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***"ANIQ INTEGRALLAR"* MAVZUSIDA YOZGAN**

REFERATI

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ANNOTATSIYA:

Ushbu, aniq integrallar mavzusidagi referatda, aniq integralga olib keladigan geometrik masalalar, aniq integral tushunchasi, aniq integral xossalari, integrallanuvchi funksiyalar sinfi atroflicha yoritilgan hamda Riman bo'yicha integrallanuvchi va integrallanmaydigan funksiyalarga misollar keltirilgan shuningdek, yetarlicha hajmda misol va masalalar yechib ko'rsatilgan.

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1. Aniq integral tushunchasi

**1.1^0. Segmentni bo`laklash.** Biror $[a,b]\subset R$ segment beril-gan bo`lsin. Bu segmentning quyidagi

$$a=x_0<x_1<x_2<...x_{n-1}<x_n=b$$

munosabatda bo`lgan

$$x_0,\ x_1,\ x_2,...x_{n-1},\ x_n \qquad (1)$$

nuqtalari to`plamini olaylik.

Ravshanki, (1) to`plam $[a,b]$ segmentni

$$B_1=[x_0,x_1]\ ,\ B_2=[x_1,\ x_2\ ],\ ...\ ,\ B_n=[x_{n-1},x_n]$$

bo`laklarga ajratadi.

1-ta`rif. Ushbu

$$a=x_0<x_1<x_2<...x_{n-1}<x_n=b$$

munosabatda bo`lgan

$$x_0,\ x_1,\ x_2,...x_{n-1},\ x_n$$

nuqtalar to`plami $[a,b]$ segmentni bo`laklash deyiladi va

$$P=\{x_0,\ x_1,\ x_2,...x_{n-1},\ x_n\}$$

kabi belgilanadi.

Bunda har bir x_k $(k=0,1,2,...,n)$ nuqta $[a,b]$ segmentning bo`luvchi nuqtasi, $[x_k,x_{k+1}]$ $(k=0,1,2,\ ...\ ,n-1)$ segment esa $P$ bo`laklashning oralig`i deyiladi.

Quyidagi

$$\lambda_p=\max\{\Delta x_k\}\quad ,\qquad \Delta x_k=x_{k+1}-x_k$$

miqdor P bo`laklashning diametri deyiladi.

Masalan, $[a,b]=[0,1]$ bo`lganda quyidagi

$$0,\ \frac{2}{10},\ \frac{4}{10},\ \frac{6}{10},\ \frac{8}{10},\ \frac{10}{10}=1\ ;$$

$$0,\ \frac{1}{10},\ \frac{4}{10},\ \frac{5}{10},\ \frac{6}{10},\ \frac{10}{10}=1$$

nuqtalar sistemasi $[0,1]$ segmentning

$$P_1=\left\{0,\ \frac{2}{10},\ \frac{4}{10},\ \frac{6}{10},\ \frac{8}{10},1\right\},$$

$$P_2=\left\{0,\ \frac{1}{10},\ \frac{4}{10},\ \frac{5}{10},\ \frac{6}{10},1\right\}$$

bo`laklashlarini hosil qiladi. Ularning diametrlari mos ravishda

$$\lambda_{p_1}=\frac{1}{5}\ ,\qquad \lambda_{p_2}=\frac{2}{5}$$

bo`ladi.

YUqoridagi keltirilgan ta`rif va misollardan ko`rina-diki, $[a,b]$ segmentning turli usular bilan istalgan sondagi bo`laklashlarini tuzish mumkin. Bu bo`laklashlardan iborat to`plamni bilan belgilaymiz:

$$=\{P\}.$$

**1.2⁰. Darbu hamda integral yig`indilar.** $f(x)$ funksiya $[a,b]$ da aniqlangan va chegaralangan bo`lsin. Unda

$$\exists m \in R, \quad \exists M \in R, \quad \forall x \in [a,b] \quad : \quad m \le f(x) \le M$$

bo`ladi.

Aytaylik,

$$P = \{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$$

$[a,b]$ segmentning biror bo`laklashi bo`lsin. U holda bu bo`lak-lashning har bir $[x_k, x_{k+1}]$ $(k = 0,1,2, ..., n-1)$ oralig`ida

$$\begin{aligned} m_k &= \inf\{f(x)\}, \quad x \in [x_k\ x_{k+1}], \\ M_k &= \sup\{f(x)\}, \quad x \in [x_k\ x_{k+1}] \end{aligned} \qquad (k = 0,1,2, ..., n-1)$$

mavjud bo`lib

$$\inf_{x \in [a,b]}\{f(x)\} \le m_k \le M_k \le \sup_{x \in [a,b]}\{f(x)\} \tag{2}$$

bo`ladi.

2-ta`rif. Ushbu

$$s = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashiga nisbatan Darbuning quyi yig`indisi deyiladi.

Ravshanki, bu yig`indi $f(x)$ funksiyaga hamda $[a,b]$ ning $P$ bo`laklashiga bog`liq bo`ladi:

$$s = s(f; P) \quad .$$

3-ta`rif. Ushbu

$$S = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashiga nisbatan Darbuning yuqori yig`indisi deyiladi.

Bu yig`indi $f(x)$ funksiyaga hamda $[a,b]$ ning $P$ bo`lak-lashiga bog`liq bo`ladi:

$$S = S(f; P) \quad .$$

Endi har bir $k \in \{0,1,2, ..., n-1\}$ ning qiymatida $[x_k, x_{k+1}]$ segmentda ixtiyoriy ξ_k nuqtani tayinlaymiz: $\xi_k \in [x_k, x_{k+1}]$ $(k = 0,1,2, ..., n-1)$. Natijada $[a,b]$ ning P bo`laklashiga nisbatan

$$\{\xi_0, \xi_1, \xi_2, ..., \xi_{n-1}\}$$

nuqtalar to`plami hosil bo`ladi. Bu nuqtalardagi $f(x)$ funksiyaning

$$f(\xi_k) \qquad (k = 0,1,2, ..., n-1)$$

qiymatlari yordamida ushbu

$$\sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

yig`indini tuzamiz.

4-ta`rif. Quyidagi

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashiga nisbatan integral yig`indisi deyiladi.

Integral yig`indi, $f(x)$ funksiyaga, $P$ bo`laklashga hamda har bir $[x_k, x_{k+1}]$ da olingan ξ_k nuqtalarga bog`liq bo`ladi:

$$\sigma = \sigma(f; P; \xi_k).$$

Ravshanki, $\xi_k \in [x_k \ , \ x_{k+1}]$ uchun

$$m_k \le f(\xi_k) \le M_k$$

bo`lib, ayni paytda

$$s(f;P) \le \sigma(f;P;\xi_k) \le S(f;P) \qquad (3)$$

tengsizliklar bajariladi.

1-misol. Ushbu

$$f(x) = |x|$$

funksiyaning $[-1, \ 1]$ segmentda quyidagi

$$P = \left\{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

bo`laklashga nisbatan Darbu yig`indilari hamda

$$\xi_k = x_k \qquad (k = 0, 1, 2, \dots, 7)$$

deb, integral yig`indi topilsin.

◄ Berilgan $f(x) = |x|$ funksiya uchun $[-1, \ 1]$ segmentning

$$P = \left\{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

bo`laklashida

$$m_0 = \frac{3}{4}, \quad m_1 = \frac{1}{2}, \quad m_2 = \frac{1}{4}, \quad m_3 = 0 \ ,$$

$$m_4 = 0 \ , \quad m_5 = \frac{1}{4}, \quad m_6 = \frac{1}{2}, \quad m_7 = \frac{3}{4}$$

$$M_0 = 1, \ M_1 = \frac{3}{4}, \ M_2 = \frac{1}{2}, \ M_3 = \frac{1}{4},$$

$$M_4 = \frac{1}{4}, \ M_5 = \frac{1}{2}, \ M_6 = \frac{3}{4}, \ M_7 = 1$$

hamda

$$\xi_0 = -1, \ \xi_1 = -\frac{3}{4}, \ \xi_2 = -\frac{1}{2}, \xi_3 = -\frac{1}{4},$$

$$\xi_4 = 0, \ \xi_5 = \frac{1}{4}, \ \xi_6 = \frac{1}{2}, \ \xi_7 = \frac{3}{4},$$

$$f(\xi_0) = 1, \ f(\xi_1) = \frac{3}{4}, \ f(\xi_2) = \frac{1}{2}, \ f(\xi_3) = \frac{1}{4},$$

$$f(\xi_4) = 0, \ f(\xi_5) = \frac{1}{4}, \ f(\xi_6) = \frac{1}{2}, f(\xi_7) = \frac{3}{4}$$

bo`ladi.

Endi $\Delta x_k = \dfrac{1}{4} \quad (k = 0, 1, 2, \dots, 7)$ bo`lishini e`tiborga olib topamiz:

$$s(f;P)=(\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+0+0+\frac{1}{4}+\frac{1}{2}+\frac{3}{4})\cdot\frac{1}{4}=\frac{3}{4},$$

$$S(f;P)=(1+\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{3}{4}+1)\cdot\frac{1}{4}=5,$$

$$\sigma(f;P;\xi_k)=(1+\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+0+\frac{1}{4}+\frac{1}{2}+\frac{3}{4})\cdot\frac{1}{4}=1. \blacktriangleright$$

**1.3^0. Aniq integral ta`rifi.** Faraz qilaylik, $f(x)$ funksiya $[a,b]$ da berilgan va chegaralangan bo`lsin. Unda $[a,b]$ oraliqning har qanday P bo`laklashi hamda har qanday $\xi_k$ $(\xi_k \in [x_k, x_{k+1}]$ , $k=0,1,2,\ldots,n-1)$ larda yuqoridagi (2) va (3) munosabatlar o`rinli bo`lib,

$$(b-a)\cdot\inf_{[a,b]}\{f(x)\}\le s(f;P)\le\sigma(f;P;\xi_k)\le$$
$$\le S(f;P)\le(b-a)\cdot\sup_{[a,b]}\{f(x)\} \qquad (4)$$

bo`ladi.

Endi $[a,b]$ segmentning bo`laklashlar to`plami $=\{P\}$ ning har bir $P\in$ bo`laklashga nisbatan $f(x)$ funksiyaning Darbu yig`indilari $s(f,P)$ va $S(f;P)$ ni tuzib , ushbu

$$\{s(f;P)\}\ ,\ \{S(f;P)\}$$

to`plamlarni qaraymiz. Bu to`plamlar (4) munosabatga ko`ra chegaralangan bo`ladi.

**5-ta`rif.** $\{s(f;P)\}$ to`plamning aniq yuqori chegarasi $f(x)$ funksiyaning $[a,b]$ oraliqdagi quyi integrali deyiladi va

$$\int_{\underline{a}}^{b} f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_{\underline{a}}^{b} f(x)dx=\sup_{P}\{s(f;P)\}.$$

**6-ta`rif.** $\{S(f;P)\}$ to`plamning aniq quyi chegarasi $f(x)$ funksiyaning $[a,b]$ oraliqdagi yuqori integrali deyiladi va

$$\int_{a}^{\overline{b}} f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_{a}^{\overline{b}} f(x)dx=\inf_{P}\{S(f;P)\}.$$

7-ta`rif. Agar $f(x)$ funksiyaning quyi hamda yuqori integrallari bir-biriga teng

$$\int_{\underline{a}}^{b} f(x)dx=\int_{a}^{\overline{b}} f(x)dx$$

bo`lsa, $f(x)$ funksiya $[a,b]$ oraliq bo`yicha integrallanuvchi (Riman ma`nosida integrallanuvchi) deyiladi.

Bunda quyi hamda yuqori integrallarning umumiy qiymati $f(x)$ funksiyaning $[a,b]$ oraliq bo`yicha aniq integrali (Riman integrali) deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

a son integralning quyi chegarasi, b son esa integralning yuqori chegarasi, $[a,b]$ segment integrallash oralig`i deyiladi.

Eslatma. YUqorida keltirilgan $f(x)$ funksiyaning integrali ta`rifiga binoan integral

$$\int_a^b f(x)dx$$

o`zgarmas sonni ifodalaydi. Binobarin, integral ostida o`zgaruvchining qanday yozilishiga bog`liq bo`lmaydi:

$$\int_a^b f(x)dx = \int_a^b f(t)dt\ .$$

2-misol. $f(x) = C \quad , \quad C \in R \ , \quad x \in [a,b]$ bo`lsin.

Bu funksiyaning integrallanuvchanligi aniqlansin.

◄ $[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$$

bo`laklashini olib, unga nisbatan Darbu yig`indilarini topamiz:

$$s(C;P) = \sum_{k=0}^{n-1} C \cdot \Delta x = C \cdot (b-a)\ , \quad S(C;P) = \sum_{k=0}^{n-1} C \cdot \Delta x_k = C \cdot (b-a)\ .$$

Bundan

$$\operatorname*{Sup}_P \{s(C;P)\} = C \cdot (b-a)\ , \qquad \inf_P \{S(C;P)\} = C \cdot (b-a)$$

bo`lib,

$$\underline{\int_a^b} C \cdot dx = \overline{\int_a^b} C \cdot dx$$

bo`lishi kelib chiqadi.

Demak, $f(x) = C$ funksiya $[a,b]$ da integrallanuvchi va

$$\int_a^b C \cdot dx = C \cdot (b-a).$$

Xususan , $f(x) = 1$ bo`lganda

$$\int_a^b dx = b - a$$

bo`ladi. ►

3-misol. $f(x) = D(x)$, $x \in [0,1]$ bo`lsin. Bu Dirixle funk-tsiyasini $[0,1]$ da integrallanuvchilikka tekshirilsin.

◄ $[0,1]$ segmentning ixtiyoriy P bo`laklashiga nisbatan Dirixle funksiyasining Darbu yig`indilari

$$s(D;P) = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k = 0, \qquad S(D;P) = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k = b - a.$$

bo`lib,

$$\operatorname*{Sup}_P \{s(D;P)\} = 0 \quad , \quad \inf_P \{S(D;P)\} = b - a$$

bo`ladi . Demak,

$$\underline{\int_0^1} D(x)dx = 0, \qquad \overline{\int_0^1} D(x)dx = b - a \quad ,$$

$$\underline{\int_0^1} D(x)dx \neq \overline{\int_0^1} D(x)dx.$$

Dirixle funksiyasi integrallanuvchi emas. ►

**1.4^0. Integral yig`indining limiti.** Faraz qilaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo`lib, u shu segmentda chegaralangan bo`lsin.

$[a,b]$ segmentni biror

$$P = \{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$$

bo`laklashini olamiz.

Ma`lumki, $f(x)$ funksiyaning bu bo`laklashga nisbatan integral yig`indisi

$$\sigma(f;P;\xi_k) = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

bo`ladi.

**8-ta`rif.** Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $[a,b]$ segmentni diametri $\lambda_P < \delta$ bo`lgan har qanday P bo`laklashi uchun tuzilgan $\sigma(f;P;\xi_k)$ yig`indi ixtiyo-riy ξ_k nuqtalarda

$$|\sigma(f;P;\xi_k) - J| = \left| \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k - J \right| < \varepsilon$$

tengsizlikni bajarsa, J son $\sigma(f;P;\xi_k)$ yig`indining $\lambda_P \to 0$ dagi limiti deyiladi va

$$\lim_{\lambda_P \to 0} \sigma(f;P;\xi_k) = J$$

kabi belgilanadi.

Bu ta`rifni quyidagicha ham aytish mumkin:

Agar

$$\forall \varepsilon > 0 \quad , \quad \exists \delta > 0 \quad , \quad \forall P \in \rho \quad , \quad \lambda_P < \delta \quad , \; \forall \xi_k$$

uchun

$$|\sigma(f;P;\xi_k) - J| < \varepsilon$$

bo`lsa, u holda

$$\lim_{\lambda_P \to 0} \sigma(f;P;\xi_k) = J$$

deyiladi.

**9-ta`rif.** Agar $\lambda_P \to 0$ da $f(x)$ funksiyaning integral yig`indisi $\sigma(f;P;\xi_k)$ chekli J limitga ega bo`lsa, $f(x)$ funksiya $[a,b]$ segmentda integrallanuvchi (Riman ma`nosida integral-lanuvchi) deyiladi, J soniga esa $f(x)$ funksiyaning $[a,b]$ segment bo`yicha aniq integrali deyiladi. Uni

$$\int_a^b f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_a^b f(x)dx = \lim_{\lambda\to 0}\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k .$$

4-misol. $f(x)=x$, $x\in[a,b]$ bo`lsin. Bu funksiyaning aniq integrali topilsin.

◄ $[a,b]$ segmentning ixtiyoriy

$$P=\{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$$

bo`laklashini olib unga nisbatan $f(x)=x$ funksiyaning integ-ral yig`indisini tuzamiz:

$$\sigma(f;P;\xi_k)=\sum_{k=0}^{n-1}\xi_k\cdot\Delta x_k \quad ,$$

bunda

$$\Delta x_k = x_{k+1}-x_k \quad , \quad x_k \le \xi_k \le x_{k+1} \quad (k=0,1,,2,...,n-1).$$

Endi

$$x_k \le \xi_k \le x_{k+1}$$

tengsizliklarni $\Delta x_k > 0$ ga ko`paytirib, hosil bo`lgan

$$x_k\cdot\Delta x_k \le \xi_k\cdot\Delta x_k \le x_{k+1}\cdot\Delta x_k$$

tengsizliklarni k ning $0,1,2, ..., n-1$ qiymatlari bo`yicha had-lab qo`shib

$$\sum_{k=0}^{n-1} x_k\cdot\Delta x_k \le \sum_{k=0}^{n-1}\xi_k\cdot\Delta x_k \le \sum_{k=0}^{n-1} x_{k+1}\cdot\Delta x_k ,$$

ya`ni

$$\sum_{k=0}^{n-1} x_k\cdot\Delta x_k \le \sigma(f;P;\xi_k) \le \sum_{k=0}^{n-1} x_{k+1}\cdot\Delta x_k \qquad (3)$$

bo`lishini topamiz.

Bu tengsizliklardagi

$$\sum_{k=0}^{n-1} x_k\cdot\Delta x_k \quad , \quad \sum_{k=0}^{n-1} x_{k+1}\cdot\Delta x_k$$

yig`indilarni Δx_k lar orqali ifodalaymiz:

$$\sum_{k=0}^{n-1} x_k\cdot\Delta x_k = \sum_{k=0}^{n-1} x_k(x_{k+1}- \ x_k) = \frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}^2- \ x_k{}^2) - \frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}- \ x_k)^2 =$$

$$=\frac{1}{2}(x_n{}^2 - x_0{}^2) - \ \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2 = \frac{b^2-a^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2,$$

$$\sum_{k=0}^{n-1} x_{k+1}\Delta x_k = \sum_{k=0}^{n-1}(x_k+\Delta x_k)\cdot\Delta x_k = \sum_{k=0}^{n-1} x_k\cdot\Delta x_k + \sum_{k=0}^{n-1}\Delta x_k{}^2 = \frac{b^2-a^2}{2} + \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2.$$

Natijada (3) tengsizliklar ushbu

$$\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2 \le \sigma(f;P;\xi_k) \le \frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2$$

ko`rinishga keladi. Bu munosabatdan

$$\left|\sigma(f;P;\xi_k)-\frac{b^2-a^2}{2}\right| \le \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2.$$

bo`lishi kelib chiqadi.

Ravshanki,

$$\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k{}^2 \le \lambda_P\,\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k = \frac{b-a}{2}\lambda_P\ .$$

Demak, $\forall\varepsilon>0$ songa ko`ra $\delta=\dfrac{2\varepsilon}{b-a}$ deyilsa, u holda $\lambda_P<\delta$ bo`lgan ixtiyoriy P bo`laklash va ixtiyoriy ξ_k larda

$$\left|\sigma(x;P;\xi_k)-\frac{b^2-a^2}{2}\right| = \left|\sum_{k=0}^{n-1}\xi_k\cdot\Delta x_k-\frac{b^2-a^2}{2}\right| < \varepsilon$$

bo`ladi. Bu esa

$$\lim_{\lambda_p\to 0}\sigma(x;P;\xi_k)=\frac{b^2-a^2}{2}=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\xi_k\cdot\Delta x_k=\frac{b^2-a^2}{2}$$

bo`lishini bildiradi. Demak,

$$\int_a^b x\,dx=\frac{b^2-a^2}{2}. \ \blacktriangleright$$

SHunday qilib, $f(x)$ funksiyaning aniq integrali ikki xil ta`riflanadi. Bu ta`riflar ekvivalent ta`riflar bo`ladi. (Qaralsin, [1] 9-bob)

Odatda, $[a,b]$ segment bo`yicha integrallanuvchi funksiya-lar to`plami $R([a,b])$ kabi belgilanadi:

$f(x)\in R([a,b]) \Leftrightarrow f(x)$ funksiya $[a,b]$ da integrallanuvchi.

2. Funksiyaning integrallanuvchilik mezoni (kriteriysi)

**2.1^0. Darbu yig`indilarining xossalari.** $f(x)$ funksiya $[a,b]$ segmentda berilgan va chegaralangan bo`lib,

$$P=\{x_0,x_1,x_2,\dots,x_{n-1},x_n\}$$

$[a,b]$ ning biror bo`laklashi bo`lsin. Ravshanki, bu holda $f(x)$ funksiyaning Darbu yig`indilari

$$s(f,P)=\sum_{k=0}^{n-1}m_k\cdot\Delta x_k\ ,$$

$$S(f;P)=\sum_{k=0}^{n-1}M_k\cdot\Delta x_k$$

mavjud bo`ladi, bunda

$$m_k=\inf\{f(x)\}\ ,\quad x\in[x_k,x_{k+1}]\ ,$$

$$M_k=\sup\{f(x)\}\ ,\quad x\in[x_k,x_{k+1}]\ ,$$

$$\Delta x_k=x_{k+1}-x_k\ ,\quad k=0,1,2,\dots,n-1.$$

1) $[a,b]$ segmentning ixtiyoriy P bo`laklashiga nisbatan tuzilgan $f(x)$ funksiyaning Darbu yig`indilari uchun

$$(b-a)\cdot\inf_{[a,b]}\{f(x)\}\leq s(f;P)\leq S(f;P)\leq(b-a)\cdot\sup_{[a,b]}\{f(x)\}$$

bo`ladi.

◄ Bu munosabat 32-ma`ruzadagi (3) tengsizliklardan kelib chiqadi. ►

Aytaylik,

$$P=\{x_0,x_1,x_2,\ldots,x_{n-1},x_n\}$$

$[a,b]$ segmentning biror bo`laklashi bo`lsin. Bu bo`laklashning bo`luvchi nuqtalari x_k $(k=0,1,2,\ldots,n)$ qatoriga yangi bo`luvchi nuqtalarni qo`shib, $[a,b]$ segmentning boshqa P' bo`laklashini hosil qilamiz. Uni $P\subset P'$ kabi belgilaymiz.

2) $[a,b]$ segmentining ixtiyoriy P va P' bo`laklashlari $(P\subset P')$ uchun

$$s(f;P)\leq s(f;P')\ ,$$
$$S(f;P)\geq S(f;P')$$

munosabatlar o`rinli bo`ladi.

◄ $[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,x_2,\ldots,x_{n-1},x_n\}$$

bo`laklashini olaylik. Soddalik uchun $P'$ bo`laklash P ning barcha bo`luvchi nuqtalari hamda qo`shimcha bitta x' nuqtadan yuzaga kelgan bo`lsin. Bu x' nuqta x_k hamda x_{k+1} nuqtalar orasida joylashsin:

$$x_k<x'<x_{k+1}.$$

Demak,

$$P'=\{x_0,x_1,x_2,\ldots,x_k,x',x_{k+1},\ldots,x_{n-1},x_n\}.$$

Bu bo`laklashlarga nisbatan Darbuning quyi yig`indilarini yozamiz:

$$s(f;P)=m_0\Delta x_0+m_1\Delta x_1+\ldots+m_k\Delta x_k+\ldots+m_{n-1}\Delta x_{n-1}$$
$$s(f;P')=m_0\Delta x_0+m_1\Delta x_1+\ldots$$
$$\ldots+\left[m_k^1\cdot(x^1-x_k)+m_k^{11}\cdot(x_{k+1}-x^1)\right]+\ldots+m_{n-1}\cdot\Delta x_{n-1}\ ,$$

bunda,

$$m_k^1=\inf\{f(x)\}\quad,\quad x\in[x_k,x']\ ,$$
$$m_k^{11}=\inf\{f(x)\}\ ,\quad x\in[x',x_{k+1}].$$

Endi $m_k^1\geq m_k$, $m_k^{11}\geq m_k$ bo`lishini e`tiborga olib topamiz:

$$s(f;P')-s(f;P)=m_k^1(x'-x_k)+m_k^{11}(x_{k+1}-x')-$$
$$-m_k\Delta x_k\geq m_k(x'-x_k)+m_k(x_{k+1}-x')-m_k\Delta x_k=0.$$

Keyingi munosabatdan

$$s(f;P)\leq s(f;P')$$

bo`lishi kelib chiqadi.

Xuddi shunga o`xshash,

$$S(f;P)\geq S(f;P')$$

bo`lishi isbotlanadi. ►

3) $[a,b]$ ning ixtiyoriy P_1 va P_2 $(P_1\in\ \ P_2\in\ \ )$ bo`laklash-larga nisbatan Darbu yig`indilari uchun

$$s(f;P_1)\leq S(f;P_2)$$

tengsizlik o`rinli bo`ladi.

◄ P_1 va P_2 bo`laklashlarning barcha bo`luvchi nuqtalari yordamida $[a,b]$ ning P' bo`laklashini hosil qilamiz. Ravshanki,

$$P_1 \subset P', \ P_2 \subset P'$$

bo`ladi.

YUqorida keltirilgan 1) va 2) xossalardan foydalanib topamiz:

$$s(f;P_1) \le s(f;P') \le S(f;P') \le S(f;P_2). \ ►$$

Natija. $[a,b]$ segmentda chegaralangan ixtiyoriy $f(x)$ funksiya uchun

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$$

bo`ladi.

◄ SHartga ko`ra $f(x)$ funksiya $[a,b]$ da chegaralangan. Demak,

$$\underline{\int_a^b} f(x)dx = \sup_P \{s(f;P)\},$$

$$\overline{\int_a^b} f(x)dx = \inf_P \{S(f;P)\}$$

integrallar mavjud.

YUqoridagi 3) xossa hamda aniq chegara ta`riflaridan

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$$

bo`lishi kelib chiqadi. ►

2.2⁰ Integrallanuvchilik mezoni (kriteriysi). endi $[a,b]$ segmentda berilgan va chegaralangan $f(x)$ funksiyaning aniq integralining mavjudligi masalasini qaraymiz.

1-teorema. $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo`li-shi uchun $\forall \varepsilon > 0$ son olinganda ham $[a,b]$ segmentning shunday $P$ bo`laklashi topilib, unga nisbatan

$$S(f;P) - s(f;P) < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

Bu teorema quyidagicha ham ifodalanishi mumkin:

$$f(x) \in R([a,b]) \Leftrightarrow \forall \varepsilon > 0, \ \exists P \in \{P\}: S(f;P) - s(f;P) < \varepsilon.$$

◄ **Zarurligi.** Aytaylik, $f(x) \in R([a,b])$ bo`lsin. Ta`rifga binoan

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$$

bo`ladi.

Ixtiyoriy musbat ε sonni olaylik. Unda quyi va yuqori integrallarning ta`riflariga ko`ra

$$\exists P_1 \in \{P\}: \quad \underline{\int_a^b} f(x)dx - s(f;P_1) < \frac{\varepsilon}{2} \quad ;$$

$$\exists P_2 \in \{P\}: \quad S(f;P_2) - \overline{\int_a^b} f(x)dx < \frac{\varepsilon}{2}$$

bo`ladi.

Endi $[a,b]$ segmentning P_1 va P_2 bo`laklashlarning barcha bo`luvchi nuqtalaridan $[a,b]$ ning P bo`laklashini hosil qilamiz.

Ravshanki, $P_1 \subset P$, $P_2 \subset P$ bo`ladi. Darbu yig`indilarining 1) va 2) xossalaridan foydalanib P bo`laklash uchun

$$\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < s(f;P_1) \le s(f;P) \le S(f;P) \le S(f;P_2) <$$

$$< \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$$

bo`lishini topamiz.

Keyingi munosabatlardan

$$S(f;P) - s(f;P) < \varepsilon$$

bo`lishi kelib chiqadi.

Etarliligi. Aytaylik,

$$\forall \varepsilon > 0 \text{ , } \exists P \in \{P\} : \quad S(f;P) - s(f;P) < \varepsilon$$

bo`lsin. Unda yuqorida keltirilgan natijaga ko`ra

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$$

bo`lib,

$$s(f;P) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le S(f;P)$$

bo`ladi. Bu tengsizliklardan

$$0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le S(f;P) - s(f;P)$$

bo`lishi kelib chiqadi.

Demak,

$$\forall \varepsilon > 0 \ : \ 0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon$$

Keyingi tengsizlikdan topamiz:

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

Demak, $f(x) \in R([a,b])$ ►

(Aniq integralning mavjudligi haqidagi teoremani quyida-gicha ham aytsa bo`ladi:

$f(x)$ funksiya $[a,b]$ da integrallanuvchi bo`lishi uchun $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilib, $[a,b]$ segmentni diametri $\lambda_p < \delta$ bo`lgan har qanday P bo`laklashga nisbatan

$$S(f;p) - s(f;p) < \varepsilon$$

tengsizlikni bajarilishi zarur va etarli)

Avvalgidek $f(x)$ funksiyaning $[x_k, x_{k+1}]$ $(k = 0,1,2, \ldots, n-1)$ ora-liqdagi tebranishini ω_k orqali belgilaymiz.

U holda

$$S(f;P)-s(f;P)=\sum_{k=0}^{n-1}(M_k-m_k)\cdot\Delta x_k=\sum_{k=0}^{n-1}\omega_k\Delta x_k$$

bo`lib, 1-teorema quyidagicha ifodalanadi:

$f(x)$ funksiya $[a,b]$ da integrallanuvchi bo`lishi uchun $\forall\varepsilon>0$ son olinganda ham $[a,b]$ segmentning shunday $P$ bo`laklashi topilib, unga nisbatan

$$\sum_{k=0}^{n-1}\omega_k\cdot\Delta x_k<\varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

Demak,

$$f(x)\in R([a,b])\Leftrightarrow\forall\varepsilon\in 0\ ,\ \exists P\in\{P\}:\ \sum_{k=0}^{n-1}\omega_k\cdot\Delta x_k<\varepsilon$$

3. Integrallanuvchi funksiyalar sinfi

3.1^0. Uzluksiz funksiyalarning integrallanuvchiligi. Aytaylik, $f(x)$ fuksiya $[a,b]$ oraliqda aniqlangan bo`lsin.

1-teorema. Agar $f(x)$ fuksiya $[a,b]$ da uzluksiz bo`lsa, u shu $[a,b]$ da integrallanuvchi, ya`ni

$$C[a,b]\subset R([a,b])$$

bo`ladi.

◄ Modomiki, $f(x)\in C[a,b]$ ekan, u Kantor teoremasiga ko`ra $[a,b]$ oraliqda tekis uzluksiz bo`ladi. Kantor teoremasining natijasiga ko`ra, $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topiladiki, $[a,b]$ oraliqni uzunliklari $\delta$ dan kichik bo`lgan bo`laklarga ajralganda har bir bo`lakdagi fuksiyaning tebranishi

$$\omega_\kappa<\frac{\varepsilon}{b-a}$$

bo`ladi. Unda $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo`lgan har qanday P bo`laklashda

$$S(f;P)-s(f;P)=\sum_{k=0}^{n-1}\omega_k\Delta x_k<\frac{\varepsilon}{b-a}\sum_{k=0}^{n-1}\Delta x_k=\varepsilon$$

bo`ladi. Demak, $f(x)\in R([a,b])$. ►

3.2^0. Monoton fuksiyalarning integrallanuvchiligi.

2-teorema. Agar $f(x)$ fuksiya $[a,b]$ segmetda chegaralan-gan va monoton bo`lsa, u shu segmentda integrallanuvchi bo`ladi.

◄ Aytaylik, $f(x)$ fuksiya $[a,b]$ segmentda o`suvchi bo`lib, $f(a)<f(b)$ bo`lsin.

$\forall\varepsilon>0$ sonni olib, unga ko`ra $\delta>0$ ni

$$\delta=\frac{\varepsilon}{f(b)-f(a)}$$

deymiz.

U xolda $[a,b]$ segmentning diametri $\lambda_P<\delta$ bo`lgan ixtiyoriy $P$ bo`laklash uchun

$$S(f;P)-s(f;P)=\sum_{k=0}^{n-1}\left(M_k-m_k\right)\cdot\Delta x_k=\sum_{k=0}^{n-1}\left[f(x_{k+1})-f(x_k)\right]\cdot\Delta x_k\le$$

$$\leq \lambda_P \sum_{k=0}^{n-1}\left[f(x_{k+1})-f(x_k)\right]=\lambda_P\cdot\left[f(b)-f(a)\right]<\frac{\varepsilon}{f(b-f(a)}\cdot\left[f(b)-f(a)\right]=\varepsilon$$

bo`ladi. Demak, $f(x)\in R([a,b])$. ►

3.3^0. Uziladigan fuksiyalarning integrallanuvchiligi.

3-teorema. Agar $f(x)$ fuksiya $[a,b]$ segmentda chegaralan-gan va shu segmentning chekli sondagi nuqtalarida uzilishga ega bo`lib, qolgan barcha nuqtalarda uzluksiz bo`lsa, fuksiya $[a,b]$ da integrallanuvchi bo`ladi.

◄ $f(x)$ fuksiya $[a,b]$ da chegaralangan bo`lsin. Demak,

$$\exists C\in R,\ \ \forall x\in[a,b]:\ \left|f(x)\right|\leq C \qquad (C>0)$$

bo`ladi.

Soddalik uchun, $f(x)$ funksiya $[a,b]$ segmentning faqat bitta x^* $(x^*\in[a,b])$ nuqtasida uzilishga ega bo`lib, qolgan barcha nuqtalarda uzluksiz bo`lsin. $\forall\varepsilon>0$ sonni olib, unga ko`ra $\delta>0$ sonni

$$\delta=\frac{\varepsilon}{16C}$$

deymiz.

x^* nuqtaning δ atrofi $(x^*-\delta,x^*+\delta)$ ni olib, ushbu

$$[a,b]\setminus(x^*-\delta,x^*+\delta)$$

to`plamni qaraymiz. Bu to`plamda $f(x)$ funksiya uzluksiz bo`lib, Kantor teoremasiga binoan u tekis uzluksiz bo`ladi. U holda shunday $\gamma>0$ son topiladiki,

$$\forall x',x''\in\left[a,x^*-\delta\right],\quad \left(\forall x',x''\in\left[x^*+\delta,b\right]\right)$$

lar uchun $\left|x'-x''\right|<\gamma$ bo`lishidan

$$\left|f(x')-f(x'')\right|<\frac{\varepsilon}{2(b-a)}$$

bo`lishi kelib chiqadi.

Endi $[a,b]$ segmentni diametri $\lambda_P<\min(\delta,\gamma)$ bo`lgan ixtiyoriy $P$ bo`laklashini olib, unga nisbatan

$$\sum_{k=0}^{n-1}\omega_k\cdot\Delta x_k \tag{1}$$

yig`indini tuzamiz.

Bu yig`indining har bir hadida $[x_k,x_{k+1}]$ $(k=0,1,2,\ldots,n-1)$ oraliqlarning uzunliklari Δx_k lar qatnashadi.

(1) yig`indining ushbu

$$[x_k,x_{k+1}]\bigcap(x^*-\delta,x^*+\delta)=\emptyset$$

munosabat bajariladigan $[x_k,x_{k+1}]$ ga mos hadlaridan tuzilgan yig`indini

$$\sum_k{}'\ \omega_k\cdot\Delta x_k$$

bilan, qolgan barcha hadlardan (bunday hadlar uchun

$$[x_k,x_{k+1}]\bigcap(x^*-\delta,x^*+\delta)\neq\emptyset$$

yoki

$$[x_k, x_{k+1}]\bigcap\{x^* - \delta\} \neq \emptyset$$

yoki

$$[x_k, x_{k+1}]\bigcap\{x^* + \delta\} \neq \emptyset$$

bo`ladi) tashkil topgan yig`indini

$$\sum_k{}'' \omega_k \cdot \Delta x_k$$

bilan belgilaymiz.

Natijada

$$\sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k = \sum_k{}' \omega_k \cdot \Delta x_k + \sum_k{}'' \omega_k \cdot \Delta x_k$$

bo`lib, tenglikning o`ng tomondagi qo`shiluvchilar uchun

$$\sum_k{}' \omega_k \cdot \Delta x_k \leq \frac{\varepsilon}{2(b-a)} \cdot \sum_k{}' \cdot \Delta x_k \leq \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2},$$

$$\sum_k{}'' \omega_k \cdot \Delta x_k \leq 2C \cdot \sum_k{}'' \cdot \Delta x_k \leq 2 \cdot C \cdot 4\delta = 8C \cdot \frac{\varepsilon}{16} = \frac{\varepsilon}{2}$$

bo`ladi. Demak,

$$\sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Bu esa $f(x)$ fuksiyaning $[a,b]$ da integrallanuvchi ekani-ni bildiradi. ►

4. Aniq integrallarning xossalari

4.1^0. Integralning chiziqlilik hamda additivlik xossalari.

1-xossa. Agar $f(x) \in R([a,b])$ va $C \in R$ bo`lsa, u holda $(C \cdot f(x)) \in R([a,b])$ bo`lib,

$$\int_a^b C \cdot f(x)dx = C\int_a^b f(x)dx$$

bo`ladi.

◄ $f(x) \in R([a,b])$ va $C \in R$ bo`lsin. Aniq integral ta`rifiga ko`ra

$$\lim_{\lambda_P \to 0} \sigma(f, P, \xi_k) = \int_a^b f(x)dx$$

bo`ladi.

Ravshanki,

$$\sigma(C \cdot f(x; P; \xi_k)) = C\sigma\big(f; P; \xi_k\big),$$

$$\lim_{\lambda_P \to 0} \sigma(C \cdot f(x; P; \xi_k)) = C \cdot \lim_{\lambda_P \to 0} \sigma(f, P, \xi_k).$$

Demak,

$$(C \cdot f(x)) \in R([a,b])$$

va

$$\int_a^b C \cdot f(x)dx = C\int_a^b f(x)dx. ►$$

2-xossa. Agar

$$f(x) \in R([a,b]),\ g(x) \in R([a,b])$$

bo`lsa, u holda

$$(f(x)+g(x)) \in R([a,b])$$

bo`lib,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

bo`ladi (additivlik hossasi)

◄ Aniq integral ta`rifiga ko`ra

$$\lim_{\lambda_P \to 0} \sigma(f,P,\xi_k) = \int_a^b f(x)dx,$$

$$\lim_{\lambda_P \to 0} \sigma(f,P,\xi_k) = \int_a^b g(x)dx$$

bo`ladi.

Ravshanki,

$$\sigma(f+g,P,\xi_k) = \sigma(f,P,\xi_k) + \sigma(g,P,\xi_k).$$

Limitga ega bo`lgan fuksiyalar haqida teoremadan foydalanib, $(f(x)+g(x)) \in R([a,b])$ va

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

bo`lishini topamiz. ►

3-xossa. Agar

$$f(x) \in R([a,c]),\ f(x) \in R([c,b])$$

bo`lsa, u holda

$$f(x) \in R([a,b])$$

bo`lib,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b g(x)dx$$

bo`ladi.

◄ Aytaylik, $a < c < b$ bo`lib, $f(x) \in R([a,c])$ va $f(x) \in R([c,b])$ bo`lsin. U holda $\forall \varepsilon > 0$ son olinganda ham $[a,c]$ oraliqning $\lambda_{P_1} < \delta_1$ bo`lgan $P_1$ bo`laklashi topiladiki

$$S(f;P_1) - s(f;P_1) < \frac{\varepsilon}{2}, \qquad (1)$$

shuningdek $[c,b]$ oraliqning $\lambda_{P_2} < \delta_2$ bo`lgan $P_2$ bo`laklashi topiladiki,

$$S(f;P_2) - s(f;P_2) < \frac{\varepsilon}{2}$$

bo`ladi.

Endi $[a,b]$ oraliqning diametri $\lambda_{P_3} < \delta = \min(\delta_1, \delta_2)$ bo`lgan ixtiyoriy $P_3$ bo`laklashini olamiz. Bu P_3 bo`laklashning bo`luvchi nuqtalari qatoriga c nuqtani qo`shib $[a,b]$ ning yangi $P$ bo`laklashini hosil qilamiz. Unga nisbatan $f(x)$ fuksiya-ning Darbu yig`indilari

$$S(f;P), \qquad s(f;P)$$

bo`lsin.

P bo`laklashning $[a,c]$ va $[c,b]$ dagi bo`luvchi nuqtalari mos ravishda ularning P_1' hamda P_2' bo`laklashlarini yuzaga keltiradi. Ravshanki, bu $P_1'$ va $P_2'$ bo`laklashlarga nisbatan quyidagi tengsizliklar o`rinli bo`ladi:

$$S(f;P_1') - s(f;P_1') < \frac{\varepsilon}{2},$$
$$S(f;P_2') - s(f;P_2') < \frac{\varepsilon}{2}.$$

Ayni paytda,

$$S(f;P) = S(f;P_1') + S(f;P_2'),$$
$$s(f;P) = s(f;P_1') + s(f;P_2')$$

bo`ladi. Bu munosabatlardan

$$S(f;P) - s(f;P) = \left(S(f;P_1') - s(f;P_1')\right) +$$
$$+ \left(S(f;P_2') - s(f;P_2')\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

bo`lishi kelib chiqadi. Demak, $f(x) \in R([a,b])$.

$f(x)$ fuksiyaning $[a,b]$, $[a,c]$, $[c,b]$ oraliqlar bo`yicha $P$ bo`laklashga nisbatan integral yig`indilari

$$\sum_{[a,b]} f(\xi_k)\cdot \Delta x_k\,, \qquad \sum_{[a,c]} f(\xi_k)\cdot \Delta x_k\,, \qquad \sum_{[c,b]} f(\xi_k)\cdot \Delta x_k$$

bo`lib,

$$\sum_{[a,b]} f(\xi_k)\cdot \Delta x_k = \sum_{[a,c]} f(\xi_k)\cdot \Delta x_k + \sum_{[c,b]} f(\xi_k)\cdot \Delta x_k$$

bo`ladi. Integral ta`rifidan foydalanib topamiz:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

SHunga o`xshash $c < a < b,\ a < b < c$ bo`lgan hollarda ham xossaning o`rinli bo`lishi isbotlanadi.►

4-xossa. Agar $f(x) \in R([a,b])$, $g(x) \in R([a,b])$ bo`lsa, u holda $f(x)\cdot g(x) \in R([a,b])$ bo`ladi.

◄ Modomiki, $f(x)$ va $g(x)$ fuksiyalar $[a,b]$ da integral-lanuvchi ekan, unda

$$S(f;P) - s(f;P) < \frac{\varepsilon}{2M'} \qquad (M' = \sup f(x), x \in [a,b])$$
$$S(g;P) - s(g;P) < \frac{\varepsilon}{2M} \qquad (M = \sup g(x),\ x \in [a,b])$$

bo`ladi.

Aytaylik, $\forall x \in [a,b]$ da $f(x) \ge 0, \quad q(x) \ge 0$ bo`lsin. U holda $\forall x \in [x_k; x_{k+1}]$ uchun

$$0 \le m_k \le f(x) \le M_k',\ m_k = \inf f(x),\ M_k' = \sup f(x);$$
$$0 \le m_k' \le g(x) \le M_k,\ m_k' = \inf g(x),\ M_k = \sup g(x)$$

bo`lib, ulardan

$$0 \le m_k \cdot m_k' \le f(x)\cdot g(x) \le M_k \cdot M_k',$$

bo`lishi kelib chiqadi. Ayni paytda,

$$m_k^0=\inf\{f(x)\cdot g(x)\},\ \ M_k^0=\sup\{f(x)\cdot g(x)\}$$

lar uchun

$$m_k\cdot m_k'\le m_k^0\le M_k^0\le M_k\cdot M_k'$$

bo`lib,

$$M_k^0-m_k^0\le M_k\cdot M_k'-m_k\cdot m_k'=$$
$$=M_k'(M_k-m_k)+m_k(M_k'-m_k')$$

bo`ladi.

Endi $M\ge M_k$, $M'\ge M_k'$ ekanini etiborga olib, topamiz;

$$S(f\cdot g;P)-s(f\cdot g;P)=\sum_{k=0}^{n-1}\left(M_k^0-m_k^0\right)\le$$
$$\le M'\cdot\sum_{k=0}^{n-1}\left(M_k-m_k\right)\cdot\Delta x_k+M\cdot\sum_{k=0}^{n-1}\left(M_k'-m_k'\right)=$$
$$=M'\left(S(f;P)-s(f;P)\right)+M'\left(S(g;P)-s(g;P)\right)<$$
$$<M'\frac{\varepsilon}{2M'}+M\cdot\frac{\varepsilon}{2M}<\varepsilon.$$

Demak, bu holda $f(x)\cdot g(x)\in R([a,b])$.

Aytaylik, $f(x)$ va $g(x)$ fuksiyalar $[a,b]$ da ixtiyoriy integrallanuvchi fuksiyalar bo`lsin.

Ravshanki, $\forall x\in[a,b]$ da

$$f(x)-\inf f(x)=f(x)-m\ge 0,$$
$$g(x)-\inf g(x)=g(x)-m'\ge 0$$

bo`ladi.

Endi $f(x)\cdot g(x)$ fuksiyani quyidagicha yozib olamiz:

$$f(x)\cdot g(x)=\left(f(x)-m\right)\left(g(x)-m'\right)+mg(x)+m'f(x)-mm'.$$

Bu tenglikning o`ng tomonidagi har bir qo`shiluvchi $[a,b]$ da integrallanuvchi bo`lganligi sababli $f(x)\cdot g(x)$ ham $[a,b]$ da integrallanuvchi bo`ladi. ►

Natija. Agar $f(x)\in R([a,b])$ bo`lsa, u holda $[f(x)]^n\in R([a,b])$ bo`ladi, bunda $n\in N$.

4.2⁰. Integralning tengsizliklar bilan bog`langan xossalari.

1-xossa. Agar $f(x)\in R([a,b])$ bo`lib, $\forall x\in[a,b]$ da $f(x)\ge 0$ bo`lsa, u holda

$$\int_a^b f(x)dx\ge 0$$

bo`ladi.

◄ Integralning ta`rifiga ko`ra

$$\lambda_P\to 0 \text{ da } \sum_{k=0}^{n-1}f(\xi_k)\cdot\Delta x_k\to\int_a^b f(x)dx$$

bo`ladi. U holda, $\forall x\in[a,b]$ da $f(x)\ge 0$ bo`lishidan

$$\sum_{k=0}^{n-1}f(\xi_k)\cdot\Delta x_k\ge 0$$

bo`lib, undan

$$\int_a^b f(x)dx \geq 0$$

bo`lishi kelib chiqadi. ►

1-natija. Agar $f(x) \in R([a,b])$, $g(x) \in R([a,b])$ bo`lib, $\forall x \in [a,b]$ da $f(x) \leq g(x)$ bo`lsa, u holda

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

bo`ladi.

◄ Ravshanki,

$$f(x) \in R([a,b]) \ , g(x) \in R([a,b]) \Rightarrow (g(x) - f(x)) \in R([a,b])$$

bo`lib,

$$g(x) - f(x) \geq 0 \Rightarrow \int_a^b (g(x) - f(x))dx \geq 0 \Rightarrow \int_a^b g(x)dx - \int_a^b f(x)dx \geq$$

$$\geq 0 \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

bo`ladi. ►

2-natija. Agar $f(x) \in R([a,b])$, $g(x) \in R([a,b])$ bo`lsa, u holda

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx} \cdot \sqrt{\int_a^b g^2(x)dx} \qquad (2)$$

bo`ladi.

◄ Ixtiyoriy $\alpha \in R$ uchun

$$\int_a^b \left(f(x) - \alpha \cdot g(x)\right)^2 dx \geq 0$$

bo`lib,

$$\alpha^2 \int_a^b g^2(x)dx - 2\alpha \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \geq 0$$

bo`ladi. Kvadrat uchhadning diskriminanti musbat bo`lmagan-ligi sababli

$$\left(\int_a^b f(x)g(x)dx\right)^2 - \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx \leq 0,$$

ya`ni,

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx} \cdot \sqrt{\int_a^b g^2(x)dx}$$

bo`ladi. ►

(2) tengsizlik Koshi-Bunyakovskiy tengsizligi deyi-ladi.

2-xossa. Agar $f(x) \in R([a,b])$ bo`lsa, $|f(x)| \in R([a,b])$ bo`lib,

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

bo`ladi.

◄ $f(x) \in R([a,b])$ bo`lsin. Integrallanuvchilik mezoniga ko`ra, $\forall \varepsilon > 0$ olinganda ham $[a,b]$ segmentning shunday P bo`laklashi topiladiki, unga nisbatan

$$S(f;P) - s(f;P) = \sum_{k=0}^{n-1} \omega_k \Delta x_k < \varepsilon$$

bo`ladi, bunda $\omega_k - f(x)$ funksiyaning $[x_k, x_{k+1}]$ dagi tebranishi.

Ravshanki, $\forall x', x'' \in [a,b]$ uchun

$$\left| |f(x')| - |f(x'')| \right| \le |f(x') - f(x'')|$$

bo`lib, undan

$$\sup \left| |f(x')| - |f(x'')| \right| \le \sup |f(x') - f(x'')|$$

bo`lishi kelib chiqadi.

Demak,

$$\omega_k^{'} \le \omega_k$$

bo`ladi, bunda $\omega_k^{'} - |f(x)|$ funksiyaning $[x_k, x_{k+1}]$ dagi tebranishi. SHularni e`tiborga olib,

$$S(|f|;P) - s(|f|;P) = \sum_{k=0}^{n-1} \omega_k^{'} \cdot \Delta x_k \le \sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k < \varepsilon$$

bo`lishini topamiz. Demak, $|f(x)| \in R([a,b])$.

$f(x)$ va $|f(x)|$ funksiyalarning integral yig`indilari uchun

$$\left| \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k \right| \le \sum_{k=0}^{n-1} |f(\xi_k)| \cdot \Delta x_k ,$$

bo`lib, $\lambda_p \to 0$ da limitga o`tish natijasida

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)| dx$$

bo`lishi kelib chiqadi. ►

**4.3⁰. O`rta qiymat haqidagi teoremalar.** Aytaylik, $f(x)$ funksiya $[a,b]$ da berilgan va chegaralangan bo`lsin. U holda $m = \inf\{f(x)\}$, $M = \sup\{f(x)\}$ $(x \in [a,b])$ mavjud va $\forall x \in [a,b]$ uchun

$$m \le f(x) \le M$$

tengsizliklar o`rinli bo`ladi.

1-teorema. Agar $f(x) \in R([a,b])$ bo`lsa, u holda shunday o`zgarmas $\mu (m \le \mu \le M)$ son mavjudki,

$$\int_a^b f(x)dx = \mu \cdot (b-a)$$

bo`ladi.

◄ Ravshanki,

$$m \le f(x) \le M \Rightarrow \int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx \Rightarrow$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Keyingi tengsizliklardan

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M$$

bo`lishi kelib chiqadi.

Agar

$$\mu = \frac{\int_a^b f(x)dx}{b-a}$$

deyilsa, undan

$$\int_a^b f(x)dx = \mu \cdot (b-a)$$

bo`lishini topamiz. ►

3-natija. Agar $f(x) \in C[a,b]$ bo`lsa, u holda shunday $\theta \in [a,b]$ topiladiki,

$$\int_a^b f(x)dx = f(\theta)\cdot(b-a)$$

bo`ladi.

◄ Bu tasdiq yuqoridagi teorema va uzluksiz funksiya-ning xossasidan kelib chiqadi.►

2-teorema. Agar $f(x) \in R([a,b]),\ g(x) \in R([a,b])$ bo`lib, $[a,b]$ da $g(x)$ funksiya o`z ishorasini o`zgartirmasa, u holda shunday o`zgarmas $\mu(m \le \mu \le M)$ son mavjudki,

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \tag{3}$$

bo`ladi.

◄ Aytaylik, $\forall x \in [a,b]$ da $g(x) \ge 0$ bo`lsin. Unda ravshanki,

$$m \le f(x) \le M \Rightarrow mg(x) \le f(x)\cdot g(x) \le Mg(x)$$

bo`ladi.

Bu munosabatdan hamda aniq integral xossalaridan foydalanib topamiz:

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

a) $\int_a^b g(x)dx = 0$ bo`lsin. U holda

$$\int_a^b f(x)g(x)dx = 0$$

bo`lib, ixtiyoriy $\mu(m \le \mu \le M)$ da (3) o`rinli bo`ladi.

b) $\int_a^b g(x)dx > 0$ bo`lsin. U holda

$$m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

bo`lib,

$$\mu = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

deyilsa, undan

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$$

bo`lishi kelib chiqadi. ►

4-natija. Agar $f(x) \in C[a,b]$ bo`lib, $g(x) \in R([a,b])$ va $g(x)$ funksiya $[a,b]$ da o`z ishorasini o`zgartirmasa, u holda shunday $\theta \in [a,b]$ topiladiki,

$$\int_a^b f(x)g(x)dx = f(\theta)\int_a^b g(x)dx$$

bo`ladi.

MUSTAQIL YECHISH UCHUN MASHQLAR

1. $f(x)$ funksiyaning $[a,b]$ da chegaralanganligi uning $[a,b]$ da integrallanuvchi bo`lishining zaruriy sharti ekani isbot-lansin.

2. Aytaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da berilgan va chegaralangan bo`lib, $P$ esa $[a,b]$ ning ixtiyoriy bo`laklashi bo`lsin. Agar $\forall x \in [a,b]$ da $f(x) \le g(x)$ bo`lsa,

$$s(f,p) \le s(g,p)$$
$$S(f,p) \le S(g,p)$$

bo`lishi isbotlansin.

3. $[a,b]$ segmentning ixtiyoriy bo`laklashi uchun

$$s(\alpha\ f(x)+\beta;P)=\alpha\ s(f;P)+\beta(b-a)\ ,$$
$$S(\alpha\ f(x)+\beta;P)=\alpha\ S(f;P)+\beta(b-a)$$

bo`lishi isbotlansin, bunda $\alpha,\beta \in R$

4. Agar $f(x)\in C[a,b]$ bo`lsa, u holda $[a,b]$ ning ixtiyoriy bo`laklashi uchun Darbuning quyi va yuqori yig`indilari $f(x)$ funksiyaning integral yig`indilari bo`lishi isbotlansin.

5. Aytaylik,

$$f(x)=\begin{cases}\sin\dfrac{1}{x}, & \text{агар}\quad 0<x\le 1\\ 0, & \text{агар}\quad x=0\end{cases}$$

bo`lsin. $f(x)\in R([0,1])$ bo`lishi isbotlansin.

6. $y=f(x)$ funksiya $[a,b]$ da integrallanuvchi bo`lib, uning qiymatlari $[c,d]$ ga tegishli bo`lsin. Agar $\Phi(y)$ funksiya $[c,d]$ da uzluksiz bo`lsa, u holda murakkab funksiya $\varphi(f(x))$ ning $[a,b]$ da integrallanuvchi bo`lishi isbotlansin.

7. Ushbu

$$\frac{\pi}{4}\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{1+\cos^2 x}\,dx\le\frac{\pi\cdot\sqrt{6}}{8}$$

tengsizlik isbotlansin.

8. Ushbu

$$\int_a^b\frac{|x|}{x}dx=|b|-|a|$$

tenglik isbotlansin.

9. $f(x)\in C((-\infty,+\infty))$ bo`lib, u $T$ $(T\ne 0)$ davrli funksiya bo`lsin. U holda $\forall\ a\in R$ uchun

$$\int_a^{a+T}f(x)dx=\int_0^T f(x)dx$$

bo`lishi isbotlansin.

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