O’ZBEKISTON RESPUBLIKASI

OLIY VA O’RTA MAXSUS TA'LIM VAZIRLIGI

ZAHIRIDDIN MUHAMMAD BOBUR NOMIDAGI

ANDIJON DAVLAT UNIVERSITETI

Matematika kafedrasi

Qo’lyozma huquqida

Nozimjonova Hilola Murodiljon qizi

KARRALI INTEGRALLAR NAZARIYASI HAQIDA

5460100 – matematika

ta’lim yo’nalishi bo’yicha

bakalavr akademik darajasini olish uchun yozilgan

BITIRUV MALAKAVIY ISH

Ish rahbari: fizika-matematika fanlari nomzodi dotsent Akbarova S.X.

ANDIJON-2015 yil

Kirish

Kadrlar tayyorlash milliy dasturiga muvofiq, keyingi yillarda ta'lim jarayonining mazmunini tubdan takomillashtirish, o'qituvchilarning mehnatini moddiy va ma'naviy rag'batlantirish bo'yicha katta ishlar qilinmoqda, ilm – fanning yanada rivojlanishiga katta e'tibor berilmoqda.

O'zbekiston Respublikasi mustaqilikka erishganidan so'ng, ta'lim – tarbiya tizimidagi islohotlar boshlangandan keyingi yillarda prezidentimiz I.Karimov, jahon tajribasi va hayotda o'zini ko'p bor oqlagan haqiqatatdan kelib chiqib, agar bu maqsadlarimizni muvaffaqiyatli ravishda amalga oshira olsak, tez orada hayotimizda ijobiy ma'nodagi "portlash effekti"ga, ya'ni, yangi ta'lim modelining kuchli samarasiga erishamiz, degan fikrni bildirgan edi.

Boshqa tabiiy fanlar qatori matematika fani o'zining ko'plab amaliy tadbiqlariga ega bo'lgan murakkab fanlardan biri bo'lib, bu sohada muhim ilmiy yangiliklar kashf qilinmoqda.

Ko'p o'zgaruvchili funksiyalarning integrallari bilan bog'liq masalalarni hal etishda karrali integrallarni o'rganish matematika va fanning boshqa tarmoqlarida katta ahamiyatga egadir. Karrali integrallar nazariyasida xuddi aniq integrallar nazariyasidagi kabi, karrali integralning mavjudligi, xossalari, ularni hisoblash hamda integralning tadbiqlari egri chiziqli integrallar va aniq integral bo'yicha ma'lumotlar asosida o'rganiladi.

Oddiy va karrali integrallar nazariyasi Fixtengolts G.M.[1-3], Azlarov T., Mansurov X.[4], [5], Shokirova X.[6], Ilin V.A., Poznyak E.G.[7], [8], Kudryavtsev L.D[9] va boshqa olimlar tomonidan yaratilgan darsliklarda keng yoritib berilgan.

Karrali integrallar nazariyasi xususiy hosilali differensial tenglamalar uchun chegaraviy masalalarini o'rganishda ham katta ahamiyatga ega bo'lib, jumladan, karrali integrallarni egri chiziqli integrallar bilan bog'lovchi formulalar keng qo'llaniladi [11]- [12].

Ushbu bitiruv malakaviy ishida matematik analiz kursidagi muhim mavzulardan biri karrali integrallar nazariyasi o'rganilgan bo'lib, u kirish qismi, ikkita bob, xulosa va foydalanilgan adabiyotlar ro'yxatidan tuzilgan.

I-bob to'rtta paragrafdan iborat bo'lib, bunda egri chiziqli integrallar, ikki karrali integrallar va ularni hisoblash bo'yicha ma'lumotlar berilgan.

1.1-§. da egri chiziqli integrallar haqida asosiy tushunchalar keltirilgan.

I. Birinchi tur egri chiziqli integral. Bizga R^2 tekislikda (K) uzluksiz egri chiziq berilgan bo'lsin. (K) chiziqda berilgan ixtiyoriy «nuqta funktsiyasini» olamiz : $f(M) = f(x,y)$ va (K) egri chiziqni $A_i A_{i+1}$ elementar yoylarga bo'lib va unda $\forall$ $M_i(\xi_i,\eta_i)$ nuqta tanlab, $f(M_i)=f(\xi_i,\eta_i)$ hisoblaymiz va quyidagi yig'indini tuzamiz:

$$\sum_{i=0}^{n-1} f(M_i)\sigma_i = \sum_{i=0}^{n-1} f(\xi_i \eta_i)\sigma_i;$$

bu integral yigindini ifodalaydi. Bu yerda σ_i - $A_i A_{i+1}$ yoy uzunligi.

T a' r I f. Agar $\lambda = max\sigma_i$ nolga intilganda bu integral yig'indi (K) egri chiziqni bo'linishiga ham, $M_i \in A_i A_{i+1}$ nuqtalarni ham tanlanishiga bog'liq bo'lmagan holda I chekli limitga ega bo'lsa, u holda bu limit, $f(M) = f(x,y)$ funksiyadan (K) yo'l yoki egri chiziq bo'yicha olingan *birinchi tur egri chiziqli integral* deyiladi, va quyidagicha belgilanadi:

$$I = \int_{(K)} f(M)ds = \int_{(K)} f(x,y)ds \tag{1}$$

(bu yerda s- (K) chiziqning yoy uzunligi va ds σ_i elementar uzunliklarni bildiradi).

Fazoviy (K) chiziq bo'lgan holda ham integral tushunchasini kiritishimiz mumkin:

$$\int_{(K)} f(M)ds = \int_{(K)} f(x,y,z)ds.$$

1) (K): $x = x(s), y = y(s), (o \le s \le S)$ chiziq parametrik holda berilganda

2)(K) : $x=\varphi(t), y=\psi(t)(t_0 \le t \le T)$ chiziq ixtiyoriy parametrik tenglamalar bilan berilganda

3)(K) : $y=y(x), a \le x \le b$ oshkor ko'rinishda berilganda (1) egri chiziqli integralni oddiy aniq integralga mos ravishda quyidagicha bog'langan:

$$\int_{(K)} f(M)ds = (R)\int_{o}^{S} f(x(s), y(s))ds, \tag{2}$$

$$\int_{(K)} f(M)ds = (R)\int_{t_0}^{T} f(\varphi(t), \psi(t)\sqrt{[\varphi'(t)]^2 + [\varphi'(t)]^2 dt}, \tag{3}$$

(bu yerda φ va ψ funksiyalar φ' va ψ' hosilalari bilan uzluksiz va faraz qilamiz, chiziqda karrali nuqtalar yo'q)

$$\int_{(K)} f(M)ds = (R)\int_{a}^{b} f(x_, y(x)\sqrt{1+[y^1(x)]^2})dx. \tag{4}$$

II.Ikkinchi tur egri chiziqli integrallar. (AB) yopiq bo'lmagan uzluksiz egri chiziq berilgan bo'lib, bu chiziqda biror $f(x,y)$ funksiya berilgan bo'lsin. Egri chiziqni $A_i(x_i\,,\ y_i)$ nuqtalar bilan qisimlarga ajratamiz. $A_i\,A_{i+1}$ chiziq oralig'ida ixtiyoriy $M_i = M_i(\xi_i\,,\ \eta_i)$ nuqtani tanlaymiz va ushbu

$$\sigma = \sum_{i=0}^{n-1} f(M_i)\Delta\, x_i = \sum_{i=0}^{n-1} f(\xi_i\,, \eta_i)\,\Delta x_i \tag{5}$$

yig'indini tuzamiz, bu yerda $x_{i+1} - x_i = \Delta x_i$.

T a' r i f. Agar $\mu = max\ A_i\ A_{i+1}$ nolga intilganda σ yig'indi, egri chiziqni bo'linishiga ham, M_i nuqtalarni tanlanishiga bog'liq bo'lmagan holda I - chekli limitiga ega bo'lsa, u holda bu limit (AB) yo'l yoki egri chiziq bo'yicha $f(M)dx$ dan olingan ikkinchi tur egri chiziqli integral deyiladi va quyidagicha belgilanadi.

$$I = \int_{(AB)} f(M)dx = \int_{(AB)} f(x,y)dx \tag{6}$$

Xuddi shunday

$$I^* = \int\limits_{(AB)} f(M)dy = \int\limits_{(AB)} f(x,y)dy \ . \qquad (7)$$

Egri chiziqli integralning umumiy ko'rinishi esa quyidagicha:

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy =$$

$$= \int\limits_{(AB)} P(x,y)dx + \int\limits_{(AB)} Q(x,y)dy \qquad (8)$$

(AB) fazoviy (masalan, yopiq bo'lmagan) chiziqda ikkinchi tur egri chiziqli integral tushunchasini xuddi yuqoridagidek kiritish mumkin:

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy + R(x,y,z)dz =$$

$$= \int\limits_{(AB)} P(x,y)dx + \int\limits_{(AB)} Q(x,y)dy + \int\limits_{(AB)} R(x,y)dz$$

Ravshanki, oddiy aniq integralning sodda xossalari qaralayotgan egri chiziqli integrallar uchun ham o'rinli.

Keyingi 1.2-§. da ikki karrali integrallar va ularning xossalari va ularni hisoblash masalalari o'rganilgan.

1. Ikki karrali integral ta'rifi. Bizga ma'lumki, egri chiziqli trapetsiyaning yuzasi haqidagi masala oddiy aniq integral tushunchasiga olib keladi. Shunga o'xshash, silindrik jismning hajmi haqidagi masala esa ikki karrali (aniq) integral tushunchasiga olib keladi.

(P) sohada $f(x,y)$ funksiya aniqlangan bo'lsin. (P) sohani chekli sondagi (P_1), (P_2),…, (P_n) sohalarning egri chiziqlari bilan bo'lamiz. Bu qism sohalar bog'langan yoki bog'lanmagan bo'lsin. (P_i) i-elementar sohada ixtiyoriy (ξ_i, η_i) nuqtani olamiz, va quyidagi yig'indini tuzamiz

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i) P_i$$

$f(x, y)$ funksiya uchun (P) sohada integral yig'indi deb ataymiz. Bu yerda P_i - (P_i) sohaning yuzasi.

λ orqali (P_i) qism sohalar diametrlarining eng kattasini belgilaymiz.

Agar λ→0 da σ integral yig'indi (P) sohani (P_i) qismlarga bo'lish usuliga, (ξ_i, η_i) nuqtaning tanlanishiga bog'liq bo'lmagan holda chekli limitga ega bo'lsa,

$$\mathrm{I} = \lim_{\lambda \to 0} \sigma,$$

u holda bu limit $f(x, y)$ funksiyaning (P) sohada *ikki karrali integrali* deyiladi va

$$\mathrm{I} = \iint_{(P)} f(x, y)\, dP$$

kabi belgilanadi.

Nihoyat, ikki karrali integralni hisoblash masalalari, ya'ni 1)to'g'ri to'rtburchakli sohada ikki karrali integralni takroriy integralga keltirish; 2)egri chiziqli soha bo'lgan holda ikki karrali integralni takroriy integralga keltirish; va ikki karrali integrallarni hisoblashga doir misollar yechib ko'rsatilgan.

II-bobda uch karrali va ko'p karrali integrallar haqida tushuncha berilgan. 2.1-§ da uch karrali integrallar bo'yicha ba'zi ma'lumotlar keltirilgan.

2.2-§.da esa matematik analiz kursidagi muhim formulalar, Grin, Stoks va Ostrogsadskiy formulalari keltirib chiqarilgan.

Bu formulalar bizga ma'lumki, soha bo'yicha olingan karrali integrallarni shu soha chegarasi bo'yicha olingan integral bilan bog'lovchi formulalardir.

1. Grin formulasi. $(PQ): y = y_0(x)(a \le x \le b)$ va $(SR): y = Y(x)(a \le x \le b)$ egri chiziqlar va y- o'qiga parallel ikkita PS va QR kesmalardan iborat (L) kontur bilan chegaralangan (D) – egri chiziqli trapetsiyadan iborat sohani qaraymiz (II.2.1-§., 1 - rasm).

Faraz qilamiz, (D) sohada $P(x,y)$ funksiya berilgan bo'lib, u bu sohada o'zining $\frac{\partial P}{\partial y}$ hosilasi bilan birgalikda uzluksiz bo'lsin.

U holda quyidagi tenglik

$$\iint\limits_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int\limits_{(L)} Pdx + Qdy. \tag{11}$$

Grin formulasi deyiladi.

E s l a t m a. Grin formulasi bir yoki bir nechta bo'lakli-silliq konturlar bilan chegaralangan, ixtiyoriy (D) soha uchun o'rinli.

2-punktda egri chiziqli integrallar yordamida yuzani hisoblashda Grin formulasining tadbig'i keltirilgan.

2. Stoks formulasi. Ushbu tenglik

$$\int\limits_{(L)} Pdx + Qdy + Rdz =$$

$$= \iint\limits_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx$$

Stoks formulasi deyiladi. Bu yerda $(S) \subset R^3$ - karrali nuqtalarga ega bo'lmagan silliq sirt, (L) - (S) sirtning chegarasi, bo'lakli silliq kontur.

3. Ostrogradskiy formulasi. Bu formula uch karrali integrallar nazariyasidagi muhim formula bo'lib, u R^2 tekislikdagi soha bo'yicha olingan ikki karrali integral bilan soha konturi bo'yicha egri chiziqli integralni bog'lovchi Grin formulasining analogidir. Bu formula fazoviy soha bo'yicha uch karrali integralni, soha chegarasi bo'yicha sirt integrali bilan bog'laydi va u quyidagi ko'rinishda bo'ladi

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = \iint\limits_{(S)} Pdydz + Qdzdx + Rdxdy. \tag{12}$$

bu yerda $(V) \subset R^3$-silliq (S) sirt bilan chegaralangan soha.

(12) formula 2-tur sirt integralining bu sirt bilan chegaralangan jism bo'yicha olingan uch karrali integral orqali umumiy ko'rinishini ifodalaydi.

Grin, Stoks va Ostrogradskiy formulalari, biror geometrik shakl bo'yicha integralni, bu shakl chegarasi bo'yicha olingan integral orqali ifodalaydi.
Grin formulasi ikki o'lchamli fazo holiga mos keladi, Stoks formulasi ham – yana ikki o'lchamli, lekin fazo"egri chizig'i " holiga, Ostrogradskiy formulasi esa – uch o'lchamli fazo holiga mos keladi.

Oxirgi 2.3-§ ko'p karrali integrallarni o'rganishga bag'ishlangan.

Oddiy, ikkikarrali va uch karrali integrallarni aniqlashda biz kesma uzunligi, tekis figura yuzasi, fazoviy jism hajmi tushunchalarini qo'llaganimiz kabi, m-karrali integralni ta'riflashda m-o'lchovli sohaning hajmi tushunchasidan foydalanamiz. Sodda m-o'lchovli soha uchun uning hajmi tushunchasini kiritaylik: Quyidagi

$$[a_1, b_1, a_2, b_2, \dots, a_m, b_m] \qquad (1)$$

m-o'lchovli to'g'riburchakli parallelepipedning hajmi deb, uning o'lchamlari ko'paytmasini ataymiz:

$$(b_1 - a_1)(b_2 - a_2) \cdot \dots \cdot (b_m - a_m).$$

Tushunarliki, chekli sondagi shunday parallepiped lardan tuzilgan jism hajmi deb faraz qilinadi. Hajm, jism parallelepipedlarga qanday yoyilganligiga bog'liq emasligini elementar ko'rsatish mumkin.

(V) sohadam o'zgaruvchili $f(x_1, x_2, \dots, x_m)$ funksiya berilgan bo'lsin, u holda , bu sohani elementar qismlarga yoyib va bizga ma'lum bo'lgan amallarni takrorlab, m-karrali integral tushunchasiga kelamiz:

$$I = \underbrace{\overbrace{\int \dots \int}^{m}}_{(V)} f(x_1, x_2, \dots, x_m) dx_1 dx_2 \dots dx_m. \qquad (2)$$

Integral ostidagi funksiya uzluksiz bo'lgan holda u mavjud bo'ladi.

m=3-o'lchovli fazoda o'rinli bo'lgan formulalarga o'xshash quyidagi formulalar o'rinli:

$$I = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \dots \int_{a_m}^{b_m} f(x_1, x_2, \dots, x_m) dx_m. \tag{3}$$

Ushbu

$$x_1^0 \leq x_1 \leq X_1 \ , \ x_2^0(x_1) \leq x_2 \leq X_2(x_1), \dots,$$

$$x_m^0(x_1, \dots, x_{m-1}) \leq x_m \leq X_m(x_1, \dots, x_{m-1}),$$

tengsizliklar bilan aniqlangan, nisbatan umumiy ko'rinishdagi sohalar uchun esa

$$I = \int_{x_1^0}^{X_1} dx_1 \int_{x_2^0(x_1)}^{X_2(x_1)} dx_2 \dots \int_{x_m^0(x_1,\dots,x_{m-1})}^{X_m(x_1,\dots,x_{m-1})} f(x_1, x_2, \dots, x_m) dx_m.$$

formula o'rinli.

Ostrogradskiy 1834 yili birinchi marta alohida $x_1, x_2, \dots, x_m$ o'zgaruvchilar bo'yicha integrallash chegarasini aniqladi, unda

$$L(x_1, x_2, \dots, x_m) \leq 0$$

ko'rinishdagi tengsizlikni qanoatlantiruvchi $x_1, x_2, \dots, x_m$ o'zgaruvchilar bo'yicha olingan ko'p karrali integrallarni hisoblash keltirladi.Bu yerda biz ixtiyoriy sondagi o'zgaruvchilar holida va (m-1)-o'lchamli yopiq sirt bo'yicha olingan integral bilan bu sirt chegaralagan jismi bo'yicha olingan m-karrali integralni bog'lovchi bizga ma'lum bo'lgan (m=3-o'lchovli fazoda)

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dx_1 dx_2 dx_3 =$$

$$= \iint\limits_{(S)} P dx_2 dx_3 + Q dx_3 dx_1 + R\, dx_1 dx_2$$

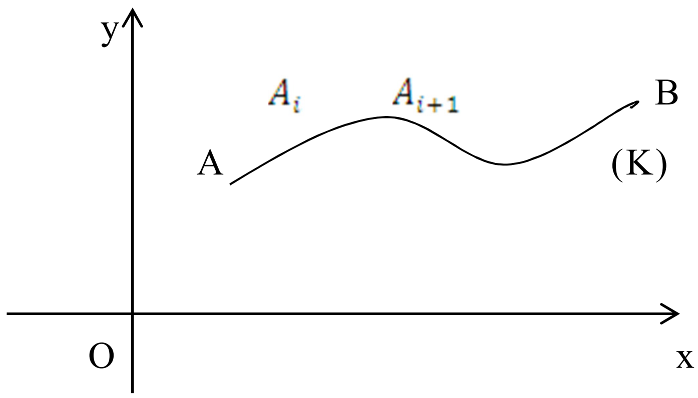
Ostrogradskiy formulasini umumlashtiruvchi formula o'rinli.

I-Bob. Asosiy tushunchalar

Ushbu bob matematik analiz kursidagi egri chiziqli integrallar, ikki karrali integrallar va ularni hisoblash masalalarini o'rganishga bag'ishlangan.

1.1-§. Egri chiziqli integrallar

I.Birinchi tur egri chiziqli integral. Mexanik masalani qaraylik: (K) uzluksiz tekis egri chiziq berilgan bo'lsin. Bu chiziq ustida massa joylashgan bo'lib, chiziqning barcha M nuqtalarida uning $\rho(M)$ zichligi ma'lum. Butun (K) egri chiziqning m massasini aniqlaymiz.



Shu maqsadda chiziqni oxirlari A va B nuqtalar orasida ixtiyoriy $A_1, A_2, \ldots, A_{n-1}$ nuqtalarini qo'yamiz va $A_0=A$, $A_n=B$ deb belgilaymiz. Aniqlik uchun, bu nuqtalar A dan B ga yo'nalishida nomerlangan A_i, A_{i+1} yoyida biror M_i nuqtani olib, bu nuqtadan $\rho(M_i)$ zichlikni hisoblaymiz. Taxminan barcha izlanayotgan massa uchun esa

$$m = \sum_{i=1}^{n-1} \rho(M_i)\sigma_i$$

ifodaga ega bo'lamiz. σ_i -$A_i A_{i+1}$ yoy uzunligi, $m_i = \rho(M_i)\sigma_i$ bu yoyning massasi.

λ - orqali σ_i uzunliklardan eng kattasini belgilab, limitga o'tamiz:

$$\lim_{\lambda \to 0} m = \lim_{\lambda \to 0} \sum_{i=1}^{n-1} \rho(M_i)\sigma_i$$

Endi (K) chiziqda berilgan ixtiyoriy «nuqta funktsiyasini» olamiz : $f(M) = f(x,y)$ va ko'rsatilgan jarayonni takrorlaymiz: (K) egri chiziqni $A_i A_{i+1}$

elementar yoylarga bo'lib va unda $\forall M_i(\xi_i, \eta_i)$ nuqta tanlab, $f(M_i) = f(\xi_i, \eta_i)$ ni hisoblaymiz va quyidagi yig'indini tuzamiz.

$$\sum_{i=0}^{n-1} f(M_i)\sigma_i = \sum_{i=0}^{n-1} f(\xi_i \eta_i)\sigma_i ;$$

bu integral yigindini ifodalaydi.

T a' r I f. Agar $\lambda = max\sigma_i$ nolga intilganda bu integral yig'indi (K) egri chiziqni bo'linishiga ham, $M_i \in A_iA_{i+1}$ nuqtalarni ham tanlanishiga bog'liq bo'lmagan holda I chekli limitga ega bo'lsa, u holda bu limit, $f(M) = f(x,y)$ funksiyadan (K) yo'l yoki egri chiziq bo'yicha olingan *birinchi tur egri chiziqli integral* deyiladi, va quyidagicha belgilanadi:

$$I = \int_{(K)} f(M)ds = \int_{(K)} f(x,y)ds \qquad (1)$$

(bu yerda s- (K) chiziqning yoy uzunligi va ds σ_i elementar uzunliklarni bildiradi).

Shuni ta'kidlash kerakki, keltirilgan ta'rifda (K) yo'lning yo'nalishi hech qanday ro'l o'ynamaydi. Agar, masalan, bu chiziq yopiq emas va (AB) va (BA) – turli yo'nalishli chiziqlar bo'lsa, u holda

$$\int_{(AB)} f(M)ds = \int_{(BA)} f(M)ds$$

bo'ladi.

Xuddi shunday, biz fazoviy (K) chiziq bo'lgan holda ham integral tushunchasini kiritishimiz mumkin:

$$\int_{(K)} f(M)ds = \int_{(K)} f(x,y,z)ds.$$

Endi (1) egri chiziqli integralni oddiy aniq integralga bog'lashni ko'rib chiqamiz. Faraz qilamiz, (K) chiziqda yo'nalish ixtiyoriy ravishda o'rnatilgan (ikkita mumkin bo'lgan yo'nalishdan biri) bo'lsin, shuning uchun M nuqtani chiziqdagi holati, boshlang'ich A nuqtadan hisoblangan s=AM yoy uzunligi bilan aniqlangan bo'lsin. U holda (K) chiziq parametrik holda

$$x = (s), y = y(s), (o \leq s \leq S)$$

tenglamalar ko'rinishida ifodalanadi, chiziq nuqtalarida berilgan $f(x, y)$ funksiya esa, $f(x, y), y(s))$ s - ga bog'liq murakkab funksiyaga keltiriladi.

Agar s_i (i=0,1,…,n) orqali AB yoydagi tanlangan bo'linish nuqtalariga mos yoyning qiymatini belgilasak, u holda, ravshanki, $\sigma_i = S_{i\pm1} - S_i = \Delta S_i$ orqali M_i nuqtani aniqlovchi s -ni qiymatini belgilab (ravshanki, $s_i \leq \overline{s_1} \leq s_{i+1}$), ko'ramizki, egri chiziqli integral uchun integral yig'indi

$$\sum_{i=o}^{n-1} f(M_i)\sigma_i = \sum_{i=o}^{n-1} f(x(\overline{s_i}), y(\overline{s_i}))\Delta S_i$$

bir vaqtda oddiy integral uchun integral yig'indi bo'ladi, shuning uchun quyidagiga ega bo'lamiz:

$$\int_{(K)} f(M)ds = (R)\int_o^S f(x(s), y(s))ds, \tag{2}$$

bunda integrallardan birining mavjudligi boshqasini mavjudligini bildiradi. Bu yerda (R) belgisi, integral oddiy Riman ta'rifi bilan tushunilishini ko'rsatadi.

Endi (K) chiziq ixtiyoriy parametrik tenglamalar bilan berilgan bo'lsin:

$$x = \varphi(t), y = \psi(t)(t_0 \leq t \leq T),$$

Bu yerda φ va ψ funksiyalar φ' va ψ' hosilalari bilan uzluksiz; faraz qilamiz, chiziqda karrali nuqtalar yo'q. U holda chiziq to'g'rilanuvchi va agar

s =s(t)

yoyni o'sishi t - parametrni o'sishiga mos kelsa, u holda (1,1-tom)

$$s' = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}$$

(3) integralda o'zgaruvchini almashtirib, quyidagini olamiz:

$$\int_{(K)} f(M)ds = (R)\int_{t_0}^T f(\varphi(t), \psi(t)\sqrt{[\varphi'(t)]^2 + [\varphi'(t)]^2 dt} \tag{3}$$

Shunday qilib, birinchi tur egri chiziqli integralni hisoblash uchun integral ostidagi funktsiyada x va y o'zgaruvchilar parametr orqali koordinatalar ifodalari bilan almashtiriladi, ds ko'paytuvchi esa parametrni funksiyasi sifatida yoy differentsiali bilan almashtiriladi.

Egri chiziq $y = y(x), a \leq x \leq b$ oshkor ko'rinishda berilgan bo'lsa, (3) formula

$$\int_{(K)} f(M)ds = (R)\int_a^b f(x, y(x)\sqrt{1+\left[y^1(x)\right]^2})dx \tag{4}$$

ko'rinishini oladi.

Bu foqmulalarni qo'llanilishiga misol keltiramiz. Quyidagi

$$I = \int_{(K)} xyds,$$

integralni hisoblang, bu yerda

$(K): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning birinchi chorakda yotgan $\frac{1}{4}$ qismi.

Y e ch i sh. Ellipsning parametrik tenglamasini yozamiz: x=acost, y= bsint.

U holda: $x_t^{'} = -a\sin t, y_t^{'} = b\cos t$ bo'lib,

$$\sqrt{x_t^{'2} + \gamma_t^{'2}} = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$$

I integralni (4) formula bo'yicha hisoblash mumkin, ya'ni

$$I = \int_{(K)} xyds = \int_o^{\pi/2} a\cos t \cdot b\sin \cdot \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}dt. \tag{5}$$

Endi bu yerda sin2t=2sintcost, cos2t=$\cos^2$t-$\sin^2$t keltirish formulalardan foydalansak, (5) dan quyidagini

$$\text{I} = \int_o^{\Pi/2} \frac{ab}{2}\sin 2t\sqrt{a^2 \cdot \frac{1-cos2b}{2} + b^2\frac{1-\cos 2t}{2}}dt = \frac{ab}{2}\int_o^{\overline{\pi/2}} \sin 2t\sqrt{\frac{a^2+b^2}{2} + \frac{b^2-a^2}{2}\cos 2t dt}$$

hosil qilamiz. Bu yerda belgilash kiritamiz:

cos2t=z, u holda -2sin2tdt=dz, yoki $\sin 2tdt = -\frac{1}{2}dz.$

Demak , bu belgilashga ko'ra

$$\mathrm{I}=\frac{ab}{4}\int_{-1}^{1}\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2z}{2}}dz=\frac{ab}{4}\cdot\frac{2}{b^2-a^2}\int_{-1}^{1}\sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}}zd(\frac{b^2-a^2}{2}z)=\frac{ab}{2}\cdot\frac{1}{b^2-a^2}$$

$$\cdot\frac{2}{3}\left(\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}z\right)^{3/2}\Big|_{-1}^{1}=$$

$$=\frac{ab}{3}\frac{1}{(b-a)(b+a)}\left[\left(\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\right)^{3/2}-\left(\frac{a^2+b^2}{2}-\frac{b^2-a^2}{2}\right)^{3/2}\right]=$$

$$=\frac{ab}{3}\cdot\frac{1}{(b-a)(b+a)}\left(b^3-a^3\right),$$

yoki $\quad I=\dfrac{ab}{3}\dfrac{1}{(b-a)(b+a)}(b-a)(b^2+ab+a^2)=\dfrac{ab}{3}\cdot\dfrac{a^2+ab+b^2}{a+b}.$

II.Ikkinchi tur egri chiziqli integrallar.

1.(AB) yopiq bolmagan uzluksiz egri chiziq berilgan bo'lsin va bu chiziqda biror f (x , y) funksiya berilgan bo'lsin. Egri chiziqni $A_i(x_i\,,\ y_i)$ nuqtalar bilan qisimlarga ajratamiz. $A_i\,A_{i+1}$ chiziq oraliqida ixtiyoriy $M_i=(\xi_i\,,\ \eta_i)$ nuqtani tanlaymiz va bunda funksiya qiymatini $f\ (M_i)=f(\xi_i\,,\ \eta_i)$ hisoblaymiz. Lekin bu qiymatni $A_i\,A_{i+1}$ yoy uzunligiga emas, balki bu yoyning x – o'qiga proeksiyasi kattaligiga, ya'ni $x_{i+1}-\ x_i=\Delta x_i$ ga ko'paytiramiz; keyin esa ushbu

$$\sigma=\sum_{i=0}^{n-1}f(M_i)\Delta\,x_i=\ \sum_{i=0}^{n-1}f(\xi_i\,,\eta_i)\ \Delta x_i \tag{6}$$

yig'indini tuzamiz.

T a' r i f. Agar $\mu=max\ A_i\ A_{i+1}$ nolga intilganda σ yig'indi, egri chiziqni bo'linishiga ham, M_i nuqtalarni tanlanishiga bog'liq bo'lmagan holda I - chekli limitiga ega bo'lsa, u holda bu limit (AB) yo'l yoki egri chiziq bo'yicha $f(M)dx$ dan olingan ikkinchi tur egri chiziqli integral deyiladi va quyidagicha belgilanadi.

$$I=\int\limits_{(AB)}f(M)dx=\int\limits_{(AB)}f(x,y)dx \tag{7}$$

Shunga o'xshash, ikkinchi tur egri chiziqli integralni olamiz:

$$I^*=\int\limits_{(AB)}f(M)dy=\int\limits_{(AB)}f(x,y)dy \tag{8}$$

Agar (AB) chiziqda ikkita P(M)=P(x,y) va Q (M)=Q(x,y) funksiyalar aniqlangan bo'lsa va quyidagi integrallar mavjud bo'lsa

$$\int\limits_{(AB)} P(M)dx = \int\limits_{(AB)} P(x,y)dx,$$

$$\int\limits_{(AB)} Q(M)dy = \int\limits_{(AB)} Q(x,y)dy,$$

u holda ularning yig'indisi egri chiziqli integral («umumiy ko'rinishdagi») deyiladi va

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy = \int\limits_{(AB)} P(x,y)dx + \int\limits_{(AB)} Q(x,y)dy \qquad (9)$$

deb olinadi.

Biz ko'rdikki, integrallash amalga oshiriladigan (AB) yo'lning yo'nalishi, *birinchi tur integral* holida hech qanday ro'l o'ynamaydi yoki $A_i\, A_{i+1}$ yoy uzunligi σ_i bu yo'nalishga bog'liq emas. *Ikkinchi tur integral* holida esa bunday emas: $A_i\, A_{i+1}$ yoyning o'sha yoki boshqa o'qlardan biriga proeksiyasi yoy yo'nalishiga bog'liq va bu yo'nalishning teskarisiga o'zgarishi bilan ishorasini o'zgartiradi. Shunday qilib, ikkinchi tur integrallar uchun

$$\int\limits_{(BA)} f(x,y)dx = -\int\limits_{(AB)} f(x,y)dx$$

bo'ladi va shunga o‘xshash

$$\int\limits_{(BA)} f(x,y)dy = -\int\limits_{(AB)} f(x,y)dy$$

bo'lib, o'ng tomondagi integrallarning mavjudligidan chap tomondagi integrallarning mavjudligi chiqadi va aksincha.

2. (AB) fazoviy (masalan, yopiq bo'lmagan) chiziqda ikkinchi tur egri chiziqli integral tushunchasini kiritamiz. Agar $f(M) = f(x\,,y\,,z)$ funksiya bu

chiziq nuqtalarida berilgan bo'lsa, u holda quyidagi ko'rinishdagi ikkinchi tur integrallar aniqlanadi:

$$\int_{(AB)} f(M)dx = \int_{(AB)} f(x,y,z)dx,$$

$$\int_{(AB)} f(M)dy = \int_{(AB)} f(x,y,z)dy, \quad \int_{(AB)} f(M)dz = \int_{(AB)} f(x,y,z)dz.$$

Nihoyat, ushbu («umumiy ko'rinishdagi») integral ham qaraladi.

$$\int_{(AB)} Pdx + Qdy + Rdz = \int_{(AB)} Pdx + \int_{(AB)} Qdy + \int_{(AB)} Rdz$$

Bu yerda yana integrallash yo'nalishini o'zgarishi integralni ishorasini o'zgartiradi.

Ko'rinib turibdiki, oddiy aniq integralning sodda xossalari qaralayotgan egri chiziqli integrallar uchun ham o'rinli.

3.Endi ikkinchi tur egri chiziqli integrallarni mavjudligi va hisoblash masalalarini ko'rib chiqamiz.

(K) = (AB) egri chiziq

$$x = \varphi(t), \; y = \psi(t) \tag{10}$$

parametrik tenglamalari bilan berilangan bo'lib, φ va ψ uzluksiz, va t parametr α va β o'zgarganda egri chiziq A danB gacha yo'nalishda ifodalanadi. $f(x\,,y\,)$ funksiyani (AB) chiziq ustida yana uzluksiz deb faraz qilamiz.

(7) integralda, yana qo'shimcha shartni, ya'ni $\varphi^{'}(t)$ hosilaning mavjud va uzluksiz deb olamiz. U holda (7) egri chiziqli integral mavjud va quyidagi

$$\int_{(AB)} f(x,y)dx = (R)\int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\varphi^{'}(t)dt$$

(8) integral uchun ham xuddi shunday uning mavjudligini o'rnatish va ushbu

$$\int_{(AB)} f(\mathrm{x},\mathrm{y})dy = (R)\int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\varphi'(t)dt$$

formulani isbotlash mumkin, bunda $\psi'(t)$uzluksiz (yoki bo'lakli uzluksiz) hosilani mavjud deb faraz qilinadi.

Nihoyat, quyidagi

$$\int_{(AB)} P(x,y)dx + Q(x,y)dy$$

umumiy ko'rinishdagi integralni qaraymiz, bu yerda P va Q uzluksiz funksiyalar. Bu holda (AB) egri chiziqqa shunday shart qo'yamizki, (10) funksiyalar uzuluksiz yoki hech bo'lmaganda, bo'lakli-uzluksiz hosilalarga ega bo'lsin. Bu farazlarda ushbu formula o'rinli

$$\int_{(AB)} P(x,y)dx + Q(x,y)dy =$$

$$= (R)\int_{\alpha}^{\beta}\left[P\big(\varphi(t),\psi(t)\big)\varphi'(t) + Q\big(\varphi(t),\psi(t)\big)\varphi'(t)\,\right]dt. \qquad (11)$$

Egri chiziqli integral ta'rifi va bu yerda ko'rsatilgan uni oddiy aniq integralga keltirish usuli, agar (10) chiziqlarda yo'nalish avvalgidek t – parametrni α dan β gacha monoton o'zgarishi bilan aniqlansa, bu chiziqlar o'zini-o'zi kesgan holga, bevosita qo'llanilishi mumkin.

Endi (7) integral

$$y = y(x)\,, \quad a \le x \le b$$

oshkor ko'rinishdagi tenglama bilan berilgan egri chiziq bo'yicha olinsin.

U holda egri chiziqqa uning uzluksizligidan tashqari hech qanday shartlarsiz,

$$\int_{(AB)} f(x,y)dx = (R)\int_{a}^{b} f\big(x,y\,(x)\big)dx \qquad (12)$$

tenglikka ega bo'lamiz. Xuddi shunday, agar (9) integral

$$x = x(y) \qquad , \quad \mathrm{c} \le \mathrm{y} \le d$$

oshkor ko'rinishda berilgan uzluksiz chiziq bo'yicha olinsa, u holda

$$\int_{(AB)} f(x,y)dy = (R)\int_{c}^{d} f(x(y), y\,)dy$$

1-m i s o l. Quyidagi egri chiziqli integralni hisoblang:

$$\int_{(L)} 2xydx + x^2dy,$$

bu yerda (L) yo'l O(0,0) va A(1,1) nuqtalarni tutashtiruvchi yo'l bo'lib,

a) $y = x$ to'g'ri chiziq b) $y = x^2$ parabola

c) $x = y^2$ parabola d) $y = x^3$ kubik parabola

a) $y = x \Rightarrow dy = dx$ bo'lgani uchun, u holda

$$\int_{(L)} 2xydx + x^2dy = \int_0^1 2x^2dx + x^2dx = \int_0^1 3x^2dx = x^3|_0^1 = 1$$

b) $y = x^2 \Rightarrow dy = 2xdx,$ bu holda

$$\int_{(L)} 2xydx + x^2dy = \int_0^1 (2x^3 + 2x^3)\,dx = \int_0^1 4x^3dx = x^4|_0^1 = 1;$$

c) $x = y^2 \Rightarrow dx = 2ydy,$

$$\int_{(L)} 2xydx + x^2dy = \int_0^1 (4y^4 + y^4)dy = \int_0^1 5y^4\,dy = y^5\,|_0^1 = 1;$$

d) $y = x^3 \Rightarrow dx = 3x^2 = dy$

$$\int_{(L)} 2xydx + x^2dy = \int_0^1 (2x^4 + 3x^4)dx = \int_0^1 5x^4\,dx = x^5|_0^1 = 1.$$

2-m i s o l.Quyidagi egri chiziqli integralni hisoblang:

$$F = \int_{(L)} xydx + (y - x)dy$$

bu yerda (L) 1- misoldagi kabi.

$$a) \qquad F = \int_0^1 x^2dx = \frac{x^3}{3}\;|_0^1 = \frac{1}{3};$$

$$b) \qquad F=\int_0^1 (x^3+(x^2-x)2x)dx=\int_0^1 (3x^3-2x^2)dx=$$

$$=\left(\frac{3}{4}x^4-\frac{2x^3}{3}\right)\Bigg|_0^1=\frac{1}{12};$$

$$c) \qquad F=\int_0^1 \big(2y^4+(y-y^2)\big)dy=\left(\frac{2y^5}{5}+\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_0^1=\frac{17}{30};$$

$$\text{g)} \qquad F=\int_0^1 (x^4+(x^3-x)\,3x^2)dx=\left(\frac{x^5}{5}+\frac{x^6}{2}-\frac{3x^4}{4}\right)\Bigg|_0^1=\frac{-1}{20}.$$

III. Birinchi tur va ikkinchi tur egri chiziqli integrallar orasidagi boglanish.

T a' r i f. Agar φ(t) va φ(t) funksiyalar bir vaqtda nolga aylanmaydigan, uzluksiz hosilalarga ega bo’lsa, u holda x= φ(t), y= φ(t) egri chiziq - *silliq chiziq* deyiladi.

(К)=(AB) silliq egri chiziqni qaraymiz va parametr sifatida s=AM yoyni tanlab, uni x=x(s), y=y(s) (0≤s≤S)tenglamalar bilan ifodalaymiz, bu yerda x(s),y(s) funksiyalar $x^1(s)$,$y^1(s)$ uzluksiz hosilalarga ega. Agar α orqalix o’q bilan yoyning o’sishi tomoniga yo’nalgan urinma orasidagi burchakni belgilasak, u holda

$$\cos\alpha= x'(s),\ \sin\alpha=y'(s)$$

bo’ladi.

Agar(К) egri chiziqda uzluksiz f(м)= f(x,y) funksiya berilgan bo’lsa, u holda ketma-ket quyidagiga ega bo’lamiz:

$$\int\limits_{(К)} f(M)dx=\int\limits_0^S f(x(s),y(s))\cdot x^1(s)ds=\int\limits_0^S f(x(s),y(s))\cos\alpha ds=\int\limits_{(K)} f(M)\cos\alpha ds$$

va ikkinchi tur egri chiziqli integral birinchi tur egri chiziqli integralga bog’landi.

Shunga o’xshash

$$\int\limits_{(К)} f(M)dy=\int\limits_{(K)} f(M)\sin\alpha ds$$

Agar (К) egri

chiziqda ikkita uzluksiz P(M)=P(x,y) va Q(M)=Q(x_1y) funksiyalar berilgan bo'lsa, u holda

$$\int_{(K)} Pdx + Qdy = \int_{(K)} (P\cos\alpha + Q\sin\alpha)ds$$

Xuddi shuningdek, fazoviy egri chiziq bo'yicha olingan egri chiziqli integrallar uchun

$$\int_{(K)} Pdx + Qdy + Rdz = \int_{(K)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

formula hosil bo'ladi, bu yerda cosα,cosβ,cosγ- urinmaning yo'naltiruvchi kosinuslari bo'lib, urinmani yo'nalishi integrallash yo'li yo'nalishiga mos bo'ladi.

1.2-§. Ikki karrali integrallar va ularning xossalari

1. Ikki karrali integral ta'rifi. Bizga ma'lumki, egri chiziqli trapetsiyaning yuzasi haqidagi masala oddiy aniq integral tushunchasiga olib keladi. Shunga o'xshash, silindrik jismning hajmi haqidagi masala esa ikki karrali (aniq) integral tushunchasiga olib keladi.

(P) sohada $f(x, y)$ funksiya aniqlangan bo'lsin. (P) sohani chekli sondagi (P_1), (P_2),…, (P_n) sohalarning egri chiziqlari bilan bo'lamiz. Bu qism sohalar bog'langan yoki bog'lanmagan bo'lsin. (P_i) i-elementar sohada ixtiyoriy (ξ_i, η_i) nuqtani olamiz, bu nuqtada funksiyani $f(\xi_i, \eta_i)$ qiymatini mos sohaning P_i yuzasiga ko'paytiramiz va barcha shunga o'xshash ko'paytmalarni qo'shamiz. Olingan yig'indini

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i) P_i$$

f(x,y) funksiya uchun (P) sohada integral yig'indi deb ataymiz.

λ orqali (P_i) qism sohalar diametrlarining eng kattasini belgilaymiz.Agar λ→0 da σ integral yig'indi (P) sohani (P_i) qismlarga bo'lish usuliga, (ξ_i, η_i)nuqtaning tanlanishiga bog'liq bo'lmagan holda chekli limitga ega bo'lsa

$$\mathrm{I} = \lim_{\lambda \to 0} \sigma,$$

u holda bu limit $f(x, y)$ funksiyaning (P)sohada ikki karrali integrali deyiladi va

$$\mathrm{I} = \iint\limits_{(P)} f(x, y)\, dP$$

kabi belgilanadi. Integralga ega funksiya integrallanuvchi deyiladi.

2. Ikki karrali integralni mavjud bo'lish sharti. Integrallanuvchi funksiya chegaralangan bo'lishi zarur. Haqiqatan, aks holda (P) sohani ixtiyoriy berilgan usulda qismlarga bo'lishda (ξ_i, η_i) nuqtalarni tanlash hisobiga integral yig'indini ixtiyoriy katta qilish mumkin.

Berilgan $f(x, y)$ funksiyani oldindan chegaralangan deb faraz qilamiz: $m \le f(x, y) \le M$.

Bir ozgaruvchili funksiya holidagi kabi, bu yerda yana Darbuning quyi va yuqori yig'indilarini kiritamiz:

$$s = \sum_{i=1}^{n} m_i P_i \,, \qquad \text{S}= \sum_{i=1}^{n} M_i P_i \,,$$

bu yerda m_i va M_i, mos ravishda $f(x, y)$ funksiyaning (P_i) sohadagi aniq quyi va aniq yuqori chegaralarini bildiradi.

(P) sohani qismlarga bo'lishning berilgan usulida, (ξ_i, η_i) nuqtani tanlashga bog'liq bo'lmagan holda, ushbu

$$s \leq \sigma \leq S$$

tengsizlik bajariladi. Lekin bu nuqtalarni tanlash hisobiga $f(\xi_i, \eta_i)$ qiymatlarni $m_i(M_i)$ ga yetarlicha yaqin qilish mumkin, shu bilan birga σ yig'indini *s(S)* ga yetarlicha yaqin qilish mumkin. Shunday qilib, Darbuning yuqori va quyi yig'indilari mos ravishda, sohaning o'sha bo'linish usuliga mos, integral yig'indining yuqori va quyi chegaralari bo'ladi.

Darbu yig'indilari uchun quyidagi xossalar o'rinli.

1^0. Boshlang'ich bo'linish chiziqlariga yangi chiziqlar qo'shish bilan (P_i) qismlarga keyingi bo'lishda, Darbuning quyi yig'indisi kamaymaydi, yuqori yig'indisi esa o'smaydi.

2^0. Har bir Darbu quyi yig'indisi, (P) sohaning hech bo'lmaganda, boshqa bo'linish usuliga mos har bir yuqori yig'indisidan katta emas.

Yana, aniq chegaralarni mavjudligi o'rnatiladi $I_* = sup\{s\}$, $I^* = inf\{S\}$ va quyidagi tengsizlik bajariladi: $s \leq I_* \leq I^* \leq S$.

Quyidagi teorema o'rinli.

T e o r e m a. Ikki karrali integralning mavjud bo'lishi uchun

$$\lim_{\lambda \to 0} (S - s) = 0$$

tenglikni bajarilishi zarur va yetarli, yoki

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} w_i P_i = 0, \qquad (1)$$

bu yerda w_i $f(x, y)$ funksiyaning (P_i) qism sohadagi $M_i - m_i$ tebranishi.

3.Integrallanuvchi funksiyalar sinfi. Integrallanish alomati yordamida quyidagilarni isbot qilish mumkin.

I. (P) sohada uzluksiz har qanday $f(x,y)$ funksiya integrallanuvchi. Haqiqatan, agar $f(x,y)$ funksiya yopiq (P) sohada uzluksiz bo'lsa, u holda tekis uzluksizlik xossasiga ko'ra, har bir $\varepsilon > 0$ songa shunday $\delta > 0$ topiladiki, (P) sohaning diametri δ dan kichik ixtiyoriy qismida, funksiyaning tebranishi ε dan kichik bo'ladi. Endi (P) soha diametrlari δ dan kichik (P_i) qismlarga yoyilgan bo'lsin. U holda barcha tebranishlar $w_i < \varepsilon$, va

$$\sum_{i=1}^{n} w_i P_i < \varepsilon \sum_{i=1}^{n} P_i = \varepsilon P,$$

bu yerdan teoremadagi (1) shartning bajarilishi kelib chiqadi. Demak, berilgan funksiyaning integrallanuvchi.

Integrallanuvchi funksiyar sinfini kengaytirish maqsadida quyidagi lemmani keltiramiz.

L e m m a.(P) sohada yuzasi nolga teng biror (L) chiziq berilgan bo'lsin. U holda har bir $\varepsilon > 0$ son uchun shunday $\delta > 0$ topiladiki, (P) soha δ dan kichik diametrli qismlarga yoyilganda, ulardan (L) bilan umumiy nuqtalarga ega bo'lganlarining yuzalarini yig'indisi ε dan kichik bo'ladi.

E s l a t m a. (L) nol yuzali chiziq bo'lsa, u holda uni yuzasi ε dan kichik bo'lgan (Q) ko'pburchak bilan o'rash mumkin.

II. Agar chegaralangan $f(x,y)$ funksiya faqat chekli sondagi nol yuzali chiziqlarda uzilishga ega bo'lsa, u holda u integrallanuvchi.

4. Integrallanuvchi funksiyalar va ikki karrali integrallar xossalari.

1^0.Agar (P) da integrallanuvchi $f(x,y)$ funksiyaning qiymatlarini ixtiyoriy ravishda biror nol yuzali (L) chiziqda o'zgartirilsa (bunda o'zgartirilgan funksiya ham chegaralangan bo'lishi kerak), u holda hosil bo'lgan funksiya yana (P) da integrallanuvchi,va uning integrali $f(x,y)$ funksiyadan olingan integralga teng. Shunday qilib, ikki karrali integralning mavjudligi va qiymati, integral ostidagi funksiyalarning chekli sondagi nol yuzali chiziqlarda qabul qiladigan qiymatlariga bog'liq emas.

2^0. Agar $f(x,y)$ funksiya berilgan (P) soha nol yuzali (L) chiziq bilan ikkita (P') va (P'') sohalarga yoyilgan bo'lsa, u holda $f(x,y)$ funksiyaning butun (P) sohada integrallanuvchiligidan (P') va (P'') qism sohalarda integrallanuvchiligi kelib chiqadi va aksincha, har ikki (P') va (P'') sohalarda integrallanuvchiligidan (P) sohada integrallanuvchiligi kelib chiqadi. Bunda

$$\iint\limits_{(P)} f(x,y)dP = \iint\limits_{(P')} f(x,y)dP' + \iint\limits_{(P'')} f(x,y)dP''.$$

3^0. Agar (P) da integrallanuvchi $f(x,y)$ funksiyani k o'zgarmas songa ko'paytirilsa, u holda o'lingan funksiya yana integrallanuvchi bo'ladi, va bunda

$$\iint\limits_{(P)} kf(x,y)dP = k\iint\limits_{(P)} f(x,y)dP.$$

4^0. Agar $f(x,y)$ va $g(x,y)$ funksiyalar (P) sohada integrallanuvchi bo'lsa, u holda $f(x,y) \pm g(x,y)$ funksiya ham integrallanuvchi bo'lib,

$$\iint\limits_{(P)} f(x,y) \pm g(x,y)\ dP = \iint\limits_{(P)} f(x,y)dP \pm \iint\limits_{(P)} g(x,y)dP.$$

5^0. Agar (P) da integrallanuvchi $f(x,y)$ va $g(x,y)$ funksiyalar uchun $f(x,y) \le g(x,y)$ tengsizlik bajarilsa, u holda

$$\iint\limits_{(P)} f(x,y)dP \le \iint\limits_{(P)} g(x,y)dP.$$

6^0. $f(x,y)$ funksiyaning integrallanuvchi bo'lgan holda $|f(x,y)|$ funksiya ham integrallanuvchi bo'ladi, va quyidagi tengsizlik o'rinli.

$$\left|\iint\limits_{(P)} f(x,y)dP\right| \le \iint\limits_{(P)} |f(x,y)|dP.$$

7^0. Agar (P) da integrallanuvchi $f(x,y)$ funksiya $m \le f(x,y) \le M$ tengsizlikni qanoatlantirsa, u holda

$$mP \le \iint_{(P)} f(x,y)dP \le MP. \qquad (2)$$

Bu tengsizlik ushbu

$$mP \leq \sum_{i=1}^{n} f\left(\xi_i, \eta_i\right) P_i \leq MP$$

tengsizlikdan limitga o’tish bilan hosil qilinadi.

(2) tengsizlikni barcha qismlarini P ga bo’lsak:

$$m \leq \frac{\iint_{(P)} f(x,y)dP}{P} \leq M$$

va

$$\mu = \frac{\iint_{(P)} f(x,y)dP}{P}$$

deb belgilasak, u holda (2) tengsizlikni boshqacha yozilishini olamiz

$$\iint_{(P)} f(x,y)dP = \mu P \ (m \leq \mu \leq M), \qquad (3)$$

bu o’rta qiymat haqidagi teoremani ifodalaydi.

Endi xususan, faraz qilamiz, $f(x,y)$ funksiya (P) da uzluksiz, va m va M sifatida uning (P) sohadagi eng kichik va eng katta qiymatlarini olamiz – Veyershtrass teoremasiga ko’ra [1,1-to’m,136-p.] ular mavjud. U holda ma’lum Boltsano’-Koshi teoremasiga ko’ra [1,1-to’m,134-p.] m va M qiymatlarni qabul qiluvchi $f(x,y)$ funksiya, har bir oraliq qiymat orqali o’tishi kerak. Shunday qilib, barcha holda (P) sohada shunday $(\bar{x},\bar{y})$ nuqta topiladiki, $\mu = f(\bar{x},\bar{y})$ bo’ladi, va (3) formula

$$\iint_{(P)} f(x,y)dP = f(\bar{x},\bar{y})P \qquad (4)$$

ko’rinishni oladi.

III. Ikki karrali integralni hisoblash.

1.To’g’ri to’rtburchakli sohada ikki karrali integralni takroriy integralga keltirish. Integrallash sohasi $(P) = [a,b;c,d]$ to’g’ri to’rtburchak sohadan iborat bo’lsin.

T e o r e m a. Agar $(P) = [a, b; c, d]$ sohada aniqlangan $f(x, y)$ funksiya uchun ushbu

$$\iint_{(P)} f(x, y) dP \qquad (1)$$

ikki karrali integral mavjud va x ning $[a, b]$ dagi har bir o'zgarmas qiymatida

$$I(x) = \int_c^d f(x, y) dy \qquad (a \leq x \leq b) \qquad (2)$$

oddiy integral mavjud bo'lsa, u holda quyidagi

$$\int_a^b dx \int_c^d f(x, y) dy \qquad (3)$$

takroriy integral ham mavjud bo'ladi va

$$\iint_{(P)} f(x, y) dP = \int_a^b dx \int_c^d f(x, y) dy \qquad (4)$$

tenglik o'rinli.

I s b o t. (P) to'g'ri to'rtburchakni aniqlovchi $[a, b]$ va $[c, d]$ oraliqlarni, bo'linish nuqtalarini qo'yib, bo'laklarga bo'lamiz:

$$x_0 = a < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_n = b,$$

$$y_0 = c < y_1 < \cdots < y_k < y_{k+1} < \cdots < y_m = d.$$

U holda (P) to'g'ri to'rtburchak qism to'g'ri to'rtburchaklarga bo'linadi:

$$\left(P_{i,k}\right) = [x_i, x_{i+1}; y_k, y_{k+1}]$$

$$(i = 0,1,2, \dots, n-1; k = 0,1, \dots, m-1).$$

$m_{i,k}$ va $M_{i,k}$ orqali, mos ravishda, $f(x, y)$ funksiyaning $\left(P_{i,k}\right)$ to'g'ri to'rtburchakdagi aniq quyi va aniq yuqori chegaralarini belgilaymiz, shuning uchun bu to'g'ri to'rtburchakni barcha (x, y) nuqtalari uchun

$$m_{i,k} \leq f(x, y) \leq M_{i,k}.$$

$[x_i, x_{i+1}]$ oraliqda x ni ixtiyoriy fiksirlab: $x = \xi_i$, va y bo'yicha y_k dan y_{k+1} gacha integrallab, quyidagiga ega bo'lamiz

$$m_{i,k} \Delta y_k \leq \int_{y_k}^{y_{k+1}} f\left(\xi_i, y\right) dy \leq M_{i,k} \Delta y_k,$$

bu yerda $\Delta y_k = y_{k+1} - y_k$; (2) integralni butun $[c, d]$ oraliq bo'yicha mavjud deb faraz qilingani uchun, y bo'yicha integral mavjud bo'ladi. O'xshash tengsizliklarni k bo'yicha 0 dan $m-1$ gacha qo'shib, quyidagini olamiz

$$\sum_{k=0}^{m-1} m_{i,k}\Delta y_k \le I\left(\xi_i\right) = \int_c^d f\left(\xi_i, y\right) dy \le \sum_{k=0}^{m-1} M_{i,k}\Delta y_k.$$

Agar bu tengsizliklarning barcha qismlarini $\Delta x_i = x_{i+1} - x_i$ ga ko'paytirsak va i bo'yicha 0 dan $n-1$ gacha qo'shsak, u holda

$$\sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{i,k}\Delta y_k \le \sum_{i=0}^{n-1} I\left(\xi_i\right)\Delta x_i \le \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} M_{i,k}\Delta y_k$$

hosil bo'ladi. Biz o'rtada I(x) funksiya uchun $\sum_{i=0}^{n-1} I\left(\xi_i\right)\Delta x_i$ integral yig'indini oldik. Chetki hadlar esa (1) ikki karrali integral uchun

$$s = \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{i,k}\Delta y_k \text{ va } \quad S = \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} M_{i,k}\Delta y_k$$

Darbu yig'indilarini ifodalaydi. Haqiqatan, $\Delta x_i \Delta y_k = P_{i,k}$ $\left(P_{i,k}\right)$ to'g'ri to'rtburchakning yuzasi bo'lgani uchun, masalan, quyidagiga egamiz

$$\sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{i,k}\Delta y_k = \sum_{i=0}^{n-1}\sum_{k=0}^{m-1} m_{i,k}\Delta x_i \Delta y_k = \sum_{i,k} m_{i,k} P_{i,k} = s.$$

Shunday qilib,

$$s \le \sum_{i=0}^{n-1} I\left(\xi_i\right)\Delta x_i \le S.$$

Agar endi barcha Δx_i va Δy_k bir vaqtda nolga intilsa, u holda (1) integralni mavjudligiga ko'ra, har ikki s va S yig'indilar unga intiladi. Bunday holda $\sum_{i=0}^{n-1} I\left(\xi_i\right)\Delta x_i$ yig'indi ham (1) integralga intiladi:

$$lim \sum_{i=0}^{n-1} I\left(\xi_i\right)\Delta x_i = \iint_{(P)} f(x, y) dP,$$

ya'ni (1) integral bir vaqtning o'zida $I(x)$ funksiyadan olingan integralga teng bo'ladi:

$$\iint\limits_{(P)} f(x,y)dP = \int\limits_a^b I(x)dx = \int\limits_a^b dx \int\limits_c^d f(x,y)dy,$$

isbot tugadi.

x va y o'zgaruvchilarni rolini almashtirib, (4) bilan birgalikda

$$\iint\limits_{(P)} f(x,y)dP = \int\limits_c^d dy \int\limits_a^b f(x,y)dx \tag{4'}$$

formulani isbot qilish mumkin, bunda $y = const$ da

$$\int\limits_a^b f(x,y)dx$$

integral mavjud deb faraz qilinadi.

E s l a t m a.Agar (1) ikki karrali integral bilan birgalikda ushbu

$\int_c^d f(x,y)dy\ (x = const)$ va $\int_a^b f(x,y)dx\ (y = const)$

oddiy integrallar ham mavjud bo'lsa, u holda (4) va (4') formulalar bir vaqtda o'rinli bo'ladi, bu yerdan

$$\int\limits_a^b dx \int\limits_c^d f(x,y)dy = \int\limits_c^d dy \int\limits_a^b f(x,y)dx \tag{5}$$

tenglikka ega bo'lamiz.

Ikki karrali integralni ko'pincha takroriy integral bilan o'xshash quyidagicha belgilanadi

$$\int_a^b \int_c^d f(x,y)dydx \text{ yoki } \int_c^d \int_a^b f(x,y)dxdy.$$

Yana

$$\int_a^b \left\{\int_c^d f(x,y)dy\right\} dx \text{ yoki } \int_c^d \left\{\int_a^b f(x,y)dx\right\} dy$$

kabi yozish mumkin.

Misol. Ushbu

$$I = \int\limits_0^1 \int\limits_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dxdy$$

integralni hisoblaymiz.

Integral ostidagi

$$f(x,y) = \frac{x}{(1+x^2+y^2)^{3/2}}$$

funksiya (P)=[0,1;0,1] sohada uzluksiz. Berilgan ikki karrali integral ham, va

$$\int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dx$$

Integral ham mavjud. Yuqorida keltirilgan teoremaga ko'ra,

$$\int_0^1 \left[\int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dx \right] dy$$

integral mavjud bo'ladi

$$\iint\limits_{(P)} f(x,y) dP = \int_c^d dy \int_a^b f(x,y) dx \qquad (4')$$

formula bo'yicha berilgan integralni quyidagicha yozib olamiz:

$$I = \int_0^1 \left[\int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dx \right] dy,$$

bu yerda avval ichki integralni hisoblasak,

$$\int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dx = \frac{1}{2} \int_0^1 \frac{d(1+x^2+y^2)}{(1+x^2+y^2)^{3/2}} =$$

$$= -\frac{1}{(1+x^2+y^2)^{1/2}} \Big|_0^1 = \frac{1}{\sqrt{1+y^2}} - \frac{1}{\sqrt{2+y^2}},$$

shuning uchun

$$I = \int_0^1 \left(\frac{1}{\sqrt{1+y^2}} - \frac{1}{\sqrt{2+y^2}} \right) dy = ln \frac{y+\sqrt{1+y^2}}{y+\sqrt{2+y^2}} \Big|_0^1 =$$

$$= ln \frac{2+\sqrt{2}}{1+\sqrt{3}} - ln \frac{1}{\sqrt{2}} = ln \frac{2+2\sqrt{2}}{1+\sqrt{3}}.$$

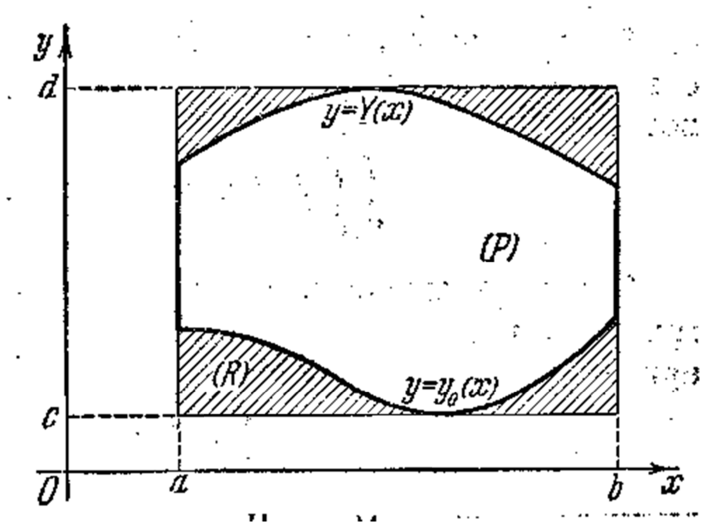
Shunday qilib,

$$\int_0^1 \int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}} dx dy = ln \frac{2+2\sqrt{2}}{1+\sqrt{3}}.$$

2. Egri chiziqli soha bo'lgan holda ikki karrali integralni takroriy integralga keltirish. (P) soha, quyidan va yuqoridan ikkita

$$y = y_0(x), \qquad y = Y(x)\ (a \le x \le b)$$

uzluksiz chiziqlar bilan, yon tomondan – ikkita $x = a$ va $x = b$ ordinatalar bilan chegaralangan bo'lsin (1-rasm)



1-rasm

Quyidagi teorema o'rinli.

T e o r e m a. Agar (P) sohada aniqlangan $f(x,y)$ funksiya uchun,

$$\iint_{(P)} f(x,y)dP$$

ikki karrali integral va x ning $[a,b]$dagi har bir o'zgarmas qiymatida

$$I(x) = \int_{y_0(x)}^{Y(x)} f(x,y)dy$$

oddiy integral mavjud bo'lsa, u holda

$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x,y)dy$$

takroriy integral ham mavjud bo'ladi va ushbu

$$\iint_{(P)} f(x,y)dP = \int_a^b dx \int_{y_0(x)}^{Y(x)} f(x,y)dy \tag{6}$$

tenglik bajariladi.

Bu teorema 1-punktda keltirilgan holga keltirish bilan isbotlanadi.

Agar (P) soha boshqa ko'rinishdagi egri chiziqli trapetsiyadan iborat va

$$x = x_0(y), \qquad x = X(y)\ (c \leq y \leq d)$$

chiziqlar va $y = c,\ y = d$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, u holda (6) ning o'rniga

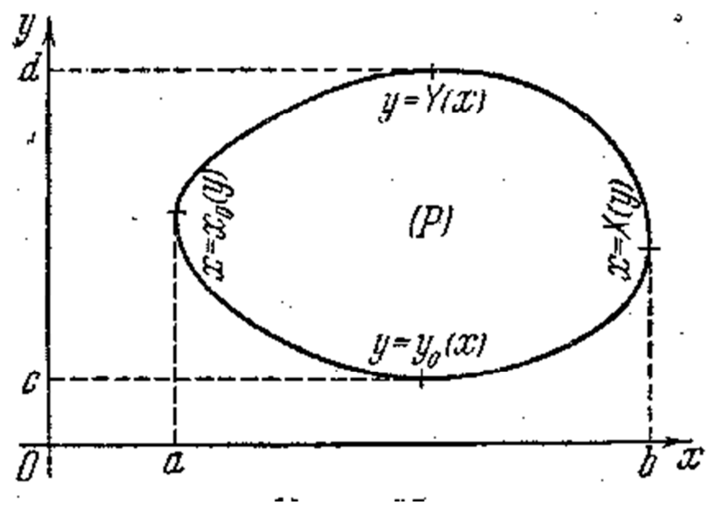
$$\iint_{(P)} f(x,y)dP = \int_c^d dy \int_{x_0(y)}^{X(y)} f(x,y)dx \ , \qquad (6')$$

bunda ikki karrali integral bilan birgalikda, $y = const$ da x bo'yicha oddiy integral mavjud deb faraz qilinadi.

E s l a t m a. Agar (P) soha konturi ordinatalar o'qiga parallellar kabi, abtsissalar o'qiga parallellar bilan ikkita nuqtada kesishsin. U holda

$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x,y)dy = \int_c^d dy \int_{x_0(y)}^{X(y)} f(x,y)dx$$

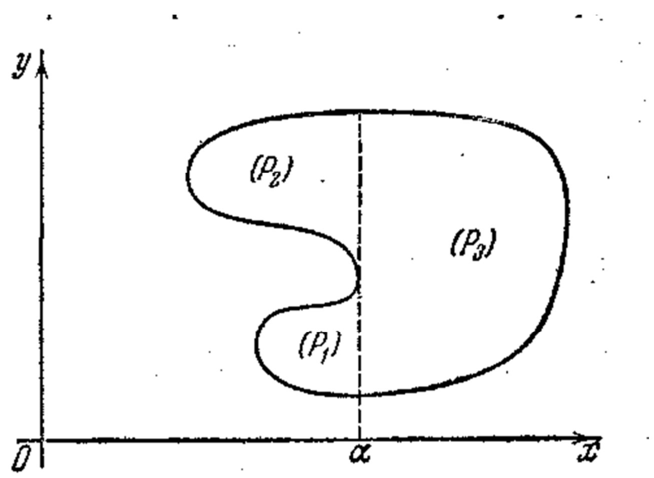
tenglik hosil bo'ladi. Bu – 1-p.dagi (5) formulaga o'xshash formuladir.



2-rasm

Agar $f(x,y)$ funksiya (P) sohada uzluksiz bo'lsa, u holda ikki karrali va oddiy integrallar mavjud, va (6) yoki (6') formulani, (P) sohaning turiga qarab, ikki karrali integralni hisoblashga qo'llash mumkin.

(P) soha murakkab kontur bo'lgan holda uni chekli sondagi qismlarga yoyiladi. Masalan, (P) figurani $x = \alpha$ to'g'ri chiziq uchta (P_1) , (P_2) va (P_3) qismlarga ajratsin (3- rasm). U holda izlangan integral bu qismlar bo'yicha olingan integrallarni yig'indisini ifodalaydi.



3-rasm

3. Ikki karrali integrallarni hisoblashga doir misollar.1) Quyidagi

$$I = \iint\limits_{(P)} e^{-y^2}\, dxdy$$

ikki karrali integralni hisoblaymiz, bu yerda
$(P) = \{(x,y) \in R^2 : 0 \le x \le y, 0 \le y \le 1\}$.

Y e ch i sh. (4') tenglikka asosan

$$I = \int\limits_0^1 \left[\int\limits_0^y e^{-y^2} dx \right] dy$$

bo'ladi, bu yerda o'ng tomondagi integrallarni hisoblasak,

$$\int_0^y e^{-y^2} dx = ye^{-y^2},$$

$$\int_0^1 ye^{-y^2} dy = \frac{1}{2}\int_0^1 e^{-y^2} d(y^2) = -\frac{1}{2}e^{-y^2}|_0^1 = -\frac{1}{2}e^{-1} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

Shunday qilib, berilgan integralning qiymati:

$$I = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

2) $(P) = \{(x,y) \in R^2 : 0 \le x \le 1, 0 \le y \le 1 - x\}$ bo'lsin. U holda

$$I = \iint\limits_{(P)} xy\, dxdy$$

ikki karrali integralni hisoblaymiz.

Y e ch i sh. (4) ga asosan

$$I=\int_0^1 x\left[\int_0^{1-x} ydy\right]dx,$$

bu yerda

$$\int_0^{1-x} ydy = y^2|_0^{1-x} = (1-x)^2$$

bo'lgani uchun,

$$\int_0^1 x\left[\int_0^{1-x} ydy\right]dx = \int_0^1 x(1-x)^2 dx = \int_0^1 (x-2x^2+x^3)dx =$$

$$=\left(\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{2}\right)\Big|_0^1 = \frac{1}{24}$$

Demak,

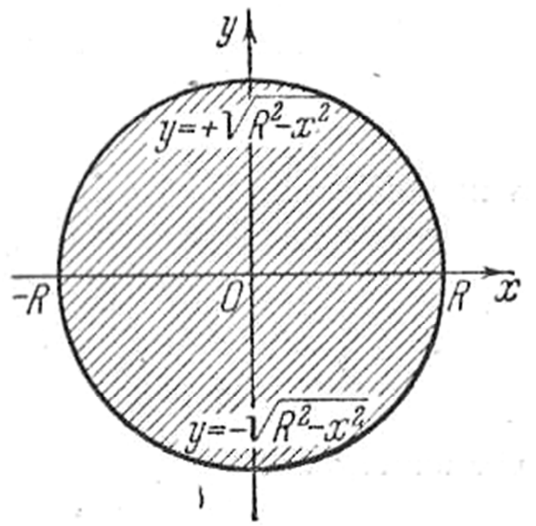
$$I=\frac{1}{24}.$$

3) Quyidagi

$$I=\iint\limits_{(P)} y^2\sqrt{R^2-x^2}\,dP$$

integralni qaraymiz, bu yerda (P) soha markazi koordinatalar boshida bo'lgan R radiusli doira(4-rasm)

Y e ch i sh. (P) soha konturining tenglamasi: $x^2+y^2=R^2$, bu yerdan $y=\pm\sqrt{R^2-x^2}$. Ravshanki, $y=+\sqrt{R^2-x^2}$ yuqori yarim aylananing tenglamasi, $y=-\sqrt{R^2-x^2}$ esa quyi yarim aylana tenglamasi bo'ladi. Demak, o'zgarmas $x\in[-R,R]$ da y o'zgaruvchi $-\sqrt{R^2-x^2}$ dan $+\sqrt{R^2-x^2}$ gacha o'zgaradi.



4-rasm

(6) formulaga ko'ra, integral ostidagi funksiya y bo'yicha juft funksiya ekanini hisobga olib, quyidagini hosil qilamiz

$$I=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}y^2\sqrt{R^2-x^2}dy=$$

$$=2\int_{-R}^{R}\sqrt{R^2-x^2}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}y^2dy$$

Endi ichki integralni hisoblaymiz:

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}y^2dy=\frac{1}{3}(R^2-x^2)^{\frac{3}{2}}$$

Keyin – juftlikni hisobga olib,

$$I=\frac{2}{3}\int_{-R}^{R}(R^2-x^2)^2dx=\frac{4}{3}\int_{0}^{R}(R^2-x^2)^2dx$$

yoki

$$I=\frac{4}{3}\int_{0}^{R}(R^4-2R^2x^2+x^4)dx=\frac{4}{3}\left(R^5-\frac{2}{3}R^5+\frac{1}{5}R^5\right)=\frac{32}{45}R^5.$$

(6') formula bo'yicha hisoblash, xuddi shunga o'xshash olib boriladi.

4) $y=0, x=1,\ y=x$ to'g'ri chiziqlar bilan hosil qilingan uchburchak soha bo'yicha ushbu

$$I = \iint \sqrt{4x^2 - y^2}\, dxdy$$

integralni hisoblaymiz.

Yechish. (6) formula bo'yicha

$$I = \int_0^1 dx \int_0^x \sqrt{4x^2 - y^2} dy$$

bo'lib, ichki integral quyidagiga teng bo'ladi

$$\int_0^x \sqrt{4x^2 - y^2} dy = \left(\frac{y}{2}\sqrt{4x^2 - y^2} + 2x^2 arcsin\frac{y}{2x}\right)\Big|_{y=0}^{y=x} =$$

$$= \frac{x}{2}\sqrt{3x^2} + 2x^2 arcsin\frac{1}{2} = \frac{\sqrt{3}}{2}x^2 + \frac{\pi}{3}x^2 = \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)x^2$$

va nihoyat

$$I = \int_0^1 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)x^2 dx = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right).$$

(6') formuladan ham foydalanib, hisoblashlarni bajarish mumkin edi, lekin bu holda nisbatan murakkab integrallarni hisoblashga to'g'ri keladi.

II-Bob. Uch karrali va ko'p karrali integrallar

2.1-§. Uch karrali integrallar

1.Ikki o'zgaruvchili funksiya uchun Riman integrali tushunchasi bilan oldingi paragraflarda o'rganib chiqdik. Endi mazkur paragrafda xuddi shunga o'xshash uch o'zgaruvchili funksiya uchun ham bu tushunchani kiritamiz. Ikki karrali integralda keltirilgan barcha mulohazalar uch karrali integral uchun ham qaytariladi, ya'ni integrallash sohasining bo'linishini olish, bo'laklarda ixtiyoriy nuqta tanlab olib, integral yig'indini tuzish va boshqalar.

$(V)-\ R^3$ fazodagi biror chegaralangan, hajmga ega bo'lgan soha bo'lsin. Bu sohada berilgan $f(x,y,z)$ funksiyani qaraymiz. (V) sohaning P bo'linishini va bu bo'linishning har bir $(V_k), k=\overline{1,n}$ bo'lagida ixtiyoriy (ξ_k, η_k, ζ_k) nuqtani olamiz va $f(x,y,z)$ funksiyaning integral yig'indisi yoki Riman yig'indisi deb ataluvchi ushbu

$$\sigma=\sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\cdot V_k$$

yig'indini tuzamiz, bu yerda V_k -(V_k) ning hajmi.

(V) sohaning shunday

$$P_1, P_2, \dots, P_m, \dots \tag{1}$$

bo'linishlarini qaraymizki, bu bo'linishlarning diametrlaridan iborat quyidagi

$$\lambda_{P_1}, \lambda_{P_2}, \dots, \lambda_{P_m}, \dots$$

ketma-ketlik nolga intilsin: $\lambda_{P_m}\to 0$. Endi har bir P_m bo'linishlarga nisbatan quyidagi

$$\sigma_m=\sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\cdot V_k$$

integral yig'indini tuzamiz va

$$\sigma_1, \sigma_2, \dots, \sigma_m, \dots \tag{2}$$

ketma-ketlikni hosil qilamiz.

1-*T a'ri f.* Agar (V) sohaning har qanday (1)-$\{P_m\}$ bo'linishlar ketma-ketligi olinganda ham, unga mos $\{\sigma_m\}$ - integral yig'indi qiymatlaridan iborat ketma-ketlik (ξ_k, η_k, ζ_k) nuqtalarni tanlab olinishiga bog'liq bo'lmagan holda bitta I songa intilsa, bu I son σ yig'indining limiti deyiladi:

$$\lim_{\lambda_P \to 0} \boldsymbol{\sigma} = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot V_k = \boldsymbol{I}.$$

2-*T a'ri f.* Agar $\lambda_P \to 0$ da σ - $f(x,y,z)$ funksiyaning integral yi g'indisi chekli I limitga ega bo'lsa, u holda $f(x,y,z)$ funksiya (V) sohada Riman ma'nosida integrallanuvchi deyiladi va I -son $f(x,y,z)$ funksiyaning (V) soha bo'yicha uch karrali integrali(Riman integrali) deb ataladi va u quyidagicha belgilanadi:

$$\iiint\limits_{(V)} f(x,y,z)dV.$$

Shunday qilib,

$$\iiint\limits_{(V)} f(x,y,z)dV = \lim_{\lambda_P \to 0} \boldsymbol{\sigma} = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot V_k.$$

2. Faraz qilamiz, (V) sohada aniqlangan $f(x,y,z)$ funksiya, shu sohada chegaralangan bo'lsin, ya'ni

$$m \leq f(x,y,z) \leq M, \forall (x,y,z) \in V.$$

$(P) - (V)$ sohaning bo'linishlar to'plami bo'lsin. Bun to'plamning har bir bo'linishiga nisbatan $f(x,y,z)$ funksiyaning Darbu yig'indilarini tuzamiz:

$$s_p(f) = \sum_{i=1}^{n} m_k V_k \,, S_p(f) = \sum_{i=1}^{n} M_k V_k.$$

Ko'rinib turibdiki, $\{s_p(f); S_p(f)\}$ to'plamlar chegaralangan.

3-*T a'ri f.* $\{s_p(f)\}$ va $\{S_p(f)\}$ to'plamlarning mos ravishda aniq yuqori va aniq quyi chegarasi $f(x,y,z)$ funksiyaning quyi va yuqori uch karrali integrali deb ataladi: quyi uch karrali integral

$$\underline{I} = \underline{\iiint_{(V)}} f(x,y,z)dV$$

va yuqori uch karrali integral

$$\overline{I} = \overline{\iiint_{(V)}} f(x,y,z)dV$$

kabi belgilanadi.

4-*T a'ri f.* Agar $f(x,y,z)$ funksiyaning quyi va yuqori uch karrali integrallari bir-biriga teng bo'lsa, u holda $f(x,y,z)$ funksiya (V) sohada integrallanuvchi deyiladi va ularning umumiy qiymati

$$I = \underline{\iiint_{(V)}} f(x,y,z)dV = \overline{\iiint_{(V)}} f(x,y,z)dV$$

bu funksiyaning uch karrali integrali (Riman integrali)deb ataladi:

$$\iiint_{(V)} f(x,y,z)dV = \underline{\iiint_{(V)}} f(x,y,z)dV = \overline{\iiint_{(V)}} f(x,y,z)dV.$$

Teorema (uch karrali integralning mavjudligi haqida). $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lishi uchun $\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$

topilib, (V) sohaning diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'linishga nisbatan Darbu yig'indilari

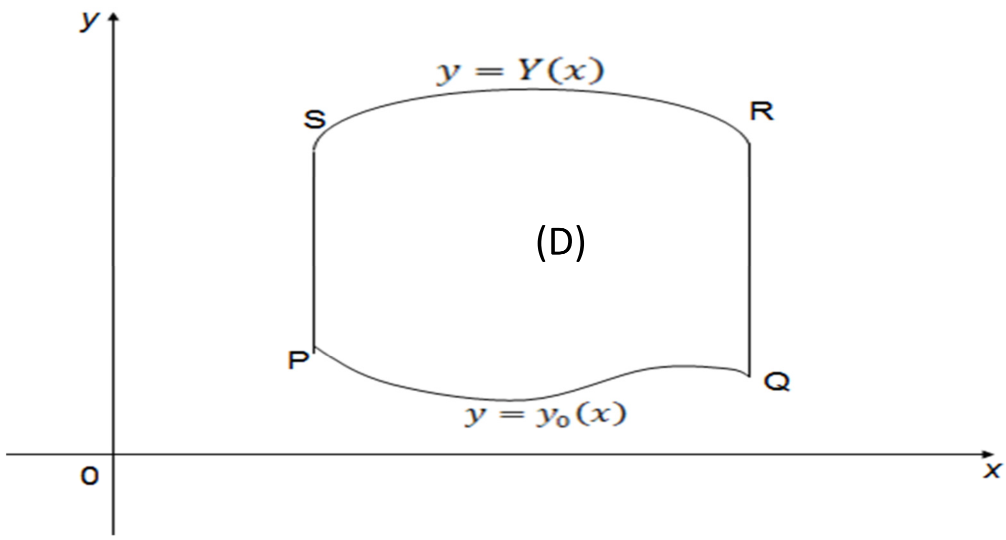
$$S_p(f) - s_p(f) < \varepsilon$$

tengsizlikni qanoatlantirishi zarur va yetarli.

2.2-§. Grin, Stoks va Ostrogsadskiy formulalari

I. Grin formulasi. Soha bo'yicha olingan ikki karrali integralni shu soha chegarasi bo'yicha olingan egri chiziqli integral bilan bog'lovchi formula Grin formulasidir.

1. $(PQ): y = y_0(x)(a \leq x \leq b)$ va $(SR): y = Y(x)(a \leq x \leq b)$ egri chiziqlar va y- o'qiga parallel ikkita PS va QR kesmalardan iborat (L) kontur bilan chegaralangan (D) – egri chiziqli trapetsiyadan iborat sohani qaraymiz (1- rasm).



1-rasm

Faraz qilamiz, (D) sohada $P(x, y)$ funksiya berilgan bo'lib, u bu sohada o'zining $\frac{\partial P}{\partial y}$ hosilasi bilan birgalikda uzluksiz bo'lsin.

Endi quyidagi ikki karrali integralni hisoblaymiz:

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy$$

I-b.,1.3-§ da berilgan (6) formulaga ko'ra

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy = \int\limits_{a}^{b} dx \int\limits_{y_0(x)}^{Y(x)} \frac{\partial P}{\partial y} dy.$$

Bu yerda ichki integral $P(x,y)$ boshlang'ich funksiya yordamida oson hisoblanadi:

$$\int\limits_{y_0(x)}^{Y(x)} \frac{\partial P}{\partial y} dy = P(x,y)|_{y=y_0(x)}^{y=Y(x)} = P\big(x,Y(x)\big) - P\big(x,y_0(x)\big).$$

Shunday qilib,

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy = \int\limits_a^b P\big(x,Y(x)\big)dx - \int\limits_a^b P\big(x,y_0(x)\big)dx.$$

Bu yerda endi har ikki integrallarni egri chiziqli integrallar bilan almashtirish mumkin. I-b.1.1-§ da berilgan (12) formulaga asosan

$$\int\limits_a^b P\big(x,Y(x)\big)dx = \int\limits_{(SR)} P(x,y)dx,$$

$$\int\limits_a^b P\big(x,y_0(x)\big)dx = \int\limits_{(PQ)} P(x,y)dx.$$

Bu yerdan

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy = \int\limits_{(SR)} P(x,y)dx - \int\limits_{(PQ)} P(x,y)dx =$$

$$= \int\limits_{(SR)} P(x,y)dx + \int\limits_{(QP)} P(x,y)dx.$$

(D) sohaning butun (L) konturi bo'yicha integralni hosil qilish uchun, olingan tenglikning o'ng tomoniga yana quyidagi

$$\int_{(PS)} P(x,y)dx \quad \text{va} \quad \int_{(RQ)} P(x,y)dx$$

integrallarni qo'shamiz. Ko'rinib turibdiki, (PS) va (RQ) kesmalar x – oqiga perpendikulyar bo'lgani uchun, bu integrallar nolga teng. Shunday qilib,

$$\iint_{(D)} \frac{\partial P}{\partial y} dxdy =$$

$$= \int_{(PS)} P(x,y)dx + \int_{(SR)} P(x,y)dx + \int_{(RQ)} P(x,y)dx + \int_{(QP)} P(x,y)dx.$$

Bu tenglikning o'ng tomoni (D) sohani chegarasi bo'lgan butun yopiq (L) kontur bo'yicha integralni ifodalaydi, lekin manfiy yo'nalishda. Demak, olingan formulani quyidagicha yozish mumkin:

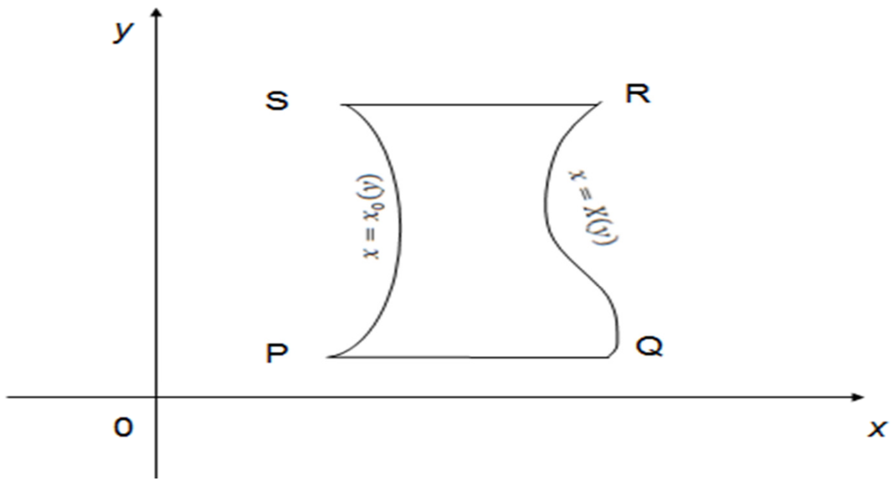
$$\iint_{(D)} \frac{\partial P}{\partial y} dxdy = -\int_{(L)} P(x,y)dx. \qquad (1)$$

Xuddi shunga o'xshash

$$\iint_{(D)} \frac{\partial Q}{\partial x} dxdy = \int_{(L)} Q(x,y)dy \qquad (2)$$

formula o'rinli, bu yerda Q funksiya (D) sohada o'zining $\frac{\partial Q}{\partial x}$ xususiy hosilasi bilan birgalikda uzluksiz deb faraz qilinadi. Bunda avval (D) soha sifatida 2-rasmda tasvirlangan egri chiziqli trapetsiya olinadi. U

$(PS): x = x_0(y)(c \le y \le d)$ va $(QR): x = X(y)(c \le y \le d)$ egri chiziqlar va x-o'qiga parallel ikkita PQ va SR kesmalar bilan chegaralangan (2-rasm).



2-rasm

Keyin formula, xuddi yuqoridagidek, x – o'qiga parallel to'g'ri chiziqlarga, bu ko'rinishdagi chekli sondagi egri chiziqli trapetsiyalarga yoyiladigan soha holida umumlashtiriladi.

Nihoyat, agar (D) soha bir vaqtda ikkala shartlarni qanoatlantirsa, ya'ni chekli sondagi birinchi turdagi egri chiziqli trapetsiyalarga yoyilgani kabi, chekli sondagi ikkinchi turdagi egri chiziqli trapetsiyalarga yoyilsa, u holda bu soha uchun har ikki (1) va (2) formulalar o'rinli, albatta, bunda P, Q va ularning $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ hosilalari uzluksiz deb faraz qilinadi. (2) dan (1) formulani ayirib, quyidagini olamiz

$$\iint\limits_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int\limits_{(L)} Pdx + Qdy. \qquad (3)$$

Bu esa *Grin formulasi* deyiladi.

Eslatma. Grin formulasi bir yoki bir nechta bo'lakli-silliq konturlar bilan chegaralangan, ixtiyoriy (D) soha uchun o'rinli.

2.Egri chiziqli integrallar yordamida yuzani ifodalash. Yuzani hisoblashda Grin formulasining tadbig'ini o'rganamiz.

Agar (3) formulada P va Q funksiyalarni shunday tanlansaki, bunda

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ ifoda 1 ga teng bo'lsa, u holda ikki karrali integral (D) figuraning D yuzasiga keltiriladi, va biz figuraning (L) konturi bo'yicha olingan, egri chiziqli integral yordamida bu yuzaning ifodasini olamiz. Demak, $Q = x, P = 0$ deb olib, quyidagini olamiz

$$D = \int_{(L)} xdy. \tag{4}$$

$Q = 0, P = -y$ bo'lganda

$$D = -\int_{(L)} ydx. \tag{5}$$

$Q = \frac{1}{2}x, P = -\frac{1}{2}y$ bo'lgan holda, formula nisbatan qulaydir:

$$D = \frac{1}{2}\int_{(L)} xdy - ydx. \tag{6}$$

Misol. Yarim o'qlari a va b bo'lgan ellipsning yuzini toping.

Ellipsning parametrik tenglamalarini qo'llaymiz:

$$x = acost, \quad y = bsint \ (0 \le t \le 2\pi).$$

(6) formulaga ko'ra,

$$D = \frac{1}{2}\int_0^{2\pi} xdy - ydx = \frac{1}{2}\int_0^{2\pi} acost\ bcostdt - bsint\ (-asint)dt =$$

$$= \frac{ab}{2}\int_0^{2\pi} (cos^2 t + sin^2 t)dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

II. Stoks formulasi. Mazkur punktda Grin formulasining umumlashmasi bo’lgan sirt integrali bilan egri chiziqli integralni bog’lovchi formulani keltirib chiqaramiz.

Faraz qilamiz, (S) - sirt silliq va karrali nuqtalarga ega bo’lmasin: U bo’lakli silliq (L) kontur bilan chegaralangan bo’lsin.

(S) sirtni o’z ichiga oluvchi biror fazoviy sohada $P(x, y, z)$ funksiya berilgan bo’lib, u bu sohada o’zining xususiy hosilalari bilan uzluksiz bo’lsin. U holda quyidagi

$$\int_{(L)} Pdx = \iint_{(S)} \frac{\partial P}{\partial z} dzdx - \frac{\partial P}{\partial y} dxdy, \tag{1}$$

formula o’rinli.

Avval (L) chiziq bo’yicha egri chiziqli integralni (Δ) chiziq bo’yicha interalga almashtiramiz:

$$\int_{(L)} Pdx = \int_{(\Delta)} P \cdot \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right). \tag{2}$$

Bu tenglikni (Δ) chiziqni ushbu

$$u = u(t), v = v(t) \ (\alpha \le t \le \beta)$$

parametric ifodasini, u orqali esa - (L) chiziqnikini

$$x = x\big(u(t), v(t)\big), \quad y = y\big(u(t), v(t)\big), \qquad z = z\big(u(t), v(t)\big)$$

kiritib, oson tekshirish mumkin. U holda ikkala integral bitta o’sha parameter bo’yicha oddiy integralga keladi:

$$\int_{\alpha}^{\beta} P \cdot \left(\frac{\partial x}{\partial u} \cdot \frac{du}{dt} + \frac{\partial x}{\partial v} \cdot \frac{dv}{dt}\right) dt.$$

Endi (2) ni o'ng tomonidagi integralga Grin formulasini qo'llaymiz:

$$\int\limits_{(\Delta)} P\cdot\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)=\iint\limits_{(\Delta)}\left\{\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right)-\frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)\right\}dudv\,. \tag{3}$$

Oxirgi integral ostidagi ifodadan qyuidagini olamiz:

$$\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right)-\frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)=$$

$$=\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial P}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v}+P\frac{\partial^2 x}{\partial v\partial u}-$$

$$-\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial P}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u}-P\frac{\partial^2 x}{\partial v\partial u}=$$

$$=\frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)-\frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right).$$

Endi buni (3) tenglikka qo'ysak, ushbu ikki karrali integralga kelamiz:

$$\iint\limits_{(\Delta)}\left\{\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right)-\frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)\right\}dudv$$

$$=\iint\limits_{(\Delta)}\left\{\frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)-\frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right\}dudv=$$

$$=\iint\limits_{(\Delta)}\left\{\frac{\partial P}{\partial z}B-\frac{\partial P}{\partial y}C\right\}dudv. \tag{4}$$

Ushbu

$$\iint\limits_{(S)} Pdydz+Qdzdx+Rdxdy=$$

$$= \iint\limits_{(S)} (Pcos\alpha + Qcos\beta + Rcos\gamma)dS,$$

bu yerda (S) sirt tomoniga mos yo’naltiruvchi kosinuslar, formula ikkinchi va birinchi tur sirt integrallarini bog’lovchi umumiy formula bo’lib, bizga ma’lumki, sirtning tanlangan tomonini xarakterlovchi, yonaltiruvchi kosinuslar, quyidagi formulalar orqali aniqlanadi

$$cos\alpha = \frac{A}{+\sqrt{A^2 + B^2 + C^2}}, \qquad cos\beta = \frac{B}{+\sqrt{A^2 + B^2 + C^2}},$$

$$cos\gamma = \frac{C}{+\sqrt{A^2 + B^2 + C^2}}.$$

Boshqa tomondan u, v parametrlar bo’yicha ikki karrali integralga o’tishda, dS elementni $+\sqrt{A^2 + B^2 + C^2}dudv$ ifoda bilan almashtiriladi. Nihoyat, ushbu

$$\iint\limits_{(S)} Pdydz + Qdzdx + Rdxdy = \iint\limits_{(S)} (AP + BQ + CR)dudv. \qquad (4')$$

O’ng tomonda, P, Q, R funksiyalarda x, y, z o’rniga ularning u, v orqali ifodalari qo’yilgan deb faraz qilinadi.

(4’) formulaga asosan,

$$\iint\limits_{(\Delta)} \left\{\frac{\partial P}{\partial z} B - \frac{\partial P}{\partial y} C\right\} dudv$$

ikki karrali integralni sirtni tanlangan tomoni bo’yicha olingan

$$\iint\limits_{(S)} \frac{\partial P}{\partial z} dzdx - \frac{\partial P}{\partial y} dxdy$$

sirt integraliga oson almashtirish mumkin. Shu bilan (1) tenglik isbotlandi.

Xuddi shunga o'xshash, quyidagi tengliklarni olamiz:

$$\int\limits_{(L)} Qdy = \iint\limits_{(S)} \frac{\partial Q}{\partial x} dxdy - \frac{\partial Q}{\partial z} dydz, \tag{1'}$$

$$\int\limits_{(L)} Rdz = \iint\limits_{(S)} \frac{\partial R}{\partial y} dydz - \frac{\partial R}{\partial x} dzdx, \tag{1''}$$

bu yerda $Q, R - x, y, z$ ga bog'liq yangi funksiyalar bo'lib, ular P funksiyaga qo'yilgan shartlarni qanoatlantiradi.

(1), $(1')$ va $(1'')$ uchala tengliklarni qo'shib, quyidagi nisbatan umumiy ko'rinishdagi formulani olamiz:

$$\int\limits_{(L)} Pdx + Qdy + Rdz =$$

$$= \iint\limits_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx. \tag{5}$$

Bu tenglik **Stoks formulasi** deyiladi.

Agar (S) sirtning bo'lagi sifatida xy tekislikdagi (D)soha olinsa,

$z = 0$ bo'lib, u holda quyidagi formula hosil qilinadi

$$\int\limits_{(L)} Pdx + Qdy =$$

$$= \iint\limits_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

bu esa ma'lumki, Grin formulasidir. Shunday qilib, oxirgi formula Stoks formulasining xususiy holidan iborat.

Nihoyat, Stoks formulasida ikkinchi tur sirt integrali birinchi tur sirt integrali bilan almashtirlishi mumkin. U holda bu formula quyidagi

$$\int\limits_{(L)} Pdx + Qdy + Rdz =$$

$$= \iint\limits_{(S)} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) cos\gamma\right] dS,$$

ko'rinishga ega bo'lib, $cos\alpha, cos\beta, cos\gamma$ sirtni tanlangan tomoniga mos normalning yo'naltiruvchi kosinuslari.

Shunday qilib, Stoks formulasi (S) sirt bo'yicha olingan II-tur sirt integrali bilan shu sirtning chegarasi bo'yicha olingan egri chiziqli integralni bog'lovchi formuladir.

Stoks formulasini qo'llashga misol keltiramiz.

Misol. $P = y^2 + z^2,\ Q = z^2 + x^2,\ R = x^2 + y^2$ bo'lsin. (S) sirt sifatida

$$x^2 + y^2 + z^2 = 2Rx (R > r, z > 0)$$

sferadan

$$x^2 + y^2 = 2rx$$

silindr bilan kesilgan olamiz.

Egri chiziqni ushbu

$$x = r(1 + cost), y = rsint, z = \sqrt{2r(R - r)}\sqrt{1 + cost} (0 \le t \le 2\pi)$$

parametrik ifodasiga o'tib, egri chiziqli integral uchun oddiy integral ko'rinishdagi yetarlicha murakkab ifodani topamiz:

$$I = \int_0^{2\pi} \left\{ [r^2 sin^2 t + 2r(R-r)(1+cost)](-rsint) \right.$$
$$+ [2r(R-r)(1+cost) + r^2(1+cost)^2]rcost$$
$$\left. + [r^2(1+cost)^2 + r^2 sin^2 t] \cdot \frac{1}{2}\sqrt{\frac{2r(R-r)}{1+cost}}(-sint) \right\} dt.$$

Figurali qavslardagi dt ga ko'paytirilgan 1- va 3- qo'shiluvchilar $f(cost)dcost$ ko'rinishga ega bo'lib, ulardan olingan integral kosinusni davriyligiga asosan, nolga teng:

$$\int_0^{2\pi} \left\{ [r^2 sin^2 t + 2r(R-r)(1+cost)](-rsint) + [r^2(1+cost)^2 + r^2 sin^2 t] \right.$$
$$\left. \cdot \frac{1}{2}\sqrt{\frac{2r(R-r)}{1+cost}}(-sint) \right\} dt =$$

$$= \int_0^{2\pi} \left\{ [r^2(1-cos^2 t) + 2r(R-r)(1+cost)]r \right.$$
$$\left. + [r^2(1+cost)^2 + r^2(1-cos^2 t)] \cdot \frac{1}{2}\sqrt{\frac{2r(R-r)}{1+cost}} \right\} d(cost) = 0,$$

ikkinchi integral esa

$$\int_0^{2\pi} [2r(R-r)(1+cost) + r^2(1+cost)^2 rcost] dt = 2\pi R r^2.$$

Shunday qilib,

$$I = 2\pi R r^2.$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=x-y, \qquad \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=y-z, \quad \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=z-x$$

ekanini hisoba olib, quyidagi

$$2\iint\limits_{(S)}(x-y)dxdy+(y-z)dydz+(z-x)dzdx$$

2- tur sirt integralini avval 1-tur integralga almashtiramiz:

$$2\iint\limits_{(S)}(x-y)dxdy+(y-z)dydz+(z-x)dzdx=$$

$$=2\iint\limits_{(S)}[(y-z)cos\alpha+(z-x)cos\beta+(x-y)cos\gamma]dS.$$

$$cos\alpha=\frac{x-R}{R}, \qquad cos\beta=\frac{y}{R}, \qquad cos\gamma=\frac{z}{R}$$

bo’lgani uchun, u holda bu ifodalarni o’rniga qo’yib, keying qisqartirishlarni bajaramiz va quyidagi ko’rinishdagi integralga kelamiz:

$$2\iint\limits_{(S)}[(y-z)cos\alpha+(z-x)cos\beta+(x-y)cos\gamma]dS=$$

$$=2\iint\limits_{(S)}\left[(y-z)\frac{x-R}{R}+(z-x)\frac{y}{R}+(x-y)\frac{z}{R}\right]dS=$$

$$=2\iint\limits_{(S)}(z-y)dS.$$

Sirtni xz tekislikka nisbatan simmetikligiga ko’ra,

$$\iint_{(S)} ydS = 0.$$

Qolgan integralni yana 2-tur integralga almashtiramiz:

$$2\iint_{(S)} (z-y)dS = 2\iint_{(S)} zdS = 2R\iint_{(S)} dxdy = 2\pi Rr^2.$$

III. Ostrogradskiy formulasi. Bizga ma'lumki, ikki karrali integrallar nazariyasidagi muhim formula, R^2tekislikdagi soha bo'yicha olingan ikki karrali integral bilan soha konturi bo'yicha egri chiziqli integralni bog'lovchi Grin formulasining analogi, uch karrali integrallar nazariyasidagi formula Ostrogradskiy formulasi bo'lib, u fazoviy soha bo'yicha uch karrali integralni, soha chegarasi bo'yicha sirt integrali bilan bog'lovchi formuladir.

(V)-jismni qaraymiz, u "silindrik g'o'la" ni ifodalab, quyidan va yuqoridan, mos ravishda

$$(S_1) \quad z = z_0(x,y)$$

$$(S_2) \quad z = Z(x,y) \quad (z_0 \leq Z)$$

sirtlar bilan chegaralangan bo'lib, xy tekislikka yuzasi nolga teng bo'lakli – silliq yopiq (K) chiziq bilan chegaralangan biror (D) sohaga proeksiyalangan; yon tomondan (V)-jism, tashkil qiluvchilari z o'qiga parallel bo'lgan (S_3) silindrik sirt bilan chegaralangan.

Faraz qilamiz, (V) sohada biror $R(x,y,z)$ funksiya aniqlangan bo'lib, u o'zining $\frac{\partial R}{\partial z}$ hosilasi bilan (V) sohada va chegarasida uzluksiz bo'lsin. U holda quyidagi formula o'rinli

$$\iiint_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint_{(S)} Rdxdy \qquad (1)$$

bo'lib, (S) jismni chegaralovchi sirt, va o'ng tomondagi integral uni yuqori tomoni bo'yicha olingan.

Haqiqatan ham, quyidagi

$$\iiint\limits_{(V)} f(x,y,z)\,dV = \iint\limits_{(D)} dxdy \int\limits_{z_0}^{Z(x,y)} f(x,y,z)dz$$

formulaga asosan

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint\limits_{(D)} dxdy \int\limits_{z_0}^{Z(x,y)} \frac{\partial R}{\partial z} dz =$$

$$= \iiint\limits_{(D)} R(x,y,Z(x,y))\,dxdy - \iiint\limits_{(D)} R(x,y,z_0(x,y))\,dxdy$$

ifodani olamiz.

Agar sirt integrallarini kiritsak, u holda quyidagi

$$\iint\limits_{(S)} f(x,y,z)dxdy = (R) \iint\limits_{(D)} f\big(x,y,z(x,y)\big)dxdy,$$

$$\iint\limits_{(S)} f(x,y,z)dxdy = -(R) \iint\limits_{(D)} f\big(x,y,z(x,y)\big)dxdy$$

formulalarga asosan

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint\limits_{(S_2)} R(x,y,z)dxdy + \iint\limits_{(S_1)} R(x,y,z)dxdy$$

bo'lib, o'ng tomondagi 1-integral (S_2) sirtning yuqori tomoni bo'yicha olingan, 2 - integral esa (S_1) sirtning quyi tomoni bo'yicha olingan integral. Bu tenglikni o'ng tomoniga (S_3) sirtning yuqori tomoni bo'yicha olingan ushbu

$$\iint\limits_{(S_3)} R(x,y,z)dxdy$$

integralni qo'shsak, tenglik o'zgarmaydi, chunki, bu integral nolga teng. Shunday qilib,

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz =$$

$$= \iint\limits_{(S_2)} R(x,y,z)dxdy + \iint\limits_{(S_1)} R(x,y,z)dxdy + \iint\limits_{(S_3)} R(x,y,z)dxdy,$$

va biz Ostrogradskiy formulasini xususiy holini ifodalovchi (1) formulaga kelamiz.

(1)formula, umuman ixtitoriy bo'lakli-silliq sirt bilan chegaralangan jism uchun ham o'rinli.

(1) formulaga o'xshash quyidagi formulalar ham o'rinli:

$$\iiint\limits_{(V)} \frac{\partial P}{\partial x} dxdydz = \iint\limits_{(S)} Pdydz, \tag{2}$$

$$\iiint\limits_{(V)} \frac{\partial Q}{\partial y} dxdydz = \iint\limits_{(S)} Qdzdx, \tag{3}$$

bu yerda P va Q funksiyalar (V) sohada o'zining $\frac{\partial P}{\partial x}$ va $\frac{\partial Q}{\partial y}$ hosilalari uzluksiz.

(1),(2),(3) formulalarni qo'shib, biz umumiy *Ostrogradskiy formulasiga* kelamiz:

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = \iint\limits_{(S)} Pdydz + Qdzdx + Rdxdy. \tag{4}$$

Bu yopiq sirtning yuqori tomoni bo'yicha olingan, 2-tur sirt integralining bu sirt bilan chegaralangan jism bo'yicha olingan uch karrali integral orqali umumiy ko'rinishini ifodalaydi.

Ostrogradskiy formulasini 1-tur sirt orqali ko'rinishini quyidagicha yoziladi:

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz =$$

$$= \iint\limits_{(S)} (Pcosa + Qcos\beta + Rcos\gamma)dS,$$

$\alpha = n\hat{\ }x,\ \beta = n\hat{\ }y, \gamma = n\hat{\ }z$ - koordinata o`qlari bilan(S) sirtga n - tashqi normal orasidagi burchaklar.

E s l a t m a. Grin, Stoks va Ostrogradskiy formulalari biror geometrik shakl bo'yicha integralni, bu shakl chegarasi bo'yicha olingan integral orqali ifodalaydi.

Grin formulasi ikki o'lchamli fazo holiga mos keladi, Stoks formulasi ham – yana ikki o'lchamli, lekin fazo"egri chizig'i " holiga, Ostrogradskiy formulasi esa – uch o'lchamli fazo holiga mos keladi.

Integral hisobning asosiy formulasiga

$$\int_a^b f'(x)dx = f(b) - f(a)\ `$$

biz bu formulalarning - bir o'lchamli fazo uchun biror analogi sifatida qarashimiz mumkin.

2.3-§. Ko'p karrali integrallar

1. m-o'lchovli jism hajmi va m-karrali integral. Analiz va uning tadbiqlari zaririyati o'rganilgan aniq integrallarning oddiy, ikkikarrali va uch karrali tiplari bilan tugallanmaydi.

Oddiy, ikkikarrali va uch karrali integrallarni aniqlashda biz kesma uzunligi, tekis figura yuzasi, fazoviy jism hajmi tushunchalarini qo'llaganimiz kabi, m-karrali integralni ta'riflashda m-o'lchovli sohaning hajmi tushunchasidan foydalanamiz. Sodda m-o'lchovli soha uchun uning hajmi tushunchasini kiritaylik: Quyidagi

$$[a_1, b_1, a_2, b_2, \dots, a_m, b_m] \qquad (1)$$

m-o'lchovli to'g'riburchakli parallelepipedning hajmi deb, uning o'lchamlari ko'paytmasini ataymiz:

$$(b_1 - a_1)(b_2 - a_2) \cdot \dots \cdot (b_m - a_m).$$

Tushunarliki, chekli sondagi shunday parallepiped lardan tuzilgan jism hajmi deb faraz qilinadi. Hajm, jism parallelepipedlarga qanday yoyilganligiga bog'liq emasligini elementar ko'rsatish mumkin.

Bunday m-o'lchovli (V) jismga kiruvchi va undan chiquvchi " parallelepiped" jismlarni qarab, (V) jism uchun V hajmi tushunchasini qurish mumkin. Biz faqat hajmi mavjud bo'lgan jismlarni qaraymiz: hajm silliq yoki bo'lakli-silliq sirtlar bilan chegaralangan jism uchun mavjud bo'ladi, xususan, sodda m-o'lchovli sohalar uchun: m-o'lchovli piramida

$$x_1 \geq 0, x_2 \geq 0, \dots, x_m \geq 0, \qquad x_1 + x_2 + \dots + x_m \leq h$$

va m-o'lchovli sfera

$$x_1^2 + x_2^2 + \dots + x_m^2 \leq r^2$$

uchun, biz quyida ularning hajmini hisoblaymiz.

Bu yerda silliq sirt deb, m-o'lchovli fazodagi m-1 parametrli m parametric tenglamalar bilan aniqlangan sirtni ataymiz, bunda tenglamalarda qatnashgan parametrlarning funksiyasi o’zining xususiy hosilalari bilan uzluksiz, hosilalarning (m-1)-tartibli matritsalarini determinantlari bir vaqtda nolga aylanmasligi kerak.

(V) sohadam o’zgaruvchili $f(x_1, x_2, \dots, x_m)$ funksiya berilgan bo’lsin, u holda , bu sohani elementar qismlarga yoyib va bizga ma’lum bo’lgan amallarni takrorlab, m-karrali integral tushunchasiga kelamiz:

$$I = \underbrace{\int \dots \int}_{(V)}^{m} f(x_1, x_2, \dots, x_m) dx_1 dx_2 \dots dx_m. \qquad (2)$$

Integral ostidagi funksiya uzluksiz bo’lgan holda u mavjud bo’ladi.

Bunday integralni hisoblash kichik karrali integrallardan oddiy integrallargacha hisoblashga keltiriladi. Integrallash sohasi (V) (1) to’g’ri burchakli parallelepipedni ifodalaganda m=3-o’lchovli fazoda o’rinli bo’lgan

$$\iiint_{(V)} f(x_1, x_2, x_3) dV = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \int_{a_3}^{b_3} f(x_1, x_2, x_3) dx_3 \qquad (*)$$

formulaga o’xshash formula o’rinli:

$$I = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \dots \int_{a_m}^{b_m} f(x_1, x_2, \dots, x_m) dx_m. \qquad (3)$$

Ushbu

$$x_1^0 \le x_1 \le X_1 \ , \ x_2^0(x_1) \le x_2 \le X_2(x_1), \dots,$$

$$x_m^0(x_1, \dots, x_{m-1}) \le x_m \le X_m(x_1, \dots, x_{m-1}),$$

tengsizliklar bilan aniqlangan, nisbatan umumiy ko’rinishdagi sohalar uchun

m=3-o’lchovli fazoda o’rinli bo’lgan

$$\iiint_{(V)} f(x_1, x_2, x_3) dV = \int_{x_1^0}^{X_1} dx_1 \int_{x_2^0(x_1)}^{X_2(x_1)} dx_2 \int_{x_3^0(x_1, x_2)}^{X_3(x_1, x_2)} f(x_1, x_2, x_3) dx_3 \qquad (**)$$

formulaga o'xshash formula o'rinli:

$$I = \int\limits_{x_1^0}^{X_1} dx_1 \int\limits_{x_2^0(x_1)}^{X_2(x_1)} dx_2 \dots \int\limits_{x_m^0(x_1,\dots,x_{m-1})}^{X_m(x_1,\dots,x_{m-1})} f(x_1, x_2, \dots, x_m) dx_m.$$

Xuddi shuningdek(sohalarni bo'lmagan mos ko'rinishi uchun, uni har bir holda qurish qiyin emas) m=3-o'lchovli fazoda berilgan

$$\iiint\limits_{(V)} f(x_1, x_2, x_3) dV = \int\limits_{x_1^0}^{X_1} dx_1 \iint\limits_{(P_{x_1})} f(x_1, x_2, x_3) dx_2 dx_3,$$

$$\iiint\limits_{(V)} f(x_1, x_2, x_3) dV = \iint\limits_{(D)} dx_1 \int\limits_{x_3^0(x_1,x_2)}^{X_3(x_1,x_2)} f(x_1, x_2, x_3) dx_3$$

formulalarga o'xshash boshqa formulalar ham o'rinli, bunda m-karrali integral kichik karrali m ta integrallarning ketma-ket hisoblashga keltiriladi. Bu yerda (P_{x_1})???, (D)???.

Bularning barchasi xuddi m=2 yoki m=3 hollar kabi isbotlanadi.

E s l a t m a. Ostrogradskiy 1834 yili birinchi marta alohida $x_1, x_2, \dots, x_m$ o'zgaruvchilar bo'yicha integrallash chegarasini aniqlanadi, unga

$$L(x_1, x_2, \dots, x_m) \leq 0$$

ko'rinishdagi tengsizlikni qanoatlantiruvchi $x_1, x_2, \dots, x_m$ o'zgaruvchilar bo'yicha olingan ko'p karrali integrallarni hisoblash keltirladi.Bu yerda biz ixtiyoriy sondagi o'zgaruvchilar holida va (m-1)-o'lchamli yopiq sirt bo'yicha olingan integral bilan bu sirt chegaralagan jismi bo'yicha olingan m-karrali integralni bog'lovchi bizga ma'lum bo'lgan (m=3-o'lchovli fazoda)

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dx_1 dx_2 dx_3 =$$

$$= \iint\limits_{(S)} P dx_2 dx_3 + Q dx_3 dx_1 + R\, dx_1 dx_2$$

Ostrogradskiy formulasini umumlashtiruvchi formulani topamiz:

2.M i s o l l a r.1) T_m m-o'lchovli piramida hajmini toping:

$$(T_m): x_1 \geq 0, \ldots, x_m \geq 0, x_1 + \cdots + x_m \leq h.$$

Yechish. Quyidagiga egamiz

$$T_m = \int \ldots \int dx_1 dx_2 \ldots dx_m = \int\limits_0^h dx_1 \int\limits_0^{h-x_1} dx_2 \ldots \int\limits_0^{h-x_1-\cdots-x_{m-1}} dx_m$$

(T_m)

Endi bu oddiy integrallarda o'zgaruvchilarni ketma-ket ushbu

$$x_1 = h\mathfrak{z}_1 \ , x_2 = h\mathfrak{z}_2 \ , \ldots, x_m = h\mathfrak{z}_m$$

formulalar bo'yicha almashtirb, va agar α_n orqali berilgan ingeralga o'xshash,lekin h =1 ga mos integralning qiymatini belgilasak, quyidagi natijaga kelamiz

$$T_m = h^m \int\limits_0^1 d\mathfrak{z}_1 \int\limits_0^{1-\mathfrak{z}_1} d\mathfrak{z}_2 \ldots \int\limits_0^{1-\mathfrak{z}_1-\cdots-\mathfrak{z}_{m-1}} d\mathfrak{z}_m =$$

$$= h^m \overbrace{\int \cdots \int}^{m} d\mathfrak{z}_1 \ldots d\mathfrak{z}_m = \alpha_m h^m.$$

$$\mathfrak{z}_1 \geq 0, \ldots, \mathfrak{z}_m \geq 0$$

$$\mathfrak{z}_1 + \cdots + \mathfrak{z}_m \leq 1$$

Boshqa tomondan, quyidagiga ega bo'lamiz(olingan natijani qo'llagan holda)

$$\alpha_m = \int_0^1 d\mathfrak{z}_m \overbrace{\int \cdots \int}^{m-1} d\mathfrak{z}_1 \dots d\mathfrak{z}_{m-1} =$$

$$\mathfrak{z}_1 \geq 0, \dots, \mathfrak{z}_{m-1} \geq 0$$

$$\mathfrak{z}_1 + \cdots + \mathfrak{z}_{m-1} \leq 1 - \mathfrak{z}_m$$

$$= \alpha_{m-1} \int_0^1 (1 - \mathfrak{z}_m)^{m-1} d\mathfrak{z}_m = \frac{\alpha_{m-1}}{m}.$$

Topilgan recurrent munosabat ($\alpha_1 = 1$ ni hisobga olgan holda) bizga quyidagini beradi

$$\alpha_m = \frac{1}{m!},$$

shuning uchun, nihoyat

$$T_m = \frac{h^m}{m!}.$$

2)Quyidagi

$$(V_m): x_1^2 + x_2^2 + \cdots + x_m^2 \leq R^2.$$

m-o'lchovli sferaning V_m hajmini toping.

Yechish.Ushbu integralni hisoblahs talab etiladi:

$$V_m = \overbrace{\int \cdots \int}^{m} dx_1 dx_2 \dots dx_m.$$

$$x_1^2 + x_2^2 + \cdots + x_m^2 \leq R^2$$

Quyidagicha

$$x_1 = R\mathfrak{z}_1 \ , x_2 = R\mathfrak{z}_2 \ , \dots, x_n = R\mathfrak{z}_n$$

deb olsak, u holda

$$V_m = R^m \overbrace{\int \cdots \int}^{m}_{\mathfrak{z}_1^2+\mathfrak{z}_2^2+\cdots+\mathfrak{z}_m^2 \le 1} d\mathfrak{z}_1 d\mathfrak{z}_2 \dots d\mathfrak{z}_m \,,$$

yoki

$$V_m = \beta_m R^m,$$

bu yerda sonly koffitsiyent β_m radiusi 1 ga teng m-o'lchovli sferaning hajmi.

β_m ni aniqlash uchun almashtirish bajaramiz

$$\beta_m = \overbrace{\int \cdots \int}^{m}_{\mathfrak{z}_1^2+\mathfrak{z}_2^2+\cdots+\mathfrak{z}_m^2 \le 1} d\mathfrak{z}_1 d\mathfrak{z}_2 \dots d\mathfrak{z}_m =$$

$$= \int_{-1}^{1} d\mathfrak{z}_m \overbrace{\int \cdots \int}^{m-1}_{\mathfrak{z}_1^2+\mathfrak{z}_2^2+\cdots+\mathfrak{z}_{m-1}^2 \le 1-\mathfrak{z}_m^2} d\mathfrak{z}_1 d\mathfrak{z}_2 \dots d\mathfrak{z}_{m-1}.$$

Ichki integral radiusi $\sqrt{1-\mathfrak{z}_m^2}$ ga teng (m-1)-o'lchovli sferaning hajmi va, ravshanki,u quyidagiga teng:

$$\overbrace{\int \cdots \int}^{m-1}_{\mathfrak{z}_1^2+\mathfrak{z}_2^2+\cdots+\mathfrak{z}_{m-1}^2 \le 1-\mathfrak{z}_m^2} d\mathfrak{z}_1 d\mathfrak{z}_2 \dots d\mathfrak{z}_{m-1} = \beta_{m-1}(1-\mathfrak{z}_m^2)^{\frac{m-1}{2}}.$$

Bu qiymatni o'rniga qo'yib, yana recurrent munosabatga kelamiz

$$\beta_m = 2\beta_{m-1}\int_0^{\frac{\pi}{2}} sin^m\theta d\theta$$

yoki

$$\beta_m = \beta_{m-1} \cdot \sqrt{\pi} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}.$$

$\beta_1 = 2$ bo'lgani uchun, u holda sodda hisoblashdan so'ng

$$\beta_m = \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}. \qquad (5)$$

Haqiqatan ham

$$\Gamma(a+1) = a\Gamma(a), 0 < a \leq 1$$

formulaga asosan,

$$\beta_2 = \beta_1 \cdot \sqrt{\pi} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} = 2\sqrt{\pi} \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)},$$

bu yerda $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(1) = \Gamma(2) = 1$ ekanini hisobga olib,

$$\beta_2 = \frac{\pi}{\Gamma(2)} = \pi$$

ni hosil qilamiz. Xuddi shunday quyidagi

$$\Gamma(a+n) = (a+n-1)(a+n-2) \cdot \ldots \cdot (a+1)a\Gamma(a), (\forall a \ -\text{uchun})$$

$$\Gamma(n+1) = n!$$

formulalarga ko'ra,

$$\beta_3 = \beta_2 \cdot \sqrt{\pi} \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}+1\right)},$$

$$\beta_4 = \beta_3 \cdot \sqrt{\pi}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} = \frac{\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}+1\right)}\sqrt{\pi}\ \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} =$$

$$= \frac{\pi^2}{\Gamma(3)} = \frac{\pi^2}{2},$$

va hakozo,

$$\beta_{m-1} = \frac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}+1\right)}$$

tenglikka ko'ra

$$\beta_m = \beta_{m-1} \cdot \sqrt{\pi}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} = \frac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}+1\right)} \cdot \sqrt{\pi}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} =$$

$$= \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}.$$

Izlangan hajm esa

$$V_m = \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}R^m$$

bo'ladi.

m- juft va toq bo'lgan hollarda ushbu formulalar olinadi:

$$V_{2n} = \frac{\pi^n}{n!}R^{2n}, \quad V_{2n+1} = \frac{2(2\pi)^n}{(2n+1)!!}R^{2n+1}.$$

Xususan, V_1, V_2, V_3 uchun, tabiiyki, bizga ma'lum bo'lgan qiymatlarni topamiz:

$$V_1 = 2R, \quad V_2 = \pi R^2, \quad V_3 = \frac{4}{3}\pi R^3$$

XULOSA

Ko’p o’zgaruvchili funksiyalarning integrallari bilan bog’liq masalalarni hal etishda karrali integrallarni o’rganish matematika va fanning boshqa tarmoqlarida katta ahamiyatga egadir. Karrali integrallar nazariyasida xuddi aniq integrallar nazariyasidagi kabi, karrali integralning mavjudligi, xossalari, ularni hisoblash hamda integralning tadbiqlari egri chiziqli integrallar va aniq integral bo’yicha ma’lumotlar asosida o’rganiladi.

Ushbu bitiruv malakaviy ishida o’zining muhim tadbiqlariga ega bo’lgan karali integrallar nazariyasi haqida o’rganilgan.

Bunda matematik analiz kursidan ma’lum bo’lgan egri chiziqli integrallar, ikki karrali integrallar, uch karrali integrallar va ko’p karrali integrallar ularni hisoblash bo’yicha ma’lumotlar keltirilgan.

Karrali integrallar nazariyasi xususiy hosilali differensial tenglamalar uchun chegaraviy masalalarini o’rganishda ham katta ahamiyatga ega bo’lib, jumladan, karrali integrallarni egri chiziqli integrallar bilan bog’lovchi formulalar Grin, Ostrogradskiy formulalari yordamida tor tebranish tenglamasi, issiqlik tarqalish tenglamasi uchun qo’yilgan chegaraviy masalalarni yechimini aniqlashda keng qo’llaniladi.

Xulosa qilib aytganda, ushbu ishda o’rganilgan karrali integrallar nazariyasi mavzusi amaliy ahamiyatga ega bo’lgan, matematik analiz fanidagi muhim mavzulardan bo’lib, undan universitet fizika, matematika yo’nalishlarida tahsil olayotgan talabalar foydalanishlari mumkin.

Adabiyotlar

1.Фихтенгольц Г.М. Математик анализ асослари, I-том,Тошкент, Ўзбекистон, 1964.

2. Фихтенгольц Г.М. Математик анализ асослари, II-том,Тошкент, Ўзбекистон, 1964.

3.Фихтенгольц Г.М. Курс дифференциального и интегрального исчисления, т.3. –М.,Наука,1966.

4.Азларов Т., Мансуров Х. Математик анализ, I –қисм,Тошкент, Ўқитувчи, 1986.

5.Азларов Т., Мансуров Х. Математик анализ, II –қисм,Тошкент, Ўзбекистон, 1995.

6.Шокирова Х. Каррали ва эгри чизиқли интеграллар, Тошкент, Ўзбекистон,1992.

7. Ильин В.А., Позняк Э.Г. Основы математического анализа, т. I, М.,Наука, 1971. (Ўзбек тилига таржима қилинган).

8. Ильин В.А., Позняк Э.Г. Основы математического анализа, т. II, М.,Наука, 1980.

9. Кудрявцев Л.Д. Курс математического анализа, т. I, II, М.,Высшая школа, 1981.

10. Демидович Б.П. Сборник задач и упражнений по математическому анализу, М., Наука, 1972.

11.Салоҳиддинов М.С. Математик физика тенгламалари, Т., Ўзбекистон, 2002.

12.Акбарова М.Х., Акбарова С.Х. Математик физика тенгламларидан масалалар тўплами. Андижон, АДУ, 2002.

Internet ma'lumotlari

1.**http://mschool**. kubsu. ru/math1/integral/dvoynoy/dvoynoy. htm.

2.www.ziyonet.uz

3.www.ref.uz

MUNDARIJA