_1 > n_2 > \cdots > n_t, a_1 \neq 0, a_2 \neq 0, \ldots, a_t \neq 0)$$

bo'lsa, u holda uning imprimitivlik indeksi h quyidagi ayormalarning eng katta umumiy bo'luvchisiga teng bo'ladi:

$$n - n_1, n_1 - n_2, \ldots, n_{t-1} - n_t \qquad (4.63)$$

haqiqatan, frobenus teoremasiga ko'ra, A matritsaning spektri λ-kompleks tekkislikni λq0 nuqta atrofida $\frac{2\pi}{h}$ burchakka burishda o'ziga o'tadi. Shuning uchun $\Delta\ (\lambda$) ko'phad qandaydir g(M) ko'phaddan

$$\Delta\ (\lambda) = g(\lambda^h)\lambda^n$$

Formula yordamida hosil qilinishi kerak. Bundan, kelib chiqadiki, h (4.63) ayirmalarning EKUBi d gat eng bo'ladi.

Teorema 4.8. $\mathrm{A} \geq 0$ matritsa faqat va faqat shu holda primitive bo'ladiki, qachonki A matritsaning qandaydir darajasi musbat bo'lsa:

$$A^p > 0 \ \ (p > 1) \qquad (4.64)$$

Isboti. Agar $A^p > 0$ bo'lsa, u holda A matritsa yoyilmaydigan bo'ladi, chunki A matritsaning yoyiluvchanligidan A^p matritsaning yoyiluvchanligi kelib chiqadi. A matritsa uchun hq1 bo'ladi, chunki, aks holda A^p musbat matritsa

$$\lambda_1^p, \lambda_2^p, \ldots, \lambda_h^p$$

h ta r^p maksimal mudulli harakteristik sonlarga ega bo'ladi. Bu perron teoremasiga ziddir.

Endi aksincha bo'lsin, ya'ni A primitiv matritsa berilgan bo'lsin.

$$A^p = \sum_{k=1}^{s} \frac{1}{(m_k - 1)!} \left[\frac{C(\lambda)\lambda^p}{\varphi^k(\lambda)}\right]^{m_k - 1}_{\lambda = \lambda_{\mathrm{k}}} \qquad (4.65)$$

bu yerda

$$\varphi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots . (\lambda - \lambda_s)^{m_s} \quad (\lambda_j \neq \lambda_f, \mathrm{j} = \mathrm{f\ da})$$

A-matritsaning minimal ko'phadi,

$$\varphi^k(\lambda) = \frac{\varphi(\lambda)}{(\lambda - \lambda_k)^{m_k}} \, (k = 1,2, \dots, s) \; C(\lambda)$$

esa keltirilgan, yopishgan matritsa

$$C(\lambda) = \left((\lambda E - A)^{-1} \varphi(\lambda)\right)$$

bu holda

$$\lambda_1 = \mathrm{r} > |\lambda_2| > \cdots > |\lambda_s|, \mathrm{m} = 1 \tag{4.66}$$

deb olish mumkin bo'lib,

$$A^p = \frac{C(r)}{\acute{\varphi}(r)} \; r^p - \sum_{k=1}^{s} \frac{1}{(m_k - 1)!} \left[\frac{C(\lambda)\lambda^p}{\varphi^k(\lambda)}\right]_{\lambda = \lambda_k}^{m_k - 1}$$

bo'ladi. Bundan (4.65) ga asosan

$$\lim_{p \to \infty} \frac{A^p}{r^p} = \frac{C(r)}{\acute{\varphi}(r)} \tag{4.67}$$

bo'ladi.

Ikkinchi tomondan C(r)>0 va $\acute{\varphi}(r) > 0$. Shuning uchun

$$\lim_{p \to \infty} \frac{A^p}{r^p} > 0$$

bo'lib, qandaydir 1 dan boshlab, A^p>0 bo'ladi.

Eslatma. Agar A matrirsa primitive va $A^p > 0$ bo'lsa, u holda barcha m>p uchun $A^m > 0$ bo'ladi, A matritsa nolli qatorni o'zida saqlamaydi.

Natija. Primitiv matritsaning darajasi har doim yoyilmaydigan va primitiv bo'ladi.

Lemma4.4. agar A-primitiv matritsa bo'lsa, u holda ixtiyoriy ikkita i,k indekslar uchun shunday $i, i_1, i_2, \dots, i_s, k \; (s \geq 0)$ indekslar zanjiri mavjudki, unda

$$a_{ii_1} > 0, a_{i_1 i_2} > 0, \dots, a_{i_s k} > 0$$

bo'ladi.

Bunday zanjirlarni A matritsada i dan k ga olib boradi deb aytamiz. S+1 son zanhirning uzunligi deyiladi. i dan k ga olib boruvchi eng qisqa zanjirda barcha zanjirlar juft-jufti bilan har xil bo'ladi.

Lemmani isbotlash uchun $s \geq 0$ deb olish yetarli bo'lib, unda

$$A^{s+1} = \left\| a_{ik}{}^{s+1} \right\|_{i,k=1}^{n} > 0$$

bo'lishi kerak. U holda

$$\sum_{i_1, i_2, \dots, i_s = 1}^{n} a_{ii_1} a_{i_1 i_2} \dots a_{i_s k} - a_{ik}^{s+1} > 0$$

va barcha qo'shiluvchilar manfiymas, u holda ularning hech bo'lmaganda bittasi musbat bo'ladi. U so'ralayotgan indekslar zanjirini beradi.

Teorema 4.9. Agar $\mathrm{A} \geq 0$ -yoyilmaydigan matritsa bo'lib, uning qandaydir darajasi A^{q} yoyiluvchi bo'lsa, u holda A^{q} daraja to'la yoyiluvchi, ya'ni A^{q} ni qatorlarini A^{q} daraja to'la yoyiluvchi, ya'ni A^{q} ni qatorini almashtirib, quyidagi ko'rinishda tasvirlash mumkin:

$$\mathrm{A}^{\mathrm{q}} = \{\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_{\mathrm{d}},\} \qquad (4.68)$$

bu yerda $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_{\mathrm{d}}$-yoyilmaydigan matritsalar. Bu matritsalar bir xil maksimal harakteristik songa ega. Shu bilan birga d son q va h sonlarning eng katta umumiy bo'luvchisi bo'lib, bu yerda h son A matritsaning imprimitivlik indeksi.

Isboti. A matritsa yoyilmaydigan bo'lgani uchun Fobenus teoremasiga ko'ra, r maksimal harakteristik songa a va A^{T} matritsalarning musbat xos vektorlari mos keladi. Ammo bu musbat vektorlar $\lambda = r^q$ harakteristik songa A^q va $(\mathrm{A}^{\mathrm{q}})^{\mathrm{T}}$ matritsalar uchun ham xos vektorlar bo'ladi. Shuning uchun ham A^{q} darajaga teorem 4.7| ni qo'llab, bu darajani (4.68) ko'rinishda tasvirlaymiz. Bu yerda $A_1, A_2, \dots, A_d$ lar yoyilmaydigan matritsalar bo'lib, r^q maksimal harakteristik songa ega bo'ladi. Ammo A matritsa r maksimal modulli quyidagi h ta harakteristik songa ega :

$$r, r\varepsilon, \dots, r\varepsilon^{h-1} (\varepsilon = e^{\frac{2\pi}{h}i}),$$

shuning uchun A^q matritsa ham quyidagi h ta maksimal modulli harakteristik ega;

$$r^q, r^q\varepsilon^q, \dots, r^q\varepsilon^{q(h-1)}$$

bo'lib, d ta son r^q ga teng bo'ladi. Bu faqat d son q va h larning eng katta umumiy bo'luvchisi bo'lgandagina mumkin. Teorema isbotlandi.

Agar teorema4.9 da q=h deb olsak, quyidagi natijani hosil qilamiz.

Natija. Agar A-h impitivlik indeksli impitiv matritsa bo'lsa, u holda A^h daraja bir hil maksimal harakteristik songa ega bo'lgan h ta primitive matritsalarga yoyiladi.

§6. To'la manfiymas matritsalar

Ta'rif 4.4. To'g'ri to'rtburchakli

$$A = \|a_{ik}\| \quad (i = 1,2, \dots, m; k = 1,2, \dots, n)$$

matritsa to'la manfiymas (to'la musbat) deyiladi, agarda bu matritsaning ixtiyoriy tartibli barcha minorlari manfiymas (musbat) bo'lsa:

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} \geq 0 \quad (\text{mos ravishda} > 0)$$

$$\left(1 \leq \begin{matrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{matrix} \leq n; p = 1,2, \dots, \min(m, n)\right)$$

biz to'la manfiymas va to'la musbat kvadrat matritsalarni qarash bilan chegaralanamiz.

Misol .

1. Vandermondning umumlashgan matritsasi

$$V = \left\| a_i^{\alpha k} \right\|_{i,k=1}^{n} (0 < a_1 < a_2 < \cdots < n; \alpha_1 < \alpha_2 < \cdots < \alpha_n)$$

to'la musbat bo'ladi.

2. Yakobincha matritsa

$$J = \left\| \begin{matrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & \dots & 0 \\ 0 & c_2 & a_3 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdot & c_{n-1} a_n \end{matrix} \right\|$$

to'la manfiymas bo'lishi uchun uning barcha bosh minorlari va b,c elementlari manfiymas bo'lishi zarur va yetarli.

To'la manfiymas A matritsa uchun quyidagi muhim determinant tengsizlik o'rinli

$$A\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} \leq$$

$$\leq A\begin{pmatrix} 1 & 2 & \cdots & p \\ 1 & 2 & \dots & p \end{pmatrix} A\begin{pmatrix} p+1 & \dots & n \\ p+1 & \dots & n \end{pmatrix} (p < n) \tag{4.69}$$

bu tengsizlikni quyidagi lemmadan foydalanib isbotlaymiz:

Lemma4.5. Agar a-to'la manfiymas matritsada qandaydir bosh minor nolga teng bo'lsa, u holda bu minorni o'rab turuvchi ixtiyoriy bosh minor nolga teng bo'ladi.

Faraz qilaylik a matritsaning barcha bosh minorlari noldan farqli bo'lsin, chunki birorta bosh minorlari nolga teng bo'lsa, yuqoridagi lemmaga asosan $|A| = 0$ bo'lib, bu holda (4.69) tengsizlikning bajarilishi ravshan.

n=2 da (4.69) tengsizlikning o'rinliligi bevosita tekshiriladi:

$$A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \leq a_{11}a_{22} \text{ chunki } a_{12} \geq 0, a_{21} \geq 0.$$

$n > 2$ da (4.69) tengsizlikni barcha n dan kichik tartibli matritsalar uchun o'rinli deb olamiz. Bundan tashqari, umumiylikni buzmasdan p>1 deb hisoblashimiz mumkin, chunki, aks holda, satr va ustunlarni teskari nomerlash hisobiga p va n-p larning rollarini almashtirishimizga to’g’ri keladi.

$$D = \|d_{ik}\|$$

$$d_{ik} = A\begin{pmatrix} 1 & 2 & \dots & p-1 & i \\ 1 & 2 & \dots & p-1 & k \end{pmatrix}, \quad (i, k = 1, p+1, \dots, n)$$

matritsani qaraymiz. Ikki marta Silvestr ayniyatdan va n dan kichik tartibli matritsalar uchun (4.69) tengsizlikni ko’plab, quyidagiga ega bo’lamiz.

$$A\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = \frac{D\begin{pmatrix} p & p+1 & \dots & n \\ p & p+1 & \dots & n \end{pmatrix}}{\left[A\begin{pmatrix} 1 & 2 & \dots & p-1 \\ 1 & 2 & \dots & p-1 \end{pmatrix}\right]^{n-p}}$$

$$\leq \frac{d_{pp} D\begin{pmatrix} p+1 & \dots & n \\ p+1 & \dots & n \end{pmatrix}}{D\begin{pmatrix} p & p+1 & \dots & n \\ p & p+1 & \dots & n \end{pmatrix}}$$

$$= \frac{A\begin{pmatrix} 1 & 2 & \dots & p \\ 1 & 2 & \dots & p \end{pmatrix} A\begin{pmatrix} 1 & 2 & \dots & p-1 & p+1 & \dots & n \\ 1 & 2 & \dots & p-1 & p+1 & \dots & n \end{pmatrix}}{A\begin{pmatrix} 1 & 2 & \dots & p-1 \\ 1 & 2 & \dots & p-1 \end{pmatrix}} \qquad (4.70)$$

$$\leq A\begin{pmatrix} 1 & 2 & \dots & p \\ 1 & 2 & \dots & p \end{pmatrix} A\begin{pmatrix} p+1 & \dots & n \\ p+1 & \dots & n \end{pmatrix}$$

demak, (4.69) tengsizlik o’rinli.

Ta’rif 4.5. $A = \|a_{ik}\|_{i,k=1}^{n}$ matritsaning

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} \quad \left(1 \leq \begin{matrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{matrix} \leq n\right) \qquad (4.71)$$

minori deyarli *bosh minor* deyiladi, agarda $i_1 - k_1, i_2 - k_{2,} \dots, i_p - k_p$ ayirmalarning faqat bittasi noldan farqli bo’lsa.

Yuqorida keltirilgan barcha xulosalar o’z kuchida qoladi, agarda “A-to’la manfiymas matritsa” shartini undan kuchsizroq bo’lgan “A matritsada barcha bosh va deyarli bosh minorlar manfiymas” shart bilan almashtirilsa.

Mashqlar:

1. To'la manfiymas matritsalarga misollar keltiring va ularni deyarli bosh minorlarini ajrating.
2. Agar $A \geq 0,\ B > C\ va\ AB$ aniqlangan bo'lsa, u xolda $AB \geq AC$ ekanligini isbotlang.
3. Agar $A \geq 0,\ B > C\ va\ AB = 0$ bo'lsa, u xolda A=0 ekanligini isbotlang.
4. Agar A keltiriluvchi matritsa bo'lsa, ixtiyoriy butun musbat p soni uchun A^p matritsa ham keltiriluvchi ekanligini isbotlang.
5. Agar

$$A = \begin{Vmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{Vmatrix} \text{ va } x = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix}$$

   bo'lsa, $y \leq Ax$ shartni qanoatlantiruvchi $y \geq 0$ vektorlar to'plamini yozing. $px \leq Ax$ shartni qanoatlantiruvchi eng katta p sonini toping.
6. Agar $A \in R_{nxn}$ manfiymas matritsa bo'lib, $\sigma_j = \sum_{k=1}^{n} a_{jk}$ bo'lsa u xolda quyidagini isbotlang

$$\min\sigma_j \leq \lambda \leq \max\sigma_j,$$

   bu erda $\lambda -$A matritsaning spektral radiusiga teng bo'lgan xaqiqiy xos qiymati.
7. Agar A primitive matritsa bo'lib, p musbat butun son bo'lsa, u xolda A^p matritsani primitive ekanligini isbotlang.
8. Agar $A \geq 0$ keltirilmaydigan matritsa bo'lib, $\varepsilon > 0$ bo'lsa, u xolda xos qiymatlarni qarab chiqish yordamida $\varepsilon I + A$ matritsani primitive ekanligini isbotlang.

V-BOB.
XOS QIYMATLARNI REGULYARLIGI VA LOKALLIGINING HAR-XIL KRITERIYALARI.

§1. Adamarning regulyarlik kriteryasi va uning umumlashgani.

$$A = \|a_{ik}\|_{i,k=1}^{n}$$

ixtiyoriy kompleks elementli $n \times n$ o’lchovli matritsa berilgan bo’lsin.

Faraz qilaylik bu matritsa xos matritsa, ya’ni $|A| = 0$ bo’lsin. U holda $|x_k| > 0$ maksimum bilan $x_1, x_2, \dots, x_n$ sonlar mavjud bo’lib, quyidagi tenglik o’rinli bo’ladi:

$$\sum_{j=1}^{n} a_{kj} x_j = 0 \qquad (5.1)$$

ammo bu holda

$$|a_{kk}||x_k| \le \sum_{j=1}^{n} |a_{kj}||x_j| \le |x_k| \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}|$$

bo’lib buni $|x_k|$ ga qisqartirsak,

$$|a_{kk}| \le |x_k| \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}| \qquad (5.2)$$

hosil bo’ladi. Shuning uchun, agar

$$H_i = |a_{ii}| - \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| > 0 \quad i = (1,2,\dots,n) \qquad (5.3)$$

Adamar sharti bajarilsa, u holda (5.2) ko’rinishdagi tengsizlik o’rinli emas va demak, A matritsa regulyar (xosmas), ya’ni $|A| \ne 0$ bo’ladi.

Shunday qilib quyidagi teorema o’rinli:

Teorema 5.1: (Adamar teoremasi).

Agar $A = \|a_{ik}\|_{i,k=1}^{n}$ matritsa uchun (5.3) tengsizliklar bajarilsa, u holda A matritsada xosmas bo’ladi.

$H_i > 0$ shart a_{ii} dioganal elementning moduli i-satr elementar modullari yig'indisidan katta (qat'iy) ekanligini bildiradi. Bunday a_{ii} element (dominiruyushiy) ustunlik qiluvchi (o'zining satri uchun) deyiladi.

Adamar sharti A matritsaning barcha dioganal elementlari (o'zining satri uchun ustunlik qiluvchi) bo'lishini talab qiladi.

Eslatma 1.

Agar (5.3) Adamar sharti bajarilsa, $mod|A|$ uchun quyidagi baho o'rinli:

$$mod|A| \geq H_1 H_2 \dots H_n > 0 \qquad (5.4)$$

(5.4) shartni o'rinli ekanligiga ishonch hosil qilish uchun

$$F = \|f_{ij}\|_{i,j=1}^{n}, |f_{ij}| = \frac{a_{ij}}{H_i} \quad (i,j = 1,2,\dots,n) \qquad (5.5)$$

Yordamchi matritsani kiritamiz. bunda quyidagi tenglik o'rinli bo'ladi:

$$|f_{ii}| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |f_{ij}| = 1 \;\; i = (1,2,\dots,n) \qquad (5.6)$$

λ_0 bilan bu matritsaning qandaydir harakteristik sonini belgilaymiz. λ_0 songa $|x_k| > 0$ maksimalli $(x_1, x_2, \dots, x_n)$ xos vector mos keladi. U holda

$$\lambda_0 \, x_k = \sum_{j=1}^{n} f_{kj} \, x_j \qquad (5.7)$$

bu tengsizlikdan

$$|\lambda_0|\,|x_k| > |f_{kk}|\,|x_k| - \sum_{\substack{j=1 \\ j\neq k}}^{n} |f_{kj}|\,|x_j| > |x_k| \left(|f_{kk}| - \sum_{\substack{j=1 \\ j\neq k}}^{n} |f_{kj}| = |x_k| \right)$$

buni $|x_k|$ ga qisqartirib,

$$|\lambda_0| > 1$$

ni topamiz. Ammo $|F|$ aniqlovchi F matritsaning harakteristik sonlari ko'paytmasiga teng. Ularning har biri 1 dan kichik emas. Shuning uchun

$$mod|F| \geq 1 \qquad (5.8)$$

ikkinchi tomondan

$$|F| = \frac{|A|}{H_1 \cdot H_2 \dots H_n} \tag{5.9}$$

(5.8) va (5.9) dan izlanayotgan (5.4) tengsizlik kelib chiqadi.

Eslatma 2.

$|A| = |A^T|$ ekanligidan a matritsani A^T matritsa bilan almashtirib, A matritsaning xosmasligidan yetarli shartini ustunlar uchun adamar shartlari ko'rinishida quyidagicha hosil qilamiz:

$$G_i = |a_{ii}| - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}| > 0 \quad (i = 1,2, \dots, n) \tag{5.10}$$

bu shartlar bajarilganda (5.4) ning o'rniga quyidagiga ega bo'lamiz:

$$mod|A| \geq G_1 G_2 \dots . G_n \tag{5.11}$$

C-ixtiyoriy xosmas $n \times n$ o'lchovli matritsa bo'lsin. U hold A va Ac matritsalar bir vaqtda xosmas bo'ladi. shuning uchun (5.3) , (5.10) shartlarda, shuningdek (5.4) va (5.11) baholarda A matritsani AC matritsa bilan almashtirish mumkin. C matritsani variatsiyalab, har xil o'zaro ekvivalent bo'lmagan xosmaslikning yetarli shartlarini, shuningdek $|A|$ uchun (5.4) va (5.11) ga o'xshash baholarni hosil qilamiz. Xususiy holda, C matritsani tanlash hisobiga ustunlarni ixtiyoriy almashtirishni amalga oshirish mumkin. u holda (5.3) shartlarning o'rniga quyidagi shartlarni hosil qilamiz:

$$H_i = |a_{i\mu_i}| - \sum_{\substack{j=1 \\ j \neq \mu_i}}^{n} |a_{ij}| > 0 \quad (i = 1,2, \dots, n) \tag{5.12}$$

bu yerda $(\mu_1, \mu_2, \dots, \mu_n)$- fiksirlangan, ammo 1,2,…n indekslarning ixtiyoriy ixtiyoriy o'rin almashgani.

Boshqacha aytganda

$$A = |a_{ij}|_{i,j=1}^{n}$$

matritsa xosmas bo'ladi, agarda uning har bir satrda ustunlik qiluvchi (dioganalda bo'lishi shart emas) elment bo'lib, va bu n ta ustunlik qiluvchi elementlar har xil ustunlarda joylashgan bo'lsa.

Shunga o'xshash tasdiq ustunlar uchun ham o'rinli.

Endi quyidagi kuchsizlangan Adamar shartlari bajarilsin:

$$H_i = |a_{ii}| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \geq 0 \quad (i = 1,2,\dots,n) \tag{5.13}$$

U holda har bir dioganal element o'zining satri uchun kuchsiz domirlovchi bo'ladi.

Faraz qilaylik, A matritsa xos va Ax=0 bo'lib, $x = (x_1, x_2, \dots, x_n) \neq 0$ vektor ustun $|x_k|$ maksimal ustunli P ta x_k elementlarga ega va avval p<n bo'lsin x vector koordinatalarini qayta nomerlab, modul bo'yicha maksimal bo'lganlarini birinchi p ta koordinatalarga keltiramiz:

$$|x_1| = |x_2| = \dots = |x_p| > |x_j|, \quad (j = p+1,\dots,n)$$

Bunda Ax=0 tenglik saqlanadi, agarda A matritsaning satr va ustunlarida qandaydir almashtirishni amalga oshirsak. Shundan so'ng quyidagini yozish mumkin:

$$a_{kk}x_k = -\sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}x_j \quad (k = 1,2,\dots,p)$$

bundan,

$$|a_{kk}||x_k| \leq (\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|)|x_k| + \sum_{j=p+1}^{n} |a_{kj}||x_j| \leq |x_k| \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| , \tag{5.14}$$

$$(k = 1,2,\dots,n)$$

Buni $|x_k|$ ga qisqartirib, quyidagini hosil qilamiz:

$$|a_{kk}| \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| \quad (k = 1,2,\dots,n) \tag{5.15}$$

bu munosabatni (5.13) ga qo'yib, xulosa qilamizki (5.15) va (5.14) da tenglik belgisi o'rinli bo'ladi. Bu esa faqat

$$\sum_{j=p+1}^{n} |a_{kj}| = 0 \;\; k = (p+1,\dots,n)$$

dagina bajariladi, ya'ni

$$A = \begin{pmatrix} A_1 & \overbrace{0}^{p\ ta} \\ A_3 & A_4 \end{pmatrix} \tag{5.16}$$

ko'rinishda bo'ladi.

Ammo (5.16) ko'rinishdagi matritsa yoyiluvchi deyiladi. Shunday qilib, p<n da A –yoyiluvchi matritsa bo'ladi.

Agar p=n bo'lsa, u holda barcha (5.15) munosabatlarda (5.13)da tenglik belgisi o'rinli bo'ladi.

Biz bunday xulosaga, A –xos matritsa deb olib, keldik.

Shunday qilib, quyidagi Adamar teoremasiga aniqlik kirituvchi quyidagi teoremani isbotladik.

Teorema 5.2. (Olga Tauski teoremasi).

Agar a - yoyilmaydigan matritsa uchun (5.13) Adamarning kuchsizlantirilgan shartda > belgi o'rinli bo'lsa, u holda A matritsa xosmas bo'ladi.

Bu teoremada $H_i \geq 0 \ (i = 1,2, \dots, n)$ shartlarni $G_i > 0 \ (i = 1,2, \dots, n)$ shartlar bilan almashtirish mumkin.

§2. Matritsa normasi.

Har bir $x \in R^n$ vektorga bitta manfiymas $\|x\|$ sonni mos qo'yamiz. R^n fazodagi ixtiyoriy x,y vektorlar va λ - ixtiyoriy skalyar uchun quyidagi shartlar bajariladi:

1. $\|x + y\| \leq \|x\|+\|y\|$
2. $\|\lambda x\| = |\lambda|\|x\|$,
3. $\|x\| > 0$ agarda $x \neq 0$ bo'lsa.
4. Shartda $\lambda = 0$ desak, $\|x\|$=0 dagina bajarilishi kelib chiqadi.

Bundan tashqari 2. Shartdan kelib chiqadiki, ixtiyoriy $x, y \in R^n$ vektorlar uchun

$$\|x - y\| \geq \|x\| - \|y\|$$

Quyidagi normalarni kiritish mumkin; vektorlarning "kubik" normasi

$$\|x\|_1 = \max_{1 \le i \le n} |x_i| \qquad (5.17)$$

yoki “oktaedrli” norma

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad (5.17')$$

“Ermitcha” norma

$$\|x\|_3 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad (5.17'')$$

Osongina tekshirib ko’rish mumkinki, bu normalar 1.,2., 3. Shartlarni qanoatlantiradi.

Endi $m \times n$ o’lchovli to’g’ri to’rtburchakli A matritsani va shu bilan birga uni y=Ax chiziqli alamshtirishni qaraymiz. $x, \in R^n$, $y \in S^m$ vektorlar.

Bu fazolarda vektorlarning $\|x\|_R = \|x\|$ va $\|g\|_s = \|y\|$ normalarni kiritib, A matritsaning normasini quyidagicha aniqlaymiz:

$$\|A\| = \sup_{x \in R, x \neq 0} \frac{\|Ax\|_s}{\|x\|_R} \qquad (5.18)$$

Normaning ta’rifidan quyidagi munosabat kelib chiqadi:

$$\|Ax\|_s \le \|A\| \|x\|_R \qquad (5.18')$$

$$\|A + B\| \le \|A\| + \|B\| \qquad (5.19)$$

$$\|\lambda A\| = |\lambda| \|A\| \qquad (5.19')$$

pxn o’lchovli B matritsa R^n ni S^p ga akaslantirsin, u holda AB matritsa R^n ni T^mga akslantiradi. R^n , S^p , T^m fazolarda vector normalarini va ular yordamida matritsalar normalarini kiritib, quyidagi tengsizlikka kelamiz:

$$\|AB\| \le \|A\| \|B\| \qquad (5.19')$$

Masalan, agar “kubik” vector normalar, (5.17) dan kelib chiqsak, u holda $A = \|a_{ik}\| \; (i = 1,2, \dots, m; k = 1,2, \dots, n)$ matritsaning normasi quyidagicha aniqlanadi:

$$\|A\| = \max_{1\le i\le m} \sum_{k=1}^{n} |a_{ik}| \qquad (5.20)$$

Haqiqqatan bu holda

$$\|Ax\|_1 = \max_{1\le i\le m} \sum_{k=1}^{n} |a_{ik}\, x_k| \le$$

$$\le \max_{1\le i\le m} |x_k| \max_{1\le i\le m} \sum_{k=1}^{n} |a_{ik}|$$

Shuning uchun

$$\frac{\|Ax\|}{\|x\|} \le \max_{1\le i\le m} \sum_{k=1}^{n} |a_{ik}|$$

Agar “oktaedrli” vector normadan kelib chiqasak

$$\|x\|_2 = \sum_{k=1}^{n} |x_k| ; \qquad \|y\|_2 = \sum_{i=1}^{m} |y_i|$$

U holda

$$\|A\| = \max_{1\le k\le n} \sum_{i=1}^{m} |a_{ik}| \qquad (5.20')$$

Endi “ermitcha” vektor normalarni,

$$\|x\|^2 = \sum_{k=1}^{n} |x_k|^2, \ \|y\|^2 = \sum_{k=1}^{n} |y_i|^2$$

qaraymiz. U holda $S = AA^*$ ermitcha musbat matrirsani kiritib, quyidagiga ega bo’lamiz:

$$\|Ax\|^2 = y * y = x^*Ax^*Ax = x^*Ax^*Ax = x^*Sx, \quad \|x\|^2 = x^*x.$$

Ammo bu holda

$$\|A\|^2 = \max_{x\ne 0} \frac{x^*Sx}{x^*x} = S,$$

bu yerda S - AA^* matritsaning maksimal harakteristik soni. Bu holda

$$\|A\| = \sqrt{S} \qquad (5.20'')$$

Endi x va y ustun vektor uchun har xil normani kiritamiz. Masalan,

$$\|x\|_2 = \sum_{k=1}^{n} |x_k|, \|y\|_1 = \max_{1\le i\le n} |y_i|$$

bo’lsin u holda

$$\|Ax\|_1 = \max_{1\le i\le m}\left|\sum_{k=1}^{n} a_{ik}x_k\right| \le \sum_{k=1}^{n}|x_k| = a\|x\|_2$$

bu yerda $a = \max_{1\le i\le n}|a_{ik}|$. Ikkinchi tomondan, agar $a = a_{pq}$ bo'lsa, u holda x_q ni $a_{pq}x_q = a|x_q|$ shartni qanoatlantiradigan qilib tanlab, $j \ne q$ da $x_j = 0$ deb olib $\|Ax\|_1 = a\|x\|_2$ tenglikka ega bo'lamiz.

Shunday qilib, bu holda

$$\|Ax\| = \max_{1\le i\le n}|a_{ik}| \tag{5.20'''}$$

bo'ladi.

§3. Adamar kriteriyasini blok matritsalarga kengaytirish.

$n\times n$ o'lchovli A matritsa $\mathbf{s^2}$ ta $n_\alpha \times n_p$ o'lchovli $A_{\alpha\beta}$ bloklarga ajratilgan bo'lsin $(\alpha,\beta = 1,2,..,s)$

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix} \tag{5.21}$$

bu holda R^n fazo s ta $R^{n\alpha}\,(\alpha = 1,2,...,s)$ qism fazolarga ajraladi. Ixtiyoriy $x \in R^n$ vector uchun quyidagi yoyilma o'rinli bo'ladi.

$$x = \sum_{\alpha=1}^{s} x_\alpha\,(x_\alpha \in R^{n\alpha}, \alpha = 1,2,..,s) \tag{5.21'}$$

$R^{n\alpha}$ qism fazolarda vector normalar kiritamiz. $A_{\alpha\beta}$ blok matritsa R^{n_β} qism fazoni $R^{n\alpha}$ qism fazoga akslantiradi, u holda u quyidagi norma bilan aniqlanadi:

$$\|A_{\alpha\beta}\| = \sup_{\substack{x_\beta\in R^{n_\beta} \\ x_\beta\ne 0}} \frac{\|A_{\alpha\beta}x_\beta\|R^{n\alpha}}{\|x_\beta\|R^{n\beta}} \tag{5.22}$$

xususiy holda $A_{\alpha\alpha}$ kvadrat matritsaning normasi quyidagicha aniqlanadi:

$$\|A_{\alpha\alpha}\| = \sup_{\substack{x_\alpha\in R^{n_\alpha} \\ x_\alpha\ne 0}} \frac{\|A_{\alpha\alpha}x_\alpha\|}{\|x_\alpha\|} \tag{5.22'}$$

Agar $|A_{\alpha\alpha}| \ne 0$ bo'lsa, u holda $|A_{\alpha\alpha}| > 0$ bo'ladi. U holda $(5.22')$ dan quyidagi kelib chiqadi:

$$\|A_{\alpha\alpha}^{-1}\| = \sup_{\substack{x_\alpha \in R^{n_\alpha} \\ x_\alpha \neq 0}} \frac{\|x_\alpha\|}{\|A_{\alpha\alpha} x_\alpha\|}$$

Demak,

$$\|A_{\alpha\alpha}^{-1}\|^{-1} = \inf_{\substack{x_\alpha \in R^{n_\alpha} \\ x_\alpha \neq 0}} \frac{\|A_{\alpha\alpha} x_\alpha\|}{\|x_\alpha\|} \quad (5.23)$$

bu tenglikning o'ng tomoni $A_{\alpha\alpha}$ -xos matritsa bo'lgan holda ham ma'noga ega.

Endi $|A| = 0$ va $x \neq 0$ da $Ax = 0$ bo'lsin. (5.21) va $(5.21')$ dan kelib chiqib, quyidagini yozaolmaymiz:

$$-A_{\alpha\alpha} x_\alpha = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{s} A_{\alpha\beta} x_\beta \, (\alpha = 1,2,..,s) \quad (5.24)$$

bundan, $(5.18')$ va (5.19) ga asosan

$$\|A_{\alpha\alpha} x_\alpha\| \leq \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta} x_\beta\| \leq \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta}\| \|x_\beta\|, (\alpha = 1,2,..,s) \quad (5.25)$$

ikkinchi tomondan, (5.23) dan kelib chiqadiki,

$$\|A_{\alpha\alpha}^{-1}\|^{-1} \|x_\alpha\| \leq \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta} x_\beta\| \leq \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta}\| \|x_\beta\|, (\alpha = 1,2,..,s) \quad (5.26)$$

§1 dagidek, α indeksni shunday tanlaymizki, unda $\|x_\alpha\|$ eng katta qiymatga ($\|x_\beta\|$ bilan solishtirganda, $\beta \neq \alpha$) ega bo'lsin va (5.26)ning o'ng tomonidagi barcha $\|x_\beta\|$ larni $\|x_\alpha\|$ bilan alamshtiramiz. So'ngra $\|x_\alpha\|$>0 ga qisqartirib, quyidagini hosil qilamiz:

$$\|A_{\alpha\alpha}^{-1}\|^{-1} \leq \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta}\| \quad (5.27)$$

Shuning uchun, "adamarning blokli shartlari" bajarilganda

$$\|A_{\alpha\alpha}^{-1}\|^{-1} = \sum_{\substack{\beta=1 \\ \beta \neq 0}}^{s} \|A_{\alpha\beta}\| > 0 \ (\alpha = 1,2,..,s) \quad (5.28)$$

bo'lib, (5.27) munosabat bo'lishi mumkin emas, ya'ni A-xos matritsa bo'lishi mumkin emas.

biz quyidagi teoremaga keldik:

Teoerema 5.3. Agar Adamarning blokli sharti (5.28) bajarilsa, u holda A – xosmas matritsa bo'ladi.

$n_1 = n_2 = ... = n_s = 1$ xususiy holda bu teorema Adamar teoremasiga o'tadi, agarda R bir o'lchovli qism fazolarga normani $\|x_\alpha\| = |x_\alpha|$ $(\alpha = 1,2,..,s)$ dek tanlasak.

§4. Fidlerning regulyarlik kriteryasi.

$n \times n$ o'lchovli A matritsa (5.21) blok ko'rinishda tasvirlangan bo'lsin. Uning uchun haqiqiy elementli $s \times s$ o'lchovli sonni matritsani kiritamiz:

$$G = \begin{pmatrix} \|A_{11}{}^{-1}\|^{-1} & -|A_{12}| & \dots & -\|A_{1s}\| \\ -\|A_{21}\| & \|A_{22}{}^{-1}\|^{-1} & \dots & -|A_{2s}| \\ \ddots & \ddots & \ddots & \ddots \\ -\|A_{s1}\| & -\|A_{s2}\| & \dots & \|A_{ss}{}^{-1}\|^{-1} \end{pmatrix}$$

bu matritsaning barcha dioganal elementlari manfiymas bo'lib, dioganalda yotmagan elementlari musbatmasdir. Ma'lumki dioganalda yotmagan barcha elementlari musbatmas bo'lib, barcha bosh minorlari musbat bo'lgan matritsa M-matritsa deyiladi.

Teoerema 5.4. (Fiddler teoremasi). Agar $s \times s$ o'lchovli G matritsa M –matritsa bo'lsa, u holda $n \times n$ o'lchovli A matritsa regulyar bo'ladi.

Isboti. Faraz qilaylik $|A| = 0$ bo'lsin. U holda $x \neq 0$ da $Ax = 0$ bo'ladi. (5.21) va $(5.21')$ ga asosan (5.26)ni hosil qilib, uni quyidagicha yozamiz:

$$\|A_{\alpha\alpha}{}^{-1}\|^{-1}\|x_\alpha\| - \sum_{\substack{\beta=1 \\ \beta\neq 0}}^{s} \|A_{\alpha\beta}\|\|x_\beta\| \leq 0(\alpha = 1,2,..,s) \qquad (5.30)$$

Avval barcha $\|x_\alpha\| > 0$ bo'lsin. U holda (5.30) da $\|x_\alpha\|$ oldidagi koeffisientlarni ortirib, ya'ni $\|A_{\alpha\alpha}{}^{-1}\|^{-1}$ ni qandaydir $\tilde{G}_{\alpha\alpha} \geq \|A_{\alpha\alpha}{}^{-1}\|^{-1}$ bilan alamshtirib, (5.30) tengsizliklardan quyidagi tengliklar sistemasini hosil qilamiz:

$$\tilde{g}_{\alpha\alpha}\|x_\alpha\| - \sum_{\substack{\beta=1\\ \beta\neq 0}}^{s}\|A_{\alpha\beta}\|\|x_\beta\| = 0 (\alpha = 1,2,..,s)$$

buni matritsali ko'rinishda quyidagicha yozamiz:

$$\tilde{G}\xi = 0,$$

bu yerda

$$\tilde{G} = \begin{pmatrix} \tilde{G}_{11} & -|A_{12}| & \dots & -\|A_{1s}\| \\ -\|A_{21}\| & \tilde{G}_{22} & \dots & -|A_{2s}| \\ \ddots & \ddots & \ddots & \ddots \\ -\|A_{s1}\| & -\|A_{s2}\| & \dots & \tilde{G}_{ss} \end{pmatrix},$$

$$0 \neq \xi = (\|x_1\|,.....,\|x_s\|)^T.$$

Bundan

$$|\tilde{G}| = 0.$$

Ikkinchi tomondan, M –matritsaning ta'rifiga ko'ra

$$|\tilde{G}| \geq |G| > 0$$

ekanligi kelib chiqadi. Biz $|A| = 0$ deb faraz qilib, qarama-qarshilikka keldik.

Agar qandaydir $\|x_\alpha\| = 0$ bo'lsa, u holda (5.30) munosabatlardan faqat $\|x_\alpha\| > 0$ shartni qanoatlantiruvchi α qiymatlariga moslarini olamiz.

Yuqoridagi mulohazalrni so'zma-so'z takrorlab, $|G|$ ning o'rniga G matritsaning qandaydir bosh minorini olib, yana qarama-qarshilikka kelamiz. Teorema to'la isbotlandi.

§5. Gershgoran doirasi va boshqa lokallashtirish sohalari.

$$A = \|a_{ik}\|_{i,k=1}^{n}$$

ixtiyoriy $n \times n$ o'lchovli, kompleks elementli matritsa bo'lib, λ uning qandaydir harakteristik soni bo'lsin. U holda $A - \lambda E$ xos matritsa bo'lib, uning uchun Adamarning barcha shartlari bajarilishi mumkin emas, ya'ni quyidagi munosabatlardan xech bo'lmaganda bittasi o'rinli bo'lishi kerak:

$$|a_{ii}-\lambda| < \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| (i=1,2,..,n) \tag{5.31}$$

(5.31) munosabatlarning har biri λ-kompleks tekkislikdagi a_{ii} markazli, $\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ radiusli qandaydir doirani aniqlaydi. Biz gershgorin tomonidan 1931 yilda yaratilgan teoremaga keldik.

Teorema 5.5. (Gershgorin teoremasi).

$$A = \|a_{ik}\|_{i,k=1}^{n}$$

matritsaning har bir λ harakteristik soni har doim (5.31) doiralarning birida joylashgan bo'ladi.

Shunday qilib, (5.31) Gershgorin doiralari barcha harakteristik sonlari yotadigan sohani beradi.

Teorema 5.6.

$$A = \{A_{\alpha\beta}\}_{\alpha,\beta}^{s} = 1$$

blok ko'rinishda tasvirlangan $n \times n$ o'lchovli A matritsaning har bir λ harakteristik soni quyidagi sohalarning hech bo'lmaganda bittasiga tegishli bo'ladi:

$$\|(A_{\alpha\alpha}-\lambda E)^{-1}\|^{-1} \leq \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{s} \|A_{\alpha\beta}\| \ (\alpha=1,2,..,s) \tag{5.32}$$

Shuningdek, x_α quyidagi sohalarning hech bo'lmaganda bittasiga tegishli bo'ladi:

$$\left\|\left(A_{\alpha\alpha}-\lambda E_{\alpha}{}^{-1}\right)\right\|^{-1} \leq \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{s} \|A_{\beta\alpha}\|, (\alpha=1,2,..,s) \tag{5.32$'$}$$

bu yerda E_α birlik matritsa bo'lib, $A_{\alpha\alpha}$ matritsalar bir xil tartibli $(\alpha=1,2,..,s)$.

Fiddler kriteriyasidan kelib chiqib, qanday lokallashtirish sohasini hosil qilish mumkinligini aniqlaymiz. $c_1, c_2,.., c_s$ manfiymas sonlar shunday tanlanganki, unda

$$G = \begin{pmatrix} c_1 & -\|A_{12}\| & \dots & -\|A_{1s}\| \\ -\|A_{21}\| & c_2 & \dots & -|A_{2s}| \\ \vdots & \vdots & \ddots & \vdots \\ -\|A_{s1}\| & -\|A_{s2}\| & \dots & c_s \end{pmatrix} \qquad (5.33)$$

matritsa kuchsizlantirilgan M – matritsa bo'lsin, ya'ni dioganalda yotmagan elementlari musbatmas bo'lib, barcha bosh minorlari manfiymas bo'lsin. Faraz qilaylik, qandaydir λ sonda quyidagi s ta tengsizlik bajarilsin:

$$\left\| \left(A_{\alpha\alpha} - \lambda E_{\alpha}{}^{-1} \right) \right\|^{-1} > c_{\alpha} \ (\alpha = 1,2,..,s) \qquad (5.34)$$

u holda, (5.33) matritsada C_{α} ni $\left\| \left(A_{\alpha\alpha} - \lambda E_{\alpha}{}^{-1} \right) \right\|^{-1}$ bilan alamashtirib, barcha dioganal elementlarni orttiramiz va M – matritsani hosil qilamiz:

$$\begin{pmatrix} \|(A_{11} - \lambda \mathrm{E}_1)^{-1}\|^{-1} & -\|A_{12}\| & \dots & -\|A_{1s}\| \\ -\|A_{21}\| & \|(A_{22} - \lambda \mathrm{E}_2)^{-1}\|^{-1} & \dots & -|A_{2s}| \\ \vdots & \vdots & \ddots & \vdots \\ -\|A_{s1}\| & -\|A_{s2}\| & \dots & \|(A_{ss} - \lambda \mathrm{E}_s)^{-1}\|^{-1} \end{pmatrix}$$

ammo bu holda Fidler teoremasiga ko'ra $|A - \lambda E| \neq 0$ va λ son A matritsaning harakteristik soni bo'lmaydi.

Shuning uchun, A matritsaning ixtiyoriy λ harakteristik soni uchun (5.34) tengsizlikning hech bo'lmaganda bittasi bajarilmaydi, ya'ni quyidagi munosabatlardan biri bajariladi:

$$\left\| \left(A_{\alpha\alpha} - \lambda E_{\alpha}{}^{-1} \right) \right\|^{-1} \leq c_{\alpha} \ (\alpha = 1,2,..,s) \qquad (5.35)$$

(5.35) dagi s ta sohani birlashtirish, maxsus tanlangan $c_1, c_2, .., c_s$ manfiymas parametrlarga bog'liq bo'lgan Fidlerning lokallashtirish sohasini tashkil qiladi.

Teorema 5.7.(Fidler teoremasi). Agar $c_1, c_2, .., c_s$ manfiymas sonlar shunday tanlangan bo'lib, (5.33) matritsa kuchsizlangan M matritsa bo'lsa, u holda A matritsaning har bir λ harakteristik soni s ta (5.35) sohalarning hech bo'lmaganda bittasiga tegishli bo'ladi.

Misol sifatida quyidagi 4-tartibli simmetrik matritsani qaraymiz:

$$A=\begin{Vmatrix} 0 & 4 & 1 & -1 \\ 4 & 0 & -1 & 1 \\ 1 & -1 & -1 & 15 \\ -1 & 1 & 15 & -1 \end{Vmatrix}$$

A-simmetrik matritsa bo'lgani uchun uning barcha harakteristik sonlari haqiqiy bo'ladi. Shuning uchun λ-kompleks tekislikdagi lokallashtirish sohasi o'rniga λ-haqiqiy o'qdagi sohalar kesishishidan hosil bo'lgan kesmani qarash mumkin.

I. Gershgoren sohasi quyidagi segmantdan iborat bo'lib,

$$-18 \le \lambda \le 16$$

bu segment Gershgorinning qolgan segmentlarini o'zida saqlaydi.

II. A matritsani quyidagicha 4 ta blokka ajratamiz:

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ A_{11}=\begin{Vmatrix} 0 & 4 \\ 4 & 0 \end{Vmatrix},\quad A_{22}=\begin{Vmatrix} -1 & 15 \\ 15 & -1 \end{Vmatrix},$$

$$A_{21}=A_{12}=\begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix}$$

Bu holda

$$(A_{11}-\lambda \mathrm{E}_1)^{-1}=\frac{1}{\lambda^2-16}\begin{Vmatrix} -\lambda & -4 \\ -4 & -\lambda \end{Vmatrix}$$

$$(A_{22}-\lambda \mathrm{E}_2)^{-1}=\frac{1}{(\lambda+1)^2-15^2}\begin{Vmatrix} -1-\lambda & -1-\lambda \\ -15 & -15 \end{Vmatrix}$$

R_1 va R_2 fazolarning quyidagi uch xil ko'rinishdagi normalarini qaraymiz:

a) R_1 va R_2 da kubik normalarni;

b) R_1 da kubik, R_2 da esa oktaedrli normalarni

c) R_1 da oktaedrli, R_2 da esa kubik normalarni.

a) barcha bloklar bo'yicha normalar (5.20′) formuladan aniqlanadi:

$$\|A_{12}\|=2,\quad \|A_{21}\|=2,\quad \left\|(A_{11}-\lambda E_1)^{-1}\right\|=\left\||\lambda|-4\right|,\ \left\|(A_{22}-\lambda E_2)^{-1}\right\|=\left\||\lambda+1|-15\right|$$

Gershgorinning blokli sohasi

$$\left\||\lambda|-4\right|\le 2,\quad \left\||\lambda+1|-15\right|\le 2$$

Quyidagi 4 ta intervaldan iborat bo’ladi:

$$-18 \le \lambda \le 16, -6 \le \lambda \le -2, 2 \le \lambda \le 6, 12 \le \lambda \le 16 \qquad \text{(IIa)}$$

b) bu holda $\left\|(A_{11} - \lambda E_1)^{-1}\right\|^{-1}$ va $\left\|(A_{22} - \lambda E_2)^{-1}\right\|^{-1}$ lar uchun ifodalar o’zgarmay qoladi, ammo

$$\|A_{12}\| = \max_x \frac{|x_1 - x_2|}{|x_1| + |x_0|} = 1, \quad \|A_2\| = \max_x \frac{2|x_1 - x_2|}{\max\limits_{1 \le i \le 2} |x_i|} = 4$$

Gershgorinning blokli sohasi

$$\left| |\lambda| - 4 \right| \le 1, \quad \left| |\lambda + 1| - 15 \right| \le 4$$

bo’lib, quyidagi 4 ta intervaldan iborat

$$-20 \le \lambda \le -12, -5 \le \lambda \le -3, 3 \le \lambda \le 5, 10 \le \lambda \le 18 \qquad \text{(IIb)}$$

c) bu holni avvalgi holdan farqi shuki, bunda

$$\|A_{12}\| = 1, \quad \|A_{21}\| = 4,$$

bo’ladi. Shuning uchun Gershgorin sohasi.

$$\left| |\lambda| - 4 \right| \le 4, \quad \left| |\lambda + 1| - 15 \right| \le 1$$

bo’lib, quyidagi 3 ta intervalga bo’linadi:

$$-17 \le \lambda \prec -15, \; -8 \le \lambda \le 15, \quad 13 \le \lambda \le 15 \qquad \text{(IIc)}$$

Bu sohalarning kesishmasi lokallashtirish sohasini beradi:

$$-17 \le \lambda \prec -15, \quad -5 \le \lambda \le -3, 3 \le \lambda \le 5,$$

III. Fidler kriteriyasini qo’llashda yana a), b), c) normalarni qaraymiz:

$$G = \begin{pmatrix} c_1 & -\|A_{12}\| \\ -\|A_{21}\| & c \end{pmatrix} = \begin{Vmatrix} c_1 & -2 \\ -2 & c_2 \end{Vmatrix} \qquad |G| = c_1 c_2 - 4 \ge 0$$

c_1 va c_2 larning eng kichik qiymatlarini hosil qilish maqsadida $c_1 c_2 = 4$ deb olamiz.

$$\left| |\lambda| - 4 \right| \le c_1, \quad \left| |\lambda + 1| - 15 \right| \le c_2$$

Fiddler sohasi $c_1 = c_2 = 2$ da (IIa) soha bilan, $c_1 = 1$, $c_2 = 4$ da (IIb) soha bilan, $c_1 = 4$ $c_2 = 1$ da esa (IIb) soha bilan ustma-ust tushadi. Fiddler sohasi quyidagi 4 ta intervaldan iborat bo’lib, bitta musbat parametrga bog’liq bo’ladi, chunki $c_1 = \dfrac{4}{c_\lambda}$.

$$-16-c_2 \le \lambda \le -16+c_2, \quad -4-c_1 \le \lambda \le -4+c_1, \tag{III}$$

$$4-c_1 \le \lambda \le 4+c_1, \quad 14-c_2 \le \lambda \le 14+c_2,$$

Barcha Fidler sohalari kesishmasini aniqlash mumkin. Buning uchun quyidagi miqdorlarni qaraymiz:

1) $-16-c_2 = -4+c_1,$

2) $-16+c_2 = -4-c_1,$

3) $-16+c_2 = -4+c_1,$

4) $4-c_1 = 14-c_2,$

5) $4+c_1 = 14-c_2,$

6) $4+c_1 = 14+c_2,$

$c_1 c_2 = 4$ tenglikdan foydalanib, eng kichik musbat ildizli 6 ta kvadrat tenglamani hosil qilamiz:

1) $c_2^2 - 12c_2 - 4 = 0, z_1 = -6+\sqrt{40} = 0{,}3246...,$

2) $c_2^2 - 12c_2 + 4 = 0, z_2 = 6-\sqrt{32} = 0{,}3431...,$

3) $c_1^2 - 12c_1 - 4 = 0, z_3 = z_1 = -6+\sqrt{40} = 0{,}3246...,$

4) $c_1^2 + 10c_1 - 4 = 0, z_4 = -5+\sqrt{29} = 0{,}3852...,$

5) $c_1^2 - 10c_1 + 4 = 0, z_5 = 5-\sqrt{21} = 0{,}4174...,$

6) $c_2^2 + 10c_2 - 4 = 0, z_6 = z_4 = -5+\sqrt{29} = 0{,}3852...,$

Lokallashtirish sohasi quyidagi 4 ta segmentdan iborat bo'ladi:

$$-16-z_1 \le \lambda \le -16+z_2 \ , \quad -4-z_2 \le \lambda \le -4+z_1$$

$$4-z_4 \le \lambda \le 4+z_5, \quad 14-z_5 \le \lambda \le 14+z_4$$

Mashqlar:

1. Adamar va Fidler kriteriyalaridan foydalanib quyidagi matritsalarning regulyarligini tekshiring:
2.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad \begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix}13 & 16 & 16\\ -5 & -7 & -6\\ -6 & -8 & -7\end{pmatrix} \qquad \begin{pmatrix}3 & 0 & 8\\ 3 & -1 & 6\\ -2 & 0 & -5\end{pmatrix} \qquad \begin{pmatrix}0 & 3 & 3\\ -1 & 8 & 6\\ 2 & -14 & -10\end{pmatrix}$$

$$\begin{pmatrix}4 & 5 & -2\\ -2 & -2 & 1\\ -1 & -1 & 1\end{pmatrix} \qquad \begin{pmatrix}9 & 22 & -6\\ -1 & -4 & 1\\ 8 & 16 & -5\end{pmatrix} \qquad \begin{pmatrix}8 & 30 & -14\\ -5 & -19 & 9\\ -6 & -23 & 11\end{pmatrix}$$

$$\begin{pmatrix}-2 & 8 & 6\\ -4 & 10 & 6\\ 4 & -8 & -4\end{pmatrix} \qquad \begin{pmatrix}3 & 7 & -3\\ -2 & -5 & 2\\ -4 & -10 & 3\end{pmatrix} \qquad \begin{pmatrix}-1 & 1 & 1\\ -5 & 21 & 17\\ 6 & -26 & -21\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 2\\ 3 & -3 & 6\\ 2 & -2 & 4\end{pmatrix} \qquad \begin{pmatrix}-4 & 2 & 10\\ -4 & 3 & 7\\ -3 & 1 & 7\end{pmatrix}$$

3. $A = \begin{bmatrix} 0 & 1 & 0 & i\\ 1 & 6 & 1 & 1\\ i/2 & i & 5 & 0\\ 0 & 1/2 & 1/2 & -2\end{bmatrix}$ bo'lsa, A va A^T matritsalarning uchu Gershgorin doirasini toping. Agar S=diag{1,1,1,4}, bo'lsa SAS^{-1} ni xisoblang va A matritsa $|z+2| \le \frac{1}{2}$ doirada xaqiqiy xos qiymatga ega ekanligini xosil qiling.
4. Agar $a_{jj} > \sum_k' |a_{jk}| = p_j$ j=1,2,…,n bo'lsa, u xolda A matritsa xosmas bo'lishini ko'rsating.
5. Agar A=B+C, B=diag{1,2,2} va j,k=1,2,3 uchun $|c_{jk}| \le \varepsilon < \frac{1}{6}$ bo'lsa, u xolda A matritsani $|z - 1 - c_{11}| \le 13\varepsilon^2$ doirada xos qiymati mavjud ekanligini isbotlang.
6. $\|A - B\| \ge \|A\| - \|B\|$ ekanligini isbotlang.
7. Xaqiqiy qiymatli $\sum_{i,j} |a_{ij}|$ funksiya matritsa normasi bo'lishini isbotlang.
8. Ixtiyoriy matritsa normasi uchun quyidagilarni isbotlang.

   $\|I\| \ge 1, \|A^n\| \le \|A\|^n, \ \|A^{-1}\| \ge \frac{1}{\|A\|}$

VI-BOB.
MATRITSALI TENGLAMALAR.

Bu bobda matritsalar nazariyasi va ularning tadbiqlari masalalarida uchraydigan matritsali tenglamalarning ba'zi ko'rinishlari qarab chiqiladi.

§1. $AX = XB$ tenglama

Bizga quyidagi matritsali tenglama berilgan bo'lsin:

$$AX = XB \tag{6.1}$$

bu yerda $A = \| a_{ij} \|_{i,j=1}^{m}$, $B = \| b_{kl} \|_{k,l=1}^{n}$, $X = \| x_{jk} \|$, $(j = 1,2,...,m;\ k = 1,2,...,n)$

A va B matritsalarning kompleks sonlar maydonidagi elementar bo'luvchilarini yozib chiqamiz:

$$(A): (\lambda - \lambda_1)^{p_1}, (\lambda - \lambda_2)^{p_2}, ..., (\lambda - \lambda_u)^{p_u}, (p_1 + p_2 + ... + p_u = m)$$

$$(B): (\lambda - \mu_1)^{q_1}, (\lambda - \mu_2)^{q_2}, ..., (\lambda - \mu_v)^{q_v}, (q_1 + q_2 + ... + q_u = n)$$

Bu elementlar bo'luvchilarga mos holda A va B matritsalarni quyidagicha Jordanning normal ko'rinishiga keltiramiz:

$$A = U\tilde{A}U^{-1}, \quad B = V\tilde{B}V^{-1} \tag{6.2}$$

bu yerda U va V lar mos ravishda m va n tartibli, xosmas kvadrat matritsalar, $\tilde{A}$ va $\tilde{B}$ quyidagicha Jordan ko'rinishidagi matritsalar:

$$A = \{\lambda_1 E^{(p_1)} + H^{(p_1)}, \lambda_2 E^{(p_2)} + H^{(p_2)}, ..., \lambda_u E^{(p_u)} + H^{(p_u)}\},$$

$$B = \{\mu_1 E^{(q_1)} + H^{(q_1)}, \mu_2 E^{(q_2)} + H^{(q_2)}, ..., \mu_u E^{(q_v)} + H^{(q_v)}\}. \tag{6.3}$$

(6.2) ni (6.1) ga qo'yib,

$$U\tilde{A}U^{-1}X = XV\tilde{B}V^{-1}$$

ni hosil qilamiz. Bu tenglikni chapdan U^{-1} ga o'ngdan V ga ko'paytirib,

$$AU^{-1}XV = U^{-1}XVB \tag{6.4}$$

tenglikni hosil qilamiz. Izlanayotgan X matritsa o'rniga

$$\tilde{X} = U^{-1}XV \tag{6.5}$$

matritsani kiritib, (6.4) ni quyidagicha yozamiz:

$$\tilde{A}\tilde{X} = \tilde{X}\tilde{B} \tag{6.6}$$

Biz (6.1) matritsali tenglamani xuddi shunday ko'rinishdagi (6.6) tenglama bilan almashtirdik, ammo (6.6) dagi berilgan matritsalar Jordanning normal ko'rinishiga ega.

$\widetilde{X}$ matritsani bloklarga ajratamiz:

$$\widetilde{X} = \left(X_{\alpha\beta}\right) \qquad \left(\alpha = 1,2,3,\dots,u; \beta = 1,2,3,\dots,v\right)$$

bu yerda $X_{\alpha\beta} - p_\alpha x q_\beta$ o'lchovli to'g'ri to'rtburchakli matritsa.

Blok matritsani kvazidiogonal matritsaga ko'paytirish qoidasidan foydalanib, (6.6) tenglamaning chap va o'ng tomonlarida ko'paytirish amalini bajaramiz. Natijada bu tenglama $u \cdot v$ ta matrissali tenglamalarga ajraladi:

$$\left[\lambda_\alpha E^{(p_\alpha)} + H^{(p_\alpha)}\right] X_{\alpha\beta} = X_{\alpha\beta}\left[\mu_\beta E^{(q_\beta)} + H^{(q_\beta)}\right],$$
$$\alpha = 1,2,\dots,u; \qquad \beta = 1,2,\dots,v.$$

Bularni quyidagicha yozamiz:

$$\left(\mu_\beta - \lambda_\alpha\right) X_{\alpha\beta} = H_\alpha X_{\alpha\beta} - X_{\alpha\beta} G_\beta \qquad \left(\alpha = 1,2,\dots,u; \beta = 1,2,\dots,v\right) \tag{6.7}$$

bu yerda

$$H_\alpha = H^{(p_\alpha)}, \; G_\beta = H^{(q_\beta)}, \left(\alpha = 1,2,\dots,u; \beta = 1,2,\dots,v\right) \tag{6.8}$$

(6.7) tenglamalardan birini olib qaraymiz. Bunda 2 xol bo'lishi mumkin.

1. $\lambda_\alpha \neq \mu_\beta$. (6.7) ning ikkala tomonini $\mu_\beta - \lambda_\alpha$ ga ko'paytirib,o'ng tomonidagi har bir hadda $(\mu_\beta - \lambda_\alpha)\ X_{\alpha\beta}$ ko'paytuvchini $H_\alpha X_{\alpha\beta} - X_{\alpha\beta} G_\beta$ bilan almashtiramiz.Bu jarayonni $r-1$ marta takrorlab, quyidagi tenglamani hosil qilamiz:

$$(\mu_\beta - \lambda_\alpha)^r \; X_{\alpha\beta} = \sum_{\sigma+\tau=r} \left(-1\right)^\tau H_\alpha^\sigma X_{\alpha\beta} G_\beta^\tau \tag{6.9}$$

(6.8) ga asosan

$$H_\alpha^{p_\alpha} = G_\beta^{q_\beta} = 0 \tag{6.10}$$

Agar (6.9) da $r > p_\alpha + q_p - 1$ deb olsak, u holda (6.9) ning o'ng tomonidagi yig'indining har bir hadida

$$\sigma \geq p_\alpha \; , r \geq q_\alpha$$

munosabatlarning hech bo’lmaganda bittasi bajarilib, (6.10) ga asosan $H_\alpha^\sigma = 0$ yoki $G_\beta^\tau = 0$ bo’ladi. Bundan $\mu_\beta \neq \lambda_\alpha$ bo’lgani uchun quyidagini topamiz:

$$X_{\alpha\beta} = 0 \tag{6.11}$$

2. $\lambda_\alpha = \mu_\beta$. Bu holda (6.7) dan quyidagi hosil bo’ladi:

$$H_\alpha X_{\alpha\beta} = X_{\alpha\beta} G_\beta \tag{6.12}$$

H_α va G_β matritsalarning diogonal ostidagi birinchi elementlari birga teng bo’lib, qolgan barcha elementlari nollardan iborat. H_α va G_β matritsalar strukturasining bu spetsifikasini hisobga olib,

$$X_{\alpha\beta} = \left\| \xi_{ik} \right\| \qquad \left(i = 1,2,\ldots, p_\alpha; k = 1,2,\ldots, q_\beta\right)$$

deb olib, (6.12) matritsali tenglamani unga ekvivalent bo’lgan quyidagi skalyar munosabatlar sistemasi bilan almashtiramiz:

$$\xi_{i+1,k} = \xi_{i,k-1} \qquad \left(\xi_{i0} = \xi_{p\alpha} = 0; i = 1,2,\ldots, p_\alpha; k = 1,2,\ldots, q_\beta\right) \tag{6.13}$$

(6.13) tengliklar quyidagini bildiradi.

1) $X_{\alpha\beta}$ matritsaning diogonaliga paralel bo’lgan chiziqqa o’zaro teng elementlar yotadi.

2) $\xi_{21} = \xi_{31} = \cdots = \xi_{p_\alpha 1} = \xi_{p_\alpha 2} = \cdots = \xi_{p_\alpha q_\beta - 1} = 0$

$p_\alpha = q_\beta$ bo’lsin . Bu holda $X_{\alpha\beta}$ kvadrat matritsa bo’lib u 1) va 2) shartga ko’ra quyidagicha bo’ladi :

$$X_{\alpha\beta} = \left\| \begin{matrix} c_{\alpha\beta} & c'_{\alpha\beta} & \ldots & c_{\alpha\beta}^{(p_\alpha - 1)} \\ 0 & c_{\alpha\beta} & \ldots & . \\ \ldots & \ldots & \ldots & . \\ \ldots & \ldots & \ldots & c'_{\alpha\beta} \\ 0 & \ldots & 0 & c_{\alpha\beta} \end{matrix} \right\| = T_{p_\alpha} \tag{6.14}$$

bu yerda $c_{\alpha\beta}$, $c'_{\alpha\beta}$, ... , $c_{\alpha\beta}^{(p_\alpha)}$ -ixtiyoriy parametrlar .

$p_\alpha < q_\beta$ da

$$X_{\alpha\beta} = \left(\overbrace{0}^{q_\beta - p_\alpha} , \mathrm{T}_{p_\alpha} \right) \tag{6.15}$$

bo’lib , $p_\alpha > q_\beta$ da esa,

$$X_{\alpha\beta} = \begin{pmatrix} & T_{p_\alpha} \\ 0 & \}\, p_\alpha - q_\beta ta \end{pmatrix} \qquad (6.16)$$

(6.14), (6.15) va (6.16) matritsalar to'g'ri yuqori uchburchak fo'rmaga ega deyiladi. $X_{\alpha\beta}$ dagi ixtiyoriy parametrlar soni p_a va q_β sonlarning kichigiga teng. Quyida keltirilgan sxema $X_{\alpha\beta}$ matritsani $\lambda_\alpha = \mu_\beta$ dagi strukturasini ko'rsatadi, (bu yerda ixtiyoriy parametrlar a,b,c,d orqali belgilangan) :

$$X_{\alpha\beta} = \begin{Vmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{Vmatrix}, \quad (p_\alpha = q_\alpha = 4),$$

$$X_{\alpha\beta} = \begin{Vmatrix} 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{Vmatrix}, \quad (p_\alpha = 3, q_\alpha = 5),$$

$$X_{\alpha\beta} = \begin{Vmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \qquad (p_\alpha = 5, q_\alpha = 3).$$

$\tilde{X}$ matritsadagi ixtiyoriy parametrlar sonini aniqlash uchun $d_{\alpha\beta}(\lambda)$ bilan $(\lambda - \lambda_0)^{p_\alpha}$ va $(\lambda - \mu)^{q_\beta}$ elementar bo'luvchilarning eng katta bo'luvchisini, $\delta_{\alpha\beta}$ bilan esa $d_{\alpha\beta}(\lambda)$ ko'phadning darajasini belgilaymiz. $\lambda_\alpha \neq \mu_\beta$ holda $\delta_{\alpha\beta} = 0$, $\lambda_\alpha = \mu_\beta$ xolda esa, $\delta_{\alpha\beta} = \min(p_{\alpha,} q_\beta)$ bo'ladi. Shunday qilib, xar ikkala xolda ham $X_{\alpha\beta}$ dagi ixtiyoriy parametrlar soni $\delta_{\alpha\beta}$ ga teng bo'lib, $\tilde{X}$ dagi ixtiyoriy parametrlar soni quyidagi formula bilan aniqlanadi:

$$N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta}$$

Bu olingan natijalarni quyidagi teorema ko'rinishida ifodalash mumkin:

Teorema 6.1. $Ax = xB$,

bu yerda

$$A = \| a_{ik} \|_{i,k=1}^{m} = U\tilde{A}U^{-1} = U\{\lambda_1 E^{(p_1)} + H^{(p_1)}, \dots, \lambda_n E^{(p_n)} + H^{(p_u)}\}U^{-1}$$

$$B = \| b_{ik} \|_{ik=1}^{n} = V\tilde{B}V^{-1} = V\{\mu_1 E^{(q_1)} + H^{(q_1)}, \dots, \mu_v E^{(q_v)} + H^{(q_v)}\}V^{-1},$$

tenglamaning yechimi

$$X = UX_{\tilde{A}\tilde{B}}V^{-1} \tag{6.17}$$

formula bilan berilib, $X_{\tilde{A}\tilde{B}} - \tilde{A}\tilde{X} = \tilde{X}\tilde{B}$ tenglamaning umumiy yechimi bo’ladi va $X_{\tilde{A}\tilde{B}}$ quyidagicha bloklarga ajraladi:

$X_{\tilde{A}\tilde{B}} = (X_{\alpha\beta}),$ $X_{\alpha\beta} - p_\alpha x q_\beta$ o’lchovli matritsa $(\alpha = 1,2,...,u; \beta = 1,2,...,v)$, agar $\lambda_\alpha \neq \mu_\beta$ bo’lsa , $X_{\alpha\beta} = 0$ bo’lib, $\lambda_\alpha = \mu_\beta$ da $X_{\alpha\beta}$ ixtiyoriy to’g’ri yuqori uchburchak matritsa bo’ladi.

$X_{\tilde{A}\tilde{B}}$ shuningdek, X N ta ixtiyoriy $c_1, c_2, ..., c_n$ parametrlar orqali chiziqli bog’lanadi, ya’ni

$$X = \sum_{i=1}^{n} C_i X_i \tag{6.18}$$

bu yerda

$$N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta} , \tag{6.19}$$

bu yerda $\delta_{\alpha\beta} - (\lambda - \lambda_\alpha)^{p_\alpha}$ va $(\lambda - \mu_\beta)^{q_\beta}$ larning eng katta umumiy bo’luvchisining darajasi.

Agar A va B matritsalar umumiy xarakteristik songa ega bo’lmasa, u holda $N = 0,\ X = 0,$ yani (6.1) tenglama faqat $X = 0$ nolli yechimga ega bo’ladi.

(6.1) matritsali tenglama $m \bullet n$ ta chiziqli bir jinsli tenglamalar sistemasiga ekvivalent bo’lib, x_{jk} $(j = 1,2, ... , m; k = 1,2, ..., n)$ noma’lumlar izlanayotgan X matritsaning elementlari bo’ladi:

$$\sum_{j=1}^{m} a_{ij} x_{jk} = \sum_{k=1}^{n} x_{ik} b_{kk} \qquad (i = 1,2,...,m; \quad k = 1,2,...,n) \tag{6.20}$$

§ 2. $A = B$ bo’lgan hususiy hol. O’rin almashinuvchi matritsalar.

(6.1) tenglamaning quyidagicha hususiy holini qaraymiz:

$$Ax = xA \tag{6.21}$$

Bu yerda $A = \| a_{ik} \|_{i,k=1}^{m}$ -berilgan, $X = \| x_{ik} \|_{i,k=1}^{n}$-izlanayotgan matritsa. Bu holda Frobenusning quyidagi masalasiga kelamiz:berilgan A matritsa bilan o’rin almashinuvchi bo’lgan barcha X matritsalarni aniqlang.

A martisani quyidagicha Jordonning normal formasiga keltiramiz:

$$A = U\tilde{A}U^{-1} = U\left\{\lambda_1 E^{(p_1)} + H^{(p_1)}, \ldots, \lambda_u E^{(p_u)} + H^{(p_u)}\right\}U^{-1} \tag{6.22}$$

U holda (6.17) formulada $V = U, \tilde{B} = \tilde{A}$ deb olib, $X_{\tilde{A}\tilde{A}}$ ni $X_{\tilde{A}}$ orqali belgilab, (6.21) tenglamaning barcha yechimlarini, ya'ni A matritsa o'rin almashinuvchi barcha matritsalarni quyidagicha ko'rinishda hosil qilamiz:

$$X = UX_{\tilde{A}}U^{-1} \tag{6.23}$$

bu yerda $X_{\tilde{A}}$ orqali $\tilde{A}$ matritsa bilan o'rin almashinuvchi barcha martisalarni belgilangan .Avvalgi paragrfdagi kabi, $X_{\tilde{A}}$ matritsa U^2 blo'klarga ajraladi:

$$X_{\tilde{A}} = \left(X_{\alpha\beta}\right)_{\alpha,\beta=1}^{u}$$

Bu ajralish $\tilde{A}$ Jordan matritsani bloklarga ajralishiga mos bo'lib, $X_{\alpha\beta}$ matritsa $\lambda_\alpha \neq \lambda_\beta$ da nol matritsa, $\lambda_\alpha = \lambda_\beta$ da ixtiyoriy to'g'ri yuqori uchburchak matritsa bo'ladi.

Misol uchun A matritsaning elementar bo'luvchilari

$$(\lambda - \lambda_1)^4, (\lambda - \lambda_1)^3, (\lambda - \lambda_2)^2, (\lambda - \lambda_2)$$

bo'lganda $X_{\tilde{A}}$ matritsaning elementlarini yozamiz.

Bu holda $X_{\tilde{A}}$ matritsa quyidagicha ko'rinishga ega bo'ladi:

$$\begin{Vmatrix}
a & b & c & d & e & f & g & & 0 & 0 & 0 \\
0 & a & b & c & 0 & e & f & & 0 & 0 & 0 \\
0 & 0 & a & b & 0 & 0 & e & & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & n & k & l & m & p & q & & 0 & 0 & 0 \\
0 & 0 & n & k & 0 & m & p & & v & 0 & 0 \\
0 & 0 & 0 & n & 0 & 0 & m & & 0 & 0 & 0 \\
 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & r & s & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & r & 0 \\
 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z
\end{Vmatrix}$$

a,b,c,d,e,f,g,h,k,l,m,p,q,r,s,t,w,z-ixtiyoriy parametrlar.

$X_{\tilde{A}}$ matritsadagi parametrlar soni $N=\sum_{\alpha,\beta=1}^{u}\delta_{\alpha\beta}$ bo'lib, $\delta_{\alpha\beta}$ $(\lambda-\lambda_\alpha)^{p_\alpha}$ va $(\lambda-\lambda_\beta)^{q_\beta}$ ko'phadning eng katta umumiy bo'luvchisining darajasini bildiradi.

A matritsaning $i_1(\lambda), i_2(\lambda), \ldots, i_t(\lambda), i_{t+1}(\lambda)=\ldots=i_n(\lambda)=1$ invariant ko'phadlarni qaraymiz. Bu ko'phadlarning darajalarini mos ravishda $n_1 \geq n_2 \geq \ldots \geq n_t > n_{t+1} = \ldots = 0$ lar orqali belgilaymiz. Ma'lumki, har bir invariant ko'phad bir nechta o'zaro tub bo'lgan elementar bo'luvchilar ko'paytmasidan iborat bo'ladi, u holda N uchun formulani quyidagicha yozish mumkin:

$$N=\sum_{g,j=1}^{t}\gamma_{gj} \tag{6.24}$$

bu yerda $\gamma_{gj}-i_g(\lambda)$ va $i_j(\lambda)$ ko'phadlar eng katta umumiy bo'luvchisining darajasi $(g, j=1,2,\ldots,t)$. Ammo $i_g(\lambda)$ va $i_j(\lambda)$ ko'phadlarning eng katta bo'luvchisi ularning biri bo'ladi, shuning uchun $\gamma_{gj}=\min(n_g, n_j)$. Bundan quyidagini hosil qilamiz:

$$N=n_1+3n_2+\ldots+(2t-1)n_t$$

N soni A matritsa bilan o'rin almashinuvch chiziqli bog'liq bo'lmagan matritsa soni bo'ladi. Biz quyidagi teoremaga keldik:

Teorema 6.2. $A=\| a_{ik} \|_{i,k=1}^{n}$ matritsa bilan o'rin almashinuvch bo'lgan, chiziqli bog'liqmas matritsalar soni quyidagi fo'rmula bilan aniqlanadi:

$$N=n_1+3n_2+\ldots+(2t-1)n_t \tag{6.25}$$

bu yerda $n_1, n_2, ..., n_t$ -mos ravishda $i_1(\lambda), i_2(\lambda), \ldots, i_t(\lambda)$ o'zgarmas bo'lmagan, A matritsaning invariant ko'phadlari darajalari.

$$n=n_1+n_2+\ldots+n_t \tag{6.26}$$

(6.25) va (6.26) dan:

$$\mathrm{N} \geq n, \tag{6.27}$$

bo'lib, tenglik belgisi $t=1$ da, ya'ni A matritsaning barcha elementar bo'luvchilari juft-jufti bilan o'zaro tub bo'lganda bajariladi.

$g(\lambda)$- λ ning qandaydir ko'phadi bo'lsin. U holda $g(A)$ matritsada A matritsa bilan o'rin almashinuvchi bo'ladi. Teskari savol tug'iladi:Qanday holda ihtiyoriy matritsa A matritsa bilan o'rin almashinuvchi bo'lib, A matritsaning ko'phadi sifatida tasvirlanadi.Bu holda $E, A, A^2, \ldots, A^{n_t - 1}$ chiziqli bog'liqmas matritsalar chiziqli kombinasiyasidan iborat bo'lgan matritsa A matritsa bilan o'rin almashinuvchi bo'ladi.

Qaralayotgan holda $N = n_t \le n$ bo'lib, (6.27) ga asosan $N = n_t = n$ ni hosil qilamiz. Shunday qilib, quyidagini hosil qildik.

Natija. A matritsa bilan o'rin almashinuvchi bo'lgan barcha matritsalar faqat va faqat $n_1 = n$ ya'ni A matritsaning barcha elementar bo'luvchilari juft-jufti bilan o'zaro tub bo'lgandagina ular A matritsaning ko'phadi sifatida tasvirlanadi.

Natija. A matritsa bilan o'rin almashinuvchi bo'lgan barcha matritsalar faqat va faqat shu holda, qachonki $n_1 = n$ ya'ni $\lambda E - A$ ning barcha elementar bo'luvchilari o'zaro tub bo'lganda bitta va faqat bitta C matritsaning ko'phadi sifatida tasvirlanadi. Bu holda A matritsa bilan o'rin almashinuvchi barcha matritsalar A matritsaning ko'phadi sifatida tasvirlanadi.

Teorema 6.3. Agar ikki A va B matritsalar o'rin almashinuvchi bo'lib, ularning biri, masalan, A quyidagicha kvazidiogonal ko'rinishga ega bo'lsa,

$$A = (A_1, A_2), \qquad (6.28)$$

bu yerda A_1 va A_2 umumiy harakteristik songa ega emas, u holda ikkinchi B matritsa ham xuddi shunday kvazidiognal ko'rinishga ega bo'ladi:

$$B = (B_1 B_2) \qquad (6.29)$$

Isboti. B matritsani (6.28) kvazidiognal ko'rinishga mos bloklarga ajratamiz:

$$B = \begin{pmatrix} B_1 & x \\ y & B_2 \end{pmatrix},$$

B_1, B_2 mos ravishda A_1, A_2 bilan bir hil o'lchovli.

$AB = BA$ ekanligidan, quyidagi to'rta matritsali tengliklarni hosil qilamiz:

$$1.\ A_1B_1 = B_1A_1, \quad 2.\ A_1x = xA_2, \quad 3.\ A_1y = yA_1, \quad 4.\ A_2B_2 = B_2A_2; \quad (6.30)$$

A_1 va A_2 lar umumiy harakteristik songa ega bo'lmaganliklari uchun (6.30) dagi 2. va 3. tenglamlar $x = 0$ va $y = 0$ yechimga ega. (6.30) ning 1. va 4. tengliklaridan A_1 va B_1 , A_2 va B_2 matritsalar o'zaro o'rin almashinuvchi ekanligi kelib chiqadi.

Teorema-6.3´. Agar $R^n = I_1^{n_1} + I_2^{n_2}$ bo'lib, $I_1^{n_1}$ va $I_2^{n_2}$ $\overline{A}$-operatorga nisbatan invariant qism fazolar bo'lsa va bu qism fazolar minimal ko'phadlar o'zaro tub bo'lsa, u holda bu $I_1^{n_1}$ va $I_2^{n_2}$ qism fazolar $\overline{A}$-operator bilan o'rin almashinuvchi bo'lgan $\overline{B}$ ixtiyoriy chiziqli operatorga nisbatan ham invariant qism fazolar bo'ladi.

Natija . Oddiy strukturali o'rin almashinuvchi matritsalarni bir vaqtda bitta o'xshash almashtirish bilan diogonal ko'rinishga keltirish mumkin.

§ 3. $Ax - xB = C$ tenglama.

Quyidagi matritsali tenglama berilgan bo'lsin.

$$Ax - xB = C\,, \tag{6.31}$$

bu yerda $A = \| a_{ik} \|_{i,k=1}^{m}$, $B = \| b_{kl} \|_{k,l=1}^{n}$ mos ravishda m va n tartibli berilgan kvadrat matritsalar, $C = \left\| c_{j,k} \right\|$- berilgan matritsa, $X = \left\| x_{jk} \right\| - m \times n$ o'lchovli to'g'ri to'rtburchakli, izlanayotgan matritsa. (6.31) tenglama X matritsaning elementlariga nisbatan $m \cdot n$ ta skalyar tenglamalar sistemasiga ekvivalent:

$$\sum_{j=1}^{m} a_{ij} x_{jk} - \sum_{l=1}^{n} x_{ie} b_{ek} = c_{ik} \quad (i = 1,2,\dots,m; k = 1,2,\dots,n).$$

Bunga mos bir jinsli tenglamalar sistemasi

$$\sum_{j=1}^{m} a_{ij} x_{jk} - \sum_{l=1}^{n} x_{ie} b_{ek} = 0 \quad (i = 1,2,\dots,m; k = 1,2,\dots,n).$$

bo'lib, matritsalar ko'rinishda quyidagicha yoziladi:

$$Ax - xB = C \tag{6.32}$$

Shunday qilib, agar (6.32) tenglama faqat nolli yechimga ega bo'lsa, u holda (6.31) tenglama yagona yechimga ega.Ammo §1 da ko'rsatilganidek, (6.32) tenglama faqat va faqat A va B matritsalar umumiy harakteristik songa ega bo'lmaganlardagina nolli yechimga ega bo'ladi. Demak, agar A va B matritsalar umumiy xarakteristik songa ega bo'lmasalar, u holda (6.31) tenglama yagona echimga ega bo'ladi, agar A va B matritsalar umumiy harakteristik songa ega bo'lsalar, u holda ozod had C matritsaga bog'liq holda ikki hol bo'ldi: (6.31) tenglama umuman yechimga ega emas, yoki quyidagi formula bilan aniqlanuvchi cheksiz ko'p yechimga ega: $X = X_0 + X_1$, bu yerda X_0-(6.31) tenglamaning hususiy yuchimi, X_1 esa (6.32) tenglama umumiy yechimi.

§4 . $f(x)=0$ skalyar tenglama.

Avval quyidagi tenglamani qaraymiz:

$$g(X)=0,$$

bu yerda $g(\lambda)=(\lambda-\lambda_1)^{\alpha_1}(\lambda-\lambda_2)^{\alpha_2}\dots(\lambda-\lambda_k)^{\alpha_k}-\lambda$ ning berilgan ko'phadi, X izlanayotgan $n-$tartibli kvdrat matritsa.

X matritsaning minimal ko'phadi, ya'ni birinchi $i_1(\lambda)$ invariant ko'phad $g(\lambda)$ ning bo'luvchisi bo'lishi kerak,u holda X matritsaning elementar bo'luvchilari quyidagi ko'rinishga ega bo'lishi kerak:

$$\left(\lambda-\lambda_{i_1}\right)^{p_i}\left(\lambda-\lambda_{i_2}\right)^{p_{i2}}\dots\left(\lambda-\lambda_{i_\lambda}\right)^{p_{i_\gamma}}\begin{pmatrix} i_1, i_2, \dots, i_\gamma = 1,2,\dots,h, \\ p_{i_1} \le p_{i_2} \le \dots \le p_{i_h} \le a_{i_n}, \\ p_{i_1}+p_{i_2}+\dots+p_{i_\gamma}=n \end{pmatrix},$$

$i_1, i_2, \dots, i_\gamma$ indekslar ichida o'zaro tenglari ham bo'lishi mumkin, X izlanayotgan matritsaning berilgan tartibi.

Izlanayotgan X matritsa quyidagi ko'rinishda tasvirlanadi:

$$X = T\left\{\lambda_{i_1}E^{(p_{i_1})}+H^{p_{i_1}},\dots,\lambda_{i_\gamma}E^{(p_{i_\gamma})}+H^{(p_{i_\gamma})}\right\}T^{-1} \qquad (6.34)$$

bu yerda $T-n-$tartibli ixtiyoriy xosmas matritsa, (6.33) tenglamaning yechimlar to'plamidagi berilgan tartibli izlanayotgan matritsa (6.34) formula bo'yicha chekli sondagi o'zaro o'hshash matritsalarga yoyiladi.

Misol 1. Quyidagi tenglama berilgan bo'lsin.

$$X^m = 0 \qquad (6.35)$$

Agar matritsaning qandaydir darajasi nolga teng bo'lsa, u holda matritsa nil'potent matritsa deyiladi.Darajasi nolga teng bo'lgan matritsaning eng kichik daraja ko'rsatkichi uning nil'potentlik indeksi deyiladi.

(6.35) tenglamaning yechimi nil'potentlik indeksi $\mu \le m$ bo'lgan barcha nil'potent matritsalar bo'lib, barcha $n-$tartbli yechimlar quyidagicha ifodalanadi:

$$X = T\left\{H^{(p_1)}, H^{(p_2)}, \dots, H^{(p_\gamma)}\right\}T^{-1} \quad \begin{pmatrix} p_1, p_2, \dots, p_\gamma \le m \\ p_1 + p_2, + \dots, + p_\gamma = n \end{pmatrix}, \qquad (6.36)$$

bu yerda T -ixtiyoriy xosmas matritsa.

Misol 2: Quyidagi tenglamani qaraymiz:

$$X^2 = X \qquad (6.37)$$

Bunday tenglamani qanoatlantiruvchi matritasalar idempotent matritsalar deyiladi. Idempotent matritasalarning elementar bo'luvchilari λ yoki $\lambda - 1$ bo'ladi.Shuning uchun idempotent matritsalarni 0 yoki 1 xarakteristik sonli oddi strukturali (yani diogonal ko'rinishga keltiriladigan) matritsa sifatida aniqlash mumkin. Berilgan $n-$tartbli barcha idempotent matritsalar quyidagi ko'rinishga ega:

$$x = T\underbrace{\{1,1,\dots,1,0,\dots 0\}}_{n\,ta}T^{-1} \qquad (6.38)$$

bu yerda T -ixtiyoriy xosmas matritsa.

Eng quyidagi umumiy tenglamani qaraymiz:

$$f(X) = 0, \qquad (6.39)$$

bu yerda $f(\lambda) - \lambda$ argumentli ko'mpleks tekslikning qandaydir G sohasidagi regulyar funksiya.Izlanayotgan $X = \|x_{ik}\|_{i,k=1}^{n}$ yechimdan uning xarakteristik sonlari G sohasida yotishini talab qilamiz.

$f(\lambda)$ funksiyaning G sohasida yotuvchi barcha nollari va ularning karralilarini yozamiz:

$$\lambda_1, \lambda_2, \dots,$$
$$a_1, a_2, \dots$$

Avvalgi holdagi kabi X matritsaning har bir elementar bo'luvchisi

$$(\lambda-\lambda_i)^{p_i} \qquad (p_i \le a_i)$$

ko'rinishida bo'lishi kerak,shunuing uchun.

$$X = T\left\{\lambda_{i1}E^{(p_{i1})} + H^{(p_{i1})},\ldots,\lambda_{i_\gamma}E^{(p_{i\gamma})} + H^{(p_{i_\gamma})}\right\}T^{-1} \tag{6.40}$$

$$\begin{pmatrix} i_1, i_2, \ldots, i_\gamma = 1,2,\ldots, p_{i_1} \le a_{i_2}, p_{i_h} \le a_{i_n}, \ldots, p_{i_\gamma} \le a_{i_\gamma}; \\ p_{i_1} + p_{i_2} + \ldots + p_{i_\gamma} = n \end{pmatrix},$$

bu yerda T -ixtiyoriy xosmas matritsa.

§5. Matritsali ko'phadli tenglamalar.

Quyidagi tenglamani qaraymiz:

$$A_0x^m + A_1x^{m-1} + \ldots + A_m = 0, \tag{6.41}$$

$$y^mA_0 + y^{m-1}A_1 + \ldots + A_m = 0 \tag{6.42}$$

bu yerda $A_0, A_1, \ldots, A_m$ -berilganlar, x va y izlanayotgan $n-$tartibli kvadrat matritsalar. Avvalgi paragrafdagi (6.33) tenglama (6.41), (6.42) tenglamalarning xususiy holi bo'lib, $A_i = \alpha_i E, \alpha_i, i = 1,2,\ldots,m$ sonlar bo'lganda (6.41), (6.42) dan (6.33) kelib chiqadi.

Quyidagi teorema (6.41),(6.42) va (6.33) tenglamalar o'rtasidagi bog'lanishni o'rnatadi.

Teorema 6.4.(6.41) va (6.42) tenglamalarning har bir yechimi

$$g(x) = 0 \tag{6.43}$$

bu yerda

$$g(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \ldots + A_m. \tag{6.44}$$

skalyar tenglamaning qanoatlantiradi .

Isbot. $F(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \ldots + A_m$ matritsali ko'phadni qaraymiz. U xolda (6.41) va (6.42) tenglamalar $F(x) = 0 \quad \hat{F}(y) = 0$ ko'rinishda yoziladi.

Umumlashgan Bezu teoremasiga asosan, agar X va Y bu tenglamalarning yechimi bo'lsa, $F(\lambda)$ ko'phad chapdan $\lambda E - X$ ga o'ngdan $\lambda E - Y$ da bo'linadi:

$$F(\lambda) = Q(\lambda)(\lambda E - X) = (\lambda E - Y)Q_1(\lambda)$$

Bundan,

$$g(\lambda) = |F(\lambda)| = |Q(\lambda)|\Delta(\lambda) = |Q(\lambda)|\Delta_1(\lambda) \tag{6.45}$$

bu yerda $\Delta(\lambda)=|\lambda E-X|$, $\Delta_1(\lambda)=|\lambda E-Y|$ lar mos ravishda X va Y matritsalarning xarakteristik ko'phadi. Gamelton – Keli teoremasiga asosan

$$\Delta(\mathrm{X})=0,\ \Delta_1(Y)=0$$

Shuning uchun (6.45) dan $g(\mathrm{X})=g(y)=0$ kelib chiqadi .

Biz (6.41) tenglamaning har bir yechimi darajasi $m\cdot n$ dan katta bo'lmagan

$$g(\lambda)=0$$

skalyar tenglamani qanoatlantirishni isbotladik. Ammo bu tenglamani berilgan $n-$tartibli matritsali yechimlar to'plami o'zaro o'xshash bo'lgan matritsalarning chekli sondagi sinfidan iborat bo'ladi . Shuning uchun (6.41) tenglamaning barcha yechimlarini quyidagi ko'rinishdagi matritsalar orasidan izlash kerak:

$$T_i D_i T_i^{-1}, \tag{6.46}$$

bu yerda D_i-ma'lum matritsa bo'lib, uni normal Jordan formaga ega deb hisoblash mumkin; T_i-ixtiyoriy n-tartibli xosmas matritsa, $i=1,2,,...,n.$ (6.46) matritsani (6.41) dagi X ning o'rniga qo'yib T_i ni shunday tanlaymizki, unda (6.41) tenglama qanoatlantirilsin. Har bir T_i uchun quyidagicha chiziqli tenglamani hosil qilamiz:

$$A_0 T_i D_i^m + A_1 T_i D_i^{m-1} + ... + A_m T_i = 0 \qquad (i=1,2,\ ....\ ,n) \tag{6.47}$$

(6.47) tenglamanidagi T_i yechimni topish uchun taklif qilinadigan yagona usul shundan iboratki, matritsani tenglamani T_i matritsa elementlariga nisbatan bir jinsli chiziqli tenglamalar sistemasi bilan almashtirishdir. (6.47) tenglamaning har bir T_i yechimini (6.46) ga qo'yib, (6.41) tenglamaning yechimini hosil qilamiz. Xuddi shunday mulohazalarni (6.42) tenglama uchun ham yuritish mumkin.

Ko'rinib turibdiki, Gamelton Keli teoremasi Teorema-6.4 ning xususiy xoli bo'lib, ixtiyoriy A kvadrat matritsa

$$\lambda E-A=0$$

tenglamani qanoatlantiradi. Shuning uchun teorema-6.4 ga asosan

$$\Delta(A)=\left|\lambda E - A\right| = 0.$$

Teorema 6.4 ni quyidagicha umumlashtirish mumkin:

Teorema 6.5.(Fillips teoremasi).Agar juft-jufti bilan o'zaro o'rin almashinuvchi bo'lgan n-tartibli $X_0, X_1,..., X_m$ matritsalar,

$$A_0 X_0 + A_1 X_1 + ... + A_m X_m = 0 \tag{6.48}$$

(bu yerda $A_0, A_1,..., A_m$-berilgan, n-tartibli kvadrat matritsalar) matritsali tenglamani qanoatlantirsa, u holda bu $X_0, X_1,..., X_m$ matritsalar quyidagi skalyar tenglamani qanoatlantiradi

$$g(X_0, X_1,..., X_m) = 0\ , \tag{6.49}$$

bu yerda

$$g(\xi_0, \xi_1,..., \xi_m) = \left|A_0\xi_0 + ... + A_m\xi_m\right| \tag{6.50}$$

Isboti. $F(\xi_0, \xi_1,..., \xi_m) = \left\| f_{ik}(\xi_0, \xi_1,..., \xi_m)\right\|_{i,k=1}^{n} = A_0\xi_0 + A_1\xi_1 + ... + A_m\xi_m$ deb olamiz, bu yerda $\xi_0, \xi_1,..., \xi_m$-skalyar o'zgaruvchilar bo'lib, $f_{ik}(\xi_0, \xi_1,..., \xi_m)$-shu skalyar o'zgaruvchilarga nisbatan chiziqli forma (i,k=1,2,...,n).

$\hat{F}(\xi_0, \xi_1,..., \xi_m) = \left\| \hat{f}_{ik}(\xi_0, \xi_1,..., \xi_m)\right\|_{i,k=1}^{n}$ orqali F matritsa uchun yopishgan matritsani belgilaymiz. Bu yerda $\hat{f}_{i,k}$ $\left|F(\xi_0, \xi_1,..., \xi_m)\right|$ aniqlovchidagi $f_{k,i}$ elementning algebraik to'ldiruvchisi (i,k=1,2,...,n). U holda $\hat{F}$ matritsaning har bir $\hat{f}_{i,k}$ (i,k=1,2,...,n) elementi $\xi_0, \xi_1,..., \xi_m$ larga nisbatan $m-1$ darajali bir jinsli ko'phad bo'ladi, shuning uchun $\hat{F}$ ni quyidagi ko'rinishda yozish mumkin:

$$\hat{F} = \sum_{j_0 + j_1 + ... + j_m} = n - 1^{F_{j_0, j_1,..., j_m}\xi_0^{j_0}\xi_1^{j_1}...\xi_m^{j_m}}$$

bu yerda $F_{j_0, j_1,..., j_m}$-qandaydir n-tartibli o'zgarmas matritsa.

$\hat{F}$ matritsaning ta'rifidan, quyidagi ayniyat kelib chiqadi:

$$\hat{F}F = g(\xi_0, \xi_1,..., \xi_m)E$$

Bu ayniyatni quyidagicha yozamiz:

$$\hat{F} = \sum_{j_0 + j_1 + ... + j_m} =_{n-1} F_{j_0, j_1,..., j_m}(A_0\xi_0 + A_1\xi_1 + A_m\xi_m)\xi_0^{j_0}\xi_1^{j_1}...\xi_m^{j_m} = g(\xi_0, \xi_1,..., \xi_m)E \tag{6.51}$$

(6.51) ayniyatning chap tomonidan o'ng tomoniga o'tish qavslarni ochish va o'xshash hadlarga keltirish yo'li bilan amalga oshiriladi. Bunda $\xi_0,\xi_1,...,\xi_m$ larni o'zaro o'rinlarini almashtirishga to'g'ri kelib, bularni A_i va $F_{j_0,j_1,...,j_m}$ matritsali koeffisientlar bilan o'rinlarini almashtirmaslikka to'g'ri keladi. Shuning uchun (6.51) o'z kuchida qoladi, agar $\xi_0,\xi_1,...,\xi_m$ larni o'zaro o'rin almashinuvchi $X_0,X_1,...,X_m$ matritsalar bilan almashtirsak:

$$\sum_{j_0+j_1+...+j_m} =_{n-1} F_{j_0,j_1,...,j_m}(A_0X_0+A_1X_1+A_mX_m)X_0^{j_0}X_1^{j_1}...X_m^{j_m}=g(X_0,X_1,...,X_m) \quad (6.52)$$

Ammo bunda $A_0X_0+A_1X_1+A_mX_m=0$ shart bajarilishi kerak bo'ladi. U holda (6.52) dan $g(X_0,X_1,...,X_m)=0$ ni xosil qilamiz.

Eslatma1. Agar (6.48) tenglama

$$A_0X_0+A_1X_1+A_mX_m=0 \quad (6.53)$$

tenglam bilan almashtirilsa, teorema 6.5 o'z kuchida qoladi.

Haqiqatan, teorema6.5 ni

$$A^T{}_0X_0+A^T{}_1X_1+A^T{}_mX_m=0$$

tenglamaga qo'llash mumkin, so'ngra bu tenglamadan hadlab, transponirlangan matritsaga o'tih mumkin.

Eslatma 2. Agar $X_0,X_1,...,X_m$ lar sifatida $X^m,X^{m-1},...,X,E$ larni olsak, Teorema 6.4 , Teorema 6.5 ning xususiy holi sifatida kelib chiqadi.

§6. Hosmas matritsadan m-darajali ildiz chiqarish.

Quyidagi tenglamani qaraymiz:

$$X^m=A \quad (6.54)$$

bu yerda A-berilgan, X-izlanayotgan n-tartibli matritsalar, m-butun musbat son.

Bu paragrafda $|A|\neq 0$ bo'lgan xolni qaraymiz. Bu holda A matritsaning barcha xarakteristik sonlari noldan farqli bo'ladi.

A matritsaning elementa bo'luvchilarini

$$(\lambda-\lambda_1)^{p_1},(\lambda-\lambda_2)^{p_2},...,(\lambda-\lambda_u)^{p_u} \quad (6.55)$$

lar orqali belgilab, A matritsani quyidagicha Jordan formasiga keltiramiz:

$$A = U\tilde{A}U^{-1} = U(\lambda_1 E_1 + H_1, ..., \lambda_u E_u + H_u)U^{-1} \qquad (6.56)$$

Izlanayotgan X matritsaning xarakteristik sonlarini m-darajali A matritsaning xarakteristik soniga teng bo'lgani uchun X matritsaning xarakteristik sonlari ham noldan farqli bo'ladi. Shuning uchun bu xarakteristik sonlarda $f(\lambda) = \lambda^m$ dan olingan hosila nolga aylanmaydi.Ammo bu holda X matritsaning elementar bo'luvchilari X matritsani m-darajaga ko'targanda yoyilmaydi. Bundan kelib chiqadiki, X matritsaning elementar bo'luvchilari quyidagilar bo'ladi:

$$(\lambda - \xi_1)^{p_1}, (\lambda - \xi_2)^{p_2}, ..., (\lambda - \xi_u)^{p_u} \qquad (6.57)$$

bu yerda $\xi_j^m = \lambda_j$ $\xi_j = \sqrt[m]{\lambda_j}$ $(j = 1,2,...,n)$.

Endi $\sqrt[m]{\lambda_j E_j + H_j}$ ni quyidagicha aniqlaymiz. λ-tekislikda markazi λ_j nuqtada bo'lib, nolni o'z ichiga olmaydigan doira olamiz. Bu doirada $\sqrt[m]{\lambda}$ funksiyani $m$ta ajralgan tarmoqlariga ega bo'lamiz. Bu tarmoqlarini doira markazida qabul qiladigan qiymatlariga qarab ajratish mumkin. $\sqrt[m]{\lambda}$ bilan λ_j nuqtada izlanayotgan X matritsaning ξ_j xarakteristik soni bilan ustma-ust tushadigan qiymatni qabul qiluvchi tarmoqni belgilaymiz va shu tarmoqdan kelib chiqib, quyidagi qator yordamida $\sqrt[m]{\lambda_j E_j + H_j}$ dan olingan funksiyani aniqlaymiz:

$$\sqrt[m]{\lambda_j E_j + H_j} = \lambda_0^{\frac{1}{m}} E_j + \frac{1}{m}\lambda_0^{\frac{1}{m}-1} H_j + \frac{1}{2!}\frac{1}{m}\left(\frac{1}{m} - 1\right)\lambda_0^{\frac{1}{m}-2} H_j^2 + ... \qquad (6.58)$$

Qaralayotgan $\sqrt[m]{\lambda}$ funksiyadan λ_j olingan hosila nolga teng emas, u holda (6.58) matritsa faqat bitta $(\lambda - \lambda_j)^{p_j}$ elementar bo'luvchiga ega bo'lib,

$$\xi_j = \sqrt[m]{\lambda_j}, (j = 1,2,...,n).$$

Bundan kelib chiqadiki,

$$\left\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, ..., \sqrt[m]{\lambda_u E_u + H_u}\right\}$$

kvazidioganal matritsa (6.57) , ya'ni izlanayotgan X matritsa elementar bo'luvchilariga ega. Shuning uchun shunday T xosmas matritsa mavjudki, unda

$$X = T\left\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, ..., \sqrt[m]{\lambda_u E_u + H_u}\right\} T^{-1} \qquad (6.59)$$

T matritsani aniqlash uchun $(\sqrt[m]{\lambda})^m = \lambda$ ayniyatdagi λ ning o'rniga $\lambda_j E_j + H_j$, $(j = 1,2,...,n)$ ni qo'yib,

$(\sqrt[m]{\lambda_j E_j + H_j})^m = \lambda_j E_j + H_j, (j = 1,2,...,n)$ ni hosil qilamiz.

(6.54) va (6.59) dan quyidagi kelib chiqadi:

$$A = T\{\lambda_1 E_1 + H_1, \lambda_2 E_2 + H_2, ..., \lambda_u E_u + H_u\} T^{-1} \qquad (6.60)$$

(6.56) va (6.60) dan quyidagini topamiz:

$$T = UX_{\tilde{A}} \qquad (6.61)$$

bu yerda $X_{\tilde{A}}$ - $\tilde{A}$ matritsa bilan o'rin almashinuvchi, ixtiyoriy xosmas matritsa ($X_{\tilde{A}}$ ning ifodasi **§**2 da keltirilgan).

(6.61) ni (6.59) ga qo'yib, (6.54) tenglamani barcha yechimlarini o'z ichiga oluvchi formulani hosil qilamiz:

$$X = UX_{\tilde{A}}\left\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, ..., \sqrt[m]{\lambda_u E_u + H_u}\right\} X_{\tilde{A}}^{-1} U^{-1} \qquad (6.62)$$

(6.54) tenglamaning barcha yechimlarini A matritsaning m darajali ildizi deb, $\sqrt[m]{A}$ ko'pqiymatli simvol bilan belgilaymiz. $\sqrt[m]{A}$ umumiy holda A matritsaning funksiyasi bo'lmaydi, ya'ni A ning ko'phadi ko'rinishida tasvirlanmaydi.

Eslatma. Agar A matritsaning barcha elementar bo'luvchilari juft-jufti bilan o'zaro tub, ya'ni $\lambda_1 \lambda_2, ..., \lambda_u$ sonlar har xil bo'lsa, $X_{\tilde{A}}$ matritsa quyidagicha kvazidiogonal ko'rinishga ega bo'ladi:

$$X_{\tilde{A}} = \{X_1, X_2, ..., X_u\}$$

bu yerda X_j matritsa $\lambda_j E_j + H_j$ matritsa bilan o'rin almashinuvchi, demak $\lambda_j E_j + H_j$ ning ixtiyoriy funksiyasi bilan, xususiy holda $\sqrt[m]{\lambda_j E_j + H_j}$ bilan o'rin almashinuvchi bo'ladi $(j = 1,2,...,n)$. Shuning uchun bu holda (6.62) quyidagicha bo'ladi:

$$X = U\left\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, ..., \sqrt[m]{\lambda_u E_u + H_u}\right\} U^{-1}$$

Shunday qilib, agar A matritsaning elementar bo'luvchilari juft-jufti bilan o'zaro tub bo'lsa, $X = \sqrt[m]{A}$ uchun formulada faqat diskret ko'pqiymatlilik bo'ladi. Bu holda ixtiyoriy $\sqrt[m]{A}$ qiymatni A ning ko'phadi sifatida tasvirlash mumkin.

Misol. Quyidagi matritsaning barcha kvadrat ildizlarini toping:

$$A = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

ya'ni $X^2 = A$ tenglamani barcha yechimlarini toping.

Bu holda A matritsaning Jordonnning normal formasiga ega. Shuning uchun (6.62) da $A = \tilde{A}$, $U = E$ deb olish mumkin $X_{\tilde{A}}$ matritsa quyidagi ko'rinishda bo'ladi:

$$X_{\tilde{A}} = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{Vmatrix},$$

bu yerda a, b, c, d, e -ixtiyoriy parametrlar.

(6.62) formula quyidagi ko'rinishni oladi:

$$X = \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix} \begin{bmatrix} \varepsilon & \frac{\varepsilon}{2} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix}^{-1}, \quad \varepsilon^2 = \eta^2 = 1 \qquad (6.63)$$

X ni o'zgartirmay, (6.62) formulada $X_{\tilde{A}}$ ni shunday skalyarga ko'paytirish mumkinki, unda $\left|X_{\tilde{A}}\right| = 1$ bo'ladi. Bu qaralayotgan holda $a^2 e = 1$ tenglikka olib keladi, bundan $e = a^{-2}$.

$X_{\tilde{A}}^{-1}$ matritsaning elementlarini hisoblaymiz. Buning uchun $X_{\tilde{A}}$ koeffisiyentilaridan tuzilgan matritsani chiziqli almashtirishni yozamiz:

$$y_1 = ax_1 + bx_2 + cx_3,$$

$$y_2 = ax_2,$$

$$y_3 = dx_2 + a^{-2}x_3.$$

Bu sistemani X_1, X_2, X_3 ga nisbatan yechib, quiydagi teskari almashtirishni hosil qilamiz:

$$x_1 = a^{-1}y_1 - (a^{-2}b - cd)y_2 - acy_3,$$

$$x_2 = a^{-1}y_2,$$

$$x_3 = -ady_2 + a^2 y_3.$$

Bundan ,

$$X_{\tilde{A}}^{-1} = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & a^{-2} \end{Vmatrix}^{-1} = \begin{Vmatrix} a^{-1} & cd - a^{-2}b & -ac \\ 0 & a^{-1} & 0 \\ 0 & -ad & a^2 \end{Vmatrix}$$

bo'lib, (6.63) dan

$$X = \begin{Vmatrix} \varepsilon & (\varepsilon-\eta)acd + \dfrac{\varepsilon}{2} & a^2c(\eta-\varepsilon) \\ 0 & \varepsilon & 0 \\ 0 & (\varepsilon-\eta)da^{-1} & \eta \end{Vmatrix} = \begin{Vmatrix} \varepsilon & (\varepsilon-\eta)vw + \dfrac{\varepsilon}{2} & (\eta-\varepsilon)v \\ 0 & \varepsilon & 0 \\ 0 & (\varepsilon-\eta)w & \eta \end{Vmatrix}, \qquad (6.64)$$

$$v = a^2c, \quad w = a^{-1}d$$

Demak, X yechim ikkita v va w ixtiyoriy parametrlar va ikkita ε va η ixtiyoriy belgilarga bog'liq.

§7. Xos matritsadan m-darajali ildiz chiqarish.

Bu paragrafda $|A| = 0$ holni qaraymiz.

Bu holda ham $|A| \neq 0$ holdagi kabi A matritsani quyidagicha Jordonning normal formasiga keltiramiz:

$$A = U(\lambda_1 E^{(p_1)} + H^{(p_1)}, \dots, \lambda_u E^{`(p_u)} + H^{(p_u)}, H^{(q_1)}, H^{(q_2)}, \dots, H^{(q_t)})U^{-1}, \qquad (6.65)$$

bu yerda $(\lambda - \lambda_1)^{p_1}, \dots, (\lambda - \lambda_u)^{p_u}$ - A matritsaning nolmas xarakteristik sonlariga mos elementar bo'luvchilari, $\lambda^{q_1}, \lambda^{q_2}, \dots, \lambda^{q_t}$ esa nolli xarakteristik sonlarga mos elementar bo'luvchilari.

U holda

$$A = U\{A_1, A_2\}U^{-1} \qquad (6.66)$$

bu yerda

$$A_1 = \{\lambda_1 E^{(p_1)} + H^{(p_1)}, \dots, \lambda_u E^{`(p_u)} + H^{(p_u)}\}, A_2 = \{H^{(q_1)}, H^{(q_2)}, \dots, H^{(q_t)}\} \qquad (6.67)$$

ko'rinib turibdiki, A_1-xosmas matritsa, ya'ni $|A_1| \neq 0$, A_2 esa nilpotentlik indeksi $\mu = \max(q_1, q_2, ..., q_t)$ bo'lgan nilpotent matritsa, ya'ni $A_2^{\mu} = 0$.

Berilgan (6.54) tenglamadan kelib chiqadiki, A matritsa izlanayotgan X matritsa bilan o'rin almashinuvchi, demak, unga o'xshash bo'lgan quyidagi matritsalar bilan ham o'rin almashinuvchi:

$$U^{-1}AU = \{A_1, A_2\} \text{ va } U^{-1}XU \qquad (6.68)$$

§2 dagi teorema 6.3 da isbotlanganidek, (6.68) matritsalarning o'rin almashinuvchanligidan va A_1 va A_2 matritsalarni umumiy xarakteristik sonlarga ega emasligidan kelib chiqadiki, (6.68) ning ikkinchi matritsasi mos ravishda kvazidiogonal formaga ega

$$U^{-1}XU = \{X_1, X_2\} \qquad (6.69).$$

(6.54) tenglamadagi A va X matritsalarni ularga o'xshash bo'lgan

$$\{A_1, A_2\} \text{ va } \{X_1, X_2\}$$

matritsalar bilan almashtirib, (6.54) tenglamani quyidagi ikkita tenglama bilan almashtiramiz:

$$X_1^m = A_1, \qquad (6.70)$$

$$X_2^m = A_2 \qquad (6.71)$$

$|A_1| \neq 0$ bo'lgani uchun (6.70) tenglamaga avvali paragrafdagi natijalarni qo'llab, X_1 ni (6.62) formula bo'yicha topamiz:

$$X_1 = X_{A_1}\left\{\sqrt[m]{\lambda_1 E^{(p_1)} + H^{(p_2)}}, ..., \sqrt[m]{\lambda_u E^{(p_u)} + H^{(p_u)}}\right\} X_{A_1}^{-1} \qquad (6.72)$$

Shunday qilib, (6.71) tenglamani qarash qoldi, ya'ni

$$A_2 = \left\{H^{(q_1)}, H^{(q_2)}, ..., H^{(q_t)}\right\} \qquad (6.73)$$

Jordonning normal formasiga ega bo'lgan , $\mu = \max(q_1, q_2, ..., q_t)$ nilpotentlik indeksili A_2 nilpotentlik matritsaning m-darajali barcha ildizlarini tanish bilan shug'ullanamiz. $A_2^{\mu} = 0$ va (6.71) dan

$$X_2^{m\mu} = 0$$

Bu oxirgi tenglikdan ko'rinadiki, X_2 izlanayotgan matritsa Y nilpotentlik indeksli nilpotent matritsa bo'lib, $m(\mu-1) \le Y \le m\mu$. X_2 matritsani Jordon formasiga o'tkazamiz:

$$X_2 = T\left\{H^{(v_1)}, H^{(v_2)}, ..., H^{(v_t)}\right\}T^{-1}, \qquad (6.74)$$

$$(v_1, v_2, ..., v_s \le y).$$

(6.74) tenglikni ikkala tomonini m-darajaga ko'tarib, quyidagini hosil qilamiz:

$$A_2 = X_2^m = T\left\{[H^{(v_1)}]^m, [H^{(v_2)}]^m, ..., [H^{(v_t)}]^m\right\}T^{-1} \qquad (6.75)$$

$[H^{(v)}]^m$ matritsa qanday elementar bo'luvchilarga ega ekanligini aniqlaymiz.

$\overline{H} - \overline{e}_1, \overline{e}_2, ..., \overline{e}_v$ bazisli v-o'chovli vektor fazodagi $H^{(v)}$matritsali chiziqli operator bo'lsin. $H^{(v)}$matritsaning ko'rinishinidan kelib chiqadiki,

$$\overline{H}\overline{e}_1 = 0, \overline{H}\overline{e}_2 = \overline{e}_1, ..., \overline{H}\overline{e}_v = \overline{e}_{v-1} \qquad (6.76)$$

Bu tengliklar ko'rsatadiki, $\overline{H}$ operator uchun $\overline{e}_1, \overline{e}_2, ..., \overline{e}_v$ vektorlar λ^v elementar bo'luvchiga mos Jordancha vektorlar zanjirini tashkil qiladi.

(6.76) tengliklarni quyidagicha yozamiz:

$$\overline{H}\overline{e}_j = \overline{e}_{j-1} \quad (j = 1,2,...,v, \quad \overline{e}_0 = 0).$$

Bundab ko'rinadiki,

$$\overline{H}^m\overline{e}_j = \overline{e}_{j-m} \quad (j = 1,2,...,v, \quad \overline{e}_0 = \overline{e}_1 = ... = \overline{e}_{-m+1} = 0) \qquad (6.77)$$

v sonini quyidagicha yozamiz:

$$v = km + r \quad (r < m),$$

bu yerda k, r butun manfiymas sonlar. $\overline{e}_1, \overline{e}_2, ..., \overline{e}_v$ bazis vektorlarni quyidagicha joylashtiramiz:

$$\begin{matrix} \overline{e}_1 & \overline{e}_2 & \cdots & \overline{e}_m \\ \overline{e}_{m+1} & \overline{e}_{m+2} & \cdots & \overline{e}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{e}_{(k-1)m+1} & \overline{e}_{(k-1)m+2} & \cdots & \overline{e}_{km} \\ \overline{e}_{km+1} & \overline{e}_{km+2} & \cdots & \overline{e}_{km+x} \end{matrix} \qquad (6.78)$$

Bu jadvalda m ta ustun bo'lib, birinchi r ta ustunda $k+1$ ta vektor, qolgan ustunlarda k ta vektorlar bor. (6.77) tengliklardan ko'rinadiki (6.78) jadvalning har bir ustunidagi vektorlar sistemasi $\overline{H}^m$ operatorga nisbatan vektorlarning Jordan zanjirini tashkil etadi. Agar (6.78) dagi vektorlarni satrlar bo'yicha ketma-ket nomerlanishini ustunlar bo'yicha qilib olsak, u holda hosil qilingan yangi bazisda $\overline{H}^m$ operatorning matritsasi quyidagicha normal Jordan formasiga ega bo'ladi:

$$\left\{\underbrace{H^{(k+1)},\dots,H^{k+1}}_{r\ ta},\ \underbrace{H^{(k)},\dots,H^{(k)}}_{m-r\ ta}\right\},$$

bundan,

$$[H^{(v)}]^m = P_{v,m}\left\{H^{(k+1)},\dots,H^{(k+1)},H^{(k)},\dots,H^{(k)}\right\}P_{v,m}^{-1}, \qquad (6.79)$$

bu yerda bir bazisdan boshqa bazisga o'tish matritsasi $P_{v,m}$ quyidagi ko'rinishda bo'ladi:

$$P_{v,m} = \left\| \begin{matrix} \overbrace{1 \quad 0 \quad \dots \quad 0}^{m\ ta} & 0\dots\dots \\ 0 \quad 0 \quad \dots \quad 0 & 1\dots\dots \\ \dots \quad \dots \quad \dots & \dots\dots\dots\dots \\ 0 \quad 0 \quad \dots \quad 0 & \\ 0 \quad 1 \quad \dots \quad 0 & \\ \dots\dots\dots\dots\dots\dots & \end{matrix} \right\| \qquad (6.80)$$

$\overline{H}^v$ matritsa bitta λ^v elementar bo'luvchiga ega bo'lib, $\overline{H}^v$ ni m-darajaga ko'targanda bu elementar bo'luvchi deyiladi.(6.79) dan ko'rinadiki $[H^{(v)}]^m$ matritsa quyidagi elementar bo'luvchilarga ega:

$$\left\{\underbrace{\lambda^{(k+1)},\dots,\lambda^{k+1}}_{r\ ta},\ \underbrace{\lambda^{(k)},\dots,\lambda^{(k)}}_{m-r\ ta}\right\}$$

Endi (6.75) tenglikka qaytib,

$$v_i = k_i m + r_i,\ (0 \le r_i < m,\ k_i > 0,\ i = 1,2,\dots,s) \qquad (6.81)$$

deb olamiz.

U holda (6.79) ga asosan (6.75) ni quyidagicha yozamiz:

$$A_2 = X_2^m = TP\left\{\underbrace{H^{(k_1+1)},\dots,H^{(k_1+1)}}_{r_1\ ta},\underbrace{H^{(k_1)},\dots,H^{(k_1)}}_{m-r_1\ ta},\underbrace{H^{(k_2+1)},\dots,H^{(k_2+1)}}_{r_2\ ta},\underbrace{H^{(k_2)},\dots,H^{(k_2)}}_{m-r_2\ ta}\right\}P^{-1}T^{-1} \qquad (6.82)$$

bu yerda $P = \{P_{v_1,m}, P_{v_2,m},\dots,P_{v_s,m}\}$.

(6.82) ni (6.73) bilan solishtirib, ko'ramizki,

$$H^{(k_1+1)},...,H^{(k_1+1)},H^{(k_1)},...,H^{(k_1)},H^{(k_2+1)},...,H^{(k_2+1)},H^{(k_2)},...,H^{(k_2)}... \qquad (6.83)$$

kataklar tartibigacha aniqlikda

$$H^{(q_1)},H^{(q_2)},...,H^{(q_t)} \qquad (6.84)$$

kataklar bilan ustma-ust tushishi kerak.

$\lambda^{v_1},\lambda^{v_2},...,\lambda^{v_3}$ elementar bo'luvchilarni X_2 uchun mumkin bo'lgan deb aytamiz, agarda matritsani m-darajaga ko'targandan co'ng bu elementar bo'luvchilar yoyilib, A_2 matritsaning berilgan $\lambda^{q_1},\lambda^{q_2},...,\lambda^{q_t}$ elementar bo'luvchilari sistemasini yuzaga keltirsa. Elementar bo'luvchilarning mumkin bo'lgan sistemasi soni har doim chekli bo'ladi, chunki

$$\max(v_1,v_2,...,v_3)\le m\mu,\ v_1+v_2+...+v_3=n_2, \qquad (6.85)$$

bu yerda n_2 - A_2 matritsaning (darajasi) tartibi.

Har bir mumkin bo'lgan elementar bo'luvchilar sistemasi $\lambda^{v_1},\lambda^{v_2},...,\lambda^{v_3}$ uchun (6.71) tenglamaning mos yechimi mavjud ekanligini ko'rsatamiz va bu yechimlarni aniqlaymiz . Bu holda shunday almashtiruvchi Q matritsa mavjudki, unda

$$\{H^{(k_1+1)},...,H^{(k_1+1)},H^{(k_1)},...,H^{(k_1)},H^{(k_2+1)},...,H^{(k_2+1)},H^{(k_2)},...,H^{(k_2)}...\}=Q^{-1}A_2Q \qquad (6.86)$$

Q matritsa kvazidiogonal matritsadagi kataklarning o'rin almashinishini amalga oshiradi. Shuning uchun Q matritsani ma'lum deb xisoblaymiz. (6.86) ga asosan (6.82) dan quyidagini hosil qilamiz:

$$A_2=TPQ^{-1}A_2QP^{-1}T^{-1}.$$

Bundan, $$TPQ^{-1}=X_{A_2}$$

yoki $$T=X_{A_2}QP^{-1} \qquad (6.87)$$

bu yerda X_{A_2} - A_2 matritsa bilan o'rin almashinuvchi ixtiyoriy matritsa.

(6.87) ni (6.74) ga qo'yib, quyidagiga ega bo'lamiz:

$$X_2=X_{A_2}QP^{-1}\{H^{(v_1)},H^{(v_2)},....,H^{(v_s)}\}PQ^{-1}X_{A_2}^{-1} \qquad (6.88)$$

(6.69), (6.72) va (6.88) dan barcha izlangan yechimlarni o'zida saqlovchi umumiy formulani hosil qilamiz:

$$X = U\{X_{A_1}, X_{A_2}QP^{-1}\}$$
$$\left\{\sqrt[m]{\lambda_1 E^{(p_1)} + H^{(p_2)}},....,\sqrt[m]{\lambda_u E^{(p_u)} + H^{(p_u)}}, H^{(v_1)},..., H^{(v_s)} X_{A_1}^{-1} PQ^{-s} X_{A_2}^{-1}\right\} U^{-1} \quad (6.89)$$

Shuni aytib o'tish kerakki, xos matritsaning m-darajali ildizi har doim ham mavjud bo'lavermaydi. Uni mavjudligi X_2 matritsa uchun mumkin bo'lgan elementar bo'luvchilar sistemasini mavjudligiga bog'liq.

Tekshirib ko'rish mumkinki,

$$X^m = H^{(p)}$$

tenglama $m > 1, p > 1$ da yechimga ega emas.

Misol. $$A = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

matritsani kvadrat ildizdan chiqaring, ya'ni

$$X^2 = A$$

tenglamani barcha yechimlarini toping.

Bu holda $A = A_2, X = X_2$, $m = 2, t = 2, q_1 = 2, q_2 = 1$.X matritsa faqat bitta λ^3 elementar bo'luvchiga ega bo'lishi mimkin. Shuning uchun $s = 1, v_1 = 3, k_1 = 1, r_1 = 1$ va

$$P = P_{3,2} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} = P^{-1}, \quad Q = E$$

Bundan tashqari, avvalgi misoldagi kabi (6.88) formulada

$$X_{A_2} = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & a^{-2} \end{Vmatrix}, \quad X_{\tilde{A}}^{-1} = \begin{Vmatrix} a^{-1} & cd - a^{-2}b & -ac \\ 0 & a^{-1} & 0 \\ 0 & -ad & a^2 \end{Vmatrix}$$

deb olish mumkin.

Bu formuladan quyidagini hosil qilamiz:

$$X = X_2 = X_{A_2} = X_{A_2} P^{(1)} H^{(3)} P X_{A_2}^{-1} = \begin{Vmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 \end{Vmatrix},$$

bu yerda $\alpha = -ca^{-1} - a^2 d$ va $\beta = a^3$-ixtiyoriy parametrlar.

§8.Matritsa logarifmi.

Quyidagi matritsali tenglamani qaraymiz:

$$e^X = A \tag{6.90}$$

Bu tenglamaning barcha yechimlarini A matritsaning logarifmi (natural) deb, uni $\ln A$ dek belgilaymiz.

A matritsaning λ_j xarakteristik soni X matritsaning ξ_j xarakteristik soni bilan $\lambda_j = e^{\xi_j}$ formula orqali bog'langan, shuning uchun, agar (90) tenglama yechimga ega bo'lsa, u holda A matritsaning barcha xarakteristik sonlari noldan farqli bo'li, A xosmas matritsa $(|A| \neq 0)$ bo'ladi. Demak, $(|A| \neq 0)$ shart (6.90) tenglama yechimi mavjudligi uchun zarurdir. Bu shart yetarli ham bo'lishini keyinroq ko'ramiz.

$|A| \neq 0$ bo'lsin. A matritsaning elementar bo'luvchilarini yozamiz:

$$(\lambda - \lambda_1)^{p_1}, (\lambda - \lambda_2)^{p_2}, ..., (\lambda - \lambda_u)^{p_u}$$

$$(\lambda_1, \lambda_2, ..., \lambda_u \neq 0, p_1 + p_2 + ... + p_u = n) \tag{6.91}$$

Bu elementar bo'luvchilarga mos ravishda A matritsani Jordonning normal ko'rinishiga keltiramiz:

$$A = U\tilde{A}U^{-1} = U\left\{\lambda_1 E^{(p_1)} + H^{(p_1)}, ..., \lambda_u E^{(p_u)} + H^{(p_u)}\right\}U^{-1}. \tag{6.92}$$

Ma'lumki, e^{ξ} funksiyaning xosilasi ξ ning barcha qiymatlarida noldan farqli, shuning uchun X matritsadan $e^X = A$ matritsaga o'tishda elementar bo'luvchilar yoyilmaydi, ya'ni X matritsa quyidagi elementar bo'luvchilarga ega bo'ladi:

$$(\lambda - \xi_1)^{p_1}, (\lambda - \xi_2)^{p_2}, ..., (\lambda - \xi_u)^{p_u}, \tag{6.93}$$

bu yerda $\lambda_j = e^{\xi_j}$ $j = 1,2,...,u$, ya'ni ξ_j $\ln \lambda_j$ ning qiymatlaridan biri bo'ladi.

λ o'zgaruvchili kompleks tekislikda markazi λ_j nuqtada bo'lgan $|\lambda_j|$ dan kichik radiusli doirani olamiz va $f_j(\lambda) = \ln \lambda$ orqali $\ln \lambda$ funksiyaning qaralayotgan doiradagi shunday tarmog'ini olamizki, u λ_j nuqtada X matritsani

$\xi_j \; j=1,2,...,u$ xarakteristik soniga teng qiymatlarni qabul qilsin. Shundan so'ng faraz qilamiz:

$$\ln(\lambda_j E^{(p_j)} + H^{(p_j)}) = f_j(\lambda_j E^{(p_j)} + H^{(p_j)}) = \ln \lambda_j E^{(p_j)} + \lambda_j^{-1} H^{(p_j)} + ... \quad (6.94)$$

$\ln \lambda$ dan olingan hosila (λ tekislikning chekli qismida) hech qayerda nolga aylanmaydi, u holda (6.94) matritsa faqat bitta $(\lambda - \xi_j)^{p_j}$ elementar bo'luvchiga ega bo'ladi. Bunga asosan quyidagi kvazidiogonal matritsa

$$\left\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\right\} \quad (6.95)$$

va X izlanayotgan matritsa xuddi shu elementar bo'luvchiga ega. Shuning uchun shunday T matritsa mavjudki, unda

$$X = T\left\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\right\}T^{-1} \quad (6.96)$$

T matritsani aniqlash uchun quyidagini qaraymiz:

$$A = e^X = T\left\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\right\}T^{-1} \quad (6.97)$$

(6.97) va (6.92) larni solishtirib, quyidagini topamiz:

$$T = UX_{\tilde{A}} \quad (6.98)$$

bu yerda $X_{\tilde{A}}$ - $\tilde{A}$ matritsa bilan o'rin almashinuvchi ixtiyoriy matritsa. (6.98) dan topilgan T uchun ifodani (6.96) ga qo'yib, matritsaning barcha logarifmini o'z ichiga oluvchi umumiy formulani hosil qilamiz:

$$X = UX_{\tilde{A}}\left\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\right\}X_{\tilde{A}}^{-1}U^{-1} \quad (6.99)$$

Eslatma.Agar A matritsaning barcha elementar bo'luvchilari o'zaro tub bo'lsa, u holda (6.99) formulaning o'ng tomonidagi $X_{\tilde{A}}$ va $X^{-1}{}_{\tilde{A}}$ ko'paytuvchilarni tashlab yuborish mumkin.

Qachon haqiqiy xosmas A matritsa haqiqiy X logarifmga ega bo'lishini aniqlaymiz. Izlanayotgan matritsa $\rho + i\pi$ xarakteristik songa mos bir nechta elementar bo'luvchilarga ega bo'lsin: $(\lambda - \rho - i\pi)^{q_1},...,(\lambda - \rho - i\pi)^{q_t}$. X matritsa haqiqiy bo'lgani uchun u qo'shma elementar bo'luvchilarga ham ega : $(\lambda - \rho + i\pi)^{q_1},...,(\lambda - \rho + i\pi)^{q_t}$. X matritsadan A matritsaga o'tishda elementar bo'luvhchilar yoyilmaydi, ammo $\rho + i\pi$, $\rho - i\pi$ xarakteristik sonlar $e^{\rho + i\pi} = e^{\rho - i\pi} = e^{\rho} = \mu > 0$ son bilan almashadi. Shuning uchun A matritsa elementar bo’luvchilari sistemasida manfiy xarakteristik songa mos keluvchi har bir elementar bo’luvchi (agar mavjud bo’lsa) juft son marta takrorlanadi. Endi bu zaruruy shartni etarli ham ekanligini isbotlaymiz, ya’ni A- haqiqiy xosmas matritsa faqat va faqat shu holda X haqiqiy logarifmga ega bo’ladi, agarda A matritsa manfiy xarakteristik sonlarga mos elementar bo’luvchilarga ega bo’lmasa, yoki (agar mavjud bo’lsa) bunday elementar bo’luvchilar juft son marta takrorlansa.

Haqiqatan, bu shart bajarilgan bo’lsin. U holda (6.94) formulaga mos (6.95) kvazidiogonal matritsada λ_i haqiqiy va musbat bo’lgan kataklarda $\ln\lambda_i$ uchun haqiqiy qiymatlarini olamiz, agar qandaydir katak kompleks λ_n ga ega bo’lsa, u holda xuddi shunday o’lchovli boshqa katak topilib, bu katakda $\ln\lambda$ va $\ln\lambda_g$ uchun kompleks qo’shma qiymatni olamiz. Har bir katak shartga ko’ra (6.98) da juft son marta takrorlanadi. U holda bu kataklarning yarmida $\ln\lambda_k = \ln|\lambda_k| + i\pi$, qolgan yarmida esa $\ln\lambda_k = \ln|\lambda_k| - i\pi$ deb olamiz. U holda kvazidiogonal matritsaning diogonalidagi elementlari yoki haqiiqy, yoki qo’shma kompleks bo’ladi. Ammo bunday kvazidiogonal matritsa har doim haqiqiy matritsaga o’xshash bo’ladi. Shuning uchun , shunday T_1 $(|T_1| \neq 0)$ xosmas matritsa mavjud bo’ladiki, unda

$$X_1 = T_1\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\}T_1^{-1}$$

matritsa haqiqiqy bo’ladi . Ammo, bu holda quyidagi matritsa ham haqiqiy bo’ladi:

$$A_1 = e^{X_1} = T_1\{\ln\{\lambda_1 E^{(p_1)} + H^{(p_1)}),...,\ln(\lambda_u E^{`(p_u)} + H^{(p_u)})\}T_1^{-1} \qquad (6.100)$$

(6.100) va (6.92) lardan ko’rinadiki, A va A_1 matritsalar o’zaro o’xshash. Ammo ikkita o’zaro o’xshash matritsalar, qandaydir haqiqiy xosmas W -matritsa yordamida bir-biriga almashtirilishi mumkin:

$$A = WA_1W^{-1} = We^{X_1}W^{-1} = e^{WX_1W^{-1}}:$$

U holda $X = WX_1W^{-1}$ matritsa A matritsaning izlangan haqiqiqy logarifmi bo’ladi.

Mashqlar.

1. Agar $A = PJP^{-1}$ va $B = Q\hat{J}Q^{-1}$ bo'lib, $J, \hat{J}$ - matritsalar Jordonning normal formasida bo'lsa, u xolda $Y = P^{-1}XQ$ bo'lganda AX+XB=0 (X uchun) va JY+Y $\hat{J}$=0 (Y uchun) ekvivalent ekanligini isbotlang.
2. Agar A matritsa har-xil xos qiymatlarga ega bo'lsa, u xolda A bilan o'rin almashinuvchi matritsa soda matritsa bo'ladi.
3. Agar f funksiya A matritsa spektorida aniqlangan bo'lsa, u xolda f(A) va A matritsalar o'zaro o'rin almashinuvchi bo'lishini aniqlang.
4. Agar A- har-xil xos qiymatli normal matritsa bo'lsa, u xolda A bilan o'zaro o'rin almashinuvchi har bir matritsa normal bo'lishini isbotlang.
5. Agar JY=YJ va J=$\lambda I_n + H_n$ bo'lsa, u xolda

$$Y = \left\| \begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots & y_n \\ 0 & y_1 & y_2 & \cdots & y_{n-1} \\ \cdots & \cdots & \cdots\ \cdots & \cdots. & \cdots. \\ & & & y_1 & y_2 \\ 0 & & \cdots. & 0 & y_1 \end{array} \right\|$$

ekanligini ko'rsating.

VII. BOB.
KVADRATIK FORMALAR VA ULARNING TADBIQLARI.

§1.1. Kvadratik formalarda o'zgaruvchilarni almashtirish.

Ta'rif 7.1. Kvadratik forma deb, n ta $x_1, x_2, \ldots, x_n$ o'zgaruvchilarga nisbatan ikkinchi darajali bir jinsli ko'pxadga aytiladi.

Kvadratik formalarni xar doim

$$\sum_{i,k}^{n} a_{ik} x_i x_k \qquad (a_{ik} = a_{ik}; i, k = 1.2...., n)$$

ko'rinishida tasvirlash mumkin, bu yerda

$$A = (a_{ik})_{i,k=1}^{n}$$

simmetrik matritsa.

$$x = (x_1, x_2, \ldots, x_n)^T$$

deb belgilasak, kvadratik formani quyidagicha yozishimiz mumkin bo'ladi:

$$x^T A x = \sum_{i,k=1}^{n} a_{i,k} x_i x_k \,. \qquad (7.1)$$

Agar A xaqiqiy simmetrik matritsa bo'lsa, u holda (7.1) forma xaqiqiy deyiladi.

A matritsaning aniqlovchisi

$$|A| = |a_{ik}|_{i,k=1}^{n}$$

(7.1) kvadratik formaning determinanti deyiladi. Agar $|A| = 0$ bo'lsa, (7.1) forma singulyar deyiladi.

Har bir kvadratik formaga quyidagicha bichiziqli forma mos keladi:

$$x^T A y = \sum_{i,k=1}^{n} a_{i,k} x_i y_k \,, \qquad (7.2)$$

bu yerda $A = (a_{ik})_{i,k=1}^{n}$, $x^T = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)^T$.

Agar $x^1, x^2, ..., x^e, y^1, y^2, ..., y^m$ lar ustun matritsalar bo'lib, $c_1, c_2, ..., c_e, d_1, d_2 ..., d_m$ lar skalyar sonlar bo'lsa, quyidagi tenglik o'rinli bo'ladi:

$$\left(\sum_{i=1}^{e} c_i x^i\right)^T A\left(\sum_{j=1}^{m} d_j y^j\right) = \sum_{i=1}^{e}\sum_{j=1}^{m} c_i d_j \left(x^i\right)^T A y^j \qquad (7.3)$$

Agar n o'lchovli evklid fazosida $\overline{A}$ operator berilgan bo'lib, bu operatorga qandaydir $\overline{e}_1, \overline{e}_2, ..., \overline{e}_n$ ortonormallashgan bazisda

$$A = \left(a_{i,k}\right)^n_{i,k=1}$$

matritsa mos kelsa, u holda ixtiyoriy

$$\overline{x} = \sum_{i=1}^{n} x_i \overline{e}_i \qquad \overline{y} = \sum_{i=1}^{n} y_i \overline{e}_i$$

vektorlar uchun quyidagi ayniyat o'rinli bo'ladi

$$\overline{x}^T A\overline{y} = \left(\overline{Ax}, \overline{y}\right) = \left(\overline{x}, \overline{Ay}\right)$$

Xususiy holda

$$\overline{x}^T A\overline{x} = \left(\overline{Ax}, x\right) = \left(\overline{x}, \overline{Ax}\right),$$

bu yerda $a_{i,k} = \left(\overline{Ae}_i, e_k\right), \left(i, k = 1,2,...,k\right)$.

Endi o'zgaruvchilarni

$$x_i = \sum_{k=1}^{n} t_{ik}\xi_k, i = 1,2,...,n \qquad (7.4)$$

yoki

$$\overline{x} = T\overline{\xi}, \quad T = \left(t_{ik}\right)^n_{i,k=1}, \quad \overline{x} = \left(x_1, x_2, ..., x_n\right)^T, \ \overline{\xi} = \left(\xi_1, \xi_2, ..., \xi_n\right)^T \qquad (7.4^1)$$

ko'rinishda almashtirganimizda (7.1) kvadratik forma koeffitsenlaridan tuzigan matritsa qanday o'zgarishini qarab chiqamiz.

Buning uchun (7.4^1) ni (7.1) ga qo'yib, quyidagicha hosil qilamiz:

$$x^T Ax = \left(T\xi\right)^T AT\xi = \xi^T T^T AT\xi = \xi^T \widetilde{A}\xi,$$

bu yerda

$$\tilde{A} = T^T A T \tag{7.5}$$

(7.5) formula

$$\xi^T A \xi = \sum_{i,k=1}^{n} \tilde{a}_{i,k} \xi_i \cdot \xi_k$$

almashgan kvadratik formaning

$$\tilde{A} = \left(\tilde{a}_{i,k}\right)^n{}_{i,k=1}$$

matritsasini ifodalaydi. (7.5) dan

$$\left|\tilde{A}\right| = |A| \cdot |T|^2 \tag{7.6}$$

kelib chiqadi.

Ta'rif 7.2. (7.5) tenglik bilan bog'langan $(|T| \neq 0)$ ikkita A va $\tilde{A}$ matritsalar kongruent deyiladi.

Shunday qilib, har bir kvadratik forma bilan, juft-jufti bilan kongruent bo'lgan matritsalar sinfi bog'langan. Bu matritsalar bir hil rangga ega bo'lib, bu rang qaralayotgan kvadratik formaning rangi bo'ladi. Bu matritsalar sinifi uchun rang invariant bo'ladi.

§2. Inertsiya qonuni

Har bir $x^T Ax$ kvadratik formani cheksiz ko'p usul bilan quyidagi ko'rinishga keltirish mumkin:

$$x^T Ax = \sum_{i=1}^{r} a_i X^2{}_i, \tag{7.7}$$

bu yerda $a_i \neq 0, i = 1,2, \dots, r$ va $X_i = \sum_{k=1}^{n} a_{ik} x_k, \quad i = 1,2,\dots,r$

$x_1, x_2, \dots, x_n$ o'zgaruvchilarning o'zaro chiziqli bog'liq bo'lmagan haqiqiy chiziqli formalari $(r \leq n)$.

$\xi_1, \xi_2, \dots, \xi_n$ - yangi o'zgaruvchilarning birinchi r tasi $x_1, x_2, \dots, x_n$ o'zgaruvchilar bilan

$$\xi_i = X_i, \quad i = 1,2,\dots,r$$

formulalar bilan bog'langan xosmas almashtirishni qaraymiz. U holda yangi o'zgaruvchilarda

$$x^T Ax = \xi^T \widetilde{A} \xi = \sum_{i=1}^{r} a_i \xi^2{}_i$$

bo'lib,

$$\widetilde{A} = diag(a_{1,} a_{2,} ..., a_r, 0, ..., 0)$$

bo'ladi.

Ammo $\widetilde{A}$ matritsaning rangi r ga teng. Demak, (7.7) ko'rinishdagi kvadratlar soni har doim formaning rangiga teng bo'ladi.

Quyidagi teorema (7.1) kvadratik formani har xil usullar bilan (7.7) ko'rinishga keltirganimizda, nafaqat kvadratlar soni, balki musbat va manfiy kvadratlar soni ham o'zgarmasligini ko'rsatadi.

Teorema 7.1. (Kvadratik formalarning inertsiya qonuni). (7.1) haqiqiy kvadratik formani (7.7) – o'zaro bog'liq bo'lmagan kvadratlar yig'indisi ko'rinishida ifodalashda musbat kvadratlar soni va manfiy kvadratlar soni ko'rsatilgan ko'rinishga keltirish usuliga bog'liq emas.

Isboti. (7.1) kvadratik forma (7.7) korinish bilan birga yana quyidagicha o'zaro bog'liq bo'lmagan kvadratlar yig'indisi ko'rinishiga keltirilgan bo'lib,

$$x^T Ax = \sum_{i=1}^{r} b_i Y_i^2,$$

$$a_1 > 0, a_2 > 0, ..., a_h > 0, a_{h+1} < 0, ..., a_r < 0,$$

$$b_1 > 0, b_2 > 0, ..., b_g > 0, b_{g+1} < 0, ..., b_r < 0$$

bo'sin. Faraz qilaylik, h≠g, masalan h<g. U holda

$$\sum_{i=1}^{r} a_i X_i^2 = \sum_{i=1}^{r} b_i Y_i^2 \qquad (7.8)$$

ayniyatda $x_1, x_2, ..., x_n$ o'zgaruvchinlarga $X_{h+1}, ..., X_r$ formalarning xech bo'lmaganda bittasi nolga aylanmaydigan va quyidagi r-(g-h) ta tenglamalar sistemasini qanoatlantiruvchi qiymatlar beramiz:

$$X_1 = 0, X_2 = 0, ..., X_n = 0, Y_{g+1} = 0, ..., Y_2 = 0 \qquad (7.9)$$

O'zgaruvchilarning bunday qiymatlarida (7.8) ayniyatning chap tomoni

$$\sum_{j=n+1}^{r} a_j x_j^{\ 2} < 0$$

ga, o'ng tomoni esa

$$\sum_{k=1}^{g} b_k y_k^{\ 2} > 0$$

ga teng bo'ladi.

Shunday qilib, h≠g degan farazimiz bizni qarama-qarshilikka olib keladi.

Ta'rif 7.3. (7.1) kvadratik formani o'zaro bog'liq bo'lmagan kvadratlar yig'indisi ko'rinishida ifodalaganimizdagi musbat kvadratlar soni π bilan manfiy kvadratilar soni γ ning ayirmasi σ shu kvadratik formaning signaturasi deyiladi.

Demak, $r = \pi + \gamma$, $\sigma = \pi - \gamma$

(7.7) dagi koeffitsentlarni $\sqrt{|a_i|}$ ko'rinishda olib, X_i larning tarkibiga kiritish mumkin bo'lgani uchun quyidagi tenglikni yozishimiz mumkin:

$$x^T A x = X_1^{\ 2} + X_2^{\ 2} + ... + X_\pi^{\ 2} - X_{\pi+1}^{\ 2} - ... - X_r^{\ 2} \qquad (7.10)$$

(7.10) ifodada $\xi_i = X_i, \quad i = 1,2,...,r$ deb olib (7.1) formani quyidagicha kanonik ko'rinishga keltiramiz:

$$\xi^T \widetilde{A} \xi = \xi_1^{\ 2} + \xi_2^{\ 2} + ... + \xi_\pi^{\ 2} - \xi_{\pi+1}^2 - ... - \xi_r^2 \qquad (7.11)$$

Bundan teorema 7.1 ga asosan quyidagicha xulosa qilamiz: Ixtiyoriy A haqiqiy simmetrik matritsa elementlari 1,-1 va 0 lardan iborat bo'lgan diogonal matritsaga kongruentdir, ya'ni

$$A = T^T diog(\underbrace{1,1,\dots 1}_{\pi ta}, \underbrace{-1,-1,...,-1}_{Yta}, 0,...0)T \qquad (7.12)$$

§3. Lagranj metodi.

Kvadratik formani kvadratlar yig'indisiga keltirishning Lagranj metodini qarab chiqamiz.

Quyidagi kvadratik forma berilgan bo'lsin

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik} x_i x_k$$

Quyidagi ikkita xolni qaraymiz:

1). Qandaydir g (1≤g ≤n) uchun diogonal koeffitsent $a_{gg} \neq 0$. U holda

$$x^T Ax = \frac{1}{a_{gg}} \left(\sum_{k=1}^{n} a_{gk} x_k \right)^2 + x^T A_1 x \qquad (7.13)$$

deb olib, bevosita tekshirib ko'rishimiz mumkinki, $x^t A_1 x$ kvadratik forma x_g o'zgaruvchini o'zida saqlamaydi. Bu usul kvadratik formadan kvadratlarni ajratib olish usuli deyilib, A matritsaning dioganal elementlari noldan farqli bo'lganda, har doim uni qo'llash mumkin.

2). $a_{gg} = 0$, $a_{hh} = 0$, $a_{gh} \neq 0$, Bu holda (7.1) ni quyidagicha o'zgartiramiz:

$$x^T Ax = \frac{1}{2a_{hg}} \left[\sum_{k=1}^{n} (a_{gk} + a_{hk}) x_k \right]^2 - \frac{1}{2a_{hg}} \left[\sum_{k=1}^{n} (a_{gk} - a_{hk}) x_k \right]^2 + x^T A_2 x \qquad (7.14)$$

Quyidagi

$$\sum_{k=1}^{n} a_{gk} x_k , \qquad \sum_{k=1}^{n} a_{hk} x_k ,$$

formalar chiziqli bog'liq emas, chunki birinchisi x_h ni o'zida saqlab, x_g ni saqlamaydi, ikkinchisi esa x_g ni o'zida saqlab, x_h ni saqlamaydi. Shuning uchun (7.14) ga kvadrat qavslardagi formalar chiziqli bog'liq emas.

Shunday qilib, $x^T Ax$ kvadratik formadan ikkita chiziqli bog'liq bo'lmagan kvadratlarni ajratib oldik. Bu kvadratlarning har biri x_g va x_n o'zgaruvchilarni o'zida saqlaydi, $x^T A_2 x$ forma esa bu o'zgaruvchilarni o'zida saqlamaydi.

Bu usulni ketma-ket qo'llab, $x^T Ax$ ni kvadratlar yig'indisiga keltirish mumkin.

(7.13) va (7.14) formulalarni mos ravishda quyidagicha ham yozish mumkin.

$$x^T Ax = \frac{1}{4a_{gg}}\left(\frac{\partial(x^T Ax)}{\partial x_g}\right)^2 + x^T A_1 x \qquad (7.13')$$

(1.13^1)

$$x^T Ax = \frac{1}{8a_{gh}}\left[\left(\frac{\partial(x^T Ax)}{\partial x_g} + \frac{\partial(x^T Ax)}{\partial x_h}\right)^2 - \left(\frac{\partial(x^T Ax)}{\partial x_g} - \frac{\partial(x^T Ax}{\partial x_h}\right)^2\right] + x^T A_2 x \qquad (7.14')$$

Misol 7.1 Quyidagi kvadratik formani kvadratlar yig'indisi ko'rinishida ifodalang:

$$x^T Ay = 4x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4x_1x_2 - 4x_1x_3 + 4x_1x_4 + 4x_2x_3 - 4x_3x_4$$

Avval $(7.13')$ formulani qo'llaymiz. (g=1)

$$\frac{\partial(x^T Ay)}{\partial x_1} = 8x_1 - 4x_2 - 4x_3 + 4x_4$$

$$x^T Ax = \frac{1}{16}(8x_1 - 4x_2 - 4x_3 + 4x_4)^2 + x^T A_1 x = (2x_1 - x_2 - x_3 + x_4)^2 + x^T A_1 x$$

bu yerda $x^T A_1 x = 2x_2x_3 + 2x_2x_4 - 2x_3x_4$.

Bu formaga $(7.14')$ formulani qo'llaymiz: $(g = 2,\ h = 3)$

$$x^T A_1 x = \frac{1}{8}(2x_2 + 2x_3)^2 - \frac{1}{8}(2x_3 - 2x_2 + 4x_4)^2 + x^T A_2 x = \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_3 - x_2 + 2x_4)^2 + x^T A_2 x$$

bu yerda $x^T A_2 x = 2x_4^2$.

Demak,

$$x^T Ax = (2x_1 - x_2 - x_3 + x_4)^2 + \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_3 - x_2 + 2x_4)^2 + 2x_4^2$$

bo'lib, r=4, σ=2, $\pi = 3$, γ =1 bo'ladi.

§4. Yakobi formulasi

(7.1) kvadratik formaning rangini r bilan belgilab, A matritsaning k-tartibli minoralarini

$$D_k = A\begin{pmatrix} 1 & 2 & ... & k \\ 1 & 2 & ... & k \end{pmatrix} \neq 0, \text{ k=1,2…,r} \tag{7.15}$$

deb olamiz. Bundan, $a_{11} = D_1 \neq 0$ bo'ladi, u holda Lagranj metodi yordamida $x^T Ax$ formadan bitta kvadrat ajratib, quyidagini hosil qilamiz:

$$x^T Ax = \frac{1}{a_{11}}(a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n)^2 + x^T A_1 x, \tag{7.16}$$

bu yerda

$$x^T A_1 x = \sum_{i,k=2}^{n} a_{ik}^{(1)} x_i x_k, \quad a_{ik}^{(1)} = a_{k,i}^{(1)}, \ i,k = 1,2,...,n \tag{7.17}$$

x_1 o'zgaruvchini o'zida saqlamaydi. (7.16) tenglikdan kelib chiqadiki, $x^T A_1 x$ ning koeffitsientlari

$$a_{i,k}^{(1)} = a_{i,k} - \frac{a_{1i}a_{1k}}{a_{11}}, \ i,k = 1,2,...,n \tag{7.18}$$

formulalar bilan aniqlanadi. U holda bu koeffitsientlar quyidagi matritsaning mos elementlari bilan ustma-ust tushadi:

$$G_1 = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22}^{(1)} & ... & a_{2n}^{(1)} \\ \dots & \dots & \dots & \dots \\ 0 & a_{n2}^{(1)} & ... & a_{nn}^{(1)} \end{pmatrix}$$

Bu matritsa esa A matritsaga Gauss usulining birinchi bosqichini qo'llab hosil qilingan.

Shunday qilib, Lagranj metodi bo'yicha bitta kvadrat ajratish jarayoni mazmun jihatidan Gauss algoritmining birinchi bosqichi bilan ustma-ust tushadi. Ikkinchi kvadratni ajratib olish uchun Gauss algoritmining ikkinchi bosqichini bajarish kerak bo'ladi va hokazo.

$A = (a_{ik})_{i,k=1}^{n}$ simmetrik matritsaga r ta bosqichdan iborat bo'lgan Gauss algoritmini to'la qo'llab, quyidagi matritsani hosil qilamiz:

$$G_r = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} & a_{1r+1} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & \dots & a_{22}^{(1)} & a_{2r+1}^{(1)} & \dots & a_{2n}^{(1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{rr}^{(r-1)} & a_{r,r+1}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

Bunga mos holda $x^T Ax$ kvadratik forma quyidagicha kvadratlar yig'indisi ko'rinishida ifodalanadi:

$$x^T Ax = \sum_{k=1}^{r} \frac{1}{a_{kk}^{(k-1)}} (a_{kk}^{(k-1)} x_k + \dots + a_{kn}^{(k-1)} x_n)^2, \tag{7.19}$$

$$\left(a_{1j}^{(0)} = a_{1j,}\ j = 1,2,\dots,n\right)$$

O'zaro bog'liq bo'lmagan chiziqli formalar uchun

$$X_k = a_{kk}^{(k-1)} x_k + \dots + a_{kn}^{(k-1)} x_n\, (a_{1k}^{(0)} = a_{1k,} k = 1,2,\dots,r) \tag{7.20}$$

qisqa belgilashlar kiritamiz.

$$a_{kk}^{(k-1)} = \frac{D_k}{D_{k-1}}, k = 1,2,\dots,r; D_0 = 1, a_k^{(0)} = a_{11} \tag{7.21}$$

ekanligini e'tiborga olsak, (7.19) ni quyidagicha yozishimiz mumkin:

$$x^T Ax = \sum_{k=1}^{r} \frac{D_{k-1}}{D_k} X_k^2 \quad (D_0 = 1) \tag{7.22}$$

Bu formulalar Yakobi formulalari deyiladi.

Yakobi formulalaridagi X_k chiziqli formalar koeffitsientlari uchun quyidagi tengliklar o'rinli:

$$a_{k-q}^{(k-1)} = \frac{A\begin{pmatrix} 1 \dots k-1k \\ 1 \dots k-1q \end{pmatrix}}{A\begin{pmatrix} 1 \dots k-1 \\ 1 \dots k-1 \end{pmatrix}}, k = 1,2,\dots,r \tag{7.23}$$

$X_k (k = 1,2,\dots,r)$ lar o'rniga

$$Y_k = D_{k-1} X_k \quad (k = 1,2,\dots,r; D_0 = 1) \tag{7.24}$$

chiziqli bog’liq bo’lmagan formalarni kiritib, Yakobi formulalarini quyidagicha yozish mumkin:

$$x^T Ax = \sum_{k=1}^{r} \frac{Y_k}{D_{k-1}D_k} \qquad (7.25)$$

Bu yerda

$$Y_k = C_{kk}x_k + C_{k,k+1}X_{k+1} + \ldots + C_{kn}x_n, k = 1,2,\ldots,r, \qquad (7.26)$$

$$C_{kq} = A\begin{pmatrix} 1 & 2 & \ldots & k-1 & k \\ 1 & 2 & \ldots & k-1 & q \end{pmatrix}, \ (q = k, k+1,\ldots,n; \ \ k = 1,2,\ldots,r) \qquad (7.27)$$

(7.25) – Yakobi formulalaridan quyidagi teoremaning o’rinli ekanligi kelib chiqadi:

Teorema 7.2. Agar rangi r ga teng bo’lgan

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik}x_i x_k$$

kvadratik forma uchun

$$D_k = A\begin{pmatrix} 1 & 2 & \ldots & k \\ 1 & 2 & \ldots & k \end{pmatrix} \neq 0, \ \ k = 1,2,\ldots,r \qquad (7.28)$$

bo’lsa, u holda musbat kvadratlar soni π va manfiy kvadratlar soni γ mos ravishda

$$1, D_1, D_2, \ldots, D_r \qquad (7.29)$$

qatordagi P-o’zgarmas ishoralar soni va V- o’zgaruvchan ishoralar soni bilan ustma-ust tushadi, yani

$$\pi = 1(1, D_1, D_2, \ldots, D_r), \ \gamma = V(1, \ \ D_1, D_2, \ldots, D_r)$$

va signatura

$$\sigma = r - 2V(1, D_1, D_2, \ldots, D_r) \qquad (7.30)$$

bo’ladi.

Misol 7.2. Quyidagi kvadratik formani kvadratlar yig’indisi ko’rinishida yozing:

$$x^T Ax = x_1^2 + 3x_2^2 - 3x_4^2 - 4x_1x_2 + 2x_1x_3 - 2x_1x_4 - 6x_2x_3 + 8x_2x_4 + 2x_3x_4$$

$$A=\begin{pmatrix} 1 & -2 & 1 & -1 \\ -2 & 3 & -3 & 4 \\ 1 & -3 & 0 & 1 \\ -1 & 4 & 1 & -3 \end{pmatrix}$$

matritsani Gauss formasiga keltiramiz.

$$G=\begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Bundan, r=2 $a_{11}=1$, $a_{22}^{(1)}=-1$ ekanligi kelib chiqadi. U holda (7.19) formulaga asosan

$$x^T Ax = (x_1 - 2x_2 + x_3 - x_4)^2 - (-x_2 - x_3 + 2x_4)^2$$

hosil bo'ladi.

§5. Kvadratik formalarning ishoralari

Ta'rif 7.4

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik} x_i x_k$$

haqiqiy kvadratik forma manfiymas (musbatmas) deyiladi, agarda o'zgaruvchilarning ixtiyoriy haqiqiy qiymatlarida

$$x^T Ax \geq 0 \quad (x^T Ax \leq 0) \tag{7.31}$$

bo'lsa.

Bu holda A simmetrik matritsa yarim musbat aniqlangan (yarim manfiy aniqlangan) deyiladi.

Ta'rif 7.5

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik} x_i x_k$$

haqiqiy kvadratik forma musbat aniqlangan (manfiy aniqlangan) deyiladi, agarda o'zgaruvchilarning ixtiyoriy noldan farqli (x≠0) qiymatlarida

$$x^T Ax > 0 \quad (x^T Ax < 0) \qquad (7.32)$$

bo'lsa.

Bu holda A matritsa musbat aniqlangan (manfiy aniqlangan) deyiladi.

Musbat aniqlangan (manfiy aniqlangan) formalar sinifi manfiymas (musbatmas) formalar sinifining qismi bo'ladi.

Manfiymas kvadratik forma o'zaro bog'liqmas kvadratlar yig'indisi ko'rinishida quyidagicha ifodalangan bo'lsin.

$$x^T Ax = \sum_{i=1}^{r} a_i X_i \qquad (7.33)$$

(7.33) da barcha kvadratlar musbat bo'lishi kerak, ya'ni

$$a_i > 0, i = 1,2,\dots,r \qquad (7.34)$$

Haqiqatdan, agar birorta $a_i < 0$ bo'lsa, $x_1, x_2, \dots x_n$ larning shunday qiymatlarini tanlashimiz mumkinki, unda

$$X_1, = \dots = X_{i-1} = X_{i+1} = \dots = X_r = 0, X_i \neq 0$$

bo'lib, $x^T Ax$ manfiy bo'lib qoladi. Aksincha, (7.33) va (7.34) dan $x^T Ax$ formaning musbatligi kelib chiqadi.

Shunday qilib, manfiymas kvadratik formalar $\sigma = r \quad (\pi = r,\ \nu = 0)$ tengliklar bilan haraktrlanadi.

Agar $x^T Ax -$ musbat aniqlangan bo'lsa, u holda u manfiymas forma ham bo'lib, (7.33) va (7.34) shartlar bajariladi. Kvadratik formaning musbat aniqlanganligidan r=n ekanligi kelib chiqadi. Haqiqatdan, agar r<n bo'lsa, $x_i, x_2, \dots, x_n$ o'zgaruvchilarni bir vaqtda nolga teng bo'lmagan qiymatlarini tanlash mumkin bo'ladiki, unda barcha X_i lar nolga teng bo'lib, $x^T Ax = 0$ bo'ladi. Bu (7.32) shartga ziddir. Aksincha, agar (7.33) da r=n bo'lib, (7.34) bajarilsa, $x^T Ax$ forma musbat aniqlangan bo'ladi.

Boshqacha aytganda, manfiymas kvadratik forma faqat va faqat singulyar bo'lmagandagina musbat aniqlangan bo'ladi.

Teorema 7.3. (7.1) kvadratik forma musbat aniqlangan bo'lishi uchun

$$D_1 = a_{11} > 0,\quad D_2 = \begin{vmatrix} a_{11} a_{12} \\ a_{12} a_{22} \end{vmatrix} > 0, \ldots, D_n = |A| > 0 \qquad (7.35)$$

tengsizliklarni bajarilishi zarur va yetarlidir.

Isboti. (7.35) shartlarni yetarli ekanligi (7.25) Yakobi formulalaridan kelib chiqadi. (7.35) shartlarni zarurligini quyidagicha ko'rsatamiz.

$x^T Ax$ formaning musbat aniqlanganligidan kelib chiqadiki, quyidagi qirqib olingan

$$x^T A_p x = \sum_{i,k=1}^{p} a_{ik} x_i x_k, \text{ (p=1,2,…,n)}$$

forma ham musbat aniqlangan bo'ladi. Ammo, bu holda barcha formalar singulyar bo'lmasligi, ya'ni

$$D_p = |A_p| \neq 0 \text{ (p=1,2,…,n)}$$

bo'lishi kerak.

Endi biz Yakobining (7.25) formulalarini (r=n) da qo'llash imkoniyatiga ega bo'lamiz. Bu formulalarning o'ng tomonidagi barcha kvadratlar musbat bo'lishi kerak, u holda

$$D_1 > 0, D_1 D_2 > 0, \ldots, D_{n-1} D_n > 0$$

Bundan (7.35) shartlarning zarurligi kelib chiqadi.

Natija: Musbat aniqlangan

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik} x_i x_k$$

kvadratik formaning koeffitsientlaridan tuzilgan A matritsani barcha bosh minorlari musbat , ya'ni

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} > 0 \quad (1 \le i_1 \le i_2 \ldots \le i_p \le n; \quad p = 1,2,\ldots,n \qquad (7.36)$$

Eslatma. Bosh minorlar ketma-ketligining manfiymas, ya'ni

$$D_1 \ge 0, D_2 \ge 0, \ldots, D_n \ge 0$$

ekanligidan $x^T Ax$ formaning manfiymas ekanligi kelib chiqmaydi.

Teorema 7.4. (7.1) kvadratik forma manfiymas bolishi uchun uning matritsasini barcha bosh minorlari manfiymas, ya'ni

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} \geq 0 \ \ (1 \leq i_1 \leq i_2 \leq \dots \leq i_p \leq n\ ;\ p = 1,2,\dots,n$$

bo'lishi zarur va yetarlidir.

Isboti: Quyidagi yordamchi formani qaraymiz

$$x^t A_\varepsilon x = x^t Ax + \varepsilon \sum_{i=1}^{n} x_i^2 \ \ (\varepsilon > 0).$$

Bundan, $\lim_{\varepsilon \to 0}(x^T A_\varepsilon x) = x^T Ax$ kelib chiqadi.

$x^T Ax$ formaning manfiymasligidan $x^T A_\varepsilon x$ formaning musbat aniqlanganligi kelib chiqadi, shuning uchun quyidagi tengsizlik o'rinli bo'ladi.

$$A_\varepsilon \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} > 0 (1 \leq i_1 \leq i_2 \dots \leq i_P \leq n; \ \ P = 1,2 \dots, n)$$

Bundan, $\varepsilon \to 0$ da limitga o'tib, (7.36) shartni xosil qilamiz.

Aksincha (7.36) shart bajarilsin. Bundan kelib chiqadiki,

$$A_\varepsilon \begin{pmatrix} i_1 & i_2 \dots i_p \\ i_1 & i_2 \dots i_p \end{pmatrix} = \varepsilon^p + \dots \geq \varepsilon^p > 0 (1 \leq i_1 \leq \dots \leq i_P \leq n; \ \ P = 1,2 \dots, n)$$

Ammo, bu holda teorema 7.3 ga asosan

$$x^T A_\varepsilon x > 0 \quad (x \neq 0)$$

Bundan $\varepsilon \to 0$ da limitga o'tib,

$$x^T A_\varepsilon x > 0$$

ni xosil qilamiz.

Kvadratik formaning musbatmaslik va manfiy aniqlanganlik shartlarini, mos ravishda (7.35) va (7.36) tengsizliklarni $-x^T Ax$ formaga qo'llab xosil qilamiz.

Teorema. 7.5 $x^T Ax$ kvadratik forma manfiy aniqlangan bo'lishi uchun

$$D_1 < 0, D_2 > 0, D_3 < 0, (-1)^n D_n > 0 \qquad (7.35')$$

tengsizliklarning bajarilishi zarur va yetarli.

Teorema. 7.6 $x^T Ax$ kvadratik forma musbatmas bo'lishi uchun

$$(-1)^p A\begin{pmatrix} i_1 & i_2 \dots i_p \\ i_1 & i_2 \dots i_p \end{pmatrix} \geq 0 \ \ (1 \leq i_1 \leq \dots \leq i_P \leq n; \quad P = 1,2 \dots, n) \qquad (7.36')$$

tengsizliklarni bajarilishi zarur va yetarli.

§6. Kvadratik formalarni bosh o'qlarga keltirish.

Quyidagi ixtiyoriy haqiqiy kvadratik formani qaraymiz

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik} x_i x_k \ .$$

Uning matritsasi $A = (a_{ik})_{i,k=1}^{n}$ haqiqiy, simmetrik bo'ladi. Shuning uchun φ qandaydir Λ haqiqiy diogonal matritsaga ortogonal- o'xshash bo'ladi, ya'ni shunday Q haqiqiy ortogonal matritsa mavjudki, unda

$$\Lambda = \mathrm{Q}^{-1} AQ \ \ , \ \left(\Lambda = (\lambda_i \delta_{ik})_{i,k=1}^{n}, \ QQ^T = E\right) \qquad (7.37)$$

bo'ladi. Bu yerda $\lambda_1, \lambda_2, \dots, \lambda_n - A$ matritsaning harakteristik sonlari.

Ortogonal matritsa uchun $Q^{-1} = Q^T$ bo'lgani uchun (7.37) dan kelib chiqadiki, $x^T Ax$ forma o'zgaruvchilarni

$$x = Q\xi \quad (QQ^T = E),$$

$$x_i = \sum_{k=1}^{n} q_{ik}\, \xi_k \left(\sum_{j=1}^{n} q_{ij}\, q_{kj} = \delta_{ik}, i, k, = 1,2, \dots, n\right) \qquad (7.38)$$

Orotgonal almashtirishda quyidagi ko'rinishga keladi:

$$\xi^T \Lambda \xi = \sum_{i=1}^{n} \lambda_i \xi_i \qquad (7.39)$$

Teorema. 7.7. $x^T Ax -$ haqiqiy kvadratik formani har doim ortogonal alamshtirish yordamida (7.39) kanonik ko'rinishga keltirish mumkin bo'lib, $\lambda_1, \lambda_2, \dots, \lambda_n$ lar A matritsaning xarakteristik sonlari bo'ladi.

Kvadratik formani ortogonal almashtirish yordamida kanonik ko'rinishga keltirish uni bosh o'qlarga keltirish deyiladi. Bunday nomlanish shu bilan bog'liqki,unda

$$\sum\nolimits^{n}{}_{i,k=1} a_{ik}\, x_i x_k = c \quad (c = const \neq 0) \qquad (7.40)$$

ikkinchi tartibli gipersirt tenglamasi o'zgaruvchilarni (7.38) ortogonal almashtirishda quyidagi kanonik ko'rinishni oladi:

$$\sum_{i=1}^{n} \xi_i \frac{\xi_i^2}{a_i^2} = 1 \ \left(\frac{\xi_i}{a_i^2} = \frac{\lambda_i}{c};\ \ \varepsilon_i = \pm 1;\ \ i = 1,2,\dots,n\right) \quad (7.41)$$

(7.39) formuladan kelib chiqadiki, $x^T Ax$ formaning rangi r, A matritsaning noldan farqli xarakteristik sonlari soniga teng bo'lib, σ signatura A matritsaning musbat va manfiy xarakteristik sonlari sonining ayirmasiga teng bo'ladi.

Bundan, hususiy xolda quyidagi tasdiq kelib chiqadi.

Agar kvadratik forma koeffitsientlari uzluksiz o'zgarganda uning rangi o'zgarmasa, u xolda koeffitsientlarni bunday o'zgarishida uning signaturasi ham o'zgarmay qoladi.

(7.39) formuladan yana kelib chiqadiki, A haqiqiy simmetrik matritsa yarim musbat aniqlangan (musbat aniqlangan) bo'ladi, faqat va faqat shu holdaki, qachonki, A matritsaning barcha xarakteristik sonlari manfiymas (musbat) bo'lsa, ya'ni u quyidagi ko'rinishda ifodalansa

$$A = Q(\lambda_i \delta_{ik})_{i,k=1}^{n} Q^{-1} [\lambda_i \geq 0 \ \ (\lambda_i > 0),\ \ i = 1,2,\dots n] \quad (7.42)$$

$$F = Q(\sqrt{\lambda_i}\delta_{ik})_{i,k=1}^{n} Q^{-1} \quad (7.43)$$

yarim musbat aniqlangan (musbat aniqlangan) matritsa A yarim musbat aniqlangan (musbat aniqlangan) matritsaning kvadrat ildizi bo'ladi:

$$F = \sqrt{A} \quad (7.44)$$

§ 7. Kvadratik formalar dastasi.

Ikkita

$$x^T Ax = \sum_{i,k=1}^{n} a_{ik}\, x_i x_k \quad va \qquad x^T Ax = \sum_{i,k=1}^{n} b_{ik}\, x_i x_k$$

haqiqiy kvadratik formalar yordamida tuzilgan $x^T Ax - \lambda x^T Bx$ (λ −parametr) forma kvadratik formar dastasi deyiladi.

Agar $x^T Bx$ forma musbat aniqlangan bo'lsa, u holda $x^T Ax - \lambda x^T Bx$ dasta regulyar deyiladi.

$$|A - \lambda B| = 0$$

tenglama $x^T Ax - \lambda x^T Bx$ kvadratik formalar dastasining xarakteristik tenglamasi deyiladi.

Bu tenglamaning qandaydir ildizini λ_0 bilan belgilaymiz. $A - \lambda_0 B$ matritsa xos matritsa bo'lgani uchun shunday $z = (z_1, z_2, \dots, z_n)^t \neq 0$ ustun mavjudki, unda

$$(A - \lambda_0 B)z = 0$$

yoki

$$Az = \lambda_0 \mathrm{B}z \quad (\mathrm{z} \neq 0)$$

bo'ladi.

λ_0 soni $x^T Ax - \lambda x^T Bx$ dastaning xarakteristik soni deyilib, $z-$ mos bosh ustun yoki bu dastaning bosh vektori deyiladi.

Teorema. 7.8. Kvadratik formalarning

$$x^T Ax - \lambda x^T Bx$$

regulyar dastasini

$$|A - \lambda B| = 0$$

xarakteristik tenglamasi xar doim $z^k = (z_{1k}, z_{2k}, \dots, z_{nk})(k = 1,2, \dots, n)$

$$Az^k = \lambda_\mathrm{k} Bz^k (k = 1,2, \dots, n) \tag{7.45}$$

bosh vektorlar mos keluvchi, n ta $\lambda_\mathrm{k}(\mathrm{k} = 1,2, \dots, \mathrm{n})$ haqiqiy xarakteristik ildizlarga ega. Bu z^k bosh vektorlarni shunday tanlash mumkinki, unda

$$(z^i)^T Bz^k = \delta_{ik} \quad (i, k = 1,2, \dots, n) \tag{7.46}$$

munosabat bajariladi.

Isboti. (7.45) tenglikni quyidagicha yozish mumkin

$$B^{-1}Az^k = \lambda_k z^k = \delta_{ik} \;\; (i, k = 1,2, \dots, n) \tag{7.45'}$$

Shunday qilib, teorema 7.8. ga ko'ra

$$D = B^{-1}A \tag{7.47}$$

matritsa quyidagilarga ega:

1) oddiy strukturaga;

2) $\lambda_1, \lambda_2, \dots, \lambda_\mathrm{n}$ xaqiqiy xarakteristik sonlarga;

3)bu xarakteristik sonlarga mos kelib, (7.46) munosabatni qanoatlantiruvchi $z^1, z^2, \dots, z^n$ xos ustunlar (vektorlar) ga.

$D = B^{-1}A$ matritsa ikkita simmetrik matritsalarning ko'paytmasidan iborat bo'lib, o'zi simmetrik bo'lmasligi mumkin. Shuning uchun $D^T = AB^{-1}$ bo'ladi. $F = \sqrt{B}$ deb olib, (7.47) tenglikdan quyidagini xosil qilamiz:

$$D = F^{-1}SF, \tag{7.48}$$

bu yerda

$$S = F^{-1}AF^{-1} \tag{7.48'}$$

simmetrik matritsa. D matritsani S simmterik matritsaga o'xshash ekanligidan 1) va 2) tasdiqlar kelib chiqadi. $u^k(k = 1,2,\dots,n)$ orqali S simmetrik matritsa xos vektorlari normalangan sistemasini belgilaymiz:

$$Su^k = \lambda_k u^k(k = 1,2,\dots,n),\ (u^k)^T u^l = \delta_{ke}(k,l = 1,2,\dots,n) \tag{7.49}$$

va

$$u^k = Fz^k(k = 1,2,\dots,n) \tag{7.50}$$

deb olib, (7.48), (7.48'), (7.49), (7.50) tengliklardan quyidagini topamiz:

$$Dz^k = \lambda_k z^k, \qquad (z^k)^T Bz^l = \delta_{kl}, \quad k,l = 1,2,\dots,n$$

ya'ni 3) tasdiq isbotlandi va teorema 7.8. to'la isbotlandi.

(7.46) dan $z^1, z^2, \dots, z^n$ ustunlarni chiziqli bog'liqmasligi kelib chiqadi.

$$\sum_{k=1}^{n} c_k z^k = 0 \tag{7.51}$$

bo'lsin. U xolda ixtiyoriy $i(1 \le i \le n)$ uchun (7.46) ga asosan

$$0 = (z^l)^T B\left(\sum_{k=1}^{n} c_k z^k\right) = \sum_{k=1}^{n} c_k z_l^T B z_k = c_l$$

bo'ladi. Shunday qilib, (7.51) da barcha $c_l(l = 1,2,\dots,n)$ nolga teng va $z^1, z^2, \dots, z^n$ ustunlar orasida hech qanday chiziqli bog'liqlik mavjud emas.

(7.46) munosabatni qanoatlantiruvchi $z^1, z^2, \dots, z^n$ bosh ustunlardan tuzilgan

$$Z = (z^1, z^2, \dots, z^n) = (z_{ik})_{i,k=1}^{n}$$

matritsani $x^T Ax - \lambda x^T Bx$ formalar dastasi uchun bosh matritsa deyiladi. Z matritsa xosmas ($|z| \neq 0$) matritsa bo'ladi, chunki uning ustunlari chiziqli bog'lanmagan.

(7.45) ning ikkala tomonini chapdan z^{l^T} satr matritsaga ko'paytirib, quyidagini xosil qilamiz

$$z^{i^T}Az^k = \lambda_k z^{i^T}Bz^k = \lambda_k \delta_{i,k}(i,k = 1,2,\dots,n) \qquad (7.52)$$

$Z = (z^1, z^2, \dots, z^n)$ bosh matritsani kiritib, (7.46) va (7.52) ni quyidagi ko'rinishda ifodalashimiz mumkin.

$$Z^T AZ = \left(\lambda_k \delta_{i,k}\right)_{i,k=1}^n, \ \ Z^T BZ = E \qquad (7.53)$$

(7.53) formulalardan ko'rinadiki,

$$x = Z\xi \qquad (7.54)$$

xosmas almashtirish $x^T Ax$ va $x^T Bx$ kvadratik formalarni

$$\sum_{k=1}^n \lambda_k\, \xi_k^2 \quad \text{va} \quad \sum_{k=1}^n \xi_k^2 \qquad (7.55)$$

kvadratlar yig'indisiga keltiradi.

(7.54) almashtirishning bu xossasi Z bosh matritsani xarakterlaydi. Xaqiqatan, (7.54) almashtirish $x^T Ax$ va $x^T Bx$ formalarni (7.55) kanonik ko'rinishga keltirsin. U holda (7.53) tenglik o'rinli bo'lib, Z matritsa ustunlari uchun (7.46) va (7.52) tengliklar o'rinli bo'ladi.

(7.53) dan Z ni xosmas ($|z| \neq 0$) matritsa ekanligi kelib chiqadi. (7.52) tenglikni quyidagicha yozamiz:

$$z^{i^T}(Az^k - \lambda_k Bz^k) = 0 \ \ (i = 1,2,\dots,n), \qquad (7.56)$$

bu yerda $k-$ ixtiyoriy fiksirlangan qiymatga ega $1 \leq k \leq n$. (7.56) tengliklar sistemasini bitta tenglikka keltirish mumkin

$$Z^T(Az^k - \lambda_k Bz^k) = 0,$$

bundan, $Z^T -$ xosmas bo'lgani uchun

$$Az^k - \lambda_k Bz^k = 0$$

ni, ya'ni ixtiyoriy k uchun (7.45) ni xosil qilamiz.

Demak, $Z -$ bosh matritsa. Shunday qilib, quyidagi teoremani isbotladik.

Teorema 7.9. Agar

$$Z = (z_{ik})_{i,k=1}^n$$

matritsa $x^T Ax - \lambda x^T Bx$ formalar regulyar dastasining bosh matritsasi bo'lsa, u xolda

$$x = Z\xi \tag{7.57}$$

almashtirish $x^T Ax$ va $x^T Bx$ formalarni bir vaqtda mos ravishda

$$\sum_{k=1}^{n} \lambda_k \xi_k^2 \qquad va \qquad \sum_{k=1}^{n} \xi_k^2 \tag{7.58}$$

kvadratlar yig'indisiga keltiradi, bu yerda $\lambda_1, \lambda_2, \dots, \lambda_n$ lar

$$x^T Ax - \lambda x^T Bx$$

dastaning Z matritsa $z^1, z^2, \dots, z^n$ ustunlariga mos keluvchi xarakteristik sonlari.

Aksincha, qandaydir (7.57) almashtirish $x^T Ax$ va $x^T Bx$ kvadratik formalarni bir vaqtda (7.58) ko'rinishga keltirsa, u holda $Z = (z_{ik})_{i,k=1}^{n}$ matritsa formalarning

$$x^T Ax - \lambda x^T Bx$$

dastasi bosh matritsasi bo'ladi.

Misol. 7.3. Umumlashgan koordinatalar sistemasida

$$2x^2 - 2y^2 - 3z^2 - 10yz + 2xz - 4 = 0 \tag{7.59}$$

ikkinchi tartibli sirt tenglamasi va

$$2x^2 + 3y^2 + 2z^2 + 2xz = 1 \tag{7.60}$$

birlik sfera tenglamasi berilgan. (7.59) tenglamani bosh o'qlarga keltirish talab qilinadi.

Bu holda

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -5 \\ 1 & -5 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Dastaning xarakteristik tenglamasi quyidagi ko'rinishga ega bo'ladi:

$$\begin{vmatrix} 2-2\lambda & 0 & 1-\lambda \\ 0 & -2-3\lambda & -5 \\ 1-\lambda & -5 & -3-2\lambda \end{vmatrix} = 0 \tag{7.61}$$

Bu tenglama $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = -4$ uchta ildizga ega. $\lambda_1 = 1$ xarakteristik songa mos bosh vektor koordinatalarini $u,\ \vartheta,\ \omega$ bilan belgilaymiz. $u,\ \vartheta,\ \omega$

miqdorlarni koeffitsientlari (7.61) aniqlovchi elementlari bilan ustma-ust tushuvchi quyidagi sistemadan topamiz.

$$0\cdot u+0\cdot \vartheta+0\cdot \omega=0$$
$$0\cdot u-5\cdot \vartheta-5\cdot \omega=0$$
$$0\cdot u-5\cdot \vartheta-5\cdot \omega=0.$$

Bundan,

$$\vartheta+\omega=0$$

$\lambda=1$ xarakteristik songa ikkita ortonormallangan bosh vektor javob berishi kerak. Birinchi vektor koordinatalarini ixtiyoriy ravishda $\vartheta+\omega=0$ shartni qanoatlantiradigan qilib olamiz. Ularni $u=0,\ \ \vartheta,\ \ \omega=-\vartheta$ dek tanlaymiz.

Ikkinchi bosh vektor koordinatalarini

$$u',\qquad \vartheta',\qquad \omega'=-v'$$

dek tanlab, ortogonallik sharti $(z^{T}Bz^{2}=0)$ ni yozamiz.

$$2uu'+3\vartheta\vartheta'+2\omega\omega'+u\omega'+u'\omega=0$$

Bundan, $u'=5\vartheta'$ kelib chiqadi. Shunday qilib, ikkinchi bosh vektor koordinatalarini $u'=5\vartheta',\ \ \vartheta',\ \ \omega'=-\vartheta'$ bo'ladi.

Shuningdek xarakteristik aniqlochida $\lambda=-4$ deb olib, mos bosh vektor koordinatalarini

$$u'',\ \ \vartheta''=-u'',\ \ \omega''=-3u''$$

ko'rinishda aniqlaymiz.

$\vartheta,\ \vartheta'\ va\ u''$ miqdorlar bosh vektor koordinatalari $x^{T}Bx=1$ birlik sferani qanoatlantirishi kerak degan shartdan topiladi, ya'ni (7.60) tenglamadan topiladi. Bundan quyidagini topamiz:

$$\vartheta=\frac{1}{\sqrt{5}},\quad \vartheta'=\frac{1}{3\sqrt{5}},\quad u''=-\frac{1}{3}$$

Shuning uchun bosh matritsa quyidagicha bo'ladi,

$$Z=\begin{pmatrix} 0 & \frac{\sqrt{5}}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{1}{3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{\sqrt{5}} & -\frac{1}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix}$$

mos koordinatalarni almashtirish $x = Z\xi$ (7.59) va (7.60) tenglamalarni quyidagicha kanonik ko'rinishga keltiradi:

$$\xi_1^2 + \xi_2^2 - 4\xi_3^2 - 4 = 0 \qquad va \qquad \xi_1^2 + \xi_2^2 + \xi_3^2 = 1.$$

Birinchi tenglamani quyidagicha yozish mumkin

$$\frac{\xi_1^2}{4} + \frac{\xi_2^2}{4} - \frac{\xi_3^2}{1} = 1$$

Bu bir yaproqli aylanma giperboloid tenglamasi bo'lib, uning haqiqiy o'qi ikkiga, mavxum o'qi birga teng. Aylanish o'qi orti koordinatalari z matritsa uchinchi ustunidan aniqlanadi, ya'ni ular $-\frac{1}{3}, \ \frac{1}{3}, \ \frac{2}{3}$ ga teng. Qolgan ikkita ortogonal o'qlar koordinatalari birinchi va ikkinchi ustunlarda beriladi.

§8. Formalar regulyar dastasi harakteristik sonlarining ekstremal xossasi.

Bizga

$$x^T A x = \sum_{i,k=1}^{n} a_{ik}\, x_i x_k \qquad va \qquad x^T B x = \sum_{i,k=1}^{n} b_{ik}\, x_i x_k$$

kvadratik formalar berilgan bo'lib, $x^T Bx -$ musbat aniqlanga bo'lsin. $x^T Ax - \lambda x^T Bx -$ regulyar dasta xarakteristik sonlari

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_n \tag{7.62}$$

shartni qanoatlantirsin. Bu xarakteristik sonlarga mos bosh vektorlarni

$$z^k = (z_{1k}, z_{2k}, \dots, z_{nk})^t \ \ (k = 1,2,\dots,n)$$

bilan belgilaymiz.

O’zgaruvchilarni bir vaqtda nolga teng bo’lmagan $(x \neq 0)$ barcha mumkin bo’lgan qiymatlarini qarab, formalarni $\frac{x^T Ax}{x^T Bx}$ nisbatining eng kichik qiymati (minimumi) ni aniqlaymiz. Buning uchun

$$x = Z\xi \quad (x_i = \sum_{i,k=1}^{n} z_{ik}\, \xi_k, i = 1,2,\dots,n)$$

almashtirish yordamida yangi $\xi_1, \xi_2, \dots, \xi_n$ o’zgaruvchilarga o’tish qulaydir. Bu yerda z berilgan dastaning bosh matritsasi. Yangi o’zgaruvchilarda qaralayotgan formalar nisbati quyidagi ko’rinishni oladi:

$$\frac{x^T Ax}{x^T Bx} = \frac{\lambda_1 \xi_1^2 + \lambda_2 \xi_2^2 + \dots + \lambda_n \xi_n^2}{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2} \qquad (7.63)$$

Son o’qida $\lambda_1, \lambda_2, \dots, \lambda_n$ sonlarga mos n ta nuqtalar olib, bu nuqtalarga mos ravishda $m_i = \xi_i^2, \quad i = 1,2,\dots,n$ massalarni qo’yamiz. U holda, (7.63) formulaga asosan $\frac{x^T Ax}{x^T Bx}$ nisbat bu sonli nuqtalarning massalar markazi bo’ladi. Shuning uchun

$$\lambda_1 \leq \frac{x^T Ax}{x^T Bx} \leq \lambda_n$$

munosabat o’rinli bo’ladi.

Bu tengsizlikning birinchi qismida qachon tenglik bajarilishini aniqlaymiz. Buning uchun (7.62) da teng xarakteristik sonlarni ajratamiz.

$$\lambda_1 = \dots = \lambda_{p_1} < \lambda_{p_1+1} = \dots \lambda_{p_1+p_2} < \cdots \qquad (7.64)$$

λ_1 nuqtadan boshqa nuqtalardagi massalar nolga teng ya’ni

$$\xi_{P_1+1} = \dots = \xi_N = 0$$

bo’lgandagina va faqat shu holda og’irlik markazi shu λ_1 nuqtaga tushadi. Bu holda mos x bosh ustunlar, $z^1, z^2, \dots, z^n$ larning chiziqli kombinatsiyasidan iborat bo’ladi. Ammo, bu barcha ustunlar λ_1 ga teng xarakteristik songa javob beradi, u holda $x \quad \lambda = \lambda_1$ uchun bosh ustun (vektor) bo’ladi.

Shunday qilib biz quyidagi teoremani isbotladik.

Teorema.7.10 $\qquad x^T Ax - \lambda x^T Bx$

regulyar dastaning eng kichik xarakteristik soni

$$\lambda_1 = min\frac{x^T Ax}{x^T Bx}$$

bu minimum λ_1 xarakteristik son uchun bosh bo'lgan vektorlardagina erishiladi.

Shunga o'xshash ,,minimallik" xarakteristikasini keyingi λ_2 xarakteristik son uchun ham berish uchun z^1 vektorga ortogonal bo'lgan, ya'ni

$$z^{1^T} Bx = 0$$

tenglamani qanoatlantiruvchi barcha x vektorlarni qarash bilan chegaralanamiz.

Bunday vektorlar uchun

$$\frac{x^T Ax}{x^T Bx} = \frac{\lambda_2 \xi_2^2 + \cdots + \lambda_n \xi_n^2}{\xi_2^2 + \cdots + \xi_n^2}$$

bo'lib,

$$min\frac{x^T Ax}{x^T Bx} = \lambda_2 > \left[z^{1^T} Bx = 0\right]$$

bo'ladi. Bunda tenglik belgisi, faqat λ_2 xarakteristik son uchun bosh bo'lib, z^1 ga ortogonal bo'lgan vektorlardagina bajariladi.

Boshqa xarakteristik sonlarga ham o'tib, quyidagi teoremaga kelamiz.

Teorema.7.11 Ixtiyoriy p $(1 \leq p \leq n)$ uchun (7.62) qatordagi λ_p xarakteristik son $\frac{x^T Ax}{x^T Bx}$ nisbat minimumidan iborat, ya'ni

$$\lambda_p = min\frac{x^T Ax}{x^T Bx}$$

bo'lib, x vektor $z^1, z^2, \dots, z^{p-1}$ ortonormallangan bosh vektorlarga ortogonal, ya'ni

$$z^{1^T} Bx = 0, z^{2^T} Bx = 0, \dots, \ z^{p-1^T} Bx = 0 \qquad (7.65)$$

bo'ladi.

Bunda (7.65) shartlarni qanoatlantirib, λ_p xarakteristik son uchun bosh vektor bo'lgan vektorlardagina minimumga erishiladi.

Teorema 7.11ni qo'llashning noqulayligi shundan iboratki, unda λ_p xarakteristik son avvalgi $z^1, z^2, \dots, z^{p-1}$ bosh vektorlarga bog'liq bo'lib, bu bosh vektorlar ma'lum bo'lgandagina teoremani qo'llash mumkin. Bundan tashqari bosh vektorlarni tanlash ma'lum ixtiyoriylikka ega.

Bu noqulaylikdan qutilish uchun $x_1, x_2, \dots, x_n$ o’zgaruvchilarga qo’yilgan bog’lanishlar haqida tushuncha kiritamiz.

$x_1, x_2, \dots, x_n$ o’zgaruvchilarning

$$L_k(x) = l_{1k}x_1 + l_{2k}x_2 + \dots + l_{nk}x_n, \quad (k = 1,2,\dots,n) \qquad (7.65')$$

chiziqli formasi berilgan bo’lsin.

$x_1, x_2, \dots, x_n$ o’zgaruvchilarga yoki x vektorga $L_1, L_2, \dots, L_n$ h ta bog’lanishlar qo’yilgan deyiladi, agarda faqat

$$L_k(x) = 0 \quad (k = 1,2,\dots,n) \qquad (7.65'')$$

tenglamalar sistemasini qanoatlantiruvchi o’zgaruvchilargina qaralsa.

$(7.65')$ dagi belgilashlarni saqlab qolgan holda ixtiyoriy chiziqli forma uchun quyidagicha belgilash kiritamiz:

$$\tilde{L}_1(x) = z^{k^T}Bx \quad (k = 1,2,\dots,n) \qquad (7.66)$$

Bundan tashqari $(7.65'')$ bog’lanishlar qo’yilgan vektorlar uchun $min\frac{x^TAx}{x^TBx}$ ni quyidagicha belgilaymiz:

$$\mu\left(\frac{A}{B};\ L_1, L_2, \dots, L_n\right).$$

Bu belgilashlarda (1,64) quyidagicha yoziladi

$$L_p = \mu\left(\frac{A}{B};\ \tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_{p-1}\right)\ (p = 1,2,\dots,n). \qquad (7.67)$$

Endi

$$L_1(x) = 0, L_2(x) = 0, \dots, L_{p-1}(x) = 0, \qquad (7.68)$$

va

$$\tilde{L}_{p+1}(x) = 0, \dots, \tilde{L}_n(x) = 0 \qquad (7.88)$$

bog’lanishlarni qaraymiz. (7.68) va (7.69) dagi bog’lanishlar soni n dan kichik bo’lgani uchun bu bog’lanishlarning xammasini qanoatlantiruvchi $x^{(1)} \neq 0$ vektor mavjud bo’ladi. (7.69) bog’lanishlar x vektorni $z^{p+1}, \dots, z^n$ bosh vektorlar bilan ortogonalligini ifodalaydi, shuning uchun $x^{(1)}$ vektor koordinatalarida $\xi_{p+1} = \dots = \xi_n = 0$ bo’lib, (7.63) ga asosan

$$\frac{x^{(1)^T}Ax^{(1)}}{x^{(1)^T}Bx^1} = \frac{\lambda_1\xi_1^2 + \dots + \lambda_p\xi_p^2}{\xi_1^2 + \dots + \xi_2^2} \leq \lambda_p$$

bo’ladi. Ammo bu holda

$$\mu(\frac{A}{B}; L_1, L_2, \dots, L_{p-1}) \leq \frac{x^{(1)^T} A x^{(1)}}{x^{(1)^T} B x^1} \leq \lambda_p$$

Bu tengsizlik (7.67) bilan birga qaralganda ko’rinadiki, μ miqdor $L_1, L_2, \dots, L_{p-1}$ bog’lanishlarda λ_p dan ortmay, $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_{p-1}$ maxsus bog’lanishlar olinganda λ_p ga erishadi.

Shunday qilib, quyidagi teorema isbotlandi.

Teorema. 7.12. Agar biz ixtiyoriy p−1 uchun $L_1, L_2, \dots, L_{p-1}$ bog’lanishlarda

$$\frac{x^T A x}{x^T B x} -$$

ikkita forma nisbati minimumini qarab, bog’lanishlarni variatsiyalasak, u holda bu minimumlarning maksimumi λ_p ga teng, ya’ni

$$\lambda_p = max\mu\left(\frac{A}{B}; L_1, L_2, \dots, L_{p-1}\right)(p = 1,2, \dots, n) \qquad (7.70)$$

bo’ladi.

Teorema. 7.11. $\lambda_1, \lambda_2, \dots, \lambda_n$ xarakteristik sonlarni “minimumlik” xarakteristikasini, Teorema. 7.12 esa “maksimal – minimallik” xarakteristikasini beradi.

$$x^T A x - \lambda x^T B x$$

dastadagi $x^T A x$ formani $-x^T A x$ forma bilan almashtirganimizda dastaning barcha xarakteristik sonlari ishorasini almashtirilib, ularga mos bosh vektorlar o’zgarmay qoladi. Shunday qilib,

$$-x^T A x - \lambda x^T B x$$

dastaning xarakteristik sonlari $\lambda_n \leq -\lambda_{n-1} \leq \dots \leq -\lambda_1$ bo’ladi.

Bundan tashqari,

$$\gamma\left(\frac{A}{B}; L_1, L_2, \dots, L_h\right) = max\frac{x^T A x}{x^T B x} \qquad (7.71)$$

belgilash kiritib, variatsiyalanuvchi vektorlarga $L_1, L_2, \dots, L_h$ bog’lanishlar qo’yilgan holda quyidagilarni yozishimiz mumkin

$$\mu\left(-\frac{A}{B}; L_1, L_2, \dots, L_h\right) = -\gamma(\frac{A}{B}; L_1, L_2, \dots, L_h)$$

va

$$max\mu\left(-\frac{A}{B}; L_1, L_2, \dots, L_h\right) = -min\gamma\left(\frac{A}{B}; L_1, L_2, \dots, L_h\right).$$

Shuning uchun,

$$\frac{x^T Ax}{x^T Bx}$$

nisbatga 7.10, 7.11, 7.12 teoremalarni qo'llab, (7.64), (7.67), (7.70) formulalar o'rniga mos ravishda quyidagilarni xosil qilamiz:

$$\lambda_p = max\frac{x^T Ax}{x^T Bx},$$

$$\lambda_{n-p+1} = \gamma\left(\frac{A}{B}; \tilde{L}_n, \tilde{L}_{n-2}, \dots, \tilde{L}_{n-p+2}\right),$$

$$\lambda_{n-p+1} = \min\left(\frac{A}{B}; L_1, L_2, \dots, L_{p-1}\right), \qquad (p = 2, \dots, n).$$

Bu formulalar $\lambda_1, \lambda_2, \dots, \lambda_n$ sonlarni mos ravishda "maksimallik" va "minimal - maksimallik" xossalarini aniqlaydi. Bularni teorema ko'rinishida quyidagicha ifodalaymiz.

Teorema. 7.13.

$$x^T Ax - \lambda x^T Bx$$

formalar regulyar dastasi xarakteristik sonlari

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$$

bo'lib, dastaning chiziqli bog'liq bo'lmagan bosh vektorlari $z^1, z^2, \dots, z^n$ bo'lsin. U holda

1) $\lambda_n -$ eng katta xarakteristik son formalar nisbati maksimumi, ya'ni

$$\lambda_n = max\frac{x^t Ax}{x^t Bx} \qquad (7.72)$$

bo'lib, bu maksimumga faqat dastaning λ_n xarakteristik soniga mos vektorlaridagina erishiladi.

2) Oxiridan $p-$ xarakteristik son λ_{n-p+1} $(2 \le p \le n)$ variatsiyalanuvchi x vektorga

$$z^{n^T}Bx = 0, \quad z^{n-1^T}Bx = 0, \quad z^{n-p+1^T}Bx = 0 \qquad (7.73)$$

bog'lanishlar qo'yilgan shartda shu formalar nisbati maksimumi bo'ladi

$$\lambda_{n-p+1} = max\frac{x^TAx}{x^TBx} \qquad (7.74)$$

ya'ni

$$\lambda_{n-p+1} = \gamma\left(\frac{A}{B}; \tilde{L}_n, \tilde{L}_{n-1}, \dots, \tilde{L}_{n-p+2}\right) \qquad (7.75)$$

Bu maksimumga faqat dastaning λ_{n-p+1} xarakteristik soniga mos va (7.73) shartlarni qanoatlantiruvchi bosh vektorlaridagina erishiladi.

3) Agar

$$L_1(x) = 0, \dots, L_{p-1}(x) = 0 \quad (2 \leq p \leq n)$$

bog'lanishlardagi

$$\frac{x^TAx}{x^TBx}$$

formalar nisbati maksimumida bog'lanishlar variatsiyalansa, u holda bu maksimumlarning eng kichik qiymati (minimumi) λ_{n-p+1} ga teng, ya'ni

$$\lambda_{n-p+1} = min\gamma\left(\frac{A}{B}; L_1, L_2, \dots, L_{p-1}\right) \qquad (7.76)$$

Quyidagi h ta o'zaro bog'liq bo'lmagan bog'lanishlar berilgan bo'lsin

$$L_1^0(x) = 0, L_2^0(x) = 0, \dots, L_h^0(x) = 0 \qquad (7.77)$$

U holda bu bog'lanishlar yordamida $x_1, x_2, \dots, x_n$ o'zgaruvchilarni h tasini qolganlari orqali ifodalash mumkin. Qolgan o'zgaruvchilarni $v_1, v_2, \dots, v_{n-h}$ lar bilan belgilaymiz. Shuning uchun (7.77) bog'lanishlar qo'yilganda

$$x^TAx - \lambda x^TBx$$

formalarning regulyar dastasi

$$v^TA^0v - \lambda v^TB^0v$$

dastaga o'tib, $v^TB^0v -$ yana musbat aniqlangan forma bo'ladi. Keying dasta faqat $n-h$ ta o'zgaruvchiga bog'liq bo'lgani uchun u

$$\lambda_1^0 \leq \lambda_2^0 \leq \dots \leq \lambda_{n-h}^0 \qquad (7.78)$$

$n-h$ ta xarakteristik songa ega bo'ladi.

(7.77) bog'lanishlarni qo'yganda barcha o'zgaruvchilarni $n-h$ ta o'zaro bog'liq bo'lmagan $v_1, v_2, \dots, v_{n-h}$ lar bilan xar xil ifodalash mumkin. Ammo (7.78) xarakteristik sonlar bu xar xillikka bog'liq bo'lmay, to'la aniqlangan qiymatlarga ega bo'ladi. Bu hech bo'lmaganda xarakteristik sonlarning minimal – makimal xossalaridan kelib chiqadi

$$\lambda_1^0 = min\frac{v^T A^0 v}{v^T B^0 v} = \mu\left(\frac{A}{B}; L_1^0, L_2^0, \dots, L_h^0\right) \qquad (7.79)$$

va umumiy xolda

$$\lambda_p^0 = max\mu\left(\frac{A^0}{B^0}; L_1, L_2, \dots, L_{p-1}\right) =$$

$$= max\mu\left(\frac{A}{B}; L_1^0, L_2^0, \dots, L_h^0;\ L_1, L_2, \dots, L_{p-1}\right) \qquad (7.80)$$

shu bilan birga (7.80) formulada faqat $L_1, L_2, \dots, L_{p-1}$ bog'lanishlar variatsialanadi.

Teorema 7.14. Agar

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \qquad x^T A x - x^T B x$$

formalarning regulyar dastasi xarakteristik sonlar bo'lib,

$$\lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_{n-h}^0$$

esa shu dastaga h ta o'zaro bog'liq bo'lmagan bog'lanishlar qo'yilgandagi xarakteristik sonlar bo'lsa, u holda

$$\lambda_p \leq \lambda_p^0 \leq \lambda_{p+h} \ \ (p = 1,2, \dots, n-h) \qquad (7.81)$$

Isboti. $\lambda_p \leq \lambda_p^0$ tengsizlik (7.70) va (7.80) formulalardan kelib chiqadi. Xaqiqatan, yangi bog'lanishlar qo'shilganda

$$\mu\left(\frac{A}{B}; L_1, \dots, L_{p-1}\right)$$

minimum miqdori ortadi yoki o'zgarmay qoladi. Shuning uchun

$$\mu\left(\frac{A}{B}; L_1, \dots, L_{p-1}\right) \leq \mu\left(\frac{A}{B}; L_1^0, \dots, L_h^0;\ L_1, \dots, L_{p-1}\right)$$

bo'lib, bundan

$$\lambda_p = max\mu\left(\frac{A}{B}; L_1, \dots, L_{p-1}\right) \leq \lambda_p^0 = \max\left(\frac{A}{B}; L_1^0, \dots, L_h^0;\ L_1, \dots, L_{p-1}\right)$$

kelib chiqadi.

(7.81) tengsizlikning ikkinchi qismi quyidagi munosabatga ko'ra o'rinli bo'ladi.

$$\lambda_p^0 = max\mu\left(\frac{A}{B}; L_1^0, \dots, L_h^0;\ L_1, \dots, L_{p-1}\right)$$

$$\leq max\mu\left(\frac{A}{B};\ L_1, \dots, L_{p-1}, L_p, \dots, L_{p+h-1}\right) = \lambda_{p+h.}$$

Bu yerda, birinchi qismda faqat $L_1, \dots, L_{p-1}$ bog'lanishlar variatsialanadi, ammo $L_p, \dots, L_{p+h-1}$ lar fiksirlangan $L_1^0, \dots, L_h^0$ bog'lanishlar bilan almashtiriladi.

$$x^T Ax - \lambda x^T Bx \text{ va } x^T \tilde{A}x - \lambda x^T \tilde{B}x \tag{7.82}$$

formalarning regulyar dastalari berilgan bo'lib, $x \neq 0$ da

$$\frac{x^T Ax}{x^T Bx} \leq \frac{x^T \tilde{A}x}{x^T \tilde{B}x}$$

bo'lsin. U holda

$$max\mu\left(\frac{A}{B};\ L_1, \dots, L_{p-1}\right) \leq max\mu\left(\frac{\tilde{A}}{\tilde{B}};\ L_1, \dots, L_{p-1}\right) \quad (p = 1,2, \dots n)$$

bo'ladi. Shuning uchun (7.82) dagi dastalarining xarakteristik sonlarini mos ravishda

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \text{ va } \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$$

lar bilan belgilab,

$$\lambda_p \leq \lambda_p \quad (p = 1,2, \dots, n)$$

tengsizlikka ega bo'lamiz.

Demak, quyidagi teorema o'rinli

Teorema 7.15. Agar (7.82) regulyar dastalar berilgan bo'lib,

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \text{ va } \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$$

mos ravishda ularning xarakteristik sonlari bo'lsa, u holda

$$\frac{x^T Ax}{x^T Bx} \leq \frac{x^T \tilde{A}x}{x^T \tilde{B}x} \tag{7.83}$$

ayniy munosabatdan

$$\lambda_p \leq \tilde{\lambda}_p \qquad (p = 1,2, \dots, n) \tag{7.84}$$

kelib chiqadi.

(7.83) tengsizlikda

$$x^T B x = x^T \tilde{B} x$$

bo'lgan xususiy xolni qaraylik. Bu holda

$$x^T \tilde{A} x = x^T A x$$

manfiymas kvadratik forma bo'lib, o'zaro bog'liq bo'lmagan musbat kvadratlar ko'rinishida yozilishi mumkin:

$$x^T \tilde{A} x = x^T A x + \sum_{i=1}^{r} [\mathrm{X_i(x)}]^2$$

U holda r o'zaro bog'liq bo'lmagan

$$X_1(x) = 0,\ X_2(x) = 0, \dots, X_r(x) = 0,$$

bog'lanishlar qo'yganimizda

$$x^T A x \text{ va } x^T \tilde{A} x$$

formalar ustma – ust tushib, (7.82) dastalar bir hil

$$\lambda_1^0 \le \lambda_2^0 \le \cdots \le \lambda_{n-2}^0$$

xarakteristik ildizlarga ega bo'ladi:

(7.82) dagi xar bir dastaga teorema 7.14 ni qo'llab,

$$\tilde{\lambda}_p \le \lambda_p^0 \le \lambda_{p+2} \qquad (p = 1,2, \dots, n-2)$$

ga ega bo'lamiz. Bunga (7.84) ni birlashtirib, quyidagi teoremaga kelamiz.

Teorema 7.16. Agar (7.82) dagi dastalar uchun

$$x^T \tilde{A} x = x^T A x + \sum_{i=1}^{r} [\mathrm{X_i(x)}]^2$$

bu yerda $X_i(x) (i = 1,2, \dots, r)$ - o'zaro bog'liq bo'lmagan chiziqli formalar, (7.84) shart bajarilib,

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \text{ va } \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_n$$

mos ravishda ularning xarakteristik sonlari bo'lsa, u holda

$$\lambda_p \le \tilde{\lambda}_p \le \lambda_{p+r} \quad (p = 1,2, \dots, n) \qquad (7.85)$$

tengsizlik o'rinli bo'ladi.

Xuddi shuningdek quyidagi teoremani ham isbotlash mumkin.

Teorema 7.17. Agar

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \text{ va } \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_n$$

lar mos ravishda

$$x^T Ax - \lambda x^T Bx \text{ va } x^T Ax - \lambda x^T \tilde{B}x$$

dastalarning xarakteristik sonlari bo'lib,

$$x^T \tilde{B}x$$

forma

$$x^T Bx$$

formaga r ta musbat kvadratlarni qo'shib hosil qilinsa, u holda

$$\lambda_{p-r} \leq \tilde{\lambda}_p \leq \lambda_p \quad (p = 1,2, \dots, n) \tag{7.86}$$

tengsizlik o'rinli bo'ladi.

Eslatma. Agar $r \neq 0$ chekli bo'lsa, teorema (7.16) va teorema (7.17) larga mos ravishda qandaydir p da

$$\lambda_p < \tilde{\lambda}_p \text{ va } \tilde{\lambda}_p < \lambda_p$$

ga ega bo'lamiz.

§9. Kvadratik formalar ustida amallar.

Quyidari n o'zgaruvchili kvadratik formani qaraymiz:

$$f(x, x) = x^T Ax, \tag{7.87}$$

bu yerda $x = (x_1, x_2, \dots, x_n)^T, A = (a_{i,j})_{i,j=1}^n, a_{i,j} = a_{j,i}$. $\alpha \in R$- ixtiyoriy xaqiqiy son.

Ta'rif 7.6 $f(x, x)$ kvadratik formani haqiqiy songa ko'paytmasi deb, quyidagi tenglik bilan aniqlanuvchi kvadratik formaga aytiladi:

$$\alpha f(x, x) = x^T(\alpha A)x = \sigma\left(\sqrt{|\alpha|}x\right)^T A\left(\sqrt{|\alpha|}x\right), \tag{7.88}$$

bu yerda $\sigma = \begin{cases} 1 \text{ agar } \alpha \geq 0 \text{ bo'lsa}, \\ -1 \text{ agar } \alpha > 0 \text{ bo'lsa}. \end{cases}$

(7.88) tenglikdan ko'rinadiki,

$$\alpha f(x, x) = \sigma f\left(\sqrt{|\alpha|}x, \sqrt{|\alpha|}x\right), \tag{7.89}$$

Xaqiqatan,

$$\alpha f(x,x) = x^T(\alpha A)x = \alpha \sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n} a_{ij}\, x_i x_j = \sigma \cdot |\alpha| \sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n} a_{ij}\, x_i x_j$$

$$= \sigma \sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n} a_{ij} \left(\sqrt{|\alpha|}x_i\right)\left(\sqrt{|\alpha|}x_j\right) = \sigma\left(\sqrt{|\alpha|}x\right)^T A\left(\sqrt{|\alpha|}x\right)$$

$$= \sigma f\left(\sqrt{|\alpha|}x, \sqrt{|\alpha|}x\right),$$

Ta'rif 7.7. $f(x,x) = x^T Ax$ va $g(x,x) = x^T Bx$ kvadratik formalarni yig'indisi deb quyidagi kvadratik formaga aytiladi:

$$\mathrm{f(x,x) + g(x,x) = x^T(A + B)x} \tag{7.90}$$

1. Kvadratik formalar ustida aniqlangan qo'shish va songa ko'paytirish amallari quyidagi xossalarga ega bo'ladi.

1) $f(x,x) + g(x,x) = g(x,x) + f(x,x)$,

2) $\left(f(x,x) + g(x,x)\right) + q(x,x) = f(x,x) + (g(x,x) + q(x,x))$,

3) $f(x,x) + 0 = f(x,x)$ sharti qanoatlantiruvchi $x^T 0x$=0 element mavjud,

4) Ixtiyoriy $f(x,x)$ uchun $f(x,x) + \left(-f(x,x)\right) = 0$ shartni qanoatlantiruvchi - $f(x,x)$ qarama-qarshi element mavjud,

5) $\alpha\left(f(x,x) + g(x,x)\right) = \alpha f(x,x) + \alpha g(x,x)$,

6) $(\alpha \pm \beta)f(x,x) = \alpha f(x,x) \pm \beta fc(x,x)$,

7) $(\alpha\beta)f(x,x) = \alpha(\beta f(x,x) = \beta\left(\alpha f(x,x)\right)$,

8) $1 \cdot f(x,x)$= $f(x,x)$,

Haqiqatan,

1) $f(x,x) + g(x,x) = x^T Ax + x^T Bx = x^T(A + B)x = x^T(B + A)x = x^T Bx + x^T Ax = g(x,x) + f(x,x)$,

2)$\left(f(x,x) + g(x,x)\right) + q(x,x) = (x^T Ax + x^T Bx) + x^T Qx = x^T\left((A + B) + Q\right)x = x^T(A + (B + Q)x = x^T Ax + x^T(B + Q)x = x^T Ax + (x^T Bx + x^T Qx) = f(x,x) + (g(x,x) + q(x,x))$,

3) $f(x,x) + 0 = x^T Ax + x^T 0x = x^T(A + 0)x = x^T Ax = f(x,x)$,

4) $f(x,x)+\big(-f(x,x)\big)=x^TAx+(-x^TAx)=x^T(A-A)x=x^T0x=0,$

5) $\alpha\big(f(x,x)+g(x,x)\big)=\alpha(x^TAx+x^TBx)=\alpha x^T(A+B)x=$

$=x^T(\alpha A+\alpha B)=\alpha x^TAx+\alpha x^TAx+\alpha x^TBx=\alpha f(x,x)\pm\alpha g(x,x),$

6) $(\alpha+\beta)f(x,x)=x^T(A+B)AX=x^T(\alpha A+\beta A)x=$

$=x^T(\alpha A+\beta A)x=x^T\alpha Ax+x^T\beta Ax=\alpha x^TAx+\beta x^TAx=\alpha f(x,x)+$
$\beta f(x,x),$

7) $(\alpha\cdot\beta)f(x,x)(\alpha\cdot\beta)x^TAx=x^T(\alpha\beta)Ax=x^T\alpha(\beta A)x=$

$=\alpha x^T\beta Ax=\alpha\big(\beta f(x,x)\big).$

8) $1\cdot f(x,x)$=$1\cdot x^TAx$ =$x^TAx=\ f(x,x).$

Demak, n o’zgaruvchili kvadratik formalar to’plami chiziqli fazo tashkil qiladi.

2. Agar $f(x,x)=x^TAx$ kvadratik forma aniq (o’zgarmas) ishorali bo’lsa, u xolda $f(x,x)=x^T\alpha Ax$ kvadratik forma ham aniq (o’zgarmas) ishorali bo’lib, $\alpha>0$ da ularning ishoralari bir xil, $\alpha<0$ da esa har- xil bo’ladi.

Bu tasdiqning to’g’riligi A va $\alpha\cdot A$ matritsalarni bir vaqtda aniq (o’zgarmas) ishorali ekanligidan kelib chiqadi.

3. Agar $f(x,x)$ va $g(x,x)$ regulyar kvadratik formalar bir xil aniq (o’zgarmas) ishorali bo’lib, $\alpha>0$ bo’lsa, $f(x,x)+\alpha g(x,x)$ kvadratik formalar dastasi ham huddi shunday aniq (o’zgarmas) ishorali bo’ladi.

Haqiqatan, aniqlik uchun $f(x,x)$ va $g(x,x)$ kvadratik formalar musbat aniqlangan bo’lsin deb olamiz. $f(x,x)>0$ va $g(x,x)>0$ bo’lib, bundan $\alpha>0$ da $\alpha g(x,x>0)$ bo’lgani uchun $f(x,x)+\alpha g(x,x)>0$ ekanligi kelib chiqadi.

Endi $f(x,x)=x^TAx$ va $g(y,y)=y^TBy,$ $x=(x_1,x_2,\dots,x_n)^T,$ $y=(y_1,y_2,\dots,y_m)^t,$ $A=(a_{ij})_{i,j=1}^n,$ $a_{ij}=a_{ji};$ $B=\big(b_{k,e}\big)_{k,e=1}^m,$ $b_{k,e}=b_{e,k}$ kvadratik formalarni qaraymiz.

Agar $m=n$ bo’lib, x va y chiziqli bog’langan, ya’ni $\lambda\neq0$ mavjudki, unda $\lambda x+y=0$ bo’lsa,

$f(x,x)+g(y,y)=f(x,x)+g(-\lambda x,-\lambda x)=f(x,x)+\lambda^2 g(x,x)=$
$=x^T(A+\lambda^2 B)x$

kvadratik formalar dastasi xosil bo'ladi.

$m\neq n$ bo'lganda vektor koordinatalarini 0 lar bilan to'ldirib, $m=n$ holga keltirishimiz mumkin.

Agar x va y vektorlar chiziqli bog'lanmagan bo'lsa, $f(x,x)+g(y,y)$ yig'indini quyidagi ko'rinishdagi bitta kvadratik forma orqali ifodalash mumkin.

$$f(x,x)+g(y,y)=x^TAx+y^TBy=(x^T,y^T)\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (7.91)$$

(7.91) kvadratik forma $n+m$ o'zgaruvchili bo'lib, unga mos matritsa $n+m$ tartibli bo'ladi.

§10 n-o'zgaruvchili kvadratik formalarni ikki o'zgaruvchili kvadratik formalar yig'indisi shaklida yozish.

Ma'lumki, kvadratik formalar amaliy ahamiyatga ega bo'lib, bunda kvadratik formalarning ishoralarini aniqlash muhim ahamiyatga ega. Ammo kvadratik formadagi o'zgaruvchilar soni ortib borishi bilan unga mos matritsaning tartibi ortib borib, unga Silvestr kriteriysini qo'llash yoki uni harakteristik sonlarini topish qiyinlashib boradi. SHuning uchun maqsadimiz berilgan kvadratik formani imkon qadar kamroq o'zgaruvchili kvadratik formalar yig'indisi ko'rinishida ifodalashdan iborat.

Teorema 7.18 (7.87) kvadratik forma uchun $n>1$ da quyidagi tenglik o'rinli

$$f(x,x)=x^TAx=$$
$$=\frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{\substack{j=2\\ i<j}}^{n}(x_i,x_j)\begin{pmatrix}a_{ii} & (n-1)a_{ij}\\ (n-1)a_{ij} & a_{jj}\end{pmatrix}\begin{pmatrix}x_i\\ x_j\end{pmatrix} \qquad (7.92)$$

Isboti. (7.92) tenglikni isbotlash uchun avval o'zgaruvchilar soni $n=2$ va $n=3$ bo'lgan xususiy xollarni qarab chiqamiz.

1. $n=2$ bo'lsin, u holda (7.92) quyidagicha yoziladi.

$$f(x,x) = x^T A x = (x_1, x_2)\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2.2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. $n = 3$ bo'lsin, u holda quyidagiga ega bo'lamiz:

$$f(x,x) = x^T A x = a_{1,1} \cdot x_1^2 + a_{2.2} \cdot x_2^2 + a_{3,3} \cdot x_3^2 +$$

$$+2a_{1,2}x_1x_2 + 2a_{1,3}x_1x_3 + 2a_{2.3}x_2x_3 =$$

$$= \frac{1}{2}a_{1,1}x_1^2 + 2a_{1,2}x_1x_2 + \frac{1}{2}a_{2.2}x_2^2 + \frac{1}{2}a_{1,1}x_1^2 + 2a_{1,3}x_1x_3 +$$

$$+\frac{1}{2}a_{3,3}x_3^2 + \frac{1}{2}a_{2.2}x_2^2 + 2a_{2.3}x_2x_3 + \frac{1}{2}a_{3,3}x_3^2 =$$

$$= \frac{1}{2}\left(a_{1,1}x_1^2 + 4a_{1,2}x_1x_2 + a_{2.2}x_2^2\right) + \frac{1}{2}\left(a_{1,1}x_1^2 + 4a_{1,3}x_1x_3 + a_{3,3}x_3^2\right) +$$

$$+\frac{1}{2}\left(a_{2.2}x_2^2 + 4a_{2.3}x_2x_3 + a_{3,3}x_3^2\right) =$$

$$= \frac{1}{2}(x_1, x_2)\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2.2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(x_1, x_3)\begin{pmatrix} a_{1,1} & a_{1,3} \\ a_{1,3} & a_{3,3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} +$$

$$+\frac{1}{2}(x_2, x_3)\begin{pmatrix} a_{2.2} & a_{2.3} \\ a_{2.3} & a_{3,3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= \frac{1}{2}\sum_{\substack{i=1 \\ i}}^{2}\sum_{\substack{j=2 \\ i<j}}^{3}\left(x_i, x_j\right)\begin{pmatrix} a_{i,i} & a_{i,j} \\ a_{i,j} & a_{j,j} \end{pmatrix}\begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

Endi umumiy holni qaraymiz.

$$f(x,x) = x^T A x = \sum_{i=1}^{n}\sum_{j=n}^{n} a_{ij}\, x_i x_j = \sum_{i=1}^{n} a_{ij}\, x_i^2 + 2\sum_{i=1}^{n-1}\sum_{\substack{i=2 \\ i<j}}^{n} a_{ij}\; x_i x_j$$

Ohirgi ifodadagi birinchi yig'indini quyidagicha o'zgartirib yozamiz:

$$\sum_{i=1}^{n} a_{ij}\, x_i^2 = a_{11}x_1^2 + a_{2.2}x_2^2 +, \dots, a_{n,n}x_n^2 =$$

$$= \underbrace{\frac{a_{1,1}}{n-1}x_1^2 + \dots + \frac{a_{1,1}}{n-1}x_1^2}_{\text{n-1ta}} + \underbrace{\frac{a_{2.2.}}{n-1}x_2^2 + \dots + \frac{a_{2.2.}}{n-1}x_2^2}_{\text{n-1ta}} +$$

$$+ \dots + \underbrace{\frac{a_{n,n}}{n-1}x_n^2 + \dots + \frac{a_{n,n}}{n-1}x_n^2}_{\text{n-1ta}} =$$

$= \frac{1}{n-1}\left(a_{1,1}x_1^2 + a_{2.2}x_2^2\right) + \frac{1}{n-1}\left(a_{1,1}x_1^2 + a_{1,3}x_3^2\right) + \frac{1}{n-1}\left(a_{1,1}x_1^2 + a_{n,n}x_n^2\right) +$

$+ \frac{1}{n-1}\left(a_{2.2}x_2^2 + a_{3,3}x_3^2\right) + \frac{1}{n-1}\left(a_{2.2}x_2^2 + a_{4,4}x_4^2\right) + \cdots + \frac{1}{n-1}\left(a_{2.2}x_2^2 +\right.$

an,nxn2+

$+\ldots+\frac{1}{n-1}\left(a_{n-1,n-1}x_{n-1}^2 + a_{n-1,n}x_n^2\right) =$

$$= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{\substack{i=1 \\ i<j}}^{n}\left(a_{i,i}x_i^2 + a_{j,j}x_j^2\right)$$

Bundan,

$$f(x,x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{\substack{j=2 \\ i<j}}^{n}(a_{i,i}x_i^2 + 2(n-1)a_{i,j}x_ix_j + a_{j,j}x_j^2) =$$

$$= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{\substack{j=2 \\ i<j}}^{n}\left(x_i, x_j\right)\begin{pmatrix} a_{i,i} & n-1a_{i,j} \\ n-1a_{i,j} & a_{j,j} \end{pmatrix}\begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

ni hosil qilamiz.

Musbat (manfiy) aniqlangan kvadratik formalar yig'indisi musbat (manfiy) aniqlangan ekanligidan, (7.92) tenglikka asosan (7.87) kvadratik forma uchun quyidagilarni hosil qilamiz:

Agar (7.87) kvadratik forma uchun

$$a_{i,i}a_{j,j} - a_{i,j}^2(n-1)^2 \geq 0, \;\; i = 1,2.\ldots,n-1; \;\; j = 2.3,\ldots,n; \;\; i < j \qquad (7.93)$$

shartlar bajarilib,

$$a_{i,i} > 0 \quad \left(a_{i,i} < 0\right) \; i = 1,1,\ldots,n \qquad (7.94)$$

bo'lsa, u manfiymas (musbatmas) bo'ladi.

Agar (7.93) shartlarning kamida bittasi uchun

$$a_{i,i}a_{j,j} - (n-1)^2a_{ij}^2 > 0; \;\; 1 \leq i \leq n-1, \;\; 2 \leq j \leq n, i < j \qquad (7.95)$$

bo'lib, (7.94) shartlar bajarilsa, (7.87) kvadratik forma musbat (manfiy) aniqlangan bo'ladi.

Ko'plab mehanik sistemalarning harakat tenglamalari to'rtinchi tartibli differentsial tenglamalar sistemasi orqali ifodalanishini hisobga olib, to'rt o'zgaruvchili kvadratik formalar musbat (manfiy) aniqlanganlik shartlarini keltiramiz:

$$f(x,x) = a_{1,1}x_1^2 + a_{2.2}x_2^2 + a_{3,3}x_3^2 + a_{4,4}x_4^2 + 2a_{1,2}x_1x_2 + 2a_{1,3}x_1x_3 +$$
$$+2a_{1,4}x_1x_4 + +2a_{2.3}x_2x_3 + +2a_{2.4}x_2x_4 + +2a_{3,4}x_3x_4 \qquad (7.96)$$

kvadratik formani qaraymiz. Bu kvadratik formaning matritsasi

$$A=\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,2} & a_{2.2} & a_{2.3} & a_{2.4} \\ a_{1,3} & a_{2.3} & a_{3,3} & a_{3,4} \\ a_{1,4} & a_{2.4} & a_{3,4} & a_{4,4} \end{pmatrix}$$

bo'lib, (7.93), (7.94), (7.95) shartlarga ko'ra (7.96) kvadratik forma musbat aniqlangan bo'lishi uchun

$$a_{i,i} > 0, \quad i = 1,2,3,4; \qquad (7.97)$$

$$a_{1,1}a_{2.2} - 9a_{1,2}^2 > 0, \quad a_n a_{3,3} - 9a_{1,3}^2 \geq 0,$$
$$a_{1,1}a_{4,4} - 9a_{1,4}^2 \geq 0, \quad a_{2.2}a_{3,3} - 9a_{2.3}^2 \geq 0, \qquad (7.98)$$
$$a_{2.2}a_{4,4} - 9a_{2.4}^2 \geq 0, \quad a_{3,3}a_{3,3} - 9a_{3,4}^2 \geq 0,$$

Manfiy aniqlangan bo'lishi uchun esa (2.12) shartlar bilan bir vaqtda

$$a_{i,i} < 0, \qquad i = 1,2.3,4, \qquad (7.99)$$

shartlar bajarilishi yetarlidir.

Eslatma: (7.98) shartlardagi qat'iy tengsizlik, qolgan ixtiyoriy tengsizlik bilan almashtirilishi mumkin.

Agar $\lambda_1, \lambda_2, \dots, \lambda_n$, lar matritsaning xarakteristik sonlari bo'lib, $A = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ bo'lsa, (7.92) forma quyidagi ko'rinishni oladi:

$$f(x,x) = x^T A x = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{\substack{j=2 \\ i<j}}^{n}\left(x_i, x_j\right)\begin{pmatrix}\lambda_i & 0 \\ 0 & \lambda_j\end{pmatrix}\begin{pmatrix}x_i \\ x_j\end{pmatrix} \qquad (7.100)$$

§11. Erkinlik darajasi n bo'lgan sistemalarning kichik tebranishlari.

n ta erkinlik darajasi bo'lgan konservativ mexanik sistemani o'zining turug'un muvozanat xolati yaqinidagi erkin tebranishlarini qaraymiz.

Sistemani muvozanat xolatdan og'ishini o'zaro bog'liq bo'lmagan $q_1, q_2, \dots q_n$ umumlashgan kordinatalar yordamida beramiz. Bunda muvozanat xolatga $q_1 = 0,\ q_2 = 0, ...,\ q_n = 0$ mos keladi. U holda sistemaning kinetik energiyasi $\dot{q}_1, \dot{q}_2, \dots \dot{q}_n$ umumlashgan tezliklarning kvadratik formasi ko'rinishida tasvirlanadi.

$$T = \sum_{i,k=1}^{n} b_{ik}\left(q_1, q_2, \dots, q_n\right)\dot{q}_i\ \dot{q}_k$$

$b_{ik}\left(q_1, q_2, q \dots q_n\right)$ koeffitsentlari $q_1, q_2, \dots q_n$ larning darajalari bo'yicha qatorga yoyib,

$$b_{ik}\left(q_1, q_2, q \dots q_n\right) = b_{ik} + \dots \qquad \left(i, k = 1,2, \dots n\right)$$

Bu yoyilmaning faqat b_{ik} o'zgarmas xadlarini olib, quyidagiga ega bo'lamiz:

$$T = \sum_{i,k=1}^{n} b_{ik}\dot{q}_i\ \dot{q}_k \qquad \left(b_{ik} = b_{ki}\ ,\ \ i, k = 1,2,...,n\right)$$

Kinetik energiya har doim musbat bo'lib, faqat $\dot{q}_1 = \dot{q}_2 = ... = \dot{q}_n = 0$ dagina nolga aylanadi. SHuning uchun $T = \sum_{i,k=1}^{n} b_{ik}\dot{q}_i\ \dot{q}_k$ - musbat aniqlangan kvadratik formadir.

Sistemaning potentsial energiyasi $\Pi\left(q_1, q_2, q \dots q_n\right)$ - umumlashgan koordinatalarning funktsiyasi bo'ladi. Umumiylikni buzmasdan $\Pi_0 = \Pi(0,0,...,0) = 0$ deb olamiz. U holda potentsial energiyani $q_1, q_2, \dots q_n$ larning darajasi bo'yicha qatorga yoyib, quyidagini hosil qilamiz:

$$\Pi = \sum_{i-1}^{n} a_i q_i + \sum_{i,k=1}^{n} a_{ik} q_i q_k + ...$$

ma'lumki muvozanat holatda potentsial energiya statsionar qiymat qabul qiladi, u holda

$$a_i = \left(\frac{\partial \Pi}{\partial q_i}\right)_0 = 0, \qquad (i = 1,2,...,n)$$

Bundan $q_1, q_2, \dots q_n$ larga nisbatan ikkinchi tartibli bulgan xadlarni saqlab qolib, quyidagicha ega bo'lamiz:

$$\Pi = \sum_{i,k=1}^{n} a_{ik,} q_i q_k \left(a_{ik} = a_{ki}, i, k = 1,2, \dots, n\right)$$

SHunday qilib, Π-potentsial Energiya va T- kinetik energiya quyidagi kavdratik formulalar bilan aniqlanadi:

$$\Pi = \sum_{i,k=1}^{n} a_{ik,} q_i q_k ,T = \sum_{i,k=1}^{n} b_{ik} \dot{q}_i \dot{q}_k \qquad (7.101)$$

Bu yerdagi ikkinchi kvadratik forma musbat aniqlangan.

Endi xarakat differentsial tenglamalarini Lagranjning ikkinchi tur tenglamalari ko'rinishda yozamiz:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i}(i = 1,2,\ldots,n) \qquad (7.102)$$

Bu yerda T va Π ning o'rniga ularni (7.101) dagi ifodasini qo'yib, quyidagini hosil qilamiz:

$$\sum_{k=1}^{n} b_{ik} \ddot{q}_k + \sum_{k=1}^{n} a_{ik} q_k = 0(i = 1,2,\ldots,n) \qquad (7.103)$$

Quyidagi simmetrik matritsalarni kiritib,

$$A = (a_{ik})_{i,k=1}^{n}, \ B = (b_{i,k})_{i,k=1}^{n}, \quad q^T (q_1, q_2, \ldots, q_n)$$

(7.103) tenglamalar sistemasini quyidagicha yozamiz:

$$B\ddot{q} + Aq = 0 \qquad (7.103')$$

(7.103) sistema yechimini quyidagi garmonik tebranishlar ko'rinishda izlaymiz:

$$q_1 = v_1 \sin(\omega t + \alpha), q_2 = v_2 \sin(\omega t + \alpha), \ldots, q_n = v_n \sin(\omega t + \alpha)$$

matritsani ko'rinishda

$$q = v \sin(\omega t + \alpha) \qquad (7.104)$$

Bu yerda $v = (v_1, v_2, \ldots, v_n)^T$ -o'zgarmas ampletuda vektorlari, ω-chastata, α-boshlang'ich fazo.

(7.104) ni $(7.103')$ ga qo'yib, $\sin(\omega t + \alpha)$ ga qisqartirib quyidagini xosil qilamiz:

$$Av = \lambda Bv \qquad (\lambda = \omega^2)$$

Ammo bu $x^T Ax - \lambda x^T Bx -$ kvadratik formalar singulyar dastasi xarakteristik tenglamasi $(A - \lambda B) = 0$ dan kelib chiqadigan $Az^k = \lambda_k B \, z^k (k = 1,2,\ldots,n)$

tenglamalar bilan ustma-ust tushadi. Demak, izlanayotgan amplituda vektori bosh vektor bo'lib, chastota kvadrati $\lambda = \omega^2$ $x^T Ax - \lambda x^T Bx$ formalar regulyar dastasining mos xarakteristik sonlari bo'ladi. $\Pi(q_1, q_2, ..., q_n)$ - potentsial energiya qa'tiy minimumga ega deb faraz qilamiz. U xolda Lajan-Drixle teoremasiga asosan sistemaning muvozanat xolati turg'un bo'ladi. Ikkinchi tomondan, $\Pi = q^T Aq$ kvadratik forma musbat aniqlangan bo'ladi.

Regulyar dastalar xaqidagi teoremaga asosan kvadratik formalarning regulyar dastasi

$$x^T Ax - \lambda x^T Bx$$

n ta xaqiqiy $\lambda_1, \lambda_2, ..., \lambda_n$ xarakteristik sonlarga ega va bu n ta sonlarga mos

$$v^1, v^2, ..., v^n \quad \left[v^k = (v_{1k}, v_{2k}, ...v_{nk})^T\right]$$

bosh vektorlarga ega bo'lib, ular quyidagi shartlarni qanoatlantiradi:

$$v^{i^T} Bv^k = \sum_{\mu, y=1}^{n} b_{\mu y} v_{\mu i} v_{yk} = \delta_{ik} \qquad (i, k = 1,2,...,n) \tag{7.105}$$

$x^T Ax$ formaning musbat aniqlanganligidan,

$$x^T Ax - \lambda x^T Bx$$

dastaning barcha xarakteristik sonlari musbat

$$\lambda_k > 0 \qquad (k = 1,2,...,n)$$

ekanligi kelib chiqadi. Ammo bu xolda ampletuda vektorlari

$$v^k = (v_{1k}, v_{2k}, ..., v_{nk})^T \qquad (k = 1,2,...,n)$$

ortanormallashganlik shartini qanoatlantiruvchi quyidagi n ta garmonik tebranishlar mavjud bo'ladi.

$$v^n Sin(\omega_k t + \alpha_k) \quad (\omega_k^2 = \lambda_k, \ k = 1,2,...,n) \tag{7.106}$$

$(7.103')$ ning chiziqli ekanligidan kelib chiqadiki, ixtiyoriy tebranish (7.106) garmonik tebranishlardan xosil qilinishi mumkin:

$$q = \sum_{k=1}^{n} A_k Sin(\omega_k t + \alpha_k) v^k \tag{7.107}$$

bu yerda, A_k, α_k $(k = 1,2,...,n)$ - ixtiyoriy o'zgarmaslar.

(7.107) dan quyidagilarni topamiz:

$$q_0 = \sum_{k=1}^{n} A_k Sin\alpha_k v^k \ , \qquad \dot{q}_0 = \sum_{k=1}^{n} \omega_k A_k Cos\alpha_k v^k \qquad (7.108)$$

(7.107) yechimini quyidagicha yozish mumkin:

$$q_i = \sum_{k=1}^{n} A_k Sin(\omega_k t + \alpha_k) v_{ik} \qquad (7.109)$$

Berilgan mexanik sistema chastotalarini kamaymaydigan tartibda nomerlab chiqamiz, ya'ni

$$0 < \omega_1 \le \omega_2 \le ... \le \omega_n$$

Bu bilan

$$x^T Ax - \lambda x^T Bx$$

dasta xarakteristik sonlari $\lambda_k = \omega_k^2 \quad (k = 1,2,...,n)$ ning joylashishi aniqlanadi.

$$\lambda_1 \le \lambda_2 \le .... \le \lambda_n$$

Berilgan sistemaga h ta o'zaro bog'liq bo'lmagan chekli statsionar bog'lanishlarni qo'yamiz. $q_1, q_2, \dots q_n$ og'ishlarni kichik miqdorlar deb xisoblab, bu bog'lanishlarni $q_1, q_2, \dots q_n$ larga nisbatan chiziqli deb qarash mumkin :

$$L_1(q) = 0,\ L_2(q) = 0,\ .....\ ,\ L_n(q) = 0.$$

Bu bog'lanishlar qo'yilgandan keyin qaralayotgan sistema n-h ta erkin -lik darajasiga ega bo'ladi. Bu sistemaning chastotasi

$$\omega_1^2 \le \omega_2^2 \le .... \le \omega_{n-h}^2$$

$L_1, L_2, ..., L_h$ bog'lanishli $x^T Ax - \lambda x^T Bx$ dastaning $\lambda_1^0 \le \lambda_2^0 \le ... \le \lambda_{n-h}^0$ xarakteristik sonlari bilan $\lambda_j^0 = \left(\omega_j^0\right)^2$ munosabat orqali bog'langan. SHuning uchun

$$\omega_j \le \omega_j^0 \le \omega_{j+h} \qquad (j = 1,2,...,n-h)$$

Shunday qilib, sistemaga h ta bog'lanishlarni qo'yganimizda uning chastotasi faqat ortishi mumkin, ammo yangi j- chastota ω_j^0 miqdori eski j+L – chastota ω_{j+L} miqdordan ortmaydi.

Xuddi shuningdek aytishimiz mumkinki, sistema bikirligi ortganda, ya'ni $q^T Aq$ forma ortganda potentsial energiya uchun ($q^T Bq$- forma o'zgarmaganda) chastota faqat ortishi mumkin, sistema inertsiyasi ortganda, ya'ni $q^T Bq$ forma ortganda esa kinetik energiya uchun ($q^T Aq$ forma o'zgarmaganda) chastota faqat kamayishi mumkin.

§ 12 Chiziqli yirik masshtabli sistemalar turg'unligi masalasiga bog'liq bo'lgan ba'zi teoremalar

A n- tartibli kvadrat matritsa bo'lib, A^T, $A^{\perp}, A^0$- lar bu matritsani mos ravishda bosh, bosh bo'lmagan dioganallarga va matritsa markaziga nisbatan transponirlangani bo'lsin.

Quyidagi sistemalarni qaraymiz:

$$\dot{x} = Ax, \tag{7.110}$$

$$\dot{x} = A^T x, \tag{7.111}$$

$$\dot{x} = A^{\perp} x, \tag{7.112}$$

$$\dot{x} = A^0 x, \tag{7.113}$$

$$\dot{x} = (A + A^0)x, \tag{7.114}$$

bu yerda $x = (x_1, x_2, ..., x_n)^T \in R^n$.

Teorema 7.19. Agar (7.110) sistema muvozanat xolati turg'un (asimptotik turg'un) yoki turg'unmas bo'lsa, u holda (7.111), (7.112), (7.113) sistemalar muvozanat holati ham mos ravishda turg'un (asimptotik turg'un) yoki turg'unmas bo'ladi.

Isbot: Transponirlangan matritsalarning xossalariga ko'ra quyidagiga ega bo'lamiz.

$$|A - \lambda E| = |A^T - \lambda E| = |A^{\perp} - \lambda E| = |A^0 - \lambda E|,$$

ya'ni A, A^T, $A^{\perp}, A^0$ matritsalarning xarakteristik ko'pxadlari bir xil bo'lib, bundan teorema 1 ning to'g'riligi kelib chiqadi.

Teorema 7.20. (7.110) sistema uchun

$$v(x) = x^T Px, \tag{7.115}$$

kvadratik forma ko'rinishdagi Lyapunov funktsiyasi mavjud bo'lib, P matritsa musbat aniqlangan, bosh dioganalga yoki matritsa markaziga nisbatan simmetrik bo'lsin.Agar

$$P_1 = H^{-1}P = PH^{-1} \tag{7.116}$$

matritsa musbat aniqlangan bo'lsa, u xolda (7.110) sistema muvozanat xolati turg'unligi (asimptotik turg'unligi) yoki turg'unmasligidan

$$\dot{x} = HAx \tag{7.117}$$

sistema muvozanat xolatini mos ravishda turg'unligi, (asimptotik turg'unligi) yoki turg'unmasligi kelib chiqadi, bu yerda H, HA–matritsa xosmas, musbat aniqlangan, bosh dioganalga yoki matritsa markaziga nisbatan simmetrik.

Isbot: (7.117) sistema uchun Lyapunov funktsiyasini quyidagi kvadratik forma ko'rinishda quramiz:

$$v_1(x) = x^T P_1 x\,. \tag{7.118}$$

Teorema 7.20 ning shartiga ko'ra P_1 matritsa musbat aniqlangan, shuning uchun (7.118) funktsiya ham musbat aniqlangan bo'lib, undan (7.117) sistema yordamida olingan xosila quyidagi ko'rinishga ega bo'ladi:

$$\dot{v}_1(x) = \dot{x}^T P_1 x + x^T P_1 \dot{x} = x^T\left[(HA)^T P_1 + P_1 HA\right]x = x^T(A^T H^T H^{-1}P + PH^{-1}HA)x =$$
$$= x^T(A^T P + PA)x = \dot{v}(x).$$

Bundan, $(\dot{v}_1(x))$ va $(\dot{v}(x))$ funktsiyalarning ishora aniqlanishi bir xil ekanligi kelib chiqadi. Bu teorema 7.20 ning to'g'riligini isbotlaydi.

Teorema 7.21. (7.110) sistema uchun (7.115) Lyapunov funktsiyasi tuzilgan bo'lib, musbat aniqlangan

$$P_1 = HP = PH,$$

matritsa mavjud bo'lsin, bu yerda H, RN–matritsalar xosmas, musbat aniqlangan va bosh dioganalga yoki matritsa markaziga nisbatan simmetrik bo'lsin. U holda (7.110) sistema muvozanat xolatining turg'unligidan (asimptotik turg'unligidan) yoki turg'unmasligidan

$$\dot{x} = H^{-1}Ax$$

sistema muvozanat xolatini mos ravishda turg'unligi (asimptotik turg'unligi) yoki turg'unmasligi kelib chiqadi. .

Isboti teorema 7.20 ning isboti kabidir.

Eslatma. Agar teorema 7.20 (teorema 7.21) da $P = E$ bo'lsa, u holda teorema 7.20 (teorema 7.21) $P_1 = H^{-1}$ $(P_1 = H)$ da o'z kuchida qoladi.

Teorema 7.22.

$$v(x) = x^T Ex$$

funktsiya quyidagi sistema uchun Lyapunov funktsiyasi bo'lsin

$$\dot{x} = A_0 x, \qquad (7.119)$$

bu yerda $x \in R^n$, $A_0 \in R^{n \times n}$, E- n-tartibli birlik matritsa va $H_1, H_2, ..., H_m$ matritsalar n o'lchovli, xosmas, musbat aniqlangan, bosh dioganalga yoki matritsa markaziga nisbatash simmetrik matritsalardir.

U xolda (7.119) sistema muvozanat xolatining turg'unligi (asimptotik turg'unligi) yoki turg'unmasligidan quyidagi

$$\dot{x} = Ax, \qquad (7.120)$$

sistema muvozanat xolatining mos ravishda turg'unligi (asimptotik turg'unligi) yoki turg'unmasligi kelib chiqadi. Bu yerda

$$A = \sum_{i=1}^{m} \alpha_i A_i \quad \text{yoki} \quad A = \sum_{i=1}^{m} \alpha_i A_i^*, \; A_i = H_i A_0, \; A_i^* = H_i^{-1} A_0, \; \alpha_i \geq 0, \; i = 1,2,...,m,$$

$$\sum_{i=1}^{m} \alpha_i^2 \neq 0.$$

Isboti: $v(x) = x^T Ex$ (7.119)sistema uchun Lyapunov funktsiyasi bo'lsin. U xolda

$$(\dot{v}(x)) = x^T (A_0^T + A_0)x.$$

(7.120) sistema uchun Lyapunov funktsiyasini quyidagi

$$v_1(x) = x^T H^{-1} x,$$

kvadratik forma ko'rinishda quramiz, bu yerda $H = \sum_{i=1}^{m} \alpha_i H_i$

Teorema 7.22 ning shartiga ko'ra N musbat aniqlangan va bosh dioganalga nisbatan simmetrik. Shuning uchun $v_1(x)$ funktsiya musbat aniqlangan bo'lib,

$$\left(\dot v_1(x)\right) = x^T(A^T H^{-1} + H^{-1}A)x = x^T\left(\left(\sum_{i=1}^m \alpha_i A_i^T\right)H^{-1} + H^{-1}\left(\sum_{i=1}^m \alpha_i A_i\right)\right)x =$$
$$= x^T\left(A_0^T\left(\sum_{i=1}^m \alpha_i H_i^T\right)H^{-1} + H^{-1}\left(\sum_{i=1}^m \alpha_i H_i\right)A_0\right)x = x^T(A_0^T HH^{-1} + H^{-1}HA_0)x =$$
$$= x^T(A_0^T + A_0)x = \left(\dot v(x)\right).$$

bo'ladi. Bundan, $\left(\dot v_1(x)\right)$ va $\left(\dot v(x)\right)$ funktsiyalarning ishora aniqlanishi bir xil ekanligi kelib chiqib, teorema 7.22 ning to'g'riligi isbotlanadi.

Agar teorema 7.22 da $A_k < A_j,\ k, j = 1,2,...,n,\ k \neq j$ va $\alpha_k + \alpha_j = 1,$ bo'lib, barcha qolgan $\alpha_i = 0,\ i \neq k,\ i \neq j,$ bo'lsa, u xolda A matritsa quyidagi shartni qanoatlantiradi.

$$A_k \le A \le A_j\,. \tag{7.121}$$

Endi (7.110)sistema muvozanat xolati turg'unligini ba'zi cheklanishlarda qarab chiqamiz. Faraz qilaylik, (7.110) sistemadagi A matritsa matritsa markaziga nisbatan simmetrik bo'lsin. Bu xolda (7.110) sistemani vertikal va gorizantal simmetriya o'qlari bo'yicha yoki o'zaro muvozanatlashuvchi qism sistemalar bo'yicha dekompozitsiya qilish mumkin.

Teorema 7.23. Agar musbat aniqlangan n- tartibli R kvadrat matritsa bosh va bosh bo'lmagan dioganallarga yoki matritsa markaziga nisbatan simmetrik bo'lib,

$$v(x) = x^T P x \tag{7.122}$$

funktsiya (7.114) sistema uchun Lyapunov funktsiyasi bo'lsin. U xolda bu funktsi (7.110) sistema uchun ham Lyapunov funktsiyasi bo'ladi, ya'ni (7.114) sistema muvozanat holati turg'unligi (asimptotik turg'unligi) yoki turg'unmasligidan (7.110) sistema muvozanat xolatining mos ravishda turg'unligi (asimptotik turg'unligi) yoki turg'unmasligi kelib chiqadi.

Isbot: Bevosita tekshirib ko'rib, ishonch xosil qilish mumkinki $P = P^T = P^{\perp}$ dan $P = P^0$ ekanligi kelib chiqadi. SHuning uchun teorema 7.23 ni $P = P^0$ xol uchun isbotlash yetarli. R matritsaning musbat aniqlanganligidan (7.122) funktsiyani musbat aniqlanganligi kelib chiqadi. U xolda quyidagilarga ega bo'lamiz:

$$(\dot{v}(x)) = x^T((A+A^0)^T P + P(A+A^0))x = (\dot{v}(x)) + x^T({A^0}^T P + PA^0)x =$$
$$(\dot{v}(x)) + x^T({A^0}^T P^0 + P^0 A^0)x = (\dot{v}(x)) + x^T(A^T P + PA)x =$$
$$= (\dot{v}(x)) + (\dot{v}(x)) = 2(\dot{v}(x)).$$

Bundan, $(\dot{v}(x))$ va $(\dot{v}(x))$ funktsilarni ishora aniqlanishi bir xil ekanligi kelib chiqib, teorema 7.23 isbotlanadi.

Teorema 7.24. Agar A matritsa bosh va bosh bo'lmagan dioganallarga yoki matritsa markaziga nisbatan simmetrik bo'lib,

$$v(x) = x^T Px,$$

funktsiya (7.110) sistema uchun Lyapunov funktsiyasi bo'lsin. (bu yerda P musbat aniqlangan kvadratik matritsa). U xolda $v_1(x) = x^T P^0 x$ va $v_2(x) = x^T(P+P^0)x$ funktsilar ham (7.110) sistema uchun Lyapunov funktsiyasi bo'ladi.

Isboti: P matritsaning musbat aniqlanganligidan P^0 va $P+P^0$. Matritsalarning ham musbat aniqlanganligi kelib chiqadi. Shuning uchun $v_1(x)$ va $v_2(x)$ funktsiyalar musbat aniqlangan bo'ladi. Teorema 7.24 ning shartiga ko'ra $A = A^0$. Bu shartdan quyidagilar kelib chiqadi:

$$(\dot{v}_1(x)) = x^T(A^T P^0 + P^0 A)x = x^T({A^0}^T P^0 + P^0 A^0)x = x^T(A^T P + PA)x,$$
$$(\dot{v}_2(x)) = x^T(A^T(P+P^0) + (P+P^0)A)x = x^T(A^T P + PA)x + x^T(A^T P + PA)^0 x,$$
$$(\dot{v}(x)) = x^T(A^T P + PA)x.$$

Bundan teorema7.24 ning to'g'riligi kelib chiqadi. Shuningdek,

$$\left|A^T P + PA - \lambda E\right| = \left|(A^T P + PA)^0 - \lambda E\right|.$$

Teorema 7.25. Agar n- o'lchovli A kvadrat matritsa bosh va bosh bo'lmagan dioganallarga yoki matritsa markaziga nisbatan simmetrik bo'lsa, u xolda bu matritsani musbat (manfiy) aniqlangan bo'lishi uchun quyidagi shartlarni bajarilishi yetarlidir: $n = 2k$ da

1. $\lambda_m(A_1) > 0 \;\; (\lambda_M(A_1) < 0)$

2. $\lambda_m^2(A_1) > \frac{1}{4}\lambda_M((A_2^T + A_2^0)(A_2^T + A_2^0)^T) \;\; \left(\lambda_M^2(A_1) > \frac{1}{4}\lambda_M(A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)$;

$n = 2k+1$da

3. $\lambda_m(A_1) > 0,\ a_{k+1,k+1} > 0\ (\lambda_M(A_1) < 0,\ a_{k+1,k+1} < 0)$,

4. $\lambda_m(A_1)\,a_{k+1,k+1} > |a|\ (\lambda_M(A_1)\,a_{k+1,k+1} > |a|)$,

5. $\lambda_m^2(A_1)\,a_{k+1,k+1} > |a|^2\left(\lambda_M^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T + 2\lambda_m(A_1)) + \frac{1}{4}a_{k+1,k+1}\lambda_M(A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)$,

$\left(\lambda_M^2(A_1)a_{k+1,k+1} < |a|^2(2\lambda_M(A_1) - \lambda_M^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T)) + \frac{1}{4}a_{k+1,k+1}\lambda_M((A_2^T + A_2^0)(A_2^T + A_2^0)^T)\right)$,

bu yerda A_1 va A_2 matritsalar k- tartibli bo'lib,

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^0 & A_1^0 \end{pmatrix}$$

dan aniqlanadi, A_1^0 va A_2^0 ularni mos ravishda matrittsa markaziga nisbatan transponirlanganidir.

$a = a_1 + a_2$, $a_1 = (a_{k+1,1}, a_{k+1,2}, ..., a_{k+1,k})^T$, $a_2 = (a_{k+1,k+2}, a_{k+1,k+3}, ..., a_{k+1,n})^T$ va $a_{k+1,k+1}$

mos ravishda A matritsa markazida joylashgan vektor va skalyarlardir.

Isboti: Agar matritsa bosh va bosh bo'lmagan diogonallarga nisbatan simmetrik bo'lsa, u xolda u matritsa markaziga nisbatan ham simmetrik bo'ladi. Shuning uchun teorema 7.25 ni matritsa markaziga nisbatan simmetrik bo'lgan matritsalar uchungina isbotlaymiz. $n = 2k$ bo'lsin, u xolda teorema 7.25 ning shartiga ko'ra A matritsa quyidagi ko'rinishga ega

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^0 & A_1^0 \end{pmatrix}.$$

Quyidagi A matritsaga mos keluvchi kvadratik formani qaraymiz.

$$x^T A x = x^T \begin{pmatrix} A_1 & A_2 \\ A_2^0 & A_1^0 \end{pmatrix} x \qquad (7.123)$$

$x = (y^T, z^T)^T$ kabi belgilash kiritamiz, bu yerda $y = (x_1, x_2, ..., x_k)^T$, $z = (x_{k+1}, x_{k+2}, ..., x_n)^T$. U xolda quyidagilarni xosil qilamiz

$$x^T A x = (y^T, z^T)\begin{pmatrix} A_1 & A_2 \\ A_2^0 & A_1^0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = y^T A_1 y + z^T A_2 y + y^T A_2^0 z + z^T A_1^0 z =$$

$$y^T A_1 y + y^T (A_2^T + A_2^0) z + z^T A_1^0 z \geq \lambda_m(A_1)\|y\| - \lambda_M^{\frac{1}{2}}\left((A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)\|y\|\|z\| + \lambda_m(A_1^0)\|z\|^2 =$$

$$= (\|y\| \|z\|)\begin{pmatrix} \lambda_m(A_1) & -\frac{1}{2}\lambda^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T) \\ -\frac{1}{2}\lambda^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T) & \lambda_m(A_1^0) \end{pmatrix}\begin{pmatrix} \|y\| \\ \|z\| \end{pmatrix},$$

$$x^T Ax \le \lambda_M(A_1)\|y\|^2 - \lambda_M^{\frac{1}{2}}\left((A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)\|y\|\|z\| + \lambda_M(A_1^0)\|z\| =$$

$$= (\|y\| \|z\|)\begin{pmatrix} \lambda_M(A_1) & \frac{1}{2}\lambda_M^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T) \\ \frac{1}{2}\lambda_M^{\frac{1}{2}}((A_2^T + A_2^0)(A_2^T + A_2^0)^T) & \lambda_M(A_1^0) \end{pmatrix}\begin{pmatrix} \|y\| \\ \|z\| \end{pmatrix}.$$

Bundan kelib chiqadiki, agar teorema 7.25 dagi 1. va 2. shartlar bajarilsa, u xolda (7.123) kvadratik forma va unga mos A matritsa musbat (manfiy) aniqlanadi.

$n = 2k + 1$ bo'lsin , u holds teorema 7.25 ning shartiga ko'ra A matritsa quyidagi ko'rinishga ega bo'ladi

$$A = \begin{pmatrix} A_1 & a_2 & A_2 \\ a_1^T & a_{k+1,k+1} & (a_1^T)^0 \\ A_2^0 & a_2^0 & A_1^0 \end{pmatrix}.$$

A matritsaga mos quyidagi kvadratik formani qaraymiz:

$$x^T Ax = x^T \begin{pmatrix} A_1 & a_2 & A_2 \\ a_1^T & a_{k+1,k+1} & (a_1^T)^0 \\ A_2^0 & a_2^0 & A_1^0 \end{pmatrix} x, \tag{7.124}$$

Bu yerda $x = (y^T, x_{k+1,k+1}, z_0^T)^T$, $y = (x_1, x_2, ..., x_k)^T$, $z_0 = (x_{k+2}, x_{k+3}, ..., x_n)^T$.

U xolda quyidagilarga ega bo'lamiz:

$$x^T Ax = (y^T, x_{k+1,k+1}, z_0^T)\begin{pmatrix} A_1 & a_2 & A_2 \\ a_1^T & a_{k+1,k+1} & (a_1^T)^0 \\ A_2^0 & a_2^0 & A_1^0 \end{pmatrix}\begin{pmatrix} y \\ x_{k+1,k+1} \\ z_0 \end{pmatrix} = y^T A_1 y + z_0^T (A_2^T + A_2^0) y +$$

$$+ y^T a x_{k+1,k+1} + a_{k+1,k+1} x_{k+1,k+1}^2 + z_0^T a^0 x_{k+1,k+1} + z_0^T A_1^0 z_0,$$

bu yerda $a = a_1 + a_2$.

Bundan, $\lambda_m(A_1)=\lambda_m(A_1^0),\ \ |a|=|a^0|,$ ekanligini xisobga olib, (7.124) kvadratik forma uchun quyidagi quyidan va yuqoridan olingan baxolarga ega bo’lamiz:

$$x^T Ax \ge \lambda_m(A_1)\|y\|^2 - \lambda_M^{\frac{1}{2}}\left((A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)\|y\|\|z_0\| - |a|\|y\||x_{k+1,k+1}| + a_{k+1,k+1}x_{k+1,k+1}^2 -$$
$$-|a|\|z_0\||x_{k+1,k+1}| + \lambda_m(A_1)\|z_0\|^2 =$$
$$=(\|y\|,|x_{k+1,k+1}|\|z_0\|)\begin{pmatrix} \lambda_m(A_1) & -\frac{1}{2}|a| & -\frac{1}{2}\lambda_M^{\frac{1}{2}}(A_2^T + A_2^0)(A_2^T + A_2^0)^T \\ \frac{1}{2}|a| & a_{k+1,k+1} & -\frac{1}{2}|a| \\ -\frac{1}{2}\lambda_M^{\frac{1}{2}}(A_2^T + A_2^0)(A_2^T + A_2^0)^T & -\frac{1}{2}|a| & \lambda_m(A_1) \end{pmatrix}\begin{pmatrix}\|y\| \\ |x_{k+1,k+1}| \\ \|z_0\|\end{pmatrix},$$

$$x^T Ax \le \lambda_M(A_1)\|y\|^2 - \lambda_M^{\frac{1}{2}}\left((A_2^T + A_2^0)(A_2^T + A_2^0)^T\right)\|y\|\|z_0\| + |a|\|y\||x_{k+1,k+1}| + a_{k+1,k+1}x_{k+1,k+1}^2 +$$
$$+|a|\|z_0\||x_{k+1,k+1}| + \lambda_M(A_1)\|z_0\|^2 =$$
$$=(\|y\|,|x_{k+1,k+1}|\|z_0\|)\begin{pmatrix} \lambda_M(A_1) & \frac{1}{2}|a| & -\frac{1}{2}\lambda_M^{\frac{1}{2}}(A_2^T + A_2^0)(A_2^T + A_2^0)^T \\ -\frac{1}{2}|a| & a_{k+1,k+1} & \frac{1}{2}|a| \\ \frac{1}{2}\lambda_M^{\frac{1}{2}}(A_2^T + A_2^0)(A_2^T + A_2^0)^T & \frac{1}{2}|a| & \lambda_M(A_1) \end{pmatrix}\begin{pmatrix}\|y\| \\ |x_{k+1,k+1}| \\ \|z_0\|\end{pmatrix}.$$

Bu baxolardan kelib chiqadiki, agar teorema 7.25 dagi 3., 4. va 5. shartlar bajarilsa, u xolda (7.124) kvadratik forma va unta mos A matritsa musbat (manfiy) aniqlangan bo’ladi.

Misol 7.4 Quyidagi oltinchi tartibli chiziqli sistemani qaraymiz:

$$\dot{x} = Ax, \tag{7.125}$$

bu yerda $x=(x_1,x_2,x_3,x_4,x_5,x_6)^T$,

$$A=\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 1 \\ -1 & -2 & 0 & 1 & 0 & 1 \\ 1 & 0 & -3 & -1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \\ 1 & 0 & 1 & 0 & -2 & -1 \\ 1 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

A matritsa matritsa markaziga nisbatan simmetrik bo’lgani uchun quyidagilarga ega bo’lamiz:

$$A_1 = \begin{pmatrix} -2 & 0 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}, \qquad A_1^0 = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & 0 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad A_2^0 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A_2^T + A_2^0 = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad (A_2^T + A_2^0)(A_2^T + A_2^0)^T = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix},$$

$$\lambda_M(A_1) = \lambda_M(A_1^0) = -1{,}382 < 0, \quad \lambda_m(A_1) = \lambda_m(A_1^0) = -3{,}68 < 0,$$

$$\lambda_M((A_2^T + A_2^0)(A_2^T + A_2^0)^T) = 6, \quad \lambda_M^2(A_1) = 1{,}909924 > \frac{1}{4}\lambda_M((A_2^T + A_2^0)(A_2^T + A_2^0)^T) = 1{,}5.$$

Demak, teorema 7.25 ga asosan A matritsa manfiy aniqlangan va (7.125) sistema muvozanat xolati asimptotik turg’un bo’ladi.

§13. Dempfirlanishi va bikirligi oshkor xolatda vaqtga bog’liq bo’lib, chiziqsiz bo’lgan sistema asimptotik turg’unligining yetarli shartlari

Dempfirlanishi va bikirligi oshkor xolatda vaqtga bog’liq bo’lib, chiziqsiz bo’lgan sistema toyigan xarakat tenglamasini quyidagi ko’rinishda qaraymiz:

$$\ddot{x} + \beta(t, x, \dot{x})\dot{x} + \alpha(t, x, \dot{x})x = 0, \qquad (7.126)$$

bu yerda α va β lar $t, x, \dot{x}$ xaqiqiy o’zgaruvchilarni

$$t \geq t_0, \quad x^2 + \dot{x}^2 \leq \mu \qquad (7.127)$$

soxada aniqlangan haqiqiy funktsiyalari (t_0, μ - musbat o’zgarmaslar).

α va β koefitsientlar o’zgarmas bo’lib, musbat bo’lganda $x = 0, \ \dot{x} = 0$ toyimagan xarakat asimptotik turg’un bo’ladi. Agar bu koeffitsientlar musbat xolatda qolib, o’zgaruvchi bo’lsa, u xolda ularni o’zgarish rejimi mavjudki unda xarakat turg’unmas bo’lib qoladi. α va β koeffitsientlarni o’zgarish qonuni ma’lum bo’lsa, turg’unlik masalasini qarab chiqish mumkin. Ba’zi amaliy tadbiqlarda α va β koeffitsientlarni xarakterlari ma’lum bo’lmay, ularni o’zgarish chegaralarigina ma’lum bo’ladi ya’ni (7.127) soxada bu funktsiyalar quyidagi shartlarni qanoatlantiradi:

$$a_1 \le \alpha(t,x,\dot{x}) \le a_2, \quad b_1 \le \beta(t,x,\dot{x}) \le b_2, \tag{7.128}$$

bu yerda, a_1, a_2, b_1, b_2 - berilgan musbat sonlar

Bu ko’rsatilgan chegaralarda α va β ixtiyoriy qonun bo’yicha o’zgarganda, $x=0,\ \dot{x}=0$ toyimagan xarakat asimptotik turg’un bo’ladigan yetarli shartlarni xosil qilish katta qiziqish uyg’otadi.

Qo’yilgan masalani qarab chiqish uchun

$$x = x_1, \quad \dot{x} = x_2$$

o’zgaruvchi almashtirib, (7.126) tenglamani quyidagi sistema bilan almashtiramiz:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha(t,x,\dot{x})x_1 - \beta(t,x,\dot{x})x_2 \end{aligned}$$

yoki matritsa ko’rinishida

$$\dot{y} = A(t,y)y, \tag{7.129}$$

bu yerda $y = (x_1, x_2)^T$, $A(t,y) = \begin{pmatrix} 0 & 1 \\ -\alpha(t,y) & -\beta(t,y) \end{pmatrix}$

Qat’iylik tartibi mos ravishda ε_1 va ε_2 bo’lgan $\varphi_1(t,y)$ va $\varphi_2(t,y)$ qat’iy musbat funktsiyalarni quyidagicha kiritamiz

$$\varphi_1(t,y) = m_1 - \alpha(t,y), \quad \varphi_2(t,y) = m_2 - \beta(t,y) \tag{7.130}$$

Tekshirib ko’rish mumkinki (7.128) tengsizlik bajarilganda $\varphi_1(t,y)$va $\varphi_2(t,y)$ funktsiyalar uchun quyidagi baxolar o’rinli:

$$m_1 - a_2 \le \varphi_1(t,y) \le m_1 - a_1, \quad m_2 - b_2 \le \varphi_2(t,y) \le m_2 - b_1 \tag{7.131}$$

bundan, $m_1 = a_2 + \varepsilon_1,\ m_2 = b_2 + \varepsilon_2$ ekanligi kelib chiqadi. SHuning uchun (7.131) ni quyidagicha yozish mumkin.

$$\varepsilon_1 \le \varphi_1(t,y) \le a_2 - a_1 + \varepsilon_1, \quad \varepsilon_2 \le \varphi_2(t,y) \le b_2 - b_1 + \varepsilon_1$$

(7.130) yordamida (7.129) ni quyidagi ko’rinishga keltiramiz

$$\dot{y} = A_0 y + \varphi_1(t,y)A_1 y + \varphi_2(t,y)A_2 y, \tag{7.132}$$

bu yerda $A_0 = \begin{pmatrix} 0 & 1 \\ -m_1 & -m_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$

(7.132) sistema uchun Lyapunov funktsiyasini quyidagi kvadratik forma ko'rinishida quramiz:

$$V(y) = y^T P y \tag{7.133}$$

bu yerda $P = \begin{pmatrix} \dfrac{m_1(m_1+1)}{m_2} & 1 \\ 1 & \dfrac{m_1+1}{m_2} \end{pmatrix}$

bu kvadratik forma musbat aniqlangan bo'ladi.

(7.133) funktsiyadan (7.132) sistema yordamida olingan xosila uchun quyidagi xosil qilamiz:

$$\dot{V}(y) = \dot{y}^T P y + y^T P \dot{y} = y^T (G_0 + \varphi_1(t,y) G_1 + \varphi_2(t,y) G_2) y,$$

bu yerda $G_0 = \begin{pmatrix} -2m_1 & 0 \\ 0 & -2m_1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 2 & \dfrac{m_1+1}{m_2} \\ \dfrac{m_1+1}{m_2} & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2\dfrac{m_1+1}{m_2} \end{pmatrix}$

va

$$\dot{V}(y) \le (\lambda_M(G_0) + (a_2 - a_1 + \varepsilon_1)\lambda_M(G_1) + (b_2 - b_1 + \varepsilon_2)\lambda_M(G_2)) \|y\|^T, \tag{7.134}$$

bu yerda

$$\lambda_M(G_0) = -2m_1, \quad \lambda_M(G_1) = 1 + \sqrt{1 + \left(\frac{m_1+1}{m_2}\right)^2}, \; \lambda_M(G_2) = \frac{m_1+1}{m_2} + \sqrt{1 + \left(\frac{m_1+1}{m_2}\right)^2}.$$

(7.134) baxodan ko'rinadiki, $\dot{V}(y)$ funktsiya quyidagi shart bajarilganda manfiy aniqlangan bo'ladi.

$$\lambda_M(G_0) + (a_2 - a_1 + \varepsilon_1)\lambda_M(G_1) + (b_2 - b_1 + \varepsilon_2)\lambda_M(G_2) < 0, \tag{7.135}$$

Bundan kelib chiqadiki, Lyapunovning asimptotik turg'unlik teoremasiga asosan (7.132) yoki (7.126) sistema toyimagan xarakati (7.135) shart bajarilganda asimptotik turg'un bo'ladi.

$\dfrac{m_1+1}{m_2} = k$ bo'lsin, u xolda (7.135) quyidagi ko'rinishda bo'ladi:

$$-2m_1 + (m_1 - a_1)(1 + \sqrt{1+k^2}) + (m_2 - b_1)(k - \sqrt{1+k^2}) < 0,$$

yoki

$$(m_1+m_2-a_1-b_1)\sqrt{1+k^2}<a_1+b_1k-1.$$

Bu tengsizlik $a_1+b_1k-1>0$ shartda ma’noga ega bo’ladi.

Bu tengsizlikdan $\varepsilon_1\to 0,\ \ \varepsilon_2\to 0$ va $k=\dfrac{a_2+1+\varepsilon_1}{b_2+\varepsilon_2}$ ni e’tiborga olib,

$$k=\frac{m_1+1}{m_2}=\frac{a_2+1}{b_2},\quad \sqrt{1+k^2}=\sqrt{1+\frac{(b_2+1)^2}{b_2^2}}=\frac{\sqrt{b_2^2+(a_2+1)^2}}{b_2}$$

$$a_2-a_1+b_2-b_1<\frac{1}{\sqrt{b_2^2+(a_2+1)^2}}\left[(a_2+1)b_1+(a_1-1)b_2\right] \tag{7.136}$$

$$(a_2+1)b_1+(a_1-1)b_2>0 \tag{7.137}$$

larni xosil qilamiz.Shunday qilib, $\alpha(t,x,\dot{x})$ va $\beta(t,x,\dot{x})$funktsiyalarning a_1,a_2,b_1,b_2 chegaralari (7.136) va (7.137) shartlarni qanoatlantirsa, u xolda toyimagan xarakat $x=0$, $\dot{x}=0$ asimptotik turg’un bo’ladi.

Mashqlar.

1. Quyidagi kvadratik formalarni kanonik ko’rinishga keltiring:
   a) $2x_1^2+x_2^2-4x_1x_2-4x_2x_3$
   b) $x_1^2+2x_2^2+3x_3^2-4x_1x_2-4x_2x_3$
   c) $3x_1^2+4x_2^2+5x_3^2+4x_1x_2-4x_2x_3$
   d) $2x_1^2+5x_2^2+5x_3^2+4x_1x_2-4x_1x_3-8x_2x_3$
   e) $x_1^2-2x_2^2-2x_3^2-4x_1x_2+4x_1x_3+8x_2x_3$
   f) $5x_1^2+6x_2^2+4x_3^2-4x_1x_2-4x_1x_3$
   g) $3x_1^2+6x_2^2+3x_3^2-4x_1x_2-8x_1x_3-4x_2x_3$
   h) $7x_1^2+5x_2^2+3x_3^2-8x_1x_2+8x_2x_3$
   i) $2x_1^2+2x_2^2+2x_3^2+2x_4^2-4x_1x_2+2x_1x_4+2x_2x_3-4x_3x_4$
   j) $2x_1x_2+2x_3x_4$
   k) $x_1^2+x_2^2+x_3^2+x_4^2+2x_1x_2-2x_1x_4-2x_2x_3+2x_3x_4$
   l) $2x_1x_2+2x_1x_3-2x_1x_4-2x_2x_3+2x_2x_4+2x_3x_4$

m) $x_1^2+x_2^2+x_3^2+x_4^2-2x_1x_2+6x_1x_3-4x_1x_4-4x_2x_3+6x_2x_4-2x_3x_4$

n) $8x_1x_3+2x_1x_4+2x_2x_3+8x_2x_4$

2. Agar A-kososimmetrik matritsa bo'lsa, u xolda $B=(E-A)(E+A)^{-1}$ matritsani ortoganal ekanligini isbotlang

VIII-BOB.
YIRIK MASSHTABLI SISTEMALAR TURG’UNLIGINING UMUMIY MASALASI

Ko’plab jarayonlar yirik mastabli sistemalar (Y.M.S.) orqali modellashtiriladi. Bunday sistemalarning xarakterli belgilari quyidagilardir:

1. Ko’p o’lchovlilik;

2.Sistema strukturasining ko’p karraliligi (to’rlar, daraxtlar,ierarxik strukturalar va boshqalar);

3.Sistema elementlarining ko’p bog’lamliligi (qism sistemalar orasidagi bitta darajadagi va xar xil darajali ierarxiyalar orasidagi bog’lanishlar);

4.Elementlari tabiatining ko’pxilliligi (mashinalar, avtomatlar, robotlar, odam-operatorlar);

5.Sistema xolati va tarkibi o’zgarishining ko’pkarraliligi (sistema strukturasi, tarkibi va bog’lanishlarning o’zgaruvchanligi);

6.Sistemaning ko’p kriteriyaliligi (qism sistemalar uchun har-xil lokal kriteriyalar va to’la sistema uchun globalь kriteriyalar, ularning qarama-qarshililigi).

7.Sistema yoki uning elementlari tabiatiga xos bo’lgan ba’zi bir xususiy belgilar.

Bu belgilar sistema dinamik xossalarini (turg’unlik, boshqaruvchanlik, kuzatuvchanlik, optimallik) o’rganishda ma’lum qiyinchiliklarga olib keladi.

§1. Masalaning qo’yilishi

Ma’lumki, Lyapunov funktsiyasini qurishning umumiy algaritmi mavjud bo’lmaganligi uchun yuqoridagi xarakterli belgilarga ega bo’lgan sistemalar turg’unligini taxlili qilishda xosil bo’ladigan qiyinchiliklar Lyapunovning to’g’ri usulini effektiv ravishda qo’llash imkonini bermaydi.

Shuning uchun qaralayotgan sistemani soddalashtirish yoki boshqa sistema bilan almashtirish bilan bog’liq bo’lgan bir qancha yo’nalishlar paydo qilinadi. bo'lib, bularda turg'unlik (turg'unmaslik) masalasi bevosita berilgan sistemani tekshirish yo'li bilan emas, balki bu oralik sistemalarni tekshirish yo'li bilan xal etiladi.

Yirik masshtabli sistemalar turg'unligining umumiy masalasi quyidagicha ikki xil ko'rinishda qo'yiladi.

A muammo: Y.M.S. o'zaro bog'liq bo'lgan qism sistemalarga dekompazitsiya qilingan (ajratilgan) bo'lsin. Shunday shartlarni aniqlashimiz kerakkki, bularda sistemaning Lyapunov bo'yicha turg'unligi o'zaro bog'liq bo'lmagan qism sistemalarning turg'unligidan va ular orasidagi bog'lanishlarning xossalaridan kelib chiqsin. Boshqacha aytganda, masalani bunday qo'yilishida, avval Y.M.S. ni tashkil qiluvchi har bir erkin qism sistema uchun Lyapunov funktsiyasi tuzilib, ularni turg'unligi (turg'unmasligi)ning yetarli shartlari xosil qilinadi. So'ngra bu qism sistemalar orasidagi bog'lanishlarning xossalaridan va erki n qism sistemalar turg'unligining yetarli shartlaridan foydalanib, berilgan sistema turg'unligining yetarli shartlari xosil

B muammo: Y.M.S. o'zaro bog'liq bo'lgan qism sistemalarga dekompazitsiya qilingan (ajratilgan) bo'lsin. Sistema tartibi pasayishini taminlaydigan shunday shartlarni aniqlashimiz kerakki, unda qaralayotgan sistemaning turg'unligi o'zaro bog'liq bo'lgan qism sistemalar xossalaridan kelib chiqib, bunda erkin qism sistemalar turg'unligi xaqidagi ma'lumotlardan foydalanilmaydi. Boshqacha aytganda masalaning bunday qo'yilishida, har bir qism sistema uchun Lyapunov funktsiyasi tanlanib, ular yordamida qaralayotgan sistema uchun Lyapunov funktsiyasi tuziladi va qaralayotgan sistema turg'unligining yetarli shartlari xosil qilinadi.

Bu ko'rsatilgan muammolar Y.M.S. turg'unligini tekshirish bilan shug'ulanuvchi mutaxassislar uchun bosh yo'nalishlarni aniqlab berdi. Ularni xal etish yo'lida bir qancha matematik usullar yaratildi yoki rivojlantirildi. Jumladan, Lyapunovning vektor-funktsiyasi usuli, vektorli normalar usuli, minimaksli usul, Lyapunov matritsa funktsiyasi usuli va boshqalar.

§2. Yirik masshtabli sistemalarning dekompozitsiyasi.

Faraz qilaylik (S). Y.M.S. xolati quyidagi differentsial tenglama bilan ifodalansin.

$$\frac{dx}{dt} = f(t,x), \qquad (8.1)$$

bu yerda $t \in T = [t_0,\ \infty) \subset R,\ \ x \in R^n, f \in F$ bo'lib, F oila quyidagicha aniqlanadi:

$$F = \{f^1, f^2, \dots, f^N\}, f^k: TxR^n \to \mathrm{R}^n, \qquad (8.2)$$

N-haqiqiy son.

Agar (S). Y.M.S. s ta qism sistemalardan tashkil topgan bo'lsa, u xolda (S). Y.M.S.ni (S_i) o'zaro bog'liq bo'lgan qism sistemalardan tashkil topgan deb qarash mumkin bo'lib, (S_i) o'zaro bog'liq bo'lgan qism sistemalar quyidagi tenglamalar bilan ifodalanadi:

$$\frac{dx_i}{dt} = f_i(t,x), \quad \forall\, i = 1,2,\dots,s \qquad (8.3)$$

bu yerda x va f vektorlar quyidagicha aniqlanadi:

$$x = (x_1^T, x_2^T, \dots, x_s^T)^T, \qquad f = (f_1^T, f_2^T, \dots, f_s^T)^T,$$

shuningdek,

$$f_i \in F_i,\ F_i = \left(f_i^1,\ f_i^2, \dots,\ f_i^N\right),\ f_i^k : \mathrm{Tx}R^n \to R^{n_i},$$

$$\forall\, k = 1,2,\dots,N,\ \ n_1 + n_2 + \cdots n_s = n\,,\ \ i = 1,2,\dots,s.$$

f_i funktsiyalar barcha $t \in R$ da, faqat va faqat x=0 dagina

$$f(t,x)=0$$

shartni qanoatlantiradi.

$$x^i = \left(O^T, O^T, \dots, x_i^T, O^T, \dots, O^T\right)^T, \qquad i = 1,2,\dots,s$$

belgilash kiritamiz,

$$g_i: \mathrm{Tx}R^{n_i} \to R^{n_i},$$

funktsiyani

$$g_i(t,x_i) = f_i\left(t,x^i\right)$$

tenglik bilan aniqlaymiz.

U xolda ($\hat{S}_i$) i- erkin qism sistema

$$\frac{dx_i}{dt} = g_i(t, x_i), \qquad (8.4)$$

bu yerda $x_i \in R^{n_i}$,

$$f_i^*(t, x) = f_i(t, x) - g_i(t, x_i), \qquad \forall\, i = 1,2, \dots, s \qquad (8.5)$$

ifoda yordamida aniqlangan f_i^* funktsiya (S) sistemani o'zining (S_i) i- qism sistemasiga ta'sirini ifodalaydi.

(8.4) va (8.5) ga asosan (8.3) ni quyidagicha yozish mumkin:

$$\frac{dx_i}{dt} = g_i(t, x_i) + f_i^*(t, x), \quad \forall\, i = 1,2, \dots, s \qquad (8.6)$$

Agar

$$g = (g_1^T, g_2^T, \dots, g_s^T)^T, \qquad f^* = \left(f_1^{*^T}, f_2^{*^T}, \dots, f_s^{*^T}\right)^T,$$

deb olsak, (8.6) ni quyidagicha yozish mumkin:

$$\frac{dx}{dt} = g(t, x) + f^*(t, x), \qquad (8.7)$$

(S) Y.M.S. ni (8.6) ko'rinishda dekompozitsiya qilish nazariy bo'lib, amaliy tadbiqlarda bu bir muncha noqulaydir. Masalan (8.5) tenglik yordamida f_i^* funktsiyalarni har doim ham aniqlash qulay bo'lavermaydi. Shuning uchun (S) Y.M.S. ni dekompozitsiya qilishni qulaylashtirish maqsadida har bir s ta (S_i) qism sistemalardan iborat bo'lgan Y.M.S. bilan sxs tartibli matritsa o'rtasida quyidagicha o'zaro bir qiymatli mosli o'rnatamiz:

1. (S) Y.M.S.ning $(\hat{S}_i)$ erkin qism sistemalarini matritsaning bosh dioganaliga mos qo'yamiz;

2. $(\hat{S}_i)$ qism sistemalarni $(\hat{S}_j)$ qism sistemaga ta'sirini ifodalovchi funktsiyani i-satr va j-ustun kesishgan joyga mos qo'yamiz. Natijada matritsaning bosh dioganalida erkin qism sistemalar bo'lib, bosh dioganaldan tashqarida bu erkin qism sistemalar orasidagi to'g'ri va teskari bog'lanishlar bo'ladi.

Bunday moslik o'rnatilgandan keyin (8.1) sistema quyidagi ko'rishlarning biriga keladi:

$$\frac{dx}{dt} = Ax, \qquad (8.8)$$

$$\frac{dx}{dt} = A(t)x, \qquad (8.9)$$

$$\frac{dx}{dt} = A(x), \qquad (8.10)$$

$$\frac{dx}{dt} = A(t,x), \qquad (8.11)$$

bu yerda A, A(t),A(x), A(t,x)-bosh dioganalga nisbatan simmetrik bo'lgan sxs tartibli kvadrat matritsalar.

(8.8), (8.9), (8.10), (8.11) sistemalarni dekompozitsiya qilish masalasi mos ravishda A, A(t),A(x), A(t,x)-matritsalarni blok matritsalarga ajratish masalasi bilan teng kuchli bo'lgani uchun (S) Y.M.S.ni dekompozitsiya qilish masalasi matritsani blok matritsalarga ajratish masalasiga keladi.

Misol tariqasida (8.8) sistemani ba'zi xususiy xollarda dekompazitsiya qilish usullarini qarab chiqamiz. Faraz qilaylik, A matritsa markazga nisbattan simmetrik bo'lgan sxs o'lchovli kvadrat matritsa bo'lsin. U xolda (8.8) sistemani quyidagicha usullarda dekompaziya qilish mumkin:

1.Vertikal va gorizantal simmetriya o'qlarga nisbatan dekompazitsiya qilish. Bu xolda (8.8) sistema n=2k da,

$$\begin{aligned} \dot{y} &= A_1 y + B_1 z, \\ \dot{z} &= B_1^0 y + A_1^0 z, \end{aligned} \qquad (8.12)$$

ko'rinishga , n=2k+1 da esa

$$\begin{aligned} \dot{y} &= A_1 y + a_2 x_{k+1} + B_1 z, \\ \dot{x}_{k+1} &= a_1^T y + a_{k+1,k+1} x_{k+1} + (a_1^T)^0 z \\ \dot{z} &= B_1^0 y + a_2^0 x_{k+1} + A_1^0 z, \end{aligned} \qquad (8.13)$$

ko'rinishga keladi. Bu yerda erkin qism sistemalar

$$\dot{y} = A_1 y, \qquad (8.14)$$

$$\dot{z} = A_1^0 z, \qquad (8.15)$$

$$\dot{x}_{k+1} = a_{k+1,k+1} x_{k+1}. \qquad (8.16)$$

ko'rinishlarda bo'lib, A_1, B_1- k-tartibli kvadrat matritsalar, A_1^0, B_1^0 -mos ravishda ularni markazga nisbatan transponirlangani, vektorlar esa

$$a_1 = (a_{k+1,1}, a_{k+1,2}, ..., a_{k+1,k})^T,\ a_2 = (a_{k+1,k+2}, a_{k+1,k+3}, ..., a_{k+1,n})^T,$$

$$y = (x_1, x_2, ..., x_k)^T,\ z = (x_{k+2}, x_{k+3}, ..., x_n)^T,\ x = (y^T, x_{k+1}, z^T)^T.$$

ko’rinishda aniqlangan.

(8.8) sistema muvozanat xolati turg’unligining yetarli shartlarini xosil qilish uchun, (8.12) sistema va (8.14), (8.15) qism sistemalarga mos Lyapunov matritsa funktsiyasi quyidagi ko’rinishda tanlanadi:

$$U_1(y,z)=\begin{pmatrix} v_{11}(y) & v_{12}(y,z) \\ v_{21}(y,z) & v_{22}(z)\end{pmatrix}, \quad v_{12}=v_{21}\ , \tag{8.17}$$

bu yerda $v_{11}(y)=y^T P_1 y,\ v_{22}(z)=z^T P_1^0 z,\ v_{12}(y,z)=v_{21}(y,z)=y^T P_2 z,\quad P_1,\ P_1^0$ - bosh dioganalga nisbatan simmetrik bo’lgan musbat aniqlangan matritsalar, P_2 - o’zgarmas matritsa bo’lib, bularning barchasi k-tartibli matritsalardir.

(8.13)sistema va (8.14), (8.15), (8.16) qism sistemalarga mos Lyapunov matritsa funktsiyasi esa, quyidagicha tanlanadi.

$$U_2(y,x_{k+1},z)=(v'_{ij}),\quad i,j=1,2,3,\quad v'_{ij}=v'_{ji} \tag{8.18}$$

bu yerda

$v'_{11}=v_{11}(y),\ v'_{12}=\alpha_{12}x_{k+1}y,\ \ v'_{13}=v_{12}(y,z),\ v'_{21}=v'_{12},\quad v'_{22}=\alpha_{22}x_{k+1}^2, v'_{23}=\alpha_{23}x_{k+1}z,$

$v'_{33}=v_{22}(z),\ v'_{31}=v'_{13}, v'_{32}=v'_{23}\quad , P_1, P_1^0,\ P_2$ -matritsalar (8.17) dagidek aniqlanadi,

$\alpha_{22}>0,\ \alpha_{12},\alpha_{23}$ -lar xaqiqiy sonlar

2.O’zaro muvozanatlashuvchi qism sistemalarga nisbatan dekompazitsiya qilish. Bu xolda (8.8) sistema n=2k da

$$\dot{y}_i=A_{ii}y_i+\sum_{\substack{j=1\\ i\neq j}}^{k} A_{ij}y_j\,,\quad i=1,2,...,k=\frac{n}{2} \tag{8.19}$$

ko’rinishga, n=2k+1 da esa,

$$\begin{gathered}\dot{y}_i=A_{ii}y_i+\sum_{\substack{j=1\\ i\neq j}}^{k} A_{ij}y_j+a_i x_{k+1},\quad i=1,2,...,k=\frac{n-1}{2},\\ \dot{x}_{k+1}=a_{k+1,k+1}x_{k+1}+\sum_{i=1}^{k}a_i y_i\,,\end{gathered} \tag{8.20}$$

ko’rinishga keladi. Bu yerda

$$A_{ii}=\begin{pmatrix} a_{ii} & a_{i,n+1-i} \\ a_{i,n+1-i} & a_{ii}\end{pmatrix},\qquad A_{ij}=\begin{pmatrix} a_{ij} & a_{i,n+1-j} \\ a_{i,n+1-j} & a_{ij}\end{pmatrix},$$

$$a_i = (a_{i,k+1}, a_{i,k+1})^T, \qquad a_j = (a_{k+1,j}, a_{k+1,j})^T,$$

$$y_i = (x_i, x_{n+1-i})^T, \qquad y = (y_1^T, y_2^T, ..., y_k^T)^T, \; i, j = 1,2,...,k, \; i \neq j,$$

bo’lib, A_{ii} va A_{ij} lar matritsa markaziga nisbatan simmetrik bo’lgan matritsalardir. Bu xolda i- erkin qism sistemalar

$$\dot{y}_i = A_{ii} y_i \tag{8.21}$$

ko’rinishda bo’lib, ularning muvozanat xolati turg’unligini yetarli shartlari

$$a_{ii} < 0, \; |a_{ii}| > |a_{i,n+1-i}|, \; i = 1,2,...,k \tag{8.22}$$

shartlardan,

$$\dot{x}_{k+1} = a_{k+1,k+1} x_{k+1} \tag{8.23}$$

qism sistema uchun muvozanat xolat turg’unligining yetarli sharti

$$a_{k+1,k+1} < 0. \tag{8.24}$$

dan iborat bo’ladi. (8.19) va (8.20) sistemalar, hamda (8.21), (8.23) qism sistemalar uchun yuqoridagi kabi Lyapunov matritsa funktsiyalarini tuzish mumkin.

§3. Lyapunov matritsa funktsiyasi usuli.

Lyapunovning matritsa funktsiyasi usuli Y.M.S. lar turg’unligi masalasi bilan shug’ulanuvchi mutaxassislar tomonidan yaratilgan bo’lib, uning moxiyati quyidagicha: Avval (8.4) qism sistemalarning har biri uchun $v_{ii}(t, x_i)$, i=1,2,…,s Lyapunov funktsiyalari tuzilib, (S_i) va (S_j) qism sistemalar orasidagi ta’sirlarni ifodalovchi bog’lanishlarga mos

$$v_{ij}(t, x_i, x_j) = v_{ji}(t, x_i, x_j), \quad \text{i,j=1,2,…,s}$$

funktsiyalar shunday tanlanadiki, unda

$$U(t,x) = \begin{pmatrix} v_{11}(t,x_1) & v_{12}(t,x_1,x_2) & \dots & v_{1s}(t,x_1,x_s) \\ v_{12}(t,x_1,x_2) & v_{22}(t,x_2) & \dots & v_{2s}(t,x_2,x_s) \\ \dots & \dots & \dots & \dots \\ v_{1s}(t,x_1,s_2) & v_{1s}(t,x_2,x_s) & \dots & v_{ss}(t,x_s) \end{pmatrix} \tag{8.25}$$

matritsa funktsiya musbat aniqlangan bo'lsin. So'ngra (8.6) sistema uchun Lyapunov funktsiyasi (8.25) matritsa funktsiya va $\eta = (\eta_1, \eta_2, \dots, \eta_s)^T$ o'zgarmas vektor yordamida quyidagicha tuziladi:

$$\mathrm{V}(\mathrm{t}, \mathrm{x}) = \eta^{\mathrm{T}} \mathrm{U}(\mathrm{t}, \mathrm{x}) \eta \qquad (8.26)$$

(8.25) matritsa funktsiyaning musbat aniqlanganlik shartlaridan foydalanib, (8.26) skalyar funktsiyaning musbat aniqlanganlik shartini quyidagi ko'rinishda aniqlaymiz:

$$\mathrm{V}(\mathrm{t}, \mathrm{x}) \geq \psi^{T}(\mathrm{x}) \mathrm{H}^{\mathrm{T}} \mathrm{B} \mathrm{H} \psi(\mathrm{x}), \qquad (8.27)$$

bu yerda

$$\psi^{T}(\mathrm{x}) = (|\psi_1(x)|, |\psi_2(x)|, \dots, |\psi_s(x)|), \quad H = H^T = diag(\eta_1, \eta_2, \dots, \eta_s),$$

B- o'zgarmas matritsa bo'lib, uning elementlari (8.25) matritsa funktsiya elementlarini quyidan baxolashda xosil bo'ladigan o'zgarmaslarning algebraik yig'indisidan ibarat bo'ladi. (8.27) tengsizlikdan ko'rinadiki, agar V o'zgarmas matritsa musbat aniqlangan bo'lsa, (8.26) funktsiya ham musbat aniqlangan bo'ladi.

(8.26) funktsiyadan (8.6) sistema yordamida olingan to'la xosilani yuqoridan baxolab, quyidagi tengsizlikka kelamiz.

$$\dot{V}(\mathrm{t}, \mathrm{x}) \leq \psi^{T}(\mathrm{x}) \mathrm{G} \psi(\mathrm{x}) , \qquad (8.28)$$

bu yerda G o'zgarmas matritsa bo'lib, uning elementlari (8.25) matritsa funktsiya elemenlaridan (8.4) qism sistemalar va (8.6) sistema yordamida olingan xosilalarni yuqoridan baxolash natijasida xosil bo'ladigan o'zgarmaslar va η_i, i=1,2,…,s lar ishtirokida tuzilgan ifodalardan iborat bo'ladi. (8.28) dan ko'rinadiki, $\dot{V}(\mathrm{t}, \mathrm{x})$ manfiy (yarim manfiy) aniqlangan bo'lishi uchun G- o'zgarmas matritsani manfiy (yarim manfiy) aniqlangan bo'lishi yetarlidir.

(8.27) va (8.28) tengsizliklardan foydalanib, (8.1) yoki (8.6) sistema muvozanat xolati asimptotik turg'unligi (turg'unligi)ning yetarli shartlarini quyidagicha ifodalaymiz.

Teorema 8.1. (8.1) tenglama bilan ifodalangan (S) Y.M.S. (8.6) ko'rinishda dekompozitsiya qilingan bo'lib, uning uchun (8.25) matritsa-funktsiya tuzilgan bo'lsin.

Agar V matritsa musbat aniqlangan bo'lib, G matritsa yarim manfiy (manfiy) aniqlangan bo'lsa, (8.1) sistema muvozanat xolati x=0 turg'un (asimptotik turg'un) bo'ladi.

Eslatma: A muammoni xal etishda (8.25) matritsa funktsiya bosh dioganalidagi $v_{ii}(t, x_i)$, $\mathrm{i} = 1,2, \dots, s$ elementlardan (8.4) qism sistemalar bo'yicha olingan xosilalar aloxida baxolanishi shart, chunki bu baxo yordamida $(\hat{S}_i)$, i=1,2,…,s erkin qism sistemalarning turg'unligi masalasi xal etiladi. V muammoni xal etishda esa bunday baxolash shart emas.

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