O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

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Matematik induksiya metodi

Toshkent–2008

B. Abdurahmonov. **Matematik induksiya metodi/** Toshkent, 2008 y.

Fizika –matematika fanlari doktori, professor A. A'zamov umumiy tahriri ostida.

Qo'llanmada matematik induksiya metodining qo'llanishiga doir turli matematik olimpiadalar masalalari yechimlari bilan keltirilgan.

Qo'llanma umumiy o'rta ta'lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o'quvchilari, matematika fani o'qituvchilari hamda pedagogika oliy o'quv yurtlari talabalari uchun mo'ljallangan.

Qo'llanmadan sinfdan tashqari mashg'ulotlarda, o'quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent
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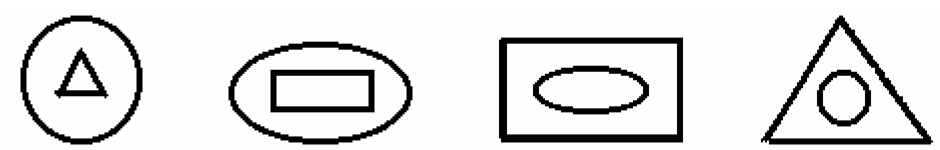
Ushbu qo'llanma Respublika ta'lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 - sonli bayyonnoma)

Qo'llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q'omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi**)**

1-§. Matematik induksiya metodi haqida

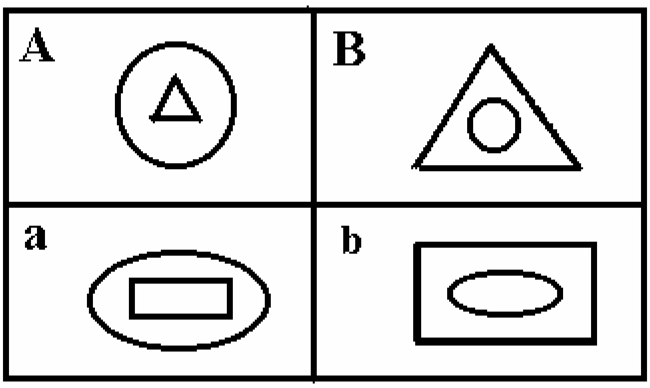
Dastlab o'xshatish tushunchasini tahlil etamiz. O'xshatish belgisi qadimgi yunonlarda boshlanishida sonlar proportsiyasi shaklida ifodalangan. Masalan, 10 : 5 =14 : 7. Keyinchalik o'xshatish so'zi shakllarga va boshqa narsalarga ham tatbiq etila boshlandi. Misollar ko'rib chiqamiz. Quidagi shakllar berilgan:

a) b) c) d)



O'xshash shakllar juftini topish talab etiladi..

Qaysi xossasiga ko'ra, ushbu juftlarni tanlash masalasi qo'yiladi. Bu masalani yechish kompyuterga yuklatildi. Kompyuter shakllarni taqqoslab, quidagi natijalarni qog'oz varag'iga chizib berdi va $A : B = a : b$ tenglikni yozdi.



Hozirgi paytda o'xshatish barcha fanlarda xizmat qiladi.

Kimyo. D.I.Mendeleyev kimyoviy elementlarning davriy sistemasini yaratdi va yangi elementlarning xossflarini o'xshatish bo'yicha ayta oldi.

Biologiya. Charl'z Darvin su'niy tanlash hodisasiga o'xshash "tabiiy tanlash" tushunchasini kiritdi.

Fizika. Tovushning havoda tarqalish qonuniyati ushbu hodisaning suv sirtida to'lqinini tarqalish hodisasiga asoslangan holda o'rnatildi.

Geologiya. Yoqutistonda olmos qazilma boyliklari topilgunga qadar Janubiy Afrika yassi tog'liklari geologik tuzilishi G'arbiy-Sibir platformasi geologik strukturasi bilan umumiy o'xshashliklari ma'lum bo'lgan. Tasodifiy holda Yoqutiston daryolaridan birida Janubiy Afrikaning olmosli yo'nalishida mavjud bo'lgan havorangli mineralga o'xshash mineral topilgan. Shundan so'ng Yoqutistonda olmos izlana boshlandi. Haqiqatdan ham u yerdan olmos topildi, keyinroq olmos boyliklari qazib olina boshlandi.

Matematikada shunday masalalar mavjudki, ba'zi farazlar yakuniy natijalarga ko'ra, noto'g'ri bo'lib chiqadi. Shunday masalalardan biri 1640 yilida tug'ilgan P.Fermaning o'ziga tegishli hisoblanadi:

u $f_n = 2^{2^n} + 1$ ko'rinishidagi natural sonlarning barchasi tub son deb faraz qilingan va faqat n = 0, 1, 2, 3, 4 lar uchun tekshirilgan. Lekin 1732 yili Leonard Eyler Pyer Fermaning farazini inkor etdi. Buning uchun u $f_5 = 2^{2^5} + 1$ soni 641 ga bo'linishini ko'rsatdi. P. Ferma nima uchun adashdi degan savol tug'iladi? Uning xatoligi shunda ediki, $f_n = 2^{2^n} + 1$ bir nechta xususiy qiymatlar uchun hisoblab (bu xususiy tasdiq), $f_n = 2^{2^n} + 1$ ning qiymati ixtiyoriy n natural son uchun tub son degan umumiy xulosaga kelgan.

L.Eyler $f(n) = n^2 + n + 41$ uchhad uchun quydagini tekshirgan. Ushbu uchhad n ning 1 dan 39 gacha qiymatlari uchun tub so bo'lgan. Lekin $n = 40$ uchun $f(n) = n^2 + n + 41$ qiymat murakkab son hisoblanadi: $f(40) = 40^2 + 40 + 41 = 1681 = 41^2$.

$f(n) = 991n^2 + 1, n \in N$ ko'rinishidagi ifoda berilgan. n sonning o'rniga 1 dan boshlab qiymat berilganda mazkur ifodaning qiymati biror sonning kvadrati bo’lmasdan, n = 12 055 735 790 331 359 447 442 538 767 bo’lgan holdagina $f(n) = 991n^2 + 1$ son to’liq kvadrat bo’ladi.

L.Eyler sodda induksiya xatolikka olib kelishi haqida haqiqtni aytgan. Matematikada cheksiz to’plam haqida mulohaza bildirilganda, chekli to’plamni tekshirish isbotlashni almashtira olmaydi.

Deduksiya va induksiya

Shunday qilib, ikkita tushunchani farqlash lozim:

1) Xususiy tasdiq; 2)Umimiy tasdiq.

Misol. Quyidagi tasdiqlardan qaysi bir xususiy, qaysi biri umumiy:

1) Nol raqami bilan tugallanuvchi son 5 ga bo’linadi? 2) 140 soni 5 ga bo’linadi?

Umumiy tasdiqdan xususiy tasdiqga o’tish **deduksiya** deyiladi**.**

Misol. Nol bilan tugallanuvchi son 5 ga bo’linganligi sababli, 140 soni 5 ga bo’linadi.

Xususiy tasdiqdan umumiy tasdiqga o’tish **induksiya** deyiladi. Induksiya ham to’g’ri, ham noto’g’ri natijaga olib kelishi mumkin.

Induksiya metodi matematikada keng qo’llaniladi, lekin unidan to’g’ri foydalanish lozim.

Tasdiq: Quyidagi uch xonali sonlar: 140, 150, 250 **5** ga bo’linadi.

Xulosa: 1) Barcha nol raqami bilan tugallanuvchi sonlar 5 ga bo’linadi (to’g’ri), 2) barcha uch xonali sonlar 5 ga bo’linadi (noto’g’ri).

Shunday savol paydo bo’ladi. To’g’ri xulosa chiqarish uchun matematikada induksiya metodidan qanday foydalanish lozim? Cheksiz sonlar hodisasini tekshirishda qaysi usullar amalga oshiriladi? Bunday usulni B.Pascal va Ya. Bernullilar taklif qilishdi. Bu usul hozirgi kunda matematik induksiya metodi deyiladi. Ushbu metodni ba’zi qadimgi grek olimlari ham foydalanishgan. Dastalab bu metod 1321 yil. Gersonid tomonidan foydalanilgan. XIX asrning ikkinchi yarmigacha bu metod asosiy isbotlash metodi hisoblangan. Shu davrdan boshlab, O.Boltsano, O.L.Koshi, K.F.Gauss, N.X.Abelning ilmiy ishlaridan so’ng, induktiv isbotlashlar o’z ahamiyatini matematikada qisman yo’qotdi.

Matematik induksiya metodini misollarda tushuntiramiz.

Berilgan. Kitob javonida kitoblar quidagicha joylashtirilgan: 1) eng chekka qismida joylashgan kitob qizil muqovada. 2) Qizil muqovali kitobning o’ng tomonida qizil muqovali kitob joylashgan.

Xulosa. Kitob javonida joylashgan barcha kitoblar qizil muqovada.

“Javonda barcha kitoblar qizil muqovada” xulosasi haqiqatdan ham to’g’ri hosoblanadi. Lekin, agar eng chekkadagi kitob qizil muqovaliligi ma’lum bo’lsa, “javondagi barcha kitoblar qizil muqovali “ degan xulosa chiqarish uchun etarli darajada emas.

Qizil muqovali kitobning o’ng tomonida joylashgan kitob qizil muqovali degan xulosa chiqarishga etarli emas (Chap ttomondagi birinchi kitob yashil muqovada ham bo’lishi mumkin).

Shuning uchun ,xulosa to’g’ri bo’lishi uchun ikkala shrt ham bajarilishi lozim. Matematika ensiklopediyasida quyidagi tushunchalar berilgan.

Matematik induksiya – matematik induksiya prinsipiga asoslangan matematik tasdiqni isbotlovchi metod:

Agar A(1) isbotlangan bo’lsa, x natural parametrga bo’g’liq $A(x)$ tasdiq isbotlangan deb hisoblanadi va ixtiyoriy n natural son uchun $A\ (n)$ to’g’ri deb faraz qilinsa, n+1 uchun $A\ (n+1)$ to’g’ri hisoblanadi.

A (1) tasdiqning isbotlanishi induksiyaning birinchi qadami hisoblanadi, $A(n)$ uchun farazdan A **(**$n+1$**)** ning isbotlanishi **induksiyali o’tish** deyiladi. Bunda induksiya parametri deyiladi, A(n+1) ni isbotlashda A(n) ni faraz qilish induktivli faraz deyiladi.

Matematik induksiya metodining mohiyati quyidagicha:

Agar tasdiqlash ketma-ketligi mavjud bo’lsa, birinchi tasdiq to’g’ri va har bir to’g’ri tasdidan so’ng to’g’ri tasdiq mavjud bo’lsa, ketma-ketlikdagi barcha tasdiq to’g’ri hisoblanadi.

Shunday qilib, matematik induksiya metodi yordamida isbotlash ikkita teoremadan iborat.

1-teorema . $n = 1$ uchun tasdiq to'g'ri.

2-teorema. Ixtiyoriy $n=k$ uchun tasdiq to'g'ri deb faraz qilinsa, u holda, navbatdagi $n=k+1$ natural son uchun tasdiq to'g'ri deb hisoblanadi.

Agar ikkala ushbu teoremalar isbotlangan bo'lsa, matematik induksiya tamoyiliga asoslangan holda, tasdiq ixtiyoriy n natural son uchun to'g'ri deb xulosa qilinadi.

Eslatma. Barcha natural sonlar uchun emas, balki n dan katta yoki teng m natural sonlar uchun induksiya bo'yicha tasdiqni isbotlash zarur bo'ladi. Bunday holda isbotlash quyidagicha bajariladi.

1-teorema. $n = m$ da tasdiq to'g'ri.

2-teorema. $n=k$ da tasdiq to'g'ri berilgan, $k \geq m$. $n = k+1$ da tasdiq o'rinli ekanligini isbotlash lozim.

2-§. Tengliklarni isbotlash.

2.1-masala. Tenglikni isbotlang

$$1 + 2 + ... + n = \frac{n(n+1)}{2}, \quad \forall n \in N. \qquad (2.1)$$

Yechilishi. $S_n = 1 + 2 + ... + n$ orqali belgilaymiz.

1-qadam. $n = 1$ da $S_1 = 1$ ga ega bo'lamiz. $n = 1$ ni (2.1.) tenglikning o'ng tomoniga qo'yamiz: $\frac{1(1+1)}{2} = 1$. $n=1$ da (2.1.) tenlikning o'ng va chap tomoni 1 ga teng ekanligini hosil qildik.

2-qadam. (2.1) tenglik $n=k$ da bajariladi deb faraz qilamiz: $S_k = 1 + 2 + ... + k = \frac{k(k+1)}{2}$. (2.1) tenglik $n=k$ uchun o'rinli ekanligini isbotlash lozim: $S_{k+1} = 1 + 2 + ... + k + (k+1) = \frac{(k+1)(k+2)}{2}$.

Haqiqatdan ham:

$$S_{k+1}=\underbrace{1+2+...+k}_{=S_k}+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}.$$

2.2-masala. Ixtiyoriy *n* natural son uchun natural qatorning dastlabki *n* ta son kvadratlar yig'indisi $\frac{n(n+1)(2n+1)}{6}$ ga tengligini isbotlash lozim.

Isbotlash. Quyidagi tenglikni isbotlang

$$S_n=1^2+2^2+...+n^2=\frac{n(n+1)(2n+1)}{6},\ \forall n\in N.$$

(2.2)

1-teorema. n=1 da (2.2) tenglikning bajarilishini tekshiramiz:

(2.2) tenglikning chap tomoni quyidagiga ega: $1^2=1$;

(2.2) tenglikning o'ng tomoni quyidagiga ega: $\frac{1(1+1)(2\cdot 1+1)}{6}=1$.

(2.2) tenglikning o'ng va chap tomoni teng, shuning uchun 1-teorema isbotlandi deb hisoblaymiz.

2-teorema. (2.2) tenglik uchun $n=k$ da quyidagi berilgan deb faraz qilaylik:

$$S_k=1^2+2^2+...+k^2=\frac{k(k+1)(2k+1)}{6}.$$

Bu tenglik $n=k+1$ uchun bajarilishini isbotlash lozim

$$S_{k+1}=1^2+2^2+...+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}.$$

Haqiqatdan: $S_{k+1}=\underbrace{1^2+2^2+...+k^2}_{=S_k}+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2=$

$$=(k+1)\frac{k(2k+1)+6(k+1)}{6}=(k+1)\frac{2k^2+7k+6}{6}=$$

$$2k^2+7k+6=0;\quad k_{1,2}=\frac{-7\pm\sqrt{49-48}}{4}=\frac{-7\pm 1}{4};$$

$$k_1=-2,\quad k_2=-\frac{3}{2}.$$

$$=\frac{(k+1)2(k+2)\left(k+\frac{3}{2}\right)}{6}=\frac{(k+1)(k+2)(2k+3)}{6}.$$ 2-teorema isbotlandi.

1- va 2- teoremalardan (2.2) tenglik ixtiyoriy n natural son uchun bajariladi.

2.3-masala. Natural *(2 n – 1)* qatorning dastlabki toq sonlar kvadratlar yig'indisi ixtiyoriy n natural son uchun $\frac{n(2n-1)(2n+1)}{3}$ ga tengligini isbotlang.

Isbotlash.

$$S_n=1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3},\ \forall n\in N \qquad (2.3)$$

tenglikni isbotlash lozim.

1-teorema. n=1 da (2.3) tenglikning bajarilishini tekshiramiz:

(2.3) tenglikning chap tomoni quyidagiga ega: $1^2=1$;

(2.3) tenglikning o'ng qismi quyidagiga ega: $\frac{1(2\cdot1-1)(2\cdot1+1)}{3}=1$.

(2.3) tenglikning o'ng va chap tomonlari teng, shuning uchun 1-teorema isbotlandi.

2-teorema. (2.3) tenglik $n=k$ da bajariladi:

$$S_k=1^2+3^2+5^2+...+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}.$$

Ushbu tenglik $n=k+1$ da bajarilishini isbotlash lozim:

$$S_{k+1}=1^2+3^2+5^2+...+(2k-1)^2+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}.$$

Haqiqatdan: $S_{k+1} = \underbrace{1^2 + 3^2 + 5^2 + ... + (2k-1)^2}_{= S_k} + (2k+1)^2 =$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 = (2k+1) \cdot \left(\frac{2k^2 + 5k + 3}{3} \right) =$$

$= \dfrac{(k+1)(2k+1)(2k+3)}{3}$. 2-teorema isbotlandi. Demak, (2.3) tenglikning ixtiyoriy *n* natural sonlar uchun bajarilishi kelib chiqadi.

2.4-masala. Natural qatorning dastlabki *n* ta beshinchi darajalari yi'g'indisi $\dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$ ga tengligini isbotlang.

Isbotlash.

$$S_n = 1^5 + 2^5 + 3^5 + ... + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}, \forall n \in N. \qquad (2.4)$$

tenglikni isbotlash lozim.

1-teorema. (2.4) tenglikning $n = 1$ bajarilishini tekshiramiz :

(2.4) tenglikning chap tomoni: $1^5 = 1$;

(2.4) tenglikning o'ng tomoni : $\dfrac{1^2(1+1)^2(2 \cdot 1^2 + 2 \cdot 1 - 1)}{12} = 1$.

(2.4) tenglikning o'ng va chap tomoni teng, shuning uchun 1-teorema isbotlandi.

2-teorema. Faraz qilaylik, (2.4) tenglik *n*=*k* da bajariladi:

$$S_k = 1^5 + 2^5 + 3^5 + ... + k^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12}.$$

$n = k+1$ da (2.4) tenglikni isbotlash lozim:

$$S_{k+1} = 1^5 + 2^5 + 3^5 + ... + (k+1)^5 = \frac{(k+1)^2(k+2)^2(2(k+1)^2 + 2(k+1) - 1)}{12}$$

Haqiqatdan:

$$S_{k+1}=\underbrace{1^5+2^5+3^5+...+k^5}_{=S_k}+(k+1)^5=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}+(k+1)^5=$$

$$=\frac{(k+1)^2}{12}\cdot\left(2k^4+14k^3+35k^2+36k+12\right)=$$

To’rtinchi darajali ko’phadni $2k^4+14k^3+35k^2+36k+12$

Ikkinchi darajali ko’phadga ajratamiz $(k+2)^2=k^2+4k+4$

$$\begin{array}{r|l}
-\;2k^4+14k^3+35k^2+36k+12 & k^2+4k+4 \\ \cline{2-2}
2k^4+8k^3+8k^2 \qquad\qquad\quad & 2k^2+6k+3 \\ \cline{1-1}
-\;6k^3+27k^2+36k+12 & \\
6k^3+24k^2+24k \qquad & \\ \cline{1-1}
-\;3k^2+12k+12 & \\
3k^2+12k+12 & \\ \cline{1-1}
0 &
\end{array}$$

$$=\frac{(k+1)^2}{12}(k+2)^2\left(2k^2+6k+3\right)=$$

$$=\frac{(k+1)^2}{12}(k+2)^2\left(2(k+1-1)^2+6(k+1-1)+3\right)=$$

$$=\frac{(k+1)^2}{12}(k+2)^2\left(2(k+1)^2+2(k+1)-1\right).$$ 2-teorema isbotlandi.

1- va 2- teoremalardan (2.4) tenglikning ixtiyoriy n natural son uchun bajarilishi ma’lum bo’ladi.

2.5-masala. Tenglikning ixtiyoriy n natural son uchun isbotlang

$$1\cdot2\cdot3+2\cdot3\cdot4+...+n\cdot(n+1)\cdot(n+2)=\frac{1}{4}n\,(n+1)(n+2)(n+3)\,. \qquad (2.5)$$

Yechilishi. $S_n=1\cdot2\cdot3+2\cdot3\cdot4+...+n\cdot(n+1)\cdot(n+2)$ orqali belgilaymiz .

1-teorema. $n = 1$ da $S_1=1\cdot2\cdot3$ ga teng. $n = 1$ni (2.5) tenglikning o'ng tomoniga qo'yamiz: $\frac{1}{4}1\cdot(1+1)\cdot(1+2)\cdot(1+3)=1\cdot2\cdot3$. Natijada $n = 1$ da (2.5) tenglikning o'ng va chap tomoni teng ekanligini hosil qilamiz. 1-teorema isbotlandi.

2-teorema. $n=k$ da (2.5) tenli bajariladi deb faraz qilaylik:

$$S_k=1\cdot2\cdot3+2\cdot3\cdot4+...+k\cdot(k+1)\cdot(k+2)=\frac{1}{4}k\cdot(k+1)\cdot(k+2)\cdot(k+3)\,.$$

$$S_{k+1}=1\cdot2\cdot3+2\cdot3\cdot4+...+(k+1)(k+2)(k+3)=\frac{1}{4}(k+1)(k+2)(k+3)(k+4)\,.$$

Tenglik to'g'riligini isbotlash lozim. Haqiqatdan:

$$S_{k+1}=\underbrace{1\cdot2\cdot3+2\cdot3\cdot4+...+k\cdot(k+1)\cdot(k+2)}_{=S_k}+(k+1)\cdot(k+2)\cdot(k+3)=$$

$$=\frac{1}{4}k\cdot(k+1)\cdot(k+2)\cdot(k+3)+(k+1)\cdot(k+2)\cdot(k+3)=$$

$$=(k+1)\cdot(k+2)\cdot(k+3)\left(\frac{1}{4}k+1\right)=\frac{1}{4}\,(k+1)(k+2)(k+3)(k+4)\,.$$

2-teorema isbotlandi. 1- va 2- teoremalardan (2.5.) tenglikning ixtiyoriy n natural son uchun bajarilishi kelib chiqadi.

2.6-masala. Tenglikni isbotlang

$$\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+...+\frac{n^2}{(2n-1)\cdot(2n+1)}=\frac{n(n+1)}{2\cdot(2n+1)},\ \forall n\in N\,. \qquad (2.6)$$

Yechilishi. $S_n=\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+...+\frac{n^2}{(2n-1)\cdot(2n+1)}$ orqali belgilaymiz.

1-teorema. $n = 1$ da $S_1 = \frac{1^2}{1 \cdot 3} = \frac{1}{3}$ ga ega bo'lamiz. (2.6) tenglikning o'ng qismiga n=1 ni qo'yib: $\frac{1(1+1)}{2 \cdot (2 \cdot 1 + 1)} = \frac{1}{3}$ ni hosil qilamiz.

Demak, $n = 1$ da (2.6) tenglikning o'ng va chap qismi teng, shuning uchun 1-teorema isbotlandi.

2-teorema. (2.6) tenglik $n = k$ da to'g'ri deb faraz qilaylik:

$$S_k = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + ... + \frac{k^2}{(2k-1) \cdot (2k+1)} = \frac{k(k+1)}{2 \cdot (2k+1)}.$$

Quyidagi tenglikning to'g'riligini isbotlash losim

$$S_{k+1} = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + ... + \frac{(k+1)^2}{(2k+1) \cdot (2k+3)} = \frac{(k+1)(k+2)}{2 \cdot (2k+3)}.$$

Haqiqatdan ham:

$$S_{k+1} = \underbrace{\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + ... + \frac{k^2}{(2k-1) \cdot (2k+1)}}_{= S_k} + \frac{(k+1)^2}{(2k+1) \cdot (2k+3)} =$$

$$= \frac{k(k+1)}{2 \cdot (2k+1)} + \frac{(k+1)^2}{(2k+1) \cdot (2k+3)} = \frac{(k+1) \cdot (2k^2 + 5k + 2)}{2 \cdot (2k+1) \cdot (2k+3)} =$$

$$= \frac{(k+1) \cdot 2 \cdot (k+2) \cdot \left(k + \frac{1}{2}\right)}{2 \cdot (2k+1) \cdot (2k+3)} = \frac{(k+1) \cdot (k+2)}{2 \cdot (2k+3)}.$$

2-teorema isbotlandi.

1- va 2- teoremalardan (2.6) tenglikning ixtiyoriy n natural son uchun bajarilishi kelib chiqadi.

2.7-masala. Tenglikning isbotlang

$$1 + 2 + 2^2 + 2^3 + ... + 2^{n-1} = 2^n - 1. \qquad (2.7)$$

Yechilishi. $S_n = 1 + 2 + 2^2 + 2^3 + ... + 2^{n-1}$ orqali belgilaymiz.

1-teorema. $n = 1$ da : $S_1 = 1$ ega bo'lamiz. $n = 1$ ni (2.7) tenglikning o'ng qismiga qo'yamiz: $2^1 - 1 = 1$. $n = 1$ da (2.7) tenglikning o'ng va chap qismlari 1 ga teng. 1-teorema isbotlandi.

2-teorema. (2.7) tenglik $n=k$ da bajariladi deb faraz qilaylik: $S_k = 1 + 2 + 2^2 + 2^3 + ... + 2^{k-1} = 2^k - 1$. Quyidagi tenglikning o'rinli ekanligini isbotlash lozim: $S_{k+1} = 1 + 2 + 2^2 + 2^3 + ... + 2^{k-1} + 2^k = 2^{k+1} - 1$.

Haqiqatdan: $S_{k+1} = \underbrace{1 + 2 + 2^2 + 2^3 + ... + 2^{k-1}}_{= S_k = 2^k - 1} + 2^k = 2^{k+1} - 1$. 2-teorema isbotlandi. 1- va 2- teoremalardan (2.7) tenglikning ixtiyoriy n natural son uchun bajarilishi kelib chiqadi.

2.8-masala. Quyidagi tenglikni isbotlang:

$$\left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 - \frac{1}{16}\right) \cdot ... \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \quad \forall n \in N : n \geq 2. \tag{2.8}$$

Yechilishi. $S_n = \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 - \frac{1}{16}\right) \cdot ... \cdot \left(1 - \frac{1}{n^2}\right)$ orqali belgilaymiz.

1-teorema. $n = 2$ da $S_2 = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$ ga ega bo'lamiz. $n = 2$ ni (2.8) tenglikning o'ng qismiga qo'yamiz: $\frac{(2+1)}{2 \cdot 2} = \frac{3}{4}$. Natijada $n = 2$ da (2.8) tenglikning o'ng va chap qismlari teng bo'ladi. 1-teorema isbotlandi.

2-teorema. (2.8) tenglik $n=k$ da bajariladi deb faraz qilaylik: $S_k = \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 - \frac{1}{16}\right) \cdot ... \cdot \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$.

Quyidagi tenglikning o'rinli ekanligini (2.8) tenglik uchun $n=k+1$ da isbotlash lozim:

$$S_{k+1} = \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 - \frac{1}{16}\right) \cdot ... \cdot \left(1 - \frac{1}{k^2}\right) \cdot \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}.$$

Haqiqatdan: $$S_{k+1}=\underbrace{\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot\left(1-\frac{1}{16}\right)\cdot\ldots\cdot\left(1-\frac{1}{k^2}\right)}_{=S_k}\cdot\left(1-\frac{1}{(k+1)^2}\right)=$$

$$=\frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2}=\frac{k+2}{2(k+1)}.$$ 2-teorema isbotlandi.

1- va 2- teoremalardan (2.8) tenglikning ixtiyoriy $n\geq 2$ natural son uchun bajarilishi kelib chiqadi.

2.9-masala. Tenglikni isbotlang

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-1}+\frac{1}{2n},\ \forall n\in N. \quad (2.9)$$

Yechilishi. $S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}$ orqali belgilaymiz.

1-teorema. $n = 1$ da $S_1=1-\frac{1}{2}=\frac{1}{1+1}$ ga ega bo'lamiz. T1-teorema isbotlandi.

2-teorema. (2.9) tenglik $n=k$ da bajariladi deb faraz qilaylik:

$$S_k=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k-1}+\frac{1}{2k}.$$

Quyidagi tenglikning o'rinli ekanligini (2.9) tenglik uchun $n=k+1$ uchun isbotlash lozim:

$$S_{k+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2k+1}-\frac{1}{2k+2}=\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k+1}+\frac{1}{2k+2}.$$

Haqiqatdan: $$S_{k+1}=\underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2k-1}-\frac{1}{2k}}_{=S_k}+\frac{1}{2k+1}-\frac{1}{2k+2}=$$

$$=\underline{\frac{1}{k+1}}+\frac{1}{k+2}+\ldots+\frac{1}{2k-1}+\frac{1}{2k}+\frac{1}{2k+1}-\underline{\frac{1}{2k+2}}=$$

$$=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k+1}+\frac{1}{2k+2}.$$ 2-teorema isbotlandi. 1-teorema va 2-teoremalardan (2.9) tenglikning ixtiyoriyn *n* natural son uchun bajarilishi kelib chiqadi.

2.10-teorema. Quyidagi tenglikni isbotlang:

$$\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n}=2\cos\frac{\pi}{2^{n+1}},\ \forall n\in N. \qquad (2.10)$$

Isboti. $S_n=\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n}$ orqali belgilaymiz.

1-teorema. *n = 1* da $S_1=\sqrt{2}=2\cos\frac{\pi}{2^{1+1}}$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. (2.10) tenglik *n*=*k* da bajariladi deb faraz qilaylik: $S_k=\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k}=2\cos\frac{\pi}{2^{k+1}}$. Quyidagi tenglikning o'rinli ekanligini (2.9) tenglik uchun *n*=*k*+1 uchun isbotlash lozim:

$$S_{k+1}=\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1}=2\cos\frac{\pi}{2^{k+2}}.$$

Haqiqatdan: $S_{k+1}=\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1}=\sqrt{2+2\cos\frac{\pi}{2^{k+1}}}=$

$$=\sqrt{2}\sqrt{1+\cos\frac{\pi}{2^{k+1}}}=\sqrt{2}\sqrt{2\cos^2\frac{\pi}{2\cdot 2^{k+1}}}=2\cos\frac{\pi}{2^{k+2}}.$$

Oxirgi tenglik to'g'ri, chunki $0<\frac{\pi}{2^{k+2}}<\frac{\pi}{2}$. 2-teorema isbotlandi. 1- va 2-teoremalardan (2.10) tenglikning ixtiyoriyn ***n*** natural son uchun bajarilishi kelib chiqadi.

2.11-masala. Quyidagi tenglikni isbotlang:

$$\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+...+\frac{1}{\ln 2^{n-1}\cdot\ln 2^{n}}=\left(1-\frac{1}{n}\right)\frac{1}{\ln^{2}2},\ \forall n\in N: n\geq 2. \quad (2.11)$$

Yechilishi. $S_n=\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+...+\frac{1}{\ln 2^{n-1}\cdot\ln 2^{n}}$ orqali belgilaymiz.

1-teorema. *n* = *2* da (2.11) tenglikning chap qismi: $\frac{1}{\ln 2\cdot\ln 4}=\frac{1}{\ln 2\cdot\ln 2^{2}}=\frac{1}{2}\cdot\frac{1}{\ln^{2}2}$ ga teng. *n* = 2 ni (2.11) tenglikning o'ng qismiga qo'yamiz: $\left(1-\frac{1}{2}\right)\frac{1}{\ln^{2}2}=\frac{1}{2}\frac{1}{\ln^{2}2}$. Natijada *n* = 2 qiymatda (2.11) tenglikning o'ng va chap qismlari teng. 1-teorema isbotlandi.

2-teorema. (2.11) tenglik *n*=*k* da bajariladi deb faraz qilaylik:

$$S_k=\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+...+\frac{1}{\ln 2^{k-1}\cdot\ln 2^{k}}=\left(1-\frac{1}{k}\right)\frac{1}{\ln^{2}2}.$$

Quyidagi tenglikning o'rinli ekanligini (2.11) tenglik uchun *n*=*k*+1 da isbotlash lozim:

$$S_{k+1}=\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+...+\frac{1}{\ln 2^{k}\cdot\ln 2^{k+1}}=\left(1-\frac{1}{k+1}\right)\frac{1}{\ln^{2}2}.$$

Haqiqatdan:

$$S_{k+1}=\underbrace{\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+...+\frac{1}{\ln 2^{k-1}\cdot\ln 2^{k}}}_{=S_k}+\frac{1}{\ln 2^{k}\cdot\ln 2^{k+1}}=$$

$$=\left(1-\frac{1}{k}\right)\frac{1}{\ln^{2}2}+\frac{1}{\ln 2^{k}\cdot\ln 2^{k+1}}=\left(1-\frac{1}{k}\right)\frac{1}{\ln^{2}2}+\frac{1}{k\cdot(k+1)\cdot\ln^{2}2}=$$

$$=\left(1-\frac{1}{k}+\frac{1}{k}-\frac{1}{k+1}\right)\frac{1}{\ln^{2}2}=\left(1-\frac{1}{k+1}\right)\frac{1}{\ln^{2}2}.$$ 2-teorema isbotlandi. 1- va 2-teoremalardan (2.11) tenglikning ixtiyoriy $n\geq 2$ natural son uchun bajarilishi kelib chiqadi.

2.12-masala. Tenglikni isbotlang

$$\sin x+\sin 2x+...+\sin nx=\frac{\sin\frac{n+1}{2}x\cdot\sin\frac{n}{2}x}{\sin\frac{1}{2}x},\ n\neq 2k\pi,\ \forall n\in N. \qquad (2.12)$$

Yechilishi. $S_n=\sin x+\sin 2x+...+\sin nx$ orqali belgilaymiz.

1-teorema. $n=1$ da $S_1=\sin x=\dfrac{\sin\frac{1+1}{2}x\cdot\sin\frac{1}{2}x}{\sin\frac{1}{2}x}$ ga ega bo'lamiz.

1-teorema isbotlandi.

2-teorema. (2.12) tenglik $n=k$ da bajariladi deb faraz qilaylik: $S_k=\sin x+\sin 2x+...+\sin kx=\dfrac{\sin\frac{k+1}{2}x\cdot\sin\frac{k}{2}x}{\sin\frac{1}{2}x}$. Quyidagi tenglikning o'rinli ekanligini (2.12) tenglik uchun $n=k+1$ da isbotlash lozim:

$$S_{k+1}=\sin x+\sin 2x+...+\sin(k+1)x=\frac{\sin\frac{k+2}{2}x\cdot\sin\frac{k+1}{2}x}{\sin\frac{1}{2}x}.$$

Haqiqatdan ham: $S_{k+1}=\underbrace{\sin x+\sin 2x+...+\sin kx}_{=S_k}+\sin(k+1)x=$

$$=\frac{\sin\frac{k+1}{2}x\cdot\sin\frac{k}{2}x}{\sin\frac{1}{2}x}+\sin\frac{2(k+1)x}{2}=\frac{\sin\frac{k+1}{2}x\cdot\sin\frac{k}{2}x}{\sin\frac{1}{2}x}+2\sin\frac{k+1}{2}x\cdot\cos\frac{k+1}{2}x=$$

$$=\frac{\sin\frac{k+1}{2}x}{\sin\frac{1}{2}x}\left(\sin\frac{k}{2}x+2\cdot\cos\left(\frac{k}{2}x+\frac{1}{2}x\right)\cdot\sin\frac{1}{2}x\right)=$$

$$=\frac{\sin\frac{k+1}{2}x}{\sin\frac{1}{2}x}\left(\sin\frac{k}{2}x+2\cdot\left(\cos\frac{k}{2}x\cdot\cos\frac{1}{2}x-\sin\frac{k}{2}x\cdot\sin\frac{1}{2}x\right)\cdot\sin\frac{1}{2}x\right)=$$

$$=\frac{\sin\frac{k+1}{2}x}{\sin\frac{1}{2}x}(\sin\frac{k}{2}x\cdot(1-2\cdot\underbrace{\sin^2\frac{1}{2}x}_{=\frac{1-\cos x}{2}})+\cos\frac{k}{2}x\cdot\underbrace{2\cdot\cos\frac{1}{2}x\cdot\sin\frac{1}{2}x}_{=\sin x})=$$

$$=\frac{\sin\frac{k+1}{2}x}{\sin\frac{1}{2}x}\left(\sin\frac{k}{2}x\cdot\cos x+\cos\frac{k}{2}x\cdot\sin x\right)=\frac{\sin\frac{k+1}{2}x\cdot\sin\frac{k+2}{2}x}{\sin\frac{1}{2}x}.$$

2-teorema isbotlandi.

1- va 2- teoremalardan (2.12) tenglikning ixtiyoriyn $\boldsymbol{n}$ natural son uchun bajarilishi kelib chiqadi.

2.13-masala. Muavr formulasini isbotlang

$$\left(r(\cos\varphi+i\sin\varphi)\right)^n=r^n(\cos n\varphi+i\sin n\varphi),\ \forall\, n\in N. \qquad (2.13)$$

1-teorema. $n = 1$da ga $\left(r(\cos\varphi+i\sin\varphi)\right)^1=r^1(\cos\varphi+i\sin\varphi)$ ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. (2.13) tenglik n=k da bajariladi deb faraz qilaylik: $\left(r(\cos\varphi+i\sin\varphi)\right)^k=r^k(\cos k\varphi+i\sin k\varphi)$. Quyidagi tenglikning o'rinli ekanligini (2.12) tenglik uchun n=k+1 da isbotlash lozim:

$$\left(r(\cos\varphi+i\sin\varphi)\right)^{k+1}=r^{k+1}(\cos(k+1)\varphi+i\sin(k+1)\varphi).$$

Haqiqatdan: $\left(r(\cos\varphi+i\sin\varphi)\right)^{k+1}=\left(r(\cos\varphi+i\sin\varphi)\right)^k\cdot\left(r(\cos\varphi+i\sin\varphi)\right)=$

$$=r^k(\cos k\varphi+i\sin k\varphi)\left(r(\cos\varphi+i\sin\varphi)\right)= \qquad \left|\ i^2=-1\ \right|$$

$$=r^{k+1}\left(\cos k\varphi\cdot\cos\varphi-\sin k\varphi\cdot\sin\varphi+i(\cos k\varphi\cdot\sin\varphi+\sin k\varphi\cdot\cos\varphi)\right)=$$

$=r^{k+1}(\cos(k+1)\varphi+i\sin(k+1)\varphi)$. 2-teorema isbotlandi.

1- va 2- teoremalardan (2.13) tenglikning ixtiyoriyn $\boldsymbol{n}$ natural son uchun bajarilishi kelib chiqadi.

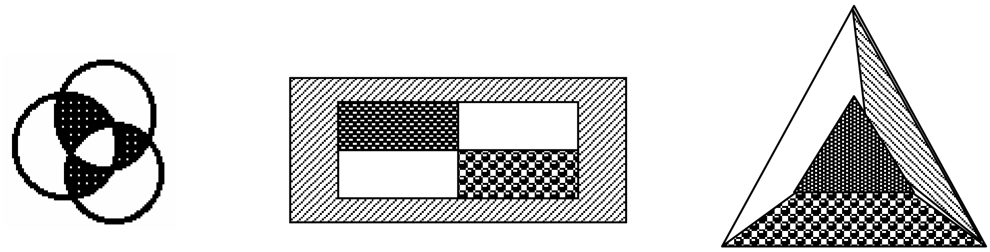
3-§. Xaritani bo’yash

Tekislikda biror geografik xarita berilgan deb faraz qilaylik.

Ta’rif. Agar ixtiyoriy davlat ma’lum bo’yoq bilan bo’yalgan, umumiy chegaraga ega bo’lgan ikkita davlat turli ranglar bilan bo’yalsa xarita *to’g’ri bo’yalgan* deyiladi.

To’g’ri bo’yalgan xaritaga ixtiyoriy geografik xarita misol bo’la oladi. Ixtiyoriy xaritani ma’lum bo’yoq bilan bo’yash mumkin, lekin bunday bo’yash tejamli bo’lmaydi. Shuning uchun quyidagi masalani qo’yish lozim bo’ladi: xaritani bo’yash uchun bo’yoqlar turlari iloji boricha kam bo’lsin.

3.1-masala.



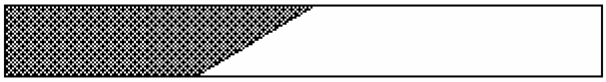
2 ta rang 3 ta rang 4 ta rang

Har qanday xaritani bo’yash uchun 5 ta turdagi bo’yoqlarni yetarli ekanligi isbotlangan.

Hozirgi kungacha to’rta bo’yoq bilan to’g’ri bo’yash mumkin bo’lgan xarita topilmagan. Birinchi marotaba ushbu holatga nemis matematigi Myobius e’tibor berdi. Shu paytgacha yirik olimlar “To’rtta bo’yoq muommosi”ni yechish uchun harakat qilishgan. To’rtta bo’yoq ixtiyoriy xaritani bo’yash yoki bo’yash mumkin emasligini isbotlashga harakat qilishgan. Lekin shu kunga qadar hech kim bajara olmagan. Qiziqarlisi shundaki, ba’zi sirtlar uchun xaritani bo’yash muommasi hal etilgan. Ixtiyoriy xaritani bo’yashda shunday sirtlar mavjudki, uni bo’yash uchun 7 ta bo’yoq etarli hisoblanadi. Shunday sirtlar ham mavjudki, 6 ta bo’yoq bilan ham xaritani bo’yash qiyin.

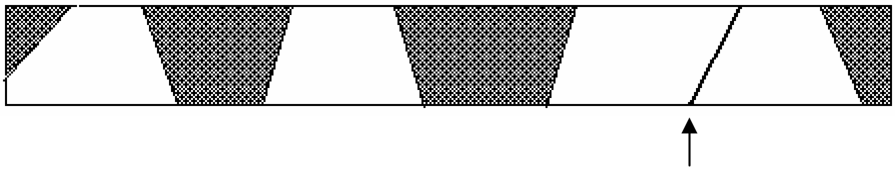
3.2-masala. To’g’ri tortburchak to’g’ri chiziq bilan n ta qismga bo’lingan deb faraz qilaylik. Bunday holda umumiy tomonga ega bo’lgan ikkita qismni turli rangdagi qora va oq bo’yoqga bo’yash mumkin.

Isbotlash. $n = 1$da quidagiga egamiz:

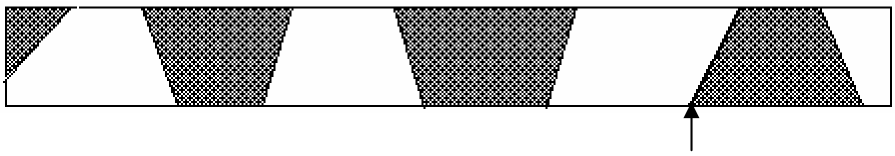


$n=k$ uchun tasdig to’g’ri deb faraz qilaylik. U holda da ham to’g’ri hisoblanadi.

$n = k + 1$ da k to’g’ri chiziqdan iborat k+1 to’g’ri chiziqli ma’lumotlardan tashkil etgan to’g’ri to’rtburchakni bo’yash lozim:



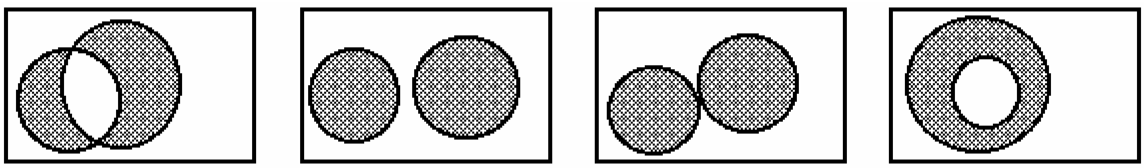
(k+1) to’g’ri chiziqdan qolgan qismlarni qarama-qarshi bo’yoqga bo’yash lozim bo’ladi:



(k+1) to’g’ri chiziq

3.3-masala. Tekislikda n ta aylana berilgan ($n \geq 2$). Ushbu aylanalardan tashkil topgan xaritada ularni ixtiyoriy joylashtirilganda ikkita bo’yoq bilan to’g’ri bo’yash mukinligini isbotlang.

Isbotlash. $n = 2$ da aylanalarning quyidagi imkoniyatlarini ko’rish mumkin:



$n = k \geq 2$ uchun tasdiq to’g’ri deb faraz qilaylik. Bunday holda tasdiq $n = k +1$ da ham to’g’ri bo’ladi. Tekislikda k+1 aylanalar berilgan deb faraz qilaylik.

4-§. Tengsizliklarni isbotlash

4.1-masala. Faraz qilaylik, a, b – katetlar uzunligi, c – to'g'ri burchak gipotenuzasining uzunligi. U holda barcha $n \geq 2$ natural sonlar uchun

$$a^n + b^n \leq c^n \qquad (4.1)$$

o'rinlidir.

1-teorema. Pifagor teoremasidan $a^2 + b^2 = c^2$ tenglik hosil qilinadi.

1-teorema isbotlandi.

2-teorema. (4.1) tengsizlik $n = k$, $k \geq 2$, da bajariladi deb faraz qilaylik,ya'ni $a^k + b^k \leq c^k$. to'gri. U holda (4.1) tengsizlik $n = k+1$: $a^{k+1} + b^{k+1} \leq c^{k+1}$ bajariladi.

Haqiqatdan:

$$a^{k+1} + b^{k+1} = a^k \cdot a + b^k \cdot b < a^k \cdot c + b^k \cdot c = c \cdot (a^k + b^k) \leq c \cdot c^k = c^{k+1}.$$

2-teorema isbotlandi.

1-teorema va 2-teoremalardan (4.1) tengsizlikning ixtiyoriy $n \geq 2$ natural son uchun o'rinli ekanligi kelib chiqadi.

4.2-masala. Agar $a \geq -1$ bo'lsa, u holda

$$(1+a)^n \geq 1 + na, \forall n \in N \qquad (4.2)$$

ekanligini isbotlang.

1-teorema. $n = 1$ da quidagiga ega bo'lamiz:

(4.2) tengsizlikning chap qismi: $(1+a)^1 = (1+a)$

(4.2) tengsizlikning o'ng qismi: $(1 + 1 \cdot a) = 1 + a$. ga teng. 1-teorema isbotlandi.

2-teorema. Faraz qilaylik, (4.2) tengsizlik $n = k$: $(1+a)^k \geq 1 + ka$ bajarilsin. U holda tengsizlik

$n = k+1$ uchun o'rinli bo'ladi: $(1+a)^{k+1} \geq 1 + (k+1)a$.

Haqiqatdan, $(1+a) \geq 0$ bo'lganda

$$(1+a)^{k+1}=(1+a)^k(1+a)\geq(1+ka)(1+a)$$
$$=1+(k+1)a+\underbrace{ka^2}_{\geq 0}\geq 1+(k+1)a$$

bajariladi.

2-teorema isbotlandi. 1- va 2- teoremalardan (4.2) tengsizlik ixtiyoriy natural son uchun bajariladi.

1-eslatma. Ma'lumki, $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ ketma-ketlik chegaraga ega bo'lib, **e** soni deyiladi:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=2,718\ldots.$$

Ketma-ketlik xossasidan:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}_{=1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e. \qquad (4.2.1)$$

ekanligi ma'lum bo'ladi.

$x_n=\left(1+\frac{1}{n}\right)^n$ $(n=1,2,...)$ ketma-ketlikning monoton o'sishini, $y_n=\left(1+\frac{1}{n}\right)^{n+1}$ $(n=1,2,...)$ ketma-ketlikning esa monoton kamayishini isbotlang.

Isbotlash. Agar $x_n < x_{n+1}$ bo'lsa, bu $\dfrac{x_{n+1}}{x_n} > 1$ tengsizlikka teng kuchli, shuning uchun $\{x_n\}$ ketma-ketlik monoton o'sadi. Quyidagi tengsizlikni isbotlash lozim:

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} > 1 \quad (n = 1, 2, ...) \cdot \qquad (4.2.2)$$

Agar $y_{n-1} > y_n$ bo'lsa, bu $\dfrac{y_n}{y_{n-1}} < 1$ tengsizlikka teng kuchli, u holda $\{y_n\}$ ketma-ketlik monoton kamayadi. Quyidagi tengsizlikni isbotlang:

$$\frac{y_n}{y_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^{n}} < 1 \quad (n = 1, 2, ...) \cdot \qquad (4.2.3)$$

(4.2.2) tengsizlikni isbotlaymiz:

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \frac{(n+2)^{n+1}\, n^n}{(n+1)^n\,(n+1)^{n+1}} \cdot \frac{n\cdot(n+1)}{n\cdot(n+1)} = \frac{((n+2)n)^{n+1}}{(n+1)^{2(n+1)}} \cdot \frac{(n+1)}{n} =$$

$$= \left(\frac{n^2+2n+1-1}{(n+1)^2}\right)^{n+1} \cdot \frac{(n+1)}{n} = \underbrace{\left(1-\frac{1}{(n+1)^2}\right)^{n+1}} \cdot \frac{(n+1)}{n} >$$

$$> \left(1-\frac{1}{(n+1)}\right) \cdot \frac{(n+1)}{n} = 1. \qquad \Rightarrow \qquad \frac{x_{n+1}}{x_n} > 1.$$

(**4.2.3**) tengsizlikni (4.2.2) tengsizlikka o'xshash isbotlanadi:

$$\frac{y_n}{y_{n-1}}=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^{n}}=\frac{(n+1)^n\cdot(n-1)^n}{(n^2-1+1)^n}\cdot\frac{n+1}{n}=\underbrace{\frac{1}{\left(1+\frac{1}{n^2-1}\right)^{n}}}_{(4.2)}\cdot\frac{n+1}{n}<$$

$$<\frac{1}{1+\frac{n}{n^2-1}}\cdot\frac{n+1}{n}=\frac{n^3+n^2-n-1}{n^3+n^2-n}<1. \quad \Rightarrow \quad \frac{y_n}{y_{n-1}}<1.$$

$x_n=\left(1+\frac{1}{n}\right)^n$ $(n=1,2,...)$ ketma-ketlik qa'tiy monoton o'sadi $y_n=\left(1+\frac{1}{n}\right)^{n+1}$ $(n=1,2,...)$ ketma-ketlik qa'tiy monoton kamayadi, (4.2.1) tengsizlikni hisobga olgan holda quyidagi tengsizlikni hosil qilamiz:

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}. \qquad (4.2.4)$$

2-eslatma. (4.2.2) tengsizlikni matematik induksiya metodi bilan isbotlaymiz. Bu tengsizlik quyidagi tengsizlikka teng kuchli:

$$\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}.$$

(4.2.2.1)

1-teorema. $n=1$ da $\left(1+\frac{1}{1}\right)^1=2<2{,}25=\left(1+\frac{1}{1+1}\right)^{1+1}$. ega bo'lamiz

2-teorema. (4.2.2.1) tengsizlikning $n=k$ da bajarilishi berilgan:

$\left(1+\frac{1}{k}\right)^{k} < \left(1+\frac{1}{k+1}\right)^{k+1}$. (4.2.2.1) tengsizlikning $n=k+1$ da bajarilishini isbotlash lozim: $\left(1+\frac{1}{k+1}\right)^{k+1} < \left(1+\frac{1}{k+2}\right)^{k+2}$.

Isbotlash. $\left(1+\frac{1}{k+1}\right)^{k+1} = \left(1+\frac{1}{k+1}\right)^{k+1} \dfrac{\left(1+\frac{1}{k+2}\right)^{k+2}}{\left(1+\frac{1}{k+2}\right)^{k+2}} =$

$$=\left(1+\frac{1}{k+2}\right)^{k+2} \frac{(k+2)^{k+1}(k+2)^{k+2}}{(k+1)^{k+1}(k+3)^{k+2}} = \left(1+\frac{1}{k+2}\right)^{k+2} \frac{((k+2)^2)^{k+1}(k+2)}{((k+1)(k+3))^{k+1}(k+3)} =$$

$$=\left(1+\frac{1}{k+2}\right)^{k+2}\left(\frac{k^2+4k+4}{k^2+4k+3}\right)^{k+1}\cdot\frac{k+2}{k+3}=$$

$$\left(1+\frac{1}{k+2}\right)^{k+2}\cdot\frac{1-\frac{1}{k+3}}{\left(1-\frac{1}{(k+2)^2}\right)^{k+1}}<$$

$$\underbrace{\left(1-\frac{1}{(k+2)^2}\right)^{k+1}}_{(4.2)} > 1-\frac{k+1}{(k+2)^2}$$

$$<\left(1+\frac{1}{k+2}\right)^{k+2}\frac{\left(1-\frac{1}{k+3}\right)}{1-\frac{k+1}{(k+2)^2}} = \left(1+\frac{1}{k+2}\right)^{k+2}\frac{(k+2)^3}{(k+3)(k^2+3k+3)} =$$

$$=\left(1+\frac{1}{k+2}\right)^{k+2}\underbrace{\frac{k^3+6k^2+12k+8}{k^3+6k^2+12k+9}}_{<1} < \left(1+\frac{1}{k+2}\right)^{k+2}.$$

2-teorema isbotlandi.

Matematik induksiya prinsipiga ko'ra (4.2.2.1) tengsizlik ixtiyoriy n natural son uchun bajariladi.

4.3-masala. Tengsizlikni isbotlang

$$(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_n)\geq 1+x_1+x_2+\ldots+x_n,\ \forall n\in N, \qquad (4.3)$$

Bu yerda $x_i, i=1,\ldots,n$, – (-1) dan katta bo'lgan bir xil ishorali son.

1-teorema. $n=1$ da $1+x_1=1+x_1$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. (4.3) tengsizlikning $n=k$ da bajarilishi berilgan:

$(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_k)\geq 1+x_1+x_2+\ldots+x_k$.

(4.3) tengsizlikning $n = k + 1$ da bajarilishini isbotlash lozim:

$$(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_k)\cdot(1+x_{k+1})\geq 1+x_1+x_2+\ldots+x_k+x_{k+1}.$$

Isbotlash.

$$\underbrace{(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_k)}_{\geq\ 1+x_1+x_2+\ldots+x_k}\cdot(1+x_{k+1})\geq(1+x_1+x_2+\ldots+x_k)\cdot(1+x_{k+1})=$$

$$=1+x_1+\ldots+x_k+x_{k+1}+\underbrace{x_1x_{k+1}+\ldots+x_kx_{k+1}}_{\geq\ 0}\geq 1+x_1+\ldots+x_k+x_{k+1}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (4.3) tengsizlik ixtiyoriy n natural son uchun bajariladi.

4.4-masala. Tengsizlikni isbotlang:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}},\quad \forall n\in N.$$

(4.4)

1-teorema. $n=1$ da: $\frac{1}{2}=\frac{1}{\sqrt{2^2}}=\frac{1}{\sqrt{4}}<\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{2\cdot 1+1}}$. 1-teorema isbotlandi.

2-teorema. $n = k$ da berilgan: $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k}<\frac{1}{\sqrt{2k+1}}$.

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k}\cdot\frac{2(k+1)-1}{2(k+1)}<\frac{1}{\sqrt{2(k+1)+1}}=\frac{1}{\sqrt{2k+3}}$$ tengsizlikni isbotlash lozim.

Isbotlash. $$\underbrace{\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k}}_{<\frac{1}{\sqrt{2k+1}}}\cdot\frac{2(k+1)-1}{2(k+1)}<\frac{1}{\sqrt{2k+1}}\frac{2k+1}{2(k+1)}\cdot\frac{\sqrt{2k+3}}{\sqrt{2k+3}}=$$

$$=\frac{1}{\sqrt{2k+3}}\cdot\frac{\sqrt{2k+1}\cdot\sqrt{2k+3}}{\sqrt{(2k+2)^2}}=\frac{1}{\sqrt{2k+3}}\cdot\underbrace{\sqrt{\frac{4k^2+8k+3}{4k^2+8k+4}}}_{<1}<\frac{1}{\sqrt{2k+3}}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko’ra (4.4) tengsizlik ixtiyoriy n natural son uchun bajariladi.

4.5-masala. Tengsizliklarni isbotlang:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n},\quad \forall n\in N:n\geq 2; \tag{4.5.1}$$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n},\quad \forall n\in N:n\geq 2. \tag{4.5.2}$$

(4.5.1) tengsizlikni isbotlaymiz.

1-teorema. $n = 2$ *da*: $1+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}}>\frac{1+1}{\sqrt{2}}=\sqrt{2}.$ ega bo’lamiz.

1-teorema isbotlandi.

2-teorema. $n = k$ d $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}>\sqrt{k}$ tengsizlik berilgan.

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$$ tengsizlikni isbotlash lozim.

Isbotlash. $$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k}+\frac{1}{\sqrt{k+1}}=$$

$= \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} > \frac{\sqrt{k^2}+1}{\sqrt{k+1}} = \sqrt{k+1}.$ 2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (4.5) tengsizlik ixtiyoriy $n \geq 2$ natural son uchun bajariladi.

(4.5.2) tengsizlikni isbotlaymiz.

2-teorema. $n = 2$ da: $1+\frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} < \frac{2+2}{\sqrt{2}} = 2\sqrt{2}$. 1-teorema isbotlandi.

2-teorema. (4.5.2) tengsizlik $n = k$ da bajarilishi berilgan:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}} < 2\sqrt{k},\ k \geq 2.$$

$n = k + 1$ da : $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$ bajariladi.

Isbotlash. $\underbrace{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}}}_{<2\sqrt{k}}+\frac{1}{\sqrt{k+1}} <$

$$< \left(2\sqrt{k}+\frac{1}{\sqrt{k+1}}\right)\cdot\frac{\sqrt{k+1}}{\sqrt{k+1}} = \sqrt{k+1}\cdot\left(\frac{2\sqrt{k^2+k}+1}{k+1}\right).$$

$\frac{2\sqrt{k^2+k}+1}{k+1} < 2$ tengsizlikning $\sqrt{4k^2+4k}+1 < 2k+2 \Leftrightarrow \sqrt{4k^2+4k} < 2k+1$ tengsizlikka teng kuchli ekanligini isbotlash lozim. Oxirgi tengsizlik quyidagi tengsizlik va tenglikdan kelib chiqadi: $\sqrt{4k^2+4k} < \sqrt{4k^2+4k+1} = 2k+1,\ \forall k \in N$.

2-teorema isbotlandi. . Matematik induksiya prinsipiga ko'ra (4.5.2) tengsizlik ixtiyoriy $n \geq 2$ natural son uchun bajariladi.

Eslatma: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}} > \underbrace{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+...+\frac{1}{\sqrt{n}}}_{n} = \frac{n}{\sqrt{n}} = \sqrt{n}$.

4.6-masala. Tengsizlikni isbotlang

$$n^{n+1} > (n+1)^n, \quad \forall n \in N : n \geq 3. \tag{4.6}$$

1-teorema. *n = 3 da*: $3^4 = 81 > 64 = 4^3$. 1-teorema isbotlandi.

2-teorema. $n = k$ da $k^{k+1} > (k+1)^k$. tengsizlikning bajarilishi berilgan.

$(k+1)^{k+2} > (k+2)^{k+1}$ tengsizlikning bajarilishini isbotlash lozim.

Isbotlash. Agar $k^{k+1} > (k+1)^k$ bo'lsa, u holda $1 > \dfrac{(k+1)^k}{k^{k+1}}$ tengsizlik bajariladi. Oxirgi tengsizlikni musbat $(k+1)^{k+2}$ songa ko'paytirib, quyidagi tenglik va tengsizlik zanjirini hosil qilamiz:

$$(k+1)^{k+2} > \frac{(k+1)^k \cdot (k+1)^{k+2}}{k^{k+1}} = \frac{((k+1)^2)^{k+1}}{k^{k+1}} = \left(\frac{k^2+2k+1}{k}\right)^{k+1} =$$

$$= \left(k+2+\frac{1}{k}\right)^{k+1} > \left| \ \frac{1}{k} > 0 \ \right| > (k+2)^{k+1}.$$ 2-teorema isbotlandi.

Matematik induksiya prinsipiga ko'ra (4.5.2) tengsizlik ixtiyoriy $n \geq 3$ natural son uchun bajariladi.

4.7-masala. Tengsizlikni isbotlang:

$$n! > \left(\frac{n}{3}\right)^n, \quad \forall n \in N; \tag{4.7.1}$$

$$n! < \left(\frac{n+1}{2}\right)^n, \quad \forall n \in N : n \geq 2; \tag{4.7.2}$$

$$n! > n^{\frac{n}{2}}, \quad \forall n \in N : n \geq 3; \tag{4.7.3}$$

$$n! > 2^{n-1}, \quad \forall n \in N : n \geq 3; \tag{4.7.4}$$

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n, \quad \forall n \in N. \tag{4.7.5}$$

(4.7.1) tengsizlikni isbotlaymiz.

1-teorema . *n = 1* da $1! > \left(\frac{1}{3}\right)^1$. Tengsizlikka ega bo'lamiz.

2-teorema. $n = k$ da (4.7.1) tengsizlik berilgan: $k!>\left(\frac{k}{3}\right)^k$.

Bu tengsizlikning $n = k+1$ bajarilishini isbotlaymiz: $(k+1)!>\left(\frac{(k+1)}{3}\right)^{(k+1)}$.

Isbotlaymiz. $(k+1)! = k!\,(k+1) > \left(\frac{k}{3}\right)^k (k+1) =$

$\left|\quad \left(\frac{k+1}{3}\right)^{(k+1)}\right.$ songa kopaytiramiz va bo'lamiz $\Big|$

$$= \left(\frac{k+1}{3}\right)^{k+1} \frac{3^{(k+1)}\cdot k^k\cdot(k+1)}{(k+1)^{(k+1)}\cdot 3^k} = \left(\frac{k+1}{3}\right)^{k+1} \frac{3}{\left(1+\frac{1}{k}\right)^k} >$$

Oraliq hisoblashlar:

$$\left(1+\frac{1}{k}\right)^k = 1+\frac{k}{k}+\frac{k(k-1)}{2!}\cdot\frac{1}{k^2}+\ldots+\frac{k(k-1)\cdot\ldots\cdot(k-k+1)}{k!}\cdot\frac{1}{k^k} =$$

$$=1+1+\frac{1}{2!}\underbrace{\left(1-\frac{1}{k}\right)}_{<1}+\ldots+\frac{1}{k!}\underbrace{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{k}\right)}_{<1} <$$

$$<1+1+\frac{1}{2!}+\underbrace{\frac{1}{3!}}_{=\frac{1}{2\cdot3}<\frac{1}{2^2}}+\ldots+\underbrace{\frac{1}{k!}}_{=\frac{1}{1\cdot2\cdot3\cdot\ldots\cdot k}<\frac{1}{2^{k-1}}} <1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}<$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}+\frac{1}{2^k}+\ldots=$$

$$=1+\frac{1}{1-\frac{1}{2}}=3 \quad \Rightarrow \quad \left(1+\frac{1}{k}\right)^k<3 \quad \Rightarrow \quad \frac{3}{\left(1+\frac{1}{k}\right)^k}>1\cdot$$

$$>\left(\frac{k+1}{3}\right)^{k+1}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (4.7.1) tengsizlik ixtiyoriy n natural son uchun bajariladi.

(4.7.2) tengsizlikni isbotlang.

1-teorema. $n = 2$ da:

(4.7.2) tensizlikningchap qismi: $2!=2$

(4.7.2) tengsizlikning o'ng qismi: $\left(\frac{2+1}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}=2,25$.

2 < 2,25 bo'lganligi sababli 1-teorema isbotlandi.

2-teorema. $n=k$ da (4.7.2) tengsizlikning bajarilishi berilgan: $k!<\left(\frac{k+1}{2}\right)^k, \quad k\geq 2$. $n = k+1$ da quyidagi tengsizlikning bajarilishini isbotlash lozim:

$$(k+1)!<\left(\frac{k+2}{2}\right)^{k+1}, k\geq 2\cdot$$

Isbotlash. $(k+1)!=k!\cdot(k+1)<\left(\frac{k+1}{2}\right)^k\cdot(k+1)=$

$\Big| \left(\frac{k+2}{2}\right)^{k+1}$ songa ko'paytiramiz va bo'lamiz $\Big|$

$$=\left(\frac{k+2}{2}\right)^{k+1}\cdot\frac{(k+1)^k\cdot(k+1)\cdot 2^{k+1}}{2^k\cdot(k+2)^{k+1}}=\left(\frac{k+2}{2}\right)^{k+1}\cdot\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}<$$

$$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}} < 1$$ tengsizlikning bajarilishini isbotlaymiz.

$$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}} = \frac{2}{\left(\dfrac{k+2}{k+1}\right)^{k+1}} = 2\cdot\frac{1}{\left(1+\dfrac{1}{k+1}\right)^{k+1}}.$$

$$\left(1+\frac{1}{k+1}\right)^{k+1} = 1+\frac{k+1}{k+1}+\underbrace{\frac{(k+1)\cdot k}{2!}\cdot\frac{1}{(k+1)^2}+....+\left(\frac{1}{k+1}\right)^{k+1}}_{>0} > 2\cdot$$

$$\Rightarrow \quad \frac{1}{\left(1+\dfrac{1}{k+1}\right)^{k+1}} < \frac{1}{2} \quad \Rightarrow \quad 2\cdot\left(\frac{k+1}{k+2}\right)^{k+1} < 2\cdot\frac{1}{2} = 1\ .$$

$$< \left(\frac{k+2}{2}\right)^{k+1}\cdot 1 = \left(\frac{k+2}{2}\right)^{k+1}.$$

2-teorema isbotlandi.

Matematik induksiya prinsipiga ko’ra (4.7.2) tengsizlik ixtiyoriy $n \geq 2$ natural son uchun bajariladi.

(4.7.3) tengsizlikni isbotlaymiz.

1-teorema. *n* = *3 da*: (4.7.3) tengsizlikning chap qismi: $3! = 6 = \sqrt{36}$.

(4.7.3) tengsizlikning o’ng qismi: $3^{\frac{3}{2}} = \sqrt{27}$.

Bo’lganligi uchun $\sqrt{36} > \sqrt{27}$ 1-teorema isbotlandi.

2-teorema. (4.7.3) tengsizlikning $n = k$ da bajarilishi berilgan: $k! > k^{\frac{k}{2}}$, $k \geq 3$. $n = k+1$ da quyidagi tengsizlikning bajarilishini isbotlaymiz :

$$(k+1)! > (k+1)^{\frac{k+1}{2}}, \quad k \geq 3.$$

Isbotlaymiz.

$$(k+1)!=k!\cdot(k+1)>k^{\frac{k}{2}}\cdot(k+1)\cdot\frac{(k+1)^{\frac{k+1}{2}}}{(k+1)^{\frac{k+1}{2}}}=(k+1)^{\frac{k+1}{2}}\cdot\frac{k^{\frac{k}{2}}\cdot(k+1)^{\frac{1}{2}}\cdot k^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}}\cdot k^{\frac{1}{2}}}>$$

(4.6) masalada $n^{n+1}>(n+1)^n,\quad \forall n\geq 3.$tengsizlik isbotlangan. Bu tengsizlikdan quyidagi tengsizlik hosil qilinadi: $n^{\frac{n+1}{2}}>(n+1)^{\frac{n}{2}},\quad \forall n\geq 3.$

$n = k$ uchun quyidagini hosil qilamiz:

$$>(k+1)^{\frac{k+1}{2}}\cdot\frac{(k+1)^{\frac{k}{2}}\cdot(k+1)^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}}\cdot k^{\frac{1}{2}}}=(k+1)^{\frac{k+1}{2}}\cdot\underbrace{\left(1+\frac{1}{k}\right)^{\frac{1}{2}}}_{>1}>\ (k+1)^{\frac{k+1}{2}}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (4.7.3) tengsizlik ixtiyoriy $n\geq 3$ natural son uchun bajariladi.

(4.7.4) tengsizlikni isbotlaymiz.

1-teorema. $n = 3$ da: (4.7.4) tengsizlikning chapqismi: $3!=6$;

(4.7.4)tengsizlikning o'ng qismi: $2^{3-1}=4$.

6 > 4 bo'ganligi sababli 1-teorema isbotlandi.

2-teorema. (4.7.4) tengsizlik $n = k$ da bajarilishi berilgan: $k!>2^{k-1},\quad \forall k\geq 3$. $n = k+1$ da tengsizlikning bajarilishini isbotlash losim: $(k+1)!>2^{k+1-1},\quad k\geq 3$.

Isbotlsh. $(k+1)!=k!\cdot(k+1)>2^{k-1}\cdot(k+1)=2^k\cdot\underbrace{\frac{k+1}{2}}_{>1}>2^k,\qquad k\geq 3da$.

2-teorema isbotlandi. . Matematik induksiya prinsipiga ko'ra (4.7.4) tengsizlik ixtiyoriy $n\geq 3$ natural son uchun bajariladi.

(4.7.5) tengsizlikni isbotlaymiz.

1-teorema. $n = 1$da $\left(\frac{1}{e}\right)^1<1!<e\left(\frac{1}{2}\right)^1$ tengsizlikka ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. (4.7.5) tengsizlik ning $n=k$ da o’rinli ekanligi berilgan: $\left(\frac{k}{e}\right)^k < k! < e\left(\frac{k}{2}\right)^k$. $n = k+1$ da tengsizlikni isbotlash lozim:

$$\left(\frac{k+1}{e}\right)^{k+1} < (k+1)! < e\left(\frac{k+1}{2}\right)^{k+1}.$$

Isbotlash. Ushbu tengsizlikning o’ng qismini isbotlaymiz.

$$(k+1)! = (k+1)\cdot k! > (k+1)\cdot\left(\frac{k}{e}\right)^k = \left(\frac{k+1}{e}\right)^{k+1}\frac{(k+1)\cdot\left(\frac{k}{e}\right)^k}{\left(\frac{k+1}{e}\right)^{k+1}} =$$

$$= \left(\frac{k+1}{e}\right)^{k+1}\frac{(k+1)\cdot k^k\cdot e^{k+1}}{(k+1)^{k+1}\cdot e^k} = \left(\frac{k+1}{e}\right)^{k+1}\cdot\frac{e}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{<e\ (4.2.4)}} > \left(\frac{k+1}{e}\right)^{k+1}$$

(4.7.5) tengsizlikning o’ng qismini isbotlaymiz.

$$(k+1)! = (k+1)\cdot k! < e\left(\frac{k}{2}\right)^k = e\left(\frac{k+1}{2}\right)^{k+1}\frac{\left(\frac{k}{2}\right)^k}{\left(\frac{k+1}{2}\right)^{k+1}} =$$

$$= e\left(\frac{k+1}{2}\right)^{k+1}\cdot\underbrace{\frac{2}{(k+1)}}_{\le 1}\cdot\underbrace{\left(\frac{k}{k+1}\right)^k}_{<1} < e\left(\frac{k+1}{2}\right)^{k+1}.$$ 2-teorema isbotlandi.

Matematik induksiya prinsipiga ko’ra (4.7.5) tengsizlik ixtiyoriy ***n*** natural son uchun bajariladi.

Eslatma: $n>1$ da (4.7.2) va $\left(1+\frac{1}{n}\right)^n < e$ tengsizlikdan foydalanib, quyidagini hosil qilamiz:

$$n! < \left(\frac{n+1}{2}\right)^n = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(\frac{n+1}{2}\right)^n}{e\cdot\left(\frac{n}{2}\right)^n} = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(1+\frac{1}{n}\right)^n}{e} < e\cdot\left(\frac{n}{2}\right)^n.$$

4.8-masala. Tengsizliklarni isbotlang:

$$x_n = \underbrace{\sqrt{2+\sqrt{2+...+.\sqrt{2}}}}_{n+1 \quad ildiz} < \sqrt{2}+1, \quad \forall n\in N; \qquad (4.8.1)$$

$$x_n = \underbrace{\sqrt{4+\sqrt{4+....+\sqrt{4}}}}_{n \quad ildiz} < 3, \quad \forall n\in N. \qquad (4.8.2)$$

(4.8.1) tengsizlikni isbotlaymiz.

1-teorema. $n = 1$ da

$x_1 = \sqrt{2+\sqrt{2}} < \sqrt{2+2\sqrt{2}+1} = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1$ ega bo’lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ uchun (4.8.1) tengsizlik berilgan :

$$x_k = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1} < \sqrt{2}+1.$$

$n = k+1$ tengsizlikni isbotlash lozim:

$$x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+2 \quad ildiz} < \sqrt{2}+1.$$

Isbotlash.

$$x_{k+1} = \sqrt{2+\underbrace{\sqrt{2+...+\sqrt{2}}}_{k+2 \quad ildiz \ <\sqrt{2}+1}} < \sqrt{2+\sqrt{2}+1} < \sqrt{2+2\sqrt{2}+1} = \sqrt{2}+1$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko’ra (4.8.1) tengsizlik ixtiyoriy ***n*** natural son uchun bajariladi.

(4.8.2) tengsizlikni isbotlaymiz.

1-teorema. $n = 1$ *da* $x_1 = \sqrt{4} < \sqrt{9} = 3$ ega bo’lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ da (4.8.1) tengsizlikning o'rinli ekanligini berilgan:

$$x_k = \underbrace{\sqrt{4+\sqrt{4+....+\sqrt{4}}}}_{k \quad ildiz} < 3.$$

$n = k+1$ da tengsizlikni isbotlash lozim:

$$x_{k+1} = \underbrace{\sqrt{4+\sqrt{4+....+\sqrt{4}}}}_{k+1 \quad ildiz} < 3.$$

Isbotlash. $x_{k+1} = \underbrace{\sqrt{4+\sqrt{4+....+\sqrt{4}}}}_{k+1 \quad ildiz} < \sqrt{4+3} = \sqrt{7} < 3$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (4.8.2) tengsizlik ixtiyoriy ***n*** natural son uchun bajariladi.

4.9-masala. Tengsizlikni isbotlang

$$\frac{5^n}{n!} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5}, \quad \forall n \in N : n \ge 6. \tag{4.9}$$

1-teorema. $n = 6$ da: $\frac{5^6}{6!} = \frac{5^5}{5!}\left(\frac{5}{6}\right)^{6-5}$. ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $\frac{5^k}{k!} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}$, $k \ge 6$. tengsizlikning bajarilishi berilgan.

Quyidagi tengsizlikning bajarilishini isbotlash lozim:

$$\frac{5^{k+1}}{(k+1)!} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

Isbotlash. $n = k \ge 6$ da quyidagiga ega bo'lamiz:

$$\frac{5^{k+1}}{(k+1)!} = \underbrace{\frac{5^k}{k!}}_{\le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}} \cdot \underbrace{\frac{5}{k+1}}_{< \frac{5}{6}} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}\frac{5}{6} = \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

2-teorema isbotlandi. . Matematik induksiya prinsipiga ko'ra (4.2) tengsizlik ixtiyoriy $n \geq 6$ natural son uchun bajariladi.

4.10-masala. Ixtiyoriy ***n*** natural son uchun

$$|\sin nx| \leq n\,|\sin x| \tag{4.10}$$

tengsizlikni isbotlang

1-teorema. $n = 1$ da : $|\sin 1x| = 1 \cdot |\sin x|$ 1-teorema isbotlandi.

2-teorema. $n = k$ da $|\sin kx| \leq k\,|\sin x|$ tengsizlikning bajarilishi berilgan.

$|\sin(k+1)x| \leq (k+1)\,|\sin x|$ tengsizlikning bajarilishini isbotlash lozim.

Isbotlash. $|\sin(k+1)x| = |\sin kx \cos x + \sin x \cos kx| \leq$

$$\leq \underbrace{|\sin kx|}_{\leq k\,|\sin x|} \cdot \underbrace{|\cos x|}_{\leq 1} + |\sin x| \cdot \underbrace{|\cos kx|}_{\leq 1} \leq (k+1)\,|\sin x|.$$

2-teorema isbotlandi.

Matematik induksiya prinsipiga ko'ra (4.10) tengsizlik ixtiyoriy ***n*** natural son uchun bajariladi.

4.11-masala. Ixtiyoriy ***n*** natural sonda

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1 \tag{4.1.1}$$

tengsizlikni isbotlash lozim.

$S_k = \dfrac{1}{k+1} + \dfrac{1}{k+2} + \ldots + \dfrac{1}{3k+1}$ orqali belgilaymiz.

1-teorema. $n = 1$ da: $S_1 = \dfrac{1}{1+1} + \dfrac{1}{1+2} + \dfrac{1}{3 \cdot 1 + 1} = \dfrac{13}{12} > 1$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ da quyidagi tengsizlikning bajarilishi berilgan:

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{3k+1} > 1.$$

Quidagi tengsizlikning bajarilishini isbotlang

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > 1.$$

Isbotlash.

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} + \left(\frac{1}{k+1} - \frac{1}{k+1}\right) =$$

$$= \underbrace{\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1}}_{=S_k > 1} + \underbrace{\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} - \frac{1}{k+1}}_{>0} > 1$$

.

" >0 " tengsizlik quyidagicha kelib chiqadi:

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} - \frac{1}{k+1} = \frac{1}{3k+2} + \frac{1}{3k+4} - \frac{2}{3k+3} =$$

$$= \frac{(3k+4)(3k+3) + (3k+2)(3k+3) - (6k+4)(3k+4)}{(3k+2)(3k+3)(3k+4)} =$$

$$= \frac{2}{(3k+2)(3k+3)(3k+4)} > 0.$$ 2-teorema isbotlandi.

Matematik induksiya prinsipiga ko'ra (4.11) tengsizlik ixtiyoriy n natural son uchun bajariladi.

5-§. Gipoteza va uning isbotlanishi

Oldingi masalalarda kim tomondandir aytilgan doimo to'g'ri deb hisoblangan gipoteza tekshirilgan. Lekin ko'p masalalarda ushbu to'g'ri gipotezani aytish qiyin. Gipotezani qurish lozim. Buning uchun n ketma – ketlikning 1, 2, 3, … qiymatlari toki etarli material to'plangunga qadar davom etadi. Qo'yilgan masalani yechishda insonning kuzatuvchanligiga bog'liq va uning qobiliyatiga ko'ra hususiy natijadan umumiy natija topiladi, ya'ni ko'p yoki oz miqdorda ishonchli gipotezani qurish losim. Shundan so'ng ushbu gipotezani tekshirishda matematik induksiya metodidan foydalaniladi. Matematik induksiya metodi gipotezada uchraydigan umumiy qonunlarni izlash imkoniyatini yaratadi va yo'lg'onni olib tashlaydi, rostni tasdiqlaydi.

5.1-masala. Faraz qilaylik, $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)}$ bo'lsin. S_n ni o'rgangan holda $S_n = \frac{n+1}{3n+1}$, gipotezani aytamiz, ya'ni

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)} = \frac{n+1}{3n+1}, \forall n \in N. \qquad (5.1)$$

Ushbu gipotezani tekshiramiz.

1-teorema. $n = 1$ da quyidagiga ega bo'lamiz:

(5.1) ning chap tomoni: $\frac{1}{1\cdot(1+1)} = \frac{1}{2}$; (5.1) ning o'ng tomoni: $\frac{1+1}{3\cdot 1+1} = \frac{1}{2}$.

1-teorema isbotlandi.

2-teorema. $n = k$ da (5.1) formula o'rinli ekanligi berilgan bo'lsin.
$n = k + 1$ da tenglikning o'rinli ekanligini tekshiramiz.

$n = k + 1$ da (5.1) formula to'g'ri deb faraz qilamiz. U holda quyidagiga ega bo'lamiz:

$$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)} = \frac{k+1}{3k+1} + \frac{1}{(k+1)(k+2)} =$$

$$=\frac{k^3+4k^2+8k+3}{(k+1)(k+2)(3k+1)}=\frac{1}{(3k+1)}\cdot\frac{k^3+4k^2+8k+3}{k^2+3k+2}=$$

$$\begin{array}{rl|l}
- & k^3+4k^2+8k+3 & k^2+3k+2 \\ \cline{3-3}
 & \underline{k^3+3k^2+2k} & k+1 \\
- & \quad k^2+6k+3 & \\
 & \quad \underline{k^2+3k+2} & \\
 & \qquad 3k+1 &
\end{array}$$

$$=\frac{1}{(3k+1)}\cdot\left(k+1+\frac{3k+1}{k^2+3k+2}\right)\neq\frac{k+1}{(3k+1)},$$

Natijada qarama-qarshilik hosil bo’ladi. Bu qarama-qarshilik (5.1) formulaning o’rinli ekanligini ko’rsatadi.

Tog’ri gipotezani aytishga harakat qilamiz. Buning uchun bir nechta S_n ni hisoblaymiz, bunda

n = 1, 2, 3, ..., larda:

$$S_1=\frac{1}{1\cdot 2}=1-\frac{1}{2}=\frac{1}{2}=\frac{1}{1+1};$$

$$S_2=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=1-\frac{1}{3}=\frac{2}{3}=\frac{2}{2+1};$$

$$S_3=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}=1-\frac{1}{4}=\frac{3}{4}=\frac{3}{3+1};$$

Ushbu tengliklarni diqqat bilan ko’rib, ixtiyoriy natural n uchun $S_n=\frac{n}{n+1}$ tenglikdagi gipotezani aytish mumkin. Matematik induksiya metodi bilan ushbu gipotazani tekshiramiz, ya’ni quyidagi tenglikni tekshiramiz:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{(n+1)},\quad \forall n\in N. \qquad (5.1.1)$$

1-teorema. $n=1$ da quyidagiga ega bo’lamiz:

(5.1.1) tenglikning chap qismi: $\frac{1}{1\cdot 2}=\frac{1}{2}$ ga teng;

(5.1.1) tenlikning o'ng qismi: $\frac{1}{1+1}=\frac{1}{2}$ ga teng. 1-teorema isbotlandi.

2-teorema. Agar (5.1.1) tenglik $n = k$ da to'g'ri bo'lsin. U holda $n = k + 1$ da ham to'g'ri bo'ladi. Haqiqatdan:

$$S_{k+1}=S_k+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{(k+2)}.$$

1- va 2-teoremalardan (5.1.1) tenglik o'rinli ekanligi kelib chiqadi.

5.2-masala. Yig'indini toping:

$$S_n=\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+...+\frac{1}{(a+n-1)(a+n)},\ a>0,\ \forall\ n\in N.$$

$n = 1, 2, 3$ da S_n ning bir nechta qiymatini topamiz:

$(n=1)$ da $S_1=\frac{1}{a(a+1)}$;

$(n=2)$ da $S_2=\frac{1}{(a+1)}\left(\frac{1}{a}+\frac{1}{a+2}\right)=\frac{2a+2}{(a+1)a(a+2)}=\frac{2}{a(a+2)}$;

$(n=3)$ da $S_3=\frac{2}{a(a+2)}+\frac{1}{(a+2)(a+3)}=\frac{1}{a+2}\cdot\frac{2(a+3)+a}{(a+3)a}=\frac{3}{a(a+3)}$.

Quyidagi gipotezani aytamiz. $a>0, \forall n\in N$ uchun quyidagi tenglik to'g'ri:

$$\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+...+\frac{1}{(a+n-1)(a+n)}=\frac{n}{a(a+n)}. \qquad (5.2)$$

Ushbu gipotezani matematik induksiya metodi bilan tekshiramiz.

1-teorema. $n = 1$ da (5.2) tenglikning chap va o'ng qismlari teng bo'ladi:

$$\frac{1}{a(a+1)}=\frac{1}{a(a+1)}.$$

2-teorema. $n = k$ da $S_k = \frac{k}{a(a+k)}$ berilgan. $S_{k+1} = \frac{k+1}{a(a+k+1)}$ ni isbotlash lozim.

Isbotlash.

$$\underbrace{\frac{1}{a(a+1)} + \frac{1}{(a+1)(a+2)} + \ldots + \frac{1}{(a+k-1)(a+k)}}_{=S_k} + \frac{1}{(a+k)(a+k+1)} =$$

$$= \frac{k}{a(a+k)} + \frac{1}{(a+k)(a+k+1)} =$$

$$= \frac{ka+k^2+k+a}{a(a+k)(a+k+1)} = \frac{k(a+k)\ +(k+a)}{a(a+k)(a+k+1)} = \frac{k+1}{a(a+k+1)} = S_{k+1}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.2) tenglikning ixtiyoriy n natural son uchun o'rinli ekanligi kelib chiqadi.

5.3-masala. Quyidagi yig'indini hisoblash uchun formulani chiqaring:

$$S_n = \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \ldots + \frac{1}{(4n-3)(4n+1)}, \ \forall n \in N.$$

Yechilishi. Berilgan yig'indida har bir kasrni quyidagi ayirma shaklida ifodalaymiz:

$$S_n = \left(1 - \frac{1}{5}\right)\frac{1}{4} + \left(\frac{1}{5} - \frac{1}{9}\right)\frac{1}{4} + \left(\frac{1}{9} - \frac{1}{13}\right)\frac{1}{4} + \ldots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)\frac{1}{4} =$$

$$= \frac{1}{4}\left(1 \underbrace{-\frac{1}{5} + \frac{1}{5}}_{=0} \underbrace{-\frac{1}{9} + \frac{1}{9}}_{=0} \underbrace{-\frac{1}{13} + \ldots + \frac{1}{4n-7}}_{=0} \underbrace{-\frac{1}{4n-3} + \frac{1}{4n-3}}_{=0} - \frac{1}{4n+1}\right) =$$

$$= \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) = \frac{1}{4}\cdot\frac{4n+1-1}{4n+1} = \frac{n}{4n+1}.$$

Matematik induksiya metodi bilan quyidagi tenglikni isbotlash lozim:

$$S_n = \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1},\ \forall n \in N.$$

5.4-masala.

1. Dastlabki *n* ta toq sonlar yig'indisini hisoblang: $1+3+5+...+(2n-1)$.

2. $\left\{ n^3 \right\}_{n=1}^{\infty}$ ketma-ketlikning dastlabki *n* ta hadlarining $S_n = 1^3 + 2^3 + 3^3 + ... + n^3,\ n = 1, 2, ...$ yig'indisini hisoblang.

1-yechimi. Izlanayatgan yig'indini S_n orqali belgilaymiz, ya'ni:

$$S_n = 1+3+5+...+(2n-1).$$

n ketma-ketlikga quyidagi qiymatlarni qo'yamiz: 1, 2, 3, 4, 5, 6

$S_1 = 1,\quad S_2 = 4,\quad S_3 = 9,\quad S_4 = 16,\quad S_5 = 25,\quad S_6 = 36.$

Bunday holda quyidagilarni hosil qilamiz:

$S_1 = 1^2,\quad S_2 = 2^2,\quad S_3 = 3^2,\quad S_4 = 4^2,\quad S_5 = 5^2,\quad S_6 = 6^2.$

Shunga asoslangan holda quyidagi gipotezani aytamiz:

ixtiyoriy $n \in N$ uchun $S_n = 1+3+5+...+(2n-1) = n^2$. (5.4.1)

Ushbu gipotezani matematik induksiya metodi bilan tekshiramiz.

1-teorema. $n = 1$ da $S_1 = 1 = 1^2$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ da (5.4.1) tenglikning bajarilishi berilgan deb faraz qilaylik. U holda tenglik $n = k + 1$ da bajariladi.

Haqiqatdan:

$$S_{k+1} = \underbrace{1+3+5+...+(2k-1)}_{=S_k} + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.4.1) tenglik ixtiyoriy *n* natural sonda bajariladi.

2-yechim. $n = 1, 2, 3, 4$ da $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$ ni topamiz:

$$S_1 = 1^3 = 1; \qquad S_2 = 1^3 + 2^3 = 9 = 3^2 = (1+2)^2;$$

$$S_3 = 1^3 + 2^3 + 3^3 = 36 = 6^2 = (1+2+3)^2;$$

$$S_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2 = (1+2+3+4)^2.$$

Quyidagi gipotezani isbotlaymiz:

$$S_n = 1^3 + 2^3 + 3^3 + ... + n^3 = (1+2+3+...+n)^2, \forall n \in N.$$

$$1+2+3+...+n = \frac{(n+1)\cdot n}{2}, \forall n \in N$$

bo'lganligi uchun

$$S_n = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{(n+1)^2 \cdot n^2}{4}, \forall n \in N \text{ bo'ladi.} \qquad (5.4.2)$$

Matematik induksiya metodi bilan (5.4.2) tenglikni isbotlaymiz.

1-teorema. $n = 1$ da (5.4.2) tenglikning chap qismi: $S_1 = 1^3 = 1$ ga teng; (5.4.2) tenglikning o'ng qismi esa: $\frac{(1+1)^2 \cdot 1^2}{4} = 1$ ga teng. 1-teorema isbotlandi.

2-teorema. (5.4.2) tenglik $n = k$ bajarilishi berilgan deb faraz qilaylik . U holda $n = k + 1$ da tenglik bajariladi.

Haqiqatdan:

$$S_{k+1} = \underbrace{1^3 + 2^3 + 3^3 + ... + k^3}_{=S_k} + (k+1)^3 = \frac{(k+1)^2 \cdot k^2}{4} + (k+1)^3 = \frac{(k+1)^2 \cdot (k+2)^2}{4}.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.4.2) tenglikning ixtiyoriy n natural sonda bajarilishi kelib chiqadi.

5.5-masala. $S_n = arctg\frac{1}{2} + arctg\frac{1}{8} + ... + arctg\frac{1}{2n^2}, \forall n \in N$ yig'indini toping.

Yechilishi. $n = 1, 2, 3$ da yig'indini topamiz:

$$S_1 = arctg\frac{1}{2} = arctg\frac{1}{1+1};$$

$$S_2 = arctg\frac{1}{2} + arctg\frac{1}{8} = arctg\frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{8}} = arctg\frac{2}{3} = arctg\frac{2}{2+1};$$

$$S_3 = arctg\frac{2}{3} + arctg\frac{1}{18} = arctg\frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\cdot\frac{1}{18}} = arctg\frac{3}{4} = arctg\frac{3}{3+1}.$$

Quyidagi gipotezani isbotlaymiz:

$$S_n = arctg\frac{1}{2} + arctg\frac{1}{8} + ... + arctg\frac{1}{2n^2} = arctg\frac{n}{n+1}, \quad \forall n \in N. \qquad (5.5)$$

Matematik induksiya metodi bilan formulani isbotlaymiz.

1-teorema. n=1 qiymatda hisoblanganda S_1 isbotlandi.

2-teorema. $n = k$ da (5.5) tenglikning bajarilishi berilgan. $n=k+1$ da tenglikning bajarilishini isbotlaymiz: $S_{k+1} = arctg\dfrac{k+1}{k+1+1}$.

Isbotlash. $$S_{k+1} = \underbrace{arctg\frac{1}{2} + arctg\frac{1}{8} + .. + arctg\frac{1}{2k^2}}_{=S_k} + arctg\frac{1}{2(k+1)^2} =$$

$$= S_k + arctg\frac{1}{2(k+1)^2} = arctg\frac{k}{k+1} + arctg\frac{1}{2(k+1)^2} =$$

$$= arctg\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{k+1}\cdot\frac{1}{2(k+1)^2}} = arctg\frac{k+1}{k+2}.$$

Bu yerda $$\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{k+1}\cdot\frac{1}{2(k+1)^2}} = \frac{(2k^2+2k+1)\cdot 2\cdot(k+1)^3}{2\cdot(k+1)^2\cdot(2k^3+6k^2+5k+2)} =.$$

$$= \frac{(2k^2+2k+1)\cdot(k+1)}{(2k^3+6k^2+5k+2)} = \frac{k+1}{k+2}.$$ Oxirgi tenglik quyidagidan kelib chiqadi:

$$-\,2k^3 + 6k^2 + 5k + 2 \;\Big|\; \underline{2k^2 + 2k + 1}$$

$$
\begin{array}{r|l}
2k^3+2k^2+k & k+2 \\
\hline
\underline{} & \\
4k^2+4k+2 & \\
-\;4k^2+4k+2 & \\
\hline
0 &
\end{array}
$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.5) tenglikning ixtiyoriy n natural sonda bajarilishi kelib chiqadi.

5.6-masala. n natural sonning qaysi qiymatlarida quyidagi tengsizliklar bajariladi:

$$2^n > n^2; \qquad (5.6.1)$$

$$3^n > n^3; \qquad (5.6.2)$$

$$2^n > 5n-3? \qquad (5.6.3)$$

(5.6.1) tengsizlikni n ning quyidagi qiymatlarida bajarilishini tekshiramiz:

$n = 1$ da: $2^1 > 1^2$; $n = 2$ da: $2^2 = 2^2$; $n = 3$ da: $2^3 = 8 < 9 = 3^2$;

$n = 4$ da: $2^4 = 16 = 4^2$; $n = 5$ da: $2^5 = 32 > 25 = 5^2$.

Hosil qilingan yuqoridagi tengsizlikdan (5.6.1) tengsizlik $n = 1$ da bajarilishi kelib chiqadi. Lekin bu tengsizlik $n = 2, 3, 4$ bajarilmaydi. $n = 5$ dan boshlab yana bajariladi.

(5.6.1) tengsizlik ixtiyoriy $n \geq 5$ da bajariladi deb faraz qilamiz. Ushbu farazni matematik induksiya metodi bilan isbotlaymiz.

1-teorema. $n = 5$ da $2^5 = 32 > 25 = 5^2$ ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ da $2^k > k^2$, $k \geq 5$ tengsizlikning o'rinli ekanligi berilgan bo'lsin. $n = k+1$ da $2^{k+1} > (k+1)^2$, $k \geq 5$ tengsizlikning o'rinli ekanligini isbotlaymiz.

Isbotlash. $2^{k+1} = 2^k \cdot 2 > k^2 \cdot 2 = k^2 + k^2 \geq$

$$k \geq 5 \Rightarrow k-1 \geq 4 \Rightarrow (k-1)^2 \geq 16 \Rightarrow k^2 - 2k + 1 \geq 16 \Rightarrow k^2 \geq 15 + 2k$$

$$\geq k^2+2k+15>k^2+2k+1=(k+1)^2.$$

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.6.1) tenglikning ixtiyoriy $n\geq 5$ natural sonda bajarilishi kelib chiqadi.

(5.6.2) tengsizlikning *n* ning quyidagi qiymatlarida bajarilishini tekshiramiz:

$n = 1$ da $3^1>1^3$; $n = 2$ da: $3^2=9>8=2^3$;

$n = 3$ da $3^3=27=3^3$; $n = 4$ da: $3^4=81>64=4^3$.

$n = 5$ da $3^5=243>125=5^3$.

Hosil qilingan tengsizlikdan (5.6) tengsizlik *n*=1 va *n*=2 da bajariladi. Bu tengsizlik *n*=3 uchun bajarilmaydi. Lekin n = 4, 5 uchun yana bajariladi. Ixtiyoriy $n\geq 4$ da (5.6.1) tengsizlik bajariladi deb faraz qilinadi. Ushbu farazni matematik induksiya metodi bilan isbotlaymiz.

1-teorema. $n = 4$ da $3^4=81>64=4^3$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $n = k$ da $3^k>k^3,\quad k\geq 4$ tengsizlikning bajarilishi berilgan.

$n = k+1$ da $3^{k+1}>(k+1)^3,\quad k\geq 4$ tengsizlikning to'gri ekanligini isbotlash lozim.

Isbotlash. $3^{k+1}=3^k\cdot 3>k^3\cdot 3=k^3+2k^3\geq$

$$k\geq 4\Rightarrow k-1\geq 3\Rightarrow (k-1)^3\geq 27\Rightarrow k^3-3k^2+3k-1\geq 27\Rightarrow k^3\geq 3k^2-3k+28$$

$$\geq k^3+2\cdot(3k^2-3k+28)\geq k^3+3k^2+3k+1+\underbrace{3k^2-9k+55}_{>0}>(k+1)^3.$$

$k\geq 4$ da $3k^2-9k=3k(k-3)>0$ ning o'rinli ekanligi kelib chiqadi.

2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.6.1) tenglikning ixtiyoriy $n\geq 4$ natural sonda bajarilishi kelib chiqadi.

(5.6.3) tengsizlikning *n* natural son qiymatlarida bajarilishini tekshiramiz:

$n = 1$ da $2^1=5\cdot 1-3=2$; $n = 2$ da $2^2=4<5\cdot 2-3=7$;

$n = 3$ da $2^3=8<5\cdot 3-3=12$; $n = 4$ da $2^4=16<5\cdot 4-3=17$;

$n = 5$ da $2^5 = 32 > 5 \cdot 5 - 3 = 22$.

Hosil qilingan yuqoridagi tengsizlikdan (5.6.3) tengsizlik n = 1, 2, 3, 4 da bajarilmaydi. Lekin n=5 bajariladi

(5.6.3) tengsizlik ixtiyoriy $n \geq 5$ da bajariladi deb faraz qilamiz. Ushbu farazni matematik induksiya metodi bilan isbotlaymiz.

1-teorema. $n = 5$ da $2^5 = 32 > 5 \cdot 5 - 3 = 22$ ga ega bo'lamiz. 1-teorema isbotlandi.

2-teorema. $n=k$ da $2^k > 5k - 3$, $k \geq 5$ tengsizlikning to'g'ri ekanliigi berilgan bo'lsin.

U holda $n = k+1$ da $2^{k+1} > 5(k+1) - 3$, $k \geq 5$ o'rinli ekanligini isbotlash lozim.

Isbotlash. $2^{k+1} = 2^k \cdot 2 > (5k-3) \cdot 2 = 5k + 5 - 3 + 5k - 3 - 5 =$

$= 5(k+1) - 3 + \underbrace{5k - 8}_{>0} > 5(k+1) - 3.$ 2-teorema isbotlandi. Matematik induksiya prinsipiga ko'ra (5.6.1) tenglikning ixtiyoriy $n \geq 5$ natural sonda bajarilishi kelib chiqadi.

7-masala. $\cos x \cdot \cos 2x \cdot \cos 4x \cdot ... \cdot \cos 2^n x$ ko'paytmani $\sin x, \sin 2^{n-1} x$ orqali ifodalang.

Gipoteza o'rnatamiz. n = 1 da: $\cos x = \dfrac{2\sin x \cdot \cos x}{2\sin x} = \dfrac{\sin 2x}{2\sin x}$;

n=2 da:

$$\cos x \cdot \cos 2x = \frac{\sin 2x}{2\sin x} \cdot \frac{2\sin 2x \cdot \cos 2x}{2\sin 2x} = \frac{\sin 4x}{2 \cdot 2\sin x} = \frac{\sin 2^2 x}{2^2 \sin x}. \qquad (5.7.0)$$

Ikkita hosil bo'lgan natijani taqqoslab, quyidagi gipotezani isbotlaymiz:

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdot ... \cdot \cos 2^n x = \frac{\sin 2^{n+1} x}{2^{n+1} \sin x}, \forall n \in N. \qquad (5.7)$$

1-teorema. $n=1$ da ushbu (5.7) tenglikning isboti (5.7.0) tenglikdan kelib chiqadi.

2-teorema. $n = k$ da ilgari surilgan gipoteza rost deb faraz qilamiz:

$\cos x \cdot \cos 2x \cdot \cos 4x \cdot ... \cdot \cos 2^k x = \dfrac{\sin 2^{k+1} x}{2^{k+1} \sin x}$. Bu tenglik $n=k+1$ da bajariladi.

Isbotlash. $\cos x \cdot \cos 2x \cdot \cos 4x \cdot ... \cdot \cos 2^k x \cdot \cos 2^{k+1} x =$

$= \dfrac{\sin 2^{k+1} x}{2^{k+1} \sin x} \cdot \dfrac{2 \cdot \cos 2^{k+1} x}{2} = \dfrac{\sin 2^{k+2} x}{2^{k+2} \sin x}$. 2-teorema isbotlandi.

1- va 2-teoremalardan (5.6.3) tenglikning ixtiyoriy n natural sonda bajarilishi kelib chiqadi.

6-§. Paskal uchburchagi.

Faraz qilaylik, a va b – haqiqiy sonlar. Turli n natural sonda $(a+b)^n$ ifodani ko'rib chiqamiz:

$n = 0$ da: $(a+b)^0 = 1.$

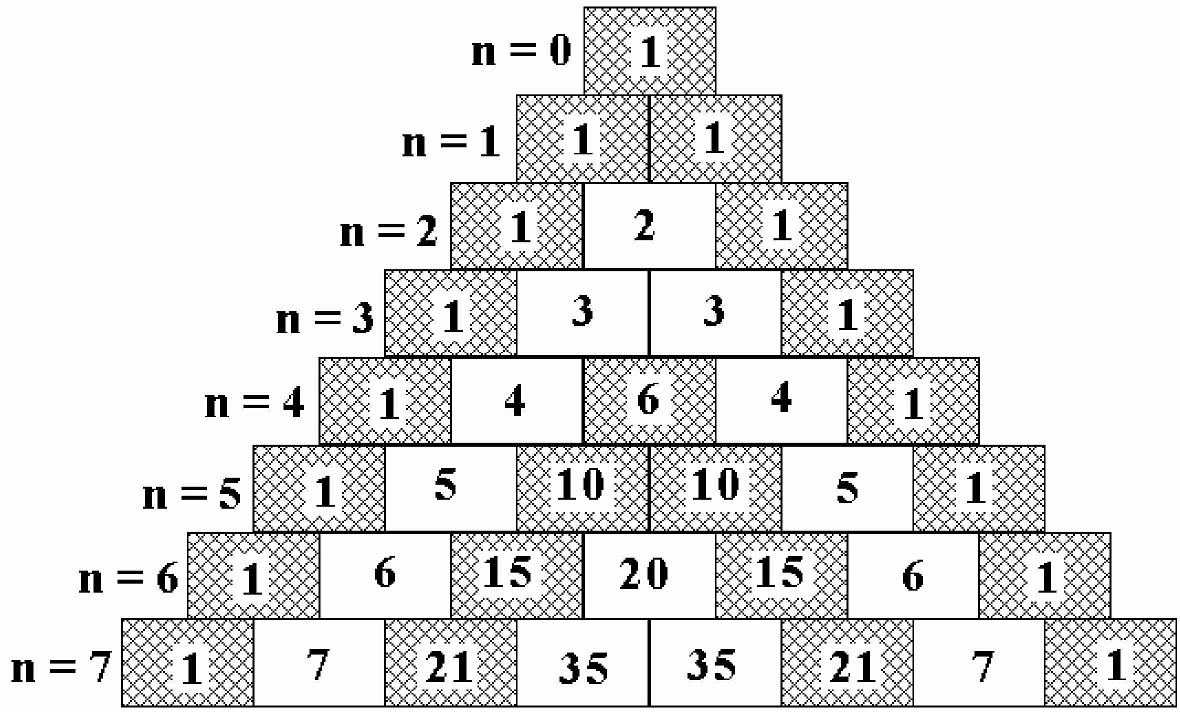
$n = 1$ da: $(a+b)^1 = (1\,a + 1\,b).$

$n = 2$ da: $(a+b)^2 = 1\,a^2 + 2\,a\,b + 1\,b^2.$

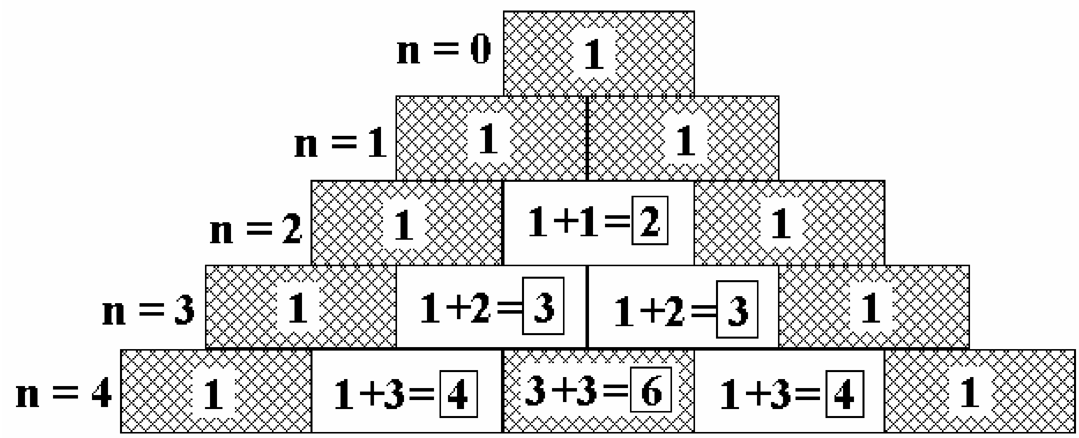
$n = 3$ da: $(a+b)^3 = 1\,a^3 + 3\,a^2\,b + 3\,a\,b^2 + 1\,b^3.$

$n = 4$ da: $(a+b)^4 = 1\,a^4 + 4\,a^3\,b + 6\,a^2\,b^2 + 4\,a\,b^3 + 1\,b^4.$

Agar koeffitsentlarni $a^{n-k}b^k, k = 0,1,2,...,n$ darajada yozsak va ularni n = 1, 2, 3, 4 da quyidagi uchburchakdagi sonlar bilan taqqoslaymiz:



U holda ushbusonlarning mos ekanligini hosil qilamiz. Yuqoridagi rasmda uchburchak quyidagi qoyda bo'yicha tashkil etiladi. Har bir qatornig chekka qismida 1 raqami joylashgan,har bir navbatdagi son oldingi qatorda joylashgan ikkita sonning yig'indisiga teng. Ushbu qoyida asosida



ushbu uchburchakning yangi qatorini ketma-ket yozish mumkin. Bunday ta'rif 1665 yil fransuz matematigi B.Paskalning "Arifmetik uchburchak haqida traktat" olimdan so'ng chop etildi. Uchburchakning shunga o'xshash variyantlari italiyan matematigi N.Tartalya, shu davrga qadar bir necha asr oldin o'rta osiyolik olim va shoir Umar Xayyom asarlarida bayon etilgan.

Fanning bir yo'nalishlaridan biri kombinatorika bo'lib, uning asosiy mohiyati quyidagicha: *m* elementlardan iborat nechta to'plam qismi mavjud va hech bo'lmaganda bittasi farqli bolgan, *n* elementlardan iborat *A* to'plamdan bitta elementni olib tashlash mumkin. Bunday to'plam qismi *n* ta elementlardan iborat *m* ta elementl ar bo'yicha **guruh** deyiladi, ularning soni C_n^m bilan belgilanadi va

$$C_n^m = \frac{n!}{m!(n-m)!}, \text{ m! } = 1\cdot 2\cdot \ldots \cdot \text{m}, \ n! = 1\cdot 2\cdot \ldots \cdot \text{n}, \ 0! = 1. \qquad (6.1)$$

formula bo'yicha hisoblanadi. Bu son Pascal uchburchagidagi sonlar bilan uzviy bog'liq. Haqiqatdan ham, $n = 0$ da $C_0^0 = 1$ ga ega bo'lamiz. Ushburchakning yuqoridagi qatori (nolinchi) bitta sondan iborat. Navbatdagi qator – ikkita sondan iborat: $C_1^0 = C_1^1 = 1$. To'rtinchi qator 5 ta sondan iborat: $C_4^0 = C_4^4 = 1,\ C_4^1 = C_4^3 = 4,\ C_4^2 = 6$.

7-§. Nyuton binomi formulasi.

7.1 masala. Nyuton binomi formulasini isbotlang

$$(a+b)^n=\sum_{m=0}^{n} C_n^m\, a^{n-m} b^m, \qquad (7.1)$$

bu yerda $C_n^m=\dfrac{n!}{m!(n-m)!}$.

1-teorema. $n=1$ da (7.1) tenglikning chap qismi $(a+b)^1$ ga teng;

Ushbu tenglikning o'ng qismi: $\sum_{m=0}^{1} C_1^m a^{1-m} b^m=\dfrac{1!}{0!1!}a+\dfrac{1!}{1!0!}b=a+b.$

2-teorema. $n=k$ da (7.1) tenglikning o'rinli ekanligi berilgan

$$(a+b)^k = \sum_{m=0}^{k} C_k^m\, a^{k-m} b^m.$$

$n=k+1$ da tenglikning o'rinli ekanligini isbotlaymiz:

$$(a+b)^{k+1}=\sum_{m=0}^{k+1} C_{k+1}^m\, a^{k+1-m} b^m.$$

Haqiqatdan: $(a+b)^{k+1}=(a+b)\cdot(a+b)^k=(a+b)\cdot\sum_{m=0}^{k} C_k^m\, a^{k-m} b^m=$

$$=\sum_{m=0}^{k} C_k^m\, a^{k+1-m} b^m+\sum_{m=0}^{k} C_k^m\, a^{k-m} b^{m+1}=$$

Ikkinchi qo'siluvchida yig'indini $m=1$ da boshlaymiz

Ikkinchi qo'shiluvchida m ning o'rniga m-1 olinadi.

$$=\sum_{m=0}^{k} C_k^m\, a^{k+1-m} b^m+\sum_{m=1}^{k+1} C_k^{m-1}\, a^{k-(m-1)} b^{m-1+1}=$$

$$=\sum_{m=0}^{k} C_k^m\, a^{k+1-m} b^m+\sum_{m=1}^{k+1} C_k^{m-1}\, a^{k+1-m} b^m=$$

Birinchi yig'indida birinchi qo'shiluvchini alohida yozamiz,

ikkinchi yig'indida – oxirgi qo'shiluvchini.

Ikkala yig'indi $m=1$ dan $m=k$ bo'yicha yi'giladi.

a, b sonlarning darajalari yig'indisi ushbu belgilar bilan mos tushadi.

$$= C_k^0 a^{k+1} + \sum_{m=1}^{k} (C_k^m + C_k^{m-1}) a^{k+1-m} b^m + C_k^k b^{k+1} =$$

$$C_k^0 = C_{k+1}^0 = C_k^k = C_{k+1}^{k+1} = 1.$$

$$C_k^m + C_k^{m-1} = \frac{k!}{m!(k-m)!} + \frac{k!}{(m-1)!(k-(m-1))!} =$$

$$= \frac{k!}{m(m-1)!(k-m)!} + \frac{k!}{(m-1)!(k+1-m)(k-m)!} =$$

$$= \frac{k!(k+1-m+m)}{m!(k+1-m)!} = \frac{(k+1)!}{m!((k+1)-m)!} = C_{k+1}^m \cdot$$

$$= \sum_{m=0}^{k+1} C_{k+1}^m a^{k+1-m} b^m .$$

2-teorema isbotlandi. 1 va 2 teoremalardan (7.1) tenglikning ixtiyoriy n uchun o'rinli ekanligi kelib chiqadi.

7.2-teorema. Leybnis formulasi. Faraz qilaylik, $u(x), v(x)$ funktsiyalar E to'olamda uzliksiz n-tartibli hosilaga ega. U holda ushbu funktsiyalarning n-tartibli hosilalari uchun Leybnis formulasi o'rinlidir:

$$(u(x) \cdot v(x))^{(n)} = \sum_{m=0}^{n} C_n^m (u(x))^{(n-m)} \cdot (v(x))^{(m)}, \quad n \in N. \qquad (7.2)$$

Isbotlash. $u(x) = u, v(x) = v$ belgilashlarni kiritamiz.

1-teorema. $n = 1$ da (7.2) tenglikning chap qismi:

$(u \cdot v)^{(1)} = u^{(1)} \cdot v + u \cdot v^{(1)}$ ga teng;

Ushbu tenglikning o'ng qismi:

$\sum_{m=0}^{1} C_1^m u^{(1-m)} v^{(m)} = \frac{1!}{0!1!} u^{(1-0)} v^{(0)} + \frac{1!}{1!0!} u^{(1-1)} v^{(1)} = u^{(1)} \cdot v + u \cdot v^{(1)}$ ga teng.

2-teorema. $n = k$ da (7.2) tenglik o'rinli ekanligi berilgan:

$$(u \cdot v)^{(k)} = \sum_{m=0}^{k} C_k^m u^{(k-m)} \cdot v^{(m)}.$$

$n = k+1$ da tenglikning o'rinli ekanligini isbotlaymiz:

$$(u \cdot v)^{(k+1)} = \sum_{m=0}^{k+1} C_{k+1}^{m} u^{(k+1-m)} \cdot v^{(m)}.$$

Haqiqatdan: $(u \cdot v)^{(k+1)} = \left((u \cdot v)^{(k)}\right)^{(1)} = \left(\sum_{m=0}^{k} C_k^m u^{(k-m)} \cdot v^{(m)}\right)^{(1)} =$

| Chiziqli kombinatsiya hosilasi hosilalarning chiziqli kombinatsiyasiga teng |

$$= \sum_{m=0}^{k} C_k^m \left(u^{(k-m)} \cdot v^{(m)}\right)^{(1)} =$$

| Differentsiyalash qoydasiga ko'ra,ikkita funktsiyaning ko'paytmasiga ega: |

$$= \sum_{m=0}^{k} C_k^m u^{(k+1-m)} \cdot v^{(m)} + \sum_{m=0}^{k} C_k^m u^{(k-m)} \cdot v^{(m+1)} =$$

| Birinchi yig'indida birinchi qo'shiluvchini alohida yozamiz, ikkinchi yig'indida – oxirgi qo'shiluvchini. |

$$= \sum_{m=1}^{k} C_k^m u^{(k+1-m)} \cdot v^{(m)} + C_k^0 u^{(k+1-0)} \cdot v^{(0)} +$$

$$+ \sum_{m=0}^{k-1} C_k^m u^{(k-m)} \cdot v^{(m+1)} + C_k^k u^{(k-k)} \cdot v^{(k+1)} =$$

| Ikkinchi qo'shiluvchida yig'indi $m = 1$ dan boshlanadi. |

Bunday holda m ning o'rniga m-1 ni olamiz ya'ni quyidagiga ega bo'lamiz:

$$\sum_{m=0}^{k-1} C_k^m u^{(k-m)} \cdot v^{(m+1)} = \sum_{m=1}^{k} C_k^{m-1} u^{(k-(m-1))} \cdot v^{(m-1+1)}.$$

$$= \sum_{m=1}^{k} \left(C_k^m + C_k^{m-1}\right) u^{(k+1-m)} \cdot v^{(m)} + C_k^0 u^{(k+1)} \cdot v + C_k^k u \cdot v^{(k+1)} =$$

(7.1) masaladagiga o'xshash bajariladi:

$$C_k^0 = C_{k+1}^0 = C_k^k = C_{k+1}^{k+1} = 1; \quad C_k^m + C_k^{m-1} = C_{k+1}^m.$$

$$= \sum_{m=1}^{k+1} C_{k+1}^m u^{(k+1-m)} \cdot v^{(m)}.$$

2-teorema isbotlandi. 1- va 2- teoremalardan (7.2) tenglikning ixtiyoriy n uchun o'rinli ekanligi kelib chiqadi.

8-§. Natural sonlarning bo'linishini isbotlash

8.1-masala. $\forall n \in N$ da quidagilarni isbotlang:

1. $3^{2n+1} + 2^{n+2}$ ifoda 7 ga bo'linadi; 2. $n^3 + 5n$ ifoda 6 ga bo'linadi.

1. 1-teorema. $n = 1$ da $3^3 + 2^3 = 27 + 8 = 35,$ ga ega bo'lamiz va 7 ga bo'linadi.

2-teorema. $3^{2k+1} + 2^{k+2}$ ifodaning 7 ga bo'linishi berilgan. Nujno dokazat' chto $3^{2k+3} + 2^{k+3}$ ifodaning 7 ga bo'linishini isbotlash lozim.

Isbotlash. $3^{2(k+1)+1} + 2^{k+1+2} = 3^{2k+1} \cdot 9 + 2^{k+2} \cdot 2 + 9 \cdot 2^{k+2} - 9 \cdot 2^{k+2} =$

$$= 9 \cdot \left(3^{2k+1} + 2^{k+2}\right) + 2^{k+2}(2-9).$$

Har bir ikkita qo'shiluvchi 7 ga bo'linadi. 2-teorema isbotlandi.

2. 1-teorema. $n = 1$ da $1^3 + 5 \cdot 1 = 6.$ ega bo'lamiz. Bunda 6 soni 6 ga bo'linadi.

2-teorema. $k^3 + 5k$ ifodaning 6 ga bo'linishi berilgan. $(k+1)^3 + 5(k+1)$ ifodaning 6 ga bo'linishini isbotlash lozim.

Isbotlash. $(k+1)^3 + 5(k+1) = k^3 + 5k + 3k(k+1) + 6$.

$k^3 + 5k$ yig'indi 6 ga faraz bo'yicha bo'linadi. Tak kak proizvedenie $k(k+1)$ ko'paytma k natural sonda 2 ga bo'linadi, u holda $3k(k+1)$ ifoda 6 ga bo'linadi.

8.2-masala. $\forall n \in N$ da quyidagilarni isbotlang:

1. $15^n + 6$ ifoda 7 ga bo'linadi; 2. $n^3 - n$ ifoda 3 ga bo'linadi;

1. 1-teorema. $n = 1$ da $15^1 + 6 = 21$ ega bo'lamiz,ya'ni 21 soni 7 ga bo'linadi.

2-teorema. $15^k + 6$ ifodaning 7 ga bo'linishi berilgan. $15^{k+1} + 6$ ifodaning 7 ga bo'linishini isbotlash lozim.

Isbotlash. $15^{k+1} + 6 = 15^k \cdot 15 + 6 =$

$$=\left(15^k+6\right)\cdot 15-15\cdot 6+6=\underbrace{\left(15^k+6\right)\cdot 15}_{7\text{ ga bo'linadi}}-\underbrace{\left(15-1\right)\cdot 6}_{7\text{ ga bo'linadi}}.$$

2-teorema isbotlandi.

2. 1-teorema. $n = 1$ da $1^3-1=0$ ga ega bo'lamiz . Bundan 0 soni 3 ga bo'linadi.

2-teorema. k^3-k ifodaning 3 ga bo'linishi berilgan. $(k+1)^3-(k+1)$ ifodaning 3 ga bo'linishini isbotlash lozim.

Isbotlash. $(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1=$

$$=\underbrace{\left(k^3-k\right)}_{3\text{ga bo'linadi}}-\underbrace{\left(k^2+k\right)\cdot 3}_{3\text{ga bo'linadi}}.$$

2-teorema isbotlandi.

8.3-masala. Har bir n natural son uchun n^7-n ga karrali ekanligini isbotlang.

1-teorema. $n = 1$ da $1^7-1=0$ son 7 ga karrali.

2-teorema. $n = k$ da k^7-k soni 7 ga bo'linishi berilgan.

$n = k+1$ da $(k+1)^7-(k+1)$ sonning 7 ga bo'lishini isbotlash losim.

Isbotlash. $n = 7$ satrdan foydalanib, Pascal uchburchgida (k+1) yig'indining 7-darajasi uchun quyidagilarni hosil qilamiz:

$$(k+1)^7-(k+1)=k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1-k-1=$$

$$=\underbrace{k^7-k}_{7\,ga\,bo'li\,nadi}+\underbrace{7k^6+21k^5+35k^4+35k^3+21k^2+7k}_{7\,ga\,bo'li\,nadi}.$$

2-teorema isbotlandi.

8.4-masala. Har bir n natural son uchun 4^n+15n ifodaning 9 ga 1 qoldiqda bo'linishini isbotlang.

Isbotlash. Masala sharti $4^n+15n-1$ sonning 9 ga karrali shartiga teng kuchli.

1-teorema. $n = 1$ da $4^1 + 15 \cdot 1 - 1 = 18$ ifoda 9 ga karrali. 1-teorema isbotlandi.

2-teorema. $n = k$ da $4^k + 15k - 1$ sonning 9 ga bo'linishi berilgan. $n = k+1$ da $4^{k+1} + 15(k+1) - 1$ sonning 9 ga bo'linishini isbotlash lozim.

Isbotlash. $4^{k+1} + 15(k+1) - 1 = 4\left(4^k + 15k - 1 - 15k + 1\right) + 15(k+1) - 1 =$

$$4\left(4^k + 15k - 1\right) - 60k + 4 + 15k + 15 - 1 = 4\underbrace{\left(4^k + 15k - 1\right)}_{9\, ga\ bo'linadi} + \underbrace{(-45k + 18)}_{9\, ga\ bo'linadi}.$$

9-§. Turli masalalar

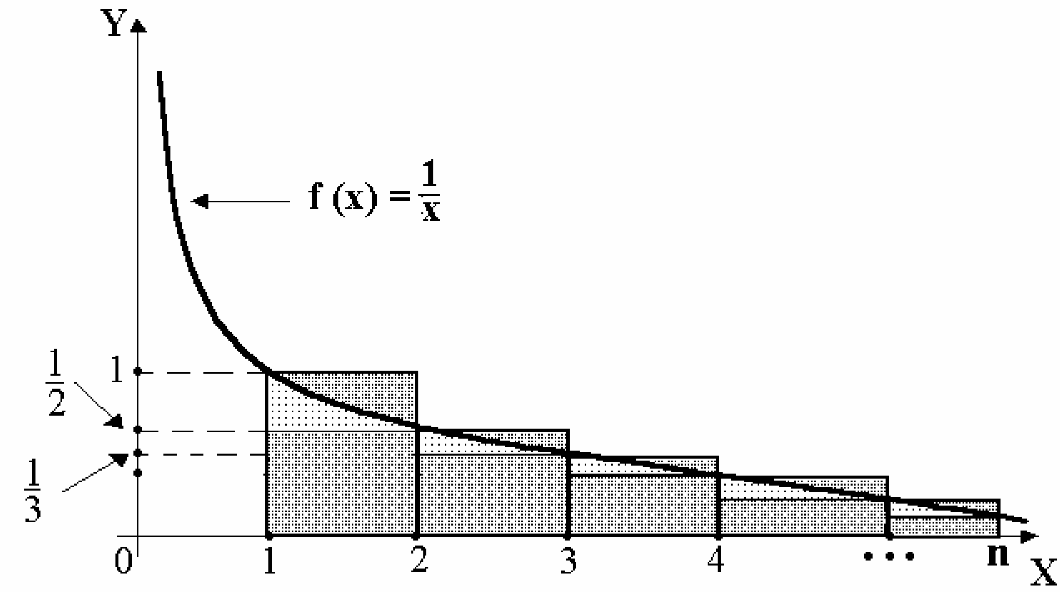
9. 1-masala. Tengsizlikni isbotlang

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \ln n + 1, \ \forall\, n \in N, \ n \geq 2. \qquad (9.1)$$

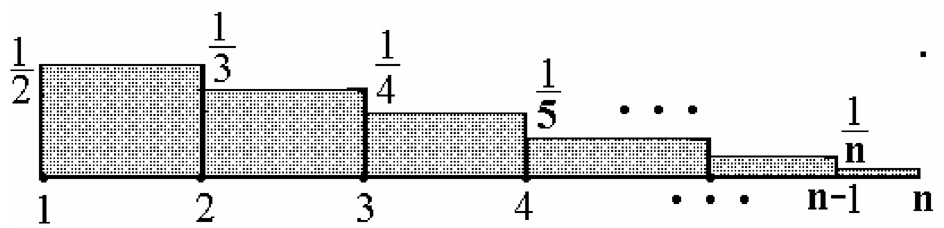
Isbotlash. Dastlab quyidagi tengsizlikni isbotlaymiz:

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}, \ \forall\, n \in N, \ n \geq 2.$$

$\int_1^n \frac{dx}{x} = \ln n$ integral $f(x) = \frac{1}{x}$ egri chiziqning $[1, n]$ oraliqda chegaralangan yuzasiga teng.

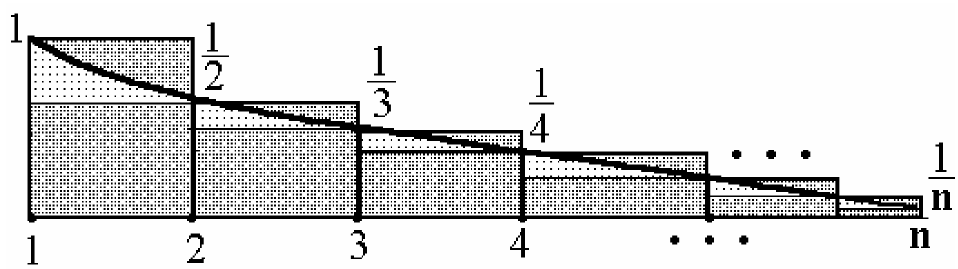


Bu yiza to'g'ri to'rtburchallar yuzalarining birlashgan yuzasidan kattadir.



$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}<\ln n \quad \forall\, n\in N,\ n\ge 2. \tag{9.1.1}$$

Bu yuza esa to'rtburchallar yuzasining birlashgan yuzasidan kichikdir.



$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-1}>\ln n, \quad \forall\, n\in N,\ n\ge 2. \tag{9.1.2}$$

Matematik induksiya metodi bilan (9.1.1) va (9.1.2) tengsizlikni isbotlaymiz:

1-teorema. Uchta rasm va [0, 1] oraliqda aniqlangan integral xossasidan quyidagi tengsizlik kelib chiqadi $\frac{1}{2}\cdot 1<\int_1^2\frac{dx}{x}=\ln 2<1\cdot 1$. Bundan n = 2 da

(9.1.1) va (9.1.2) tengsizliklarning o’rinli ekanligi tasdiqlanadi. 1-teorema isbotlandi.

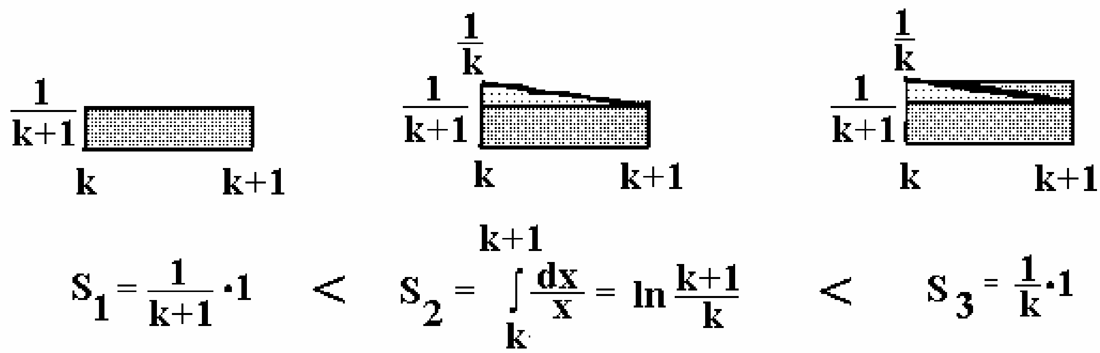
2-teorema. $n = k$ quyidagi tengsizlikning bajarilishi berilgan

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}<\ln k<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1}, \quad k \geq 2.$$

$n = k+1$ da ushbu tengsizlikning bajarilishini isbotlash lozim:

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}+\frac{1}{k+1}<\ln(k+1)<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1}+\frac{1}{k}, \quad k \geq 2. \qquad (9.1.3)$$

Isbotlash. Quyidagi shakllarning S_1 , S_2 , S_3 yuzalarini taqqoslaymiz:



(9.1.3) tengsizlikning chap qismini taqqoslaymiz:

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}+\frac{1}{k+1} < \ln k+\frac{1}{k+1}\cdot 1 < \ln k + \int_{k}^{k+1}\frac{dx}{x} =$$

$$=\ln k+\ln(k+1)-\ln k=\ln(k+1), \quad k \geq 2.$$

(9.1.3) tengsizlikning o’ng qismini taqqoslaymiz:

$$\ln(k+1)=\ln k+\ln(k+1)-\ln k=$$

$$=\ln k+\underbrace{\int_{k}^{k+1}\frac{dx}{x}}_{<\frac{1}{k}}<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1}+\frac{1}{k}, \quad k \geq 2.$$

2-teorema isbotlandi. 1 va 2-teoremalarning isbotidan ixtiyoaiy natural $n \geq 2$ uchun tasdiq o'rinli ekanligi kelib chiqadi.

9. 2-masala. Ushbu qonuniyatning tarqalish muhiti qanday:

$$1=\frac{1\cdot 1}{1};\quad 121=\frac{22\cdot 22}{1+2+1};\quad 12321=\frac{333\cdot 333}{1+2+3+2+1};$$

$$1234321=\frac{4444\cdot 4444}{1+2+3+4+3+2+1}\ \mathbf{?}$$

Gipotezani isbotlaymiz. Har bir *n* natural son uchun quyidagi tenglik to'g'ri:

$$1\cdot 2...(n-1)n(n-1)...2\cdot 1=\frac{\overbrace{nn...n}^{n}{}^2}{1+2+...+(n-1)+n+(n-1)+...+2+1}. \qquad (9.2)$$

Matematik induksiya metodi yordamida gipotezani isbotlaymiz.

1-teorema. $n = 1$ da $1=\frac{1\cdot 1}{1}$ ga ega bo'lamiz. 1-teorema isbotlandi.

(9.2) tenglikning o'ng qismi maxraji yig'indisini topamiz:

$$1+2+...+(n-1)+n+(n+1)+n+(n-1)+...+2+1=$$

$$=\frac{n\cdot(n-1)}{2}+n+\frac{n\cdot(n-1)}{2}=n^2. \qquad \text{U holda}$$

$$\frac{\overbrace{nn...n}^{n\ ta}\cdot\overbrace{nn...n}^{n\ ta}}{1+2+...+(n-1)+n+(n+1)+n+(n-1)+...+2+1}=\frac{\overbrace{nn...n}^{n\ ta}\cdot\overbrace{nn...n}^{n\ ta}}{n^2}=$$

$=\frac{\overbrace{nn...n}^{n\ ta}}{n}\cdot\frac{\overbrace{nn...n}^{n\ ta}}{n}=\overbrace{11...1}^{n\ ta}\cdot\overbrace{11...1}^{n\ ta}$. Natijada quyidagi tenglikni isbotlash lozim:

$$12...(n-1)n(n-1)...21=\overbrace{11...1}^{n\ ta}\cdot\overbrace{11...1}^{n\ ta} \qquad (9.2.1)$$

2-teorema. *n*=*k* da (9.2.1) tenglikning bajarilishi berilgan.

$$12...(k-1)k(k-1)...21=\overbrace{11...1}^{k\ ta}\cdot\overbrace{11...1}^{k\ ta}.$$

$n = k+1$ da (9.2.1) tenglikning bajarilishi ni isbotlash lozim:

$$12...k(k+1)k...21=\overbrace{11...1}^{k+1\,ta}\cdot\overbrace{11...1}^{k\,ta}.$$

Isbotlash.

$$\underbrace{(11...1)}_{k+1\,ta}\cdot\underbrace{(11...1)}_{k\,ta}=(\underbrace{(11...1)}_{k\,ta}\cdot 10+1)^2=(\underbrace{(11...1)}_{k\,ta}\cdot 10)^2+2\cdot\underbrace{(11...1)}_{k\,ta}\cdot 10+1=$$

$$=12...(k-1)k\underbrace{(k-1)k...2100}_{k+1\,ta}+\underbrace{22...20}_{k+1\,ta}+1=$$

$=12...(k-1)k(k+1)k(k-1)...21$. 2-teorema isbotlandi.

9.3-masala. Ixtiyoriy juft n natural son uchun tengsizlikni isbotlang:

$$x-\frac{x^3}{3!}+...-\frac{x^{2n-1}}{(2n-1)!}\le \sin x\le x-\frac{x^3}{3!}+...+\frac{x^{2n+1}}{(2n+1)!},\ 0\le x\le\frac{\pi}{2}. \qquad (9.3)$$

Isbotlash. Isbotlashda quyidagi ma'lumotlar talab etiladi:

$\left[0,\frac{\pi}{2}\right]$ kesmada $\sin x, \cos x$ funktsiyalar quyidagi shartlarni qanoatlantiradi:

a) Ushbu funktsiyalarning yuqori chegarasi o'zgaruvchi bo'lgan integrallar:

$$\int_0^x \sin t dt=-\cos x\ ,\ \int_0^x \cos t dt=\sin x; \qquad (9.3.1)$$

b) $$0\le\sin x\le 1,\ 0\le\cos x\le 1. \qquad (9.3.2)$$

1-teorema. n = 2 da quyidagi tengsizliklarning o'rinli ekanligini isbotlaymiz :

$$x-\frac{x^3}{3!}\le\sin x\le x-\frac{x^3}{3!}+\frac{x^5}{5!},\ 0\le x\le\frac{\pi}{2}$$

$x\in\left(0,\frac{\pi}{2}\right)$ da $\sin x=\int_0^x\cos t dt\le\int_0^x 1dt=x$ ga ega bo'lamiz, ya'ni: $\sin x\le x$. x ni

t ga almashtirish va ushbu tengsizlikni t bo'yicha 0 dan $x\in\left(0,\frac{\pi}{2}\right)$gacha

tekshirganda $\int_0^x\sin t dt\le\int_0^x t dt=\frac{x^2}{2}, \int_0^x\sin t dt=1-\cos x$ ni hosil qilamiz. U holda

$x \in \left(0, \dfrac{\pi}{2}\right)$ da $\cos x \geq 1 - \dfrac{x^2}{2}$ tengsizlikka teng kuchli $1 - \cos x \leq \dfrac{x^2}{2}$ tengsizlik bajariladi. X ni T ga almashtiruvchi va ushbu tengsizlikni T bo'yicha 0 dan $x \in \left(0, \dfrac{\pi}{2}\right)$ gacha integrallovchi protseduragani takrorlaymiz:

$$\int_0^x \cos t\,dt \geq \int_0^x \left(1 - \frac{t^2}{2}\right)dt = x - \frac{x^3}{3\cdot 2}, \quad \int_0^x \cos t\,dt = \sin x.$$

Shuning uchun

$$x - \frac{x^3}{3!} \leq \sin x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{9.3.4}$$

Yana bir bor shuni takrorlaymiz:

$$\int_0^x \left(t - \frac{t^3}{3!}\right)dt \leq \int_0^x \sin t dt = 1 - \cos x, \quad \int_0^x \left(t - \frac{t^3}{3!}\right)dt = \frac{x^2}{2} - \frac{x^4}{4!}.$$

Oldingiga o'xshash quyidagiga ega bo'lamiz:

$$1 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{4!} \quad \Leftrightarrow \quad \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{4!}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Tortinchi marotaba X ni T ga almashtirish va ushbu tengsizlikni T bo'yicha 0 dan $x \in \left(0, \dfrac{\pi}{2}\right)$ gacha integrallab, quyidagiga ega bo'lamiz:

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}. \tag{9.3.5}$$

(9.3.4) va i (9.3.5) tengsizliklardan quyidagi hosil bo'ladi:

$$x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

1-teorema isbotlandi.

2-teorema. $n = k$ (k - juft) da (9.3) tengsizlikning bajarilishi berilgan::

$$x-\frac{x^3}{3!}+...-\frac{x^{2k-1}}{(2k-1)!}\le \sin x\le x-\frac{x^3}{3!}+...+\frac{x^{2k+1}}{(2k+1)!},\ 0\le x\le \frac{\pi}{2}. \quad (9.3.6)$$

$n=k+2$ da ushbu tengsizlikninig bajarilishini isbotlash lozim.

$$x-\frac{x^3}{3!}+...-\frac{x^{2k+3}}{(2k+3)!}\le \sin x\le x-\frac{x^3}{3!}+...+\frac{x^{2k+5}}{(2k+5)!},\ 0\le x\le \frac{\pi}{2}. \quad (9.3.7)$$

Isbotlash. 1-teoremani isbotiga o'xshash to'rt marotaba X ni T ga almashtirish va T bo'yicha 0 dan $x\in\left(0,\frac{\pi}{2}\right)$ gacha mos tengsizlikni integrallash protsedurasini bajarish kerak. (9.3.6) tengsizlikning o'ng qismida x ni t ga almashtirish va T bo'yicha 0 dan $x\in\left(0,\frac{\pi}{2}\right)$ gacha ushbu tengsizlikni integrallab quyidagini hosil qilamiz:

$$1-\cos x=\int_0^x \sin t\,dt\le \int_0^x\left(t-\frac{t^3}{3!}+...+\frac{t^{2k+1}}{(2k+1)!}\right)dt=\frac{x^2}{2}-\frac{x^4}{4!}+...+\frac{x^{2k+2}}{(2k+2)}.$$

$\Rightarrow\ \cos x\ge 1-\frac{x^2}{2}+\frac{x^4}{4!}-...-\frac{x^{2k+2}}{(2k+2)}$. Ikkinchi marotaba X ni T ga almashtirish va T bo'yicha 0 dan $x\in\left(0,\frac{\pi}{2}\right)$ gacha integtallab, quyidagiga ega bo'lamiz:

$$\sin x=\int_0^x \cos t\,dt\ge \int_0^x\left(1-\frac{t^2}{2}+...-\frac{t^{2k+2}}{(2k+2)!}\right)dt=x-\frac{x^3}{3!}+...-\frac{t^{2k+3}}{(2k+3)!}.$$

Bu quyidagiga teng kuchli

$$\sin x\ge x-\frac{x^3}{3!}+...-\frac{t^{2k+3}}{(2k+3)!},\quad x\in\left(0,\frac{\pi}{2}\right). \qquad (9.3.8)$$

(9.3.6) tengsizlikning chap qismi isbotlandi. (9.3.8) tengsizlikni uchinchi marotaba X ni T ga almashtirish va T bo'yicha $x\in\left(0,\frac{\pi}{2}\right)$ gacha integrallab, quyidagiga ega bo'lamiz:

$$1-\cos x=\int_0^x \sin t dt \geq \int_0^x \left(t-\frac{t^3}{3!}+...-\frac{t^{2k+3}}{(2k+3)!}\right)dt=\frac{x^2}{2}-\frac{x^4}{4!}+...-\frac{x^{2k+4}}{(2k+4)!}.$$

Bundan $\cos x \leq 1-\frac{x^2}{2}+\frac{x^4}{4!}-...+\frac{x^{2k+4}}{(2k+4)!}$. To'rtinchi marotaba ushbu

tengsizlikni X ni T ga almashtirish va T bo'yicha $x\in\left(0,\frac{\pi}{2}\right)$ gacha integrallab,

quyidagiga ega bo'lamiz:

$$\sin x=\int_0^x \cos t dt \leq \int_0^t \left(1-\frac{t^2}{2}+...+\frac{t^{2k+4}}{(2k+4)!}\right)dt=x-\frac{x^3}{3!}+...+\frac{x^{2k+5}}{(2k+5)!}.$$

Ya'ni quyidagi tengsizlik bajariladi:

$$\sin x \leq x-\frac{x^3}{3!}+...+\frac{x^{2k+5}}{(2k+5)!}.$$

(9.3.6) tengsizlikning o'ng qismi, ushbu tengsizlik va (9.3.8) tengsizlikdan (9.3.7) tengsizlik hosil qilinadi. 2-teorema isbotlandi.

9.4-masala. Har bir natural $n>1$ uchun $2^{2^n}+1$ soni 7 raqami bilan tugallanadi.

1-teorema. n=2 da 7 raqami bilan tugallanuvchi $2^{2^2}+1=17$ hosil bo'ladi.

2-teorema. $2^{2^k}+1$ sonning 7 raqami bilan tugallanishi berilgan. $2^{2^{k+1}}+1$ sonning 7 raqami bilan tugallanishini isbotlash losim.

Isbotlash. $2^{2^k}+1$ son 7 raqami bilan tugallanganligi sabab, 2^{2^k} soni 6 raqami bilan tugaydi. Agar son 6 raqami bilan tugallansa, u holda unung kvadrati ham 6 raqami bilan tugaydi. $n=k+1$ da quyidagini hosil qilamiz: $2^{2^{k+1}}=2^{2^k\cdot 2}=2^{2^k}\cdot 2^{2^k}$ va 6 raqami bilan tugallanadi. 2-teorema isbotlandi.

9.5-masala. Ixtiyoriy n naturl son uchun

$$\left(10^n+10^{n-1}+...+1\right)\cdot\left(10^{n+1}+5\right)+1$$

soni to'liq kvadrat ekanligini isbotlang.

Isbotlash. Ushbu masalani yechishda matematik induksiya metodidan foydalanmasdan bajarish mumkin. $10^n + 10^{n-1} + ... + 1$ yig'indi *q*=10 maxrajli *n*+1 hadlardan iborat geometric progressiyani anglatadi. U holda

$$\left(10^n + 10^{n-1} + ... + 1\right) \cdot \left(10^{n+1} + 5\right) + 1 = \left(\frac{10^{n+1} - 1}{10 - 1}\right) \cdot \left(10^{n+1} + 5\right) + 1 =$$

$$= \frac{\left(10^{n+1}\right)^2 - 10^{n+1} + 5 \cdot 10^{n+1} - 5 + 9}{9} = \frac{\left(10^{n+1}\right)^2 + 2 \cdot 10^{n+1} \cdot 2 + 2^2}{9} = \left(\frac{10^{n+1} + 2}{3}\right)^2$$

ixtiyoriy *n* natural son uchun. 2-teorema isbotlandi.

Eslatma.

Yuqorida ko'rilgan masalalardan matematik induksiya metodi yordamida juda katta sinfdagi turli masalalarni yechish ko'rsatiladi. Ushbu metodni qo'llashga doir ko'p masalalarni ko'rsatish mumkin. Masalan quyidagi tengsizlikni isbotlaylik:

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n-1} + \frac{1}{2n} < \frac{1}{2}, \forall n \in N : n \geq 2 .$$

2-teorema. $n = k$ da ushbu tengsizlikning bajarilishi berilgan. $n = k+1$ da bu tengsizlikning bajarilishini isbotlash lozim.

$S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n-1} + \frac{1}{2n}$ ni belgilaymiz. U holda quyidagi hosil qilinadi:

$$S_{k+1} = \underbrace{\frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{2k-1} + \frac{1}{2k}}_{= S_k} + \frac{1}{2k+1} + \frac{1}{2k+2} < \frac{1}{2} + \frac{1}{2k+1} + \frac{1}{2k+2} .$$

Hosil bo'lgan tengsizlikning chap qismi 0,5 dan kichik bo'lganligi sababli natijaga erishib bo'lmaydi. Matematik induksiya metodini qo'llash qiyinchiliklarga olib keldi. Bu tengsizlikni boshqa usul bilan isbotlanadi. Ixtiyoriy $n \geq 2$ uchun quyidagi navbatdagi *n* -1 tengsizlik va tenglik bajariladi

$$\frac{1}{n+1} > \frac{1}{2n}, \quad \frac{1}{n+2} > \frac{1}{2n}, \quad \ldots \quad \frac{1}{2n-1} > \frac{1}{2n}, \quad \frac{1}{2n} = \frac{1}{2n}.$$

Ushbu tengsizlik va tengliklarni qo'shib talab etilayatgan quyidagi tengsizlikni hosil qilamiz

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}.$$

Mustaqil yechish uchun masalalar.

1. Arifmetik progressiyaning n- hadi quyidagi formula bilan hisoblanishini isbotlang

$a_n = a_1 + d(n-1)$, bu yerda a_1 – birinchi had, d – progressiya ayirmasi.

2. Geometrik progressiyaning n-hadi quyidagi formula bilan hisoblanishini isbotlang.

$a_n = a_1 q^{n-1}$, bu yerda a_1 – birinchi had, q – progressiya maxraji.

Quyidagilarni isbotlang:

1. $n^3 + (n+1)^3 + (n+2)^3, \forall n \in N,$ 9 ga bo'linadi.

2. $3^n - 2^n \geq n, \forall n \in N$.

3. $\left(\dfrac{n^2+1}{n^2}\right)^n \geq \dfrac{n+1}{n}, \forall n \in N$.

4. $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}, \forall n \in N$.

5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \dfrac{n \cdot (n+1) \cdot (n+2)}{3}, \forall n \in N$.

6. $\dfrac{1}{4 \cdot 5} + \dfrac{1}{5 \cdot 6} + \dfrac{1}{6 \cdot 7} + ... + \dfrac{1}{(n+3) \cdot (n+4)} = \dfrac{n}{4 \cdot (n+4)}, \forall n \in N$.

7. $\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \dfrac{1}{7 \cdot 10} + ... + \dfrac{1}{(3n-2) \cdot (3n+1)} = \dfrac{n}{(3n+1)}, \forall n \in N$.

8. $\left(1 + \dfrac{1}{n}\right)^{n+1} > \left(1 + \dfrac{1}{n+1}\right)^{n+2}, \forall n \in N$.

9. $\left(1 - \dfrac{1}{n}\right)^n < \left(1 - \dfrac{1}{n+1}\right)^{n+1}, \forall n \in N$.

10. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n! = (n+1)! - 1, \forall n \in N$.

11. $2^2 + 6^2 + ... + (4n-2)^2 = \dfrac{4n(2n-1)(2n+1)}{3}, \forall n \in N$.

12. $\frac{7}{1\cdot 8}+\frac{7}{8\cdot 15}+\frac{7}{15\cdot 22}+...+\frac{7}{(7n-6)\cdot(7n+1)}=1-\frac{1}{(7n+1)}, \forall n\in N$.

13. $\frac{1}{4\cdot 8}+\frac{1}{8\cdot 12}+\frac{1}{12\cdot 16}+...+\frac{1}{4n\cdot(4n+4)}=\frac{1}{16}-\frac{1}{16\cdot(n+1)}, \forall n\in N$.

14. $\frac{5!}{0!}+\frac{6!}{1!}+...+\frac{(5+n)!}{n!}=\frac{(5+n+1)!}{6n!}, \forall n\in N$.

15. $(1+a)^n>1+na+\frac{n(n-1)}{2}a^2,\ \forall n\in N: n\geq 3,\ a>0$.

16. $(2n-1)!< n^{2n-1},\ \forall n\in N: n\geq 2$.

17. $1^3+3^3+5^3+...+(2n-1)^3=n^2(2n^2-1), \forall n\in N$.

18. $\frac{1}{\ln 3\cdot\ln 9}+\frac{1}{\ln 9\cdot\ln 27}+...+\frac{1}{\ln 3^{n-1}\cdot\ln 3^n}=\left(1-\frac{1}{n}\right)\frac{1}{\ln^2 3},\ \forall n\in N: n\geq 2$.

19. Ixtiyoriy n natural son uchun $11^{n+2}+12^{2n-1}$ ifodaning 133 ga karrali ekanligini isbotlang.

20. $6^{2n-2}+3^{n+1}+3^{n-1}, \forall n\in N$ sonning 11 ga bo'linishini isbotlang.

21. $10^{n+1}+2, \forall n\in N$ sonning 3 ga bo'linishini isbotlang.

22. 7 tiyindan katta bo'lgan ixtiyori pullar yig'indisini 3 va 5 tiyinliklar bilan qaytimsiz to'lash mumkinligini isbotlang.

23. Faraz qilaylik ketma-ketlik quyidagicha berilgan bo'lsin:

$x_1=2; x_2=3; x_n=3x_n-2x_{n-1},\quad \forall n\in N: n\geq 3,\ \ x_n=2^{n-1}+1, \forall n\in N$

formulaning to'g'riligini isbotlang.

24. Tengsizlikni isbotlang:

a). $\left|\sin\sum_{k=1}^{n}x_k\right|\leq\sum_{k=1}^{n}\sin x_k, \forall n\in N, x_k\in[0,\pi], k=1,...,n$.

b). $\left|\sin^n x\right|\leq\left|\sin x\right|, \forall n\in N, x\in(-\infty,+\infty)$.

c). $\sin^{2n} x + \cos^{2n} x \le 1, \forall n \in N$.

25. Tengsizlikni isbotlang: $x_n = \underbrace{\sqrt{5+\sqrt{5+...+.\sqrt{5}}}}_{n \quad ildiz} < \frac{1+\sqrt{21}}{2}, \quad \forall n \in N$.

26. Tenglikni isbotlang:

$$\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + ... + \sin\frac{n\pi}{3} = 2\sin\frac{n\pi}{6} \cdot \sin\frac{(n+1)\pi}{3}, \forall n \in N.$$

27. $\frac{1}{2} + \cos x + \cos 2x + ... + \cos nx = \dfrac{\sin\frac{2n+1}{2}x}{2\sin\frac{1}{2}x}, \; n \ne 2k\pi, \; \forall n \in N$.

$\frac{1}{3}, \frac{1}{15}, \frac{1}{35}, \; ..., \frac{1}{4n^2-1}, \; ...$ ketma-ketlik berilgan. Dastlabki n hadi yig'indisini toping.

28. Har qanday n natural son uchun $n^5 - n$ ifoda 5 ga karrali ekanligini isbotlang.

$k = 2, 4$. da har bir n natural son uchun $n^k - n$ ning k ga karrali mulohazasining bajarilishini tekshiring.

29. Ixtiyoriy $n \in N$ da isbotlang:

a). $6^{2n-2} + 3^{n+1} + 3^{n-1}$ sonning 11 ga bo'linishini.

b). $5 \cdot 2^{3n-2} + 3^{3n-1}$ sonning 19 ga bo'linishini.

30. Ixtiyoriy $n \in N$ da tengsizlikning o'rinli ekanligin isbotlang:

a). $\frac{n}{2} < 1 + \frac{1}{2} + ... + \frac{1}{2^n - 1} < \frac{1}{\sqrt{3n}+1}$.

b). Agar $x_i > 0 \; (1 \le i \le n)$ i $x_1 + x_2 + ... + x_n \le \frac{1}{2}$ bo'lsa, u holda

$$(1-x_1)\cdot(1-x_2)\cdot ... \cdot(1-x_n) \ge \frac{1}{2}.$$

c). Agar $x_i > 0\ (1 \le i \le n)$ va $x_1 + x_2 + ... + x_n = 10$, bo'lsa, u holda

$$x_1 \cdot x_2 \cdot ... \cdot x_n \le \left(\frac{10}{n}\right)^n.$$

d). $$\frac{4^n}{n+1} > \frac{(2n)!}{(n!)^2}.$$

31. Ixtiyoriy $n \in N$ da tengsizlikni isbotlang:

a) $$\sqrt{x_1 \cdot x_2 \cdot ... \cdot x_n} \le \frac{x_1 + x_2 + ... + x_n}{n}, \quad x_i > 0;$$

b) $$(x_1 + x_2 + ... + x_n) \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \ge n^2, \quad x_i > 0, \quad i = 1, 2, ..., n;$$

c) $$\frac{x_1^n + x_2^n}{2} \ge \left(\frac{x_1 + x_2}{2}\right)^n, \quad x_1, x_2 > 0;$$

d) $$x^n + x^{n-2} + x^{n-4} + ... + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \ge n + 1.$$

32. $n \in N$ uchun $\lg(n+1) > \dfrac{\lg 1 + \lg 2 + ... + \lg n}{n}$ tengsizlikni isbotlang**.**

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Mundarija