

**O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA
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TA'LIM VAZIRLIGI
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UNIVERSITETI
FIZIKA-MATEMATIKA FAKULTETI**

«Amaliy matematika va matematik fizika» kafedrası
5480100 - «Amaliy matematika va informatika» yo'nalishi

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BITIRUV MALAKAVIY ISHI

Mavzu: Chegaraviy shartlarning biri spektral
parametrga bog'liq bo'lgan Shturm-Liuvill masalasi
uchun Borg teoremasining analogi.

Ish ko'rib chiqildi va himoyaga ruxsat
berildi:

«Amaliy matematika va matematik
fizika» kafedrası mudiri:

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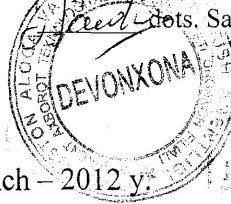
«21» 06 2012 y.

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Urganch – 2012 y.

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Kirish

1946 yil Shved matematigi G.Borg[1] tamonidan quydagi hozirda klassik teoremaqa aylangan teorema isbot qilindi.

Teorema A. Shturm-Liuvill tenglamasini

$$y'' + [\lambda - q(x)]y = 0 \quad (1)$$

quydagi chegaraviy shartlar

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0 \quad (2)$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0 \quad (3)$$

bilan qaraymiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya bo'lib, $\sin(\gamma - \beta) \neq 0$, u holda $q(x)$ potensial (1)+(2) va (1)+(3) chegaraviy masalalarning spektrlari yordamida yagona aniqlanadi.

1965 yilda esa Xitoy matematigi Li Yuex-Shena[3] quydagi teoremani isbot qilgan.

Teorema B. Quydagi chegaraviy masalani ko'rib chiqamiz:

$$y'' + [\lambda^2 - q(x)]y = 0 \quad (4)$$

$$y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0 \quad (5)$$

bu yerda $q(x)$ haqiqiy uzluksiz funksiya bo'lib, $a \neq 0$ haqiqiy son. U holda $q(x)$ potensial (4)+(5) chegaraviy masalaning spektri yordamida yagona aniqlanadi. TeoremaB da faqat bitta spektr yetarlicha hayratlanarli holat. Mazkur bitiruv malakaviy ishda teoremaA va teoremaB orasidagi bog'lanish o'rganiladi, hamda teoremaB ning umumlashmasi isbot qilinadi. Bundan tashqari (4)+(5) chegaraviy masala uchun yoyilma teoremasi o'rganiladi. Teorema B ning umumlashmasi quydagi teoremdan iborat:

Teorema C. Quydagi chegaraviy masalani ko'rib chiqamiz:

$$y'' + [\lambda^2 - q(x)]y = 0 \quad (4')$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta + f(\lambda)[y(\pi)\cos\gamma + y'(\pi)\sin\gamma] = 0 \end{cases} \quad (5')$$

bu yerda $q(x)$ haqiqiy uzluksiz funksiya bo'lib, α, β, γ haqiqiy sonlar, $\sin(\beta - \gamma) \neq 0$, $f(\lambda)$ esa tartibi 1 dan kichik bo'lgan toq butun funksiya. Bundan tashqari λ ning haqiqiy qiymatlarida $f(\lambda)$ haqiqiy qiymatlar qabul qiladi. U holda $q(x)$ potensial (4')+(5') chegaraviy masalaning spektri yordamida aniqlanadi.

1-§. Teorema B ning isboti va umumlashmasi.

Biz bu paragrafda teorema A yordamida teorema B ni va aksincha teorema B yordamida teorema A ni isbotlaymiz. teorema B ning umumlashmasi bilan teorema C ni boshqacha usulda isbotlaymiz.

$\lambda = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}^1$ deb belgilaymiz va (4) tenglamaning quydagi boshlang'ich shartlarni qanoatlantiruvchi $C(x, \lambda)$ va $S(x, \lambda)$ yechimlarni kiritib olamiz:

$$\begin{cases} C(0, \lambda) = 1, & C'(0, \lambda) = 0 \\ S(0, \lambda) = 0, & S'(0, \lambda) = 1 \end{cases} \quad (6)$$

$C(x, \lambda)$ va $S(x, \lambda)$ yechimlar har bir fiksirlangan x da λ ga nisbatan 1 tartibli butun funksiyalar (λ^2 ga nisbatan $\frac{1}{2}$ tartibli butun funksiyalar) bo'lishi bizga maxsus kurslardan malum. Bu fikrlar yechimlarning asimptotikasidan kelib chiqadi[4,5]

Biz ushbu

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) = 0, & y(\pi) = 0 \end{cases}$$

chegaraviy masalaning xos qiymatlarini ξ_n , $n \geq 1$ orqali va quydagi

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) = 0, & y'(\pi) = 0 \end{cases}$$

chegaraviy masalaning xos qiymatlarini η_n , $n \geq 0$ orqali belgilaymiz. U holda ushbu

$$S(\pi, \lambda) = \pi \prod_{n=1}^{\infty} \frac{\xi_n - \lambda^2}{n^2}, \quad (7)$$

$$S'(\pi, \lambda) = \prod_{n=0}^{\infty} \frac{\eta_n - \lambda^2}{(n + \frac{1}{2})^2} \quad (8)$$

yechimlar o'rinli bo'ladi ([5], 8-§). Xuddi shunday quydagi asimptotik formulalar ham bizga yaxshi ma'lum:

$$\xi_n = n^2 + \frac{1}{\pi} \int_0^\pi q(t) dt + \gamma_n, \quad \{\gamma_n\} \in l_2, \quad (9)$$

$$\eta_n = (n + \frac{1}{2})^2 + \frac{1}{\pi} \int_0^\pi q(t) dt + \beta_n, \quad \{\beta_n\} \in l_2, \quad (10)$$

$\omega(\lambda) = aS'(\pi, \lambda) + \lambda S(\pi, \lambda)$ orqali (4)+(5) masalaning harakteristik funksiyasini belgilaymiz. $\omega(\lambda) = 0$ tenglamaning ildizlari (4)+(5) masalaning xos qiymatlari bilan ustma-ust tushadi, $\omega(\lambda)$ funksiya birinchi tartibi butun funksiya bo'ladi.

$\omega(\lambda)$ funksiyaning ildizi borligi yoki yo'qligini tekshirish uchun osuillyatsiya

teoremasidan foydalanamiz ([5], 15-§).

(7) va (8) dan foydalanib quydagi tenglikni yozish mumkin:

$$\omega(\lambda) = a \prod_{n=0}^{\infty} \frac{\eta_n - \lambda^2}{(n + \frac{1}{2})^2} + \lambda \pi \prod_{n=1}^{\infty} \frac{\xi_n - \lambda^2}{n^2} \quad (11)$$

osulliyatsiya teoremasidan ushbu

$$\eta_0 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \dots < \eta_k < \xi_{k+1} < \eta_{k+1} < \dots$$

tengsizliklar, yani xos qiymatlarning almashinib kelishi kelib chiqadi

([5], 16-§). Agar $\sqrt{\eta_k}, \sqrt{\eta_{k+1}}$ larni (11) ga qo'ysak,

$$\begin{aligned} \omega(\sqrt{\eta_k}) &= \pi \sqrt{\eta_k} \prod_{n=1}^k \frac{\xi_n - \eta_k}{n^2} \prod_{n=k+1}^{\infty} \frac{\xi_n - \eta_k}{n^2} \\ \omega(\sqrt{\eta_{k+1}}) &= \pi \sqrt{\eta_{k+1}} \prod_{n=1}^{k+1} \frac{\xi_n - \eta_{k+1}}{n^2} \prod_{n=k+2}^{\infty} \frac{\xi_n - \eta_{k+1}}{n^2} \end{aligned}$$

bo'lgani uchun

$$\text{sign} \omega(\sqrt{\eta_k}) = (-1)^k,$$

$$\text{sign} \omega(\sqrt{\eta_{k+1}}) = (-1)^{k+1}$$

bo'ladi. Bunga ko'ra, $\omega(\lambda)$ funksiya $[\sqrt{\eta_k}, \sqrt{\eta_{k+1}}]$ kesmaning chetgi nuqtalarida turli ishorali qiymatlar qabul qiladi. Demak, bu kesmada $\omega(\lambda)$ funksiya kamida bitta nolga ega ekan. Bitiruv malakaviy ishning ikkinchi paragrafidagi taxlilga ko'ra $\omega(\lambda)$ harakteridan funksiyaning barcha ildizlari haqiqiy, oddiy bo'lib, ular uchun ushbu

$$\lambda_n = n - \frac{1}{\pi} \arctg a + \frac{1}{2n} \int_0^{\pi} q(x) dx + \bar{O}\left(\frac{1}{n}\right), \quad (|n| \rightarrow \infty) \quad (12)$$

asimptotik fo'rmla o'rinli. Bu yerda $n \in \mathbb{Z}$.

Quyidagi asimptotik fo'rmulalar bizga yaxshi ma'lum:

$$S(x, \lambda) = \frac{\sin \lambda x}{\lambda} + \underline{O}\left(\frac{e^{|\tau|x}}{\lambda^2}\right),$$

$$S'(x, \lambda) = \cos \lambda x + \underline{O}\left(\frac{e^{|\tau|x}}{\lambda}\right),$$

$$C(x, \lambda) = \cos \lambda x + \underline{O}\left(\frac{e^{|\tau|x}}{\lambda}\right),$$

$$C'(x, \lambda) = -\lambda \sin \lambda x + \underline{O}(e^{|\tau|x}). \quad (13)$$

Endi (4),(5) turidagi ikkinchi masalani ko'rib chiqamiz:

$$y'' + [\lambda^2 - q(x)]y = 0 \quad (14)$$

$$y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0 \quad (15)$$

bu masalaning xos qiymatlari (4)+(5) masalaning xos qiymatlari bilan ustma-ust tushadi deb hisoblaymiz.

(4) tenglamaning ushbu

$$\varphi(\pi, \lambda) = -a, \quad \varphi'(\pi, \lambda) = \lambda \quad (16)$$

Boshlang'ich shartlarni qanoatlantiruvchi yechimini $\varphi(\pi, \lambda)$ orqali belgilaymiz.

(14) tenglamaning

$$\tilde{C}(0, \lambda) = 1, \quad \tilde{C}'(0, \lambda) = 0,$$

$$\tilde{S}(0, \lambda) = 0, \quad \tilde{S}'(0, \lambda) = 1,$$

$$\tilde{\varphi}(\pi, \lambda) = -a, \quad \tilde{\varphi}'(\pi, \lambda) = \lambda.$$

Boshlang'ich shartlarni qanoatlantiruvchi yechimlarni mos ravishda $\tilde{C}(0, \lambda)$, $\tilde{S}(0, \lambda)$ va $\tilde{\varphi}'(\pi, \lambda)$ orqali belgilaymiz.

(4) tenglama yechimlarining Vronskiani x o'zgaruvchiga bog'liq bo'lmaganligi uchun ushbu

$$\begin{vmatrix} S(x, \lambda) & \varphi(x, \lambda) \\ S'(x, \lambda) & \varphi'(x, \lambda) \end{vmatrix} = \begin{vmatrix} S(\pi, \lambda) & \varphi(\pi, \lambda) \\ S'(\pi, \lambda) & \varphi'(\pi, \lambda) \end{vmatrix} = \begin{vmatrix} S(\pi, \lambda) & -a \\ S'(\pi, \lambda) & \lambda \end{vmatrix} = \lambda S(\pi, \lambda) + aS'(\pi, \lambda) = \omega(\lambda)$$

tengliklar o'rinli. Demak,

$$S(x, \lambda)\varphi'(x, \lambda) - S'(x, \lambda)\varphi(x, \lambda) = \omega(\lambda). \quad (17)$$

$\tilde{\omega}(\lambda) = a\tilde{S}'(\pi, \lambda) + \lambda\tilde{S}(\pi, \lambda)$ orqali (14)+(15) masalaning xarakteristik funksiyasini belgilaymiz. $\tilde{\omega}(\lambda) = 0$ tenglamaning ildizlari (14)+(15) masalaning xos qiymatlari bilan ustma-ust tushadi. Yechimlar asimptotikasiga ko'ra

$$\omega(\lambda) = a \cos \lambda \pi + \sin \lambda \pi + \underline{O}\left(\frac{e^{|\lambda|\pi}}{\lambda}\right), \quad |\lambda| \rightarrow \infty$$

va

$$\tilde{\omega}(\lambda) = a \cos \lambda \pi + \sin \lambda \pi + \underline{O}\left(\frac{e^{|\lambda|\pi}}{\lambda}\right), \quad |\lambda| \rightarrow \infty$$

asimptotikalar o'rinli.

$\omega(\lambda)$ va $\tilde{\omega}(\lambda)$ butun funksiyalarning nollari va asimptotikalari ustma-ust tushganligi sababli $\omega(\lambda) = \tilde{\omega}(\lambda)$ bo'ladi. Bunga ko'ra, xuddi yuqoridagidek qilib,

$$\tilde{S}(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{S}'(x, \lambda)\tilde{\varphi}(x, \lambda) = \omega(\lambda). \quad (18)$$

ayniyat keltirilib chiqariladi.

(17) va (18) ayniyatlarda $x = 0$ va $x = \pi$ desak,

$$\omega(\lambda) = -\varphi(0, \lambda), \quad \omega(\lambda) = \lambda S(\pi, \lambda) + a S'(\pi, \lambda) \quad (19)$$

$$\omega(\lambda) = -\tilde{\varphi}(0, \lambda), \quad \omega(\lambda) = \lambda \tilde{S}(\pi, \lambda) + a \tilde{S}'(\pi, \lambda) \quad (20)$$

kelib chiqadi.

Agar (17) va (18) ayniyatlarda $\lambda = \lambda_n$ desak, va $\omega(\lambda_n) = 0$ ni hisobga olsak, quydagilar hosil bo'ladi:

$$S(x, \lambda_n) = C_n \varphi(x, \lambda_n),$$

$$\tilde{S}(x, \lambda_n) = \tilde{C}_n \tilde{\varphi}(x, \lambda_n).$$

$C_n = \tilde{C}_n$ ekanini ko'rsatamiz. Bu tenglamalarda $x = \pi$ desak,

$$C_n = \frac{S(\pi, \lambda_n)}{\varphi(\pi, \lambda_n)} = \frac{S(\pi, \lambda_n)}{-a},$$

$$\tilde{C}_n = \frac{\tilde{S}(\pi, \lambda_n)}{\tilde{\varphi}(\pi, \lambda_n)} = \frac{\tilde{S}(\pi, \lambda_n)}{-a}$$

xosil bo'ladi.

$S(x, \lambda)$, $C(x, \lambda)$, $\tilde{S}(x, \lambda)$, $\tilde{C}(x, \lambda)$ funksiyalar λ o'zgaruvchi bo'yicha juft funksiya bo'ladi, chunki tenglamada λ o'rniga $(-\lambda)$ qo'ysak, u o'zgarmay qoladi, boshlang'ich shartlar esa, λ ga bog'liq emas. Demak,

$$S(x, -\lambda) = S(x, \lambda), \quad C(x, -\lambda) = C(x, \lambda), \quad \tilde{S}(x, -\lambda) = \tilde{S}(x, \lambda), \quad \tilde{C}(x, -\lambda) = \tilde{C}(x, \lambda)$$

bularga ko'ra, ushbu

$$S(\pi, -\lambda) = S(\pi, \lambda), \quad S'(\pi, -\lambda) = S'(\pi, \lambda),$$

$$\tilde{S}(\pi, -\lambda) = \tilde{S}(\pi, \lambda), \quad \tilde{S}'(\pi, -\lambda) = \tilde{S}'(\pi, \lambda).$$

tengliklar o'rinli bo'ladi. Demak, (19)dan

$$\omega(\lambda) = \lambda S(\pi, \lambda) + a S'(\pi, \lambda),$$

$$\omega(-\lambda) = -\lambda S(\pi, \lambda) + a S'(\pi, \lambda),$$

$$S(\pi, \lambda) = \frac{1}{2\lambda} [\omega(\lambda) - \omega(-\lambda)]$$

kelib chiqadi. Xuddi shu tarzda (20) dan

$$\tilde{S}(\pi, \lambda) = \frac{1}{2\lambda} [\omega(\lambda) - \omega(-\lambda)]$$

hosil bo'ldi. Bu tengliklarga muvofiq,

$$S(\pi, \lambda) = \tilde{S}(\pi, \lambda).$$

Demak, $C_n = \tilde{C}_n$ ekan.

Endi quydagi funksiyalarni ko'rib chiqamiz:

$$\Phi(x, \lambda) = \frac{1}{\omega(\lambda)} \left\{ \varphi(x, \lambda) \int_0^x S(t, \lambda) f(t) dt + S(x, \lambda) \int_x^\pi \varphi(t, \lambda) f(t) dt \right\}, \quad (21)$$

$$\tilde{\Phi}(x, \lambda) = \frac{1}{\omega(\lambda)} \left\{ \tilde{\varphi}(x, \lambda) \int_0^x \tilde{S}(t, \lambda) f(t) dt + \tilde{S}(x, \lambda) \int_x^\pi \tilde{\varphi}(t, \lambda) f(t) dt \right\}. \quad (22)$$

Bu yerda $f(t) \in C[0, \pi]$.

R_n orqali ularni

$$\begin{aligned} A & \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a, i \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right) \right), \\ B & \left(- \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right), i \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right) \right), \\ C & \left(- \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right), -i \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right) \right), \\ D & \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a, -i \left(n + \frac{1}{2} - \frac{1}{\pi} \arctg a \right) \right). \end{aligned}$$

nuqtalar bo'lgan kvadrat konturini belgilaymiz.

Quyidagi

$$\lim_{n \rightarrow \infty} \int_{R_n} [\Phi(x, \lambda) - \tilde{\Phi}(x, \lambda)] d\lambda = 0 \quad (23)$$

tenglik bajarilishini isbotlaymiz. (21) va (22) ko'ra

$$\begin{aligned} \Phi(x, \lambda) - \tilde{\Phi}(x, \lambda) = \\ \frac{\varphi(x, \lambda) - \tilde{\varphi}(x, \lambda)}{\omega(\lambda)} \int_0^x S(t, \lambda) f(t) dt + \frac{S(x, \lambda) - \tilde{S}(x, \lambda)}{\omega(\lambda)} \int_x^\pi \varphi(t, \lambda) f(t) dt \end{aligned}$$

bo'ladi. (13) asimptotik formulalardan

$$\frac{\varphi(x, \lambda) - \tilde{\varphi}(x, \lambda)}{\omega(\lambda)} \int_0^x S(t, \lambda) f(t) dt = \frac{O\left(\frac{e^{|\tau|x}}{\lambda^2}\right) \int_0^x \sin \lambda t f(t) dt}{a \cos \lambda \pi + \sin \lambda \pi}$$

va

$$\frac{S(x, \lambda) - \tilde{S}(x, \lambda)}{\omega(\lambda)} \int_x^\pi \varphi(t, \lambda) f(t) dt = \frac{O\left(\frac{e^{|\tau|x}}{\lambda^2}\right) O\left(e^{|\tau|(\pi-x)}\right)}{a \cos \lambda \pi + \sin \lambda \pi}$$

kelib chiqadi. Bu asimptotik formulalarga binoan

$$\int_{R_n} [\Phi(x, \lambda) - \tilde{\Phi}(x, \lambda)] d\lambda = O\left(\frac{1}{n}\right)$$

xosil bo'ladi. Bundan esa (23) tenglikka ega bo'lamiz.

Ikkinchi tamondan Koshining chegirmalar haqidagi teoremasiga ko'ra

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{R_n} [\varphi(x, \lambda) - \tilde{\varphi}(x, \lambda)] d\lambda = \\ \sum_{k=-\infty}^{\infty} \{ [\varphi(x, \lambda_k) - \tilde{\varphi}(x, \lambda_k)] \int_0^x f(t, \lambda_k) f(t) dt + [S(x, \lambda_k) - \\ \tilde{S}(x, \lambda_k)] \int_x^\pi \varphi(t, \lambda_k) f(t) dt \} \end{aligned}$$

tenglik o'rinli bo'ladi. Bu yerda $\omega(\lambda)$ funksiyaning nollari oddiy ekanligi

ishlatildi. Agar bu tenglikka

$$\varphi(x, \lambda_k) = C_k S(x, \lambda_k) \text{ va } \tilde{\varphi}(x, \lambda_k) = C_k \tilde{S}(x, \lambda_k)$$

ifodalarni qo'ysak, hamda (23) tenglikni inobatga olsak, ushbu

$$\sum_{k=-\infty}^{\infty} \frac{1}{\omega'(\lambda_k)} C_k [S(x, \lambda_k) - \tilde{S}(x, \lambda_k)] \int_0^{\pi} S(t, \lambda_k) f(t) dt = 0 \quad (24)$$

ayniyat xosil bo'ladi.

2- § da $S(t, \lambda_k), n \in \mathbb{Z}$ funksiyalarning chiziqli erkli bo'lishini ko'rsatamiz. Biz $f(t)$ funksiyani quydagi shartlarni qanoatlantiruvchi qilib tanlaymiz:

$$\int_0^{\pi} S(t, \lambda_k) f(t) dt = 0, \quad k \neq n$$

$$\int_0^{\pi} S(t, \lambda_k) f(t) dt = 1, \quad k = n.$$

Agar bu $f(t)$ funksiyani (24) tenglikka qo'ysak,

$$S(t, \lambda_k) = \tilde{S}(t, \lambda_k), \quad (n \in \mathbb{Z})$$

kelib chiqadi. Bunga ko'ra,

$$q(x) = \frac{S''(x, \lambda_n)}{S(x, \lambda_n)} + \lambda_n^2 = \frac{\tilde{S}''(x, \lambda_n)}{\tilde{S}(x, \lambda_n)} + \lambda_n^2 = \tilde{q}(x),$$

yani $q(x) = \tilde{q}(x), \quad x \in [0, \pi]$ bo'ladi.

TeoremaB isbotlandi.

TeoremaB ni quydagicha umumlashtirish mumkin.

Teorema1. Quydagi chegaraviy masalani ko'rib chiqamiz:

$$y'' + [\lambda^2 - q(x)]y = 0 \quad (25)$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta + f(\lambda)[y(\pi)\cos\gamma + y'(\pi)\sin\gamma] = 0. \end{cases} \quad (26)$$

Bu yerda $q(x)$ haqiqiy uzluksiz funksiya bo'lib, α, β, γ haqiqiy sonlar,

$\sin(\beta - \gamma) \neq 0$, $f(\lambda)$ esa tartibi 1 dan kichik bo'lgan toq butun funksiya.

Bundan tashqari λ ning haqiqiy qiymatlarida $f(\lambda)$ haqiqiy qiymatlar qabul qiladi. U holda $q(x)$ potensial (25)+(26) chegaraviy masalaning spektri yordamida yagona aniqlanadi.

Isbot. (25) tenglamaning ushbu

$$\varphi(0, \lambda) = -\sin\alpha, \quad \varphi'(0, \lambda) = \cos\alpha$$

Boshlang'ich shartni qanoatlantiruvchi yechimini $\varphi(x, \lambda)$ orqali belgilaymiz.

$\varphi(x, -\lambda) = \varphi(x, \lambda)$ ayniyat bajarilishi ravshan. Bu yechim (26) chegaraviy shartlarning birinchisini qanoatlantiradi, chegaraviy shartlarning ikkinchisiga

$\varphi(x, \lambda)$ ni qo'yib ushbu

$$\Delta(\lambda) = \varphi(\pi, \lambda)\cos\beta + \varphi'(\pi, \lambda)\sin\beta + f(\lambda)[\varphi(\pi, \lambda)\cos\gamma + \varphi'(\pi, \lambda)\sin\gamma]$$

harakteristikadan funksiyani tuzib olamiz. Agar

$$\Delta_1(\lambda) = \varphi(\pi, \lambda) \cos \beta + \varphi'(\pi, \lambda) \sin \beta,$$

$$\Delta_2(\lambda) = \varphi(\pi, \lambda) \cos \gamma + \varphi'(\pi, \lambda) \sin \gamma$$

belgilashlarni kiritamiz,

$$\Delta(\lambda) = \Delta_1(\lambda) + f(\lambda) \Delta_2(\lambda)$$

bo'ladi. $\Delta_1(\lambda)$ va $\Delta_2(\lambda)$ lar mos ravishda ushbu

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) \cos \alpha + y'(0) \sin \alpha = 0 \\ y(\pi) \cos \beta + y'(\pi) \sin \beta = 0 \end{cases} \quad (*)$$

va

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) \cos \alpha + y'(0) \sin \alpha = 0 \\ y(\pi) \cos \gamma + y'(\pi) \sin \gamma = 0 \end{cases} \quad (**)$$

masalalarning harakteristik funksiyalari ekanligi ravshan. Bundan tashqari

$\Delta_1(-\lambda) = \Delta_1(\lambda)$, $\Delta_2(-\lambda) = \Delta_2(\lambda)$ va $f(-\lambda) = -f(\lambda)$ bo'lgani uchun

$$\Delta(\lambda) = \Delta_1(\lambda) - f(\lambda) \Delta_2(\lambda)$$

bo'ladi. Demak,

$$\Delta_1(\lambda) = \frac{\Delta(\lambda) + \Delta(-\lambda)}{2},$$

$$\Delta_2(\lambda) = \frac{\Delta(\lambda) - \Delta(-\lambda)}{2f(\lambda)}.$$

Bu formulalarga binoan, $\Delta(\lambda)$ va $f(\lambda)$ malum bo'lsa $\Delta_1(\lambda)$ va $\Delta_2(\lambda)$

funksiyalar bir qiymatli ravishda topiladi. $\Delta_1(\lambda)$ va $\Delta_2(\lambda)$ tenglamalarni

yechib, mos ravishda (*) va (**) masalalarning spektrlarini topamiz. Borg

teoremasiga ko'ra, yoki teorema A ga ko'ra bu spektrlar yordamida $q(x)$

potensial yagona aniqlanadi. $\Delta(\lambda)$ harakteristik funksiya (25)+(26)

masalaning xos qiymatlari va asimptotikasi orqali yagona aniqlanganligini

hisobga olsak, $q(x)$ ning bu xos qiymatlar orqali yagona aniqlanishi kelib

chiqadi. Teorema 1 isbotlandi.

Teorema2. Agar teorema1 bajariladi deb olsak undan teorema A kelib chiqadi.

Isbot. Quydagi belgilashlarni kiritib olamiz:

$$\Delta_1(\lambda) = \varphi(\pi, \lambda) \cos \beta + \varphi'(\pi, \lambda) \sin \beta,$$

$$\Delta_2(\lambda) = \varphi(\pi, \lambda) \cos \gamma + \varphi'(\pi, \lambda) \sin \gamma$$

$$\tilde{\Delta}_1(\lambda) = \tilde{\varphi}(\pi, \lambda) \cos \beta + \tilde{\varphi}'(\pi, \lambda) \sin \beta,$$

$$\tilde{\Delta}_2(\lambda) = \tilde{\varphi}(\pi, \lambda) \cos \gamma + \tilde{\varphi}'(\pi, \lambda) \sin \gamma$$

Bu yerda $\tilde{\varphi}(x, \lambda)$ orqali

$$y'' + [\lambda^2 - \tilde{q}(x)]y = 0$$

Tenglamani

$$\tilde{\phi}(0, \lambda) = -\sin \alpha \quad \text{va} \quad \tilde{\phi}'(0, \lambda) = \cos \alpha$$

Boshlang'ich shartlarni qanoatlantiruvchi yechimi belgilangan.

Quyidagi funksiyalarni kiritib olamiz:

$$\omega(\lambda) = \Delta_1(\lambda) + a\lambda\Delta_2(\lambda),$$

$$\tilde{\omega}(\lambda) = \tilde{\Delta}_1(\lambda) + a\lambda\tilde{\Delta}_2(\lambda).$$

Agar $\tilde{\Delta}_1(\lambda) = \Delta_1(\lambda)$ va $\tilde{\Delta}_2(\lambda) = \Delta_2(\lambda)$ bo'lsa, ya'ni Borg teoremasining shartlari bajarilsa, u holda $\tilde{\omega}(\lambda) = \omega(\lambda)$ bo'ladi. Bunga ko'ra teorema 1 ning sharti bajariladi. Bu yerda $f(\lambda) = a\lambda$ deb olingan. teorema 1 ga binoan $\tilde{q}(x) = q(x)$ bo'ladi, ya'ni teorema A ning hulosasi kelib chiqadi. Teorema 2 isbotlandi.

Natija. Teorema A va teorema 1 o'zaro ekvivalent ekan.

2-§. Ixtiyoriy funksiyani (4)+(5) chegaraviy masalaning xos funksiyalari bo'yicha yoyish.

Quyidagi chegaraviy masalani ko'rib chiqamiz:

$$y'' + [\lambda^2 + \mu - \tilde{q}(x)]y = 0 \tag{27}$$

$$y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0. \tag{28}$$

Bu yerda $q(x)$ haqiqiy, uzluksiz funksiya, $a \neq 0$ va μ haqiqiy parametrlar.

(27)+(28) masalani o'rganish uchun uni boshqa bir masalalar bilan bog'laymiz. Buning uchun avvalo quyidagi shartlarni qanoatlantiruvchi qilib $M(x)$ funksiyani kiritib olamiz:

$$\begin{cases} M'' + [\mu - q(x)]M = 0 \\ M(\pi) = 1, \quad M'(\pi) = 0 \end{cases} \tag{29}$$

Ushbu

$$M'' + [\mu - q(x)]M = 0 \tag{29'}$$

tenglama va

$$M(0)=0, \quad M'(\pi)=0 \tag{30}$$

Chegaraviy shartlar yordamida xosil bo'ladigan chegaraviy masalaning spektri quyidan chegaralangan. Bu masalaning eng kichik xos qiymatini μ_0

orqali belgilasak va (29) masalada $\mu < \mu_0$ deb olsak, Shturm teoremasidan ([5],14-§)

$$M(x) > 0, x \in [0, \pi]$$

kelib chiqadi.

Endi (27)+(28) masalada quydagi almashtirishni bajaramiz:

$$y = \mu \eta, \quad \xi = \int_0^x \frac{1}{M^2(t)} dt. \quad (31)$$

Bu yerda η -yangi erkli o'zgaruvchi va $\eta = \eta(\xi)$ -yangi nomalum funksiya.

Bu almashtirish natijasida quydagi masala kelib chiqadi:

$$\eta'' + \lambda^2 M^4 \eta = 0 \quad (32)$$

$$\eta(0) = 0, \quad a\eta'(b) + \lambda\eta(b) = 0 \quad (33)$$

Bu yerda

$$b = \int_0^\pi \frac{1}{M^2(t)} dt.$$

(32)+(33) masalani o'rganish uchun ushbu

$$\begin{cases} x_1 = -\lambda \eta \\ x_2 = \eta' \end{cases} \quad (34)$$

Yangi funksiyalarni kiritib olamiz. Bu funksiyalar quydagi

$$\begin{cases} x_1 = -\lambda x_2 \\ x_2' = \lambda M^4 x_1 \end{cases} \quad (35)$$

Differensial tenglamalarni qanoatlantiradilar. Bundan tashqari ushbu

$$x_1(0) = 0, \quad a x_2(b) - x_1(b) = 0 \quad (36)$$

Chegaraviy shartlar ham bajariladi.

(35) tenglamalar sistemasini quydagi ko'rinishda yozish mumkin:

$$L_0 X = \lambda X. \quad (37)$$

Bu yerda

$$L_0 = \begin{pmatrix} 0 & \frac{1}{M^4} \frac{d}{d\xi} \\ -\frac{d}{d\xi} & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Shunday qilib yuqoridagi almashtirishlar (27)+(28) masaladagi λ ga nisbatan nochiziqlikni hamda chegaraviy shartlardan λ ni yo'qotdi.

L_0 ning teskarisi integral aperator bo'ladi. Bunga ko'ra ushbu

$$\begin{cases} x_1 = -\lambda \int_0^\xi x_2 d\xi, \\ x_2 = -\frac{\lambda}{a} \int_0^b x_2 d\xi - \lambda \int_\xi^b M^4 x_1 d\xi \end{cases}$$

(38)

Tengliklarni olamiz. (38) tengliklar yordamida aniqlangan integral operatorni J_0 orqali belgilab, (38) tenglikni tubandagi ko'rinishda yozish mumkin:
 $X = \lambda J_0 X$.

Quyidagi Gilbert fazasini kiritib olamiz:

$$H = \left\{ X = \begin{pmatrix} x_1(\xi) \\ x_2(\xi) \end{pmatrix} : \int_0^b [|x_1(\xi)|^2 + |x_2(\xi)|^2] d\xi < \infty \right\}.$$

Bu fazoda skalyar ko'paytmani ushbu

$$(XY) = \int_a^b (M^4 x_1 \bar{y}_1 + x_2 \bar{y}_2) d\xi \quad (39)$$

Tenglik yordamida kiritsak, u holda unga mos norma quydagicha bo'ladi

$$\|X\| = \sqrt{\int_0^b [M^4 |x_1|^2 + |x_2|^2] d\xi} \quad (40)$$

Normaga ekvivalent bo'ladi.

Bo'laklab integrallash qoidasini qo'llasak, hamda chegaraviy shartlarni hisobga olsak, obsalyut uzluksiz

$X = X(\xi), Y = Y(\xi)$ vektor-funksiyalar uchun ushbu

$$(L_0 X, Y) = (X, L_0 Y) \quad (41)$$

Tenglik kelib chiqadi. Bu tenglikdan barcha $X, Y \in H$ uchun ushbu

$$(J_0 X, Y) = (X, J_0 Y). \quad (42)$$

(38) ifodalarga ko'ra $J_0 X = 0$ bo'lishi uchun $X=0$ bo'lishi zarur va yetarli ekani kelib chiqadi, yani J_0 apertorning yadrosi faqat noldan iborat bo'ladi.

(42) tenglikka ko'ra J_0 apertor o'z-o'ziga qo'shma, bundan J_0 apertorning barcha xos qiymatlari haqiqiy o'qda yotishi kelib chiqadi.

J_0 integral apertorning ifodasidan uning kompakt operator ekani chiqadi. $J_0 \neq 0$ ekani ravshan. Demak, o'z-o'ziga qo'shma, kompakt, noldan farqli chiziqli operatorlarning umumiy nazariyasiga ko'ra J_0 ning normallangan xos funksiyalaridan tuzilgan sistema H Gilbert fazosida to'la ortanarmallangan sistema bo'ladi, H cheksiz o'lchamli bo'lgani uchun xos funksiyalar soni cheksizta bo'ladi. Xos qiymatlar soni ham cheksizta ekanini ko'rsatish maqsadida, xos qiymatlar oddiy bo'lishini isbotlaymiz. Agar (36)+(37) masalada λ xos qiymatga 2 ta xos funksiya mos kelsin desak va

ularning ayirmasini qarasak, bu ayirma uchun

$$x_1'' + \lambda^2 M^4 x_1 = 0$$

va

$$x_1(0) = 0, \quad x_1'(0) = 0$$

Bajarilishi kelib chiqadi. Bunga ko'ra $x_1 = 0$ va $x_2 = 0$ bo'ladi. Bu esa λ xos qiymatga mos keluvchi xos funksiyalar o'zaro proporsional ekanini ko'rsatadi. Demak, L_0 operatorning cheksiz ko'p xos qiymati bor ekan.

L_0 operatorning xos qiymatlari $\lambda_n, n \geq 1$ orqali ularga mos keluvchi normallangan xos funksiyalarni $X_n, n \geq 1$ orqali belgilaymiz. $F \in H$ ixtiyoriy funksiya bo'lsin. U holda xos funksiyalar sistemasi H Gilbert fazosida to'la bo'lgani uchun ushbu

$$F = \sum_{n=1}^{\infty} \alpha_n X_n. \quad (43)$$

Yoyilma fo'rmulasi o'rinli bo'ladi. Bu yerda

$$\alpha_n = (F, X_n) = \int_0^b \left[M^4 f_1 \overline{x_1^{(n)}} + f_2 \overline{x_2^{(n)}} \right] d\xi. \quad (44)$$

Agar, xususan, $f_2 = 0$ deb olsak, (43) ga ko'ra

$$f_1 = \sum_{n=1}^{\infty} \alpha_n X_1^{(n)},$$

$$0 = \sum_{n=1}^{\infty} \alpha_n x_2^{(n)}$$

Tengliklar xosil bo'ladi. Agar biz, bu tengliklarning birinchisida (31) almashtirishlarni qo'llasak,

$$f_1 = \sum_{n=1}^{\infty} \alpha_n \frac{-\lambda_n}{M} y_n, \quad \alpha_n = \int_0^{\pi} f_1(-\lambda_n) M y_n dx \quad (45)$$

kelib chiqadi.

Olingan natijalarni quydagi teoremlarda mujassamlashtiramiz.

Teorema 3. (36),(37) tengliklar bilan aniqlangan L_0 operator yoki shunga ekvivalent bo'lgan H Gilbert fazosida aniqlangan J_0 operator cheksizta xos qiymatga ega bo'lib, bu xos qiymatlar haqiqiy va karrasizdir. Bundan tashqari bu xos qiymatlarga mos keluvchi xos funksiyalar sistemasi H Gilbert fazosida to'la ortogonal sistemani xosil qiladi.

Teorema 4. (27), (28) chegaraviy masalacheksizta xos qiymatga ega bo'lib, bu xos qiymatlar haqiqiy va karrasizdir. Bundan tashqari bu xos qiymatlarga mos keluvchi xos funksiyalar $L_2(0, \pi)$ fazoda to'la bo'ladi, yani $L_2(0, \pi)$ fazodan olingan ixtiyoriy funksiyaning (45) ko'rinishidagi qatorga yoyish mumkin. Bu yerda biz $\mu < \mu_0$ deb hisoblaymiz. μ_0 orqali (29')+(30) chegaraviy masalaning eng kichik xos qiymati belgilangan.

Izoh. (27)+(28) chegaraviy masalani xos funksiyalari o'zaro ortogonal emas.

Izoh. Teorema 4 ni biz $\mu < \mu_0$ boʻlgan holda isbot qildik, aslida bu teorema $\mu \geq \mu_0$ boʻlgan holda ham oʻrinli boʻladi.

Xulosa.

Quyidagi

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0 \end{cases} \quad (46)$$

Masalani ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy, uzluksiz funksiya, $a \neq 0$ haqiqiy son, λ esa umuman olganda kompleks qiymat qabul qiluvchi spectral parameter. (46) chegaraviy masala cheksizta xos qiymatga ega bo'lib, bu xos qiymatlar haqiqiy va karrasizdir. Bundan tashqari $L_2(0, \pi)$ fazodan olingan ixtiyoriy funksiyaning (46) masalaning xos funksiyalari bo'yicha (45) ko'rinishidagi qatorga yoyish mumkin. (46) masalaning xos qiymatlari uchun ushbu

$$\lambda_n = n - \frac{1}{\pi} \arctg a + \frac{1}{2n} \int_0^\pi q(x) dx + O\left(\frac{1}{n}\right), \quad n \rightarrow \pm\infty \quad (47)$$

Asimptotik formula o'rinli. (46) chegaraviy masala $\{\lambda_n\}_{n=-\infty}^{\infty}$ xos qiymatlari bo'yicha bir qiymatli tiklanadi, ya'ni bu xos qiymatlar bo'yicha $q(x)$ funksiya va chegaraviy shartdagi a son yagona aniqlanadi. a sonni topish uchun (47) tenglikdan foydalanamiz:

$$\begin{aligned} -\frac{1}{\pi} \arctg a &= \lim_{n \rightarrow \infty} (\lambda_n - n), \\ a &= \operatorname{tg} \left[\lim_{n \rightarrow \infty} (n - \lambda_n) \pi \right] \end{aligned} \quad (48)$$

$q(x)$ ko'effitsientni topish uchun teorema 1 ning isbotida keltirilgan usul yordamida odatdagi Borg yagonalik teoremasiga keltiramiz. Buning uchun avvalo

$$f(\lambda) = \frac{\lambda}{a}, \quad \Delta(\lambda) = \pi(\lambda_0 - \lambda) \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\lambda_n - \lambda}{n}$$

Ushbu funksiyalarni tuzib olamiz. Bu funksiyalar yordamida

$$\begin{aligned} \Delta_1(\lambda) &= \frac{\Delta(\lambda) + \Delta(-\lambda)}{2}, \\ \Delta_2(\lambda) &= \frac{\Delta(\lambda) - \Delta(-\lambda)}{2f(\lambda)}. \end{aligned}$$

Funksiyalarni tuzib olamiz. Teorema 1 ning isbotida ko'rsatganimizdek, mos ravishda

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) = 0, \quad y'(\pi) = 0 \end{cases} \quad (49)$$

va

$$\begin{cases} y'' + [\lambda^2 - q(x)]y = 0 \\ y(0) = 0, \quad y(\pi) = 0 \end{cases} \quad (50)$$

Chegaraviy masalalarning harakteristik tenglamalari bo`ladi. Demak, $\Delta_1(\lambda) = 0$ va $\Delta_2(\lambda) = 0$ tenglamalarni yechsak, (49) va (50) masalalarning xos qiymatlarini topa olamiz. Bular yordamida $q(x)$ ni tuzish usuli [5] adabiyotda keltirilgan.

Adabiyotlar

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