O’ZBEKISTON RESPUBLIKASI

OLIY VA O’RTA MAXSUS TA’LIM VAZIRLUIGI

NAMANGAN DAVLAT UNIVERSITETI

FIZIKA – MATEMATIKA FAKUL’TETI

MATEMATIKA KAFEDRASI O’QITUVCHISI

Bohonov Zafarning

***“ Sirtning birinchi kvadratik formasi”* MAVZUSIDAGI**

REFERATI

Namangan 2013

Mavzu: Sirtning birinchi kvadratik formasi

Reja.

1. **Sirtning birinchi kvadratik formasi**
2. **Birinchi kvadratik formasining ishorasi**
3. **Sirtda yotuvchi chiziqlar orasidagi burchak**
4. **Sirt ustidagi sohaning yuzini aniqlash**
5. **Sirt koordinat chiziqlari tashkil etgan burchak**
6. **Ortogonal traektoriyalar**

Ω regulyar sirt $\vec{r} = \vec{r}(u,v)$ vektor tenglamasi orqali berilgan bo'lsin. $P(u,v)$ nuqtada

$$[r'_u, r'_v] \neq 0, \varphi_1 = d\vec{r}^{\,2} \qquad (1)$$

ifodaga sirtning birinchi kvadratik formasi deyiladi. Ko'ramizki, sirtning birinchi kvadratik formasi $\vec{r}$ - to'la differentsialiningkvadratiga teng. $d\vec{r} = \vec{r}'_u du + \vec{r}'_v dv$ bundan

$$d\vec{r}^{\,2} = \vec{r}'^2_u du^2 + 2(\vec{r}'_u \vec{r}'_v) du dv + \vec{r}'^2_v dv^2$$

$$\varphi_1 = \vec{r}'_u du^2 + 2(\vec{r}'_u, \vec{r}'_v) du dv + \vec{r}'^2_v dv^2 \qquad (2)$$

(2) sirtning har bir nuqtasida du ва dv larga nisbatan kvadratik formani ifodalaydi. $d\vec{r}'^{\,2} = ds^2$, $ds = \sqrt{d\vec{r}'^{\,2}}$ ni e'tiborga olsak,

$$ds^2 = \vec{r}'^2_u + 2(\vec{r}'_u, \vec{r}'_v) du dv + \vec{r}'^2_v dv^2 \qquad (3)$$

(3) ni sirtning chiziqli elementi deyiladi va uni birinchi kvadratik forma desa bo'ladi. quyidagi belgilashlarni kiritaylik.

$$E = \vec{r}'^2_u, \quad F = (\vec{r}'_u, \vec{r}'_v), \quad G = \vec{r}'^2_v \qquad (4)$$

$(g_{11} = \vec{r}'^2_u, g_{12} = (\vec{r}'_u, \vec{r}'_v), g_{22} = \vec{r}'^2_v)$ belgilash ham mumkin. $\varphi_1 = Edu^2 + 2Fdudv + Gdv^2$

Bundan

$$EG - F^2 > 0 \qquad (6)$$

haqiqatan ham, $\vec{r}_u^{\,2} \vec{r}_v^{\,2} - (\vec{r}_u \vec{r}_v)^2 = [\vec{r}_u \vec{r}_v]^2 > 0$. Ko'ramizki φ_1 musbat ishorali va $\vec{r}'^2_u > 0$. Ω-sirtga qarashli γ chiziq $u = u(t)$, $v = v(t)$ parametrik tenglama orqali berilgan bo'lib, $t_0 \le t \le t_1$. γ chiziqning vektor tenglamasi

$$\vec{r} = \vec{r}(u(t), v(t)) \qquad (7)$$

ko'rinishga ega. γ chiziqning $M(t_0)$ va $N(t_1)$ nuqtalari orasidagi yoy uzunligini xisoblaylik.

$$ds = |d\vec{r}| = |\vec{r}'(u(t), v(t))| dt = \sqrt{d\vec{r}^{\,2}} = \sqrt{\varphi_1} = \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt} * \frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2} dt$$

Bundan

$$s(t) = \int_{M(t_0)}^{N(t_1)} \sqrt{\varphi_1} dt = \int_{t_0}^{t_1} \sqrt{Eu'^2_t + 2Fu'_t v'_t + Gv'^2_t} dt \qquad (8)$$

(8) Ω sirtga qarashli γ silliq chiziqning $M(t_0)$ ва $N(t_1)$ nuqtalari orasidagi yoy uzunligining hisoblash formulasidir.

Ω regulyar sirtda γ silliq chiziqdan tashqari ushbu chiziq bilan kesishuvchi L silliq chiziq $u = u(\tau)$, $v = v(\tau)$ tenglama orqali aniqlangan bo'lsin. γ va L chiziqlar $P(u,v) \in \Omega$ nuqtada kesishsin.

Ta'rif: Ω regulyar sirtga qarashli γ va L silliq chiziqlar tashkil etgan burchak deb, ularning kesishish P nuqtasida γ va L chiziqlarga o'tkazilgan urinmalar tashkil etgan eng kichik burchakka aytiladi. γ chiziqqa o'tkazilgan urinma vektor

$$\frac{dr}{dt} = \vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt} \qquad (9)$$

L chiziqqa P nuqtada o'tkazilgan urinma vektor

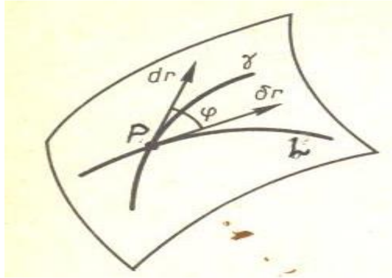
$$\frac{\delta r}{\delta \tau} = \vec{r}_u \frac{\delta u}{\delta \tau} + \vec{r}'_v \frac{\delta v}{\delta \tau} \qquad (10)$$

$d\vec{r} = \vec{r}'_u du + \vec{r}'_v dv$, $\delta\vec{r}' = \vec{r}'_u \delta u + \vec{r}'_v \delta v$

vektorlarni skalyar ko'paytmasidan

$$\cos\varphi = \frac{(d\vec{r}\,\delta\vec{r})}{\sqrt{dr^{-2}}\sqrt{\delta\tau^2}} \qquad (11)$$

kelib chiqadi.



27-chizma

$$d\vec{r}^{\,-2} = Edu^2 + 2Fdudv + Gdv^2 \,(\alpha)$$

$$\delta\vec{r}^{\,-2} = E\delta u^2 + 2F\delta u\delta v + C\delta v^2 \left(\beta\right)$$

$$(d\vec{r}\,\delta\vec{r}) = \vec{r}_u^{\,\prime 2} du\delta u + (\vec{r}_u^{\,\prime}\vec{r}_v^{\,\prime})(du\delta v + dv\delta u) + \vec{r}_v^{\,\prime 2} dv\delta v =$$

$$= Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v(\gamma)$$

formuladan foydalansak,

$$\cos\varphi = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Ddv\delta v}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}} \qquad (12)$$

formulaga ega bo'lamiz. Ω sirtga qarashli koordinat chiziqlar kesishib tashkil etgan burchak formulasini yozaylik.

$$\gamma : \begin{cases} u = t \\ v = const \end{cases} \qquad L : \begin{cases} u = const \\ v = t \end{cases} \qquad (13)$$

γ uchun $dv = 0$, L uchun $\delta u = 0$ bo'lgani uchun (12) dan

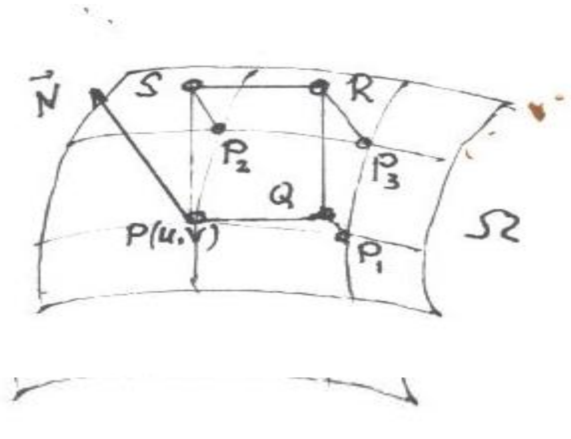
$$\cos\varphi = \frac{Fdu\delta v}{\sqrt{Edu^2}\sqrt{G\delta v^2}} = \frac{F}{\sqrt{EG}} \qquad (14)$$

(14) dan quyidagi teorema kelib chi?adi.

Teorema: Ω sirt koordinat chiziqlari ortogonal bo'lishi uchun $F = 0$ shartning bajarilishi zarur va etarlidir.

Endi $\vec{r} = \vec{r}(u, v)$ vektor ko'rinishda berilgan Ω silliq sirt sohasining yuzini hisoblash formulasini yozaylik.

Sirtning silliq chiziqlar qism yoylari bilan chegaralangan Φ sohasini ajrataylik. Φ sohani u, v parametrlarining o'zgarish sohasi deb qarash mumkin. Φ sohaning har bir nuqtasiga u, v parametrning fiksirlagan qiymatlari mos keladi va aksincha. Φ soxani "u" va "v" koordinat chiziqlar oilasi orqali egri chiziqli parallelogrammlarga ajratamiz. qarama-qarshi tomonlar juftlaridan biri "u" chiziqlar bilan, ikkinchisi "v" chiziqlar bilan chegaralanadi.



2-chizma

Chizmada $PP_1 \; P_2P_3$ эgri chiziqli parallelogramm tasvirlangan bo'lib, uchlari $P(u,v)$, $P_1(u+\Delta u, v)$, $P_2(u, v+\Delta v)$, $P_3(u+\Delta u, v+\Delta v)$ koordinatalarga ega. $\vec{N}$ -sirtning P nuqtadagi normal vektori bo'lsin. "u" va "v" chiziqlarga P nuqtada urinmalar o'tkazamiz. $PP_1 \; P_2P_3$ egri chiziqli parallelogrammni $\vec{N}$ ga parallel ravishda urinma tekislikka proektsiyalaymiz. Urinma tekislikda $PQSR$ to'?ri parallelogramm hosil bo'ladi. Uni $\vec{r}_u \; \Delta u$ va $\vec{r}_v \; \Delta v$ vektorlar bo'yicha ko'rilgan urinma tekislikka tegishli parallelogramm bo'lishidan $PP_1 \quad P_2P_3$ parallelogramm o'rniga olish mumkinligi kelib chiqadi.

$PQSR$ parallelogramm yuzini $G(g)$ orqali belgilaylik. Φ sohadagi egri chiziqli parallelogrammlarning yig'indisi $G = \sum G(g)$ bo'lsin.

Ta'rif: Sirt sohasi Φ ning yuzi deb, g soha o'lchov bo'yicha cheksiz kichrayganda

$$S = l\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum_g G(g) \qquad (15)$$

songa aytiladi.

$$G(g) = \left|\left[\vec{r}_u{}'\Delta u \quad \vec{r}_v{}'\Delta v\right]\right| = \left|\left[\vec{r}_u{}' \quad \vec{r}_v{}'\right]\right|\Delta u \Delta v \qquad (16)$$

$\vec{r}_u{}'\vec{r}_v{}'$ hosilalarning uzluksizligidan (15) limit mavjud bo'lib, ikki karrali integral ta'rifi bo'yicha

$$S = \iint_{\Phi} \left|\left[\vec{r}_u{}'\vec{r}_v{}'\right]\right| dudv \qquad (17)$$

ga tengdir.

Shunday qilib,

$$S = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum_g G(g)$$

$$\left[\vec{r}_u{}'\vec{r}_v{}'\right]^2 + \left(\vec{r}_u{}'\vec{r}_v{}'\right)^2 = \vec{r}_u{}'^2\vec{r}_v{}'^2 \qquad (18)$$

(4) dan foydalanib, (18) dan

$$\left(\left[\vec{\tau}_u{}'\vec{\tau}_v{}'\right]\right) = \sqrt{EG - F^2} \qquad (19)$$

tenglikni keltirib chiqarish mumkin.

$$S = \iint_{\Phi} \sqrt{EG - F^2}\, dudv \qquad (20)$$

(20) Ω sirt Φ sohasi yuzini hisoblash formulasi bo'lib, φ_1 forma koeffitsientlari orqali ifodalanadi.

Misol:

R radiusli sfera yuzi hisoblansin. Sferaning parametrik tenglamasi

$x = R\cos u \cos v$, $y = R\sin u \cos v$, $r = R\sin v$

$$\Phi = \left\{(u,v)\middle|\, 0 \le u \le 2\pi, -\frac{\pi}{2} \le v \le \frac{\pi}{2}\right\}$$

$\vec{r}_u{}' = -R\sin u \cos v \cdot \vec{i} + R\cos u \cos v \cdot \vec{j}$, $\vec{r}_v{}' = -R\cos u \sin v \cdot \vec{i} - R\sin u \sin v \cdot \vec{j}$,

$E = \vec{r}\,'^2(u) = R^2\cos^2 v$, $G = \vec{r}\,'^2(v) = R^2$, $F = \left(\vec{r}_u{}'\vec{r}_v{}'\right) = 0$,

$\sqrt{EG - F^2} = R^2\cos v$,

$$S = \iint_{\Phi} \sqrt{EG - F^2}\, dudv = R\int_0^{2\pi} du \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos v dv = 4\pi R^2$$

Agar sirt $z = g(x,y)$ tenglamasi orqali berilgan bo'lsa, uning yuzini hisoblash formulasi

$$S = \iint_{(\Phi)} \sqrt{1 + g_x^2 + g_y^2}\; dx\, dy \qquad (21)$$

ko'rinishda yoziladi. Isbotlashni o'quvchiga tavsiya etamiz.

Sirtda yotuvchi chiziqlar orasidagi burchak

Agar sirtning birinchi kvadratik formasi va sirt ustida bir-biri bilan kesishgan chiziqlarningtenglamalari berilgan bo`lsa, bu chiziqlar orasidagi burchakni aniqlash mumkin.

Sirt ustidagi chiziqlarning kesishgan nuqtasini *M* bilan belgilaylik. Chiziqlarning bu umumiy nuqtadagi urinma vektorlarn $dr = r_u du + r_v dv$ va $\delta r = r_u \delta u + r_v \delta v$ bo`lsin. r_u va r_v qiymatlar sirtning tenglamasilan topiladi. Shu sababli, ikkala *dr* va δr vektor uchun ham, r'_i va r'_v qiymatlar bir xildir. *dr* va δ*r* vektorlar orasidagi burchakni chiziqlar orasidagi burchak deb qabul qilamiz. Vektorlar orasidagi φ burchakni aniqlash uchun bu vektorlarni o`zaro skalyar ko`paytiramiz: $(dr \cdot \delta r) = |dr| \cdot |\delta r| \cdot \cos\varphi$. Bundan: $\cos\varphi = \dfrac{(dr \cdot \delta r)}{|dr| \cdot |\delta r|}$;

$$|dr| = ds = \sqrt{E\,du^2 + 2Fdudv + G\,dv^2}\ ;$$

$$|\delta r| = ds_1 = \sqrt{E\,\delta u^2 + 2F\,\delta u \delta v + G\delta v^2}\ .$$

Chiziqlar orasidagi burchak uchun

$$\cos\varphi = \frac{E\,du\,\delta\upsilon + F\left(du\,\delta u + dv\,\delta v\right) + G\,dv\,\delta v}{\sqrt{E\,du^2 + 2Fdu\,dv + G\,dv^2} \cdot \sqrt{E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2}}$$

formula hosil bo`ladi. Bundan $du : dv$ va yo`nalishdagi chiziqlarning ortogonalligi quyidagini beradi:

$$E\,du\,\delta u + F\left(du\,\delta v + dv\,\delta u\right) + G\,dv\,\delta v = 0$$

Demak, sirtning *M* nuqtasidagi ye, *F, G* koeffitsientlar topilsa va chiziqlar urinmalarining yo`nalishlari ($du : dv$ va δ*u*:δ*v* yoki $\frac{du}{dt}, \frac{dv}{dt}, \frac{\delta u}{dt}, \frac{\delta}{(}$ ) aniqlansa, chiziqlar orasidagi burchakni topish mumkin. Xususiy holda, egri chiziqli (*u* va *v*) koordinata chiziqlari orasidagi burchakni topaylik. Buning uchun *u* bo`yicha olingan differentsiallarni *du, dv* bilan, *v* bo`yicha olingan differentsiallarni δ*u*, δ*v* bilan belgilaymiz. *u* chiziq bo`yicha $du \neq 0,\ dv = 0$ va *v* chiziq bo`yicha $\delta i = 0,\ d\delta \neq 0$. Shu sababli, (1) formulaning ko`rinishi quyidagicha bo`ladi:

$$\cos\varphi = \frac{F}{\sqrt{EG}}$$

Koordinata chiziqlarnning ortogonallik sharti *F*=0 dir; demak, differentsiallar ko`paytmasining nolga tengligi koordinata chiziqlarining ortogonalligini bildiradi va aksincha. Misol. *x = i + v;* u = *u – v ; z=uv* sirt ustida *u*=*t*, $v = -\frac{1}{2}$ va $u = t,\ v = \frac{1}{2}t - 1$ chiziqlar berilgan. Bu chiziqlar orasidagi burchak topilsin.

Berilgan chiziqlarning kesishgan nuqtasini aniqlaymiz. Chiziqlarning berilgan tenglamalaridan *t* parametrni yo`qotish bilan hosil qilingan *u* + 2*v*= 0, *u – 2v* – 2= 0 sistemani yechib, $M_0\left(1, -\frac{1}{2}\right)$ kesishish nuqtasinn topamiz. Bu yerda $t_0$ ning qiymati 1 ga teng bo`ladi.

E, F, G koeffitsientlarni chiziqlarning kesishgan $M_0\left(1, -\frac{1}{2}\right)$ nuqtasi uchun hisoblaymiz:

$$x_i = 1,\ x_v = 1,\ E = 2 + v^2,\quad E = \frac{9}{4}$$

$$y_i = 1,\ y_v = -1,\ \ F = 1 - 1 + uv,\quad F = \frac{1}{2}$$

$z_v = u,\quad z_u = v,\quad G = 2 + u^2,\quad G = 3.$

Birinchi chiziqning tenglamasidan $\frac{du}{dt} = 1,\ \frac{dv}{dt} = -\frac{1}{2}.$

Ikkinchi chiziqning tenglamasidan $\frac{\delta u}{dt} = 1,\ \frac{\delta v}{dt} = \frac{1}{2}$. Topilgan qiymatlarni (1) formulaga qo`yamiz: $\cos\varphi = \frac{\frac{3}{2}}{\sqrt{\frac{14}{4}}\cdot\sqrt{\frac{10}{4}}} = \frac{3}{\sqrt{35}}.$

Ortogonal traektoriyalar

Sirt ustidagn M_0 $(u_0,\ v_0)$ nuqtaning atrofida bir parametrli $\varphi(u,v)+c=0$ chiziqlar oilasi berilgan va $M_0(u_0\ ,v_0)$ nuqtada $\varphi_u'^2 + \varphi_v'^2 \neq 0$ bo`lsin. Bu oilaga ortogonal bo`lgan ikkinchi oilani topamiz. Birinchi oilaning yunalishini $\varphi_v : \varphi_u$ bilan va ikkinchi oilaning yo`nalnshini $dv : du$ bilan belgilab, ortogonallik shartini yozamiz:

$$E\varphi_v\, du + F\left(\varphi_v\, dv - \varphi_u\, du\right) - G\varphi_u\, dv = 0$$

yoki

$$\left(E\varphi_v - F\varphi_u\right)du + \left(F\varphi_v - G\varphi_u\right)dv = 0\,. \qquad (1)$$

Ana shu tenglik ikkinchi oilaning differentsial tenglamasidir. Differentsial tenglamalar nazariyasidan ma`lumki, (1) ni $\mu$ $(u,\ v)$ integral kupaytuvchiga ko`pantirganda, uning chap tomoni qandaydir ψ $(u,\ v)$ funktsiining to`liq differensialiga aylanadi:

$$\mu\left\{\left(E\varphi_v - F\varphi_u\right)du + \left(F\varphi_v - G\varphi_u\right)\right\} = d\psi\,.$$

Berilgan oilaga ortogonal oilaning tenglamasi

$$\psi\left(u,v\right) + c_1 = 0$$

shakldа bo`ladi. Haqiqatan, topilgan oila chizigining yunalishi $\psi_v : \psi_u$ dir.

$$\psi_u = \mu\left(E\varphi_v - F\varphi_u\right);\quad \psi_v = \mu\left(F\varphi_v - G\varphi_u\right)$$

bo`lganidan $\mu\left\{\left(E\varphi_v - F\varphi_u\right)\psi_v + \left(F\varphi_v - G\varphi_v\right)\psi_u\right\} = 0$. Bu esa $\varphi_v$ : $\varphi_u$ va $\psi_v : \psi_u$ yo`nalishlarning ortogonallik shartidir. Endi M_0 $(u_0,\ v_0)$ nuqta atrofida eski u va v koordinat chiziqlarini $\varphi\left(u,v\right) + c = 0,\ \psi\left(u,v\right) + c_1 = 0$ dan iborat chiziqlarga almashtiramiz. Matematik analiz kursidan ma`lumki,

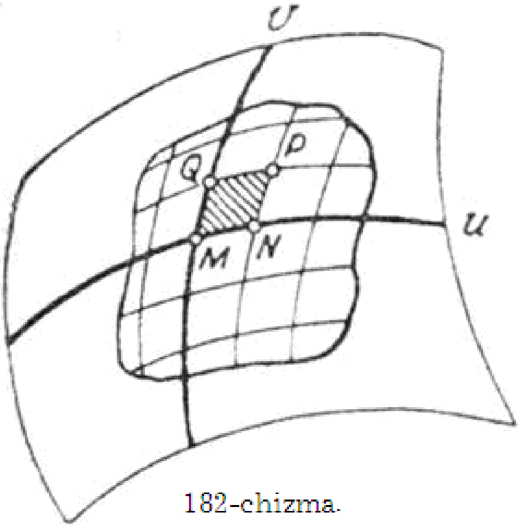
$$\begin{vmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{vmatrix} \neq 0$$

shartida, bu almashtirishni bajara olamiz. Bizning misolimizda — $\varphi_u\psi_v + \varphi_v\psi_u = E\varphi_u'^2 - 2F\varphi_u\varphi_v + G\varphi_v^2 \neq 0$. Shunday qilib, sirtning har bir nuqtasi atrofida ortogonal koordinata chiziqlari doimo mavjud bo`lib, birinchi oiladagi chiziqlarni ixtiyoriy ravishda tanlashi-miz mumkin.

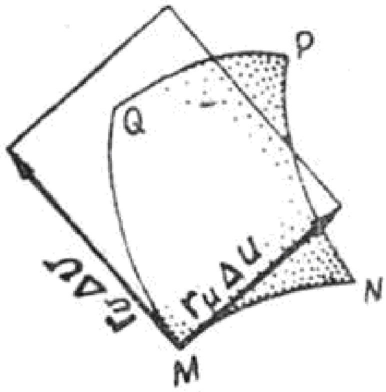
Sirt ustidagi sohaning yuzini aniqlash

Sirt ustida sodda yopiq chiziq bilan chegaralangan W soha berilgan bo`lsin. *W* sohadagi nuqtalardan soni chekli koordinata chiziqlarini o`tkazamiz. Bu soha egri chiziqli ikki xil to`rtburchak- ka ajraladi. 1) To`liq to`rtburchaklar (bunday to`rtburchaklar-ning ikki tomoni u chiziqlar bilan va qolgan ikki tomoni v chiziqlar bilan chegaralangan). 182-chizmada bunday to`rtburchaklardan biri $(\square MNPQ)$ katta qilib ko`rsatilgan.

2) T o` l i q s i z t o` r t b u r ch a k l a r. Bu ko`rinishdagi to`rtburchaklar-ning har biri sirt ustidagi sohani chegaralaydigan chiziqni ham qisman qoplaydi. To`rtburchaklarning har biri kichrayib borganda, to`liqsiz to`rt - burchaklar sohani chegaralaydigan chiziqqa yaqinlashadi va bu to`rtburchaklar-ning hammasini eni nolga intiladigan yopiq tasma ichiga joylashtirish mumkin bo`ladi. Shu sababli ikkinchi ko`rinishdagi to`rtburchaklarni e`tibor - siz qoldiramiz.



182-chizma.



183-chizma

chizmadagi egri chiziqli to`rtburchakni tekshiraylik. Bunda *MN* va *PQ* tomonlar *u* chiziqlarga qarashli. *NP* va *MQ* tomonlar esa *v* chiziqlarga qarashlidir. To`rtburchak uchlarining koordinatalari: $M(u,v),\ N(u+\Delta u,v),\quad Q(u,\ v+\Delta v),\quad P(u+\Delta u,v+\Delta v)$. Shuning uchun

$\overline{OM}=r(u,v),\ \overline{ON}=r(u+\Delta u,\ v),$

$$\overline{OP}=r(u+\Delta u,\ v+\Delta v),\ \overline{OQ}=r(u,\ v+\Delta v)\ \text{bo`lib},$$

$$\overline{MN}=r(u+\Delta u,\ v)-r(u,v).$$

Chekli orttirmalar teoremasiga asosan, $\overline{MN}=\Delta u\cdot r_u[u+\theta\Delta u,v]$.

Agar birinchi tartiblidan yuqori cheksiz kichiklarni e`tiborga olmasak. $\overline{MN}\cong r_u\Delta u$ bo`ladi. Xuddi shunga o`xshash $\overline{MQ}\cong r_u\Delta v, MN\cong\left|\overline{MN}\right|\cong|r_v|\Delta u$ va $MQ\cong\left|\overline{MQ}\right|\cong|r_v|\Delta v$ bo`lgani uchun, egri chiziqli to`rtburchakning yuzi o`rniga tomonlari $r_u\Delta u$ va $r_v\Delta v$ vektorlardan iborat to`g`ri chiziqli parallelogramm - ning yuzini olamiz, ya`ni egri chiziqli *MNPQ* to`rtburchakning yuzi deb $r_u\Delta u$ va $r_v\Delta v$ vektorlardan yasalgan parallelogrammnpng yuzini qabul qilamiz. Bu parallelogramm sirtning *M* nuqtasidagi urinma tekislikda bo`lib, uning yuzi ushbuga teng:

$$\Delta\sigma=\left|[r_u r_v]\right|\Delta u\Delta v$$

yoki

$$\Delta\sigma=\sqrt{(r_u r_u)\cdot(r_v r_v)-(r_u r_v)\cdot(r_u r_v)}\Delta u\Delta v=\sqrt{EG-F^2}\Delta u\Delta v.$$

Demak, yuzning $\Delta\sigma$ ifodasi sirtning birinchi kvadratik formasidagi *E, G, F* koeffitsientlar orqali ifodalanadi.

$\sqrt{EG-F^2}$ qiymat $\left|(r_u r_v)\right|$ ga teng bo`lgani uchun, u hamma vaqt noldan kattadir (oddiy nuqtada u noldan farqli).

Shunday qilib, har bir egri chiziqli to`rtburchakning yuzi o`rniga to`g`ri chiziqli parallelogrammning yuzini olamiz. Sohaga tegishli bo`lgan hamma parallelogrammlar yuzlarining yig`indisini tuzaylik:

$$\sum \Delta\sigma = \sum \sqrt{EG-F^2}\Delta u\Delta v. \qquad (1)$$

Har bir Δu va Δv qiymatlarni nolga intiltiramiz. Bu holda egri chiziqli to`rtburchaklarning soni cheksizlikka intilib, har birining yuzi nolga intiladi. (1) dan limit olamiz. Sirt ustidagi *W* sohaning yuzi deb (1) yig`indining limitiga aytamiz:

$$S = \lim \sum \Delta\sigma = \lim \sum \sqrt{EG-F^2}\Delta u\Delta v.$$

Karrali integrallar nazariyasiga asosan, $\lim \sum \Delta u \Delta v$ ning qiymati sirtdagi egri chiziqli koordinat sistemasiga bog`liq bo`lmasdan, faqat sohaning shakliga bog`liq bo`lib, $\sqrt{EG-F^2}$ dan olingan ikki o`lchovli integralga teng;

$$\iint\limits_W \sqrt{EG-F^2}\,du\,dv \qquad (2)$$

Natija. Sirt ustidagi egri chiziq oyining uzunligini, sirt ustidagi chiziqlar orasidagi burchakni va sohaning yuzini aniqlash uchun sirt ustidagi egri chiziqli koordinat sistemasiga nisbatan *E, F* va *G* qiymatlarni *u, v* orqali aniqlash kifoyadir. *E, F* va *G* larni aniqlash uchun, sirtning shaklini va tenglamasini bilish shart emas.

Sirt oshkor $z=f(x, u)$ oshkor tenglama bilan berilgan holda tegishli *W* sohaning yuzi uchun ushbu formula hosil qilinadi:

$$S = \iint\limits_W \sqrt{1+p^2+q^2}\,dx\,dy.$$

Misollar. $x=R\cos u\cos v$, $u=R\cos u\sin v$, $z=R\sin u$ sfera va uning ekvatori bilan chegaralapgan sohaning yuzi aniqlansin.

Ye ch i sh. Sirtning tenglamasidan sohaning *M* nuqtasi uchun *E, F* va *G* qiymatlarni aniqlaymiz: $E=R^2$, $F=0$, $G=R^2\cos^2 u$. Bularni (2) formulaga qo`yib, ikki o`lchovli integralni hisoblaymiz. Birinchi integralning chegaralari 0 dan 2π gacha va ikkinchi iitegralning chegaralari 0 dan $\frac{\pi}{2}$ gacha bo`lgani uchun:

$$S = \iint\limits_W \sqrt{EG-F^2}\,du\,dv = \int_0^{2\pi} dv \int_0^{\frac{\pi}{2}} \sqrt{R^2\cdot R\cos^2 u - 0^2}\,du\,dv = 2\pi R^2$$

Buning to`g`riligi elementar geometriya kursidan ma`lum.

Sirt ustidagi yoy, uning uzunligi (birinchi kvadratik forma), sohaning yuzi tushunchalari sof geometrik xarakterga ega bo`lgani uchun, ular sirt ustida egri chiziqli koordinatalarni tanlashga, chiziqqa qanday ichki siniq chiziq chizilishiga yoki sohani parallelogrammlarga bo`lish usuliga bog`liq emas. Bu tasdiqlarning isbotiga biz to`xtamaymiz. *N* normal-vektor ham invariant xarakterga egadir.

Endi 74-paragrafda qaralgan sirtlarga qaytamiz:

Tekislik: r = xi +yj, dr= dxi + duj

$$\hat{O}_1 = ds^2 = dx^2 + 2\cos\omega\,dx\,dy + dy^2,$$

$$E=1,\ F=\cos\omega,\ G=1,$$

$$\omega = \frac{\pi}{2} \quad \text{da} \ F=0,\ ds^2 = dx^2+dy^2.$$

Qutb koordinatalarida:

$$r = \rho e(\varphi),\ dr = d\rho e(\varphi) + \rho e\left(\varphi + \frac{\pi}{2}\right)d\varphi$$

bundan

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2.$$

Sfera: $r = a\left[\cos\theta\, e(\varphi) + \sin\theta\, k\right]$.

$$dr = a\left[-\sin\theta\, e(\varphi)\, d\theta + \cos\theta\, e\left(\varphi + \frac{\pi}{2}\right) d\varphi + \cos\theta\, d\theta\, k\right],$$

$$\hat{O}_1 = ds^2 = a^2\left[\cos^2\theta\, d\varphi^2 + d\theta^2\right].$$

Demak,

$$E = a^2\cos^2\theta,\ \ F = 0,\ G = a^2,$$

bunda F=0, ya`ni meridianlar va parallellar tikdir.

Aylanma sirt: $r = \varphi(u)e(v) + \psi(u)k$.

$$dr = \left[\varphi'(u)e(u) + \psi'(u)k\right]du + \varphi(u)e\left(v + \frac{\pi}{2}\right)dv,$$

$$r_u = \varphi'(u)\ e(v) + \psi'(u)k,\quad r_v = \varphi(u)e\left(v + \frac{\pi}{2}\right)dv$$

$$\hat{O}_1 = ds^2 = \left[\varphi'^2(u) + \psi'^2(u)\right]du^2 + \varphi^2(u)dv^2,$$

$$E = \varphi'^2(u) + \psi'^2(u),\quad F = 0,\quad G = \varphi^2(u).$$

$z = \psi(u) = u,\ x = \varphi(z)$ bo`lgan xususiy holda, ya`ni profil chiziq x=φ(z) shakldagi tenglama bilan berilganda:

$$r = \varphi(u)e(v) + uk,$$

$$r_u = \varphi'(u)e(v) + k,\quad r_v = \varphi(u)e\left(v + \frac{\pi}{2}\right)$$

bo`lib, bulardan:

$$E = 1 + \varphi'^2(u),\ \ F = 0,\ \ G = \varphi^2(u).$$

Masalan, $\rho(= x) = a\cdot ch\dfrac{u}{a}$ — zanjir chiziqni z(= i) o`qi atrofida aylantirishdan hosil qilingan

$$r = a\cdot ch\frac{u}{a}e(\varphi) + uk$$

katenoid uchun birinchi kvadratik forma

$$\hat{O}_1 = ds^2 = ch^2\frac{u}{a}du^2 + a^2ch^2\frac{u}{a}d\varphi^2. \qquad (3)$$

Aylanma sirt z=f(ρ) tenglama bilan berilsa, x=u=ρ faraz qilamiz, u vaqtda:

$$r = \rho e(v) + f(\rho)k,$$

$$r_\rho = e(v) + f'(\rho),\ \ r_v = \rho e\left(v + \frac{\pi}{2}\right)$$

$$E = 1 + f'^2(\rho),\ \ F = 0,\ \ G = \rho^2,$$

$$\hat{O}_1 = ds^2 = \left[1 + f'^2(\rho)\right]d\rho^2 + \rho^2dv^2.$$

Ikkala holda ham F=0, ya`ni koordinat chiziqlari (parallellar va meridianlar) ortogonaldir. Oxirgi tenglikka diqqat qilaylik. Agar

$$\left[1 + f'^2(\rho)\right]d\rho^2 = du^2.$$

deb faraz qilinsa, u holda *di* — profil chiziq, z=f(ρ) ning yoyi uzunligi differentsialini tasvirlaydi, bu yoy biror paralleldan boshlab hisoblanadi: ρ q sonst ga u=const mos kelib, u=cosnt ham parallellarni ifodalaydi. Demak,

$$ds^2 = du^2 + \rho^2dv^2,$$

ya'ni

$$E=1,\quad F=0,\quad G=\rho^2.$$

Masalan,

$$\rho=\frac{a}{2}\left(e^{\frac{z}{a}}+e^{-\frac{z}{a}}\right)=a\cdot ch\frac{z}{a}$$

zanjir chiziqni o`z asosi atrofida aylantirishdan hosil qilingan katenoid uchun $\rho^{2}= a^2 + u^2$ bo`lib, uning birinchi kvadratik formasi quyidagi shaklni oladi:

$$ds^2=du^2+\left(a^2+u^2\right)dv^2$$

demak,

$$E=1,\ F=0,\ G=a^2+u^2.$$

Gelikoid: $r = u\,\boldsymbol{e}(v) + av\boldsymbol{k}$.

$$r_u=e(v),\quad r_v=u\,e\left(v+\frac{\pi}{2}\right)+ak,$$

$$E=1,\quad F=0,\quad G=a^2+u^2,$$

$$ds^2=du^2+\left(a^2+u^2\right)dv^2.$$

Mashqlar

Quyidagi sirtlarning birinchi kvadratik formalari topilsin:

1) Doiraviy silindr $r=a\,e(\varphi)+v\,k$.

2) Doiraviy konus $r=\{\cos a\,e(\varphi)+\sin a\,k\}v$.

3) Uchi qutbdagi konus $r=v\,l(u)$.

4) Tor $r=(a+b\cos u)\,e(v)+b\sin u\,k.$

5) Psevdosfera $r=a\left\{\sin u\;e(v)+\left[\cos u+\ln tg\frac{u}{2}\right]k\right\}.$

6) To`g`ri chiziqli sirt $r=\rho(u)+v\,l(u)$.

Jumladan, yuqoridagi mashqda qaralgan urinmalar, bosh normallar, binormal-lar cirtlarn uchun: $\bar{\rho}(u)+c\,\bar{\tau}(u)$, $\bar{\rho}(u)+v\bar{v}(u)$, $\rho(u)-v\bar{\beta}(u)$, bunda u—tabiiy parametr.

7) $r=\sin u\,e(u)+uk$.

Javob: 1) $ds^2=a^2d\varphi^2+dv^2$; 2) $ds^2=(v\cos\alpha)^2\,d\varphi^2+dv^2$;

3) $ds^2=b^2du^2+(a+b\cos u)^2\,dv^2$; 4) $ds^2=a^2\left(\dfrac{\cos^6 u}{\sin^2 u}du^2+\sin^2 u\,dv^2\right)$;

5) $ds^2=\left(\bar{\rho}'^2+2\bar{\rho}'e'v+v^2\,l^2\right)du^2+2e\,\bar{\rho}'\,du\,dv+l^2dv^2$;

Uchta xususiy holda: $ds^2=\left(1+k^2v^2\right)du^2+2du\,dv+dv^2$;

$$ds^2=\left[1+v^2\left(k^2+\sigma^2\right)\right]du^2+dv^2;\qquad ds^2=\left(1+v^2\sigma^2\right)du^2+dv^2.$$

6) $ds^2=\left(1+\cos^2 u\right)du^2+\sin^2 u\,dv^2$.

1. Sirtning birinchi kvadratik formasi: $ds^2=du^2+sh^2u\,dv^2$ bo`lib, shu sirt ustida $u-v=0$ chiziq berilgan. Bu chiziq yoyining uzunligi topilsin.

Javob: s = sh u_2 – shu_2.

2. Katenoid ustida $u\pm v = 0$ chiziqlar berilgan. Ular orasidagi burchak topilsin.

K o` r s a t m a . Katenoid uchun $ds^2=du^2+\left(a^2+u^2\right)dv^2$.

Javob: $\cos\varphi = \dfrac{1-a^2}{1+a^2}.$

3. $r = u\,e(v) + a v k$ gelikoid ustida yotuvchi $\ln\left(u+\sqrt{u^2+a^2}\right)+v = \mathrm{co}\,nst$,

$\ln\left(u+\sqrt{u^2+a^2}\right)-v = const$ chiziqlarning ortogonalligi isbotlansin.

4. $r = a\sqrt{1+u^2}\,i + a u\, j + av\,k$ sirt va uning ustida *uq*sh*v* chiziq berilgan. Bu chiziqning tabiiy tenglamalari yozilsin.

5. Chiziqlar oilasi $\dfrac{dv}{du} = f\left(u,v\right)$ berilgan, bunda *f* (*u*, *v*) — uzluksiz funktsiya. Shu oilaga "ortogonal" oila topilsin.

E ch i sh. Ortogonallikning

$$E\,du\,\delta u + F\left(du\,\delta v + dv\,\delta u\right) + G\,dv\,\delta u = 0$$

shartidan

$$\frac{\delta v}{\delta u} = -\frac{E + F\cdot f\left(u,v\right)}{F + G\cdot f\left(u,v\right)}$$

yoki

$$\frac{\delta v}{\delta u} = f_1\left(u,v\right).$$

Bu esa birinchi tartibli differentsial tenglamadir. Uni integrallab berilgan onlaning ortogonal traektoriyalarini topamiz.

6. Sirtdagi *i* chiziqlar (*v*=const) va *v* chiziqlar (*u* = const) ning ortogonal traektoriyalari topilsin.

Javob: $E\,\delta u + F\,\delta v = 0,\; F\delta u + G\,dv = 0.$

7. Gelikoid ustida yotuvchi $\ln\left(u+\sqrt{a^2+u^2}\right)-v = 4$ chiziq koordinata chiziqlari orasidagi burchakni har bir nuqtada teng ikkiga bo`ladi. Buni isbotlang.

8. I z o t e r m i k k o o r d i n a t a l a r. Aylanma sirtning birinchi kvadratik formasi uchun yuqorida ushbu formulani bergan edik;

$$ds^2 = \left[\varphi'^2\left(u\right)+\psi'^2\left(u\right)\right]du^2 + \varphi^2\left(u\right)dv^2$$

Bu formuladagi birinchi qo`shiluvchi — meridian yoyi differentsialining kvadratini ifodalaydi, uni dl^2 bilan belgilaymiz:

$$\left[\varphi'^2\left(u\right)+\psi'^2\left(u\right)\right]du^2 = dl^2$$

u holla:

$$ds^2 = dl^2 + \varphi^2\left(u\right)dv^2 = \varphi^2\left(u\right)\left[\frac{dl^2}{\varphi^2\left(u\right)} + dv^2\right].$$

Agar

$$\int\frac{dl}{\varphi(l)} = \xi\left(u\right),\; v = \eta$$

deb faraz qilsak,

$$ds^2 = \varphi^2\left(\xi\right)\left[d\xi^2 + d\eta^2\right]$$

bo`ladi.

Sirtning bu ko`rinishdagi birinchi kvadratik formasi tekislikning to`g`ri burchakli Dekart sistemasidagi chiziqli elementdan ko`paytuvchi bilangina farqlanadi; ξ va η — sirtning *izotermik parametrlari* deyiladi. Masalan, katenoidning (3) kvadratik formasini izotermik shaklga keltirish oson:

$$ds^2 = a^2 ch^2 \xi \left(d\xi^2 + d\eta^2 \right).$$

bunda $\xi = \dfrac{u}{a}, \quad \eta = \varphi.$

9. To`g`ri chiziqli $r = \bar{\rho}(u) + v\, l(u)$ sirtning chiziqli

$$ds^2 = \left(\bar{\rho}'^2 + 2v\, l' \bar{\rho}' + v^2 l'^2 \right) du^2 + 2l\, \bar{\rho}'\, du\, dv + dv^2$$

elementini olib va unda *i* ni tabiiy parametr sifatida qabul qilib, ushbu

$$b = l' \bar{\rho}', \; c^2 = l'^2, \; \bar{l}\, \rho' = \cos\theta$$

belgilashlarni kiritsak, u holda

$$ds^2 = \left(1 + 2bv + c^2 v^2 \right) du^2 + 2\cos\theta\, du\, dv + dv^2.$$

Sirtning chiziqli elementi bu sirtdagi (u, v) va ($u+ du$, $v+dv$) nuqtalar orasidagi masofaning kvadratini beradi. Uni quyidagi shaklda yozish mumkinligi isbotlansin:

$$ds^2 = \left(dv + \cos\theta du \right)^2 + \left(\sin^2\theta + 2bv + c^2 v^2 \right) du^2 .$$

Agar $dv + \cos\theta\, du = 0$ va $b^2 + c^2 v = 0$ bo`lsa, ds^2 minimumga erishadi (nega?).

Demak, yasovchidagi qisilish nuqtasining v parametri $v = -\dfrac{b^2}{c^2}$ dan aniqlanadi. Bundan nuqtalarning geometrik o`rni qisilish chizig`idir; $b = \bar{\rho}' l' = 0$ bo`lsa, $v = 0$ dan iborat yo`naltiruvchi qisilish chizig`ini ifodalaydi (nega?).

10. To`g`ri chiziqli sirtning hamma yasovchilari tayin (yo`naltiruvchi) tekislikka parallel bo`lsa, u *Katalan sirti* deyiladi. To`g`ri chiziqli sirtning Katalan sirti bo`lish sharti topilsin. Bunday sirtning to`la aniq-lanishi uchun, unda yotuvchi ikkita (yo`naltiruvchi) chiziq bo`lishi kerakmi?

Yo`naltiruvchilari ikkita:

$$x^2 + z^2 - 2ax = 0, \qquad y = 0,$$
$$y^2 + z^2 - 2ay = 0, \qquad x = 0.$$

aylanalardan iborat Katalan sirtining tenglamasi topilsin.

Ko`rsatma. $l(u)$ yasovchi normal-vektori $n$ dan iborat tekislikka doimo parallel bo`lsa, $pl(i) = 0$, $pl'(i) = 0$, $pl''(i) = 0$ shartlar bajariladi. Bulardan:

$$l(u)\, l'(u)\, l''(u) = 0.$$

Javob: Sirt ikki bo`lakdan iborat:

$$(x^2 + y^2)^2 + z^2 - 2a(x + y) = 0 \text{ (elliptik tsilindr)},$$
$$z^4 + z^2 [(x - y)^2 — 2a(x + y)] + 4a^2 xy = 0.$$

11. Katalan sirtining[1] hamma yasovchilari ma`lum bir to`g`ri chiziq bi-lan kesisha borsa, bunday sirt *konoid* deniladi. Bu to`g`ri (yo`naltiruvchi) chiziq yo`naltiruvchi tekislikka tik bo`lsa, u holda konoid t o` g` r i k o n o i d deb ataladi. Gelikoid to`g`ri konoid bo`la oladimi? Uning striktsion chizig`i nimadan iborat? Giperbolik paraboloid konoidmi?

Konoidning shaklini aniqlash uchun yo`naltiruvchi to`g`ri chiziqni, tekis-likni va yana bitta yo`naltiruvchi chiziqni berish kerak. Masalan, a) yo`naltiruvchi to`g`ri chiziq $u=0$, $z=h$, yo`naltiruvchi tekislik YOZ va yo`naltiruvchi chiziq $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \; z = 0$; b) yo`naltiruvchi to`g`ri chiziq $x= a$, $y= 0$, yo`naltirunchi tekislik $z = 0$, yo`naltiruvchi chiziq $y^2 = 2pz$, $x = 0$.

Javob: a) $\left(1 - \dfrac{x^2}{a^2}\right)\left(\dfrac{z}{h} - 1\right)^2 - \dfrac{y^2}{b^2} = 1$; b) $a^2 y^2 = 2pz(x - a)^2$.

Foydalanilgan adabiyotlar

[1] Katalan sirti adabiyotda bazan *silidroid* deyiladi.

1. А. В. Погорелов. Геометрия. М. « Наука»   1983.
2. M.A.Sobirov va A.E. Yunupov. “ Differensial geometriya kursi”
3. A. Ya. Narmanov “ Differensial geometriya”