

Gravitating dyons with large electric charge

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Abstract

We consider non-Abelian dyons in Einstein–Yang–Mills–Higgs theory. The dyons are spherically symmetric with unit magnetic charge. For large values of the electric charge the dyons approach limiting solutions, related to the Penney solutions of Einstein–Maxwell–scalar theory.
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1. Introduction

Magnetic monopoles arise as regular non-perturbative solutions of $SU(2)$ Yang–Mills–Higgs theory [1]. Monopoles with magnetic charge $n = 1$ are spherically symmetric, monopoles with higher charge have axial symmetry [2–5] or no rotational symmetry at all [6]. When electric charge is added to magnetic monopoles, dyons arise [7,8]. These are stationary solutions with vanishing angular momentum [9,10].

The non-trivial vacuum structure of $SU(2)$ Yang–Mills–Higgs theory not only allows for magnetic monopoles, but it results in a plethora of further regular non-perturbative solutions [11–15]. The simplest of these solutions are monopole–antimonopole pairs, where a monopole and an antimonopole form an unstable equilibrium configuration [11–13].

When gravity is coupled to Yang–Mills–Higgs theory, gravitating monopoles [16,17], dyons [18], monopole–antimonopole pairs [19–22], and further configurations arise [23]. These solutions depend on a dimensionless coupling constant α , which is proportional to the square root of Newton’s constant and the Higgs vacuum expectation value. For each type of solution, a branch of gravitating solutions emerges smoothly from the corresponding flat space solution and extends up to a maximal value of α , beyond which the size of the core of the solution

would be smaller than its Schwarzschild radius [16]. At the maximal value of α this fundamental branch bifurcates with a second branch of solutions.

For monopoles and dyons, in the case of vanishing Higgs potential, this second branch reaches a critical value of α , where it bifurcates with the branch of extremal Reissner–Nordström black holes with the same charge(s) [16–18]. For monopole–antimonopole pair solutions and other composite solutions, in contrast, the second branch extends back to $\alpha = 0$, where a pure Einstein–Yang–Mills solution is reached (after scaling w.r.t. α) [24–27].

As the second monopole respectively dyon branch reaches the critical value of the coupling constant α , and merges with the extremal Reissner–Nordström branch, the space–time splits into two regions [16]. The exterior space–time of the critical solution, corresponds to the one of an extremal Reissner–Nordström black hole [16], while the interior space–time retains regularity at the center, due to the influence of the non-Abelian fields present.

Here we reconsider gravitating dyons and study the properties to the solutions in the limit of large electric charge Q . We demonstrate, that for $Q \rightarrow \infty$ we also obtain a limiting solution, which consists of two regions, a non-Abelian interior region and an Abelian exterior region. But in this case, the non-Abelian interior part has a flat metric, and the Abelian exterior part corresponds to a certain Penney solution of Einstein–Maxwell–scalar theory (after scaling w.r.t. Q) [28].

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In Section 2 we present the Einstein–Yang–Mills–Higgs action, the Ansatz, the equations of motion and the boundary conditions. We then consider the equations for large electric charge, and relate them to the equations of Einstein–Maxwell–scalar theory. We describe the relevant features of the Penney solutions in Section 3, and discuss the properties of the numerically constructed non-Abelian dyons in Section 4, where we also address their relation to the Penney solutions. We present our conclusions in Section 5.

2. Einstein–Yang–Mills–Higgs solutions

2.1. Action

We consider the $SU(2)$ Einstein–Yang–Mills–Higgs action in the limit of vanishing Higgs potential,

$$S = \int \left[\frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu \Phi D^\mu \Phi) \right] \sqrt{-g} d^4x \quad (1)$$

with curvature scalar R , $SU(2)$ field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu], \quad (2)$$

gauge potential $A_\mu = 1/2\tau^a A_\mu^a$, gauge covariant derivative

$$D_\mu = \nabla_\mu + ie[A_\mu, \cdot], \quad (3)$$

and Higgs field $\Phi = \tau^a \Phi^a$; G is Newton's constant, and e is the gauge coupling constant. Since we here consider vanishing Higgs potential, we impose a Higgs field vacuum expectation value v .

Variation of the action Eq. (1) with respect to the metric $g_{\mu\nu}$, the gauge potential A_μ^a , and the Higgs field Φ^a leads to the Einstein equations and the matter field equations, respectively.

2.2. Ansatz

We employ Schwarzschild-like coordinates and parametrize the line element by [16,18]

$$ds^2 = -A(r)^2 N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 d\Omega^2, \quad (4)$$

$$N(r) = 1 - \frac{2\mu(r)}{r},$$

with mass function $\mu(r)$. The Ansatz for the gauge potential and Higgs field is given by

$$A_\mu dx^\mu = \frac{1 - H_2(r)}{2e} (\tau_\varphi d\theta - \tau_\theta \sin\theta d\varphi) + \frac{B_1(r)}{2e} \tau_r dt, \quad (5)$$

$$\Phi = v\Phi_1(r)\tau_r,$$

where the $su(2)$ matrices τ_r , τ_θ , and τ_φ are defined as scalar products of the spatial unit vectors with the Pauli matrices $\tau^a = (\tau_x, \tau_y, \tau_z)$, and the subscripts on the functions indicate the correspondence to the functions of the more general Ansatz, necessary for the construction of monopole–antimonopole systems [14,25].

2.3. Equations of motion

We now change to dimensionless quantities, the dimensionless coordinate \tilde{r} , the dimensionless electric function \tilde{B}_1 , and the dimensionless mass function $\tilde{\mu}$,

$$\tilde{r} = evr, \quad \tilde{B}_1 = \frac{B_1}{ev}, \quad \tilde{\mu} = ev\mu, \quad (6)$$

and introduce the dimensionless coupling constant α

$$\alpha^2 = 4\pi G v^2. \quad (7)$$

Suppressing the \sim in the following, the tt and rr components of the Einstein equations yield for the metric functions the equations,

$$\mu' = \alpha^2 \left(\frac{r^2 B_1'^2}{2A^2} + \frac{B_1^2 H_2^2}{A^2 N} + N H_2'^2 + \frac{1}{2} N r^2 \Phi_1'^2 + \frac{(H_2^2 - 1)^2}{2r^2} + \Phi_1^2 H_2^2 \right), \quad (8)$$

and

$$A' = \alpha^2 r \left(\frac{2B_1^2 H_2^2}{A^2 N^2 r^2} + \frac{2K'^2}{r^2} + \Phi_1'^2 \right) A, \quad (9)$$

while the matter field equations yield

$$(AN H_2')' = A H_2 \left(\frac{H_2^2 - 1}{r^2} + \Phi_1^2 - \frac{B_1^2}{A^2 N} \right), \quad (10)$$

$$\left(\frac{r^2 B_1'}{A} \right)' = \frac{2B_1 H_2^2}{AN}, \quad (11)$$

and

$$(r^2 AN \Phi_1')' = 2A \Phi_1 H_2^2. \quad (12)$$

These equations then depend only on the dimensionless coupling constant α [16].

A particular solution of these equations is the embedded Reissner–Nordström solution with unit magnetic charge, $P = 1$, and arbitrary electric charge Q ,

$$\mu(r) = m - \frac{\alpha^2(Q^2 + 1)}{2r}, \quad A(r) = 1, \quad (13)$$

$$H_2(r) = 0, \quad B_1(r) = v - \frac{Q}{r}, \quad \Phi_1(r) = 1, \quad (14)$$

where the extremal solution satisfies

$$m = r_H = \alpha\sqrt{Q^2 + 1}. \quad (15)$$

2.4. Boundary conditions

Dyons are globally regular particle-like solutions of the $SU(2)$ Einstein–Yang–Mills–Higgs system. Regularity of the solutions at the origin then requires the boundary conditions

$$\mu(0) = 0, \quad (16)$$

and [7]

$$H_2(0) = 1, \quad B_1(0) = 0, \quad \Phi_1(0) = 0. \quad (17)$$

Asymptotic flatness of the solutions, on the other hand, implies that the metric functions A and μ both approach constants at infinity. We adopt

$$A(\infty) = 1. \quad (18)$$

The matter functions satisfy the asymptotic boundary conditions

$$H_2(\infty) = 0, \quad B_1(\infty) = \nu, \quad \Phi_1(\infty) = 1. \quad (19)$$

The dimensionless magnetic charge P is the topological charge of the solutions, while the dimensionless electric charge Q , and the dimensionless scalar charge c_H are obtained from the asymptotic expansion of the fields, thus

$$P = 1, \quad Q = -\lim_{r \rightarrow \infty} r(B_1 - \nu), \quad c_H = \lim_{r \rightarrow \infty} r^2 \partial_r \Phi_1, \quad (20)$$

and $\mu(\infty) = m$ represents the dimensionless mass of the solutions.

2.5. Scaled equations of motion

Since we are interested in the limit of large electric charge, $Q \rightarrow \infty$, we now consider the above set of equations in terms of quantities scaled by the charge Q . Thus we introduce

$$\bar{r} = \frac{r}{Q}, \quad \bar{m} = \frac{m}{Q}, \quad \bar{c}_H = \frac{c_H}{Q}. \quad (21)$$

In the limit $Q \rightarrow \infty$ we then obtain a coupled set of equations for μ , A , B_1 and Φ_1 ,

$$\bar{\mu}' = \frac{\alpha^2}{2} \bar{r}^2 \left(\frac{B_1'^2}{A^2} + N \Phi_1'^2 \right), \quad (22)$$

$$A' = \alpha^2 \bar{r} \Phi_1'^2 A, \quad (23)$$

$$\frac{\bar{r}^2 B_1'}{A} = \bar{Q} = 1, \quad (24)$$

$$\bar{r}^2 A N \Phi_1' = \bar{c}_H, \quad (25)$$

where the $'$ denotes differentiation w.r.t. \bar{r} , while the equation for H_2 decouples,

$$H_2'' = \frac{H_2(H_2^2 - 1)}{\bar{r}^2}. \quad (26)$$

The coupled set of equations Eqs. (22)–(25) corresponds precisely to the set of equations of Einstein–Maxwell–scalar theory, when the electric charge has the value $Q = 1$. We therefore now turn to the Einstein–Maxwell–scalar solutions.

3. Einstein–Maxwell–scalar solutions

3.1. Action

The matter Lagrangian of Einstein–Maxwell–scalar theory reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \Phi_1 \partial^\mu \Phi_1 \quad (27)$$

with the Abelian gauge potential $A_\mu dx^\mu$ and the scalar field Φ_1 , giving rise to the stress-energy tensor in the Einstein equations,

$$G_{\mu\nu} = 2\alpha^2 T_{\mu\nu}, \quad (28)$$

in terms of dimensionless coordinates and fields.

3.2. Penney solutions

Penney obtained static spherically symmetric solutions with electric charge Q and scalar charge c_H . He employed the metric parametrization

$$ds^2 = -e^{-a} dt^2 + e^a dR^2 + e^b d\Omega^2, \quad (29)$$

with radial coordinate R .

The Penney solutions then read

$$e^b = ((R - \rho)(R - \sigma))^{-\Lambda} \left(\frac{\sigma(R - \rho)^\Lambda - \rho(R - \sigma)^\Lambda}{\sigma - \rho} \right)^2, \quad (30)$$

$$e^b = ((R - \rho)(R - \sigma))e^a,$$

$$e^b \hat{B}_1' = Q, \quad e^{b-a} \Phi_1' = c_H, \quad (31)$$

where $A_0 = \hat{B}_1$ denotes the time component of the gauge potential.

The constants ρ , σ , and Λ of the Penney solutions satisfy the relations

$$\Lambda^2 \rho \sigma = \alpha^2 Q^2, \quad \alpha^2 c_H^2 = (1 - \Lambda^2) \left(\frac{\rho - \sigma}{2} \right)^2. \quad (32)$$

Since we want to relate the Penney solutions to the limiting dyon solutions, we must select those Penney solutions, which do not exhibit a metric singularity anywhere except the origin. Thus we now focus on the case, where in the Reissner–Nordström limit no horizons occur, but a naked singularity is present. In the Penney solutions this is achieved by the choice $\rho = \rho_0 + i\sigma_0$, $\sigma = \rho_0 - i\sigma_0$. Defining

$$\xi^2 = (R - \rho_0)^2 + \sigma_0^2, \quad \tan \Psi = \frac{\sigma_0}{R - \rho_0}, \quad (33)$$

$$\tan \varphi_0 = -\frac{\rho_0}{\sigma_0}, \quad \gamma^2 = \rho_0^2 + \sigma_0^2,$$

then yields the metric functions

$$e^a = \left(\frac{\cos(\Lambda \Psi - \varphi_0)}{\cos \varphi_0} \right)^2, \quad e^b = \xi^2 e^a \quad (34)$$

and the relations, Eq. (32),

$$\Lambda^2 = \frac{\alpha^2 Q^2}{\gamma^2}, \quad \cos^2 \varphi_0 = \frac{\alpha^2 c_H^2}{\alpha^2 Q^2 - \gamma^2}, \quad (35)$$

$$m^2 = \alpha^2 Q^2 \sin^2 \varphi_0,$$

where the mass m is obtained from the asymptotics. From $\sin^2 \varphi_0 + \cos^2 \varphi_0 = 1$ we obtain

$$\frac{1}{\Lambda^2} = 1 - \frac{c_H^2}{Q^2 - \left(\frac{m}{\alpha}\right)^2}. \quad (36)$$

Since $\Lambda^2 \geq 0$ we obtain the bound

$$c_H^2 \leq Q^2 - \left(\frac{m}{\alpha}\right)^2. \quad (37)$$

3.3. Limit of large electric charge

We now consider the limit of large charge, taking $Q \rightarrow \infty$. Then according to Eq. (35) also $\Lambda \rightarrow \infty$. Eq. (36) then requires that the bound in Eq. (37) is precisely saturated,

$$c_H^2 = Q^2 - \left(\frac{m}{\alpha}\right)^2. \quad (38)$$

To obtain the limiting Penney solution for $Q \rightarrow \infty$, we thus impose this bound. We then consider the set of equations w.r.t. the scaled coordinate $\bar{R} = R/Q$, and introduce the scaled quantities $\bar{Q} = Q/Q = 1$, $\bar{\gamma} = \gamma/Q$, and $\bar{c}_H = c_H/Q$. In particular, we reexpress the relation $\alpha^2 Q^2 = \gamma^2 \Lambda^2$ as $\alpha^2 \bar{Q}^2 = \bar{\gamma}^2 \Lambda^2$, i.e., $\bar{\gamma} \rightarrow 0$. In this limit

$$\begin{aligned} \cos^2 \varphi_0 &= \bar{c}_H^2, \\ e^a &= \left(\cos\left(\frac{\alpha \bar{c}_H}{\bar{R}}\right) + \sqrt{\frac{1}{\bar{c}_H^2} - 1} \sin\left(\frac{\alpha \bar{c}_H}{\bar{R}}\right) \right)^2, \\ \xi^2 &= Q^2 \bar{R}^2. \end{aligned} \quad (39)$$

We are interested in the outer extremum of the metric function e^a , since this is a particular point of the space–time. It will later be identified as the transition point, where the space–time of the dyons will split into a non-Abelian interior region and an Abelian exterior region.

The outer extremum of e^a occurs at \bar{R}_0 , where

$$\frac{\alpha \bar{c}_H}{\bar{R}_0} = \arctan \sqrt{\frac{1}{\bar{c}_H^2} - 1}, \quad e^{a(\bar{R}_0)} = \frac{1}{\bar{c}_H^2}. \quad (40)$$

We now impose the dyon boundary conditions on the Maxwell and scalar function,

$$\begin{aligned} \hat{B}_1(\bar{R}_0) &= 0, & \hat{B}_1(\infty) &= \nu, \\ \Phi_1(\bar{R}_0) &= 0, & \Phi_1(\infty) &= 1. \end{aligned} \quad (41)$$

Integration of the scalar field Eq. (31) then yields

$$\Phi_1(\infty) = \bar{c}_H \int_{\bar{R}_0}^{\infty} \frac{1}{\bar{R}^2} d\bar{R} = \frac{\bar{c}_H}{\bar{R}_0} = 1, \quad (42)$$

i.e., $\bar{R}_0 = \bar{c}_H$, which together with Eq. (40) leads to

$$\cos \alpha = \bar{c}_H, \quad (43)$$

relating the scaled scalar charge to the coupling constant α .

To integrate the gauge field Eq. (31), we first reexpress the metric function e^a via

$$e^a = \left(\frac{\cos[\alpha(\frac{\bar{c}_H}{\bar{R}} - 1)]}{\cos \alpha} \right)^2. \quad (44)$$

Integration then yields a relation between the asymptotic value of the gauge potential and the coupling constant α ,

$$\hat{B}_1(\infty) = \int_{\bar{R}_0}^{\infty} e^{-b} d\bar{R} = \frac{\sin \alpha}{\alpha} = \nu. \quad (45)$$

Addressing finally the transformation to Schwarzschild-like coordinates, we note that

$$A^2 N = e^{-a}, \quad r^2 = e^b, \quad dR = A dr. \quad (46)$$

The transition point r_0 is thus given by $r_0^2 = e^{b(R_0)} = R_0^2 Q^2 / c_H^2 = Q^2$, i.e., $\bar{r}_0 = 1$.

4. Results

4.1. Gravitating dyons

Gravitating dyons have been considered before [18,22]. We here reconsider gravitating dyons and address their dependence on the electric charge Q . We focus on large values of the charge and, in particular, the limit $Q \rightarrow \infty$.

For dyon solutions in flat space, the magnitude of the electric charge is uniquely determined by the asymptotic value ν of the electric component of the gauge potential. This potential parameter ν is bounded, $0 \leq \nu \leq 1$, and in the limit of vanishing Higgs potential, one has the monotonic relation between Q and ν ,

$$Q(\nu) = \frac{\nu}{\sqrt{1 - \nu^2}}. \quad (47)$$

Thus the charge diverges in the limit $\nu \rightarrow 1$.

For gravitating dyons it was also expected, that (in the limit of vanishing Higgs potential) the bound $\nu = 1$ would be reached monotonically and that it would correspond to infinite electric charge [16].

When constructing the dyon solutions numerically, however, a surprise is encountered. For fixed coupling constant α , the gravitating dyons do not vary monotonically with ν . Instead at a maximal value ν_{\max} a bifurcation is encountered. Here a second branch extends slightly backwards towards smaller values of ν . These bifurcating branches are exhibited in Fig. 1, where we demonstrate the dependence of the charge Q and the scaled mass $\bar{m} = m/Q$ on the parameter ν .

As seen in Fig. 1, the second branches rise very steeply as Q becomes large. They are confined to the intervals $\nu_{\text{cr}}(\alpha) \leq \nu \leq \nu_{\max}(\alpha)$. The lower bounds $\nu_{\text{cr}}(\alpha)$, approached in the limit of infinite charge, are very close to the upper bounds $\nu_{\max}(\alpha)$. The critical values $\nu_{\text{cr}}(\alpha)$ are exhibited in Fig. 2. We note, that for small α , the critical values $\nu_{\text{cr}}(\alpha)$ exhibit an almost quadratic dependence on α . No solutions are found beyond $\alpha = \pi/2$.

Let us now consider dyons for very large values of Q , in order to identify the solution obtained in the limit $Q \rightarrow \infty$. Clearly, the mass m diverges in the limit $Q \rightarrow \infty$. However, the scaled mass $\bar{m} = m/Q$ tends to a finite limiting value, which depends on the coupling strength α , as seen in Fig. 1. Another quantity of interest is the scalar charge c_H , determining the $1/r$ power law decay of the Higgs field of the dyon. Like the

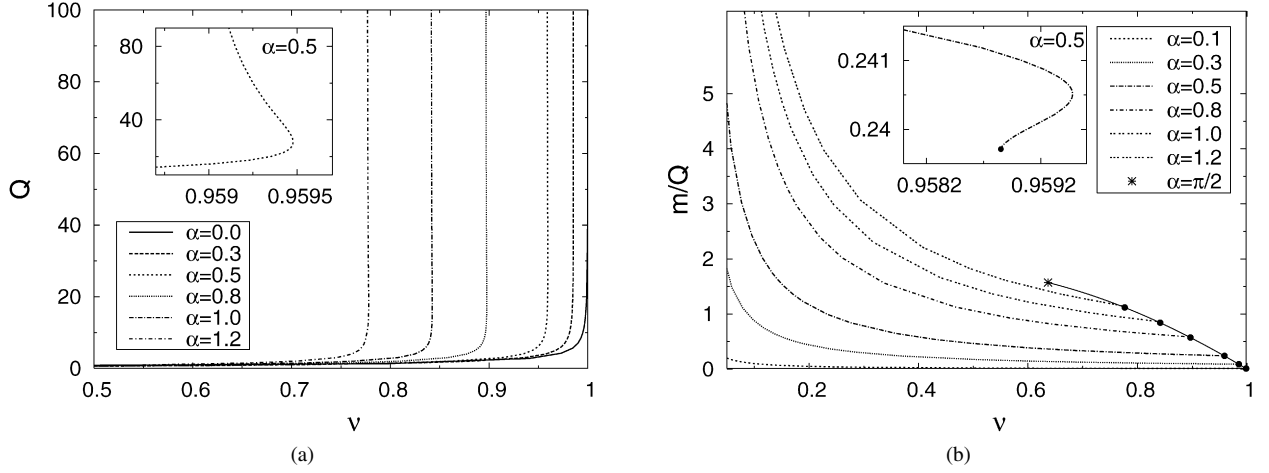


Fig. 1. Dyons: (a) electric charge $Q(\nu)$ versus ν , (b) scaled mass $\bar{m} = m/Q$ versus ν , for several values of α .

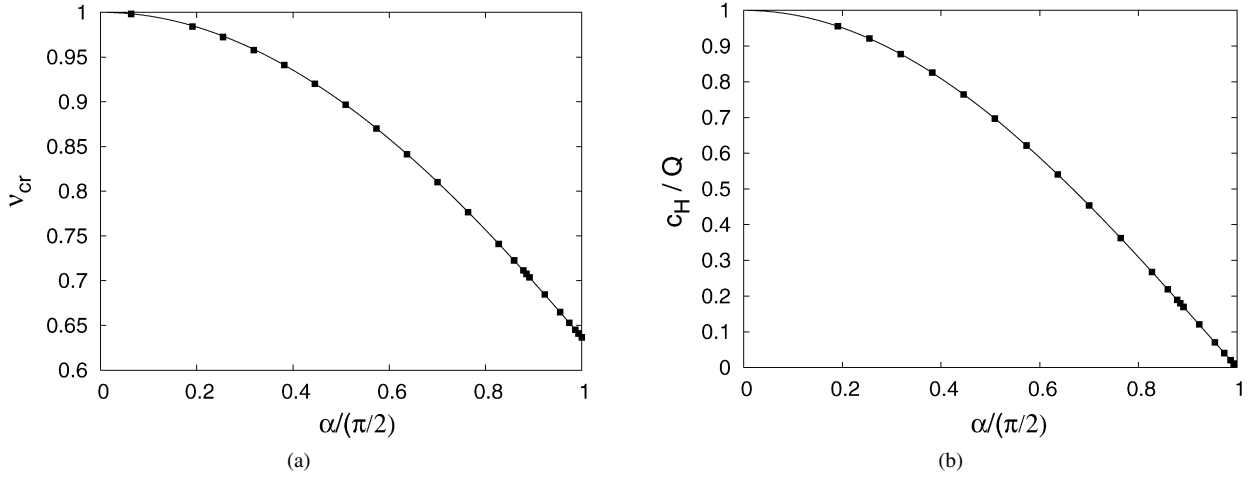


Fig. 2. Dyons: (a) critical values ν_{cr} versus α , (b) scaled scalar charge $\bar{c}_H = c_H/Q$ versus α . The exact dependence as obtained from the Penney solution is also exhibited for ν_{cr} and \bar{c}_H .

scaled mass, the scaled scalar charge $\bar{c}_H = c_H/Q$ approaches a finite limiting value, depending on the coupling strength α . In Fig. 2 \bar{c}_H is shown versus α for a very large value of the electric charge, $Q = 10000$.

4.2. Relation with the Penney solutions

To understand the limiting behaviour, we now consider the solutions themselves for very large values of Q . We exhibit in Fig. 3 the metric and matter functions in scaled Schwarzschild-like coordinates, $\bar{r} = r/Q$, for a dyon solution with very large electric charge, $Q = 10000$, at a coupling strength $\alpha = 0.5$. The solution then appears to consist of two parts, an interior part in the region $0 \leq \bar{r} \leq \bar{r}_0$, and an exterior part in the region $\bar{r}_0 \leq \bar{r} \leq \infty$, with $\bar{r}_0 = 1$.

In the interior region, $0 \leq \bar{r} \leq 1$, the limiting solution is given by

$$\begin{aligned} A(\bar{r}) &= \text{const}, & N(\bar{r}) &= 1, \\ \hat{B}_1(\bar{r}) &= \Phi_1(\bar{r}) = 0, & H_2 &= H_2(\bar{r}), \end{aligned} \quad (48)$$

i.e., the metric is flat and the time-component of the gauge potential and the Higgs field both vanish. The only non-trivial function is the spatial gauge potential function H_2 , satisfying the single decoupled equation, Eq. (26).

In the exterior region $1 \leq \bar{r} \leq \infty$, on the other hand, the spatial gauge potential function H_2 vanishes, while the metric and the other two matter functions satisfy the coupled set of equations Eqs. (22)–(25). This system represents a special case of the coupled Einstein–Maxwell–scalar equations, where the scaled electric charge has the value $\bar{Q} = 1$, studied in a different parametrization by Penney [28].

As discussed in Section 3, in the relevant Penney solutions no horizons occur, but a naked singularity is present. In the limit $Q \rightarrow \infty$, these Penney solutions precisely saturate the bound on the scalar charge, Eq. (37), i.e., they satisfy a quadratic relation between the scaled global charges,

$$\bar{m}^2 = \alpha^2(1 - \bar{c}_H^2). \quad (49)$$

Due to the dyon boundary conditions for the Maxwell and scalar functions the Penney solutions furthermore satisfy the relations Eqs. (43) and (45), i.e., the scaled scalar charge \bar{c}_H ,

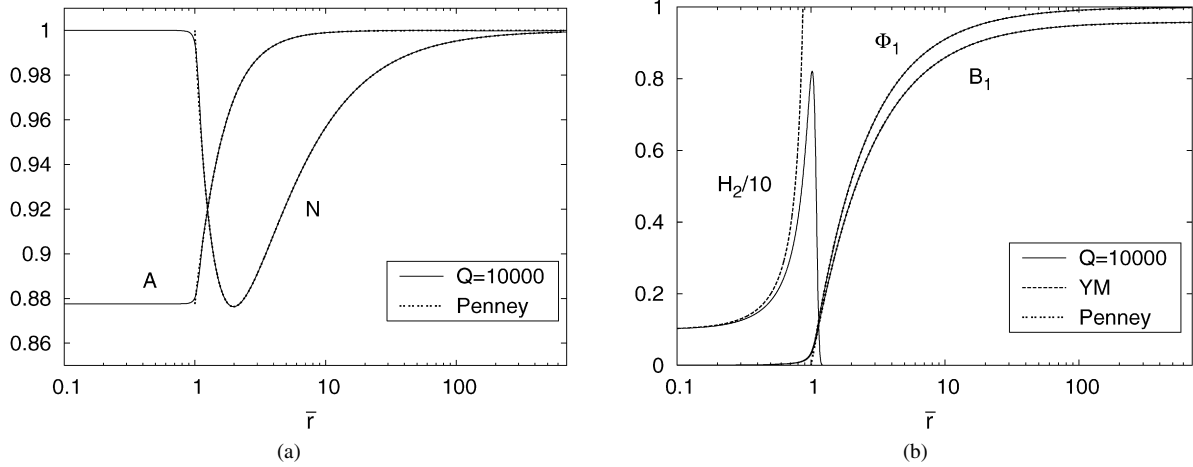


Fig. 3. Dyons: (a) metric functions A and N , (b) gauge potential and Higgs field functions \hat{B}_1 , H_2 , Φ_1 , versus scaled Schwarzschild-like coordinate $\bar{r} = r/Q$, for an almost limiting solution at $\alpha = 0.5$, $Q = 10000$. Also shown is the limiting solution: the Penney solution in the exterior, and the flat non-Abelian solution in the interior.

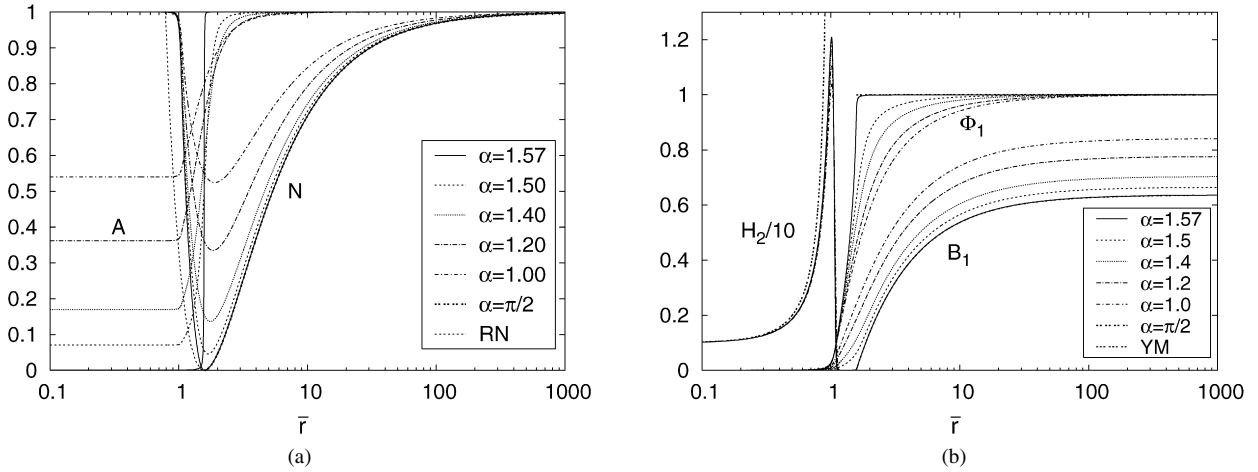


Fig. 4. Dyons: (a) metric functions A and N , (b) gauge potential and Higgs field functions \hat{B}_1 , H_2 , Φ_1 , versus scaled Schwarzschild-like coordinate $\bar{r} = r/Q$, for solutions with $Q = 10000$ and several values of α , approaching $\alpha = \pi/2$. Also shown is the limiting solution: the extremal RN solution in the exterior, and the flat non-Abelian solution in the interior.

and the potential parameter ν are given in terms of the coupling strength α ,

$$\bar{c}_H = \cos \alpha, \quad \nu = \frac{\sin \alpha}{\alpha}, \quad (50)$$

and thus the scaled mass satisfies $\bar{m} = \alpha \sin \alpha$. The validity of these relations for the limiting dyon solutions is seen in Fig. 1 for the scaled mass $\bar{m} = m/Q$ and in Fig. 2 for ν and $\bar{c}_H = c_H/Q$.

Identifying the transition point \bar{r}_0 as the outer extremum of the metric functions g_{tt} and g_{rr} of this particular Penney solution, we find $\bar{r}_0 = 1$ (Eq. (46)). Evaluating the metric functions A and N at the transition point $\bar{r}_0 = 1$ yields

$$A(\bar{r}_0) = \bar{c}_H, \quad N(\bar{r}_0) = 1. \quad (51)$$

For comparison we superimpose in Fig. 3 the limiting solutions in the two regions. In the exterior region $\bar{r}_0 \leq \bar{r} < \infty$ the limiting solution is the Penney solution with the same α and $\bar{Q} = 1$, saturating the mass bound, Eq. (49). This Penney solution also determines the constant metric functions in the interior

region $0 \leq \bar{r} \leq \bar{r}_0$, via their boundary values at the transition point \bar{r}_0 , Eq. (51). On the other hand, the boundary conditions at the origin determine the constant matter functions \hat{B}_1 and Φ_1 in the interior, and thus also at the transition point \bar{r}_0 .

In contrast the matter function H_2 is a solution of Eq. (26) [29]. Expanding the solution $H_2(\bar{r})$ of the Yang–Mills equation Eq. (26) in a power series, $H_2(\bar{r}) = 1 + h_1 \bar{r} + h_2 \bar{r}^2/2 + O(\bar{r}^3)$, shows that $h_1 = 0$ and h_2 is a free parameter, characterizing the solution. Solving the equation numerically in the interval $0 \leq \bar{r} \leq 1$ with boundary conditions $H_2(0) = 1$, $H_2(1) = h^*$, and varying h^* , we observe that $h_2(h^*)$ increases with increasing h^* and tends to a finite value $h_2^{\max} \approx 3.047$ when h^* tends to infinity. Comparison with the numerical dyon solutions for large charge Q , we find some evidence that $H_2''(0)$ indeed tends to the value h_2^{\max} for $Q \rightarrow \infty$, i.e., H_2 diverges at $\bar{r} = 1$ in the limit. Convergence is very slow, however, approximately like $1/\sqrt{Q}$.

We finally consider the range of α where dyon solutions exist. For small electric charge it was shown before [18], that

dyons exist only below a maximal value of α , e.g., $\alpha_{\max}(Q = 0) = 1.40$ and $\alpha_{\max}(Q = 1) = 1.41$. As the charge is increased further, α_{\max} increases as well. In the limit $Q \rightarrow \infty$, we obtain from the above considerations a bound for α ,

$$0 \leq \alpha \leq \frac{\pi}{2}. \quad (52)$$

Concerning the limit $\alpha \rightarrow \pi/2$ we see that the scaled scalar charge \bar{c}_H tends to zero. Thus in the limit there is no (non-trivial) scalar field in the exterior, whereas there is still an electric field. This suggests, that for $\alpha \rightarrow \pi/2$ the scaled limiting solution corresponds in the exterior to an extremal Reissner–Nordström solution with charge $Q = \alpha$. Inspection of the analytical formulae and the numerical solutions shows, that this is indeed the case. We exhibit the approach towards this limit for the metric and matter functions in Fig. 4. We note, however, that in this limit, the RN solution is approached only in the interval $\pi/2 \leq \bar{r} < \infty$. The limiting solution is discontinuous at $\pi/2$, since the metric function $A(\bar{r})$ assumes the RN value $A = 1$ only for $\bar{r} > \pi/2$, whereas it vanishes for $\bar{r} < \pi/2$, causing the metric to differ from the RN metric in this region.

Concluding, we see that the scaled limiting solution consists of two parts: an Abelian exterior Penney solution, saturating the bound Eq. (49) and determined by the value of α (with $\bar{Q} = 1$), and a non-Abelian interior part with flat metric, where the constant metric functions are determined by the exterior Penney solution at the transition point $\bar{r}_0 = 1$.

5. Conclusions

We have reconsidered dyon solutions of Einstein–Yang–Mills–Higgs theory, in the limit of vanishing Higgs potential. Dyon solutions then exist for arbitrarily large values of the charge. As the charge becomes large, the coupled system of Einstein–Yang–Mills–Higgs equations decomposes into a single equation for the magnetic gauge potential, and a coupled system of equations for the metric, the electric gauge potential and the scalar field. This system of equations is equivalent to the set of Einstein–Maxwell–scalar equations, studied by Penney in a different parametrization [28].

In the limit $Q \rightarrow \infty$, the non-Abelian dyons solutions then tend to limiting solutions, which consist of two parts, a non-Abelian interior solution with flat metric, and a gravitating Abelian exterior part. The interior solutions represent non-trivial solutions of the single equation for the magnetic gauge potential (with the remaining equations trivially satisfied). They are regular at the origin, but diverge at the transition point, where the parts are joined. The exterior solutions, on the other hand, correspond to the exterior part of particular Penney solutions, which then determine the asymptotic properties of the solutions, such as their mass and their scalar charge.

It appears interesting to also consider monopole–antimonopole pairs with large electric charge. When electric charge is added to a monopole–antimonopole pair, the monopole and antimonopole experience a repulsive force and the poles move further apart [8]. More importantly, however, the pair begins to rotate about its symmetry axis with an angular momentum

J , equal to the product of the total electric charge Q and the (magnitude of the) individual magnetic charge n of the constituent magnetic poles, $J = nQ$ [10].

Preliminary study shows, that the presence of electric charge also leads to bifurcations of the charged monopole–antimonopole solutions, with new branches of solutions arising. Considering larger values of the charge, we observe, that the values of ν , \bar{c}_H and \bar{m} appear to be consistent with the respective Penney relations. This seems surprising though, since the Penney solutions carry no angular momentum, while the charged dyons possess angular momentum $J = nQ$.

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References

- [1] G. 't Hooft, Nucl. Phys. B 79 (1974) 276; A.M. Polyakov, Pis'ma JETP 20 (1974) 430.
- [2] E.J. Weinberg, A.H. Guth, Phys. Rev. D 14 (1976) 1660.
- [3] C. Rebbi, P. Rossi, Phys. Rev. D 22 (1980) 2010.
- [4] R.S. Ward, Commun. Math. Phys. 79 (1981) 317; P. Forgacs, Z. Horvath, L. Palla, Phys. Lett. B 99 (1981) 232; M.K. Prasad, Commun. Math. Phys. 80 (1981) 137; M.K. Prasad, P. Rossi, Phys. Rev. D 24 (1981) 2182.
- [5] B. Kleihaus, J. Kunz, D.H. Tchrakian, Mod. Phys. Lett. A 13 (1998) 2523.
- [6] N.J. Hitchin, N.S. Manton, M.K. Murray, Nonlinearity 8 (1995) 661; C.J. Houghton, P.M. Sutcliffe, Commun. Math. Phys. 180 (1996) 343; C.J. Houghton, P.M. Sutcliffe, Nonlinearity 9 (1996) 385; P.M. Sutcliffe, Int. J. Mod. Phys. A 12 (1997) 4663; C.J. Houghton, N.S. Manton, P.M. Sutcliffe, Nucl. Phys. B 510 (1998) 507.
- [7] B. Julia, A. Zee, Phys. Rev. D 11 (1975) 2227; M.K. Prasad, C.M. Sommerfeld, Phys. Rev. Lett. 35 (1975) 760.
- [8] B. Hartmann, B. Kleihaus, J. Kunz, Mod. Phys. Lett. A 15 (2000) 1003.
- [9] M. Heusler, N. Straumann, M. Volkov, Phys. Rev. D 58 (1998) 105021.
- [10] J.J. van der Bij, E. Radu, Int. J. Mod. Phys. A 17 (2002) 1477; J.J. van der Bij, E. Radu, Int. J. Mod. Phys. A 18 (2003) 2379.
- [11] C.H. Taubes, Commun. Math. Phys. 86 (1982) 257; C.H. Taubes, Commun. Math. Phys. 86 (1982) 299; C.H. Taubes, Commun. Math. Phys. 97 (1985) 473.
- [12] W. Nahm, unpublished; Rüber, Thesis, University of Bonn 1985.
- [13] B. Kleihaus, J. Kunz, Phys. Rev. D 61 (2000) 025003.
- [14] B. Kleihaus, J. Kunz, Ya. Shnir, Phys. Lett. B 570 (2003) 237; B. Kleihaus, J. Kunz, Ya. Shnir, Phys. Rev. D 68 (2003) 101701(R); B. Kleihaus, J. Kunz, Ya. Shnir, Phys. Rev. D 70 (2004) 065010.
- [15] J. Kunz, U. Neemann, Ya. Shnir, Phys. Lett. B 640 (2006) 57.
- [16] K. Lee, V.P. Nair, E.J. Weinberg, Phys. Rev. D 45 (1992) 2751; P. Breitenlohner, P. Forgacs, D. Maison, Nucl. Phys. B 383 (1992) 357; P. Breitenlohner, P. Forgacs, D. Maison, Nucl. Phys. B 442 (1995) 126.
- [17] B. Hartmann, B. Kleihaus, J. Kunz, Phys. Rev. Lett. 86 (2001) 1422; B. Hartmann, B. Kleihaus, J. Kunz, Phys. Rev. D 65 (2001) 024027.
- [18] Y. Brihaye, B. Hartmann, J. Kunz, Phys. Lett. B 441 (1998) 77; Y. Brihaye, B. Hartmann, J. Kunz, N. Tell, Phys. Rev. D 60 (1999) 104016.
- [19] B. Kleihaus, J. Kunz, Phys. Rev. Lett. 85 (2000) 2430.
- [20] V. Paturyan, E. Radu, D.H. Tchrakian, Phys. Lett. B 609 (2005) 360.
- [21] B. Kleihaus, J. Kunz, U. Neemann, Phys. Lett. B 623 (2005) 171.
- [22] B. Kleihaus, J. Kunz, F. Navarro-Lérida, U. Neemann, arXiv: 0705.1511 [gr-qc].

- [23] B. Kleihaus, J. Kunz, Ya. Shnir, Phys. Rev. D 71 (2005) 024013;
J. Kunz, U. Neemann, Ya. Shnir, Phys. Rev. D 75 (2007) 125008.
- [24] R. Bartnik, J. McKinnon, Phys. Rev. Lett. 61 (1988) 141.
- [25] B. Kleihaus, J. Kunz, Phys. Rev. Lett. 78 (1997) 2527;
B. Kleihaus, J. Kunz, Phys. Rev. D 57 (1998) 834.
- [26] R. Ibadov, B. Kleihaus, J. Kunz, Ya. Shnir, Phys. Lett. B 609 (2005) 150.
- [27] B. Kleihaus, J. Kunz, K. Myklevoll, Phys. Lett. B 632 (2006) 333, hep-th/0509106.
- [28] R. Penney, Phys. Rev. 182 (1969) 1383.
- [29] G. Rosen, J. Math. Phys. 13 (1972) 595;
See also A. Actor, Rev. Mod. Phys. 51 (1979) 461.