O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI
ALISHER NAVOIY NOMIDAGI
SAMARQAND DAVLAT UNIVERSITETI

FIZIKA FAKULTETI

NAZARIY FIZIKA VA KVANT ELEKTRONIKASI KAFEDRASI

O. PARDAYEV, N.QOSIMOV, N. ESHQOBILOV, U. SAYDULLAYEV

NAZARIY MEXANIKADAN AMALIY MASHG'ULOTLAR

(USLUBIY ISHLANMALAR)

SAMARQAND -2012

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

UDK 581.01 (07)
O.Pardayev, N.Qosimov, N.Eshqobilov, U.Saydullayev
Nazariy mexanikadan masalalar yechish (uslubiy ishlanma). Samarqand,SamDU bosmaxonasi,2012,- b.,

Annotatsiya

Ushbu uslubiy ishlanma Samarqand universiteti fizika fakultetining 2-bosqich talabalari uchun nazariy mexanika fanidan o'tiladigan amaliy mashg'ulotlarda yechiladigan masalalarni o'z ichiga oladi.Unda namunaviy o'quv dasturida o'tilishi ko'zda tutilgan mavzular bo'yicha masalalar tanlab olingan va har bir amaliy mashg'ulot biror mavzuga bag'ishlangan.Mavzuning muhimligiga qarab 2 mashg'ulotda tahlil qilinadigan masalalar ham tanlangan.

Har bir mashg'ulotda berilgan mavzu bo'yicha o'tiladigan darsning maqsadi,unga oid nazariya bo'yicha qisqacha nazariya yoritilgan,namuna uchun 2-3 ta masala batafsil yechib ko'rsatilgan,shu tipdagi bir necha masalalar talabalarga mustaqil ishlash uchun qo'ldirilgan.Bu masalalarning bir qismini darsxonada,ulgurmagan taqdirda uyda ishlashga topshiriq tariqasida berilgan va ularning javoblari keltirilgan.Har bir mashg'ulot oxirida masalalarni mustaqil ishlash uchun uslubiy tavsiyalar ham keltirilgan.

Fizika fakultetida tahsil olayotgan talabalar,shuningdek nazariy mexanikadan dars beruvchi yosh o'qituvchilar uchun mo'ljallangan.

Taqrizchi- fizika –matematika fanlari nomzodi
Samarqand davlat chet tillar instituti
dotsenti Amonov A.K.

MUNDARIJA

ABBYY PDF Transformer 2.0
Click here to buy
www.ABBYY.com

1- MASHG'LOT

Mavzu: «Koordinatalar sistemasi .Dekart va silindrik koordinatalar»

Darsning maqsadi: Talabalar bu darsda ortlari harakatchan bo'lgan koordinatalr sistemasi bilan tanishadilar va ularda moddiy nuqtaning vektorli parametrlarining qanday tasvirlanishini o'rganishadi.

Qisqacha tushuncha. Dekart,silindrik va sferik koordinatalar sistemalari qaraladi.Skalyar funksiyalarni yangi o'zgaruvchilarga almashtirish talabalarga tanish bo'lsada,ko'pgina fizk kattaliklar vector kattaliklar hisoblanganliklari tufayli bu yerda o'zgaruvchilarga almashtirishdan tashqari ularni yangi ortlarga proyeksiyalashga ham to'g'ri keladi.Topshiriqlarning mazmuni shundan iborat bo'ladiki,eng soda fizik kattaliklar bo'lgan radius –vektor $\vec r$,tezlik-$\vec v$, tezlanish-$\vec w$ larni Dekart va silindrik koordinatalar sistemasi ortlariga proyeksiyasini topishdan iborat bo'ladi.

Dekart koordinatalar sistemasi: o'zgaruvchilar bo'lib radius-vektorning 3 ta o'zaro orthogonal yo'nalishlarga bo'lgan $x, ó, z$ proyeksiyalari hisoblanishadi (1-rasm). $\vec n_x, \vec n_ó, \vec n_z$ ortlar o'ng uchlikni hosil qilib,moddiy nuqta harakati mobaynida qo'zg'almas bo'lib qoladi (vaqtga bog'liq bo'lmaydi).Umumiy holda nuqta radius-vektori

$$\vec r = x\,\vec n_x + ó\,\vec n_ó + z\,\vec n_z$$

ko'rinishidagi 3 ta nolga teng bo'lmagan $x, ó, z$ proyeksiyalarga ega bo'ladi.

Vaqt bo'yicha radius –vektor $\vec r$ ni differensiyallab tezlik-$\vec v$ ni olamiz:

$$\vec v = \dot{\vec r} = \dot x\,\vec n_x + ŏ\,\vec n_ó + \dot z\,\vec n_z$$

Xuddi shunday yo'l bilan tezlanishni olamiz:

$$\vec w = \dot{\vec v} = \ddot{\vec r} = \dot x\,\vec n_x + ŏ\,\vec n_ó + \dot z\,\vec n_z$$

Silindrik koordinatalar sistemasi: bu yerda o'zgaruvchilar bo'lib hisoblanishadi:radius-vektorning xy tekisligiga proyeksiyasi bo'lgan ρ uzunlik,bu uzunlikning x-o'qi orasidagi brchak φ hamda radius-vektorning z-o'qiga proeksiyasi bo'lgan z.(2-rasm) $\vec n_\rho, \vec n_\varphi, \vec n_z$ ortlar o'ng uchlikni hosil qilib, $\vec n_\rho$ xy tekisligida yotuvchi ρ uzunlik bo'ylab, $\vec n_\varphi$ xy tekisligidagi $\vec n_\rho$ ga tik yo'nalgan va $\vec n_z$ xy tekislikka tik va z-o'qi bo'ylan yo'nalgan bo'ladi. $\vec n_\rho, \vec n_\varphi$ ortlar $\vec n_x, \vec n_y, \vec n_z$ dekart ortlari bo'yicha quyidagicha ifodalangan bo'ladi:

$$\begin{aligned} \vec n_\rho &= \cos\varphi\,\vec n_x + \sin\varphi\,\vec n_y; \\ \vec n_\varphi &= -\sin\varphi\,\vec n_x + \cos\varphi\,\vec n_y; \\ \vec n_z &= \vec n_z. \end{aligned} \tag{1}$$

Nuqta harakati vaqtida φ burchak o'zgarib turganligi tufayli $\vec n_\rho, \vec n_\varphi$ lar vaqtga bog'liq bo'ladi. $\vec n_\rho, \vec n_\varphi, \vec n_z$ larning vaqt bo'yicha hosilalari quyidagicha bo'ladi:

$$\begin{aligned} \dot{\vec n}_\rho &= -\dot\varphi\sin\varphi\,\vec n_x + \dot\varphi\cos\varphi\,\vec n_y = \dot\varphi\,\vec n_\varphi; \\ \dot{\vec n}_\varphi &= -\dot\varphi\cos\varphi\,\vec n_x - \dot\varphi\sin\varphi\,\vec n_y = -\dot\varphi\,\vec n_\rho; \\ \dot{\vec n}_z &= 0. \end{aligned} \tag{2}$$

Radius-vektorni $\vec{n}_\rho, \vec{n}_\varphi, \vec{n}_z$ ortlar bo'yicha yoyib yozsak

$$\vec{r} = \rho\,\vec{n}_\rho + z\,\vec{n}_z \qquad (3)$$

ko'rinishga ega bo'ladi.Shunday qilib radius-vektor nol bo'lmagan ikkita proyeksiyaga ega bo'ladi.

Masalalar

1-1. Zaryadlangan zarra doimiy birjinsli elektr va magnet maydoni ($\vec{E} \perp \vec{H}$) da kichik chziqli qarshilik ko'rsatuvchi ($\vec{F} = -k\,\vec{v}$) muhitda harakat qiladi.Zarraning elektr maydon yo'nalishidagi tezligi hisoblansin.

Yechish. Masala shartiga ko'ra elektr va magnet maydonlari o'zaro tik yonalishda berilgan va ular quyidagicha yo'nalgan bo'lsin:
$\vec{E} = (E_0, 0, 0)$, $\vec{H} = (0, 0, H)$ Zarraning harakati uch o'lchamli fazoda yuz beradi va uning harakat tenglamasi

$$m\ddot{\vec{r}} = -k\,\vec{v} + \frac{e}{c}\left[\vec{v} \times \vec{H}\right] + e\,\vec{E}$$

Bu tenglamani koordinata o'qlari bo'yicha proyeksiyalaymiz

$$\ddot{x} + \frac{k}{m}\dot{x} - \frac{e}{m\,c}\dot{y}H_0 - eE_x = 0\ ,$$

$$\ddot{y} + \frac{k}{m}\dot{y} + \frac{e}{m\,c}\dot{x}H_0 = 0\ ,$$

$$\ddot{z} + \frac{k}{m}\dot{z} = 0$$

Agar biz $\lambda = \dfrac{k}{m}$, $\omega = \dfrac{eH_0}{m\,c}$ belgilashlar kiritsak harakat tenglamalari quyidagi ko'rinish oladi:

$$\ddot{x} + \lambda\,\dot{x} - \omega\,\dot{y} = \frac{eE_0}{m}\ ,$$

$$\ddot{y} + \lambda\,\dot{y} + \omega\,\dot{x} = 0\ ,$$

$$\ddot{z} + \lambda\,\dot{z} = 0$$

Bu tenglamalarni yechish uchun $u = \dot{x} + i\,\dot{y}$ almashtirish o'tkazamiz va yuqoridagi tenglamalarning dastlabki ikkisi uchun umumiy tenglamaga kelamiz

$$\dot{u} + \lambda\,u + i\,\omega\,u = \frac{e\,E_0}{m}$$

Bu tenglama birjinsli bo'lmagan tenglama bo'lib,uning ikki yechimi mavjud bo'ladi
$u = u_1 + u_2$.Bunda u_1 yechim
$\dot{u}_1 + \lambda\,u_1 + i\omega\,u_1 = 0$

Birjinsli tenglamaning xususiy integrali, u_2 esa $(\lambda + i\,\omega)u_2 = \dfrac{e\,E_0}{m}$

Tenglamaning yechimi hisoblanadi va bu yechimlar quyidagi ko'rinishga ega bo'ladilar:

$$u_1 = a\,e^{-(\lambda+i\omega)t}, \qquad u_2 = \frac{e\,E_0}{m(\lambda+i\omega)}$$

yoki umumiy yechim

$$u = a\,e^{-(\lambda+i\omega)t} + \frac{e\,E_0}{m(\lambda+i\omega)} \qquad (a = A\,e^{-i\alpha},\ \alpha - \text{faza})$$

Biz axtarayotgan tezlik

$$\dot{x} = \operatorname{Re} u = A\,e^{-\lambda t}\cos(\omega t+\alpha) + \frac{e\,E_0\lambda}{m(\lambda^2+\omega^2)}$$

ko'rinishda bo'ladi,chunki

$$\operatorname{Re} u = \operatorname{Re}\frac{e\,E_0}{m(\lambda+i\omega)} = \operatorname{Re}\frac{e\,E_0}{m(\lambda+i\omega)}\cdot\frac{(\lambda-i\omega)}{(\lambda-i\omega)} = \operatorname{Re}\frac{e\,E_0\lambda - ie\,E_0\omega}{m(\lambda^2+\omega^2)} = \frac{e\,E_0\lambda}{m(\lambda^2+\omega^2)}$$

1-2. Nuqtaning harakat tenglamasi qutb koordinatalari

$$\rho = \rho(t), \quad \varphi = \varphi(t)$$

orqali berilgan bo'lsa,uning tezligi va tezlanishi topilsin.

Yechish.Qutb koordinatalar sistemasi silindrik koordinatalar sistemasining xususiy holi hisoblanib,unda $z = 0$ bo'ladi.Bu sistemada nuqta tezligi va tezlanishini toppish uchun Dekart koordinatalari x, y larning qutb koordinatalar ρ, φ bilan bog'lanishini yozamiz

$x = \rho\cos\varphi, \quad y = \rho\sin\varphi$

Bu koordinatalardan vaqt bvo'yicha birinchi hosila nuqta tezligini,ikkinchi hosila tezlanishini berishini bilamiz

$$v_x = \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi, \qquad v_y = \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi$$

Bulardan nuqta tezligining modulini topamiz

$$v = \sqrt{v_x{}^2 + v_y{}^2} = (\dot{\rho}^2 + \rho^2\dot{\varphi}^2)^{1/2}$$

Yo'naltiruvchi kosinuslar esa

$$\cos(\rho,\vec{v}) = \frac{\dot{\rho}}{v}, \qquad \cos(\varphi,\vec{v}) = \frac{\rho\dot{\varphi}}{v}$$

formulalar bilan aniqlanadi.

Demak,tezlikning qutb koordinatalarga proyeksiyalari

$v_\rho = \dot{\rho}$ – radial tezlik, $\quad v_\varphi = \rho\dot{\varphi}$ – transversal tezlik deb ataladi.

Nuqtaning tezlanishlari,qayd qilganimizdek,

$$w_x = \dot{v}_x = \ddot{x} = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\cos\varphi - \left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\sin\varphi,$$

$$w_y = \dot{v}_y = \ddot{y} = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\sin\varphi - \left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\cos\varphi$$

Demak,

$w_\rho = \ddot{\rho} - \rho\dot{\varphi}^2$ – radial tezlanish,

$w_\varphi = 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = \dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dot{\varphi}\right)$ – transversal tezlanish deb ataladi.

Yo'naltiruvchi kosinuslar esa

$$\cos(\rho,\vec{w}) = \frac{w_\rho}{w} = \frac{\ddot{\rho} - \rho\dot{\varphi}^2}{w}, \qquad \cos(\varphi,\vec{w}) = \frac{2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}}{w}$$

ko'rinishda bo'ladi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

1-3. Nuqtaning harakati qutb koordinatalarida

$$\rho = bt, \quad \varphi = \frac{k}{t} \tag{1}$$

tenglamalar orqali berilgan. Bu yerda $b, k = const$.

Nuqta trayektoriyasi, uning tezlik va tezlanishi aniqlansin.

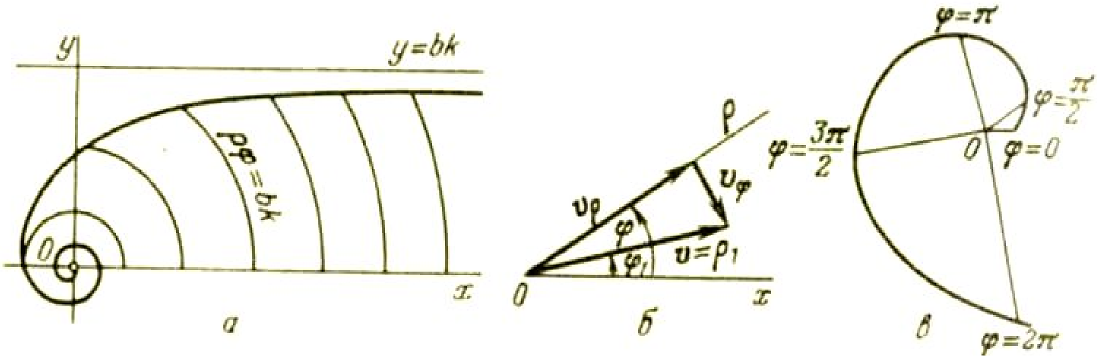
Yechish. Nuqtaning berilgan harakat tenglamalarini parametrik tenglama deyish mumkin va unda vaqt parametr bo'lib hisoblanadi. Ochiq ko'rinishda trayektoriya tenglamasi (1) tenglamalardan vaqtni yo'qotish bilan topiladi

$$\rho\varphi = kb \quad yoki \quad \rho = \frac{kb}{\varphi}$$

Bu tenglama giperbolik spiral tenglamasi bo'lib hisoblanadi. Bu tenglamada radius-vektor ρ ning φ burchakka ko'paytmasi

$$\rho\varphi = kb = const \ ,$$

yani doimiy son ekanligi ko'rinib turibdi.



Egrilikning ana shu xossasidan uning ko'rinishini chizishimiz mumkin. Haqiqatan, $\rho\varphi$ ko'paytma φ burchakka mos keladigan ρ radiusli aylana yoyining uzunligi hisoblanadi. Shunday qilib, O nuqtadan boshlab har xil radiusli aylanalar chizamiz va har bir aylanadan Ox – o'qi bo'yicha bk ga teng bo'lgan yoylar o'tkazamiz hamda bu o'qda hosil bo'lgan nuqtalar $\rho\varphi = kb$ tenglamani qanoatlantirgan bo'ladi. Bu nuqtalarning geometric o'rni giperbolik spiral hisoblanadi. Bu egriliklarning bittasi Ox – o'qiga parallel bo'lgan asimptotaga ega bo'ladi va u $y = bk$ ($\varphi \to 0$ da $\rho \to \infty$ bo'lgan) tenglamani qanoatlantiradi. Egrilikning boshqa tarmog'i cheklanmagan sonda aylanib, qutbga yaqinlashadi ($\varphi \to \infty$ da $\rho \to 0$ bo'ladi). O nuqta egrilikning asimptotik qutbi deyiladi.

Endi nuqta tezligini topamiz. Nuqta radial tezligining moduli

$v_\rho = \dot{\rho} = b,$

transversal tezlikniki esa

$$v_\varphi = \rho\dot{\varphi} = -\rho\frac{k}{t^2} \ .$$

Nuqta tezlik vektori

$$\vec{v} = v_\rho \vec{n}_\rho + v_\varphi \vec{n}_\varphi = b\vec{n}_\rho - \rho\frac{k}{t^2}\vec{n}_\varphi$$

Bu tezlik moduli esa

$$v = \sqrt{{v_\rho}^2 + {v_\varphi}^2} = \sqrt{b^2 + \rho^2\frac{k^2}{t^4}} = b\sqrt{1 + \frac{k^2}{t^2}} = \frac{b}{\rho}\sqrt{\rho^2 + b^2k^2}$$

Nuqtaning tezlanishlari oldingi masaladagidek topiladi. Radial tezlanish

$$w_\rho = \ddot{\rho} - \rho\dot{\varphi}^2 = -\frac{k^2 b^4}{\rho^3},$$

transversal tezlanish esa

$$w_\varphi = 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = \frac{2kb^3}{\rho^2} - \frac{2kb^3}{\rho^2} = 0$$

Nuqtaning mutlaq tezlanishi radius-vektor bo'ylab qutbga tomon yo'nalgan bo'ladi.

T0PSHIRIQ

Moddiy nuqta tezlik vektori va tezlanish vektorining silindrik sistema koordinatalari ortlariga bo'lgan proyeksiyarini yozing:

JAVOB:

$\vec{v}(\dot{\rho}, \rho\dot{\varphi}, \dot{z})$, $\vec{w}(\ddot{\rho} - \rho\dot{\varphi}^2, 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}, \ddot{z})$.

Uslubiy tavsiyalar:

1.Koordinata sistemalarini sinf taxtasida ularning ortlarini fazoviy va xy tekisligida ko'rsatgan holda chizib ko'rsating.Sistemalar ortlarini chizib ko'rsatish paytida,shuningdek nuqta radius-vektorini koordina sistemalari ortlari bo'yicha yozishda iloji boricha talabalar ishtirokiga erishish kerak.

2.Ko'p hollarda ortlardagi indekslarni yozishda talabalar, ularni chalkashtirib,xatolikka yo'l qo'yishadi.Shuning uchun talabalardan indekslarni aniq yozishni talab qilish lozim bo'ladi.

3.Vaqt bo'yicha o'zgaruvchi va ortlarni differensiallash ancha murakkab bo'lgani uchun har br ortdan olinadigan birinchi va ikkinchi hosilalarni blok-blok qilib bajarib olish lozim.

4.Ko'pchilik talabalar proyeksiya olish va differensiallash qoidasi sedan chiqargan bo'lishi mumkin.Shuning uchun ularni eslatish kerak bo'ladi.

ABBYY PDF Transformer 2.0
Click here to buy
www.ABBYY.com

2- MASHG'ULOT

Mavzu: «Koordinatalar sistemasi . Sferik koordinatalar»

Darsning maqsadi: Talabalar bu darsda ortlari harakatchan bo'lgan koordinatalr sistemasi bilan tanishadilar va ularda moddiy nuqtaning vektorli parametrlarining qanday tasvirlanishini o'rganishadi.

Qisqacha tushuncha. Dekart,silindrik va sferik koordinatalar sistemalari qaraladi.Skalyar funksiyalarni yangi o'zgaruvchilarga almashtirish talabalarga tanish bo'lsada,ko'pgina fizk kattaliklar vector kattaliklar hisoblanganliklari tufayli bu yerda o'zgaruvchilarga almashtirishdan tashqari ularni yangi ortlarga proyeksiyalashga ham to'g'ri keladi.Topshiriqlarning mazmuni shundan iborat bo'ladiki,eng soda fizik kattaliklar bo'lgan aaaaaaaaaradius –vektor $\vec r$,tezlik-$\vec v$, tezlanish-$\vec w$ larni sferik koordinatalar sistemasi ortlariga proyeksiyasini topishdan iborat bo'ladi.

Sferik koordinatalar sistemasi. Bunda o'zgaruvchilar: radius-vektorning uzunligi r , radius-vektorning z-o'qi bilan tashkil qilgan burchagi θ , xy tekisligiga radius-vektor proyeksiyasining x-o'qi bilan tashkil qilgan burchagi φ (3-rasm). $\vec n_x, \vec n_y, \vec n_z$ ortlar o'ng uchlikni hosil qilib,$\vec n_r, \vec n_\theta, \vec n_\varphi$ ortlardan $\vec n_r$ radius-vektor boylab yo'nalgan,$\vec n_\theta$ ort $\vec n_r$ ga tik yo'nalgan, $\vec n_\varphi$ esa xy tekisligida yotuvchi va $\vec n_\theta$ ga tik yo'nalgan bo'ladi . $\vec n_x, \vec n_y, \vec n_z$ ortlar silindrik sistema koordinatalari bo'yicha quytidagicha yoziladi:

$$\begin{cases} \vec n_r = \vec n_\rho \sin\theta + \vec n_z \cos\theta; \\ \vec n_\theta = \vec n_\rho \cos\theta - \vec n_z \sin\theta;. \\ \vec n_\varphi = \vec n_\varphi; \end{cases} \qquad \textbf{(4)}$$

Burchak θ va $\vec n_\rho$ ort nuqta harakati vaqtida o'zgarib turganligi uchun $\vec n_r, \vec n_\theta$ ortlar vaqtga bog'liq bo'ladi.(4) ni vaqt bo'yicha differensiallab,topamiz

$$\begin{cases} \dot{\vec n}_r = \dot\theta(\cos\theta\ \vec n_\rho - \sin\theta\ n_z\ ) + \sin\theta\, \dot{\vec n}_\rho = \dot\theta\, \vec n_\theta + \dot\varphi \sin\theta\, \vec n_\varphi; \\ \dot{\vec n}_\theta = -\dot\theta\ (\sin\theta\, \vec n_\rho + \cos\theta\, \vec n_z) + \cos\theta\, \dot{\vec n}_\rho = -\dot\theta\, \vec n_r + \dot\varphi \cos\theta\, \vec n_\varphi;. \\ \dot{\vec n}_\varphi = -\dot\varphi\, \vec n_\rho = -\dot\varphi\,(\sin\theta\, \vec n_r + \cos\theta\, \vec n_\theta); \end{cases} \qquad \textbf{(5)}$$

Nuqta radius-vektorini $\vec n_r,\ \vec n_\theta,\ \vec n_\varphi$ ortlar orqali yozsak

$$\vec r = r\vec n_r$$

ko'rinish oladi.Demak nuqta radius-vektorining nolga teng bo'lmagan yagona komponentasi,yani $\vec r(r,0,0)$ mavjud bo'ladi.

Masalalar

2-1. Nuqtaning harakati uning
$\rho = \rho(t),\quad \varphi = \varphi(t),\quad \theta = \theta(t)$
sferik koordinatalari bilan berilgan bo'lsa,nuqta tezligining dekart koordinatalaridagi va sferik koordinatalari sistemasining koordinatalari chiziqlariga bo'lgan proyeksiyalari hisoblansin.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Yechish. Nuqtaning dekart koordinata o’qlariga bo’lgan proyeksiyalari v_x, v_y, v_z larni toppish uchun dekart koordinatalarining sferik koordinatalari o’rtasidagi bog’lanish

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta \qquad (1)$$

ni vaqt bo’yicha differensiallaymiz

$$\begin{aligned} v_{xz} &= \dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi; \\ v_y &= \dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta + r\dot{\varphi}\sin\theta\cos\varphi; \qquad (2) \\ v_z &= \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta. \end{aligned}$$

Oxirgi formulani tezlikning sferik sistema koordinatalaridagi tashkil etuvchilari $v_\kappa, v_\theta, v_\varphi$ orqali yozish mumkin

$$\begin{aligned} v_x &= v_r\sin\theta\cos\varphi + v_\theta\cos\theta\cos\varphi - v_\varphi\sin\varphi, \\ v_y &= v_r\sin\theta\sin\varphi + v_\theta\cos\theta\sin\varphi + v_\varphi\cos\varphi, \qquad \mathbf{(3)} \\ v_z &= v_r\cos\theta - v_\theta\sin\theta \end{aligned}$$

U holda (2) va (3) larni solishtirib,topamiz

$$v_r = \dot{r},\ \ v_\theta = r\dot{\theta},\ \ v_\varphi = r\dot{\varphi}\sin\theta.$$

2-2. Nuqtaning harakati o’zining sferik koordinatalari

$$r = 2t^2,\ \ \varphi = \frac{\pi t}{4},\ \ \theta = \frac{\pi t}{4} \qquad (1)$$

orqali berilgan bo’lsa, $t = 2s$ momentdagi tezligi va tezlanishi topilsin.

Yechish. Nuqta tezligini topish uchun uning sferik koordinatalar chizig’idagi proyeksialaridan foydalanamiz (oldingi masalaga qaralsin)

$$v_r = \dot{r},\ \ v_\theta = r\dot{\theta},\ \ v_\varphi = r\dot{\varphi}\sin\theta. \qquad \mathbf{(2)}$$

(1) dagi sferik koordinatalardan vaqt bo’yicha birinchi hosila olamiz

$$\dot{r} = 4t,\quad \dot{\varphi} = \frac{\pi}{4},\quad \dot{\theta} = \frac{\pi}{4}.$$

Bu hosilalarni (2)ga olib borib qo’yamiz

$$v_r = 4t\,m/s,\ \ v_\theta = 2t^2\frac{\pi}{4} = \frac{\pi t^2}{2}m/s,\ \ v_\varphi = 2t^2\frac{\pi}{4}\cdot\sin\frac{\pi}{4} = \frac{\pi t^2}{2}\sin\frac{\pi t}{4}\ \ m/s.$$

yoki $t = 2s$ ekanligini hisobga olsak

$$v_r = 8m/s,\quad v_\theta = 2\pi\ m/s,\quad v_\varphi = 2\pi\ \ m/s$$

qiymatlarga ega bo’ladi.Tezlik moduli esa

$$v = \sqrt{v_r^{\,2} + v_\theta^{\,2} + v_\varphi^{\,2}} = \sqrt{8^2 + (2\pi)^2 + (2\pi)^2} \approx 12\,m/s.$$

Nuqtaning tezligini toppish uchun harakatni dekart koordinatalar sistemasida yozish qulayroqdir:

$$\begin{aligned} x &= r\sin\theta\cos\varphi = 2t^2\sin\frac{\pi t}{4}\cos\frac{\pi t}{4}, \\ y &= r\sin\theta\sin\varphi = 2t^2\sin\frac{\pi t}{4}\sin\frac{\pi t}{4}, \qquad (3) \\ z &= r\cos\theta = 2t^2\cos\frac{\pi t}{4}. \end{aligned}$$

(3) ning birinchi va ikkinchi tenglamalarida trigonometriya formulalaridan foydalanib,(3)ni qayta yozamiz

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$x = t^2 \sin\frac{\pi t}{2}, \quad y = t^2(1-\cos\frac{\pi t}{2}), \;\; z = 2t^2 \cos\frac{\pi t}{4}. \qquad (4)$$

(4) dagi koordinatalardan vaqt bo'yicha birinchi va ikkinchi hosilalar olamiz:

$$\dot{x} = 2t\sin\frac{\pi t}{2} + t^2\frac{\pi}{2}\cos\frac{\pi t}{2}, \; \dot{y} = 2t(1-\cos\frac{\pi t}{2}) + t^2\frac{\pi}{2}\sin\frac{\pi t}{2}, \; \dot{z} = 4t\cos\frac{\pi t}{2} - 2t^2\frac{\pi}{4}\sin\frac{\pi t}{4}.$$

$$\begin{aligned}
\ddot{x} &= 2\sin\frac{\pi t}{2} + 2t\frac{\pi}{2}\cos\frac{\pi t}{2} + \frac{\pi}{2}(2t\cos\frac{\pi t}{2} - t^2\frac{\pi}{2}\sin\frac{\pi t}{2}),\\
\ddot{y} &= 2(1-\cos\frac{\pi t}{2}) + 2t\frac{\pi}{2}\sin\frac{\pi t}{2} + \frac{\pi}{2}(2t\sin\frac{\pi t}{2} + t^2\frac{\pi}{2}\cos\frac{\pi t}{2}),\\
\ddot{z} &= 4\cos\frac{\pi t}{2} - 4t\frac{\pi}{4}\sin\frac{\pi t}{4} - 2\frac{\pi}{4}(2t\sin\frac{\pi t}{4} + t\,\frac{\pi}{4}\cos\frac{\pi t}{4}).
\end{aligned} \qquad (5)$$

Tezlik va tezlanish ifodalariga vaqtning qiymati (t=2 s) ni qo'yamiz

$$\dot{x} = -2\pi, \; \dot{y} = 8, \; \dot{z} = -2\pi$$

$$\ddot{x} = -4\pi, \; \ddot{y} = 4-\pi^2, \; \ddot{z} = -4\pi.$$

Bulardan tezlanish va tezlik modullari

$$w = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = \sqrt{(-4\pi)^2 + (4-\pi^2)^2 + (-4\pi)^2} \approx 18{,}7 m/s,$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{(-2\pi)^2 + 64 + (-2\pi)^2} \approx 12 m/s.$$

Bulardan ko'ramizki,dekard koordinatalar sistemasi o'qlaridagi tezlik vektorining qiymati sferik koordinatalar sistemasidagi qiymatiga mos tushar ekan.Shunday ham bo'lishi kerak,chunki ikki koordinatalar sistemasi ham qo'zg'almasdir.

T0PSHIRIQ

Moddiy nuqta tezlik vektori va tezlanish vektorining sferik sistema koordinatalari ortlariga bo'lgan proyeksiyarini yozing:

JAVOB:

$$\vec{v}\,(\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\varphi});$$

$$\vec{w}\,(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta, r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta + r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\,\dot{\theta}\,\dot{\varphi}\cos\theta).$$

Uslubiy tavsiyalar:

1.Koordinata sistemalarini sinf taxtasida ularning ortlarini fazoviy va xy tekisligida ko'rsatgan holda chizib ko'rsating.Sistemalar ortlarini chizib ko'rsatish paytida,shuningdek nuqta radius-vektorini koordina sistemalari ortlari bo'yicha yozishda iloji boricha talabalar ishtirokiga erishish kerak.

2.Ko'p hollarda ortlardagi indekslarni yozishda talabalar, ularni chalkashtirib,xatolikka yo'l qo'yishadi.Shuning uchun talabalardan indekslarni aniq yozishni talab qilish lozim bo'ladi.

3.Vaqt bo'yicha o'zgaruvchi va ortlarni differensiallash ancha murakkab bo'lgani uchun har br ortdan olinadigan birinchi va ikkinchi hosilalarni blok-blok qilib bajarib olish lozim.

4.Ko'pchilik talabalar proyeksiya olish va differensiallash qoidasi sedan chiqargan bo'lishi mumkin.Shuning uchun ularni eslatish kerak bo'ladi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

3-MASHG'ULOT

Mavzu: «Egrichiziqli koordinatalar sistemasi.Lame koeffitsiyentlari»

Darsning maqsadi. Talabalarga egrichiziqli koordinatalar sistemasida ishlashni va Lame koeffitsiyentlarni turli koordinatalar sistemasi uchun topishni o'rgatish.

Qisqacha tushuncha. Harakatning yoki koordinatalar sistemasining berilish usuliga qarab tezlik va tezlanish proyeksiyalarini hisoblash formulalari turlicha bo'lishi mumkin.Tezlik va tezlanishni hisoblash usulini umumlshtirish uchun bir-biriga orthogonal bo'lgan q, q_2, q_3 egrichiziqli koordinatalar kiritiladi.Dekart koordinatalaridan egrichiziqli koordinatalarga o'tish formulasi quyidagicha bo'ladi:

$$x = x(q_1, q_2, q_3)\,, \quad y = y(q_1, q_2, q_3)\,, \qquad z = z(q_1, q_2, q_3)$$

Bulardan vaqt bo'yicha differensial olib,dekart koordinatalarida tezlik proyeksiyalarini topamiz

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3\,,$$

$$\dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 + \frac{\partial y}{\partial q_3}\dot{q}_3\,,$$

$$\dot{z} = \frac{\partial z}{\partial q_1}\dot{q}_1 + \frac{\partial z}{\partial q_2}\dot{q}_2 + \frac{\partial z}{\partial q_3}\dot{q}_3\,.$$

Bu yerda

$$\dot{q}_1 = \frac{d\,q_1}{d\,t}, \quad \dot{q}_2 = \frac{d\,q_2}{d\,t}, \qquad \dot{q}_3 = \frac{d\,q_3}{d\,t}.\text{-}$$

umumlashgan tezliklar deb ataladi.Tezlik modulining kvadrati uchun

$$v^2 = v_x{}^2 + v_y{}^2 + v_z{}^2 = H_1\,\dot{q}_1{}^2 + H_2\,\dot{q}_2{}^2 + H_3\,\dot{q}_3{}^2$$

Ifodani olamiz.Biz bu yerda umumlashgan tezliklarning ortogonalligini hisobga oldik.Oxirgi formuladagi H_1, H_2, H_3 lar koeffitsiyentlar bo'lib,Lame koeffitsiyentlari deyiladi va ular quyidagicha aniqlanadi:

$$H_1 = \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2}\;;$$

$$H_2 = \sqrt{\left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2}\;;$$

$$H_3 = \sqrt{\left(\frac{\partial x}{\partial q_3}\right)^2 + \left(\frac{\partial y}{\partial q_3}\right)^2 + \left(\frac{\partial z}{\partial q_3}\right)^2}\;;$$

Egrichiziqli koordinatalarda izlanayotgan tezlik proyeksiyalari quyidagicha yoziladi;

$$v_1 = H_1\,\dot{q}_1\,, \;\; v_2 = H_2\,\dot{q}_2\,, \;\;\; v_3 = H_3\,\dot{q}_3\,,$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Masalalar

3-1. Sferik koordinatalarda tezli proyeksiyalari va tezlanish proyeksiyalari aniqlansin.

Yechish. Umumlshgan koordinatalar rolini bu masalada

$q_1 = r,\ \ q_2 = \theta,\ \ q_3 = \varphi$

sferik koordinatalar o'ynaydi.Dekard koordinatalarining sferik koordinatalar bilan bog'lanishi bizga ma'lum

$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta$

Bundan biz Lame koeffitsiyentlarini topamiz

$$H_1 = \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2} = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = 1,$$

$$H_2 = \sqrt{\left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2} = \sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta} = r,$$

$$H_3 = \sqrt{\left(\frac{\partial x}{\partial q_3}\right)^2 + \left(\frac{\partial y}{\partial q_3}\right)^2 + \left(\frac{\partial z}{\partial q_3}\right)^2} = \sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi + 0} = r\sin\theta.$$

Demak, egrichiziqli koordinatalarda izlanayotgan tezlik proyeksiyalari quyidagicha bo'lar ekan:

$$v_1 = H_1\dot{q}_1 = \dot{r},\ \ v_2 = H_2\dot{q}_2 = r\dot{\theta},\ \ \ v_3 = H_3\dot{q}_3 = r\sin\theta\cdot\dot{\varphi}.$$

Sferik koordinatalar o'qlarida tezlanishning proyeksiyalari

$$w_i = \frac{1}{H_i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right) \qquad (i = 1,2,3)$$

Bu yerda $T-$kinetik energiya

$$T = \frac{1}{2}v^2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\varphi}^2\right)$$

Har bir umumlashgan koordinatalar bo'yicha $\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)$ ayirmani topamiz:

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)\Big|_{i=1} = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1}\right) = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r}\right) = \ddot{r} - r\dot{\theta}^2 - r^2\sin^2\theta\cdot\dot{\varphi}^2.$$

$H_1 = H_r = 1$ bo'lgani uchun

$$w_r = \frac{1}{H_r}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r}\right) = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r}\right) = \ddot{r} - r\dot{\theta}^2 - r^2\sin^2\theta\cdot\dot{\varphi}^2.$$

Xuddi shu yo'l bilan topamiz

$$w_\theta = \frac{1}{H_\theta}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}\right) = \frac{1}{r}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\cdot\dot{\varphi}^2.$$

$$w_\varphi = \frac{1}{H_\varphi}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi}\right) = \frac{1}{r\sin\theta}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi}\right) = r\sin\theta\cdot\ddot{\varphi} + 2\dot{r}\dot{\varphi}\sin\theta + 2r\cos\theta\cdot\dot{\theta}\dot{\varphi}.$$

3-2. Dekart koordinatalari toroidal koordinata (ρ,φ,ψ) lar orqali quyidagicha
$x=(a+\rho\cos\varphi)\cos\psi,\quad y=(a+\rho\cos\varphi)\sin\psi,\ z=\rho\sin\varphi$
ifodalansa, bu koordinatalar sistemasi o'qlarida tezlik va tezlanish proyeksiyalari hisoblansin.

Yechish. Lame koeffitsiyentlarini hisoblaymiz:

$$H_\rho=\sqrt{\left(\frac{\partial x}{\partial\rho}\right)^2+\left(\frac{\partial y}{\partial\rho}\right)^2+\left(\frac{\partial z}{\partial\rho}\right)^2}=\sqrt{\cos^2\varphi\cos^2\psi+\cos^2\varphi\sin^2\psi+\sin^2\varphi}=1\ ,$$

$$H_\varphi=\sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2+\left(\frac{\partial z}{\partial\varphi}\right)^2}=\rho\sqrt{\sin^2\varphi\cos^2\psi+\sin^2\varphi\sin^2\psi+\cos^2\varphi}=\rho,$$

$$H_\psi=\sqrt{\left(\frac{\partial x}{\partial\psi}\right)^2+\left(\frac{\partial y}{\partial\psi}\right)^2+\left(\frac{\partial z}{\partial\psi}\right)^2}=\sqrt{a^2\sin^2\psi+a^2\cos^2\psi}+\rho\sqrt{\cos^2\varphi\sin^2\psi+\cos^2\varphi\cos^2\psi}=$$
$$=a+\rho\cos\varphi.$$

U holda tezlik proyeksiyalari toroidal koordinatalarda quyidagicha bo'ladi:
$v_\rho=H_\rho\dot\rho=\dot\rho,\quad v_\varphi=H_\varphi\dot\varphi=\rho\dot\varphi,\qquad v_\psi=H_\psi\dot\psi=(a+\rho\cos\varphi)\cdot\dot\psi$

Tezlik moduli

$$v=\sqrt{\dot\rho^2+\rho^2\dot\varphi^2+(a+\rho\cos\varphi)^2\dot\psi^2}\,.$$

Nuqtaning kinetik energiasi

$$T=\frac{1}{2}v^2=\frac{1}{2}(\dot\rho^2+\rho^2\dot\varphi^2+(a+\rho\cos\varphi)^2\dot\psi^2)$$

ko'rinishga ega bo'lgani uchun oldingi masaladagidek,tezlanish proyeksiyalari quyidagicha bo'ladi:

$$w_\rho=\frac{1}{H_\rho}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot\rho}-\frac{\partial T}{\partial\rho}\right)=\ddot\rho-\rho\dot\varphi^2-(a+\rho\cos\varphi)\dot\psi^2\cos\varphi,$$

$$w_\varphi=\frac{1}{H_\varphi}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot\varphi}-\frac{\partial T}{\partial\varphi}\right)=\rho\ddot\varphi+2\dot\rho\dot\varphi+(a+\rho\cos\varphi)\dot\psi^2\sin\varphi,$$

$$w_\psi=\frac{1}{H_\psi}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot\psi}-\frac{\partial T}{\partial\psi}\right)=\frac{1}{(a+\rho\cos\varphi)}\frac{d}{dt}\left[(a+\rho\cos\varphi)^2\dot\psi\right]=$$
$$=\ddot\psi(a+\rho\cos\varphi)+2\dot\psi\,(\dot\rho\cos\varphi-\rho\dot\varphi\sin\varphi).$$

1-2-TOPSHIRIQLAR

1.Silindrik koordinatalar o'qlariga tezlik proyeksiyasini toping.
2. Silindrik koordinatalar o'qlariga tezlanish proyeksiyasini toping.

JAVOBLAR:

1. $v_\rho=\dot\rho,\ v_\varphi=\rho\dot\varphi,\quad v_z=\dot z;\qquad 2.\ \vec w(\ddot\rho-\rho\dot\varphi^2,\ \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot\varphi),\ddot z).$

Uslubiy tavsiyalar:
1.Berilgan egrichizqli koordinatalar sistemasi uchun Lame koeffitsiyentlarini aniqlash kerak.
2.Egrichiziqli koordinata sistemasining ortogonalligini tekshirib chiqish lozim.
3.Egrichiziqli koordinatalarda nuqta tezligi proyeksiyasi va tezlanishi proyeksiyalarini hisoblash kerak.

4-MASHG'ULOT
Mavzu: «Moddiy nuqtaning impuls momenti»

Darsning maqsadi. Talabalarga moddiy nuqtaning impuls momenti va uning turli koordinata sistemalarida ko'rinishi to'g'risida ma'lumot berish.Impuls momentini o'rganish kelgusida atom fizikasini,kvant mexanikasini o'rganishda talabalarga qo'l keladi.

Qisqacha tushuncha. Moddiy nuqtaning impuls momenti (ayrim adabiyotlarda harakat miqdor momenti deb ham yuritiladi) vector kattalik bo'lib,u

$$\vec{l} = [\vec{r}\times\vec{p}] = m[\vec{r}\times\vec{v}] = m\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

formula bilan aniqlanadi.Markaziy-simmetrik maydonda impuls momenti harakat integrali (saqlanuvchan kattalik) hisoblanadi.Uning harakat integrali bo'lishligi fazoning izotroplik xossasi bilan bog'langan bo'ladi (fazoda barcha yo'nalishlar bir xildir).

Misollar

4-1. Impuls momenti komponentalari l_x , l_y , l_z larni silindrik va sferik koordinatalar orqali ifodalang.

Yechish. Dekart (x,y,z) va silindrik koordinatalar quyidagicha bog'langan:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

Dastlab l momentning komponentalarini yozaylik

$$l_x = [\vec{r}\times\vec{p}]_x = y\,p_z - z\,p_y = m(y\dot{z} - z\dot{y});$$
$$l_y = [\vec{r}\times\vec{p}]_y = z\,p_x - x\,p_z = m(z\dot{x} - x\dot{z});$$
$$l_z = [\vec{r}\times\vec{p}]_z = x\,p_y - y\,p_x = m(x\dot{y} - y\dot{x}).$$

Endi biz tezlik komponentalarini topamiz

$$\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi, \quad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi, \quad \dot{z} = \dot{z}.$$

U holda

$$l_x = m\,(r\sin\varphi\cdot\dot{z} - z\dot{r}\sin\varphi - zr\dot{\varphi}\cos\varphi) = m\sin\varphi\,(r\dot{z} - z\dot{r}) - mzr\dot{\varphi}\cos\varphi;$$
$$l_y = m\,(z\dot{r}\cos\varphi - zr\dot{\varphi}\sin\varphi - r\dot{z}\cos\varphi) = m\cos\varphi(z\dot{r} - r\dot{z}) - mzr\dot{\varphi}\sin\varphi;$$
$$l_z = m\,(r\dot{r}\cos\varphi\sin\varphi + r^2\dot{\varphi}\cos^2\varphi - r\dot{r}\sin\varphi\cos\varphi + r^2\dot{\varphi}\sin^2\varphi) = mr^2\dot{\varphi}.$$

Bu misoldan o'zimizga bir foydali munosabatni keltirib chiqarishimiz mumkin.Buning uchun moddiy nuqtaning silindrik sistemadagi Lagranj funksiyasini yozamiz

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - U(x,y,z) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r,\varphi)$$

Bundan umumlashgan impulsni topaylik

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\dot{\varphi}$$

Topilganni $l_z = mr^2\dot{\varphi}$ bilan solishtirsak,ularning o'zaro teng ekanligini ko'ramiz:

$$p_\varphi = l_z .$$

Endi biz sferik koordinatalarda impuls momenti komponentalarini topamiz.Yana biz dekart koordinatalari va sferik (r,θ,φ) koordinatalar o'rtasidagi bog'lanishni yozamiz:

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

Bizga $\dot x, \dot y, \dot z$ tezliklarni toppish kerak bo'ladi.Ular quyidagicha bo'ladi:

$$\dot x = \dot r\sin\theta\cos\varphi + r\dot\theta\cos\theta\cos\varphi - r\dot\varphi\sin\theta\sin\varphi;$$

$$\dot y = \dot r\sin\theta\sin\varphi + r\dot\theta\cos\theta\sin\varphi + r\dot\varphi\sin\theta\cos\varphi;$$

$$\dot x = \dot r\cos\theta - r\dot\theta\sin\theta .$$

Navbatma-navbat impuls momentining har bir komponentasi hisoblansa,biz ular uchun quyidagilarga ega bo'lamiz:

$$l_x = m(y\dot z - z\dot y) = -mr^2\,(\dot\theta\sin\varphi + \dot\varphi\sin\theta\cos\theta\cos\varphi);$$

$$l_y = m(y\dot z - z\dot y) = mr^2\,(\dot\theta\cos\varphi - \dot\varphi\sin\theta\cos\theta\sin\varphi);$$

$$l_x = m(y\dot z - z\dot y) = mr^2\dot\varphi\sin^2\theta .$$

Silindrik sistema uchun qilganimizdek,sferik sistemada moddiy nuqta uchun Lagranj funksiyasini yozamiz

$$L = \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2) - U(x,y,z) = \frac{m}{2}(\dot r^2 + r^2\dot\varphi^2\sin^2\theta + r^2\dot\theta^2) - U(r,\theta,\varphi)$$

Bundan yana umumlshgan impulsni topamiz va p_φ, l_z larning o'zaro teng ekanligini ko'ramiz.

1-2-TOPSHIRIQLAR

Ellips bo'ylab harakat qilayotgan moddiy nuqta uchun

1) impuls momentining harakat integrali ekanligi isbotlansin;

2) impuls momenti va sektorli tezlik o'rtasida $l = 2m\dot f$ munosabat mavjudligi isbotlansin (bu yerda $\dot f = \frac{ds}{dt}$ – sektorli tezlik).

3-4-TOPSHIRIQLAR

Impuls momentining mutlaq kattaligi aniqlansin:

3) silindrik koordinatalarda;

4) sferik koordinatalarda.

JAVOBLAR:

3. $l^2 = m^2r^2\dot\varphi^2(r^2 + z^2) + m^2(r\dot z - z\dot r)^2 .$
4. $l^2 = m^2r^4(\dot\theta^2 + \dot\varphi^2\sin^2\theta).$

Uslubiy tavsiyalar:

1. Impuls momentini kuch momenti bilan aralshtirmaslik lozim.
2. Impuls momenti vector kattalik bo'lgani uchun uning komponentalarini topishda ikki vektorning vektorli ko'paytmasini ochish qoidasidan foydalanish kerak.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

5-MASHG'ULOT
Mavzu: «Moddiy nuqta harakat qonunlari va trayektoriyasi»

Darsning maqsadi: Harakatning berilgan qonunlari bo'yicha nuqta harakat trayektoriyasini,tezligi va tezlanishini hisoblashga qaratilgan masalalar yechish.

Qisqacha tushuncha. Nuqtaning trayektoriyasini aniqlashni talab qilgan masalalar ikki variantda yechilishi mumkin:

1.Agar harakat qonuni berilgan bo'lsa,bu qonunlardan vaqtni yo'qish bilan osongina trayektoriya topiladi.Bu yo'l bilan nuqta uchta koordinatalari o'rtasida bog'lanish topiladi.Bu holda nuqta tezligi va tezlanishi berilgan harakat qonunlarini vaqt bo'yicha differensiallab topiladi.Bu holda koordinata sistemalariga mos tezlik va tezlanish proyeksiyalari oldingi mavzuda topilgani kabi topiladi.

2.Agar harakat tenglamalari ma'lum bo'lsa,nuqta tezligi va harakat qonuni vektorli ko'rinishdagi harakat tenglamalarini integrallash yo'li bilan topiladi.Topilgan vektorli $\vec{r}(t)$ funksiya tanlab olingan koordinata sistemasining ortiga proyeksiyalanadi.Yoki berilgan tenglamalarni proyeksiyalarini topib,keyin esa ularni integrallash mumkin.Ikkala holda ham trayektoriya uchun tenglama topiladi.

Masalalar

5-1.Moddiy nuqtaning harakat qonuni

$$\begin{cases} x = a\cos\omega t, \\ y = a\sin\omega t, \\ z = 0 \end{cases}$$

(bu yerda a, ω =const) ko'rinishda berilsa,nuqtaning trayektoriyasi, tezligi va tezlanishi topilsin.

Yechish. Qoidaga binoan berilgan harakat tenglamalaridan vaqtni yo'qotish yo'li bilan trayektoriya tenglamasini topamiz.Buning uchun har ikkala tenglamani kvadratga ko'tarib,ularning char va o'ng tomonlarini mos ravishda qo'shamiz:

$$x^2 = a^2\cos^2\omega t,$$

$$y^2 = a^2\sin^2\omega t,$$

$$x^2 + y^2 = a^2\cos^2\omega t + a^2\sin^2\omega t = a^2\left(\sin^2\omega t + \cos^2\omega t\right) = a^2,$$

Demak,moddiy nuqta radiusi a gat eng bo'lgan aylana bo'ylab harakat qilar ekan.Nuqtaning tezligi va tezlanishi uning harakat qonunini vaqt bo'yicha bir va ikki marta differensiallab topiladi:

$$v_x = \frac{dx}{dt} = -a\omega\sin\omega t,$$

$$v_y = \frac{dy}{dt} = a\omega\cos\omega t,\ v_z = \frac{dz}{dt} = 0$$

Nuqtaning to'liq tezligi:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t} = a\omega.$$

$$w_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -a\omega^2\cos\omega t,$$

$$w_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = -a\omega^2\sin\omega t,$$

$$w_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} = 0$$

Bulardan to'liq tezlanish

$$w = \sqrt{w_x^2 + w_y^2} = \sqrt{a^2\omega^4\cos^2\omega t + a^2\omega^4\sin^2\omega t} = a\omega^2.$$

5-2. N nuqta

$$x = a\cos(kt - \varepsilon), \quad y = b\cos kt \qquad (1)$$

tenglamalari bo'yicha tebranma harakat qiladi.Bu nuqtaning trayektoriyasi aniqlansin. ε ning qanday qiymatlarida nuqta trayektoriyasi parabolaga aylanadi? Boshlang'ich vaqt momentida nuqta tezligi topilsin.

Yechish. Nuqta trayektoriyasini toppish uchun harakat tenglamalaridan vaqtni yo'qatamiz

$$\frac{x}{a} = [\cos 2kt\cos\varepsilon + \sin kt\sin\varepsilon] = [\cos^2 kt\cos\varepsilon - \sin^2 kt\cos\varepsilon + 2\sin kt\cos kt\sin\varepsilon] \qquad (2)$$

Harakatning ikkinchi tenglamasidan

$$\cos kt = \frac{y}{b}, \quad \sin kt = \sqrt{1 - \frac{y^2}{b^2}}. \qquad (3)$$

(3)ni (2) ga qo'yib,nuqta trayektoriya tenglamasini topamiz:

$$\frac{x}{a} = \left(2\frac{y^2}{b^2} - 1\right)\cos\varepsilon + 2\frac{y}{b}\sin\varepsilon\sqrt{1 - \frac{y^2}{b^2}} \qquad (4)$$

Nuqtaning harakat tenglamalaridan shu narsa ma'lumki, ε ning istalgan qiymatlarida x, y koordinatalar mos ravishda $\pm a, \pm b$ qiymatlardan ortiq bo'lmaydi.Shunday qilib,N nuqta trayektoriyasi tomonlari $2a$ va $2b$ bo'lgan to'g'riburchak ichida yotgan bo'ladi.(4) tenglama ε ning $\varepsilon_1 = 0$ va $\varepsilon_2 = \pi$ qiymatlarida parabolaga aylanadi:

$$y^2 = \frac{b^2}{2}\left(1 \pm \frac{x}{a}\right)$$

Endi biz nuqtaning tezligini topamiz.Tezlikning proyeksiyalari harakat tenglamalaridan vaqt bo'yicha hosila olib topiladi:

$$v_x = \dot{x} = -2ak\sin(2kt - \varepsilon), \quad v_y = \dot{y} = -bk\sin kt.$$

Tezlikning bu komponentalarining boshlang'ich vaqt momenti t=0 dagi qiymatlari

$$v_x(t=0) = -2ka\sin\varepsilon, \quad v_y(t=0) = 0.$$

Nuqta tezligi moduli $v = 2k\sin\varepsilon$ ga teng bo'ladi.

1-3 TOPSHIRIQLAR

Agar harakat qonuni

1. $\begin{cases} x = a\cos\omega t; \\ y = b\sin\omega t \ ; \\ z = 0 \\ a, b, \omega = const. \end{cases}$ 2. $\begin{cases} \rho = \rho_0 \ ; \\ \varphi = \dot\varphi_0 t + \varphi_0; \\ z = \dot z_0 t + z_0 \ ; \\ \rho, \varphi_0, \dot\varphi_0, z_0, \dot z_0 = const \end{cases}$ 3. $\begin{cases} r = \dot r_0 t + r_0 \ ; \\ \theta = \dfrac{\pi}{2}; \\ \varphi = \dot\varphi_0 t + \varphi_0; \\ r_0, \dot r_0, \varphi_0, \dot\varphi_0 = const \end{cases}$

ko'rinishda berilsa,nuqta trayektoriyasi,tezligi va tezlanishi topilsin.
Barcha trayektoriyalar ucho'lchamli fazoda tasvirlansin.

4-5 TOPSHIRIQLAR

Kuchlanganligi

4. $\vec{E} = \vec{E}_0 \cos\omega t, \vec{v}_0 \perp \vec{E}_0$;

5. $\vec{E} = \vec{E}_0$, $\vec{v}_0$ *tezlik* $\vec{E}_0$ *ga nisba*tan α *burchakostidayo'na*lg*an*

bo'lgan elektr maydonida massasi m, zaryadi e bo'lgan zarra harakat qilayotgan bo'lsa,uning trayektoriyasi topilsin.Bu yerda $\vec{E}_0, \vec{v}_0, \omega = const$

JAVOBLAR:

1. xy tekisligida yotuvchi ellips $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \quad z = 0;$

$$\begin{cases} v_x = -a\omega\sin\omega t; \\ v_y = b\omega\cos\omega t; \\ v_z = 0 \end{cases} \qquad \begin{cases} w_x = -a\omega^2\cos\omega t; \\ w_y = -b\omega^2\sin\omega t; \\ w_z = 0; \end{cases} \qquad \vec{w} = -\omega^2\vec{r}.$$

2. $\rho = \rho_0, \dfrac{z - z_0}{\dot z_0} = \dfrac{\varphi - \varphi_0}{\dot\varphi_0}$ qonuniga ega bo'lgan vintli chiziq**.**

$$\begin{cases} v_\rho = 0 \ ; \\ v_\varphi = \rho_0\dot\varphi_0 \ ; \\ v_z = \dot z_0 ; \end{cases} \qquad \begin{cases} w_\rho = -\rho_0\dot\varphi_0 \ ; \\ w_\varphi = 0; \\ w_z = 0; \end{cases} \qquad \vec{w} = -\rho\dot\varphi_0\vec{n}_\rho .$$

3. $\theta = \dfrac{\pi}{2}; \quad \dfrac{r - r_0}{\dot r_0} = \dfrac{\varphi - \varphi_0}{\dot\varphi_0}$ qonuniga ega bo'lgan spiral.

$$\begin{cases} v_r = \dot r_0 \ ; \\ v_\theta = 0; \\ v_\varphi = r\dot\varphi_0 = (\dot r_0 t + r_0)\dot\varphi_0 \ ; \end{cases} \qquad \begin{cases} w_r = -r\dot\varphi^2{}_0 = -(\dot r_0 t + r_0); \\ w_\theta = 0; \\ w_\varphi = 2\dot r_0\dot\varphi_0 \ ; \end{cases}$$

4. $x = -\dfrac{E_0}{m\omega^2}\cos\omega\dfrac{y - y_0}{v_0} + x_0 , \quad z = z_0 .$

5. $x = \frac{E_0 (y - y_0)^2}{2 v_0^2 \sin^2 \alpha} + (y - y_0) ctg\alpha + x_0, \quad z = z_0.$

Uslubiy tavsiyalar:

1.Har bir masalada ishlatiladigan koordinata sistemasining tanlab olinishini asoslash lozim.Talabalar takliflarini ham qarab chiqish kerak.

2.Trayektoriya tenglamasi va tezlik,tezlanish vektorlari proyeksiyalari qaralayotgan koordinata sistemasida chiqarilishi lozimligini tushuntirish kerak bo'ladi.Trayektoriya chizmasi moddiy nuqta harakatining oddiy ucho'lchamli fazodagi harakatiga mos kelmog'i lozim.

3.Oldingi mashg'ulotda qatnashmagan talabani darsda qatnashgan talaba yoniga o'tqazish lozim.

6-MASHG’ULOT
Mavzu: «Moddiy nuqta harakat qonunlari va trayektoriyasi»

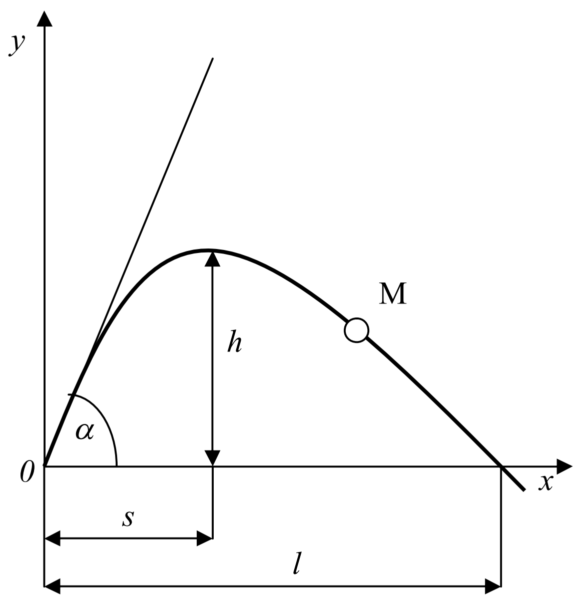
Darsning maqsadi: Harakatning berilgan qonunlari bo’yicha nuqta harakat trayektoriyasini,tezligi va tezlanishini hisoblashga qaratilgan masalalar yechish.

Qisqacha tushuncha. Nuqtaning trayektoriyasini aniqlashni talab qilgan masalalar ikki variantda yechilishi mumkin:

1.Agar harakat qonuni berilgan bo’lsa,bu qonunlardan vaqtni yo’qish bilan osongina trayektoriya topiladi.Bu yo’l bilan nuqta uchta koordinatalari o’rtasida bog’lanish topiladi.Bu holda nuqta tezligi va tezlanishi berilgan harakat qonunlarini vaqt bo’yicha differensiallab topiladi.Bu holda koordinata sistemalariga mos tezlik va tezlanish proyeksiyalari oldingi mavzuda topilgani kabi topiladi.

2.Agar harakat tenglamalari ma’lum bo’lsa,nuqta tezligi va harakat qonuni vektorli ko’rinishdagi harakat tenglamalarini integrallash yo’li bilan topiladi.Topilgan vektorli $\vec{r}(t)$ funksiya tanlab olingan koordinata sistemasining ortiga proyeksiyalanadi.Yoki berilgan tenglamalarni proyeksiyalarini topib,keyin esa ularni integrallash mumkin.Ikkala holda ham trayektoriya uchun tenglama topiladi.

Masalalar



6-1. Gorizontga α burchak ostida tashlangan M nuqta (havo qarshiligi hisobga olinmasin)

$$x = v_0 \cos\alpha \cdot t \qquad (1),$$

$$y = v_0 \sin\alpha \cdot t - gt^2/2 \qquad (2)$$

tenglamalarga asoslanib harakat qiladi. Bu yerda v_0, α, g – doimiy kattaliklar.

Trayektoriya tenglamasi,nuqtaning boshlang’ich holatiga nisbatan eng baland ko’tarilishi (h),eng baland ko’;tarilish holatigacha gorizont bo’ylab masofa (s) hamda gorizont bo’ylab eng uzoq uchish masofasi (l) hisoblansin.

Yechish. Trayektoriya tenglamasining ochiq ko’rinishini toppish uchun nuqtaning harakat tenglamalaridan vaqtni yo’qatamiz

$$t = \frac{x}{v_0 \cos\alpha} \qquad (3)$$

Buni ikkinchi tenglamaga qo’yamiz:

$$y = tg\alpha \cdot x - \frac{g}{2 v_0^2 \cos^2\alpha} x^2 \qquad (4)$$

Analitik geometriyadan ma'lumki,oxirgi tenglama trayektoriya tenglamasi bo'lib,u y-o'qiga parallel bo'lgan simmetriya o'qli parabola tenglamasi bo'ladi.Haqiqatan, y ning har bir qiymatiga x ning ikkita qiymati mos keladi.Bu parabola koordinata boshidan o'tadi,chunki $x=0, y=0$ koordinatalar qiymatlari uning tenglamasini qanoatlantiradi.

Nuqtaning gorizontga nisbatan eng yuqori balandlikka ko'tarilishini toppish uchun qoidaga ko'ra y ning ekstremal qiymatini topishimiz lozim bo'ladi.Buning uchun y dan x bo'yicha hosila olib,uni nolga tenglashtiramiz:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=0,$$

lekin bizda $\frac{dx}{dt}\neq 0$.Shuning uchun $\frac{dy}{dt}=0$ bo'lishi lozim.

$$\frac{dy}{dt}=v_0\sin\alpha-g\,t_1=0 \tag{5}$$

Demak, y o'zining ekstremal qiymatiga t_1 ning

$$t_1=\frac{v_0\sin\alpha}{g} \tag{6}$$

qiymatida erishadi.Vaqtning bu qiymatini (2) tenglamaga qo'yib,eng yuqori ko'tarilish balandligini topamiz

$$h=y_{\max}=\frac{v_0^{\,2}\sin^2\alpha}{g}-\frac{g\,v_0^{\,2}\sin^2\alpha}{2g^2}=\frac{1}{2}\frac{v_0^{\,2}\sin^2\alpha}{g}$$

y ning bu ekstremal qiymati minimum bo'lmasdan,maksimum bo'ladi,chunki $t=t_1$ da y dan olingan ikkinchi hosila manfiy bo'ladi:

$$\ddot{y}\big|_{t=t_1}=-g$$

Nuqtaning eng yuqoriga ko'tarilish balandligiga to'g'ri keluvchi abscissa qiymatini aniqlash uchun (6) ni (1) tenglamaga qo'yish kerak

$$s=x_1=v_0\cos\alpha\frac{v_0\sin\alpha}{g}=\frac{v_0^{\,2}\sin 2\alpha}{2g}$$

Gorizontal bo'yicha uchish uzoqligi l (4) trayektoriya tenglamasidan $y=0$ bo'lganda topiladi:

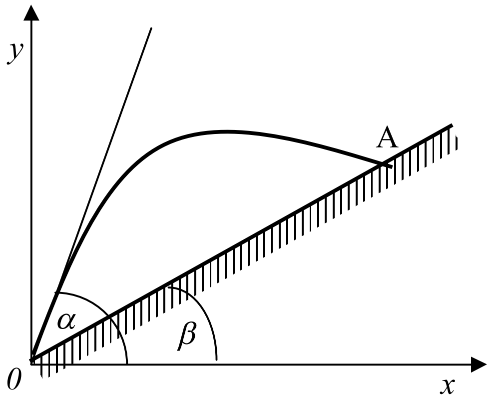
$$x\cdot tg\alpha-\frac{g\,x^2}{2v_0^{\,2}\cos^2\alpha}=0$$

Bu tenglikdan x ning ikkita qiymatini topamiz:

$$x_0=0,\;\; x_2=l=\frac{v_0^{\,2}\sin 2\alpha}{g}$$

Bu qiymatlarning birinchisi nuqtaning boshlang'ich holatiga to'g'ri kelsa,ikkinchisi gorizontal bo'yicha uchish uzoqligini ko'rsatadi.Biz l, s qiymatlarini solishtirib, $l=2s$,yani nuqtaning eng yuqori holatiga gorizontal uzoqlikning yarmida erishishini ko'ramiz.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com



6-2. Sirti gorizontga nisbatan β burchakka og'gan tepalik pastida joylashgan quroldan otilgan snaryad

$$x = v_0 \cos\alpha \cdot t \qquad (1),$$

$$y = v_0 \sin\alpha \cdot t - gt^2/2 \qquad (2)$$

tenglamalar bo'yicha harakat qiladi.Bu yerda α – gorizont va snaryadning uchib chiqish yo'nalishi o'rtasidagi burchak.

Snaryadning OA yo'nalishi bo'yicha eng uzoqqa uchishi uchun u qanday α burchak ostida otilishi kerak?

Yechish. Oldingi masaladagidek,(1) va (2) lardan trayektoriya tenglamasini topamiz $y = tg\alpha \cdot x - \dfrac{g}{2v_0^2\cos^2\alpha}x^2$ (3)

Tepalik sirtining xy vertical tekislikka proyeksiyasi hisoblangan OA to'g'ri chiziqning tenglamasi bo'lib

$$y = x\,tg\,\beta \qquad (4)$$

tenglama hisoblanadi.Snaryad yerga A nuqtada tushadi va (3) va (4) ordinatalar o'zaro teng bo'ladi

$$x\,tg\,\beta = tg\alpha \cdot x - \frac{g}{2v_0^2\cos^2\alpha}x^2$$

Bundan

$$x = (tg\,\alpha - tg\,\beta)\frac{2v^2{}_0}{g}\cos^2\alpha = \frac{v_0^2}{g}(\sin 2\alpha - 2\cos^2\alpha \cdot tg\beta).$$

α burchakka bog'liq ravishda eng uzoqqa uchishni aniqlash uchun x dan α bo'yicha hosila olib,uni nolga tenglashtirish lozim

$$\frac{dx}{d\alpha} = \frac{v_0^2}{g}(2\cos 2\alpha + 2\sin 2\alpha \cdot tg\,\beta) = 0$$

Bundan

$$ctg\,2\alpha = -tg\,\beta,$$

yoki

$$2\alpha = \frac{\pi}{2} + \beta, \quad \alpha = \frac{1}{2}\left(\frac{\pi}{2} + \beta\right)$$

TOPSHIRIQ

Nuqta harakati silindrik koordinatalarda

$$\rho = \rho_0(1 + \beta t), \ \varphi = \varphi_0(1 + \beta t), \ z = z_0(1 + \beta t)$$

vaqtning funksiyalari tariqasida berilgan bo'lsa,nuqtaning trayektoriyasi va trayektoriya bo'ylab harakat qonunini aniqlang.

JAVOB:

$\dfrac{z}{z_0} = \dfrac{\rho}{\rho_0}$ – vint chizig'i, $s = \beta t\sqrt{\rho_0^2 + \rho^2{}_0\varphi^2{}_0 + z_0^2}$ -harakat qonuni.

Uslubiy tavsiyalar:

1.Har bir masalada ishlatiladigan koordinata sistemasining tanlab olinishini asoslash lozim.Talabalar takliflarini ham qarab chiqish kerak.

2.Trayektoriya tenglamasi va tezlik,tezlanish vektorlari proyeksiyalari qaralayotgan koordinata sistemasida chiqarilishi lozimligini tushuntirish kerak bo'ladi.Trayektoriya chizmasi moddiy nuqta harakatining oddiy ucho'lchamli fazodagi harakatiga mos kelmog'i lozim.

3.Oldingi mashg'ulotda qatnashmagan talabani darsda qatnashgan talaba yoniga o'tqazish lozim.

7-MASHG'ULOT
Mavzu: «Bir o'lchamli erkin harakat»

Darsning maqsadi. Moddiy nuqtaning erkin harakatning ikki sohasini ajratib turgan chegaradan o'tish holini qarab chiqish.Talabalarni berilgan potensial maydonda nuqtaning bir o'lchamli harakat xarakterini tadqiq qilishni o'rgatish.

Qisqacha tushuncha. Potensial energiya U=const bo'lgan moddiy nuqta erkin harakatining har bir sohasida E energiya, $\vec{p}$ impuls va $\vec{M}$ impuls momenti saqlanuvchan kattaliklar bo'lib hisoblanadi.Nuqta sohalarni ajratib turgan chegaradan o'tishida bu chegaraga tik bo'lgan yo'nalishda fazoning birjinsligi buziladi.Harakatning bir-biriga bog'liq bo'lmagan integrallari bo'lib faqat energiya va ajralish chegarasiga parallel yo'nalishda impuls vektorining komponentasi hisoblanadi.Bu saqlanish qonunlari qaralayotgan ikki sohani ajratib turgan chegaradan moddiy nuqta o'tayotganida kattaliklarning va tezlik yonalishining o'zgarishini aniqlash imkonini beradi (yorug'likning ikki muhit chegarasidan o'tishdagi sinish qonuniga o'xshash).

2.Bir o'lchamli harakatni tadqiq qilishda potensial energiya U ning x koordinataga qanday bog'liqligi muhim ahamiyatga ega: nuqta energiyasi harakat integrali sanaladi- E=const.Nuqta harakati faqatgina $E \geq U(x)$ bo'lganda mumkin bo'ladi. $E = U(x)$ tenglamaning ildizlari E energiyalik nuqtaning burilish nuqtalarini aniqlaydi.Nuqta (x_1, x_2) burilish nuqtalari orasidagi potensial o'rada harakat qilsa,uning harakati finitli harakat hisoblanadi.U holda $E \geq U(x)_{\min}$ finitli harakat energiyasining ruxsat etilgan qiymatlari sohasini aniqlaydi ($U(x)_{\min}$ –potensial o'raning «tubi»).Harakat $x \geq x_3$ sohada sodir bo'lsa,moddiy nuqta $+\infty$ gacha ketishi mumkin-harakat infinitli bo'ladi (xuddi shunday -∞ga ham ketishi mumkin).Bir o'lchamli harakat uchun

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}} + t_0$$

o'rinli bo'lgani uchun potensial o'rada nuqtaning harakat qilish davri x_1 nuqtadan x_2 nuqtagacha va teskari harakat vaqti tariqasida aniqlanadi:

$$T = 2\sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$

Masalalar

7-1. Massasi $m = 1$ nuqtaning $U(x) = x$ potensial maydonda harakatqilsa,uning burilish nuqtalarini toping va harakatni integrallang. Boshlang'ich vaqt momentida $x(0) = 1, \dot{x}(0) = 0$.

Yechish. Nuqta bir o'lchamli,yani x –o'qi bo'ylab harakat qiladi.Uning uchun burilish nuqtasi to'liq energiyasi

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$E = T + U(x) = \frac{1}{2}\dot{x}^2 + x$ ning burilish nuqtasida potensial energiyasi $U(x) = x$ ga tengligi bilan aniqlanadi:

$$\frac{1}{2}\dot{x}^2 + x = x$$

Bundan $\frac{1}{2}\dot{x}^2 = 0$ yoki $x(t) = const$ ekanligini topamiz.Boshlang'ich shartga ko'ra $const = 1$.Demak, $x(t) = 1$ va nuqtaning to'liq energiyasi $E = 1$.U holda nuqtaning potensial maydondagi harakat vaqti

$$t = \sqrt{\frac{1}{2}}\int\frac{dx}{\sqrt{E - U(x)}} = \sqrt{\frac{1}{2}}\int\frac{dx}{\sqrt{1-x}} = -\sqrt{2(1-x)} + const$$

Boshlang'ich moment $t = 0$ da $x(0) = 1$ bo'lgani uchun $const = 0$ bo'ladi. Demak, $t = -\sqrt{2(1-x)}$. Bu tenglikning har ikkala tomonini kvadratga ko'tarib, $x = 1 - \frac{1}{2}t^2$ ekanligini topamiz. Shu holga mos keladigan harakat tenglamasi $\ddot{x} = -1$ ni yechib,yana biz olgan natijaga kelish mumkin.Haqiqatan,bu tenglamani vaqt bo'yicha integrallasak, $\dot{x} = -t + const$. Boshlang'ich shartdan $const = 0$ va $\dot{x} = -t$. Bu tenglikni yana bir bor integrallab, $x(t) = -\frac{t^2}{2} + const$. $t = 0$ da $const = 1$. Demak, $x = 1 + \frac{1}{2}t^2$.

Quyida keltirilayotgan rasmlarda berilgan potensial maydondagi harakat va nuqtaga ta'sir qilayotgan kuch yo'nalishi ko'rsatilgan.

rasm

7-2. $U(x) = \frac{1}{2}kx^2$ potensial maydonda nuqta harakat qonuni aniqlansin.

Yechish. Berilgan potensialni

$$t = \sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E - U(x)}}$$

formulaga qo'yamiz:

$$t = \sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E - \frac{1}{2}kx^2}} = -\sqrt{\frac{m}{k}}\arccos\left(x(t)\sqrt{\frac{k}{2E}}\right) + const.$$

Bundan

$x = \sqrt{\frac{2E}{k}}\cos\left(\sqrt{\frac{k}{m}}t + \alpha\right)$.Bu yerda α-boshlang'ich shartdan topiladigan doimiy son.Masalan, $t = 0$ vaqt momentida $x(0) = 0$ deb hisoblasak, $\alpha = \frac{\pi}{2}$ bo'lishi kerak.

7-3. $U(x) = x^2$ maydonda $m = 1$ massaga ega bo'lgan moddiy nuqta uchun burilish nuqtasi, tebranish davri topilsin va harakatning finitlik sharti ko'rsatilsin.

Yechish. Harakatning finitlik sharti $E \geq U_{\min}$ dan foydalanib,
$U(x) = U(0) = 0$
ega bo'lamiz.Demak, $E \geq U$ bo'lib, moddiy nuqta finitli harakatlanishini ko'ramiz.Bu nuqtaning berilgan potensial maydonda burilish nuqtasini topish uchun
$E = U(x)$
Tenglamani yechin,uning ildizlarini topamiz: $x_{1,2} = \pm\sqrt{E}$.

Moddiy nuqtaning berilgan maydonda tebranish davri

$$T = \frac{2}{\sqrt{2}} \int_{-\sqrt{E}}^{+\sqrt{E}} \frac{dx}{\sqrt{E - x^2}} = \frac{2}{\sqrt{2}} \int_{-\sqrt{E}}^{+\sqrt{E}} \frac{d\left(\frac{x}{\sqrt{E}}\right)}{\sqrt{1 - \left(\frac{x}{\sqrt{E}}\right)^2}} = \frac{2}{\sqrt{2}} \arcsin \frac{x}{\sqrt{E}} \Big|_{-\sqrt{E}}^{+\sqrt{E}} =$$

$$= \frac{2}{\sqrt{2}} \left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{2\pi}{\sqrt{2}}.$$

yoki

$$T = \pi\sqrt{2}$$

1-TOPSHIRIQ

1.Massasi m,tezligi v_1 bo'lgan zarra potensial energiyasi $U_1 = const$ 1-yarimfazodan
potensial energiyasi $U_2 = const$ 2-yarimfazoga o'tganda v_2 tezlik qiymati va $n = \frac{\sin\theta_1}{\sin\theta_2}$ sindirish ko'rsatgichi aniqlansin.Bu yerda θ_1, ϑ_2 lar $\vec{v}_1, \vec{v}_2$ tezliklarning 1- va 2-yarimfazolar ajralish tekisligiga o'tkazilgan normal o'rtalardagi burchaklar.

2-4-TOPSHIRIQLAR

Massasi m=1 bo'lgan moddiy nuqta

2. $U(x) = x^3$;
3. $U(x) = tg^2 x$;
4. $U(x) = -\frac{1}{x^2}$

potensial maydonlarda harakat qilsa,harakatning finitlik sharti ko'rsatilsin,burilish nuqtalari va tebranish davri hisoblansin.

5-TOPSHIRIQ

Massasi m=2 bo'lgan moddiy nuqtaning $U(x) = -tg^2 x$ potensial maydonda harakat tenglamasini ntegrallang (t=0 boshlang'ich vaqtda x=0 , $\dot{x} = 2$)

JAVOBLAR:

1. $v_2 = v_1\sqrt{1 + 2\frac{u_1 - u_2}{mv_1^2}}, \quad n = \frac{v_2}{v_1}\sqrt{1 + 2\frac{u_1 - u_2}{mv_1^2}}.$

2. $E \geq 0$, $x_{1,2} = \pm\sqrt{E}$, $T = \pi\sqrt{2}$.

3. $E \geq 0$, $x_{1,2} = \pm arctg\sqrt{E}$, $T = \dfrac{\pi\sqrt{2}}{\sqrt{E+1}}$.

4. $E \prec 0$, $x_{1,2} = \pm\sqrt{\dfrac{1}{|E|}}$, $T = \dfrac{2\sqrt{2}}{|E|}$.

5. $t = \dfrac{1}{\sqrt{3}}\arcsin\dfrac{\sqrt{3}}{2}x$, $\left(x = \dfrac{2}{\sqrt{3}}\sin\sqrt{3}t\right)$

Uslubiy tavsiyalar:

1.Birinchi masalada qanday harakat integrallari mavjud bo'lishligini muxokama qilmoq zarur.

2.Bir o'lchamli harakatlarga bag'ishlangan masalalarda

a) potensial energiya grafigini chizish;

b) burilish nuqtalarini aniqlash;

c) berilgan turdagi harakat amalgam oshayotgan energiyaning qiymatlari mumkin bo'lgan sohani aniqlash;

d) berilgan energiyali nuqta finitli harakati davrini hisoblash zarur bo'ladi.

3.3-masalada $\sin\alpha$ ni $tg\alpha$ orqali ifodalashga to'g'ri keladi. Buning uchun quyidagi shakllardan foydalanish maqsadga muvofiqdir:

8-MASHG'ULOT

Mavzu: «Markaziy kuch maydonida harakat va uning xarakterini aniqlash»

Darsning maqsadi. Talabalar $U_{eff}(r)$-effektiv potensialning grafigini chizishni va undan berilgan markaziy kuch maydonlarida zarra harakat xarakterini tahlil qilishda foydalanishni o'rganishi lozim.

Qisqacha tushuncha. Zarraning markaziy kuch maydonida harakati paytida uning energiyasi va impuls momenti saqlanuvchan (harakat integrali) bo'ladi: $E = const, \vec{l} = const.$ Harakat kuch markazidan o'tuvchi $\vec{l}$ ga tik bo'lgan tekislikda sodir bo'ladi.Kuch markaziga qutb (z=0 bo'lgandagi silindrik) koordinata sistemasi boshini joylashtirib,topamiz

$$l = mr^2\dot{\varphi}, \quad E = \frac{m}{2}\dot{r}^2 + U_{eff}(r), \tag{1}$$

$$U_{eff}(r) = \frac{l^2}{2mr^2} + U(r), \tag{2}$$

Bu yerda $\frac{l^2}{2mr^2}$ - markazdan qochma energiya deyiladi, $\dot{\varphi} = \frac{d\varphi}{d\,t}$ – burchakli tezlik.

Shunday qilib,zarraning radial harakati $U_{eff}(r)$ maydondagi bir o'lchamli harakatiga mos keladi.Shuning uchun $E, \vec{l}$ ruxsat etilgan harakat sohalari $E > U_{eff}(r)$ sharti bilan aniqlanadi,burilish nuqtalari ko'rganimizdek, $E = U_{eff}(r)$ shartdan topiladi.Ma'lumki,markazdan qochma energiya kuch markazi atrofida potensial to'siqni yaratadi. $U(r)$ning berilishiga qarab uch turdagi harakatning sodir bo'lishi mumkin bo'ladi:: $r = r_{\min}$ dan $r \to \infty$ gacha infinitli harakat, $r_{\min}$ dan $r_{\max}$ gacha burilish nuqtalari orasida finitli harakat,va harakatningruxsat etilgan sohasi bo'lib $r = r_{\max}$ radiusli aylana hisoblanganda markazga tushib qolish.Saqlanuvchan energiya va impuls momentining mumkin bo'lgan qiymatlari har bir konkret holda $U_{eff}(r)$ grafikasining ko'rinishidan topiladi. r_1 va r_2 o'rtasidagi masofani bosib o'tish uchun ketgan vaqt

$$t = \pm\sqrt{\frac{m}{2}}\int_{r_1}^{r_2}\frac{dr}{\sqrt{E - U_{caì}(r)}}, \tag{3}$$

Formula bilan hisoblanadi.Bu yerda $\pm$ ishora $\frac{dr}{dt}$ radail harakat tezligining ishorasiga mos keladi.

Masalalar

8-1. $U(r) = \frac{1}{2r^2}$ maydonda $m = 1$ massaga ega bo'lgan zarra uchun burilish nuqtalari topilsin va harakatning xarakteri tekshirilsin.

Yechish. Burilish nuqtalarini aniqlash uchun (2) tenglikni hisobga olgan holda $E = U_{eff}(r)$ tenglamadan foydalanamiz

$$Ueff(r) = \frac{l^2}{2mr^2} + \frac{1}{2r^2} = E\,.$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Masala shartiga ko'ra $m=1$,yani

$$\frac{l^2}{2r^2}+\frac{1}{2r^2}=E$$

yoki

$$2r^2E=l^2+1,$$

bundan

$$r=\sqrt{\frac{l^2+1}{2E}}$$

ekanligini topamiz. l^2 albatta musbat hisoblanadi.Agar $E\prec 0$ bo'lsa, r kompleks son bo'lib qoladi..Shuning uchun $E\succ 0$ bo'lishi talab qilinadi,demak,zarra harakati infinitli harakat bo'ladi.

8-2. Massasi m bo'lgan moddiy nuqta $U(r)=\frac{\alpha}{r}$ (bu yerda $\alpha>0$) potensial maydonda harakat qilsa,uning markazga eng yaqin va markazdan eng uzoq joylashgan holati aniqlansin.

Yechish. Berilgan maydonda nuqta uchun effektiv potensial energiya

$$U_{eff}(r)=\frac{l^2}{2mr^2}+\frac{\alpha}{r}$$

Shaklga ega bo'ladi.Burilish nuqtalarini topamiz:

$$E=\frac{l^2}{2mr^2}+\frac{\alpha}{r}$$

Bundan

$$2mEr^2-2m\alpha r-l^2=0$$

kvadrat tenglamaga kelamiz va undan

$$r_{\min}=\frac{\alpha}{2E}\left(1+\sqrt{1+\frac{El^2}{m\alpha^2}}\right),\quad r_{\max}=\frac{\alpha}{2E}\left(1-\sqrt{1+\frac{El^2}{m\alpha^2}}\right)$$

ekanligini topamiz.

1-5-TOPSHIRIQLAR

Massasi $m=1$ bo'lgan zarra uchun quyidagi maydonlarda burilish nuqtalari hisoblansin va harakatning xarakteri tekshirilsin:

1. $U(r)=\frac{1}{2}r^2$
2. $U(r)=\left(\frac{1}{r^2}+r^2\right)\frac{1}{2}$
3. $U(r)=\frac{1}{2r^4}$
4. $U(r)=-\frac{\alpha^2}{2r^2},\ \alpha=const$

5. $U(r) = -\dfrac{\alpha^2}{r^2}$, $\alpha = const$

Eslatma**:** 5-masalada burilish nuqtasi energiyaning $E = 0$ qiymati uchun axtarilsin.

6-7-TOPSHIRIQLAR

4-masalada $l^2 \prec \alpha^2$ va 5-masalada $E = 0$ bo'lganida zarraning $r = r_{\max}$ nuqtadan markazga tushib qolish vaqti aniqlansin.Markazga tushishda $\dfrac{dr}{dt} \prec 0$ bo'ladi deb olinsin.

JAVOBLAR:

1. $r_{\min} = \sqrt{\dfrac{l^2+1}{2E}}$. Harakat finitli, l – istalgan

2. $r_{1,2} = \sqrt{E \pm \sqrt{E^2 - l^2}}$. Harakat finitli, $E \geq |l|$, l- istalgan.

3. $r_{1,2} = \sqrt{E \pm \sqrt{E^2 - (l^2+1)}}$. Harakat finitli, $E \geq \sqrt{l^2+1}$, l-istalgan.

4. $r_{\min} = \sqrt{\dfrac{l^2 + \sqrt{l^4 + 16E}}{4E}}$. Harakat infinitli, $E \succ 0$, l-istalgan.

5. a) $r_{\min} = \sqrt{\dfrac{l^2 - \alpha^2}{2E}}$. Harakat infinitli, $E \succ 0$, $l^2 \succ \alpha^2$.

b) $r_{\max} = \sqrt{\dfrac{\alpha^2 - l^2}{2|E|}}$. $E \succ 0$, $l^2 \prec \alpha^2$ bo'lganda markazga tushish

$E \geq 0$, $(l^2 \prec \alpha^2)$ bo'lgan holda burilish nuqtasi mavjud bo'lmaydi, $r = 0$ dan to $r = \infty$ gacha zarra to'siqsiz harakat sodir etadi;

c) $E \geq 0$ $(l^2 = \alpha^2)$ bo'lganda radius-vektor bo'yicha erkin harakat.

6. a) $E \succ (U_{eff})_{\max}$ bo'lganda $r = 0$ dan to $r = \infty$ gacha to'siqsiz harakat;

b) $E = (U_{eff.})_{\max}$ bo'lganda r_0 radiusli aylana bo'ylab harakat $\left((U_{eff})_{\max} = U_{eff}(r_0)_{\max}\right)$

c) $(U_{eff.})_{\max} \succ E \succ 0$, $r \succ r_0$ bo'lganda infinitli harakat;

d) $E \prec (U_{eff.})_{\max}$, $r \succ r_0$ bo'lganda markazga tushish. $E = 0$ bo'lganda:

$r_{\max} = \sqrt{\dfrac{\alpha}{l}\sqrt{2}}$, l-istalgan.

7. $t = \dfrac{\sqrt{\alpha^2 - l^2}}{2|E|} > 0$ (harakat kuch markazitomon bo'lganda, tezlik $\dfrac{dr}{dt} \prec 0$ va ildiz oldida minus ishora turadi va $t \succ 0$ (musbat bo'lishini) ta'minlaydi).

8. $t = \dfrac{\alpha}{l^2\sqrt{2}} \succ 0$.

Uslubiy tavsiyalar:

1.Effektiv potensial egriligi taxtada chiziladi.

2.Bu ngrafikda turli tipdagi (finitli,infinitli,markazga tushib qolish) harakat sohalari ajratilib ko'rsatiladi.

3.Burilish nuqtalari topiladi.

4.Qaralayotgan sohada harakatning berilgan tipi amalga oshayotgan holda energiya va impuls momentining qiymatlari aniqlanadi.

5. r ning radius-vektor qiymatini bildirishini bildirishini,uning manfiy va mavhum bo'la olmasligini talabalarga qayta-qayta uqdirish kerak bo'ladi.

9-MASHG'ULOT

Mavzu: «Markaziy maydonda harakatning trayektoriyasi va qonunlari»

Darsning maqsadi. Talabalarda 5-mashg'ulotdan olgan bilim va ko'nikmalarini yanada mustahkamlash. Bu bilimlarni markaziy kuch maydonida harakatiga oid yanada murakkabroq masalalarni yechishda ishlatish .

Qisqacha tushuncha. Ma'lumki, moddiy nuqtaning markaziy kuch maydonida harakat trayektoriyasi

$$\varphi-\varphi_0=\frac{l}{\sqrt{2m}}\int\frac{dr}{r^2\sqrt{E-U_{eff}(r)}}. \tag{1}$$

kvadraturadan aniqlanadi. Harakat qonunini aniqlash uchun quyidagi integral xizmat qiladi:

$$t-t_0=\sqrt{\frac{m}{2}}\int\frac{dr}{\sqrt{E-U_{eff}(r)}}. \tag{2}$$

$t=t(r)$ va $\varphi=\varphi(r)$ tenglamalarni bilgan holda moddiy nuqtaning $r=r(t)$ va $\varphi=\varphi[r(t)]$ harakat qonunlarini topish mumkin. Odatda markaziy kuch maydonida integrallash deganda $t=t(r)$ va $\varphi=\varphi(r)$ larni hisoblash tushuniladi.

Masalalar

9-1. Massasi $m=1$ bo'lgan moddiy nuqtaning $U(r)=\dfrac{1}{2r^2}$ maydonda harakat tenglamasi integrallansin ($E.>0$, l esa istalgan qiymatlar qabul qiladi).

Yechish. Ma'lumki, effektiv potensial energiya

$$U_{eff}(r)=\frac{l^2}{2mr^2}+U(r).$$

shaklda yozilar edi. Masala shartiga ko'ra bu energiyani quyidagicha yozamiz:

$$U_{eff}(r)=\frac{l^2}{2mr^2}+\frac{1}{2r^2}=\frac{l^2}{2r^2}+\frac{1}{2r^2}=\frac{l^2+1}{2r^2}.$$

Nuqta harakat qonuni (2)ga $U_{eff}(r)$ qiymatini qo'yish orqali topiladi

$$t-t_0=\sqrt{\frac{m}{2}}\int\frac{dr}{\sqrt{E-\frac{l^2+1}{2r^2}}}=\frac{\sqrt{2}}{\sqrt{2}}\int\frac{rdr}{\sqrt{2Er^2-(l^2+1)}}=$$

$$=\int\left[2Er^2-\left(l^2+1\right)\right]^{-1/2}rdr=\frac{1}{2E}\int\left[2Er^2-\left(l^2+1\right)\right]^{-1/2}d\left[2Er^2-\left(l^2+1\right)\right]=$$

$$=\frac{1}{2E}\left[2Er^2-\left(l^2+1\right)\right]^{1/2}=\frac{1}{2E}\sqrt{2Er^2-\left(l^2+1\right)}.$$

Ushbu tenglikdan radius-vektorning qiymatini topamiz

$$2E\left(t-t_0\right)=\sqrt{2Er^2-\left(l^2+1\right)} \tag{1}$$

yoki

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$(t-t_0)=\frac{1}{2E}\sqrt{2Er^2-(l^2+1)}$$

(1) ning har ikkala tomonini kvadratga ko'taramiz:

$$4E^2(t-t_0)^2=2Er^2-(l^2+1),$$

bundan

$$r^2=2E(t-t_0)^2+\frac{l^2+1}{2E}$$

Demak,

$$r=\sqrt{2E(t-t_0)^2+\frac{l^2+1}{2E}}.$$

Moddiy nuqtaning harakat trayektoriyasi uchun masala shartiga ko'ra

$$\varphi-\varphi_0=\frac{l}{2m}\int\frac{dr}{r^2\sqrt{E-U_{eff}(r)}}=\frac{l}{2m}\int\frac{dr}{r^2\sqrt{E-\left(\frac{l^2+1}{2r^2}\right)}}=$$

$$=\frac{l}{2}\int\frac{r\sqrt{2}dr}{r^2\sqrt{2Er^2-(l^2+1)}}=\frac{l\sqrt{2}}{2}\int\frac{dr}{r\sqrt{2Er^2-(l^2+1)}}=$$

$$=\frac{2l}{2\sqrt{2}}\int\frac{dr}{r\sqrt{2Er^2-(l^2+1)}}=\frac{l}{\sqrt{l^2+1}}\arccos\sqrt{\frac{l^2+1}{2E}}\cdot\frac{1}{r}.$$

Oxirgi tenglikdan radius-vektor kattaligini topamiz

$$r=\sqrt{\frac{l^2+1}{2E}}\sec\frac{\sqrt{l^2+1}}{l}(\varphi-\varphi_0).$$

9-2. Massasi $m=1$ bo'lgan zarraning

$$U(r)=\frac{1}{2}\left(\frac{1}{r^2}-\frac{1}{r}\right)$$

potensial maydondagi harakati integrallansin.

Yechish. Berilgan potensial markaziy maydonni hosil qilgani uchun bu maydonda zarraning trayektoriya tenglamasi

$$\varphi=\int\frac{\frac{l}{r^2}dr}{\sqrt{2[E-U(r)]-\frac{l^2}{r^2}}}=l\int\frac{\frac{dr}{r}}{\sqrt{2Er^2+r-(l^2+1)}}\qquad(1)$$

Bu integral $\int\frac{dx}{x\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{-c}}arctg\frac{2c+2b}{2\sqrt{-c}\sqrt{ax^2+bx+c}}$ ($c<0$ bo'lganda)

Jadval integrali hisoblanadi.Bu natijani (1) bilan solishtirib,topamiz: $x=r,\ a=2E,\ b=1,\ c=-(l^2+1)<0$,chunki l^2 (xuddi shunday $l>0$) hamma vaqt musbat qiymatlarga ega bo'ladi.Demak,(1) integral qiymati quyidagicha bo'ladi:

$$\varphi=\int\frac{\frac{l}{r^2}dr}{\sqrt{2[E-U(r)]-\frac{l^2}{r^2}}}=l\int\frac{\frac{dr}{r}}{\sqrt{2Er^2+r-(l^2+1)}}=\frac{l}{\sqrt{l^2+1}}arctg\frac{r-2(l^2+1)}{2\sqrt{l^2+1}\sqrt{2Er^2+r-(l^2+1)}}$$

1-2-TOPSHIRIQLAR

1. $E=0$, l-ning istalgan qiymati uchun $U(r)=-\frac{\alpha^2}{r^6}$ maydonda harakatlanayotgan va massasi $m=1$ bo'lgan moddiy nuqtaning trayektoriyasi topilsin Bu nuqta $r=r_{\max}$ dan $r=0$ gacha harakatlanganda qanday burchakka buriladi?

2.Massasi m ga teng bo'lgan moddiy nuqta $U(r)=\frac{\chi}{3}r^3$ $(\chi>0\ const)$ maydonda harakatlanmoqda.Agar impuls momenti l ,radiusi R, burchak tezligi ω ga teng va trayektoriyasi aylanadan iborat bo'lgan moddiy nuqtaning to'liq energiyasi hisoblansin.

JAVOBLAR:

1. $\varphi-\varphi_0=\frac{1}{2}\arcsin\frac{l\,r^2}{\alpha\sqrt{2}};\ r=\sqrt{\frac{\alpha\sqrt{2}}{l}\sin\ 2(\varphi-\varphi_0)};$

$\Delta\varphi=\frac{\pi}{4}$ (aylanishning 1/8 qismiga burilish).

2. $R=\left(\frac{l^2}{m\chi}\right)^{1/5}, E=\frac{5}{6}l\omega,\ \ \omega=\left(\frac{l\chi^2}{m^3}\right)^{1/5}$

Uslubiy tavsiyalar:

1.Bu mashg'ulotda keltirilgan masalalar oldingi mashg'ulotda keltirilgan masalalarning davomi hisoblanadi.

2.Agar masalani yechishda $t=t(r),\ \varphi=\varphi(r)$ bog'lanishlar topilgan bo'lsa,masala yechilgan hisoblanadi.Masala yechishda $r=r(t)$-zarraning radius-vektor bo'ylab harakat qonunini ham va $r=r(\varphi)-$trayektoriya tenglamasini ham topish maqsadga muvofiqdir.

3.Talabalar albatta effektiv potensial energiyaning grafigini chizishi zarur . Bu chizma harakatning finitli yoki infinitli ekanligini yaqqol ko'rsatib turadi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

10-MASHG'ULOT
Mavzu: «Kulon maydonida harakat»

Darsning maqsadi. Nyutonning tortishuv maydoni yoki Kulon elektrostatik maydonida harakat qilayotgan moddiy jism uchun kosmik tezliklarni hisoblash, Kepler qonunlaridan butunolam tortilish qonunini, butunolam tortilish qonunidan esa Kepler qonunlarini keltirib chiqarish.

Qisqacha tushuncha. Ma'lumki, tortishuv maydoni yoki Kulon maydonida jism potensial energiyasi r ga teskari, kuch esa r^2 ga teskari proporsional bo'ladi

$$U(r) = -\frac{\alpha}{r},\ \vec{F}(r) = -G\frac{mM}{r^3}\vec{r} = -\gamma\frac{m}{r^3}\vec{r}$$

Bu yerda $G-$gravitatsion doimiylik, $m, M-$mos ravishda jism va Quyosh massasi, $\gamma = GM-$Gauss doimiysi, $\alpha > 0$ (tortishuv uchun), $\alpha < 0$ (itarishuv uchun) bo'lgan sonlar. Bunday maydonda jism harakati

$$r = p(\pm 1 + e\cos\varphi)^{-1}$$

Trayektoriya bo'yicha sodir bo'ladi, bunda «+» ishora $\alpha > 0$ ga, «-» ishora esa $\alpha < 0$ ga mos keladi, $p = \frac{l^2}{m\alpha}-$orbita parametri, $e = \sqrt{1+\frac{2E}{\alpha}p}-$orbita ekssentrisiteti. Jismning burilish nuqtalari $r_{1,2} = \frac{\alpha}{2E}(1 \pm e)$, katta yarim o'q $a = \left|\frac{\alpha}{2E}\right|$- munosabatlar bilan aniqlanadi. Effektiv potensial energiya $\alpha > 0$ va $\alpha < 0$ hollarda grafikda quyidagicha tasvirlanadi:

Masalalar

10-1. Kepler qonunlari yordamida butunolam tortilish qonunini chiqaring.

Yechish. Keplerning ikkinchi qonuniga binoan planeta radius-vektori vaqtning teng oralig'ida teng yuzalar chizadi. Demak, planeta harakati tekislikda sodir bo'ladi va biz bu harakatni qutb koordinata sistemasida qaraymiz:

$$x = r\cos\varphi,\ y = r\sin\varphi.$$

Bu sistemada planeta tezlanishi

$$w = \left(\ddot{x}^2 + \ddot{y}^2\right)^{\frac{1}{2}} = \ddot{r} - r\dot{\varphi}^2$$

U holda planetaga ta'sir qiluvchi markaziy kuch

$$F_r = \mu\,(\ddot{r} - r\dot{\varphi}^2)$$

Bu yerda $\mu = \frac{m}{m+M}$-keltirilgan massa, m, M -mos ravishda planeta va Quyosh massasi. Keltirilgan massaning bu yerda ishlatilishiga sabab shuki, Keplerning birinchi qonuni bo'yicha planetalar ellips bo'yicha harakat qilib, ellipsning fokuslaridan birida Quyosh joylashgan bo'ladi. Lekin Quyosh harakati hisobga olinsa, ellipsning fokusi Quyosh markazi bilan mos tushmasdan, planeta+Quyosh sistemasining massalar markazida joylashgan bo'ladi.

Ikkinchi tomondan planetaga qo'yilgan tortishish kuchi butunolam tortishuv qonuni bilan aniqlanadi:

ABBYY PDF Transformer 2.0
Click here to buy
www.ABBYY.com

$$\vec{F} = -G\frac{mM}{r^3}\vec{r} = -\gamma\frac{m}{r^3}\vec{r}$$

Bu yerda $G, \gamma = GM$ – mos ravishda gravitatsion va Gauss doimiyliklari.Markaziy maydonda planeta impuls momenti saqlanuvchan kattalik bo’ladi:

$$l = \mu r^2\dot{\varphi} = const\,.$$

Shuning uchun

$$r^2\dot{\varphi} = C = const\,.$$

Oxirgi tenglik ikkilangan sektorli tezlik ifodasi hisoblanadi.Sektorli tezlikning vaqt bo’yicha saqlanishi Kepler ikkinchi qonunining tasdig’idir.Planeta statsionar orbita bilan harakat qilgani uchun

$$\mu(\ddot{r} - r\dot{\varphi}^2) = -\gamma\frac{m}{r^2} \qquad \text{yoki} \qquad \ddot{r} - r\dot{\varphi}^2 + \frac{m}{\mu}\frac{\gamma}{r^2} = 0 \qquad (1)$$

Agar

$$\frac{m}{\mu} = \frac{m(m+M)}{mM} = \left(1 + \frac{m}{M}\right) \equiv k$$

deb belgilasak,(1) tenglamani C doimiylik orqali

$$\ddot{r} - \frac{C^2}{r^3} + \frac{k\gamma}{r^2} = 0 \qquad (2)$$

shaklga keladi.Bu tenglama shaklini o’zgartiramiz.Buning uchun

$$\dot{\varphi} = \frac{C}{r^2},\quad \frac{dr}{dt} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{C}{r^2}\frac{dr}{d\varphi}.$$

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{C}{r^2}\frac{d}{d\varphi}\left(\frac{C}{r^2}\frac{dr}{d\varphi}\right)$$

U holda (2) tenglama

$$\frac{C}{r^2}\frac{d}{d\varphi}\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right) - \frac{C}{r^3} = -\frac{k\gamma}{r^2}$$

ko’rinish oladi.Bu tenglamani soddaroq yozish uchun $r = \frac{1}{x}$ almashtirish o’tkazamiz va quyidagilarga ega bo’lamiz:

$$\frac{dr}{d\varphi} = \frac{d}{d\varphi}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\frac{dx}{d\varphi},\quad \frac{1}{r^2}\frac{dr}{d\varphi} = -\frac{dx}{d\varphi}$$

U vaqtda harakat tenglamamiz

$$\frac{d^2x}{d\varphi^2} + x = \frac{k\gamma}{C^2}$$

ko’rinishdagi tenglamaga keladi.Bu o’zgaruvchida F_r kuch quyidagicha yoziladi:

$$F_r = -\mu C^2x^2\left(\frac{d^2x}{d\varphi^2} + x\right) \qquad (3)$$

Keplerning birinchi qonunini ishlatib,yani orbita tenglamasi

$$x = \frac{1 + e\cos(\varphi + \alpha)}{p}$$

Ekanligidan kerakli hosilalarni olib, (3) ni

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$F_r = -\frac{\mu C^2 x^2}{p}$$

shaklga keltiramiz.Agar x o'zgaruvchidan r o'zgaruvchiga qaytsak,

$$F_r = -\frac{C^2 m}{p r^2}\frac{1}{1+\frac{m}{M}} = \frac{C^2 M}{p^2 r^2}\frac{1}{1+\frac{M}{m}}$$

Quyidagicha belgilashlarni kiritamiz:

$$\frac{C^2}{p^2}\frac{1}{1+\frac{m}{M}} = \gamma_q\,,\quad \frac{C^2}{p^2}\frac{1}{1+\frac{M}{m}} = \gamma_p$$

va

$$F_r = -\gamma_q \frac{m}{r^2} = \gamma_p \frac{M}{r^2}$$

Bu tenglikdan

$$\frac{\gamma_q}{M} = \frac{\gamma_p}{m} = G$$

yangi belgilash kelib chiqadi va oxir-oqibatda

$$F_r = -G\frac{mM}{r^2}$$

butunolam tortilish qonuni kelib chiqadi.

10-2. Butunolam tortilish qonunidan Kepler qonunlarini keltirib chiqaring.

Yechish.XVII asrda astronomik kuzatish natijalarini umumlashtirib Kepler planetalar harakatining kinematik qonunlarini asoslab bergan.Bu qonunlar asoslangan davrda na dinamikaning qonunlari van a butunolam tortilish qonuni mavjud bo'lgan.Hozirgi vaqtda Kepler qonunlarini matematik asoslab berish imkoniyati mavjud.Bu masalada biz butunolam tortilish qonuni yordamida Kepler qonunlarini asoslashga harakat qilamiz.

Keplerning birinchi qonuni bo'yicha barcha planetalar ellips bo'ylab harakat qiladi,ellipsning fokuslaridan birida Quyosh joylashgan bo'ladi. Bu fakt,uni matematik asoslab o'tirmaymiz.

Biz ikkinchi va uchinchi qonunlarini asoslash uchun butunolam tortilish qonunini keltiramiz

$$F = -G\frac{mM}{r^2} = -\gamma\frac{m}{r^2},$$

bu yerda $\gamma = GM$ – Gauss doimiysi deb aytiladi, G – gravitatsiya doimiysi.Bu kuchga to'g'ri keluvchi potensial

$$U(r) = -\gamma\frac{m}{r}$$

tortishuv xarakteriga ega bo'ladi.Planeta+Quyosh sistemasi uchun Lagranj funksiyasi (bu yerda Quyosh harakati ham hisobga olinib,Quyosh ellipsning fokuslaridan birida emas,berilgan sistema massalar markazida joylashadi deb olingaganda) quyidagi ko'rinishda yoziladi:

$$L = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\varphi^2\right) + \gamma\frac{m}{r}$$

Bu yerda biz masalani qutb koordinatalar sistemasida qaradik,chunki qaralayotgan harakat $x\,y$ tekisligida sodir bo'ladi hamda φ burchak siklik koordinata hisoblanadi.Shuning uchun umumlashgan impuls harakat integrali (saqlanuvchan kattalik) bo'ladi

$$p_{\varphi} = \mu r^{2}\dot{\varphi} = const$$

Umumlashgan impuls impuls momenti l ga teng bo'ladi:

$$p_{\varphi} = l = \mu r^{2}\dot{\varphi} = const$$

Bu yerda biz markaziy maydonda yassi harakat qilayotgan bitta zarra (planeta) uchun bu qonun oddiy geometric ma'noga ega bo'lishini ko'ramiz. $\frac{1}{2}r\cdot r\,d\varphi$ (rasmga qaralsin!) ifda planetaning bir-biriga cheksiz yaqin radius-vektorlari va trayektoriya yoyi elementi o'rtasidagi sector yuzasini ifoda etadi.Agar bu yuzani

$$d\,s = \frac{1}{2}r\cdot r\,d\varphi$$

deb belgilasak,

$$l = \mu r^{2}\dot{\varphi} = 2\,\mu\frac{d\,s}{d\,t} = const$$

Bundan

$$\frac{d\,s}{d\,t} = const$$

ekanligi,yani vaqt oralig'ida chizilgan yuzaning saqlanishini ko'ramiz.
Bu Keplerning ikkinchi qonunining matematik isboti hisoblanadi.

Endi yuqorida yozilgan tenglikdan

$$ds = \frac{l}{2\,\mu}d\,t$$

ni topib,uni integrallasak

$$\int_{0}^{S} d\,s = \frac{l}{2\,\mu}\int_{0}^{T} d\,t$$

$$S = \frac{l}{2\,\mu}T$$

Tenglikka ega bo'lamiz.Bu yerda $S = \pi\,a\,b -$ ellips yuzasi, $a, b -$ ellipsning katta va kichik yarimo'qlari, $T -$ planetaning ellipsni to'la aylanish davri.

Ellips katta va kichi yarimo'qlarining uning parametri p va ekssentrisiteti e bilan bog'lanishlari

$$a = \frac{p}{1-e^{2}},\; b = \frac{p}{\sqrt{1-e^{2}}}$$

lardan foydalanib ellips yuzasini topamiz

$$S = \pi\,a\,b = \pi\frac{p^{2}}{\left(1-e^{2}\right)^{3/2}} = \pi\,a^{3/2}\sqrt{p} = \pi\,a^{3/2}\frac{l}{\sqrt{\mu\gamma}} = \pi\,a^{3/2}\frac{l}{\sqrt{\mu GM}} \qquad \left(p = \frac{l^{2}}{\mu\gamma} = \frac{l^{2}}{\mu\,G\,M}\right)$$

U holda aylanish davri uchun

$$T = \frac{2\,\mu\,S}{l} = 2\,\mu\,\pi\,a^{3/2}\frac{1}{\sqrt{\mu\,G\,M}} = 2\,\pi\,a^{3/2}\frac{1}{\sqrt{G\,(m+M)}}$$

tenglikka ega bo'lamiz.Bu tenglikning har ikkala tomonini kvadratga ko'tarib

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(m+M)}$$

nisbatga kelamiz.Quyosh harakati inobatga olinsa va massalar markazi sistemasida ellipsning katta yarimo'qiga ma'lum tuzatish kiritish lozim bo'ladi:

$a = a'(1 + \frac{m}{M}),$ a' – massalar markazi sistemasidagi katta yarimo'q.U holda biz Keplerning uchunchi qonunining matematik ifodasini olamiz

$$\frac{T^2}{a'^{3/2}} = \frac{4\pi^2(M+m)^2}{GM^3} \approx \frac{4\pi^2}{GM}(1+\frac{m}{M})$$

ko'ramizki,Keplerning uchunchi qonuni taqribiy ekan va $\frac{T^2}{a'^{3/2}}$ nisbat planeta massasiga bog'liq bo'lar ekan,lekin $\frac{m}{M}$ nisbat juda kichik qiymatga ega bo'ladi.

1-2-TOPSHIRIQLAR

1.Massasi m bo'lgan raketa R radiusli Yer atrofiga chiqarilgan bo'lsa,uning uchun birinchi va ikkinchi kosmik tezliklar topilsin.

2.Yer sirtiga urinma bo'yicha yo'nalgan v_0 tezlik bilan elliptik orbitaga chiqarilgan massasi m bo'lgan raketaning R radiusli Yer sirtidan maksimal uzoqlashi topilsin.Natijani birinchi va ikkinchi tezliklar orqali ifodalang.

JAVOBLAR:

1. $v_1 = \sqrt{gR}$, $v_2 = \sqrt{2gR}$.

2. $h = 2R\frac{v_0^2 - v_1^2}{v_2^2 - v_0^2}$

Uslubiy tavsiyalar:

1.Kosmik raketaning birinchi va ikkinchi tezliklarini aniqlashning mezonini energiyaning qiymati aniqlaydi.Birinchi kosmik tezlik raketaning minimal ruxsat etilgan finitli orbita bo'ylab harakatida amalgam oshsa,ikkinchi tezlik minimal ruxsat etilgan infinitli orbita bo'ylab

Harakatida amalgam oshadi.Birinchi tezlikda $E = (U_{eff})_{\min} = U_0$ da,ikkinchi tezlik $E = 0$ da amalgam oshadi.

11-MASHG'ULOT
Mavzu: «Ikki jism masalasi. To'qnashuvlar diagrammalari»

Qisqacha nazariya. 1. Ikki jism masalasini, bir zarraning ta'sirlashuv kuchlar maydonidagi harakati kabi deb qarash mumkin. Bunda zarraning markaziy tizimidagi trayektoriyasi keltirilgan massali zarraning trayektoriyasiga o'xshash bo'ladi,

$$\vec{\rho}_1 = \frac{m_2}{M}\vec{r},\ \vec{\rho}_2 = -\frac{m_1}{M}\vec{r},$$

bu yerda $\vec{r} = \vec{\rho}_1 - \vec{\rho}_2 = \vec{r}_1 - \vec{r}_2$ - nisbiy radius-vektor,

$\vec{\rho}_i$ -nuqtaning markaziy tizimidagi radius-vektor,

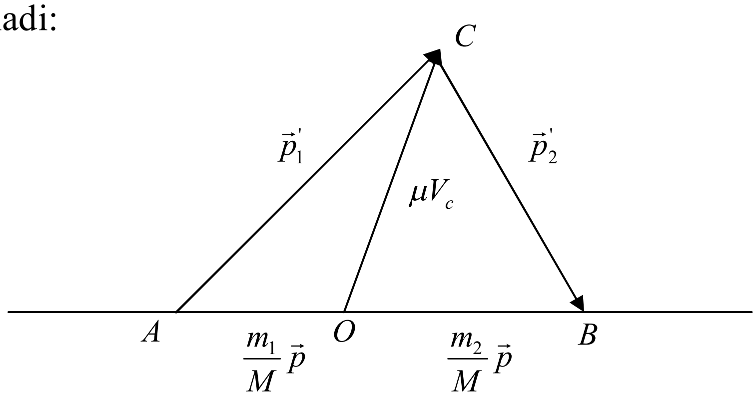
$\vec{r}_i$ -laboratoriya tizimidagi radius-vektor.

Faraz qilinadiki, tizimning inersiya markazi inersiya bo'yicha harakatlanadi. Keltirilgan massali zarraning harakati markaziy kuch maydonidagi harakat kabi tahlil qilinadi. Bu holda radial harakat

$$U_{ñàì} = U(r) + \frac{M^2}{2\mu r^2}$$

maydonida bir o'lchamli harakatga olib kelinadi. Bu yerda $\vec{M} = \vec{r} \times \mu\vec{v}$ -keltirilgan massali zarraning impuls momenti $(\vec{M} = const)$. Masalani yechish uchun $U_{ñàì}(r)$ ning grafigini chizish va berilgan radiusli aylana bo'yicha harakatga mos M impuls momenti qiymatini aniqlash kerak.

2. Umumiy holda to'qnashuv diagrammasi quyidagi sxema (*1-rasm*) bo'yicha tuziladi:



1-rasm

$OC = \mu v_0$ aylana radiusi ($v_0 = \dot{\vec{r}}$ -to'qnashuvi zarralarning nisbiy harakat tezligi) $AO = \frac{m_1}{M}\vec{P}$, $OB = \frac{m_2}{M}\vec{P}$, $\vec{P} = \vec{P}_1 + \vec{P}_2$ tizimning to'la impulsi. Agar to'la impulsning yo'nalishi zarba yo'nalishi bilan mos tushsa, unda $\angle COB=\chi$ -markaziy tazimda chetlanish burchagi, $\angle CAO=\theta_1$, $\angle CBO=\theta_2$ -laboratoriya tizimida to'qnashuvdan keyingi zarralarning chetlanish burchaklari.

$\vec{P}_1 = m_1\vec{v}_1, \vec{P}_2 = m_2\vec{v}_2$ - zarralarning to'qnashuvidan keyingi impulslari.

Masalalar

11-1. m_1 va m_2 massali moddiynuqtalarning o'zaro ta'sirlashuvidagi potensial energiyasi $U(r)=\frac{\chi}{2}(r-a)^2$ ga teng. Ular bir – biridan b masofada aylanma orbita bo'yicha harakatlansa, ularning markaziy tizimdagi tezliklari qanday bo'lishi? (a,b,χ -musbat doimiylar).

Yechish. Masala shartiga ko'ra ta'sirlashuv potensial energiya

$$U(r)=\frac{\chi}{2}(r-a)^2 \qquad (1)$$

Samarali potensial energiya esa quyidagi ko'rinishga ega:

$$U_{caì}(r)=\frac{M^2}{2\mu r^2}+\frac{\chi}{2}(r-a)^2 \qquad (2)$$

$r=b$ da $\frac{dU_{ñàì}}{dr}\Big|_{r=b}=0$ ni aniqlaymiz. Demak, (2) ifodadan quyidagini olamiz:

$$\left[-\frac{M^2}{\mu}\cdot\frac{1}{r^3}+\chi(r-a)\right]\Bigg|_{r=b}=0,$$

bu yerdan

$$M^2=\mu b^3\chi(b-a) \qquad (3)$$

Endi energiyaning saqlanish qonuniga asosan quyidagini yozamiz:

$$E=U_{ñàì}(r)^{\min}=U_{ñàì}(r=b),$$

bu yerda

$$\frac{\mu v^2}{2}+U(b)=\frac{M^2}{2\mu b^2}+\frac{\chi}{2}(b-a)^2.$$

$U(b)$ va $\frac{\chi}{2}(b-a)^2$ larning bir-biriga tengligini hisobga olsak.

$$\frac{\mu v^2}{2}=\frac{M^2}{2\mu b^2}$$

yoki

$$v^2=\frac{M^2}{\mu^2 b^2} \qquad (4)$$

(3) va (4) larni birgalikda yechib quyidagiga ega bo'lamiz:

$$v^2=\frac{\mu b^3\chi(b-a)}{\mu^2 b^2}=\frac{b\chi(b-a)}{\mu}$$

yoki

$$v=\sqrt{\frac{b\chi(b-a)}{\mu}} \qquad (5)$$

Har bir moddiy nuqtalarning tezliklari esa quyidagicha topiladi:

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$v_1 = \frac{m_2}{m_1+m_2}v = \frac{m_2}{m_1+m_2}\cdot\frac{m_1}{m_1}v = \frac{\mu}{m_1}v, \tag{6'}$$

$$v_2 = \frac{m_1}{m_1+m_2}v = \frac{m_1}{m_1+m_2}\cdot\frac{m_2}{m_2}v = \frac{\mu}{m_2}v, \tag{6''}$$

bu yerda

$\mu = \dfrac{m_1 m_2}{m_1+m_2}$ -keltirilgan massa.

(5) ni hisobga olgan holda (6’) va (6”) larni quyidagicha yozish mumkin:

$$v_1 = \frac{1}{m_1}\sqrt{\frac{\mu^2 b\chi(b-a)}{\mu}} = \frac{1}{m_1}\sqrt{b\mu\chi(b-a)}, \tag{7'}$$

$$v_2 = \frac{1}{m_2}\sqrt{\frac{\mu^2 b\chi(b-a)}{\mu}} = \frac{1}{m_2}\sqrt{b\mu\chi(b-a)}, \tag{7''}$$

(7’) va (7”) ifodalarni umumiy holda quyidagicha yozish mumkin:

$$v_i = \frac{1}{m_i}\sqrt{b\mu\chi(b-a)},\ i=1{,}2.$$

TOPSHIRIQLAR

1-3. Zarralar harakatining to’qnashuvdan keyin laboratoriya tizimida ba’zi bir xarakteristikalari olinsin va to’qnashuv diagrammalari chizilsin:

Agar 1. $m_1 = m_2 = m$, $\vec{v}_1 = \vec{v}$, $\vec{v}_2 = -\vec{v}$, markaziy tizimda chetlanish burchagi χ teng bo’lsa v_1', v_2', θ_1 va θ_2 lar topilsin.

2. $m_1 = \dfrac{m}{2}$, $m_2 = m$, $\vec{v}_1 = 2\vec{v}$, $\vec{v}_2 = \vec{v}$, markaziy tizimda chetlanish burchagi $\chi = \pi$ teng bo’lsa v_1', v_2', θ_1 va θ_2 lar topilsin.

3. $m_1 = m$, $m_2 = \dfrac{m}{3}$, $\vec{v}_1 = \vec{v}$, $\vec{v}_2 = -\vec{v}$, markaziy tizimda chetlanish burchagi χ teng bo’lsa v_1' va θ_1 lar topilsin.

4. Boshlang’ich tezligi $\vec{V}$ bo’lgan zarracha ikki qismga parchalandi. Parchalanish natijasida hosil bo’lgan zarrachalarning chiqish burchagini toping.

JAVOBLAR:

1. $v_1' = v_2' = v,\quad \theta_1 = \chi,\quad \theta_2 = \pi - \chi.$

2. $v_1' = \dfrac{2}{3}v,\quad v_2' = \dfrac{5}{3}v,\quad \theta_1 = \theta_2 = 0$

3. $v_1' = v\cos\dfrac{\chi}{2},\quad \theta_1 = \dfrac{\chi}{2}.$

4. $\theta = \dfrac{v_0\sin\theta_0}{\vec{V} + v_0\cos\theta_0}$

Uslubiy ko’rsatma: Zarralarning laboratoriya tizimida to’qnashuvdan oldingi va to’qnashuvdan keyingi implus va tezliklari tenglashtirilib olinadi

12-MASHG'ULOT
Mavzu: "Zarralarning elastik sochilishi. Effektiv kesim"

Darsning maqsadi. Berilgan potensial maydonda sochilayotgan zarra uchun differensial va to'liq effektiv kesimni hisoblashni o'rgatish

Qisqacha tushuncha. Massasi m bo'lgan zarraning qo'zg'almas kuch markazi (zarralar inertsiya markazida joylashgan) tomonidan yaratilgan $U(r)$ potensial maydonda chetlanishi qaraladi.Zarraning bu maydonda og'ish burchagi

$$\varphi = \int_{r_{\min}}^{\infty} \frac{\rho \frac{dr}{r^2}}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}}}$$

formula bilan aniqlanadi.Zarraning sochilishi esa effektiv kesim deb ataluvchi kesim bilan xarakterlanadi:

$$d\sigma = \frac{dN}{N_0} = 2\pi\rho(\theta)\,d\rho(\theta) = 2\pi\rho(\theta)\left|\frac{d\rho(\theta)}{d\theta}\right| d\theta$$

Bu yerda dN- $d\Omega = \sin\theta\, d\theta\, d\varphi$ fazoviy burchak ichida sochilayotgan zarralar soni, N_0 – vaqt birligida ko'ndalang kesimning yuza birligiga tushayotgan zarralar soni, ρ – nishon masofasi, θ – sochilish burchagi.

Masalalar

12-1.Kulon maydonida zarra sochilishning effektiv kesimi uchun Rezerford formulasining o'rinli ekanligi isbotlansin.

Yechish. Kulon maydoni

$$U(r) = \frac{\alpha}{r}$$

potensial bilan aniqlanadi Bu maydonda zarraning og'ish burchagi

$$\varphi_0 = \int_{r_{\min}}^{\infty} \frac{\rho\, dr}{r^2\sqrt{1 - \frac{\rho^2}{r^2} - 2\frac{U(r)}{mv_\infty^2}}} = -\rho\int_{r_{\min}}^{\infty} \frac{d\left(\frac{1}{r}\right)}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2\alpha}{mv_\infty^2 r}}} = \arccos\frac{\frac{\alpha}{mv_\infty^2\rho}}{\sqrt{1 + \left(\frac{\alpha}{mv_\infty^2\rho}\right)^2}}.$$

ekanligini topamiz Bu natijadan ρ ni topish qiyin emas:

$$\rho^2 = \left(\frac{\alpha}{mv_\infty^2}\right)^2 tg^2\varphi_0$$

Markaziy kuch maydonida sodir bo'ladigan sochilishni xarakterlovchi burchaklar o'rtasida

$$\theta = \frac{\pi - \varphi_0}{2}$$

munosabat mavjudligidan

$$tg^2\varphi_0 = ctg^2\frac{\theta}{2}$$

(rasm chiziladi!!!)
Tenglik o'rinli ekanligini topamiz.U holda

$$\rho^2 = \left(\frac{\alpha}{mv_\infty{}^2}\right)^2 ctg^2\frac{\theta}{2}$$

Sochilishning differensial kesimi

$$d\sigma = 2\pi\,\rho(\theta)\left|\frac{d\,\rho(\theta)}{d\,\theta}\right| d\,\theta = \pi\left(\frac{\alpha}{mv_\infty{}^2}\right)^2 \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}\,d\,\theta$$

Fazoviy burchak

$$d\Omega = \sin\theta\, d\theta\, d\,\varphi = 2\pi\sin\theta\, d\theta = 4\pi\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

ifodasidan

$$\cos\frac{\theta}{2}\,d\,\theta = \frac{d\,\Omega}{4\pi\sin\frac{\theta}{2}}$$

bo'lgani uchun

$$d\,\sigma = \left(\frac{\alpha}{2\,mv_\infty{}^2}\right)^2 \frac{d\,\Omega}{\sin^4\frac{\theta}{2}}$$

ekanligini topamiz va bu formula *Rezerford formulasi* deb ataladi.

12-2.Zarralar radiusi a ga teng bo'lgan mutlaq qattiq sharchaga urilib, undan sochilayapti. Sochilishning to'liq kesimi hisoblansin.

Yechish. Sharcha mutlaq qattiq bo'lgani uchun unga tushayotgan zarra sharchaning ichiga singib kira olmaydi,yani sharchaning potensialini bu holda ($r < a$ bo'lganda) cheksiz katta deyish mumkin. $r > a$ bo'lganda esa sharchadan burilib sochilgan zarra erkin harakat qiladi-zarraga sharchaning ichki potensiali mutlaqo ta'sir ko'rsamaydi.Demak $r > a$ bo'lganda potensial nolga teng bo'ladi.Xulosa qilib aytganda sharchaning potensialini

$$U(r) = \begin{cases} \infty, & r < a \\ 0, & r < a \end{cases}$$

ko'rinishda yoza olamiz va bu potensialda sochilish kesimini topamiz.

Agar sharchaga tushayotgan zarralar uchun nison masofa $\rho > a$ bo'lsa,sochilish sodir bo'lmaydi (zarralar sharcha bilan to'qnashmaydi).Masala shartiga ko'ra

$$\rho = a\sin\varphi_0 = a\sin\frac{\pi-\theta}{2} = a\cos\frac{\theta}{2}, \qquad d\rho = -\frac{a}{2}\sin\frac{\theta}{2}\,d\theta$$

Differensial kesim

$$d\,\sigma = 2\pi\,|\,\rho\, d\,\rho\,| = \frac{\pi\, a^2}{2}\sin\theta\, d\,\theta = \frac{a^2}{4}\,d\,\Omega.$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Bundan ko'rinadiki,massalar markazi sistemasida sochilish izotrop bo'ladi,ya... burchakka bog'liq bo'lmaydi.

$\int d\Omega = 4\pi$ bo'lganidan to'liq effektiv kesim

$$\sigma = \pi a^2$$

-sochilish kesimi sharcha kesimining yuzasiga teng bo'lar ekan.
(rasmi chiziladi)

12-3.Zarraning

$$U(r) = \frac{\alpha}{r} - \frac{\beta}{r^2}$$

potensial maydonning markaziga tushish effektiv kesimini hisoblang ($\alpha, \beta > 0$ deb olinsin).

Yechish. Berilgan holda effektiv potensial

$$U_{eff}(r) = \frac{\alpha}{r} - \frac{\beta}{r^2} + \frac{mv_\infty{}^2\rho}{2r^2} = \frac{\alpha}{r} - \frac{\beta - E\rho^2}{r^2}.$$

Agar bu yerda $\beta > E\rho^2$ bo'lsa, effektiv potensial musbat maksimumga

$$\left.\frac{dU_{eff}(r)}{dr}\right|_{r=r_0} = -\frac{\alpha}{r^2{}_0} + \frac{2(\beta - E\rho^2)}{r_0{}^3} = 0 \text{ dan } \qquad r_0 = \frac{2(\beta - E\rho^2)}{\alpha} \text{ nuqtada erishadi}$$

$$U_{eff}(r)_{\max} = \frac{\alpha^2}{2(\beta - E\rho^2)} - (\beta - E\rho^2)\frac{\alpha^2}{4(\beta - E\rho^2)^2} = \frac{\alpha^2}{4(\beta - E\rho^2)}$$

(rasmda a)-hol).Zarraning markazga tushishi uchun uning energiyasi effektiv potensialning shu maksimal qiymatidan katta bo'lmog'i kerak:

$$E > \frac{\alpha^2}{4(\beta - E\rho^2)}$$

Bundan nishon masofasining maksimal qiymatini toppish mumkin:

$$\rho_{\max}{}^2 = \frac{\beta}{E} - \frac{\alpha^2}{4E^2}$$

Aks holda zarra maydon markaziga yaqinlasha olmaydi.Endi effektiv kesimni topamiz:

$$\sigma = \pi\rho_{\max}{}^2 = \pi\left(\frac{\beta}{E} - \frac{\alpha^2}{4E^2}\right)$$

Kesim o'zining ta'rifiga ko'ra hamavaqt musbat kattalik bo'ladi.Demak zarra energiyasi

$$E > \frac{\alpha^2}{4\beta}$$

shartini bajarmog'i lozim.Agar bus hart bajarilmasa $\sigma = 0$, boshqacha aytganda sochilish sodir bo'lmaydi.

Agar $\alpha > 0, \beta < 0$ bo'lsa,effektiv potensial faqat itarish kuchiga olib keladi-markazga tushish ro'y bermaydi (rasmda b-hol).

Agar $0 < \beta < E\rho^2$ bo'lib $\alpha < 0$ bo'lsa,markazga tushish ro'y bera olmaydi-bu holda effektiv potensial r_0 nuqtada minumumga ega .Agar $0 > \beta > E\rho^2, \alpha < 0$ bo'lsa,ixtiyoriy energiyali zarra markazga tushadi.

12-4. Zarralarning R radiusli sferik jism sirtida tutib qolinishi kesimini toping.Zarralarning jism bilan o'zaro ta'sir potensial energiyasi $U(r) = -\dfrac{\alpha}{r^n}$ ga teng.

Yechish. Bunday maydonda effektiv potensial

$$U_{eff} = \frac{m v_\infty^{\ 2}\rho^2}{2r^2} - \frac{\alpha}{r^n} = \frac{T_\infty \rho^2}{r^2} - \frac{\alpha}{r^n}, \quad T_\infty = \frac{m v_\infty^{\ 2}}{2}.$$

Bu potensialning ekstremal qiymatini topamiz:

$$\frac{dU_{eff}}{dr} = -\frac{2T_\infty\rho^2}{r^3} + \frac{n\alpha}{r^{2n}} r^{n-1} = 0$$

Bundan effektiv potensialning

$$r_1 = \left(\frac{n\alpha}{2T_\infty\rho^2}\right)^{1/(n-2)}$$

nuqtada ekstremal qiymatga erishishini topamiz:

$$U_{eff}(r_1) = \frac{\alpha}{2}(n-2)\left(\frac{n\alpha}{2T_\infty\rho^2}\right)^{n/(n-2)}$$

Jism nuqta kabi bo'lganda zarralarni tutib qolish $\rho < \rho_0$ da amalgam oshgan bo'lur edi.Buning uchun $U_{eff}(r_1, \rho_0) \le T_\infty$ bajarilmog'i darkor.Oxirgi tenglikdan

$$\rho_0 = \sqrt{n}\,(n-2)^{(2-n)/2n}\left(\frac{\alpha}{2T_\infty}\right)^{1/n}$$

ekanligini topamiz.U holda

$$r_1(\rho_0) = (n-2)^{1/n}\left(\frac{2T_\infty}{\alpha}\right)^{-1/n}$$

shaklga ega bo'ladi va bu qiymatda effektiv potensial $U_{eff}(r_1)$ o'zining maksimal qiymatiga erishadi.Agarda $R. < r_1(\rho_0)$ bo'lsa,zarralarning tutilish kesimi

$\sigma = \pi\rho_0^{\ 2} = \pi\, n(n-2)^{(2-n)/n}\left(\dfrac{\alpha}{2T_\infty}\right)^{2/n}$ ko'rinishga ega bo'ladi.Agarda $R. > r_1(\rho_0)$ bo'lsa, $\rho < \rho_1$ bo'lgan barcha zarralar tutib qolinadi,bu yerda ρ_1 quyidagi tenglamadan aniqlanadi:

$$\frac{T_\infty\rho_1^{\ 2}}{R^2} - \frac{\alpha}{R^n} = T_\infty$$

va effektiv kesim

$$\sigma = \pi\rho_1^{\ 2} = \pi R^2\left(1 + \frac{\alpha}{T_\infty R^n}\right).$$

ko'rinishda bo'ladi.

1-4 TOPSHIRIQLAR

1.Mutlaq qattiq,silliq va elastik b radiusli sharcha radiusi a bo'lgan xuddi shunday sharchadan sochilganda,differensial va to'liq kesim hisoblansin.
2.Oldingi masaladagi sharchalar bir xil radiusga ega bo'lsin,yani a=b.
3.Nuqtaviy zarralar radiusi $\rho = b\sin\frac{z}{a}$ $(z \in [0 < \pi < a], a, b = const, v_\infty \uparrow\uparrow) OZ)$ bo'lgan mutlaq qattiq,silliq va elastic aylanuvchan sirtdan sochilganda,differensial va to'liq kesim topilsin.

JAVOBLAR:

1. $d\sigma = \frac{1}{4}(a+b)^2\, d\Omega\,, \quad \sigma = \pi(a+b)^2\,.$

2. $d\sigma = a^2\, d\Omega\,, \quad \sigma = 4\pi a^2\,.$

3. $d\sigma = \frac{1}{4\cos^4\frac{\theta}{2}} a^2\, d\Omega\,, \quad \sigma = \pi b^2\,.$

Uslubiy tavsiyalar:

1.Nishon masofasi va sochilish burchagining o'zaro teskari bog'lanishda ekanligini bilish uchun sochilish jarayonini aniq chiza bilish kerak;
2 Zarraning burilish nuqtalarini hamishagidek
$E = U_{eff}(r)$ tenglikdan topiladi.Agar bu tenglik kvadrat tenglamadan iborat bo'lsa,uning manfiy ildizi $r_{\min}$ ni,musbat ildizi esa $r_{\max}$ ni aniqlaydi.Berilgan topshiriqlarning barchasida markazga eng yaqin va markazdan eng uzoq nuqtalar ana shunday aniqlanadi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

13-MASHG'ULOT
Mavzu: «Lagranj funksiyasi va Lagranj tenglamalari»

Darsning maqsadi. Berilgan zarra yoki zarralar sistemasi uchun Lagranj funksiyasini va bu funksiya orqali Lagranj tenglamasini tuzishni o'rganish.
Qsqacha tushuncha. Berilgan zarra uchun Lagranj funksiyasi bu zarra kinetik va potensial energiyalar farqi tariqasida tuziladi:

$$L = T - U$$

Agar zarralar sistemasi berilgan bo'lsa ,har bir energiyalar yig'indisining ayirmasi

$$L = \sum T_n - \sum U_n$$

kabi topiladi.Bu funksiyadan umumlashgan tezlik va umumlashgan koordinata bo'yicha kerakli hosilalar olinib,Lagranj tenglamasi tuziladi:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial l}{\partial q} = 0$$

Masalalar

13-1. Massasi m bo'lgan erkin zarra uchun dekart,silindrik va sferik koordinatalarida Lagranj funksiyasi tuzilsin va Lagranj tenglamalari yozilsin.

Yechish: a) Lagranj funksiyasini avvalo Dekart koordinatalarida yozamiz

$$L = T = \frac{m}{2}(\dot{x}^2 + \breve{o}^2 + \dot{z}^2)$$

(chunki erkin zarra uchun potensial energiya $U(x, \acute{o}, z) = 0$).Lagranj tenglamalari

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = x, \acute{o}, z)$$

formula orqali aniqlanadi.Bizda

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = \frac{\partial L}{\partial \acute{o}} = \frac{\partial L}{\partial z} = 0 \quad ,$$

chunki L funksiya $x, \acute{o}, z$ koordinatalarga oshkor bog'liq emas.Shu sababli

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\frac{\partial L}{\partial \breve{o}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0$$

bo'ladi.Bulardan

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial \breve{o}} = \frac{\partial L}{\partial \dot{z}} = 0$$

ekanligini topamiz.Demak erkin harakatlanayotgan zarra uchun

$$\dot{x} = \breve{o} = \dot{z} = const$$

yoki

$$v = \sqrt{\dot{x}^2 + \breve{o}^2 + \dot{z}^2} = const ,$$

zarraning tekis harakatlanishi ko'ramiz.

b) qutb koordinatalarda Lagranj funksiyasini ifodalash uchun bu koordinatalarning Dekart koordinatalari bilan bog'lanishini yozamiz

$$x = \rho\cos\varphi, \qquad \acute{o} = \rho\sin\varphi$$

Ulardan vaqt bo'yicha olingan hosilalar

$$\dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi, \quad \dot{\acute{o}} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi$$

U holda

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2)$$

Lagranj tenglamalari esa

$$\ddot{\rho} = \rho\,\dot{\varphi}^2, \qquad \rho^2\ddot{\varphi} = 0$$

shaklda bo'ladi.

c) silindrik koordinatalar qutb koordinatalari kabi bo'ladi,faqat bu yerda $z \neq 0$. Shuning uchun Lagranj funksiyasi

$$L = \frac{m}{2}(\dot{x}^2 + \dot{\acute{o}}^2 + \dot{z}^2) = \frac{m}{2}(\dot{\rho}^2 + \rho^2\,\dot{\varphi}^2 + \dot{z}^2)$$

Lagranj tenglamalari esa

$$\ddot{\rho} = \rho\dot{\varphi}^2, \qquad \rho^2\ddot{\varphi} = 0, \qquad \ddot{z} = 0$$

shaklga ega bo'ladi.

d) sferik koordinatalar r, θ, φ va Dekart koordinatalari

$$x = r\sin\theta\cos\varphi, \qquad \acute{o} = r\sin\theta\sin\varphi, \qquad z = r\cos\theta$$

o'zaro bog'lanishda bo'lishini bilamiz.Ulardan vaqt bo'yicha olingan hosilalar

$$\dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi,$$

$$\dot{\acute{o}} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi,$$

$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

Lagranj funksiyasi esa

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)$$

Bundan Lagranj tenglamalarini topamiz

$$\ddot{r} = r(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta), \;\; \ddot{\theta} = r^2(1 + \dot{\varphi}^2\sin 2\theta), \quad \ddot{\varphi} = 0$$

13-2. Birjinsli og'irlik maydonida (og'irlik kuchi tezlanishi – g) joylashganquyidagi sistemalar uchun Lagranj funksiyasi tuzilsin:

1) yassi qo'shaloq mayatnik uchun (1-rasm);

2) m_2 massalik yassi mayatnik berilgan bo'lib,osilgan nuqtasi (m_1 massalik) gorizontal to'g'ri chiziq bo'ylab harakat qila oladi (2-rasm);

3) osilgan nuqtasi vertikal a–radiusli aylana bo'ylab doimiy γ chastota bilan tekis aylanuvchi yassi mayatnik uchun (3-rasm).

Yechish:

1) mayatnik uzunliklari $l_1 = const$, $l_2 = const$,vertical o'q bilan mos ravishda φ_1, φ_2 burchaklar tashkil etishadi.U holda m_1 uchun

$$T_1 = \frac{m_1}{2}(\dot{x}^2 + \dot{\acute{o}}^2), \qquad U_1 = -m_1 g\, l_1\cos\varphi_1,$$

$$x_1 = l_1\sin\varphi_1, \; \acute{o}_1 = l_1\cos\varphi_1; \;\; \dot{x}_1 = l_1\dot{\varphi}_1\cos\varphi_1, \quad \dot{\acute{o}}_1 = -l_1\dot{\varphi}_1\sin\varphi_1.$$

bo'ladi. m_2 massalik nuqta uchun esa

$$x_2 = l_1\sin\varphi_1 + l_2\sin\varphi_2, \acute{o}_2 = l_1\cos\varphi_1 + l_2\cos\varphi_2,$$

$$\dot{x}_2 = l_1\dot{\varphi}_1\cos\varphi_1 + l_2\dot{\varphi}_2\cos\varphi_2, \quad \dot{\acute{o}}_2 = -(l_1\dot{\varphi}_1\sin\varphi_1 + l_2\dot{\varphi}_2\sin\varphi_2)$$

Berilgan sistema uchun Lagranj funksiyasi

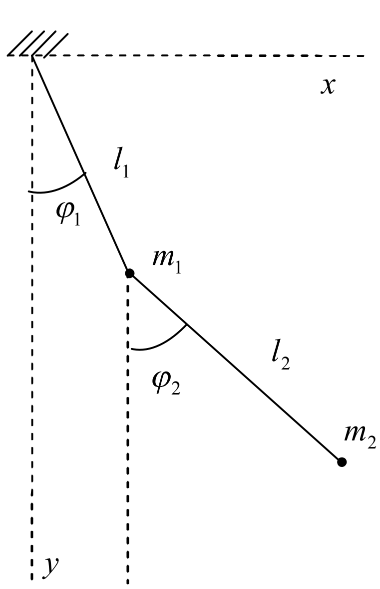
$$L = T_1 + T_2 - (U_1 + U_2)$$

formula bilan aniqlanadi.Bu yerda

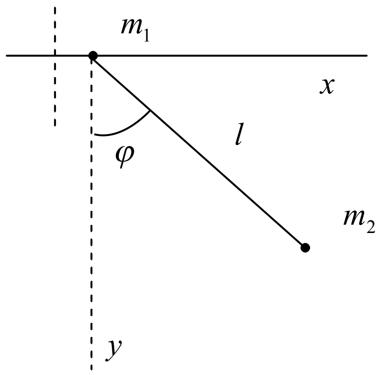
$$T_1 = \frac{m_1}{2} l_1^{\,2} \dot{\varphi}_1^{\,2}, \quad T_2 = \frac{m_2}{2} l_1 \dot{\varphi}_1^{\,2} + \frac{m_2}{2}\left[l^2{}_1 \dot{\varphi}_1^{\,2} + l^2{}_2 \dot{\varphi}_2^{\,2} + 2 l_1 l_2 \cos(\varphi_1 - \varphi_2)\right] \quad ,$$

$$U_2 = -m_2 g\,(l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

Endi biz Lagranj funksiyasini yoza olamiz



1-rasm



2-rasm

$$L = \frac{m_1 + m_2}{2} l^2{}_1 \dot{\varphi}_1^{\,2} + \frac{m_2}{2} l_2^{\,2} \dot{\varphi}_2^{\,2} + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + (m_1 + m_2) g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi_2$$

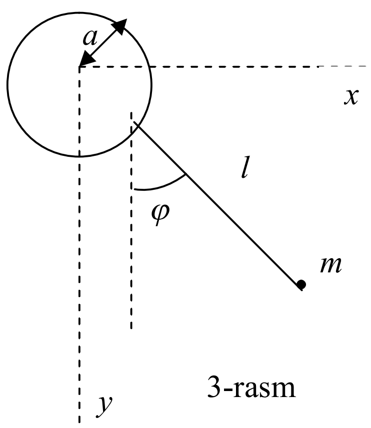
2) 2-rasmdan topamiz:

$$X = x + l \sin\varphi, \quad \acute{o} = l\cos\varphi$$

$$\dot{X} = \dot{x} + l\dot{\varphi}\cos\varphi, \quad \breve{o} = -l\sin\varphi$$

Sistemaning kinetik va potensial energiyalari

$$T = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\varphi}^2 + 2l\,\dot{x}\dot{\varphi}\cos\varphi), \quad U = -m_2 g l \cos\varphi$$



3-rasm

Lagranj funksiyasi esa

$$L = T - U = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\varphi}^2 + 2l\,\dot{x}\dot{\varphi}\cos\varphi) + m_2 g l\cos\varphi$$

1) Nuqtaning koordinatalari

$$x = a\cos\gamma\, t + l\sin\varphi, \quad \acute{o} = -a\sin\gamma\, t + l\cos\varphi$$

Tezliklari

$$\dot{x} = -a\gamma\sin\gamma\, t + l\dot{\varphi}\cos\varphi, \quad \breve{o} = -a\gamma\cos\gamma\, t - l\dot{\varphi}\sin\varphi$$

Lagranj funksiyasi

$$L = T - U = \frac{m}{2} l^2\dot{\varphi}^2 + m a^2\gamma^2 + m a l\gamma\sin(\varphi - \gamma\, t) + m g l\cos\varphi$$

ABBYY PDF Transformer 2.0
Click here to buy
www.ABBYY.com

13-3. Zarra uchun Lagranj funksiyasi

$$L = \frac{m\dot{\vec{r}}^{\,2}}{2} + \frac{q}{c}\left(\vec{A}(r,t)\cdot\dot{\vec{r}}\right) - q\varphi(r,t) \qquad (1)$$

ko'rinishida berilsa,bu zarra uchun Lagranj tenglamasi yozilsin.

Yechish: (1) funksiyada $\vec{A}(r,t),\varphi(r,t)$ – koordinata va vaqtning vektorli va skalyar funksiyalari, q,c -qandaydir doimiy parametrlar.Lagranj funksiyasidagi zarra tezligi bo'yicha chiziqli bo'lgan had umumlashgan potensial deb aytiladi.Avvalo biz (1) dagi skalyar ko'paytmani ochib

$$(\vec{A}\cdot\dot{\vec{r}}) = A_x\dot{r}_x + A_{\acute{o}}\dot{r}_{\acute{o}} + A_z\dot{r}_z = A_x\dot{x} + A_{\acute{o}}\dot{\acute{o}} + A_z\dot{z} \qquad (2)$$

(1) ni qayta yozamiz

$$L = \frac{m}{2}(\dot{x}^2 + \dot{\acute{o}}^2 + \dot{z}^2) + \frac{q}{c}(A_x\dot{x} + A_{\acute{o}}\dot{\acute{o}} + A_z\dot{z}) - q\varphi \qquad (3)$$

Endi Lagranj tenglamasining komponentalarini tuzamiz

$$\begin{gathered}
\frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x; \quad \frac{\partial L}{\partial x} = \frac{q}{c}\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_{\acute{o}}}{\partial x}\dot{\acute{o}} + \frac{\partial A_z}{\partial x}\dot{z}\right) - q\frac{\partial\varphi}{\partial x} \\
\frac{\partial L}{\partial \dot{\acute{o}}} = m\,\dot{\acute{o}} + \frac{q}{c}A_{\acute{o}}; \quad \frac{\partial L}{\partial \acute{o}} = \frac{q}{c}\left(\frac{\partial A_x}{\partial \acute{o}}\dot{x} + \frac{\partial A_{\acute{o}}}{\partial \acute{o}}\dot{\acute{o}} + \frac{\partial A_z}{\partial \acute{o}}\dot{z}\right) - q\frac{\partial\varphi}{\partial \acute{o}} \\
\frac{\partial L}{\partial \dot{z}} = m\dot{z} + \frac{q}{c}A_z\ ; \quad \frac{\partial L}{\partial z} = \frac{q}{c}\left(\frac{\partial A_x}{\partial z}\dot{x} + \frac{\partial A_{\acute{o}}}{\partial z}\dot{\acute{o}} + \frac{\partial A_z}{\partial z}\dot{z}\right) - q\frac{\partial\varphi}{\partial z}
\end{gathered} \qquad (4)$$

Lagranj tenglamasi umumiy holda

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial r} = 0$$

bo'lgani uchun

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + \frac{q}{c}\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial \acute{o}}\frac{d\acute{o}}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}\right) = m\ddot{x} + \frac{q}{c}\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial \acute{o}}\dot{\acute{o}} + \frac{\partial A_x}{\partial z}\dot{z}\right)$$

bo'ladi va (4)ni hisobga olib x komponenta uchun Lagranj tenglamasini yozamiz

$$m\ddot{x} = q\left(-\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}\right) + \frac{q}{c}\left\{\left[\frac{\partial A_{\acute{o}}}{\partial x} - \frac{\partial A_x}{\partial \acute{o}}\right]\dot{\acute{o}} + \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\dot{z}\right\} \qquad (5)$$

Qolgan komponentalar uchun ham tenglama (5) ga o'xshash bo'ladi.

$$\begin{gathered}
E_x = -\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}, \quad E_{\acute{o}} = -\frac{\partial\varphi}{\partial \acute{o}} - \frac{1}{\tilde{n}}\frac{\partial A_{\acute{o}}}{\partial t}, \quad E_z = -\frac{\partial\varphi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial z}; \\
H_x = \frac{\partial A_z}{\partial \acute{o}} - \frac{\partial A_{\acute{o}}}{\partial z}, \quad H_{\acute{o}} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad H_z = \frac{\partial A_{\acute{o}}}{\partial x} - \frac{\partial A_x}{\partial \acute{o}}
\end{gathered} \qquad (6)$$

almashtirishlarini o'tkazamiz.Agar $\vec{E} = (E_x, E_{\acute{o}}, E_z)$, $\vec{H} = (H_x, H_{\acute{o}}, H_z)$ vektorlarni kiritsak, (6) larni umumlashtirib

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{\partial\varphi}{\partial\vec{r}}, \qquad \vec{H} = rot\,\vec{A}$$

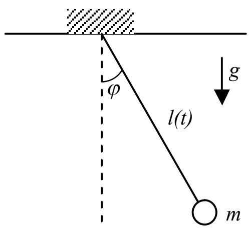
deb yoza ilamiz Lagranj tenglamasini

$$m\ddot{\vec{r}} = q\vec{E} + \frac{q}{c}\left[\dot{\vec{r}} \cdot \vec{H}\right]$$

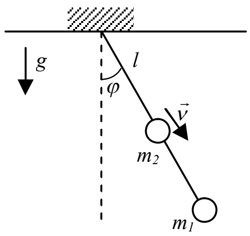
shaklga keltiramiz.Bu tenglama Lorents kuchini ifoda etadi, $\vec{E},\vec{H}$ elektromagnit maydon kuchlanganlik vektorlari, $\vec{A},\varphi$-elektromagnit maydonning vektorli va skalyar potensiallari,tenglamada ishtirok etishiga qarab c – yorug'lik tezligini, q – zaryad miqdorini ifoda etadi.

1-6-TOPSHIRIQLAR

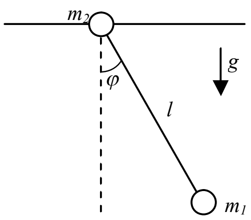
Bir jinsli og'irlik maydonida joylashgan quyidagi zarralar uchun Lagranj funksiyasi yozilsin.



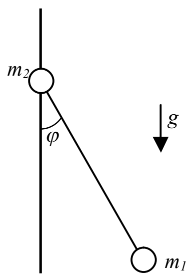
1.Mayatnik uzunligi vaqtning funksiyasi hisoblanadi,ya'ni $l = l(t)$



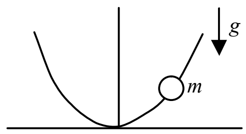
2.Mayatnik uzunligi $l = const$, m_2 massalik zarra sterjen bo'ylab $v = const$ tezlik bilan harakat qiladi va boshlang'ich holati ixtiyoriy.



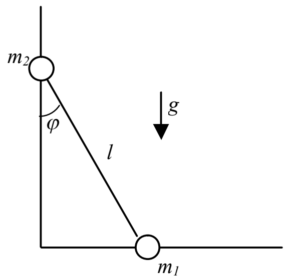
3. m_2 massalik zarra og'irlik kuchi maydoniga perpendikulyar yo'nalishda harakat qilishi mumkin.



4. m_2 massalik zarra og'irlik kuchi maydoni yo'nalishi bo'ylab harakat qilishi mumkin.



5.Zarra $z = kx^2$ parabola bo'ylab harakat qilishi mumkin.



6. m_2 massalik zarra $\vec{g}$ bo'yicha, m_1 massalik zarra $\vec{g}$ ga perpendikulyar harakat qilishi mumkin.

JAVOBLAR:

1. $L = \frac{m}{2}(\dot{l}^2 + l^2\dot{\varphi}^2) + mgl\cos\varphi$

2. $L = \frac{1}{2}(m_1 l^2 + m_2 v^2 t^2 \dot{\varphi}^2) + (m_1 l + m_2 v t) g \cos\varphi$

3. $L = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_1}{2}(l^2\dot{\varphi}^2 + 2\dot{x}\dot{\varphi}\, l\cos\varphi) + m_1 gl\cos\varphi$

4. $L = \frac{m_1 + m_2}{2}\dot{z}^2 + \frac{m_1}{2}(l^2\dot{\varphi}^2 - 2\dot{z}\dot{\varphi}\sin\varphi) + m_1 gl\cos\varphi$

5. $L = \frac{m}{2}(1 + 4kx^2)\dot{x}^2 - mgkx^2$

6. $L = \frac{1}{2}(m_1\cos^2\varphi + m_2\sin^2\varphi) l^2\dot{\varphi}^2 - m_2 gl\cos\varphi$

Uslubiy tavsiyalar:

1.Zarra yoki zarralar sistemasi holatini to'liq aniqlovchi o'zaro bog'liq bo'lmagan koordinatalar-umumlshgan koordinatalar tanlanadi;

2.Zarra yoki zarralar sistemasi dekart koordinatalari va tezliklari tanlab olingan umumlashgan koordinatalar va umumlashgan tezliklar orqali ifodalanadi;

3.Qaralayotgan sistema Lagranj funksiyasi ta'rifga ko'ra tuziladi;

4.Tuzilgan Lagranj funksiyasidan umumlashgan koordinata va umumlashgan tezlik bo'yicha hosilalaridan Lagranj tenglamasi tuziladi.

14-MASHG'ULOT

Mavzu: «Gamilton funksiyalari.Gamiltonning kanonik tenglamalari»

Darsning maqsadi. Berilgan zarra yoki zarralar sistemasi uchun Gamilton funksiyasini va bu funksiya orqali Gamilton tenglamasini tuzishni o'rganish.

Qisqacha tushuncha. Berilgan zarra uchun Gamilton funksiyasi bu zarra kinetik va potensial energiyalar yig'indisi tariqasida tuziladi:

$$H = T + U$$

Agar zarralar sistemasi berilgan bo'lsa ,har bir energiyalar yig'indisining yig'indisi

$$H = \sum T_n + \sum U_n$$

kabi topiladi.Bu funksiyadan umumlashgan tezlik va umumlashgan koordinata bo'yicha kerakli hosilalar olinib,Gamilton tenglamasi tuziladi: $\dot{p}_k = -\frac{\partial H}{\partial q_k}$, $\dot{q}_k = \frac{\partial H}{\partial p_k}$.

Masalalar

14-1.Massasi m bo'lgan zarraning biror potensial maydonda harakati uchun dekart koordinatalar sistemasida Gamilton tenglamasi tuzilsin.

Yechish.Dekart koordinatalarida Gamilton funksiyasi

$$H = T - U = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U(x, y, z)$$

Bundan impulsning mos komponentalarini topamiz:

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}.$$

Bulardan Gamilton funksiyasini tuzamiz:

$$H = \frac{1}{2m}\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right) + U(x, y, z)$$

Gamilton tenglamasini endi quyidagicha yoza olamiz:

$$p_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}, \qquad p_ó = -\frac{\partial Đ}{\partial ó} = -\frac{\partial U}{\partial ó}, \qquad p_z = -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z};$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{ó} = \frac{\partial H}{\partial p_ó} = \frac{p_ó}{m}, \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

14-2. Kvazielastik kuch $\vec{F} = -c\vec{r}$ ta'siri ostida bo'lgan m massalik erkin moddiy nuqta harakatining kanonik tenglamasi tuzilsin (bu yerda $c-$doimiy koeffitsiyent).Qarshilik kuchining harakatga ta'siri e'tiborga olinmasin.Umumlshgan koordinatalar tariqasida Dekart koordinatalari x, y, z lar olinsin.

Yechish.Erkin moddiy nuqta uchta erkinlik darajasiga ega bo'ladi.Uchta umumlashgan koordinatalar $q_1 = x,\ q_2 = y,\ \ q_3 = z$ larga uchta qo'shma umumlshgan impulslar $p_x,\ p_y,\ p_z$ mos keladi.

Umumlshgan impulslar p_x, p_y, p_z ni aniqlash uchun moddiy nuqtaning kinetik energiyasini hisoblaymiz

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Umumlashgan impulslar

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \qquad (1)$$

Nuqtaga ta'sir qiluvchi kuch $\vec{F}$ potensialli kuch bog'lanishlar esa mavjud emas,shuning uchun Gamilton funksiyasi nuqtaning to'liq mexanik energiyasiga teng bo'ladi:

$$H = T + U$$

Bu yerdagi potensial energiya potensial kuchning nuqtaning berilgan holatidan muvozanat holatigacha ko'chirishda bajargan ishiga teng bo'ladi

$$U = \int_r^0 \vec{F}\, d\vec{r} = -\int_r^0 r\, dr = \frac{c r^2}{2}$$

Biz bilamizki $r^2 = x^2 + y^2 + z^2$.Demak, $U = \frac{c}{2}\left(x^2 + y^2 + z^2\right)$

Endi biz Gamilton funksiyasini umumlashgan koordinatalar va umumlashgan impulslar orqali yoza olamiz

$$H = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{c}{2}\left(x^2 + y^2 + z^2\right) \qquad (2)$$

(1) dan umumlashgan tezliklarni umumlashgan impulslarning funksiyalari tariqasida ifoda etamiz

Topilganlarni (2) ga qo'yamiz

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{z} = \frac{p_z}{m}.$$

$$H = \frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_z{}^2) + \frac{c}{2}\left(x^2 + y^2 + z^2\right)$$

Endi biz kanonik tenglamalarni tuzamiz

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

Berilgan holda

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{z} = \frac{p_z}{m}., \quad \dot{p}_x = -c x, \quad \dot{p}_y = -c y, \quad \dot{p}_z = -c z$$

Shunday qilib biz oltita birinchi tartibli kanonik tenglamalarni oldik.Ular oltita kanonik x, y, z, p_x, p_y, p_z o'zgaruvchilarga mos keladi.Bu tenglamalarni dekart koordinatalar o'qlarida proyeksiyalangan uchta ikkinchi tartibli differensial tenglamalarga keltirish mumkin

$$m\ddot{x} = -c x, \quad m\ddot{y} = -c y, \quad m\ddot{z} = -c z$$

Oxirgi tenglamalar $\vec{F} = -c\vec{r}$ kuchning dekart koordinata o'qlariga proyeksiyalari hisoblanadi.

ABBYY PDF Transformer 2.0
Click here to buy
www.ABBYY.com

1-6- TOPSHIRIQLAR

Agar Gamilton funksiyasi quyidagi ko'rinishda berilgan bo'lsa,Gamilton tenglamalari yozilsin:

1. $H = \dfrac{p^2}{2ml^2} - mgl\cos\varphi - \dfrac{m}{2}\dot{l}^2$

2. $H = \dfrac{p^2}{2m(1+4kx^2)} + mgkx^2, \quad k = const$

3. $H = a\ln p_x + \dfrac{p_{\acute{o}}{}^2}{2x} + a - \text{`}a\ln a, \qquad a = const$

4. $H = c\sqrt{p^2 + m^2c^2}, \qquad p^2 = p_x{}^2 + p_{\acute{o}}{}^2 + p_z{}^2 \;, c = const$

5. $H = \dfrac{p_u p_v}{t u^2 v^2}, \qquad t - vaqt$

6. $H = \dfrac{p_u{}^2 + p_v{}^2}{2(u^2+v^2)} + au \;, \quad a = const$

JAVOBLAR:

1. $\dot{\varphi} = \dfrac{p^2}{ml^2}, \quad \dot{p} = -mgl\sin\varphi$

2. $\dot{x} = \dfrac{p^2}{m(1+4k^2x^2)}, \qquad \dot{p} = \dfrac{4k^2xp^2}{m(1+4k^2x^2)} - 2mgkx$

3. $\dot{x} = \dfrac{a}{p_x}, \; \dot{p}_x = \dfrac{1}{2x^2}p_{\acute{o}}{}^2; \qquad \breve{o} = \dfrac{p_{\acute{o}}}{x} \quad , \quad \dot{p}_{\acute{o}} = const$

4. $\dot{\vec{r}} = \dfrac{c\,\vec{p}}{\sqrt{p^2 + m^2c^2}}, \quad \vec{p} = const$

5. $\dot{u} = \dfrac{p_v}{t u^2 v}, \; \dot{p}_u = 2\dfrac{p_u p_v}{t u^3 v^2}, \qquad \dot{v} = \dfrac{p_u}{t u v^2}, \; \dot{p}_v = 2\dfrac{p_u p_v}{t u^2 v^3}$

6. $\begin{cases} \dot{u} = \dfrac{p_u}{u^2+v^2}, & \dot{p}_u = u\dfrac{p_u{}^2 + p^2{}_v}{(u^2+v^2)} - a \\ \dot{v} = \dfrac{p_v}{u^2+v^2}, & \dot{p}_v = v\dfrac{p_u{}^2 + p_v{}^2}{(u^2+v^2)} \end{cases}$

Uslubiy tavsiyalar:

1.Zarra yoki zarralar sistemasi holatini to'liq aniqlovchi o'zaro bog'liq bo'lmagan koordinatalar-umumlshgan koordinatalar tanlanadi;

2.Zarra yoki zarralar sistemasi dekart koordinatalari va tezliklari tanlab olingan umumlashgan koordinatalar va umumlashgan tezliklar orqali ifodalanadi;

3.Qaralayotgan sistema Gamilton funksiyasi ta'rifga ko'ra tuziladi;

4.Tuzilgan Gamilton funksiyasidan umumlashgan koordinata va umumlashgan tezlik bo'yicha hosilalaridan Gamilton tenglamasi tuziladi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

15-MASHG'ULOT
Mavzu: «Gamilton-Yakobi tenglamasi»

Darsning maqsadi. Talabalarda Gamilton-Yakobi tenglamasini tuzish va bu mavzuga oid masalalarni yechish bo'yicha ko'nikma hosil qilish.

Qisqacha tushuncha.

Gamilton-Yakoni tenglamasi quyidagi ko'rinishga ega:

$$\frac{\partial S}{\partial t}+H\left(q_1,q_2,...q_S,\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},...\frac{\partial S}{\partial q_S},t\right)=0$$

bu erda $S=\int\limits_{t_1}^{t_0}Ldt$.

Masalalar

15-1. Gamilton-Yakobi tenglamasi yordamida matematik mayatnikning kichik tebranishlar qonunini toping. Boshlang'ich momentda mayatnik tinch turibdi, u osilgan l uzunlikdagi ip vertikaldan φ_0 burchakka og'gan. Qarshilik kuchining harakatga to'sqinligi hisobga olinmasin.

Yechish. Matematik mayatnik bitta erkinlik darajasiga ega. Umumlashgan koordinata q tariqasida mayatnikning vertilaldan og'ish burchagi φ ni olamiz, yani $q=\varphi$.

Og'irlik kuchi potensialli kuch bo'lgani , cho'zilmovchi ip statsionar bog'lanishni yuzaga keltirgani uchun Gamilton funksiyasi H to'liq mexanik energiyaga teng bo'ladi:

$$H=T+U=h \tag{1}$$

Nuqta kabi massaga ega bo'lgan mayatnik kinetic energiyasi (biz qutb koordinatalar sistemasida ish ko'rmoqdamiz)

$$T=\frac{1}{2}mv^2=\frac{1}{2}ml^2\dot{\varphi}^2 \qquad (2),$$

umumlashgan impuls esa

$$p_\varphi=\frac{\partial T}{\partial\dot{\varphi}}=ml^2\dot{\varphi} \tag{3}$$

Bundan

$$\dot{\varphi}=\frac{p_\varphi}{ml^2} \tag{4}$$

Mayatnikning potensial energiyasi

$$U=mg\,\Delta x=mgl(1-\cos\varphi) \tag{5}$$

Lekin φ burchak kichik bo'lgani uchun

$$\cos\varphi=1-\frac{\varphi^2}{2}+\frac{\varphi^4}{4!}-.... \tag{6}$$

Tebranishni kichik dedik, shuning uchun (6) qatorning dastlabki ikki hadi bilan chegaralansak bo'ladi, yani

$$\cos\varphi\approx1-\frac{\varphi^2}{2}$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

U holda potensial energiya

$$U = \frac{m g l}{2} \varphi^2 \tag{7}$$

shakl oladi.Hamilton funksiyasi esa

$$H = \frac{m l^2}{2} \dot{\varphi}^2 + \frac{m g l}{2} \varphi^2 = \frac{1}{2 m l^2} p_\varphi{}^2 + \frac{m g l}{2} \varphi^2 \tag{8}$$

Eslatamizki,Gamilton-Yakobi tenglamasi yordamida kanonik tenglamalarning birinchi integrallarini aniqlash to'g'rsidagi masalaning yechimi yasovchi funksiya S ni axtarishni taqozo etadi.Bu funksiya bizga eski o'zgaruvchilar-umumlashgan koordinatalar va impulslarni yangi doimiy koordinatalar va impulslarga kanonik almashtirish imkoniyatini beradi.Berilgan holda eski φ, p_φ o'zgaruvchilarga yangi,lekin hozircha noma'lum, α, β doimiyliklar mos keladi.Mayatnik erkinlik darajasi $s=1$ bo'lgani uchun yasovchi funksiya S quyidagicha bo'ladi:

$$S = -h t + S_0(\varphi, h), \tag{9}$$

bu yerda h-to'liq mexanik energiya bo'lib,biz axtarayotgan α gat eng,yani $\alpha = h$.
(9) dagi S_0 ni hisoblash uchun Gamilton-Yakobi tenglamasidan foydalanamiz.Berilgan holda $s=1$ va $q=\varphi$ bo'lgani sabab

$$H\left(\varphi, \frac{\partial S_0}{\partial \varphi}\right) = h \tag{10}$$

(8) da p_φ ni $\frac{\partial S_0}{\partial \varphi}$ ga almashtirb va uni (10) ga kiritb,Gamilton-Yakobi tenglamasini

$$\frac{1}{2 m l^2} \left(\frac{\partial S_0}{\partial \varphi}\right)^2 + \frac{m g l}{2} \varphi^2 = h$$

ko'rinishda olamiz.Bizda S_0 faqat bitta o'zgaruvchi φ ga bog'liq bo'lgani uchun

$$\frac{\partial S_0}{\partial \varphi} = \frac{d S_0}{d \varphi}$$

bo'ladi.Demak,

$$\frac{1}{2 m l^2} \left(\frac{d S_0}{d \varphi}\right)^2 + \frac{m g l}{2} \varphi^2 = h \tag{11}$$

Shunday qilib,xususiy hosilalarda bo'lgan Gamilton-Yakobi tenglamasi berilgan holda oddiy differensial tenglamaga keltirildi.Bu tenglamani $\frac{d S_0}{d \varphi}$ ga nisbatan yechib,

$$\frac{d S_0}{d \varphi} = l \sqrt{2 m} \sqrt{h - \frac{m g l}{2} \varphi^2}$$

ga ega bo'lamiz.Uni integrallab,topamiz

$$S_0 = l \sqrt{2 m} \int \sqrt{h - \frac{m g l}{2} \varphi^2}\, d\varphi \tag{12}$$

(12) ni (9) ga qo'yamiz

$$S = -h t + l \sqrt{2 m} \int \sqrt{h - \frac{m g l}{2} \varphi^2}\, d\varphi \tag{13}$$

Moddiy sistema konservativ va $s=1$ bo'lgani uchun

$$p_\varphi = \frac{\partial S_0}{\partial \varphi} = -l\sqrt{2m}\int \frac{m g l \varphi d\varphi}{2\sqrt{h - \frac{m g l}{2}\varphi^2}}$$

$$\beta = -\frac{\partial S_0}{\partial h} = t - l\sqrt{2m}\int \frac{d\varphi}{2\sqrt{h - \frac{m g l}{2}\varphi^2}} \qquad (14)$$

Bu integrallarni integrallab,topamiz

$$p_\varphi = l\sqrt{2m\left(h - \frac{m g l}{2}\varphi^2\right)}, \qquad \beta = t - \sqrt{\frac{l}{g}}\arcsin\sqrt{\frac{m g l}{2}}\varphi \qquad (15)$$

(15) dan ko'rinadiki,yangi doimiy koordinata bo'lib to'liq mexanik energiya h, yangi doimiy impuls bo'lib h koordinataga qo'shma β hisoblanar ekan.

h va β larni aniqlash uchun mayatnik harakatining boshlang'ich shartlaridan foydalanamiz: $t=0$ bo'lganda $\varphi = \varphi_0$, $\varphi = \varphi_0$, $\dot\varphi = \dot\varphi_0 . = 0$ .Endi (3) formulani qo'llab, $p_\varphi = m l^2 \dot\varphi_0 = 0$ ekanligini topamiz.(15) formulaga

$$t = 0\ ,\ \varphi = \varphi_0\ ,\ p_\varphi = p_{\varphi_0} = 0.$$

larni qo'yib, kanonik tenglamalarning izlanayotgan birinchi integrallarini topamiz:

$$p_\varphi = m l\sqrt{g l(\varphi_0{}^2 - \varphi^2)}\ ; \qquad \varphi = \varphi_0 \cos\sqrt{\frac{g}{l}}\,t\ .$$

15-2. Massasi m bo'lgan og'irlik kuchi maydonida harakatlanayotgan erkin moddiy nuqtaning harakat qonuni trayektoriya tenglamasi topilsin.Umumlashgan koordinatalar tariqasida dekart koordinatalari x, y, z lar olinsin.Masalani Gamilton-Yakobi tenglamalari yordamida yechilsin.

Yechish. Oldingi masalada ko'rganimizdek,yasovchi funksiya yordamida Gamilton-Yakobi tenglamasi quyidagi ko'rinishda yozilar edi:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + m g z = 0 \qquad (1)$$

Berilgan holda energiya integral hisoblanganligi va x, y koordinatalar siklik koordinatalar bo'lganligi uchun o'zgaruvchilarga ajratish usulini qo'llab,yasovchi funksiya S ni quyidagicha axtaramiz:

$$S = -h t + \alpha_1 x + \alpha_2 y + S_0(z) \qquad (2),$$

Bu yerda h – to'liq mexanik energiya, α_1 , α_2 – yangi doimiy koordinatalar,ayni shu paytda integrallash doimiyliklari ham.Hozircha h, α_1 , α_2, $S_0(z)$ lar bizga noma'lum bo'lgan kattaliklar.

(2) funksiadan t, x, y, z lar bo'yicha hosila olamiz

$$\frac{\partial S}{\partial t} = -h, \qquad \frac{\partial S}{\partial x} = \alpha_1, \qquad \frac{\partial S}{\partial y} = \alpha_2, \qquad \frac{\partial S}{\partial z} = \frac{\partial S_0}{\partial z} = \frac{d S_0}{d z} \qquad (3)$$

Bu hosilalarni (1) ga qo'yamiz

$$\frac{1}{2m}\left[\alpha^2{}_1 + \alpha^2{}_2 + \left(\frac{d S_0}{d z}\right)^2\right] + m g z = h \qquad (4)$$

Shunday qilib,energiya integrali hamda i x, y ikkita siklik koordinatalar mavjudligi tufayli xususiy hosilali Gamilton-Yakobi tenglamasi oddiy differensial tenglamaga aylandi.Bu tenglamani $d S_0 / dz$ ga nisbatan yechib,

$$\left(\frac{d S_0}{d z}\right) = \sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha^2{}_2} \tag{5}$$

(5) ni integrallab,topamiz

$$S_0 = \int \sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha^2{}_2}\ dz \tag{6}$$

(6) ni (2) ga qo'yamiz

$$S = -h t + \alpha_1 x + \alpha_2 y + \int \sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha^2{}_2}\ d z \tag{7}$$

Bu yerda bizga h, α_1, α_2 lar noma'lum.

Bizdagi eski umumlashgan koordinatalar x, y, z lar , umumlashgan impulslar p_x, p_y, p_z larning yangi doimiy koordinatalar h, α_1, α_2 va yangi doimiy impulslar $\beta_1, \beta_2, \beta_3$ lar bilan bog'lanishini hosil qilamiz

$$p_x = \frac{\partial S}{\partial x} = \alpha_1, \quad p_y = \frac{\partial S}{\partial y} = \alpha_2, \quad p_z = \frac{\partial S}{\partial zx} = \sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2}, \tag{8}$$

$$\beta_1 = -\frac{\partial S}{\partial \alpha_1} = -x - \frac{\partial S_0}{\partial \alpha_1}, \quad \beta_2 = -\frac{\partial S}{\partial \alpha_2} = -y - \frac{\partial S_0}{\partial \alpha_2}, \quad \beta_3 = -\frac{\partial S}{\partial \alpha_1} = -t - \frac{\partial S_0}{\partial h}, \tag{9}$$

(2) dan kerakli hosilalarni topamiz

$$\frac{\partial S_0}{\partial \alpha_1} = \int \frac{-\alpha_1 d z}{\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2}}, \quad \frac{\partial S_0}{\partial \alpha_2} = \int \frac{-\alpha_2 d z}{\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2}},$$

$$\frac{\partial S_0}{\partial h} = \int \frac{m d z}{\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2}},$$

Bu integrallarni hisoblab,topamiz

$$\frac{\partial S_0}{\partial \alpha_1} = \frac{\alpha_1}{m^2 g}\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2}, \quad \frac{\partial S_0}{\partial \alpha_2} = \frac{\alpha_2}{m^2 g}\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2},$$

$$\frac{\partial S_0}{\partial h} = -\frac{1}{m g}\sqrt{2m(h - m g z) - \alpha_1^2 - \alpha_2^2},$$

Endi (9) tenglama quyidagicha ko'rinishga keladi:

$$\beta_1 = -x - \frac{\alpha_1}{m^2 g}\sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha_2^2},.$$

$$\beta_2 = -y - \frac{\alpha_2}{m^2 g}\sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha_2^2}, \tag{10}$$

$$\beta_3 = -t + \frac{1}{m g}\sqrt{2m(h - m g z) - \alpha^2{}_1 - \alpha_2^2}. \tag{11}$$

Shunday qilib biz eski kanonik o'zgaruvchilar bo'lmish x, y, z, p_x, p_y, p_z lar bilan yangi kanonik doimiyliklar $h, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ larni bog'lovchi oltita tenglamalar sistemasini oldik.

Integrallash doimiyliklarini aniqlash uchun harakatning boshlang'ich shartlari

$$t = 0,\ x = x_0, y = y_0, z = z_0,\ p_x = p_{x0}, p_y = p_{y0}, p_z = p_{z0}$$

dan foydalanamiz.(10) bilan aniqlanuvchi ikkita tenglama moddiy nuqtaning izlanayotgan trayektoriya tenglamalari hisoblansa,(11) tenglama vaqtga oshko0r bog'liq bo'lib,moddiy nuqtaning izlanayotgan harakat qonuni hisoblanadi.0

Oxirida biz (8),(10) va (11) tenglamalar yordamida b balandlikdan $z-$o'qi bo'ylab erkin tushayotgan moddiu nuqtaning tenglamasini olamizMa'lumki,trayektoriya tenglamasi (bizda $x-y=0$)

$$z=b-\frac{gt^2}{2}$$

Ko'rinishga ega bo'ladi.Boshlang'ich shartlar esa quyidagichadir:

$$t=0\,da\quad x=y=0, z=b\ , \dot{x}=\dot{y}=\dot{z}=0$$

Umumlashgan impulslar esa

$$p_x=m\dot{x},\quad p_y=m\dot{y},\quad p_z=m\dot{z}$$

Boshlang'ich vaqtda $p_x=0,\ \ p_y=0,\ \ p_z=0$

Bu qiymatlarni (8) ga qo'yamiz

$$0=\alpha_1,\qquad 0=\alpha_2,\qquad 0=\sqrt{2m(h-mgz)-\alpha^2{}_1-\alpha_2{}^2} \qquad (12)$$

Boshlang'ich shartlarni (10) va (11) larga qo'yib,topamiz

$$\beta_1=-\frac{\alpha_1}{m^2g}\sqrt{2m(h-mgz)-\alpha^2{}_1-\alpha_2{}^2}\ ,.$$

$$\beta_2=-\frac{\alpha_2}{m^2g}\sqrt{2m(h-mgz)-\alpha^2{}_1-\alpha_2{}^2}\ , \qquad (13)$$

$$\beta_3=\frac{1}{mg}\sqrt{2m(h-mgz)-\alpha^2{}_1-\alpha_2{}^2}\ .$$

(12) va (13) tenglamalar sistemasi yechib,topamiz

$$\alpha_1=0,\quad \alpha_2=0,\quad h=mgb,\quad \beta_1=0\ ,\ \beta_2=0\ ,\ \ \beta_3=0\ ,$$

(10-!1) larga bu qiymatlarni qo'yib,haqiqatan ham

$$x=y=0, z=b-\frac{gt^2}{2}$$

ekanligiga iqror bo'lamiz.

1-2-TOPSHIRIQLAR

Massasi m bo'lgan moddiy og'irlik kuchi maydonida nuqta harakatini tavsiflovchi Gamilton-Yakobi tenglamasi tuzilsin:

1.Umumlshgan koordinatalar qilib dekart koordinatalari x,y,z lar olinsin:

1.Umumlshgan koordinatalar qilib sferik koordinatalari ρ,θ,φ lar olinsin:

JAVOBLAR:

1. . $\frac{\partial S}{\partial t}+\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right]+mgz=0$

2. $\frac{\partial S}{\partial t}+\frac{1}{2m}\left[\left(\frac{\partial S}{\partial \rho}\right)^2+\frac{1}{\rho^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right]+mg\rho\cos\theta=0$

Uslubiy tavsiyalar:

1.Dastlab yasovchi funksiya kiritiladi va unung vaqt bno'yicha integrali hisoblanadi.
2.Gamilton-Yakobi tenglamasiga kirgan hosilalar hisoblanib,yasovchi funksiya ifodasiga qo'yiladi.
3.Oxirda topilgan tenglikka kiruvchi $\frac{dS_0}{dz}$ integrallanib, S_0 qiymati yasovchi funksiya ifodasiga qo'yladi.
4.Eski kanonik oz'garuvchilardan yangi kanonik doimiyliklarga o'tiladi va ularning ko'rinishi yuqorida keltirilgan masalalarda ko'rsatilgandek topiladi.

16-MASHG'ULOT
Mavzu: «Eyler-Lagranj tenglamasi»

Darsning maqsadi. Berilgan Lagranj funksiyasi uchun Eyler-Lagranj tenglamasini yozish texnikasini ishlab chiqish.
Qisqacha tushuncha. Ta'rifga ko'ra umumlashgan impuls

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_k},$$

umumlashgan kuch

$$Q_k \equiv \frac{\partial L}{\partial q_k}$$

formulalar bilan aniqlanadi. Eyler-Lagranj tenglamasi ana shu kattaliklar orqali quyidagicha aniqlanadi

$$\frac{dp_k}{dt} - Q_k = 0, \quad k = 1, 2......, s$$

Bu yerda s - qaralayotgan sistemaning erkinlik darajalar soni bo'lib, dissipativ kuchlar hisobga olinmaydi.

Masalalar

16-1. Agar Lagranj funksiyasi quyidagi ko'rinishga

$$L = \frac{1}{2}(m_1 + m_2)\dot{z}^2 + (m_1 - m_2)g\,z.$$

ega bo'lsa, umumlashgan impulslar, umumlashgan kuchlar topilsin, Eyler-Lagranj tenglamasi yozilsin va mumkin bo'lsa, u integrallansin.

Yechish. Umumlashgan impulsni quyidagicha aniqlaymiz:

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{z}} = (m_1 + m_2)\dot{z}\,, \quad \frac{d\,p}{d\,t} = (m_1 + m_2)\ddot{z}$$

Ta'rifga asosan umumlashgan kuch

$$Q_k = \frac{\partial L}{\partial q_k} = \frac{\partial L}{\partial z} = (m_1 - m_2)g.$$

Eyler-Lagranj tenglamasini tuzamiz:

$$(m_1 + m_2)\ddot{z} = (m_1 - m_2)g.$$

Tenglikning har ikkala tomonini $m_1 + m_2$ ga bo'lib, topamiz

$$\ddot{z} = \frac{m_1 - m_2}{m_1 + m_2}g.$$

Ushbu ifodani 2 marta integrallab quyidagiga ega bo'lamiz:

$$z = \frac{m_1 - m_2}{m_1 + m_2} \cdot \frac{gt^2}{2} + c_1 t + c_2,$$

bu yerda c_1 va c_2 lar integral doimiyliklari Odatda ular boshlang'ich shartlarga ko'ra topiladi..

16-2. Matematik mayatnik uchun Lagranj funksiyasi

$$L=\frac{1}{2}ml^2\dot{\varphi}^2+mgl\cos\varphi$$

berilgan.Eyler-Lagranj tenglamasi yozilsin va u integrallansin.

Yechish. Oldingi masaladagidek,umumlashgan impuls va umumlashgan kuchni topamiz (berilgan masalada umumlshgan koordinata bo'lib φ hisoblanadi)

$$p_\varphi=\frac{\partial L}{\partial\dot{\varphi}}=ml^2\dot{\varphi},\quad Q_\varphi=\frac{\partial L}{\partial\varphi}=-mlg\sin\varphi$$

Bulardan Eyler-Lagranj tenglamasini topamiz

$$ml^2\ddot{\varphi}+mlg\sin\varphi=0$$

Bu tenglamani har ikkala tomonini ml^2 ga bo'lamiz

$$\ddot{\varphi}+\frac{g}{l}\sin\varphi=0$$

Mayatnik kichik tebranayotgan bo'lsa, $\sin\varphi\approx\varphi$ va tenglama

$$\ddot{\varphi}+\frac{g}{l}\varphi=0$$

Bu tenglamani

$$\ddot{\varphi}+\omega^2\varphi=0$$

ko'rinishdagi tenglama qilib yozsak,

$\omega=\sqrt{\dfrac{g}{l}}$, yoki matematik mayatnikning tebranish davri uychun mashhur

$T=2\pi\sqrt{\dfrac{l}{g}}$ formulaga ega bo'lamiz.

1-6-TOPSHIRIQLAR

Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, umumlashgan impulslar, umumlashgan kuchlar topilsin., Eyler-Lagranj tenglamalari yozilsin va mumkin bo'lsa, ular integrallansin:

1. $L=\frac{m}{2}\left(\dot{l}^2+l^2\dot{\varphi}^2\right)+mgl\cos\varphi,\quad l=l(t)$ -vaqtning berilgan funksiyasi.

2. $L=\frac{1}{2}\left(m_1l^2+m_2c^2t^2\right)\dot{\varphi}^2+\left(m_1l+m_2ct\right)g\cos\varphi.$

3. $L=\frac{1}{2}(m_1+m_2)\dot{x}^2+\frac{m_1}{2}\left(l^2\dot{\varphi}^2+2\dot{x}\dot{\varphi}\,l\cos\varphi\right)+m_1gl\cos\varphi.$

4. $L=\frac{1}{2}(m_1+m_2)\dot{z}^2+\frac{m_1}{2}\left(l^2\dot{\varphi}^2+2\dot{z}\dot{\varphi}\,l\cos\varphi\right)+m_1gl\cos\varphi+(m_1+m_2)gz.$

5. $L=\frac{m}{2}\left(1+4k^2x^2\right)-mgkx^2.$

6. $\begin{cases} L=\dfrac{m_2}{2}\dot{x}^2+\dfrac{m_1}{2}\dot{x}^2\dfrac{x^2}{l^2-x^2}-m_2gx, \\ L=\dfrac{1}{2}\left(m_1\cos^2\varphi+m_2\sin^2\varphi\right)l^2\dot{\varphi}^2-m_2gl\cos\varphi \end{cases}$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

JAVOBLAR:

1. $l\ddot{\varphi} + 2\dot{l}\dot{\varphi} + g\sin\varphi = 0$

2. $\left(m_1 l^2 + m_2 c^2 t^2\right)\ddot{\varphi} + 2m_2 c^2 t\dot{\varphi} + \left(m_1 l + m_2 ct\right)g\sin\varphi = 0$

3. $$\begin{cases} \left(m_1 + m_2\right)x + m_1 l\sin\varphi = p_x t + c_2, \quad p_x = const \\ l\ddot{\varphi} + \ddot{x}\cos\varphi + g\sin\varphi = 0 \end{cases}$$

4. $$\begin{cases} \left(m_1 + m_2\right)z + m_1 l\cos\varphi = \dfrac{m_1 + m_2}{2}gt^2 + c_1 t + c_2 \\ l\ddot{\varphi} - \ddot{z}\sin\varphi + g\sin\varphi = 0 \end{cases}$$

5. $\left(1 + 4k^2x^2\right)\ddot{x} + 4k^2 x\dot{x}^2 + 2gk\,x = 0.$

6. $$\begin{cases} m_2\ddot{x} + m_1\dfrac{x^2}{l^2 - x^2}\ddot{x} + m_2 g + m_1\dfrac{l^2}{\left(l^2 - x^2\right)^2}x\dot{x}^2 = 0, \\ l\ddot{\varphi}\left(m_1\cos^2\varphi + m_2\sin^2\varphi\right) + \left(m_2 - m_1\right)l^2\dot{\varphi}^2\sin\varphi\cos\varphi - m_2 g\sin\varphi = 0. \end{cases}$$

Uslubiy tavsiyalar:

1.Umumlashgan impuls va umumlashgan kuchni topishda Lagranj funksiyasidan olinadigan hosila xususiy hosila ekanligiga Eyler –Lagranj tenglamasida differensiallash to'liq ekanli e'tibor bermoq lozim.

2.Bu mavzuda masala Eyler-Lagranj tenglamasini toppish va uni yozishdan iboratgina bo'lib qolmasdan uni yechish hamdir.Shuning uchun chiqarilgan tenglamani iloji boricha integrallash talab etiladi.

3.Mabodo tenglamada umumlashgan kuch nolga teng bo'lib chiqsa,u vaqtda o'z-o'zidan ma'lum bo'ladiki,umumlashgan impuls berilgan holda harakat integrali bo'lib hisoblanadi.

17-MASHG'ULOT
Mavzu: «Umumlashgan energiya»

Darsning maqsadi. Talabalarni umumlshgan energiya va Gamilton funksiyasini tuzish va ular o'rtasidagi farqni tushuntirish.

Qisqacha tushuncha: . Berilgan Lagranj funksiyasiga asosan umumlashgan impuls hisoblanadi:

$$p_k = p_k(q_1,...,q_S,...,\dot{q}_1,...,\dot{q}_S,t). \tag{1}$$

Ta'rifga asosan umumlashgan energiya quyidagicha topiladi:

$$H_{ue} = \sum_k p_k \dot{q}_k - L, \tag{2}$$

bu yerda $p_k \equiv \dfrac{\partial L}{\partial \dot{q}_k}$ - umumlashgan impuls.

Gamilton funksiyasi bu umumlashgan energiyani umumlashgan koordinatalar va umumlashgan impulslar (tezliklar emas) orqali ifodalanishdir. Shunday qilib, $p_k \equiv \dfrac{\partial L}{\partial \dot{q}_k}$ dan $\dot{q}_k$ topilib, H_{ue} ifodasiga qo'yiladi va H_{GF} Gamilton funksiyasi topiladi.

II. Gamilton funksiyasini bilgan holda, Gamilton tenglamasini quyidagicha aniqlash mumkin:

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}. \tag{3}$$

Masalalar

17-1. Agar Lagranj funksiyasi quyidagi ko'rinishga

$$L = \frac{1}{2}(m_1 + m_2)\dot{z}^2 + (m_1 - m_2)\, g\, z$$

ega bo'lsa, umumlashgan energiya va Gamilton funksiyasi yozilsin.

Yechish. Berilgan Lagranj funksiyasi asosida umumlashgan impulsni quyidagicha topamiz:

$$p_z = \frac{\partial L}{\partial \dot{z}} = (m_1 + m_2)\dot{z}$$

bu yerdan

$$\dot{z} = \frac{p_z}{m_1 + m_2} \tag{4}$$

(1) tenglamaga ko'ra umumlashgan energiya quyidagicha topiladi:

$$H_{ue} = (m_1 + m_2)\dot{z}^2 - \frac{1}{2}(m_1 + m_2)\dot{z}^2 - (m_1 - m_2)gz = \frac{1}{2}(m_1 + m_2)\dot{z}^2 - (m_1 - m_2)gz.$$

Endi umumlashgan energiya ifodasiga (4) dan $\dot{z}$ ni o'iymatini qo'yib, Gamilton funksiyasini quyidagicha topamiz:

$$H_{GF} = \frac{1}{2}(m_1 + m_2)\frac{p_z^2}{(m_1 + m_2)^2} - (m_1 - m_2)gz = \frac{p_z^2}{2(m_1 + m_2)} - (m_1 - m_2)gz.$$

17-2. Agar Gamilton funksiyasi quyidagi ko'rinishga

$$H = \frac{p^2}{2(m_1 + m_2)} - (m_1 - m_2)gz.$$

ega bo'lsa, Gamilton tenglamalari yozilsin.

Yechish. Ushbu berilgan Gamilton funksiyasi asosida Gamilton tenglamalarini quyidagicha topamiz:

$$\frac{dq}{dt} = \frac{dz}{dt} = \frac{\partial H}{\partial p} = \frac{p_z}{m_1 + m_2},$$

$$\frac{dp_z}{dt} = -\frac{\partial H}{\partial z} = (m_1 - m_2)g.$$

Demak, Gamilton tenglamalari

$$\dot{z} = \frac{p_z}{m_1 + m_2} \quad \text{va} \quad \dot{p}_z = (m_1 - m_2)g.$$

1-4-TOPSHIRIQLAR

Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, umumlashgan energiya va Gamilton funksiyasi yozilsin.

1. $L = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\varphi}^2\right) + mgl\cos\varphi,$

bu yerda $l = l(t)$ -vaqtning berilgan funksiyasi

2. $L = \frac{m}{2}\left(1 + 4k^2x^2\right)\dot{x}^2 - mgkx^2.$

3. $L = \frac{x\dot{y}^2}{2} + a\ln\dot{x}, \quad a = const.$

4. $L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}, \quad v^2 = v_x^2 + v_y^2 + v_z^2,\ c = const.$

JAVOBLAR:

1. $\begin{cases} H_{UE} = \frac{m}{2}l^2\dot{\varphi}^2 - mgl\cos\varphi - \frac{m}{2}\dot{l}^2, \\ H_{GF} = \frac{p^2}{2ml^2} - mgl\cos\varphi - \frac{m}{2}\dot{l}^2. \end{cases}$

2. $\begin{cases} H_{ue} = \frac{m}{2}\left(1 + 4k^2x^2\right)\dot{x}^2 + mgkx^2, \\ H_{GF} = \frac{p^2}{2m\left(1 + 4k^2x^2\right)} + mgkx^2. \end{cases}$

3. $\begin{cases} H_{ue} = \frac{1}{2}x\dot{y}^2 - a\ln\dot{x} + a, \\ H_{GF} = a\ln p_x + \frac{p^2}{2x} + a - a\ln a. \end{cases}$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$4.\begin{cases} H_{ue} = \dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}}, \\ H_{GF} = c\sqrt{p^2 + m^2c^2}. \end{cases}$$

5-10-TOPSHIRIQLAR

Agar gamilton funksiyasi quyidagi ko'rinishga ega bo'lsa, Gamilton tenglamalari yozilsin:

$$5.\ H = \frac{p^2}{2ml^2} - mgl\cos\varphi - \frac{m}{2}\dot{l}^2,$$

bu yerda $l = l(t)$ -vaqtning berilgan funksiyasi.

$$6.\ H = \frac{p^2}{2m\left(1+4k^2x^2\right)} + mgkx^2, \quad k = const.$$

$$7.\ H = a\ln p_x + \frac{p_y^2}{2x} + a - a\ln a, \quad a = const.$$

$$8.\ H = c\sqrt{p^2 + m^2c^2},\ p^2 = p_x^2 + p_y^2 + p_z^2,\ c = const.$$

$$9.\ H = \frac{p_x p_y}{tx^2y^2},\ t\text{ -vaqt}.$$

$$10.\ H = \frac{p_u^2 + p_v^2}{2\left(u^2+v^2\right)} + au, \quad a = const.$$

JAVOBLAR:

$$1.\begin{cases} \dot{\varphi} = \dfrac{p}{ml^2}, \\ \dot{p} = -mgl\sin\varphi \end{cases}$$

$$2.\begin{cases} \dot{x} = \dfrac{p}{m\left(1+4k^2x^2\right)}, \\ \dot{p} = \dfrac{4k^2p^2x}{m\left(1+4k^2x^2\right)} - 2mgkx. \end{cases}$$

$$3.\begin{cases} \dot{x} = \dfrac{a}{p_x},\ \dot{p}_x = \dfrac{1}{2x^2}p_y^2. \\ \dot{y} = \dfrac{p_y}{x}, \quad \dot{p}_y = const. \end{cases}$$

$$4.\ \dot{\vec{r}} = \frac{c\vec{p}}{\sqrt{p^2 + m^2c^2}}, \quad \vec{p} = const.$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$5.\begin{cases} \dot{x} = \dfrac{p_y}{t\,x^2 y}, \\ \dot{y} = \dfrac{p_x}{t\,x^2 y}. \end{cases} \qquad \begin{cases} \dot{p}_x = 2\dfrac{p_x p_y}{t\,x^3 y^2}, \\ \dot{p}_y = 2\dfrac{p_x p_y}{t\,x^2 y^3}. \end{cases}$$

$$6.\begin{cases} \dot{u} = \dfrac{p_u}{u^2 + v^2}, \\ \dot{v} = \dfrac{p_v}{u^2 + v^2}. \end{cases} \qquad \begin{cases} \dot{p}_u = u\dfrac{p_u^2 + p_v^2}{\left(u^2 + v^2\right)^2} - a \\ \dot{p}_v = v\dfrac{p_u^2 + p_v^2}{\left(u^2 + v^2\right)^2}. \end{cases}$$

Uslubiy tavsiyalar:

1.Mavzudagi Gamilton funksiyasi oldingi masalalarda ham qayd qilingani uchun bu mavzudagi formulalar talabalarga tanish hisoblanadi.Shuning uchun masalalarni sinf taxtasida ishlaganda talabalar ishtirokini faollashtirish lozim.

2.Umumlashgan energiya va Gamilton funksiyasini yozganda talabalar diqqatini tezlik va nuqta massasining suratdan maxrajga siljishiga diqqatni tortish kerak.

3.Lagranj funksiyasining umumlashgan energiya bilan mos tushmasligiga talabalar diqqatini tortish kerak bo'ladi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

18-MASHG'ULOT
Mavzu: «Puasson qavslari»

Darsning maqsadi. Puasson qavslari nazariy mexanikada ham,kvant mexanikasida ham katta ahamiytga ega bo'lgani uchun talabalarni bu qavslarni ocha bilishni o'rgatish.Bu qavslar bilan harakat integrallari bevosita bog'liqdir.

Qisqacha tushuncha: f_1 va f_2 fizik kattaliklar uchun Puasson qavsi quyidagi ko'rinishga ega:

$$\{f_1 f_2\} = \sum_k \left[\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial p_k} - \frac{\partial f_2}{\partial q_k}\frac{\partial f_1}{\partial p_k}\right] = -\{f_2, f_1\}, i = \overline{1, S} \tag{1}$$

bu yerda q_k, p_k - fazoviy fazaga bog'liq bo'lmagan o'zgaruvchilar,

$$\{f, f\} = 0.$$

Fizik kattaliklarning pausson qavsini topish uchun quyidagi qoidalardan foydalanish kerak:

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}, \frac{\partial p_i}{\partial p_j} = \delta_{ij}; \frac{\partial p_i}{\partial q_j} = 0. i, j = \overline{1,3}$$

Pusson qavsi uchun quyidagi xossalar mavjud:

$$\{f, g\} = -\{g, f\}. \tag{2}$$

$$\{\alpha f, g\} = \alpha\{f, g\}, \alpha = const. \tag{3}$$

$$\{f, c\} = 0, c = const. \tag{4}$$

$$\frac{\partial}{\partial t}\{f, g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}. \tag{5}$$

Ushbu f, g, h funksiyalar uchun quyidagi ayniyat o'rinlidir:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \equiv 0. \tag{6}$$

Ushbu ifodaga Pausson ayniyati deyiladi.

1. Agar $f = c_1$ va $g = c_2$ kanonik tenglamalarning birinchi integrallari bo'lsa, $\{f, g\}$ puasson qavsi ham bu tenglamalarning birinchi integrallari bo'ladi, ya'ni

$$\frac{\partial f}{\partial t} + \{f, H\} = 0; \frac{\partial g}{\partial t} + \{g, H\} = 0 \tag{7}$$

$$\{f, g\} \equiv c_3, \quad c_3 = const. \tag{8}$$

Pussonning ushbu xossasi Puasson teoremasi deb yuritiladi.

Masalalar

18-1. l_1 va l_2 impuls momentlari uchun Puasson qavslari topilsin.

Yechish. Puasson qavsining qoidasiga asosan quyidagini yozish mumkin:

$$\{l_1, l_2\} = \sum_i \left(\frac{\partial l_1}{\partial p_i}\frac{\partial l_2}{\partial x_i} - \frac{\partial l_1}{\partial x_i}\frac{\partial l_2}{\partial p_i}\right), \tag{9}$$

bu yerda

$$\vec{l} = [\vec{r}\ \vec{p}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}, \qquad (10)$$

$$\left. \begin{aligned} l_1 &= (yp_z - p_y z) = (x_2 p_3 - p_2 x_3), \\ l_2 &= (xp_z - p_x z) = (x_1 p_3 - p_1 x_3), \\ l_3 &= (xp_y - p_x y) = (x_1 p_2 - p_1 x_2). \end{aligned} \right\} \qquad (11)$$

(9) va (11) ifodalarni birgalikda qarab quyidagini hosil qilamiz:

$$\{l_1, l_2\} = \sum_i \left\{ \frac{\partial}{\partial p_i}(x_2 p_3 - x_3 p_2) \frac{\partial}{\partial x_i}(x_3 p_1 - x_1 p_3) - \frac{\partial}{\partial x}(x_2 p_3 - x_3 p_2) \frac{\partial}{\partial p_i}(x_3 p_1 - x_1 p_3) \right\} =$$
$$= x_2 p_1 - (-p_2)(-x_1) = x_2 p_1 - x_1 p_2 = -l_3.$$

Demak,

$$\{l_1 l_2\} = -l_3 \qquad \text{va} \qquad \{l_2 l_1\} = l_3$$

18-2. Gamilton tenglamalarini Puasson qavslari orqali quyidagicha yozilishi mumkinligini isbotlang:

$$\frac{d p_i}{d t} = \{p_i, H\}, \qquad \frac{d q_i}{d t} = \{q_i, H\}$$

Bu yerda H – Gamilton funksiyasi.Y

Yechish.Istalgan f fizik kattalikning vaqt bo'yicha o'zgarish qonuni

$$\frac{d f}{d t} = \frac{\partial f}{\partial t} + \{f, H\} \qquad (*)$$

tenglama bilan ifodalanishini bilamiz.Agar bizda

a) $f = p_i$ bo'lsa, umumlashgan impulsning esa vaqtning oshkormas funksiyasi ekanligidan $\frac{\partial p_i}{\partial t} = 0$ bo'ladi va (*) tenglamadan

$$\frac{d p_i}{d t} = \{p_i, H\}$$

tengligining o'rinli ekanligini ko'ramiz:

b) $f = q_i$ bo'lsin.Bu holda ham xuddi shunday $\frac{\partial q_i}{\partial t} = 0$ bo'lgani uchun

$$\frac{d q_i}{d t} = \{q_i, H\}$$

bo'ladi.

18-3. Tashqi kuch mavjud bo'lmaganida

$$f = x - \frac{p}{m} t$$

funksiyaning harakat integrali ekenligini isbotlang.

Yechish.Oldingi misolda istalgan f funksiyaning harakat qoning

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\{f,H\}$$

ko'rinishda yozilishini bilamiz.Bu tenglamaning char tomonini topaylik

$$\frac{df}{dt}=\frac{d}{dt}(x-\frac{p}{m}t)=\frac{dx}{dt}-\frac{p}{m}=v_x-\frac{mv_x}{m}=v_x-v_x=0$$

Bundan $f=const$ -

harakat integrali bo'lishligini ko'ramiz.Ikkinchi tomondan

$$\frac{\partial f}{\partial t}+\{f,H\}=0$$

ekanligini isbotlaymiz

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial t}(x-\frac{p}{m}t)=-\frac{p}{m};$$

$$\{f,H\}=\left\{(x-\frac{p}{m}t),\frac{p^2}{2m}\right\}=\frac{\partial}{\partial p}(x-\frac{p}{m}t)\cdot\frac{\partial p^2}{2m\partial x}+\frac{\partial}{\partial x}(x-\frac{p}{m}t)\cdot\frac{\partial p^2}{2m\partial p}=\frac{p}{m}$$

Demak,

$$\frac{\partial f}{\partial t}+\{f,H\}=-\frac{p}{m}+\frac{p}{m}=0$$

1-9-TOPSHIRIQLAR

Quyidagi fizik kattaliklar uchun Puasson qavslari topilsin.

1. $\{l_i,x_j\}$.
2. $\{l_i,p_j\}$. $\quad \vec{l}=\vec{r}\,x\,\vec{p},\ i,j=1,2,3,\quad l^2=l_1^2+l_2^2+l_3^2$
3. $\{l_i,l_j\}$. $\quad$ 1,2,3 x,y,z o'qlariga mos keladi.
4. $\{l_i,l^2\}$
5. $\{x_iT\},\{x_iU\},\{x_iH\}$.
6. $\{p_iT\},\{p_iU\},\{p_iH\}$.
7. $\{l_iT\},\{l_iU\},\{l_iH\}$.
8. $\{l^2T\},\{l^2U\},\{l^2H\}$.
9. $\{TU\},\{TH\},\{UH\}$.

5-9 msaslalarda $H=T+U$ -Gamilton funksiyasi, $T=\frac{p^2}{2m}$,$U=U(\vec{r})$- massasi m ga teng bo'lgan erkin moddiy nuqtaning kinetik va potensial energiyasi.

JAVOBLAR:

$\{l_ix_j\}$	x_1	x_2	x_3
l_1	0	x_3	$-x_2$
l_2	$-x_3$	0	x_1
l_3	x_2	$-x_1$	0

2.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$\{l_i p_j\}$	p_1	p_2	p_3
l_1	0	p_3	$-p_2$
l_2	$-p_3$	0	p_1
l_3	p_2	$-p_1$	0

3.

$\{l_i l_j\}$	l_1	l_2	l_3
l_1	0	l_3	$-l_2$
l_2	$-l_3$	0	l_1
l_3	l_2	$-l_1$	0

4. $\{l_i, l^2\} = 0 \quad i = 1,2,3.$

5. $\{x_i, T\} = \{x_i, H\} = \dfrac{p_i}{m}, \{x_i, U\} = 0.$

6. $\{p_i, T\} = 0, \{p_i, U\} = \{p_i, H\} = F_i = -\dfrac{\partial U_i}{\partial x_i}.$

7. $\{l_i, T\} = 0, \{l_i, U\} = \{l_i, H\} = M_i.$

8. $\{l^2, T\} = 0, \{l^2, U\} = \{l^2, H\} = 2\vec{l}\vec{M}.$

9. $\{T, U\} = \{T, H\} = -\{U, H\} = \dfrac{1}{m}\vec{p}\vec{F}.$

Bu yerda hamma joyda $\vec{F} = -\nabla U, \quad \vec{M} = \vec{r} \, x \, \vec{F}.$

Uslubiy tavsiyalar:

1.1-3 masalalar javoblarida jadvalning satrlari va ustunlari nimani anglatganligini talabalarga ko'rsatish lozim.

2.Impuls momentining barcha komponentalarini oldindan taxtaga yozib qo'yish kerak.

3. $\{l_i, l^2\}$ tipidagi Puasson qavslarini topganda $\{l_i, l^2\} = \{l_i, l_j^{\,2}\} = \sum_j 2 l_j \{l_i, l_j\}$ tenglikdan foydalanish kerak.

19-MASHG'ULOT
Mavzu: «Erkinlik daraja soni birga teng bo'lgan sistemalarda Lagranj funksiyasini garmonlashtirish»

Darsning maqsadi. Berilgan Lagranj funksiyasi uchun garmonik yaqinlashuvga ega bo'lishni talabalarga o'rgatish.

Qisqacha tushuncha. Erkin daraja soni birga teng bo'lgan tizimlarda Lagranj funksiyasining ko'rinishi quyidagicha bo'lishi mumkin:

$$L = \frac{a(q)}{2}\dot{q}^2 - u(q), \tag{1}$$

bu yerda $a(q)$-oddiy holda massaga mos keluvchi va massa rolini uynovchi ba'zi umumlashgan koordinataning funksiyasi.

Garmonik yaqinlashish g'oyasi juda ham oddiydir: $q = q_0$-kichik atrofida finitli harakatni qaraymiz. Ekstremum nuqtasidan $x = q = q_0$ kichik siljish bo'yicha Lagranj funksiyasini qatorga yozamiz. Ikkinchi tartibli haddan yuqorisining kichikligini hisobga olgan holda tashlab yuborib, garmonik yaqinlashishdagi Lagranj funksiyasini olamiz.

Yuqorida aytilganlarni bajarish uchun Lagranj funksiyasidan $q_0$li had qatnashmagan qismi, ya'ni potensial energiya qismi ajratib olinadi, ekstremum sharti $\frac{\partial u}{\partial q} = 0$ yoziladi va bu shartni qanoatlantiruvchi q_{0i} nuqta topiladi.

Keyin q_{0i} nuqtada ikkinchi hosila $\left(\frac{\partial^2 u}{\partial q^2}\right)$ hisoblanadi. Agar ikkinchi hosila ≤ 0 bo'lsa, u tashlab yuboriladi. q_{0i} nuqtada $u(q)$ minimumiga mos keluvchi hosila qiymatini k deb belgilasak: $k \equiv \left(\frac{\partial^2 u}{\partial q^2}\right)_{q_{0i}} > 0$ Kinetik energiya qismidagi $\frac{1}{2}\dot{q}^2 = \frac{1}{2}\dot{x}^2$ oldidagi koeffisiyent $a(q)q_{0i}$ nuqtada hisoblanadi va m orqali belgilanadi, ya'ni

$$m = a(q_{0i}).$$

Shunday qilib, garmonik yaqinlashishda Lagranjfunksiyasi quyidagicha yoziladi:

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right),$$

bu yerda $\omega = \sqrt{k/m}$ -garmonik tebranishning xususiy chastotasi.

Masalalar

19-1. Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, sistemaning xususiy garmonik tebranishlar chastotasi topilsin.

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2,$$

bu yerda $k, m - const$.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Yechish. Lagranj funksiyasidagi potensial energiya quyidagiga teng:

$$U(q)=\frac{1}{2}kq^2.$$

Ushbu $U(q)$ dan birinchi hosila olib, nolga tenglashtirib, potensial minimum bo'lgan nuqta topiladi, ya'ni

$$\frac{\partial U}{\partial q}_{|q=q_0}=\frac{2kq_0}{2}=kq_0=0,$$

Bundan $q_{10}=0$ ekanligini topamiz.

Ikkinchi hosila esa,

$$\frac{\partial^2 U}{\partial q^2}=k>0.$$

Demak, garmonik yaqinlashishda Lagranj funksiyasi qsyidagi ko'rinishga ega bo'ladi:

$$L=\frac{1}{2}m\dot{x}^2-\frac{k}{2}x^2,$$

bu yerda $a(q)=m,\ \dot{q}=\dot{x}.$ hamda $\omega^2=\frac{k}{m}$ yoki $\omega=(\frac{k}{m})^{\frac{1}{2}}$

19-2. Rasmda ko'rsatilgan mayatnik tebranish chastotasi topilsin.Bu mayatnik osilgan ipining uchiga m_1 massalik nuqta joylashgan va bu nuqta gorizontal yo'nalishda harakat qiladi.

Yechish. Bu mayatnik uchun umumlashgan impuls p_x saqlanuvchan kattalik hisoblanadi va u sistema to'liq impulsining gorizontal komponentasi bilan mos tushadi

$$p_x=(m_1+m_2)\dot{x}+m_2l\dot{\varphi}\cos\varphi=const \qquad (*)$$

Hamavaqt sistemani bir butunligicha tinch holatda deb hisoblash mumkin bo'lgani uchun $const=0$ .U holda yuqoridagi tenglamani integrallash quyidagini beradi:

$$(m_1+m_2)x+m_2l\sin\varphi=const$$

Bu tenglik gorizontal yo'nalishda sistema inersiya markazining harakatlanmasligini ifoda etadi.(*) tenglamadan

$$\dot{x}=\frac{m_2l}{(m_1+m_2)}\dot{\varphi}\cos\varphi$$

$x-$o'qi bo'yicha harakatlanayotgan nuqtaning kinetic energiyasi

$$T=\frac{m_1}{2}\dot{x}^2=\frac{m_1m_2l^2}{2(m_1+m_2)}\dot{\varphi}^2$$

Bu yerda biz mayatnik kichik tebranayotgani uchun $\varphi<<1$, $\cos\varphi\approx1$

Mayatnikning potensial energiyasi esa

$$U=\frac{m_2gl}{2}\varphi^2$$

Demak,sistema uchun Lagranj funksiyasi

$$L=\frac{m_1m_2l^2}{2(m_1+m_2)}\dot{\varphi}^2-m_2gl\varphi^2$$

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

Shakldа yoziladi va bu funksiya yordamida mayatnik harakat tenglamasini yozsak,u quyidagicha ko'rinishda bo'ladi:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \quad , \quad \ddot{\varphi} + \frac{g(m_1 + m_2)}{m_1 l}\varphi = 0$$

Demak,sistema tebranish chastotasi

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}$$

ko'rinishga ega bo'ladi.

1-6-TOPSHIRIQLAR

Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, tizimning garmonik tebranishda xususiy chastotasi topilsin:

$$\left.\begin{aligned}
&1.\ L = \frac{1}{2}\dot{q}^2 + a\sin bq \\
&2.\ L = \frac{1}{2q}\left(\dot{q}^2 - aq - b\right) \\
&3.\quad L = \dot{q}^2 - aq^2 e^{bq} \\
&4.\ \ L = \left(\dot{q}^2 - a\right)ch\, bq \\
&5.\ \ L = \frac{1}{q}\dot{q}^2 - \frac{aq}{\ln bq}
\end{aligned}\right\} \quad a, b - const.$$

$$6.\ \ L = \frac{1}{2}m\left(1 + 4k^2x^2\right)\dot{x}^2 - mgkx^2, \quad g, k - const.$$

JAVOBLAR:

1. $b\sqrt{a}$
2. $\sqrt{a}$.
3. $\sqrt{a}$.
4. $b\sqrt{\dfrac{a}{2}}$.
5. $\sqrt{\dfrac{a}{2}}$.
6. $\sqrt{2gk}$.

Uslubiy tavsiyalar:

1.Lagranj funksiyasidan $\dot{q}$ ni o'z ichiga olmagan hadni ajratib uni potensial energiya deb qaraymiz.

2.Potensialning ekstremallik qiymatidan muvozanat nuqtasi q_0 topiladi.

3.Potensial energiyaning ikkinchi hosilasining muvozanat nuqtasidagi qiymati topiladi.Bu hosila ≤ 0 bo'lgan nuqtalar tashlab ketiladi,noldan katta bo'lsa bu hosila k deb belgilanadi.

4.Garmonik yaqinlashuvda Lagranj funksiyasi

$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$ ko'rinishda yozilib,chastota topiladi.

20-MASHG'ULOT
Mavzu: «I. Erkinlik daraja soni birga teng bo'lgan sistemada majburiy tebranishlar»

Darsning maqsadi. Majburiy tebranishlar uchun harakat qonunini topishni va bunda boshlang'ich shartlarni hisobga olishni talabalarga o'rgatish.

Qisqacha tushuncha. Erkinlik darajasi birga teng bo'lgan sistemaning majburiy tebranishini harakat tenglamasi quyidagi ko'rinishga ega:

$$\ddot{x} + \omega^2 x = \frac{1}{m} F(t). \qquad (1)$$

Bu yerda ω erkin tebranishlar chastotasi. Ma'lumki, birjinsli bo'lmagan doimiy koeffitsiyantli chiziqli differensial tenglamaning yechimi

$$x = x_0 + x_1$$

ko'rinishdagi ikki qismdan iborat bo'lar edi, unda x_0 – birjinsli tenglamaning umumiy yechimi, x_1 – birjinsli bo'lmagan tenglamaning xuxusiy integrali.

Agar tashqi kuch vaqtning oddiy davriy funksiyasi

$$F(t) = f\cos(\alpha t + \beta)$$

Bo'lsa, umumiy yechim

$$x = a\cos(\omega t + \varphi) + \frac{f}{m(\omega^2 - \alpha^2)}\cos(\alpha t + \beta)$$

ko'rinishda bo'ladi. Agar majburlovchi kuch vaqtning ixtiyoriy funksiyasi bo'lsa, (1) tenglamani integrallash mumkin bo'ladi:

$$\frac{d}{dt}(x + i\omega x) - i\omega(x + i\omega x) = \frac{F(t)}{m} \qquad (2)$$

Agar $\xi = x + i\omega x$ belglash kiritsak, (2)

$$\frac{d\xi}{dt} - i\omega\xi = \frac{F(t)}{m}$$

shaklga keladi. Uning yechimi

$$\xi = \exp(i\omega t)\left\{\int_0^t \frac{F(t)}{m}\exp(-i\omega t)\,dt + \xi_0\right\}. \qquad (3)$$

Sistemaning boshlang'ich energiyasini nolga teng deb olib, majburlovchi kuchning $-\infty$ dan to $+\infty$ gacha vaqt ta'sirida sistemaga berilgan energiya

$$E = \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)\exp(-i\omega t)\,dt\right|^2 \qquad (4)$$

formula bilan topiladi.

Masalalar

20-1-2-3. 'Massasi" m, xususiy chastotasi ω bo'lgan sistemaning tashqi kuch $F(t)$ ta'siri ostida tebranishining chastotasi topilsin. Boshlang'ich momentda $x = \dot{x} = 0$ bo'lgan. Tashqi kuch quyidagicha berilgan:

1. $F(t) = A$, 2. $F(t) = A\cos t$, 3. $F(t) = A\sin\alpha t$.

Yechish. Tashqi kuch ta'siri ostida sistemada bo'ladigan tebranishi tenglamasi quyidagi birjinsli bo'lmagan tenglama edi:

$$\ddot{x}+\omega^2 x=\frac{F(t)}{m}$$

Berilgan uch hol uchun bu tenglamani yozamiz va ularni umumiy holda yechamiz:

1. $\ddot{x}+\omega^2 x=\frac{A}{m}$, 2. $\ddot{x}+\omega^2 x=\frac{A}{m}\cos\alpha t$, 3. $\ddot{x}+\omega^2 x=\frac{A}{m}\sin\alpha t$

Bu tenglamalarning barchasi aytganimizdek,bir jinsli bo'lmagan differensial tenglamalardir.Bu tenglamalar bir jinsli qismlarining xususiy integrallari

$1-2-3.$ $x_0=a\cos(\omega t+\varphi)$,

bu yerda φ – boshlang'ich faza Ikkinchi yechimlari esa

1. $x_1=\frac{A}{m\omega^2}$; 2. $x_1=\frac{A}{m(\omega^2-\alpha^2)}\cos\alpha t$; 3. $x_1=\frac{A}{m(\omega^2-\alpha^2)}\sin\alpha t$.

Shunday qilib,umumiy yechim

$$1.\quad x=\frac{A}{m\omega^2}+a\cos(\omega t+\varphi);\qquad 2.\quad x=\frac{A}{m(\omega^2-\alpha^2)}\cos\alpha t+a\cos(\omega t+\varphi);$$

$$3.\quad x=\frac{A}{m(\omega^2-\alpha^2)}\sin\alpha t+a\cos(\omega t+\varphi) \qquad (*)$$

Boshlang'ich shartlar asosida (*) dan

1. $a=-\frac{A}{m\omega^2}$; 2. $a=-\frac{A}{m(\omega^2-\alpha^2)}$; 3. $a=-\frac{A}{m(\omega^2-\alpha^2)}$

ekanligini topamiz.Bu yerda biz $t=0$ da $\varphi=0$ ekanligini hisobga oldik.(*) ning 1-holdagi yechim

$$x=\frac{A}{m\omega^2}(1-\cos\omega t)=\frac{2A}{m\omega^2}\sin^2\frac{\omega t}{2},$$

2-holdagi yechim

$$x=\frac{A}{m(\omega^2-\alpha^2)}(\cos\alpha t-\cos\omega t),$$

3-holdagi yechim esa

$$x=\frac{A\omega}{m(\omega^2-\alpha^2)}\left(\sin\alpha t-\frac{\alpha}{\omega}\sin\omega t\right)$$

20-4. 1.Sistemaning $F(t)=F_0 e^{-\alpha t}\cos\varphi t$ kuch ta'sirida majburiy tebranishini toping.Boshlang'ich vaqt momenti $t=0$ da sistema muvozanat holatda joylashgan ($x=0,\dot{x}=0$).

Yechish. $F(t)$ o'rniga qiymatini qo'yib yozamiz

$$\xi=\frac{F_0}{m}e^{i\omega_0 t}\int_0^t e^{-\alpha t}e^{-i\omega_0 t}\cos\varphi t\,dt=\frac{F_0}{m}e^{i\omega_0 t}\int_0^t e^{-\alpha t-i\omega_0 t}\frac{1}{2}\left(e^{i\varphi t}+e^{-i\varphi t}\right)dt=\frac{F_0}{2m}e^{i\omega_0 t}\left(\int\left(e^{[-\alpha-i(\omega_0-\varphi)]t}dt\right)+\right.$$

$$\left.+\int_0^t e^{[-\alpha-i(\omega_0+\varphi)]t}dt\right)=\frac{F_0}{2m}e^{i\omega_0 t}\left\{-\frac{e^{-[\alpha+i(\omega_0-\varphi)]t}-1}{\alpha+i(\omega_0-\varphi)}-\frac{e^{-[\alpha+i(\omega_0+\varphi)]t}-1}{\alpha+i(\omega_0+\varphi)}\right\}=$$

$$
=\frac{F_0}{2m}\left\{\frac{e^{i\omega_0 t}\left[\alpha+i(\omega_0+\varphi)+\alpha+i(\omega_0-\varphi)\right]}{\alpha^2+\varphi^2-\omega^2{}_0+2i\omega_0\alpha}\right\}-\frac{F_0}{2m}e^{-\alpha t}\left\{\frac{e^{i\varphi t}\left[\alpha+i(\omega_0+\varphi)\right]+e^{-i\varphi t}\left[\alpha+i(\omega_0-\varphi)\right]}{\alpha^2+\varphi^2-\omega^2{}_0+2i\omega_0\alpha}\right\}=
$$

$$
=\frac{F_0}{m}\left\{\frac{(\cos\omega_0 t+i\sin\omega_0 t)(\alpha+i\omega_0)}{\alpha^2+\varphi^2-\omega^2{}_0+2i\omega_0\alpha}-\frac{1}{2}e^{-\alpha t}\,\frac{\alpha(e^{i\varphi t}+e^{-i\varphi t})+i\omega_0(e^{i\varphi t}+e^{-i\varphi t})+i\varphi(e^{i\varphi t}-e^{-i\varphi t})}{\alpha^2+\varphi^2-\omega_0{}^2+2i\omega_0\alpha}\right\}=
$$

$$
\begin{aligned}
&=\frac{F_0}{m}\left\{\frac{(\cos\omega_0 t+i\sin\omega_0 t)(\alpha+i\omega_0)\left(\alpha^2+\varphi^2-\omega_0{}^2-2i\omega_0\alpha\right)}{(\alpha^2+\varphi^2-\omega_0{}^2)^2+4\alpha^2\omega_0{}^2}\right\}-\\
&-\frac{F_0}{m}e^{-\alpha t}\left\{\frac{(\alpha+i\omega_0)\cos\varphi t-\varphi\sin\varphi t}{(\alpha^2+\varphi^2-\omega_0{}^2)^2+4\alpha^2\omega_0{}^2}(\alpha^2+\varphi^2-\omega_0{}^2-2i\omega_0\alpha)\right\}
\end{aligned}
\qquad (5)
$$

Tebranish

$$
x=\frac{1}{i\omega_0}\,\mathrm{Im}\,\xi
$$

bilan aniqlangani uchun (5)dan topamiz:

$$
x=\frac{F_0}{m\left[(\alpha^2+\varphi^2-\omega_0{}^2)^2+4\alpha^2\omega_0{}^2\right]}\left[\begin{array}{l}-(\alpha^2+\omega_0{}^2-\varphi^2)\cos\omega_0 t+\dfrac{\alpha}{\omega_0}(\alpha^2+\omega_0{}^2+\varphi^2)\sin\omega_0 t-\\ -e^{-\alpha t}(\alpha^2+\varphi^2-\omega_0{}^2)\cos\varphi t+2\alpha\varphi\, e^{-\alpha t}\sin\varphi t\end{array}\right]
$$

20-5. Ossilyator $F(t)=F_0\exp(-t^2/\tau^2)$ kuch ta'sirida $t=-\infty$ dan $t=\infty$ gacha qancha maksimal energiya olgan?

Yechish. Avvalo mavzuning kirish qismida keltirilgan (4) formulaning o'ng tomonida joylashgan integralni hisoblaylik

$$
\int_{-\infty}^{\infty}F(t)\exp(-i\omega t)dt=F_0\int_{-\infty}^{\infty}\exp(-t^2/\tau^2-i\omega t)dt
$$

Bu integralda vaqtni siljitamiz:

$$
t\to t+\frac{i\omega\tau^2}{2}
$$

U holda integralimiz

$$
\int_{-\infty}^{\infty}F(t)\exp(-i\omega t)dt=F_0\int_{-\infty}^{\infty}\exp(-t^2/\tau^2-i\omega t)dt=F_0\exp(-\frac{\omega^2\tau^2}{4})\int_{-\infty}^{\infty}\exp(-t^2/\tau^2)dt=
$$

$$
=F_0\exp(-\frac{\omega^2\tau^2}{4})\tau\int_{-\infty}^{\infty}\exp(-y^2)\,dy=\sqrt{\pi}F_0\tau\exp(-\frac{\omega^2\tau^2}{2})
$$

Demak,tashqi kuch tomonidan ossilyatorga berilgan energiya

$$
E=\frac{1}{2m}\pi F^2{}_0\tau^2\exp(-\frac{\omega^2\tau^2}{2})
$$

Bu energiyaning maksimal qiymatini hisoblash uchun $\omega\tau=\beta$ deb belgilab olamiz.

$$E = \frac{1}{2m}\pi F^2{}_0 \frac{\beta^2}{\omega^2}\exp(-\frac{\beta^2}{2})$$

va uning maksimal qiymatini hisoblaymiz

$$\frac{dE}{d\beta} = C(2\beta - \beta^3)\exp(-\frac{\beta^2}{2}) = 0 .$$

Bundan $\beta = \sqrt{2}$ ekanligini topamiz.Natijada

$$E_{\max} = \frac{\pi F^2{}_0}{m\omega^2 e}$$

1-3-TOPSHIRIQLAR

Agar boshlang'ich $t = 0$ vaqt momentida $x = \dot{x} = 0 \rightarrow (\xi_0 = 0)$ va ta'sir etuvchi kuch quyidagicha ko'rinishga ega bo'lsa, erkinlik darajasi birga teng bo'lgan tizimning majburiy tebranish qonuni aniqlansin:

1. $F = const = F_0$
2. $F = at$
3. $F = F_0 e^{-\alpha t}$.

JAVOBLAR:

1. $x = \dfrac{F_0}{m\omega^2}(1 - \cos\omega t)$.
2. $x = \dfrac{a}{m\omega^2}(\omega t - \sin\omega t)$.
3. $x = \dfrac{F_0}{m[(\omega^2 + \alpha^2)]}(e^{-\alpha t} - \cos\omega t + \dfrac{\alpha}{\omega}\sin\omega t)$

Uslubiy tavsiyalar:

1.Tenglamaning xususiy yechimlari topiladi.
2.Boshlang'ich shartlar asosida amplituda va faza aniqlanadi.
3.Topilgan kattaliklar orqali umumiy yechim yozladi.

21 - MASHG'ULOT

Mavzu: «Erkinlik daraja soni birga teng bo'lgan sistemada majburiy tebranishlar. II yechimlarni ulash»

Darsning maqsadi. Majburiy tebranishlar uchun harakat qonunini topishni va bunda boshlang'ich shartlarni hisobga olishni, hamda $t<\tau$ va $t>\tau$ uchun Eyler-Lagranj tenglamasini umumiy yechimini olishni talabalarga o'rgatish.

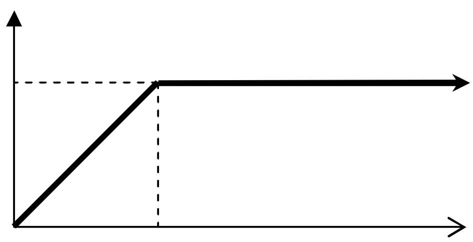
Qisqacha nazariya. Agar ichki tebranuvchi tizimga tashqi ta'sir bo'lib, vaqtning biror τ momentida birdaniga o'zgarsa, unda majburiy tebranish jarayoni uchun Eyler-Lagranj tenglamasini umumiy yechimini $t<\tau$ va $t>\tau$ uchun bir-biriga yuog'liq bo'lmagan holda topishga to'g'ri keladi. Jarayonning o'zi, tabiiyki, sakrab o'zgara olmaydi, shuning uchun $F_I(t)$ majbur etuvchi kuch uchun $t<\tau$ holda topilgan yechim. $F_2(t)$ majbur etuvchi kuch ta'sir etganda $t>\tau$ holdagi yechimga uzluksiz o'tishi kerak. Uzluksizlik talab, nafaqat x yechimga, shu bilan birga uning $\dot{x}$ birinchi hosilaga ham qo'yiladi.
Shunday qilib, $t=\tau$ nuqtada ulash sharti deb atalmish quyidagi shart bajarilishi lozim:

$$\begin{cases} x_I(\tau)=x_{II}(\tau), \\ \dot{x}_I(\tau)=\dot{x}_{II}(\tau). \end{cases} \qquad (1)$$

Bu yerda I va II sonlar - I $(t<\tau)$ va II $(t>\tau)$ uchastkalarga mos keladi. Ko'rish qiyin emaski,ulashsharti boshlang'ich qiymatlari $t<\tau$ holdagi yechim uchun va $t=\tau$ vaqt momentida $F_I(t)$ ta'sir natijasida tizimning olgan qiymati $\dot{x}_{II}(\tau)$ qiymati bo'lishi kerak.

Masalalar

21-1. Agar tizimning xususiy chastotasi ω, massasi m, $t=0$da $x=\dot{x}=0$, lekin $F(t)$ quyidagi ko'rinishda berilgan bo'lsa, $F(t)$ tashqi kuch ta'siri ostida $t>\tau$ holda erkinlik darajasi birga teng bo'lgan tizimning oxirgi tebranish amplitudasi topilsin.



Yechish. Ma'lumki, majburiy tebranishlar tenglamasini quyidagi ko'rinishga yozish mumkin:

$$\ddot{x}+\omega^2 x=\frac{F}{m} \qquad (2)$$

Masalaning berilishidagi rasmdan quyidagini topamiz:

$$\frac{F}{f}=\frac{t}{\tau}$$

bu yerdan

$$F=\frac{f}{\tau}t$$

ni olamiz.
Rasmdan ko'rinadiki,

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$F(t)=\begin{cases} I,\ F(t)=ct=\dfrac{f}{\tau}t, \\ II,\quad F(t)=f. \end{cases}$$

$F(t)=\dfrac{f}{\tau}t$ holni qarab chiqaylik. (3) ifodani (2) tenglamaga qo'yib, quyidagini olamiz:

$$\ddot{x}+\omega^2 x=\frac{f}{m\tau}t.$$

Bunday tenglamaning yechimi, bir jinsli $\ddot{x}+\omega^2 x=0$ bir jinsli,ikkinchi tartibli differensial tenglamaning yechimi $x_0=a\cos(\omega t+\varphi)$ va $x_I=At$ yechimlarning yig'indisidan iborat, ya'ni

$$x=x_0+x_I$$

Majbur etuvchi kuch ta'siridagi x_I yechimini topish uchun $x_I=At$ dan hosilalar olib, (4) tenglamaga qo'yib, quyidagini olamiz:

$$0+\omega^2 At=\frac{f}{m\tau}t, \tag{4}$$

Bu yerdan

$$A=\frac{f}{m\omega^2\tau}$$

Demak, $x_I=At=\dfrac{f}{m\omega^2\tau}t.$

Shunday qilib, yechimimiz quyidagi ko'rinishni oladi:

$$x=a\cos(\omega t+\varphi)+\frac{f}{m\omega^2\tau}t \tag{5}$$

Endi boshlang'ich shartlarni qo'llab, birinchi uchastka uchun x_I ni aniqlaymiz:

$$t=0,\quad x=\dot{x}=0,$$

Shunga ko'ra

$$\dot{x}=-\omega a\sin(\omega t+\varphi)+\frac{f}{m\omega^2\tau} \tag{6}$$

Demak,

$$\begin{cases} 0+a\cos\varphi+0, \\ 0=-\omega a\sin\varphi+\dfrac{f}{m\omega^2\tau}, \end{cases}$$

bu yerdan

$$\begin{cases} \cos\varphi=0,\quad \varphi=\dfrac{\pi}{2} \\ a=-\dfrac{f}{m\omega^2\tau} \end{cases} \tag{7}$$

(5) va (7) ifodalarni btrgalikda qarab, quyidagini hosil qilamiz:

$$x=-\frac{f}{m\omega^2\tau}\cos\left(\omega t+\frac{\pi}{2}\right)+\frac{f}{m\omega^2\tau}t$$

Yoki

$$x_I = \frac{f}{m\omega^2\tau}\left(t - \frac{1}{\omega}\sin\omega t\right). \tag{8}$$

Endi $F(t) = f$ holini qarab chiqaylik. Ushbu hol uchun differensial tenglama quyidagi ko'rinishga ega:

$$\ddot{x} + \omega^2 x = \frac{f}{m} \tag{9}$$

bu yerda

$$x_I = C \tag{10}$$

(9) va (10) tenglamalarni birgalikda yechib,quyidagini hosil qilamiz:

$$\omega^2 C = \frac{f}{m}$$

yoki

$$C = \frac{f}{m\omega^2}.$$

Demak, bu xoll usun umumiy yechim quyidagicha bo'ladi:

$$x_{II} = x_0 + x_I = a\cos(\omega t + \varphi) + \frac{f}{m\omega^2}. \tag{11}$$

Endi chegaraviy shartlardan foydalanib, oxirgi tebranish amplitudasini topamiz. Topdikki,

$$\begin{cases} x_I = \dfrac{f}{m\omega^2}\left(t - \dfrac{1}{\omega}\sin\omega t\right), \\ x_{II} = a\cos(\omega t + \varphi) + \dfrac{f}{m\omega^2}. \end{cases}$$

(1) ifodani amalga oshiramiz, ya'ni

$$\begin{cases} \dfrac{f}{m\omega^2\tau}\left(\tau - \dfrac{1}{\omega}\sin\omega t\right) = a\cos(\omega t + \varphi) + \dfrac{f}{m\omega^2}, \\ \dfrac{f}{m\omega^2}(1 - \cos\omega\tau) = -\omega a\sin(\omega\tau + \varphi), \end{cases}$$

yoki

$$\begin{cases} a\cos(\omega\tau + \varphi) = -\dfrac{f}{m\omega^3\tau}\sin\omega\tau, \\ a\sin(\omega\tau + \varphi) = -\dfrac{f}{m\omega^3\tau}(1 + \cos\omega t). \end{cases}$$

Bu tenglamalarni har ikala tomonlarini kvadratga ko'tarib, ularni mos tomonlarini qo'shib, quyidagini hosil qilamiz:

$$a^2 = \frac{2f^2}{m^2\omega^6\tau^2}(1 - \cos\omega\tau) = \frac{4f^2}{m^2\omega^6\tau^2}\sin^2\frac{\omega\tau}{2}$$

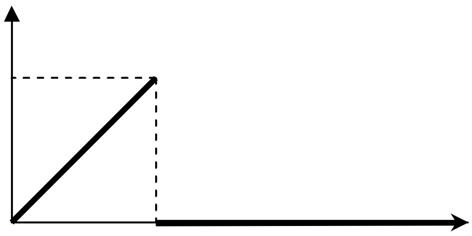
yoki

$$a = \frac{2f}{m\omega^3\tau}\sin\frac{\omega\tau}{2}.$$

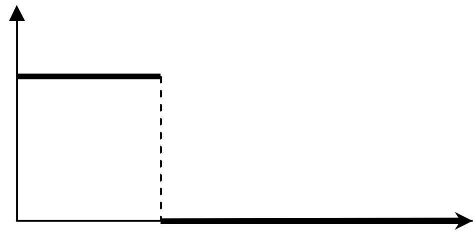
TOPSHIRIQLAR

1-2-TOPSHIRIQ. Agar tizimning xususiy chastotasi ω, massasi $m, t=0$,da $x=\dot{x}=0$ lekin quyidagi ko'rinishda berilgan bo'lsa, $F(t)$ tashqi kuch ta'siri ostida $t>\tau$ holda erkinlik darajasi birga teng bo'lgan tizimning oxirgi tebranish amplitudasi topilsin.

1.



2.



3-TOPSHIRIQ Agar ω_0 xususiy chastota va m massaga egabo'lsa, F o'zgarmas doimiy kuch ta'siri ostida erkinlik darajasi birga teng bo'lgan tizimning muvozanat holatidan siljishi va garmonik tebranish chastotasining o'zgarish topilsin.

JAVOBLAR:

1. $a=\dfrac{f}{m\omega^3\tau}\left[(\omega\tau)^2-2\omega\tau\,\sin\omega\tau+4\sin^2\dfrac{\omega\tau}{2}\right]^{1/2}.$

2. $a=2\dfrac{f}{m\omega^2}\sin\dfrac{\omega\tau}{2}.$

3. $\Delta\omega=0,\ \Delta x_0=\dfrac{F}{m\omega_0^2}.$

Uslubiy tavsiyalar:

21-1-masalaga uxshash, tenglamaning xususiy yechimlari topiladi, boshlang'ich shartlar asosida amplituda va faza aniqlanadi.Topilgan kattaliklar orqali umumiy yechim yozladi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

22 - MASHG'ULOT
Mavzu: «So'nuvchi tebranishlar»

Darsning maqsadi. Tashqi kuchlar bo'lgandagi tebranishlar uchun xarakat qonun xarakterlarini o'rgatish.

Qisqacha nazariya. Dissipativ kuchlar ta'sirida bo'lgan tizimning harakat tenglamasi garmonik yaqinlashishda quyidagi ko'rinishga ega:

$$\sum_i \left[m_{ij}\ddot{x}_j + \alpha_{ij}\dot{x}_j + k_{ij}x_j\right] = 0,\ \ l = \overline{1,S} \qquad (1)$$

Bunday bir jinsli chiziqli differensial tenglamalar tiziming xususiy yechimi quyidagi ko'rinishda axtariladi:

$$x_j = A_j e^{rt} \qquad (2)$$

Amplituda tenglamasi uchun

$$\sum_i \left[r^2 m_{ij} + r\alpha_{ij} + k_{ij}\right]A_j = 0 \qquad (3)$$

bir jinsli algebraik tenglamalar tizimini hosil qiladi. Serkulyar tenglama, bu A_j ni nolinchi yechimining mavjud bo'lishlik shartini ifodalaydi:

$$\det\left|r^2 m_{ij} + r\alpha_{ij} + k_{ij}\right| = 0 \qquad (4)$$

Bu tenglamaning yechimi $r_\alpha\left(\alpha = \overline{1,2S}\right)$ yoki haqiqiy bo'lishi mumkin yoki kompleks-qo'shma juftlikni hosil qiladi. Har qanday $r_{1,2} = -\lambda \pm i\omega$ juftga quyidagi ko'rinishdagi yechim to'g'ri keladi

$$\theta_\alpha = a_\alpha e^{-\lambda t}\cos(\omega t + \varphi), \qquad (5)$$

ya'ni qandaydir so'nuvchi tebranish. Har qaysi …haqiqiy ildizga,

$$\theta_\alpha = a_\alpha e^{-\lambda t} \qquad (6)$$

ko'rinishdagi aperiodik (davriy bo'lmagan) so'nish mos keladi. Shunday qilib, serkulyar tenglama ildizini tahlil qilib, tizimda – so'nuvchi tebranishlar, aperiodik so'nuvchi, u yoki bularning kombinasiyasi va erkin tebranishlar sodir bo'lishini aniqlash mumkin.

Eslatamizki, serkulyar tenglama yechishda …yechimning paydo bo'lishi, tizimning bir butun ilgarlanma harakatidan guvoh beradi. Dissipativ kuchlar ta'sirida bo'lgan tizim uchun Eyler-Lagranj tenglamasi quyidagicha yoziladi:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k^d = -\frac{d\Phi}{d\dot{q}_k} = -\frac{\partial \Phi}{\partial \dot{x}_k}, \qquad (7)$$

bu yerda L-Lagranjfunksiyasi, Φ-dissipativ fuknsiya, ya'ni

$$\Phi = \frac{1}{2}\sum_{ij}\alpha_{ij}\dot{x}_{ij}\dot{x}_j, \quad 2\varphi = -\frac{dE}{dt}. \qquad (8)$$

Masalalar

22-1. Agar Lagranj va dissipativ funksiyalar quyidagicha berilgan bo'lsa, tizimning harakat xarakteri o'rganilsin:

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

$$\begin{cases} L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}\left(4x_1^2 + 4x_2^2 - 2x_1x_2\right), \\ \Phi = \dot{x}_1^2 + \dot{x}_2^2. \end{cases}$$

Yechish. Eyler-lagranj tenglamasiga asosan, berilgan masala shartidan quyidagini hosil qilamiz,ya'ni (7) ifodaga asosan

$$\frac{d}{dt}\left(\dot{x}_1 + \dot{x}_2\right) - 4x_1 - 4x_2 + x_1 + x_2 = 2\dot{x}_1 + 2\dot{x}_2$$

yoki

$$\ddot{x}_1 + \ddot{x}_2 - 2\dot{x}_1 - 2\dot{x}_2 - 3x_1 - 3x_2 = 0$$

Ushbu tenglamani yechimini quyidagi ko'rinishda axtaramiz:

$$x_1 = A_1 e^{rt}, \quad x_2 = A_2 e^{rt}.$$

Bu yechimdan hosilalar olib, tenglamaga quyib, quyidagi tenglamani hosil qilamiz:

$$A_1 r^2 e^{rt} + A_2 r^2 e^{rt} - 2A_1 r e^{rt} - 2A_2 r e^{rt} - 3A_1 e^{rt} - 3A_2 e^{rt} = 0$$

yoki

$$A_1 r^2 + A_2 r^2 - 2A_1 r - 2A_2 r - 3A_1 - 3A_2 = 0$$

Bundan

$$\left(r^2 - 2r - 3\right)A_1 + \left(r^2 - 2r - 3\right)A_2 = 0$$

yoki

$$\left(r^2 - 2r - 3\right)\left(A_1 + A_2\right) = 0$$

bu yerda $A_1 + A_2 \neq 0$ deb, quyidagi tenglamani yechamiz:

$$r^2 - 2r - 3 = 0,$$

$$r_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2; \, r_1 = 3, \, r_2 = -1.$$

(5) ifodaga asosan quyidagi yechimni olamiz $\left(\lambda = 1, i\omega = 2\right)$:

$$\begin{cases} \theta_1 = a_1 e^t \cos\left(\sqrt{2} + \varphi_1\right), \\ \theta_2 = a_2 e^t \cos\left(2t + \varphi_2\right). \end{cases}$$

Demak,yechim ikkita so'nuvchi tebranishlardan iborat bo'lar ekan.

TOPSHIRIQLAR

1-2-3-TOPSHIRIQ. Agar lagranj dissipativ funksiyalar quyidagicha ko'rinishda berilgan bo'lsa, tizimning harakat xarakteri o'rganilsin:

1. $\begin{cases} L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \left(x_1^2 + x_2^2\right), \\ \Phi = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\right). \end{cases}$

2. $\begin{cases} L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{3}{2}\left(\dot{x}_1^2 + x_2^2\right) + 2x_1x_2, \\ \Phi = 2\left(\dot{x}_1^2 + \dot{x}_2^2\right). \end{cases}$

3. $\begin{cases} L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right), \\ \Phi = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1\dot{x}_2\right). \end{cases}$

4-5-**TOPSHIRIQ**. Agar osilgan nuqtada ishqalanish tufayli energiyaning yutilishi $\left(-\frac{dE}{dt}\right)$ ga teng bo'lsa, bir jinsli og'irlik maydonida harakat xarakteri o'rganilsin.

4. Ikkilangan burchak tezlik kvadratiga.
5. To'rtlangan chiziqli tezlik kvadratiga.

JAVOBLAR:

1. $\begin{cases} \theta_1 = a_1 \cos\left(\sqrt{2}t + \varphi_1\right), \\ \theta_2 = a_2 e^{-t} \cos\left(t + \varphi_2\right). \end{cases}$ θ_1 erkin va θ_2 so'nuvchi tebranish.

2. $\begin{cases} \theta_1 = a_1 e^{\left(-2+\sqrt{3}\right)t}, \\ \theta_2 = a_2 e^{\left(-2-\sqrt{3}\right)t}, \\ \theta_3 = a_3 e^{-2t} \cos\left(t + \varphi_1\right). \end{cases}$

Ikkita aperiodik so'nuvchi va bitta so'nuvchi tebranish.

3. $\begin{cases} \theta_1 = a_1 e^{-\frac{1}{2}t} \\ \theta_2 = a_2 e^{-\frac{1}{2}t} \\ \theta_3 = a_3. \end{cases}$

Ikkita aperiodik so'nuvchi va ilgarilanma ko'chish.

4. $x = ae^{-\frac{1}{4}t} \cos\left(\frac{\sqrt{7}}{4}t + \varphi\right)$. So'nuvchi tebranish.

5. $x = e^{-2t}\left(a_1 e^{\frac{7}{\sqrt{2}}t} + a_2 e^{-\frac{7}{\sqrt{2}}}\right)$.Aperiodik so'nuvchi.

Uslubiy tavsiyalar:

Eyler-Lagrang tenglamalari tuziladi va tuzilgan tenglamalar sistemasining echimi izlanadi. Topilgan echimlarda xarakat xarakteri aniqlanadi.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

23 - MASHG'ULOT

Mavzu: «Erkinlik darajasi ko'p bo'lgan tizimning erkin tebranishi. Normal koordinatalar»

Darsning maqsadi. Erkinlik darajasi ikki va undan ortiq bo'lgan sistemalarning tebranishlar uchun xarakat xarakterlarini o'rgatish.

Qisqacha nazariya. ***Normal koordinatalar.*** Ushbu mavzuning asosiy maqsadi, erkinlik darajasi ko'p bo'lgan garmonik tebranish masalalarida Eyler-Lagranj tenglamasining yechish va normal koordinatalarga o'tish yo'llarini o'rganishdan iyuoratdir.

Garmonik yaqinlashishda erkinlik darajasini S ga teng bo'lgan tizimning Lagranj funksiyasini qarab chiqamiz, ya'ni

$$L=\frac{1}{2}\sum_{i,j} m_{ij}\dot{x}_i\dot{x}_j-\frac{1}{2}\sum_{i,j} k_{ij}x_ix_j. \tag{1}$$

Masalan, agar $S=2$ bo'lsa, Lagranj funksiyasining ko'rinishi qo'yidagicha bo'ladi:

$$L=\frac{1}{2}\left[m_{11}\dot{x}_1^2+2m_{12}\dot{x}_1\dot{x}_2+m_{22}\dot{x}_2^2\right]-\frac{1}{2}\left[k_{11}x_1^2+2k_{12}x_1x_2+k_{22}x_2^2\right], \tag{2}$$

bu yerda m_{ij} - «massa» tenzori,

$$k_{ij}=\left(\frac{d^2U}{dq_idq_j}\right)_{x_i=x_j=0}.$$

Bunday Lagranj funksiyasi uchun Eyler-Lagranj tenglamasi chiziqli tebranishlar tenglamalmri hisoblanadi. Bu tenglama quyidagi ko'rinishga ega:

$$\sum_i\left[m_{ij}\ddot{x}_j+k_{ij}x_j\right]=0, \tag{3}$$

ya'ni ikkinchi tartibli, birjinsli chiziqli differensial tenglamalar tizimini hosil qiladi. Bunday tenglamalarning yechimi quyidagi kurinishda axtariladi:

$$x_j=A_je^{i\omega t}.$$

Masalan, $S=2$ bo'lganda, ushbu yechimning ko'rinishi quyidagicha bo'ladi:

$$x_i=Ae^{i\omega t},x_2=Be^{i\omega t}. \tag{4}$$

(3) va (4) tenglamalarni birgalikda yechib, A_j amplitudaga nisbatan bir jinsli, chiziqli algabraik tenglamalar tizimiga ega bo'lamiz, ya'ni

$$\sum_j A_j\left[-\omega^2m_{ij}+k_{ij}\right]=0 \tag{5}$$

Ushbu tenglamalar yechimga ega bo'lishi uchun, tizimning determinanti nolga teng bo'lishi kerak, ya'ni

$$\det\left|-\omega^2m_{ij}+k_{ij}\right|=0 \tag{6}$$

Bu tenglama serkulyar tenglama deyiladi va $\omega_\alpha\left(\alpha=\overline{1,S}\right)$ normal garmonik tebranishlarning xususiy chastotalarini aniqlash uchun xizmat qiladi. Amplitudauchun ω_α^2 ildizning aniq qiymatlarini algebraik tenglama qo'yib,bir xil chastotaga ega bo'lgan ular orasidagi bog'liqlikni topamiz:

$$\frac{A_{j\alpha}}{A_{i\alpha}} \equiv D_{j\alpha}, \left(\alpha = \overline{1, S}\right). \quad (7)$$

Unda umumiy yechim quyidagi ko'rinishni oladi:

$$x_i = \sum_{\alpha} D_{j\alpha} A_\alpha \theta_\alpha, \left(\alpha = \overline{1, S}\right), \quad (8)$$

bu yerda $\theta_\alpha = C_\alpha a^{i\omega_\alpha t} \left(\theta_\alpha = a_\alpha \cos(\omega_\alpha t + \varphi_\alpha)\right)$-normal tebranish. Masalan, $S = 2$ bo'lsa, quyidagi yechimlarga ega bo'lamiz:

$$\begin{cases} x_1 = A_1\theta_1 + A_2\theta_2, \\ x_2 = \dfrac{B_1}{A_1} A_1\theta_1 + \dfrac{B_2}{A_2} A_2\theta_2. \end{cases} \quad (9)$$

$A_1, A_2,...$kattaliklar aniqmas qoladi. Ularni Lagranj funksiyasining normallik shartidan topish mumkin. Normallashgan lagranj funksiyasi quyidagicha yoziladi:

$$L = \frac{1}{2}\sum_{\alpha} \dot{\theta}_\alpha^2 - \sum_{\alpha} \frac{\omega_\alpha^2}{2} \dot{\theta}_\alpha^2. \quad (10)$$

$$T = \frac{1}{2}\sum_{ij} m_{ij} \dot{x}_i \dot{x}_j \quad (11)$$

Kinetik energiya ifodasiga $\dot{x}_i$ ning qiymatini qo'yib, differensiallab hosil qilingan yechim x_j, normal koordinatalar orqali ifodalangan bo'ladi, $S = 2$ bo'lsa, quyidagini hosil qilamiz:

$$\begin{cases} \dot{x}_1 = A_1\dot{\theta}_1 + A_2\dot{\theta}_2, \\ \dot{x}_2 = B_1\dot{\theta}_1 + B_2\dot{\theta}_2 = \dfrac{B_1}{A_1} A_1\dot{\theta}_1 + \dfrac{B_2}{A_2} A_2\dot{\theta}_2, \end{cases} \quad (12)$$

bu yerda $\dfrac{B_\alpha}{A_\alpha} = D_\alpha$ -oldin olingan amplitudalar nisbati.

$\dot{\theta}_\alpha$ orqali topilgan kinetik energiya ifodasi $\dfrac{1}{2}\sum_{\alpha} \dot{\theta}_\alpha^2$ ga tenglashtiriladi. Shunday qilib, normallashtirish sharti aralash hadlardagi koeffisiyentlarning nolga tenglashtirishni ($\alpha \neq \beta$ da $\dot{\theta}_\alpha \dot{\theta}_\beta$ ni ichiga oluvchi) va $\dot{\theta}_\alpha^2$ oldidagi koeffisiyentning $\dfrac{1}{2}$ ga tengligini talab etishga olib keladi.

A_α kattalik, ± 1 Ahamiyatga egabo'lmagan aniqlikkacha normallashtirish shartidan aniqlanadi.

Lagranj fuksiyasini garmonizasiyalashtirish.

Odatda tizimning Lagranj funksiyasi harakat uchun yoziladi. Garmonik tebranishlar hosil bo'lishi uchun tizim muvozanat holatdan cheksiz kichik. Shu sababli, ushbu muvozanat holatidan cheksiz kichik ko'chishlar uchun Lagranj funksiyasi yozilsa, bunga Lagranj funksiyasini *garmonizasiyalashtirish* deyiladi.

Mexanik tizimning erkinlik darajasi S bo'lib, unga golonom tutib turuvchi stasionar bog'lanishlar qo'yilgan bo'lsa, u holda uning Lagranj funksiyasi quyidagi ko'rinishga ega bo'ladi:

$$L=\frac{1}{2}\sum_{i,k=1}^{S}a_{ik}(q_1,q_2,...,q_S)\dot{q}_i\dot{q}_k-U(q_1,q_2,...,q_s).$$

Garmonizasiyalashtirilgan Lagranj funksiyasini hosil qilish uchun birinchi navbatda tizimning muvozanat holati mavjudligini aniqlash lozim. Bu esa, potensial energiyaning minimal qiymatini topish bilan amalga oshiriladi, ya'ni

$$\frac{\partial U}{\partial q_i}=0,\ \frac{\partial U}{\partial q_2}=0,...,\frac{\partial U}{\partial q_S}=0;$$

$$\left.\frac{\partial^2 U}{\partial q_1^2}\right|_{q_0}=k_{11}>0.$$

Bularni yechib, muvozanat holat koordinatalari topildi.

$$\frac{\partial^2 U}{\partial q_1\partial q_2}=k_{12}=k_{21}>0,$$

$$\left.\frac{\partial^2 U}{\partial q_2^2}\right|_{q_0}=k_{22}>0$$

bo'lsa, u holda tizimda turg'un muvozanat holat mavjud deymiz. Aks holda, muvozanat holatimiz noturg'un bo'ladi. Bunday holat atrofida tebranishlar hosil bo'lmaydi.

Kinetik va potensial energiyalar garmonizasiyalashtirilib, keyin garmonizasiyalashgan Lagranj funksiyasi yoziladi. Bu esa quyidagicha bajariladi:

$$U(q_1,q_2,...q_s)=U(q_{10},q_{20},...q_{s0})+\sum_{i=1}^{s}\left.\frac{\partial U}{\partial q_1}\right|_{q_0}(q_i-q_{i0})+$$
$$+\frac{1}{2}\sum_{i,k}\frac{\partial^2 U}{\partial q_i q_k}(q_i-q_{i0})(q_k-q_{k0})+...\approx\frac{1}{2}\sum_{i,k}k_{ik}x_ix_k. \tag{13}$$

Xuddi shunday kinetik energiya ham

$$T=\frac{1}{2}\sum_{i,k=1}^{S}a_{ik}(q_1,q_2,...,q_S)\dot{q}_i\dot{q}_k\approx\frac{1}{2}\sum_{i,k}m_{ik}\dot{x}_i\dot{x}_k,\quad q_i=q_{i0}+x_0 \tag{14}$$

Demak, garmonizasiyalashtirilgan Lagranj funksiyasi quyidagi ko'rinishga ega bo'ladi:

$$L=\frac{1}{2}\sum_{i,k}m_{ik}\dot{x}_i\dot{x}_k-\frac{1}{2}\sum_{i,k}k_{ik}x_ix_k. \tag{15}$$

Masalalar

23-1. Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, erkinlik darajasi ikkiga teng bo'lgan tizimning chastotalari va normal tebranish koordinatalari topilsin:

$$L=\frac{1}{2}(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2)-\frac{1}{2}(x^2+y^2).$$

Yechish. Ushbu masalani yechish uchun Eyler-Lagranj tenglamalari tuziladi, ya'ni

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} + \frac{1}{2}\dot{y}, \quad \frac{\partial L}{\partial \dot{y}} = \dot{y} + \frac{1}{2}\dot{x};$$

$$\frac{d}{dt}\left(\dot{x} + \frac{1}{2}\dot{y}\right) = \ddot{x} + \frac{1}{2}\ddot{y}, \quad \frac{d}{dt}\left(\dot{y} + \frac{1}{2}\dot{x}\right) = \ddot{y} + \frac{1}{2}\ddot{x};$$

$$\frac{\partial L}{\partial x} = -x, \quad \frac{\partial L}{\partial y} = -y.$$

Ushbu ifodalardan Eyler-Lagranj tenglamasi quyidagicha tuziladi:

$$\begin{cases} \ddot{x} + x + \frac{1}{2}\ddot{y} = 0 \\ \frac{1}{2}\ddot{x} + \ddot{y} + y = 0 \end{cases} \qquad (16)$$

Bu tenglamalarning yechimlarini quyidagicha axtaramiz:

$$x = Ae^{i\omega t}, \quad y = Be^{i\omega t} \qquad (14)$$

(1) va (2) tenglamalarni birgalikda yechib, quyidagiga ega bo'lamiz:

$$\begin{cases} -\omega^2 A + A - \frac{\omega^2}{2}B = 0, \\ -\frac{\omega^2}{2}A + \omega^2 B + B = 0 \end{cases}$$

yoki

$$\begin{cases} (1-\omega^2)A - \frac{\omega^2}{2}B = 0, \\ -\frac{\omega^2}{2}A + (1-\omega^2)B = 0. \end{cases} \qquad (17)$$

Endi ushbu tenglamalar noldan farqli yechimga ega bo'lishi uchun determinant tuzami:

$$\Delta = \begin{vmatrix} 1-\omega^2 & -\frac{\omega^2}{2} \\ -\frac{\omega^2}{2} & 1-\omega^2 \end{vmatrix} = 0$$

Bu kasriy tenglamani yechib,chastotalarni aniqlaymiz:

$$(1-\omega^2)^2 - \frac{(\omega^2)^2}{4} = 0,$$

$$1 - 2\omega^2 + (\omega^2)^2 - \frac{(\omega^2)^2}{4} = 0,$$

$$\frac{3}{4}(\omega^2)^2 - 2\omega^2 + 1 = 0$$

yoki

$$3(\omega^2)^2 - 8\omega^2 + 4 = 0$$

$$D = 64 - 48 = 16$$

$$\omega_{1,2}^2 = \frac{8 \pm 4}{6}; \quad \omega_1^2 = 2, \quad \omega_1 = \sqrt{2}; \quad \omega_2 = \sqrt{\frac{2}{3}}.$$

A va B koeffisiyentlarni aniqlash uchun ω_1 va ω_2 larning qiymatlarini (17) tenglamaga qo'yib quyidagiga ega bo'lamiz:

$$\begin{cases} \left(1 - \omega_1^2\right)A - \dfrac{\omega_1^2}{2}B = 0, \\ -\dfrac{\omega_1^2}{2}A + \left(1 - \omega_1^2\right)B = 0. \end{cases}$$

Bu yerdan,

$$\begin{cases} -A_1 - B_1 = 0 \\ -A_1 - B_1 = 0 \end{cases} \Rightarrow D_1 = \frac{B_1}{A_1} = -1.$$

Xuddi shunday ω_2 ni qiymatlarini qo'yib quyidagini topamiz:

$$\begin{cases} \dfrac{1}{3}A_2 - \dfrac{1}{3}B_2 = 0 \\ -\dfrac{1}{3}A_2 + \dfrac{1}{3}B_2 = 0 \end{cases} \Rightarrow D_2 = \frac{B_2}{A_2} = 1.$$

Demak, harakat tenglamalarining yechimlari quyidagicha:

$$x = \mathrm{Re}\left(C_1 A_1 e^{i\omega_1 t} + C_2 A_2 e^{i\omega_2 t}\right) = A_1 A_2 \,\mathrm{Re}\left(C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}\right),$$

$$\mathrm{Re}\left(C_1 e^{i\omega_1 t}\right) = a_1 \cos(\omega_1 t + \varphi_1) \equiv \theta_1,$$

$$a_2 \cos(\omega_2 t + \varphi_2) \equiv \theta_2.$$

Shunday qilib, yechim quyidagicha bo'ladi:

$$x = A_1\theta_1 + A_2\theta_2.$$

y komponentasi uchun esa yuqoridagiga o'xshash hisoblanadi, ya'ni

$$y = \mathrm{Re}\left(C_3 B_1 e^{i\omega_1 t} + C_4 B_2 e^{i\omega_2 t}\right) = \mathrm{Re}\left(C_1 D_1 A_1 e^{i\omega_1 t} + C_2 D_2 A_2 e^{i\omega_2 t}\right),$$

yoki

$$y = -A_1\theta_1 + A_2\theta_2.$$

Normal koordinatalarni aniqlash uchun, quyidagilarni hisoblaymiz:

$$T = \frac{1}{2}\sum_{i,k} a_{ik}\dot{q}_1\dot{q}_k \equiv \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right),$$

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2\right) = \frac{1}{2}[\left(A_1\dot{\theta}_1 + A_2\dot{\theta}_2\right)^2 + \left(A_1\dot{\theta}_1 + A_2\dot{\theta}_2\right)\times$$

$$\times\left(-A_1\dot{\theta}_1 + A_2\dot{\theta}_2\right)^2 + \left(-A_1\dot{\theta}_1 + A_2\dot{\theta}_2\right)] = \frac{1}{2}\left(A_1^2\dot{\theta}_1^2 + 3A_1^2\dot{\theta}_2^2\right) =$$

$$= \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right).$$

Bu tenglikdan A_1 va A_2 lar qiymatlarini aniqlaymiz, ya'ni

$$A_1^2 = 1,\; A_1 = 1;$$

$$3A_2^2 = 1,\; A_2 = \frac{1}{\sqrt{3}}$$

A_1 va A_2 koeffisiyentlar qiymatlarini (16) va (17) ifodalarga qo'yib, quyidagi oxirgi yechimga ega bo'lamiz:

$$\begin{cases} x = \theta_1 + \dfrac{1}{\sqrt{3}}\theta_2\,, \\ y = -\theta_1 + \dfrac{1}{\sqrt{3}}\theta_2. \end{cases}$$

Bu yerdan θ_1 va θ_2 lar quyidagiga teng bo'ladi:

$$\begin{cases} \theta_1 = a_1\cos\left(\sqrt{2}t + \varphi_1\right), \\ \theta_2 = a_2\cos\left(\sqrt{\dfrac{2}{3}}t + \varphi_2\right) \end{cases}.$$

23-2. Mexanik tizimning quyidagi Lagranj funksiyasi garmonizasiyalashtirilsin:

$$L = \frac{1}{2}\left(q_1\dot{q}_1^2 + q_2\dot{q}_2^2\right) - \left(\frac{1}{q_1 q_2} + q_1 + q_2\right).$$

Yechish. Ushbu Lagranj funksiyasida potensial energiya qismi quyidagiga teng:

$$U = \frac{1}{q_1 q_2} + q_1 + q_2\,.$$

Bundan q_1 va q_2 lar bo'yicha hosilalar olib, q_{10} va q_{20} koordinatalarni aniqlaymiz, ya'ni

$$\left.\begin{aligned} \frac{\partial U}{\partial q_1} &= -\frac{1}{q_1^2 q_2} + 1 = 0 \\ \frac{\partial U}{\partial q_2} &= -\frac{1}{q_1 q_2^2} + 1 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -1 + q_1^2 q_2 = 0 \\ -1 + q_1 q_2^2 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} q_2 = \frac{1}{q_1^2} \\ -1 + q_1 q_2^2 = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow q_1 \cdot \frac{1}{q_1^4 = 1} \Rightarrow q_1^3 = 1;\; q_1^0 = 1;\; q_{20} = 1.$$

Endi k_{11}, k_{22} va $k_{11} = k_{22}$ larni aniqlaymiz. Buning uchun potensial energiyadan q_1 va q_2 koordinatalar bo'yicha ikkinchi tartibli hosilalar olamiz, ya'ni

$$\frac{\partial^2 U}{\partial q_1^2} = \frac{2}{q_1^3 q_2}, \quad \left(\frac{\partial^2 U}{\partial q_1^2}\right)_0 = 2 = k_{11} \succ 0;$$

$$\frac{\partial^2 U}{\partial q_2^2} = \frac{2}{q_1 q_2^3}, \quad \left(\frac{\partial^2 U}{\partial q_2^2}\right)_0 = 2 = k_{22} \succ 0;$$

$$\frac{\partial^2 U}{\partial q_1 \partial q_2} = \frac{2}{q_1^2 q_2^2}, \quad \left(\frac{\partial^2 U}{\partial q_1 \partial q_2}\right)_0 = 2 = k_{12} = k_{21} \succ 0.$$

$x_1 = q_1 - q_{10}$ va $x_2 = q_2 - q_{20}$ larni va (1) nazarga olib,quyidagiga ega bo'lamiz:

$$U = \frac{1}{2}\left(2x_1^2 + 2x_2^2 + 2x_1x_2\right) = x_1^2 + x_2^2 + x_1x_2.$$

Kinetik energiya qismini garmonizasiyalash uchun m_{11}, m_{22} quyidagilarga teng:

$$m_{11} = q_{10} = 1, \quad m_{22} = q_{20} = 1, \quad m_{12} = m_{21} = 0.$$

Demak, garmonizasiyalashgan kinetik energiya (2) ga asosan quyidagiga teng bo'ladi:

$$T = \frac{1}{2}\left(\dot{x}_1^2 + x_2^2\right).$$

Shunday qilib, garmonizasiyalashtirilgan Lagranj funksiyasi quyidagi ko'rinishga ega bo'lar ekan:

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \left(x_1^2 + x_2^2 + x_1x_2\right).$$

TOPSHIRIQLAR

1-2-3-**TOPSHIRIQ.**Agar Lagranj funksiyasi quyidagi ko'rinishga ega bo'lsa, erkinlik darajasi ikkiga teng bo'lgan tizimning chastotalari va normal tebranish koordinatalari topilsin:

1. $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{\omega_0^2}{2} + \alpha xy.$

2. $L = \frac{1}{2}\left(\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2\right) - \frac{1}{2}\left(\dot{x}^2 - \dot{x}\dot{y} + \dot{y}^2\right).$

3. $L = \frac{1}{2}\left(2\dot{x}^2 + 2\dot{x}\dot{y} + 2\dot{y}^2\right) - \frac{1}{2}\left(3\dot{x}^2 + 2\dot{y}^2\right).$

4-5-6-**TOPSHIRIQ.**.Mexanik tizimning quyidagi Lagranj funksiyasi garmonizasiyalashtirilsin:

4. $L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + \dot{q}_2^2\right) - \left(\ln q_1q_2\,\frac{1}{q_1} + \frac{1}{q_2}\right).$

5. $L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + \dot{q}_2^2\right) - \left(q_1^3 - q_2^3 + 3q_1q_2\right).$

6. $L = \frac{m}{2}\left(2\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right)l^2 + ml^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2) + mgl\left(2\cos\varphi_1 + \cos\varphi_2\right).$

Tizim ikkilangan yassi mayatnik ko'rinishida tasvirlangan,.

JAVOBLAR:

1. $\begin{cases} x = \dfrac{1}{\sqrt{2}}(\theta_1 + \theta_2), \\ y = \dfrac{1}{\sqrt{2}}(-\theta_1 + \theta_2) \end{cases} \qquad \begin{cases} \theta_1 = a_1\cos\left(\sqrt{\omega_0^2 + \alpha}\,t + \varphi_1\right), \\ \theta_2 = a_2\cos\left(\sqrt{\omega_0^2 - \alpha}\,t + \varphi_2\right) \end{cases}$

2. $\begin{cases} x = \theta_1 + \dfrac{1}{\sqrt{3}}\theta_2\,, \\ y = -\theta_1 + \dfrac{1}{\sqrt{3}}\theta_2. \end{cases}$ $\begin{cases} \theta_1 = a_1\cos\left(\sqrt{3}t + \varphi_1\right), \\ \theta_2 = a_2\cos\left(\dfrac{1}{\sqrt{3}}t + \varphi_2\right) \end{cases}$

3. $\begin{cases} x = \dfrac{1}{\sqrt{5}}(2\theta_1 + \theta_2), \\ y = \dfrac{1}{\sqrt{5}}(-3\theta_1 + \theta_2) \end{cases}$ $\begin{cases} \theta_1 = a_1\cos\left(\sqrt{6}t + \varphi_1\right), \\ \theta_2 = a_2\cos(t + \varphi_2) \end{cases}$

4. $L = \dfrac{1}{2}\left(\dot{x}_1^2 + \dot{x}_1\dot{x}_2 + \dot{x}_2^2\right) - \dfrac{1}{2}(x_1^2 + x_2^2).$

5. $L = \dfrac{1}{2}\left(\dot{x}_1 + \dot{x}_1\dot{x}_2 + \dot{x}_2^2\right) - 3\left(x_1^2 + x_1x_2 + x_2^2\right).$

6. $L = \dfrac{1}{2}ml^2\left[2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2\right] - \dfrac{1}{2}mgl\left[2\varphi_1^2 + \varphi_2^2\right].$

Uslubiy ko'rsatma:

1. Eyler-Lagrang tenglamalari tuziladi va defferinsial tenglamalar sistemasi echiladi. Normal koordinatalar topiladi va o'zgarmas koeffisentlarning qiymatlari topiladi.
2. Potensial va kenetik energiyalar ajratib olinadi va ularning birinchi, ikkinchi va aralash hosilalari topilib koeffesintga tenglashtiriladi.

24- MASHG'LOT
Mavzu: «Inersiya tenzori»

Qisqacha nazariya. Inersiyamomentlar tenzori komponentalarining ta'rifiga ko'ra, uni quyidagi ko'rinishda yozish mumkin:

$$I_{i,k} = \sum_i m_i \left[x_i^2 \delta_{ik} - x_i x_k\right],$$

bu yerda $i = \overline{1, N}$ -tizimning tashkil etuvchi zarralar raqami,
m_i -ularning massalari,
i -1, 2, 3 koordinata o'qlarining raqamlari,
x_{ik} - i o'qdagi zarralarning $\vec{x}_i$ radiks–vektorlar proyeksiyalari,

$$x_i^2 = x_{i1}^2 + x_{i2}^2 + x_{i3}^2,$$

$$\delta_{\alpha\beta} = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$$ Kroneker belgisi.

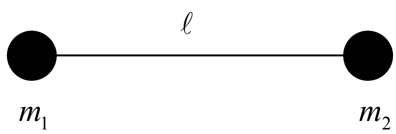
Agar qattiq jism yaxlit deb qarash mumkin bo'lsa, u holda yig'indi o'rnida jism hajmi bo'yicha integral olish mumkin:

$$I_{ik} = \int \rho\left(x_l \delta_{ik} - x_i x_k\right) dV .$$

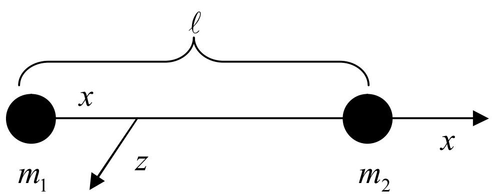
Agar koordinata tizimi sifatida markaziy tizimolingan bo'lsa, unda inersiya tenzorining komponentalari markaziy inersiya momentlari deyiladi; adabiyotlarda asosan anna shu momentlar keltiriladi. Koordinata o'qlarida diagonal ko'rinishda keltirilgan inersiya tenzori, inersiyaning asosiy o'qlari deyiladi, asosiy o'qlarga nisbatan o'qlari inersiya momentlari esa, inersiyaning asosiy momentlari deyiladi. Ba'zan inersiyaning asosiy o'qini tizimning simmetriyasi to'g'risidagi fikrlashdan birdaniga bilish mumkin.

Masalalar

24-1. Quyidagi mexanik tizimlarning asosiy inersiya mometrlari hisoblansin: Chiziqli ikki atomli molekulani.



Yechish. Molekulalar orasidagi masofa $\ell = const$ birinchi navbatda og'irlik markazini aniqlaymiz:



Ushbu molekulalarning koordinatalarini aniqlaymiz:
m_1 ning koordinatalari uchun $x_1 = -x,\ y_1 = 0,\ z_1 = 0$.
m_2 ning koordinatalai uchun $x_2 = l - x, \quad y_2 = 0, \quad z_2 = 0$.

Endi koordinatalarni aniqlash uchun quyidagi yig’indidan foydalanamiz:

$$\sum_{i=1}^{2} m_i x_i = 0$$

yoki

$$m_1 x_1 + m_2 x_2 = 0.$$

Ushbu ifodaga x_1 va x_2 larning qiymatlarini qo’yib, quyidagini topamiz:

$$-m_1 x + m_2 (l - x) = 0$$

yoki

$$m_2 l = (m_1 + m_2) x,$$

bu yerda

$$x = -\frac{m}{m_1 + m_2} l = \frac{m_2}{M} l.$$

Shunday qilib, oxiri koordinatalar uchun quyidagini aniqlaymiz:

$$x_1 = -\frac{m_2}{M} l, \quad y_1 = 0, \quad z_1 = 0;$$

$$x_2 = -\frac{m_1}{M} l, \quad y_2 = 0, \quad z_2 = 0.$$

Bularni hisobga olgan holda, inersiya momentlarining komponentalarini quyidagicha topamiz:

$$I_1 = \sum_{i=1}^{2} m_i \left(y_i^2 + z_i^2\right) = 0$$

$$I_2 = \sum_{i=1}^{2} m_i \left(x_i^2 + z_i^2\right) = \sum_{i=1}^{2} m_i x_i^2 = m_1 x_1^2 + m_2 x_2^2 = m_1 \frac{m_2^2}{M^2} l^2 + m_2 \frac{m_1^2}{M^2} l^2 = \frac{m_1 m_2}{M^2} l^2 (m_2 + m_1) =$$

$$= \frac{m_1 m_2}{M^2} l^2 M = \frac{m_1 m_2}{M} l^2,$$

$$I_3 = \sum_{i=1}^{2} m_i \left(x_i^2 + y_i^2\right) = \sum_{i=1}^{2} m_i x_i^2 = I_2.$$

Shunday qilib, quyidagi javoblarga ega bo’lamiz:

$$I_1 = 0, I_2 = I_3 = \frac{m_1 m_2}{M} l^2.$$

24-2. Quyidagi mexanik tizimlarning asosiy inersiya mometrlari hisoblansin: Yarim o’qlari a, b, c bo’lgan uch o’qli ellipsoid.

Yechish. Inersiya markazi ellipsoid markaziga, bosh inersiya o’qlari esa ellipsoid o’qlariga mos tushadi. Ellipsoid hakim bo’yicha integrallash sfera hajmi bo’yicha integrallashga keltirilishi mumkin. Buning uchun

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Ellipsoid sirtining tenglamasini

$$\xi^2 + \eta^2 + \zeta^2 = 1$$

Birlik sfera sirt tenglamasiga o’tkazuvchi $x = a\xi, y = b\eta, z = c\zeta$ koordinatalar almashtirishini bajaramiz. Shu yo’l bilan x o’qiga nisbatan inersiya momenti topiladi:

$$I_1 = \rho\iiint\left(y^2+z^2\right)dxdydz = \rho\, abc\iiint\left(b^2\eta\ \ +c^2\zeta^2\right)d\xi d\eta d\zeta = abc\frac{I^{'}}{2}\left(b^2+c^2\right),$$

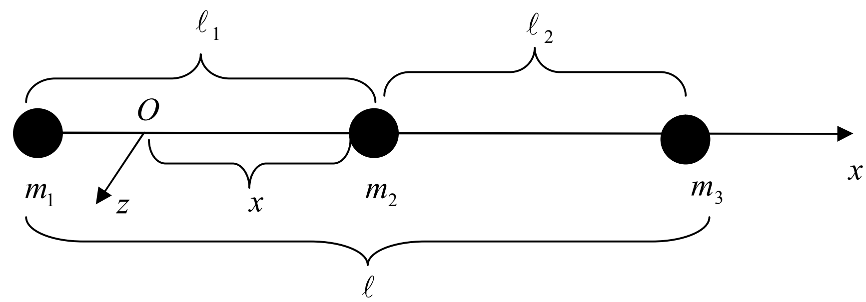
bu erda $I^{'}$-birlik radiusli sharning enarsiya momenti. Ellips hajmi $\frac{4\pi}{3}abc$ ga teng ekanligini nazarda tutsak, oxirgi tenglikdan

$$I_1 = \frac{\mu}{5}\left(b^2+c^2\right),\quad I_1 = \frac{\mu}{5}\left(a^2+c^2\right),\quad I_1 = \frac{\mu}{5}\left(b^2+a^2\right)$$

larni hosil qilamiz.

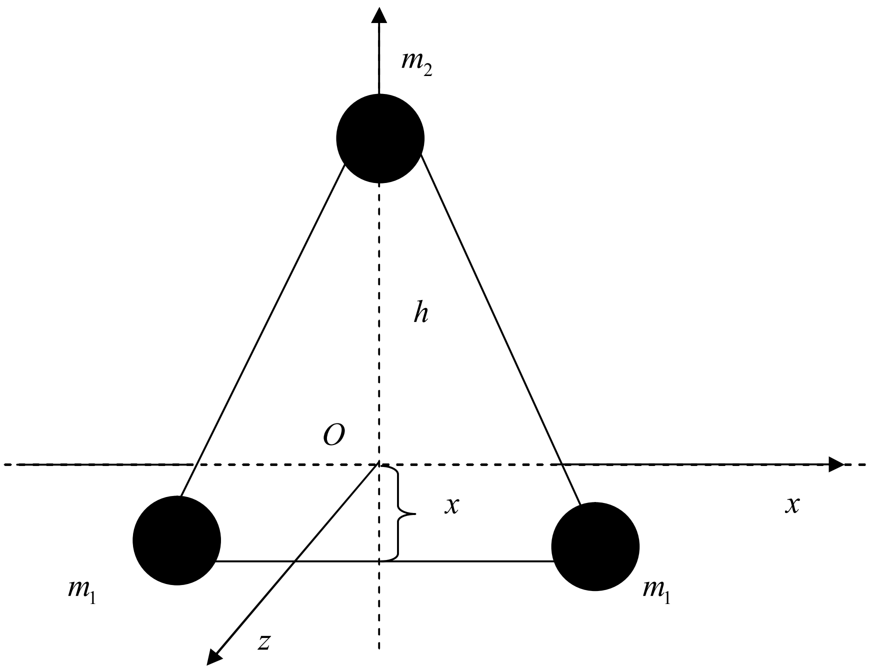
TOPSHIRIQLAR

1-TOPSHIRIQ. Quyidagi mexanik tizimning asosiy inersiya momentlari hisoblansin:Chiziqli uch atomli molekulani.

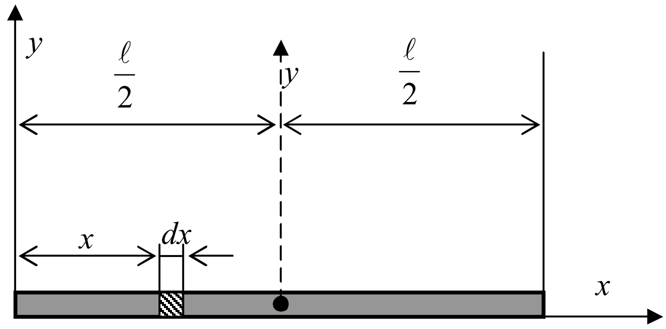


$$l_1 + l_2 = l$$

2-TOPSHIRIQ. Quyidagi mexanik tizimning asosiy inersiya momentlari hisoblansin:Teng yonli uchburchakda joylashgan uch atomli molekulani. Suv (H_2O) molekulasi.



3-TOPSHIRIQ. Quyidagi bir jinsli yaxlit jismning bosh inersiya momentlari aniqlansin. ℓ uzunlikdagi ingichka sterjen.



4-TOPSHIRIQ. Quyidagi bir jinsli yaxlit jismning bosh inersiya momentlari aniqlansin. Radiusi R va balandligi h bo'lgan doiraviy silindr.

JAVOBLAR:

1. $I_1 = I_2 = \frac{1}{M}\left(m_1 m_2 l_1^2 + m_2 m_1 l_2^2\right); \quad M = m_1 + m_2 + m_3$.

2. $I_1 = \frac{m_1 a^2}{2}, \ I_2 = 2\frac{m_1 m_2}{2m_1 + m_2}h^2, \ I_3 = I_1 + I_2$.

3. $I_1 = I_2 = \frac{1}{12}\mu\ell^2, \ I_3 = 0$.

4. $I_1 = I_2 = \frac{\mu}{4}\left(R^2 + \frac{h^2}{3}\right), \ I_3 = \frac{\mu}{2}R^2$

Uslubiy tavsiyalar:

1. 2-1-masalaning ishlanish yo'li bo'yicha hisonlanadi.
2. Teng yonli uchburchakning balandligina inersiay marakazi deb, x va y o'qlarini rasmda ko'rsatilgandek yo'naltirib, o'qlarga proeksiyalanadi.
3. sterjen yo'g'onligi hisobga olinmaydi deb olinadi. Sterjen uzunlik birligiga to'g'ri keladiga massasini ρ bilan belgilasak, $dm = \rho dx$ bo'ladi. Burun ℓ bo'yicha integrallaymiz.
4. Inersiya momentlarini yalanish o'qiga nisbatan olamiz.

ABBYY PDF Transformer 2.0 Click here to buy www.ABBYY.com

25- MASHG'LOT
Mavzu: «Qattiq jismning kinetik energiyasi»

Qisqacha nazariya.

Qattiq jismning energiyasi va aylanma harakatidagi kinetik energiyalarining yig'indisidan iborat bo'ladi, ya'ni

$$T=\frac{\mu V^2}{2}+\frac{1}{2}\sum_{i,k} I_{ik}\Omega_i\Omega_k$$

bu yerda

μ -qattiq jismning to'la massasi,

$\vec{V}=\dot{\vec{R}}$ inersiya markazining harakat tezligi,

I_{ik} -markaziy tizimda inersiya momentlari tenzorining komponentalari,

$\vec{\Omega}$ -bir lahzadagi burchak tezlik,

i,k -1, 2, 3 koordinata o'qlari raqamlari.

Agar koordinata o'qlari asosiy inersiya o'qlari bo'yicha yo'nalgan bo'lsa, unda

$$I_{ik}=I_i\delta_{ik}$$

va qattiq jismning kinetik energiyasi quyidagicha yoziladi:

$$T=\frac{MV^2}{2}+\frac{1}{2}\sum_i I_i\Omega_i^2 .$$

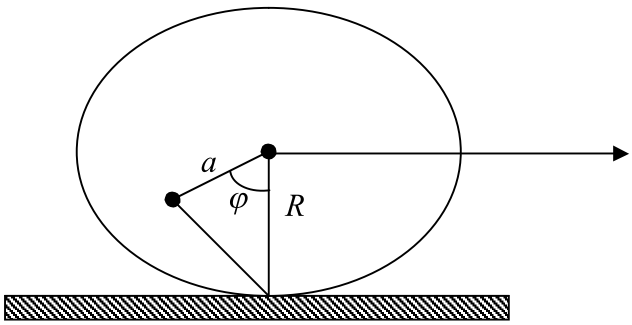
Shuni ta'kidlash foydaliki, qattiq jismning istalgan nuqtasini harakatidagi chiziqli tezligi qo'zg'almas deb qabul qilingan boshqa istalgan nuqtasiga nisbatan quyidagiga teng bo'ladi:

$$\vec{V}_i=\dot{\vec{R}}_i=\vec{\Omega}\times\vec{R}_i$$

Masalalar

25-1. Agar silindrning M massasi uning hajmi bo'yicha shunday taqsimlangan bo'lsaki, undan a masofada asosiy o'qlardan biri silindr o'qiga parallel bo'lsa, Ω burchak tezlik bilan tekislik bo'yicha dumalayotgan R radiusli silindrning kinetik energiyasi topilsin. Ushbu o'qqa nisbatan silindrning inersiya momenti I ga teng.

Yechish. Aylanishning burchak tezligi $\dot{\varphi}=\frac{d\varphi}{dt}$ ga teng. $\dot{\varphi}$ hamma parallel bo'lgan o'qlar atrofidan aylanish burchak tezligi bir xildir. Inersiya markazi lahzaviy o'qdan $b=\sqrt{a^2+R^2-2aR\cos\varphi}$ masofada turadi.



Shuning uchun, uning tezligi quyidagi ifodaga teng bo'ladi:

$$V=\dot{\varphi}\,b=\dot{\varphi}\sqrt{a^2+R^2-2aR\cos\varphi}.$$

Bu ho lda kinetik energiya quyidagicha aniqlanadi:

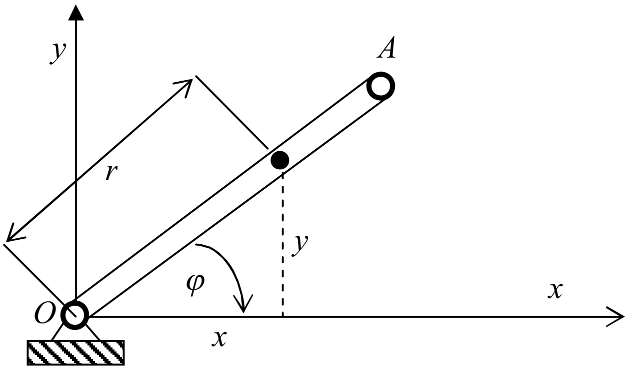
$$T=\frac{MV^2}{2}+\frac{1}{2}\sum_\alpha I_\alpha\Omega_\alpha^2=\frac{M}{2}\dot{\varphi}^2\left(a^2+R^2-2aR\cos\varphi\right)+\frac{I}{2}\dot{\varphi}^2$$

yoki

$$T=\frac{\dot{\varphi}}{2}\left[M\left(a^2+R^2-2aR\cos\varphi\right)+I\right]$$

bu yerda $\dot{\varphi}=\Omega$ burchak tezlik.

25-2. OA kulisa gorizantal tekislikda o'zining O uchidan o'tuvchi z o'q atrofida aylana oladi (rasmda yuqoridan ko'rinishi tasvirlangan). Massasi m ga teng sirpang'ich kulisa ichida harakatlana oladi. Sirpang'ichini moddiy nuqta deb qaralsin. Kulisaning z o'qqa nisbatan inersiya momenti I_z ga teng. Sistemaning kinetik energiyasi topilsin. Qarshilik kuchi hisobga olimasin.



Echish. Umumlashgan koordinatalar deb r,φ qutb koordinatalarni kiritamiz. U holda sistemaning kinetik energiyasi kulisa va sirpang'ich kinetik energiyalar yig'indisidan iborat, yani:

$$T=T_k+T_s.$$

Kulisa z o'q atrofida yalanma harakatda bo'lgani uchun

$$T_k=\frac{1}{2}I_z\dot{\varphi}^2$$

Rasmdan

$$x=r\cos\varphi,\quad y=r\sin\varphi$$

Bo'lganligida

$$\dot{x}=\dot{r}\cos\varphi-r\dot{\varphi}\sin\varphi$$
$$\dot{y}=\dot{r}\sin\varphi+r\dot{\varphi}\cos\varphi$$

Bundan sirpang'ichning tezligi va kinetik energiyasi

$$\vartheta_s=\dot{x}^2+\dot{y}^2=r^2\dot{\varphi}^2+\dot{r}^2$$

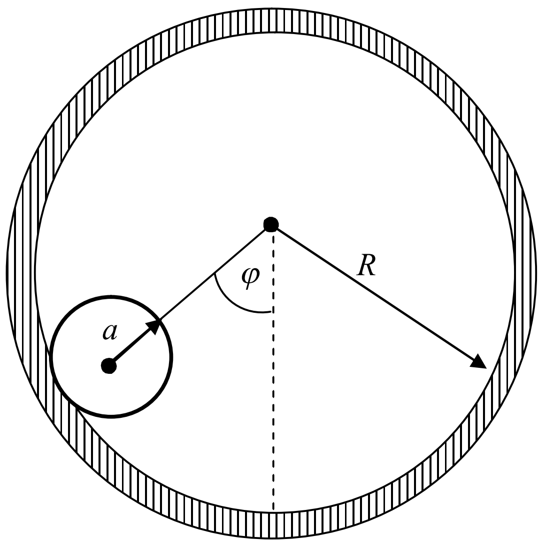
$$T_s=\frac{1}{2}m\left(r^2\dot{\varphi}^2+\dot{r}^2\right)$$

Shunday qilib sistemaning to'liq kinetik energiyasi

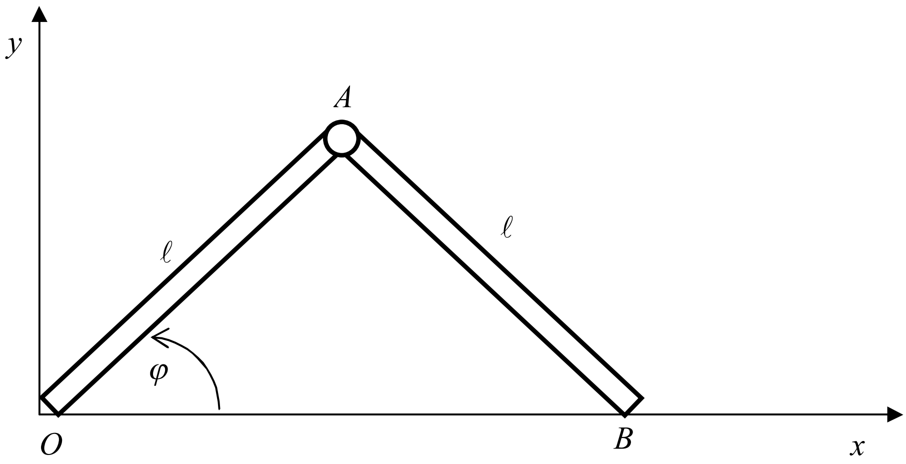
$$T=\frac{1}{2}\left(I_z+m\dot{r}^2\right)\dot{\varphi}^2+\frac{1}{2}m\dot{r}^2$$

TOPSHIRIQLAR

1-TOPSHIRIQ. Quyidagi sistemaning kinetik energiyasi topilsin: R radiusli silindrik sirtning ichki tomonida dumalayotgan a radiusli bir jinsli silindr.



2-TOPSHIRIQ. Quyidagi sistemaning kinetik energiyasi topilsin: OA va AB yupqa bir jinsli sterjenlar bo'lib, ularning uzunliklari ℓ ga teng va A nuqtaga sharnur yordamida buruktirilgan. OA sterjun O nuqta atrofida ayalanadi (rasm tekisligida). AB sterjenning B uchi esa Ox o'qi bo'ylab sirpanadi.



JAVOBLAR:

1. $T = \frac{3}{4}\mu(R-a)^2\dot{\varphi}^2$

2. $T = \frac{\mu\ell^2}{3}\left(1+3\sin^2\varphi\right)\dot{\varphi}^2$

Uslubiy tavsiyalar:

1. Dumalayotgan silindrning inersiya markazini o'q ustida joylashtitiliadi. Burchak tezlikni silindrlarning bir-biriga tegish chizig'iga mos tushadigan oniy o'q atrofidagi aylanish tezligi sifatida hisoblanadi.

2. AB sterjen inersiya markazida dekart koordinatalar sistemasi kiritiladi. OA sterjen inersiya markazining tezligi $\ell\dot{\varphi}/2$ ga teng bo'ladi.

26- MASHG'LOT
Mavzu: «Qattiq jismning harakat tenglamalari. D'Alamber prinsipi»

Qisqacha nazariya.

Qattiq jism oltita erkinlik darajasiga ega bo'lgani uchun harakatning umumiy tenglamalar sistemasi oltita mustaqil tenglamalardan iborat bo'lishi kerak. Ularni ikki vektor – impulis va jism momentidan olingan vaqt bo'yicha birinchi tartibli hosilalar ko'rinishida olish mumkin. Bulardan

birinchi tenglamasi: $\frac{d\vec{P}}{dt} = \vec{F}$,

ikkinchi tenglamasi: $\frac{d\vec{M}}{dt} = \vec{K}$,

bu erda $\vec{P} = \sum \vec{p} = \mu\vec{V}$ -jismning to'la impulisi, $\vec{F} = \sum \vec{f}$ -jismga ta'sir etuvchi to'la kuch, $\vec{K} = \sum [\vec{r}\,\vec{f}]$-kuch momenti.

Qattiq jism muvozanatda bo'lishi uchun

$$\vec{F} = \sum \vec{f} = 0$$
$$\vec{K} = \sum [\vec{r}\,\vec{f}] = 0$$

bo'lishi kerak.

Yondashuvchi jismlar harakat tenglamalarini tuzinshda reaksiya kuchlari oshkor kiritishga asoslanga d'Alamber prinsipi mavjud. Bu usulning mohiyati shundan iboratki, har bir yondashuvchi jismlar uchun

$$\frac{d\vec{P}}{dt} = \sum \vec{f}, \qquad \frac{d\vec{M}}{dt} = \sum [\vec{r}, \vec{f}]$$

tenglamalarini yozishdan iborat. Bu ta'sir etuvchi $\vec{f}$ kuchlarga qatoriga reaksiya kuchlari ham kiritiladi.

Masalalar

26-1. d'Alamber prinsipidan foydalanib, $\vec{F}$ tashqi kuch va $\vec{K}$ kuch momentlari ta'sirida tekislikda dumalayotgan bir jinsli sharning harakat tenglamalari topilsin.

Echish. $\vec{V}$ orqali ilgarilanma harakat tezligini (shar markaz tezligi) Ω bilan esa shar aylanishinig burchak tezligini belgilaymiz. Sharning sirtga tegish nuqtasining tezligini aniqlash uchun umumiy $\vec{v} = \vec{V} + [\vec{\Omega}\,\vec{r}]$ formulada $\vec{r} = -a\vec{n}$ ifodani kiritamiz. Izlanayotgan bog'lanish tenglamasi (tegish nuqtasida siroanish bo'lmasligidan)

$$\vec{V} - a[\vec{\Omega}\,\vec{r}] = 0 \tag{1}$$

ko'rinishni olamiz. Bu erda a-shar radiusi. $\vec{n}$-yodosh nuqtasida dumalash sirtiga o'tkazilgan normalning birlik vektori.

Endi (1) bog'lanish tenglamasi shar va tekislik yondoshgan nuqtada qo'yilgan reaksiya kuchini (reaksiya kuchlarini $\vec{R}$ bilan belgilaymiz) kiritib,

$\mu\frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$ (2)

$$I\frac{d\vec{\Omega}}{dt} = \vec{K} - a[\vec{n}\,\vec{R}] \tag{2}$$

tenglamalarni yozamiz.

Bu erda $\vec{P} = \mu\vec{V}$ va shar pildiroq uchun $\vec{M} = I\vec{\Omega}$ hisobga olingan)

Endi (1) bo'glanish tenglamasini vaqt bo'yicha differensiallab, $\dot{\vec{V}} = a[\dot{\Omega}\vec{n}]$ tenglikni olamiz. $\dot{\vec{V}}$ ni (1) tenglamaga qo'yib va (2) yordamida $\dot{\Omega}$ ni yo'qotib, reaksiya kuchini $\vec{F}$ va $\vec{K}$ bilan bog'livchi

$$\frac{I}{a\mu}\left(\vec{F} + \vec{R}\right) = \left[\vec{K}\,\vec{n}\right] - a\vec{R} + a\vec{n}\left(\vec{n}\,\vec{R}\right)$$

tenglamani topamiz. Bu tenglamani komponentalar bo'yicha qayta yozib va $I = \frac{2}{5}\mu a^2$ ni o'rniga qo'yib quyidagilarni olamiz:

$$R_x = \frac{5}{7a}K_y - \frac{2}{7}F_x, \qquad R_y = \frac{5}{7a}K_x - \frac{2}{7}F_y, \qquad R_z = -F_z$$

(xy sharning dumalash tekisligi). Bu ifodalarni (1) tenglamaga qo'yib, faqat muayyan tashqi kuch va momentlarni o'z ichiga olgan harakat tenglamalarini topamiz:

$$\frac{dV_x}{dt} = \frac{5}{7\mu}\left(F_x + \frac{K_y}{a}\right),$$

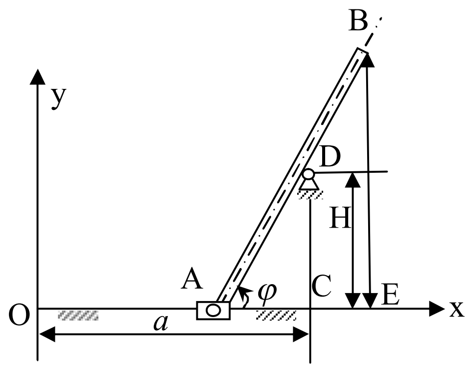
$$\frac{dV_y}{dt} = \frac{5}{7\mu}\left(F_y + \frac{K_x}{a}\right).$$

Burchak tezlikning Ω_x, Ω_y komponentalari bog'lanish tenglamalari yordamida V_x va V_y orqali ifodalanadi, Ω_z uchun esa

$$\frac{2}{5}\mu a^2 \frac{d\Omega_z}{dt} = K_z$$

tenglamaga ega bo'lamiz.

26-2. AB sterjenning A uchi v o'zgarmas tezlik bilan to'g'ri chiziqli yo'naluvchida sirpanadi va bunda sterjen harakat vaqtida D shkiftga tayanadi. Sterjen va uning B uchi harakat tenglamalari yozilsin. Sterjen uzunligi ℓ ga teng; shkif to'g'ri chiziqli yo'naltiruvchidan H balandlikda o'rnatilgan. Harakatning boshlanishida sterjenning A uchi qo'zg'almas koordinatalar sistemasi boshi O nuqta bilan ustma-ust tushadi; $OC = a$. A nuqta qutb deb olinsin.



Yechish. Berilganlar shaklda ko'rsatilgan. Sterjenning A qutb atrofida aylanish burchagini φ bilan belgilaymiz. A nuqta x o'qi bo'ylab to'g'ri chiziqli harakat qilgani uchun hamma vaqt $y_a = 0$. Endi x_A koordinatasini topamiz. Shakldan va A nuqta v tezlik bilan tekis harakat qilgani uchun

$$x_A = OA = v \cdot t$$

bo'ladi.

Shakldan: $\dfrac{H}{AC} = tg\varphi; \qquad AC = a - v\,t\,;$

$$tg\varphi = \frac{H}{a - v\,t} \text{ bundan } \varphi = arctg\frac{H}{a - v\,t}.$$

B nuqtaning koordinatalarini topamiz:

$$x_B = OA + AE, \qquad OA = v\,t; \qquad AE = \ell\cos\varphi,$$

$$\cos\varphi = \frac{1}{\sqrt{1 + tg^2\varphi}} = \frac{a - v\,t}{\sqrt{(a - \vartheta\,t)^2 + H^2}},$$

natijada

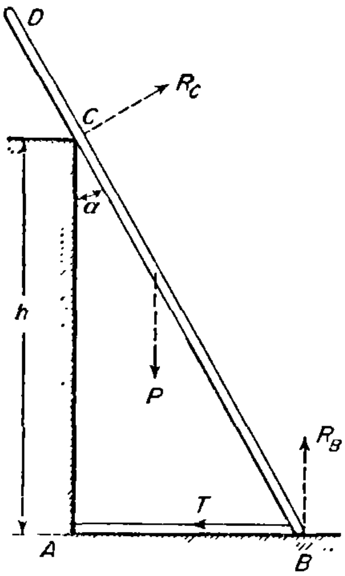
$$x_B = v\,t + \frac{a - v\,t}{\sqrt{H^2 + (a - v\,t)^2}}\,\ell\,,$$

$$y_B = BE = \ell\sin\varphi = \ell\frac{H\ell}{\sqrt{H^2 + (a - v\,t)^2}}.$$

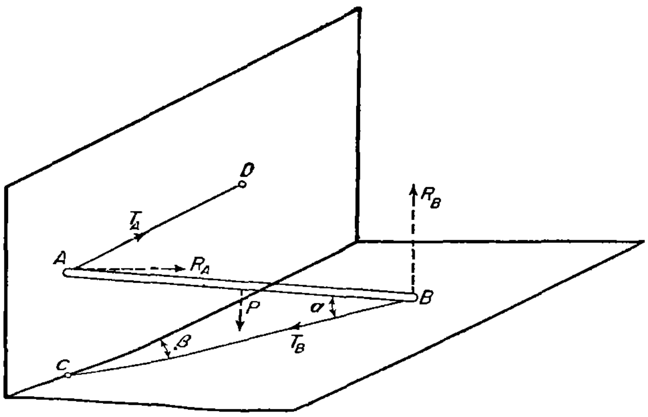
TOPSHIRIQLAR

1-TOPSHIRIQ. P og'irlik va ℓ uzunlikdagi bir jinsli BD sterjen devorga tirab qo'yilgan; uning B pastki uchi AB ip bilan tortib qo'yilgan. Tayanch reaksiyasi va ipning tarangligi aniqlansin (1-rasm).

2-TOPSHIRIQ. P og'irlikdagi AB sterjen uchlari bilan vertikal tekisliklarga tiraladi va u shu vaziyatda ikkita gorizontal AD va BC iplar bilan ushlab turiladi; BC ip AB sterjen bilan bir vertikal tekislikda yotadi. Tayanch reaksiyalari va iplarning tarangligi aniqlansin (2-rasm).

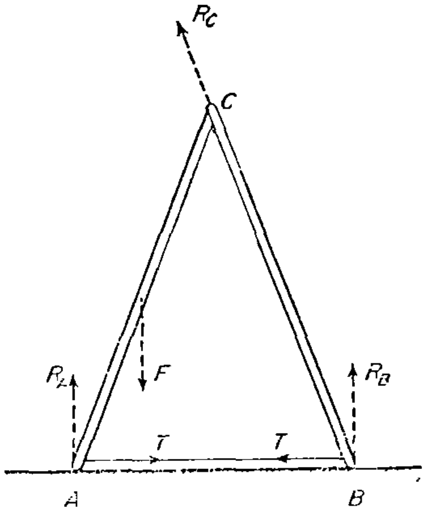


(1-rasm)



(2-rasm)

3-TOPSHIRIQ. l uzunlikdagi ikkita sterjen yuqoridan sharnur vositasida biriktirilgan, pastdan esa, AB ip bilan bir-biriga tortib qo'yilgan (3-rasm). Sterjenlardan birining o'rtasiga F kuch ta'sir ettiriladi (sterjen og'irligi hisobga olinmasin). Reaksiya kuchlari topilsin.



(3-rasm)

JAVOBLAR:

1. $R_C = \dfrac{P\ell}{4h}\sin 2\alpha$, $\quad R_B = P - R_C \sin\alpha$, $T = R_C \cos\alpha$.

2. $R_B = P$, $T_B = \dfrac{P}{2} ctg\alpha$, $\quad R_A = T_A \sin\beta$, $T_A = T_B \cos\beta$.

3. $R_A = \dfrac{3}{4}F$, $\quad R_B = \dfrac{F}{4}$, $\quad R_C = \dfrac{F}{4\sin\alpha}$, $\quad T = \dfrac{1}{4}F\, ctg\alpha$.

Uslubiy tavsiyalar:

1. Reaksiya kuchlarini sretjendan tik yuqorida va perpendikulyar qo'yib, bog'lanish tenglamalarini tuziladi.
2. Reaksiya kuchlarini tekislikka perpendikulyar qo'yib, bog'lanish tenglamalarini tuziladi.
3. Reaksiya kuchlarini tekislikka perpendikulyar qo'yib, bog'lanish tenglamalarini tuziladi.

ADABIYOTLAR

1. Ландау Л.Д., Лифшиц Е.М. Механика.-М.: Наука, 1988.- 216 с.
2. Ландау Л.Д., Лифшиц Е.М. Гидродинамика.-М.: Наука, 1988.-736 с.
3. Ландау Л.Д., Лифшиц Е.М. Теория упругости. –М.: Наука, 1987. -248 с.
4. Маркеев А.П. Теоретическая механика. –М.: Наука, 1990.-446 с.
5. Ольховский И.И. Курс теоретической механики для физиков. –М.: Наука,1970.-448 с.
6. Кудрявсев Н.В. Практикум по классической механике. –Томск, 1986.-105 с.
7. Ольховский И.И. Павленко Ю.Г., Кузменков Л.С. Задачи по теоретической механике для физиков. –М.: Изд-во Моск. ун-та, 1977.-395 с.
8. Гречко Л.Г. Сугаков В.И., Томасевич О.Ф., Федорченко А.М. Сборник задач по теоретической физике. –М.: Высшая школа, 1972.-336 с.