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«Umumiy matematika» kafedrasi

Matematika informatika ixtisosligining 4^b uzb guruh talabasi
I.Otenovning «To’r tenglamalarini yechishning to`g`ri va iteratsiya usullari» mavzusidagi

Bitiruv malakaviy ishi

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Kirish.

Ta`limni tarbiyadan, tarbiyani esa ta`limdan ajratib bo`lmaydi-bu sharqona qarash, sharqona hayot falsafasi.

Bu haqda fikr yuritganda, men Abdulla Avloniyning «Tarbiya biz uchun yo hayot yo mamot, yo najot yo halokat, yo saodat yo falokat masalasidir» degan chuqur ma`noli so`zlarini eslayman.

Shuning uchun ham mustaqillikning dastlabki yillaridanoq butun mamlakat miqyosida ta`lim va tarbiya, ilm-fan, kasb-hunar o`rgatish tizimlarini tubdan isloh qilishga nihoyatda katta zarurat sezila boshladi. Kadrlar tayyorlash milliy dasturini ishlab chiqish bilan bog`lik jarayon uzoq yillar mobaynida bu sohada bir qancha muommolar yig`ilib qolganini ko`rsatdi. Shuning uchun ham bu og`ir, mas`uliyatli, ammo hal qilishni aslo paysalga solib bo`lmaydigan ishni qadam-baqadam, izchillik bilan bajarishga bel bog`ladik. [1]

Matematika fizika tenglamalari kursini o`rganishda va fizika texnikaning juda ko`p masalalarini ikkinchi tartibli xususiy xosilali differentsial tenglamalar va ularga qo`yilgan chegaraviy shartlarda masalalarni echish usullari o`rgatiladi [2].

Ko`p xollarda xususiy xosilali differentsial tenglamalarning yechimlarini analitik usulda olish mumkin emas. Shuning uchun yechimni sonli usilda olishga harakat etamiz. Chegirmalar usuli yordamida

$\Delta u=-f$ Puasson tenglamasi uchun chegarviy masalalar, algebrik tenglamalar sistemasiga keltiriladi sistemaning tartibi to`rning ichki tugunlar soniga teng bo`lib u to`rning odimi kichiraygan sari ortadi. Ikkinchi tartibli xususiy hosilali differentsial tenglama chekli chegirmalar usuli yordamida uch nuqtali to`r tenglamalariga keltiriladi. U tenglamani oddiy iteratsiya, Zeydel va matritsaviy haydash usulidan foydalanib yechish mumkin [4].

Biz ushbu bitiruv malakaviy ishda ikkinchi tartibli xususiy hosilali differentsial tenglamalarga qo`yilgan chegaraviy masalalarni matritsaviy haydash usilidan foydalanib yechishga batafsil to`xtab o`tdik [6]. Bu bitiruv malakaviy ish uch paragrafdan yakunlash va foydalanilgan adabiyotlar tizimidan iborat bo`lib, birinchi paragrafda ikkinchi tartibli xususiy xosilali differentsial tenglamalarni sonli yechishning to`g`ri va iteratsiya usillari keltirilgan yani Puasson tenglamasi uchun chegaraviy shartlarda to`g`riburchakli oblastta yechish ko`rib o`tilgan. Ikki metod uchun ham arifmetik operatsiyalar soni $Q=0\ (N^2 \log_2 N)$, bunda N-bitta yo`nalish bo`yicha tuginlar soni.
Ikkinchi paragrafda ikki qatlamli iteratsion sxemalarga to`xtab o`tildi. Iteratsion metod qandaydir dastlabki $y_0 \in H$ yaqinlashishdan boshlab tenglamaning yechimini ketma-ket topishga yordam beradi $y_1, y_2, \ldots y_k, y_{k+1} \ldots$ unda, k-iteratsiya nomeri. y_{k+1} ning qiymati o`zidan oldingi $y_k, y_{k+1} \ldots$ iteratsiyalar yordamida topiladi. Agar y_{k+1} ni xisoblashda faqat olding y_k iteratsiya foydalanilsa, unda iteratsion metod bir qadamli, agar olding ikki iteratsiya foydalanilsa, unda iteratsiya metodi ikki qadamli deb nomlanadi. Biz quyda bir qadamli iteratsiyalik metod formasi bo`yicha ikki qatlamli sxema bilan mos kelishini ko`rsatamiz.

Bitiruv malakaviy ishning uchinchi paragrafida to`r tenglamalarini yechishning haydash usilining ayrim to`rlari keltirilgan.

1. Ya`ni kuchli o`zgaruvchi koefitsientlardagi chegirmali tenglamani yechishning oqim usili.

2. Tsikllik haydash usili. Ularga bog`liq ancha misollar keltirilgan

§1. To`g`ri metodlar.

1.1 To`g`ri va iteratsion metodlar.

Biz elliptik tenglamalar uchun chegarviy masalalarni chegirmali approksimatsiyalash natijasida chiziqli algebraik tenglamalar sistemasiga ega bo`lgan edik. (chegirmali yoki to`r tenglamalariga)

$$A(x)\cdot y(x)=\sum_{\xi\in V(x)}B(x,\xi)\cdot y(\xi)+F(x),\qquad x\in\omega_h \tag{1}$$

Bu yerda *V(x),* (2r+1) krest shablonining 2p tuginlar to`plami. Tuginning o`zi kirmaydi $\xi\neq x$.

V(x)-to`plamining x tuguni atrofi deb ataymiz. A(x) va V(x, ξ)-berilgan tenglama koefitsientlari (1) tenglamaga

$$y|_{\gamma_h}=\mu(x) \tag{2}$$

chegaraviy shart qo`shib yoziladi [3]. Ushbu sistemaning matritsasi A ning tartibi, to`rning tugunlar soni N ga teng. Misol uchun har bir g`arazsiz o`zgaruvchi x_1, x_2, …x_p ($h_1=h_2=\ldots=h_p=h$) bo`yicha h odim bilan to`rga N=0 $\left(\frac{1}{h^p}\right)$ tuginlar soni kerak, bunda p-o`lcham. Ikki va uch o`lchamda tenglamalar soni katta bo`ladi, $N\approx10^4$-$10^6$ (Mis $h=\frac{1}{100}$) sistemaning matritsasi ko`p nollik elementga ega bo`ladi, va qiyin o`zlashtiriladigan matritsaga ega bo`ladi, ya`ni matritsaning eng katta xususiy qiymatining eng kichik xususiy qiymatiga nisbati judayam katta (10^{-3}~10^{-4}).

Elliptik to`r tenglamalarining ushbu o`zgachaliklariga o`xshagan sonli yechish uchun tejamli algoritmlar ishlab chiqishni talab etadi. To`g`ri tejamli usuli juda tor, lekin ahamiyatli to`r tenglamalar sinfini yechish uchun qo`llaniladi. Bundan tashqari to`g`ri metodlar iteratsion usilda operatorni yuqori qatlamda to`rlandirish uchun ham qo`llaniladi. Hozirgi paytta Puasson tenglamasini Dekart, polyar, tsilindr, sferik, koordinatalar sistemasida chegirmalik chegaraviy masalani yechish uchun ikki tejamkor to`g`ri metod bor, ularning biri dekompozitsiya- ya`ni Gaussning ketma-ket yo`q etish usilining modifikatsiyasi. Ikkinchisi o`zgaruvchini ajratish metodi ya`ni Fur`e turlandirishiga asoslangan. Ikki metod uchun ham arifmetik operatsiyalar soni Q=0 ($N^2 \log_2 N$), bunda N-bir yo`nalish bo`yicha tugunlar soni.

Yuqori ikki metodtan tashqari yana bir to`g`ri metod-matritsaviy haydash usuli kurib o`tiladi. U elliptik chegirmali tenglamalarni soha murakkab bo`lgan holatni tekshiradi. Lekin matritsaviy haydash usili Q=O(N^4) arifmetik amalni bajarishni va katta xotirani talab etadi.

Ketma-ket yaqinlashish metodining iteratsiyasi soha erikli bo`lgan holda, o`zgaruvchini koefitsientlar bilan birga umumiy tenglamalar uchun qollaniladi. Puasson tenglamasi uchun Dirixle masalasini $\overline{G}=\{0\le x_\alpha \le l_\alpha,\ \alpha=1,2\}$ to`g`ri burchakta ko`rib o`tamiz. G-chegarasi.

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=-f(x),\ \ x=(x_1,x_2)\in G,\ \ u|_\Gamma=\mu(x) \qquad (3)$$

$\overline{G}$ da h_1, h_2 qadam bilan to`g`ri burchakli to`r tuzamiz.

$$\omega_h=\left\{x_{i_1 i_2}=(i_1 h_q, i_2 h_2)\in \overline{G}, \quad \alpha=0,1,...N_\alpha, \quad h_\alpha N_\alpha=l_\alpha, \quad \alpha=1,2\right\}$$

Mayli $\gamma_h=\{x_{i1,i2}\in G\}$ to`rning chegarasi. (3) tenglamaga mos keluvchi Dirixle masalasining chegirmali masalasi

$$\Delta u=-f(x), \quad x\in \omega_h, \quad u|_{\gamma_h} = \mu(x)$$

$$\Delta=\Delta_1+\Delta_2, \quad \Delta_\alpha u = y_{x_\alpha x_\alpha}, \quad \alpha=1,2 \qquad (4)$$

$$y_{i_1 i_2} = y(i_1 h_1, i_2 h_2)$$

ko`rinishda bo`ladi.

1.2 O`zgaruvchini ajratish metodi.

(4) masalaning yechimini bir jinsli chegaraviy shartlarda ko`rib o`tamiz.

$$\wedge u=-\varphi, \quad u|_{\gamma_h} = 0 \qquad (5)$$

bu erda φ, (4) masaladagi f tan faqat chegaradagi tugunlarda $\frac{1}{h_1^2}\mu,\ i_1=1,\ i_1=N_1-1$ qiymatga va $\frac{1}{h_2^2}\mu,\ i_2=1,\ i_2=N_2-1$ qiymatga o`zgaradi.

Mayli $\mu_k(jh_2)$ va λ_k-xos funktsiya va xos qiymat bo`lsin.

$$\wedge_2\mu_k+\lambda_k\mu_k=0, \quad h_2\le x_2 \le l_2-h_2 \qquad (6)$$

$$\mu_k(0)=\mu_k(l_2)=0$$

$$\mu_k(jh_2)=\sqrt{\frac{2}{l_2}}\sin\frac{k\pi j}{N_2}, \quad \lambda_k=\frac{4}{h_2^2\sin^2\frac{k\pi h_2}{2l_2}} \quad \lambda_k=\frac{4}{h_2^2}\sin^2\frac{k\pi h_2}{2l_2} \qquad k=1,2,3\ldots N_2-$$

ekanligini bila o`tirib (5) masalaning yechimini quydagicha qator ko`rinishda qidiramiz [3].

$$y_{ij}=\sum_{k=1}^{N_2-1}C_k(ih_1)\mu_k(jh_2)\ i=1,2,3\ldots.N_1-1, j=1,2,3\ldots N_2-1 \qquad (7)$$

bu erda S_k-Fur`e koeffitsienti $x_1=ih_1$ dan qarab. (7) ni (5) tenglamaga olib borib qo`ygandan so`ng

$\wedge y=\wedge_1 y+\wedge_2 y=$

$$\sum_{k=1}^{N_2-1}\left[\mu_k(jh_2)\wedge_1 C_k(ih_1)+C_k(ih_1)\wedge_2\mu_k(jh_2)\right]=-\sum_{k=1}^{N_2-1}\varphi_k(ih_1)\mu_k(jh_2)$$

(8)

ga ega bo`lamiz. Bu erda $\varphi_k(ih_1)$, $\varphi(x)$ funktsiyasining Fur`e koeffitsientlari. $\varphi_k(ih_1)=\sum_{i=1}^{N_2-1}\varphi(ih_1, jh_2)\cdot\mu_k(jh_2)\cdot h_2$

(6) ni hisobga olib va μ_k - funktsiyalarining ortogonalligi (8) dan S_k ni aniqlash uchun quydagi masalaga ega bo`lamiz.

$$\wedge_1 c_k-\lambda_k c_k=-\varphi_k, \qquad h_1\le x_1 \le l_1-h_1$$

$$c_k(0)= c_k(l_1)=0, \qquad k=1,2,\ldots N_2-1 \qquad (9)$$

ko`rinib turganidek $c_k(ih_1)$, $x_1=ih_1$ ning funktsiyasi sifatida har bir k uchun haydash metodi bilan topiladi. Hammasi bo`lib N_2-1 marta haydash algoritmidan foydalanish kerak. (7) formula bo`yicha $c_k(ih_1)$ ni bila o`tirib (5) masalaning yechimini topamiz.

φ_k -Fur`e koeffitsientlaridan hisoblash va u_{ij} - yechimni topish umumiy bir formula bilan hisoblab topiladi. Bu erda

$$v_j=\sum_{k=1}^{N-1} z_k \sin\frac{k\pi j}{N}, \qquad j=1,2,3\ldots N-1$$

katorning yig`indisini topish mumkin. Uning uchun Fur`e to`rlandirishining maxsus algoritmi qo`llaniladi, ya`ni barcha ko`rsatilgan yig`indi $q=5N\log_2 N$ arifmetik operatsiyada ($N=2^n$) oddiy holda $0(N^2)$ arifmetik operatsiyada hisoblanadi.

Shu bilan birga (4) chegirmalik Dirixle masalasining yechimi to`g`riburchakda $0(N_1N_2\log_2 N_2)$ arifmetik operatsiyada topiladi. Xususiy holda, o`zgaruvchini ajratish metodi dekompozitsiya metodi bilan birgalikta qo`llanilishi mumkin.

§2. Ikki qatlamli iteratsion sxemalar.

2.1. Masalaning qo`yilishi.

Mayli birinchi to`r

$$Au=f \qquad (10)$$

bir jinsli tenglama berilsin. Bunda $A:H\rightarrow H$ chiziqli operator, haqiyqiy fazoda skalyar ko`paytma va norma $\|y\|=\sqrt{(y,y)}$ aniqlangan, quydagicha faraz qilamiz $A=A^{*}>0$, $f\in H$-hohlagan vektor. Dastlab iteratsion metodning umumiy xarakteristikasiga to`xtaymiz. Iteratsion metod qandaydir dastlabki $y_0\in H$ yaqinlashishdan boshlab tenglamaning yechimini ketma-ket topishga yordam beradi $y_1,y_2,\ldots y_k,y_{k+1}\ldots$ u holda, k-iteratsiya nomeri. y_{k+1} ning qiymati o`zidan oldingi $y_k,y_{k+1}\ldots$ iteratsiyalar yordamida topiladi. Agar y_{k+1} ni hisoblashda faqatgina oldingi y_k iteratsiya foydalanilsa, u holda iteratsion metod bir qadamli, agar oldingi ikki iteratsiya foydalanilsa, unda iteratsiya metodi ikki qadamli deb ataladi. Biz quydagi bir qadamli iteratsion metod formasi bo`yicha ikki qatlamli sxema bilan mos kelishini ko`rsatamiz [7].

Hohlagan chiziqli bir qadamli iteratsion metod (10) tenglamani yechish uchun

$$B_k y_{k+1}=c_k y_k+F_k, \qquad k=0,1,2,\ldots.. \qquad (11)$$

Ko`rinishida yozilishi mumkin. Bu erda $B_k$ va $c_k$ lar N tan olingan chiziqli operatorlar, ular iteratsiya nomeri k-dan g`arazli, ol $F_k\in H$-berilgan funktsiya y_k - k-chi iteratsiya. (10) tenglamaning k-dan g`arazli aniq yechimi (11) tenglamani qanoatlandirishi shart.

$$(B_k-c_k)u=F_k \qquad \text{al ol} \qquad (B_k-c_k)A^{-1}f=F_k$$

o`rinli bo`lganda bajariladi. Bundan kelib chiqadi

1) $(B_k-c_k)^{-1}$ -teskari operator bor

2) $f=A(B_k-c_k)^{-1}F_k$.

Hamma vaqt $\tau^{-1}_{k+1}(B_k-c_k)=A$, $F_k=f\tau_{k+1}$, $k=0,1,2,\ldots$ deb olish mumkin, bunda $\tau_{k+1}>0$-sonli parametr.

Natijada ikki qatlamli iteratsion sxemaning kanonik formasini olamiz.

$$B_k\frac{y_{k+1}-y_k}{\tau_{k+1}}+Ay_k=f, \qquad k=0,1,2,..... \tag{12}$$

$k=0$ bo`lganda erkli dastlabki yaqinlashishni olamiz (nollik iteratsiya) $y_0\in H$ . $B^{-1}_k$ -teskari operator bor bo`lganlikdan (12) dan

$$y_{k+1}=y_k-\tau_{k+1}B^{-1}_k(Ay_k-f) \tag{13}$$

yaki $y_{k+1}=y_k-\tau_{k+1}B^{-1}_kr_k=y_k-\tau_{k+1}\omega_k$, bu erda $r_k=Ay_k-f$, $\omega_k=B^{-1}_kr_k$ tuzatish. Agar y_k -iteratsiya belgili bo`lsa, u holda $y_{k+1}$ (13) tenglamadan topiladi. $y_0$- ni bila o`tirib ketma-ket $y_1,y_2,\ldots$ larni topamiz.

Iteratsion metod qiymatga ega agar u yaqinlashuvchi bo`lsa ya`ni

$$\| y_k-u\|\to 0,\ k\to\infty \ \text{da} \tag{14}$$

odatta qandaydir $\varepsilon>0$ kichkina qiymat berib (10) masalaning taqribiy yechimini topamiz.

$$\| y_k-u\|\le\varepsilon\| y_0-u\| \tag{15}$$

sharti o`rinli bo`lsa hisoblashni to`xtatamiz. Bu shart amalda tekshirish uchun noqulay, sababi u belgisiz vektor, va

$$\|A y_k-f\|\le\varepsilon\| Ay_0-f\| \tag{16}$$

sharti bilan almashtirilishi mumkin, bu erda

$$r_k=Ay_k-f=Ay_k-Au$$

umumiy holda (15) ning o`rniga

$$\| y_k-u\|_D\le\varepsilon\| y_0-u\|_D \tag{17}$$

bunda $D=D^*>0$ - qandaydir operator.

$D=A^2$ deb olib (17 tengsizlikdan (16) ni olamiz $z_k= y_k-u$ qoldiq uchun tenglama yozamiz. Au=f bo`lganlikdan

$$B_k \frac{z_{k+1} - z_k}{\tau_{k+1}} + Az_k = 0, \qquad k = 0,1,2,..... \tag{18}$$

$z_0 \in H$ berilgan. Agar $B_k=B$, k-dan g`arazli bo`lmasa, unda

$\omega_k=B^{-1}r_k$ va

$$B \frac{\omega_{k+1} - \omega_k}{\tau_{k+1}} + A\omega_k = 0, \tag{19}$$

bir jinsli tenglamani qanoatlantiradi. haqiqattan ham (13) tan

$$y_{k+1}-y_k=-\tau_{k+1}B^{-1}r_k= -\tau_{k+1}\,\omega_k$$

Bu tenglikning ikki tarafini A operatorini tasir ettirip, va

$$Ay_{k+1}-Ay_k= (Ay_{k+1}-f)-(Ay_k - f)=r_{k+1}- r_k$$

$$r_{k+1}- r_k =B(B^{-1}r_{k+1}-B^{-1} r_k)=B(\omega_{k+1} -\omega_k)$$

ekanligini hisobka olib (19) tenglamani hosil qilamiz [8].

(18) dan ko`rinib turganidek

$$z_{k+1}=S_{k+1}z_k,\ S_{k+1}=E-\tau_{k+1}B^{-1}A \tag{20}$$

bu erda S_{k+1} k-qatlamdan k+1 qatlamga o`tuvchi operator.

$z_k, z_{k-1},\dots z_1$ larni ketma-ket yo`q qilib k=n-1 da

$$z_n=T_nz_0,\quad T_n=S_nS_{n-1}\dots S_2S_1 \tag{21}$$

ga ega bo`lamiz. Bu erda T_n - (18) sxemaning yechuvchi operatori (21) dan

$$\| z_n \|_D=\| T_n z_0 \|_D \le \| T_n\|_D \|z_0 \|_D \quad \text{yoki}$$

$$\| z_n \|_D \le q_n \|z_0 \|_D , \qquad q_n= \|T_n \|_D \tag{22}$$

ni topamiz. Agar $q_n \le \varepsilon$ bo`lsa iteratsiya o`rinlanish sharti to`xtatiladi. Shunday qilib, iteratsiyasi yaqinlashuvchi bo`lishini tekshirish uchun, yechuvchi T_k operatorining normasini baholash kerak. Au=f tenglamaning hohlagan $\{V_k\}$ operatorda va hohlagan $\{\tau_{k+1}\}$ parametrda yechish uchun (20) sxema aniq approksimatsiyaga ega. Lekin q_n sxemasi $\{V_k\}$ va $\{\tau_{k+1}\}$ lardan g`arazli. Shuning uchun $V_k$ va $\tau_{k+1}$ larni shunday qilib saylab o`lish kerak, (20) sxemaning yechuvchi operatori T_n ning normasi $\| T_n\|_D=q_n$ eng kichkina bo`lishi kerak. Shu bilan birga $V_k$ ni saylab o`lishda, berilgan y_k da u_{k+1} larni

$$V_k\, u_{k+1}=G`_k\ ,\ G`_k=V_k\, u_k-\tau_{k+1}(Ay_k-f),$$

tenglamadan aniqlash uchun arifmetik amallarni boricha kamaytish kerak. Iteratsion metodning asosiy masalasi shundan iborat. (12) korinishidagi hohlagan iteratsion protsesni normal kurinishda

$$B\frac{du}{dt}+Au=f,$$

nostatsionar tenglamani echishda ikki qatlamli sxema deb qarash mumkin. Bu erda t_{k+1} parametrni

$$t_{k+1}=\sum_{m=1}^{k+1}\tau_m$$

Vaqtning qadami deb qarash mumkin.

Iteratsion sxema va nostatsionar masala uchun sxema orasidagi farq quydagidan iborat.

a) (12) iteratsion sxema (10) tenglamani aniq approksimatsiyalaydi. Sababi (10) tenglamaning echimi u hohlagan V_k va τ_{k+1} larda (12) tenglamani qanoatlantiradi.

b) Aniq echimni olish τ_{k+1} parametrni va V_k operatorni saylab olish iteratsiyaning yig`ilish shartiga va arifmetik amallarning ozligiga bog`lik. (nostatsionar masalada qadamni saylab olish, approksimatsiya talabiga bog`inadi) Mayli Q(ε)- arifmetik amallarning umumiy soni bolsin, yani (12) metod yordamida (10) tenglama echimini berilgan ε>o aniqliqta olish kerak bolsin. Sxemani yani $\{\tau_k\}$ va $\{V_k\}$ larni shunday qilib saylab olish kerak yani Q(ε) eng kichik bolsin. Agar n=n(ε) ε-aniqlikda oladigan iteratsiyaning minimal soni bolsa u holda

$$Q(\varepsilon) = \sum_{k=1}^{n(\varepsilon)} Q_n = \overline{Q}_n n,$$

bunda Q_k k-chi iteratsiyani topish uchun ketadigan amallar soni. Q(3) ning minimumin topish, iteratsiya soni n(ε), va Q_n sonining minimumini topishga olib keladi.

Agar V_k=E-birlik operator bolsa unda (12) aniq iteratsion sxema deb ataladi.

$$\frac{y_{k+1} - y_k}{\tau_{k+1}} + Ay_k = f, \qquad k = 0,1,2,.....$$

hohlagan $y_0 \in H$ uchun. Agar $V_k \neq E$ bolsa (12) sxema noaniq.

2.2 Oddiy iteratsiya metodi.

$$\tau_0 = \frac{2}{\gamma_1 + \gamma_2} \tag{23}$$

parametrda oddiy iteratsiya metodi quydagi kurinishda boladi.

$$\frac{y_{k+1} - y_k}{\tau_0} + Ay_k = f, \tag{24}$$

Sababi $t_1 = \cos\frac{\pi}{2} = 0, \quad \tau_1 = \tau_0$.

Bu erda

$$q_1 = \frac{2\rho_1}{1+\rho_1^2} = \rho_0 \tag{25}$$

Bog`lanish formulasi $r_k=Ay_k–f$ uchun $y_{k+1}=Sy_k$, $S=E-\tau_0 A$ tenglamaga ega bolamiz. $T_1=S$ bolganlikdan (25) dan otish operatorining normasi $||S||=\rho_0=\frac{1-\xi}{1+\xi}$ boladi. Oddiy iteratsiya metodi boyicha n- iteratsiyadan keyin $r_n=S^n r_0$, $||r_n||\le\rho^n{}_0||r_0||$ ga ega bolamiz. $\rho^n{}_0\le\varepsilon$ sharti orinli boladi agar

$$n\ge\frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln\left(\frac{1}{\rho_0}\right)}$$ bolsa va $n\ge n_0(\varepsilon)$

$$n_0(\varepsilon)=\frac{\ln(\frac{1}{\varepsilon})}{2\xi} \qquad (26)$$

orinli.

2.3. Maksimum qiymat printsipi.

Biz endilikda doymiy koeffitsentli tenglamani qarab otamiz

$$\vartheta_t=a^2\vartheta_{xx}+\beta\vartheta_x+\gamma\vartheta$$

Bu tenglama $\vartheta=e^{\mu x+\lambda t}\cdot u$

bu erda

$$\mu=-\frac{\beta}{2a^2},\ \ \lambda=\gamma-\frac{\beta^2}{4a^2}$$

urniga qoyish yordamida

$$u_t=a^2u_{xx} \qquad (*)$$

korinishiga keltiradi. Tenglama echimining maksimal qiymat printsipi deb ataluvchi quydagi hossasini keltiramiz [3,4].

Agar u(x,t) funktsiya $0\le t\le T$ va $0\le x\le\iota$ yopiq sohada aniqlangan va uzluksiz bolsa va soxaning $0<x<\iota$ va $0<t\le T$ nuqtalarida $u_t=a^2u_{xx}$ harorat tenglamasini qanoatlantirsa u holda u(x,t)funktsiyasi uzuning maksimal

va minimal qiymatini boshlang`ich vaqt momentida yoki $x = 0, x = \ell$ chegara nuqtalarida erishadi. u(x,t)=const bolsa funktsiya harorat otkazuvchanlik tenglamasini qanoatlantiradi va uzining maksimal va minimal qiymatiga hohlagan nuqtada erishadi.

Bu teoremaga qarshi kelmaydi, sababi uning sharti buyicha agar maksimal (minimal) qiymat oblast ichida erishilsa u holda t=0 yoki x=0, x=l da erishiladi.

Bu teoremaning fizik ma`nosi tushinarlik agar temperatura chegarada va boshlangich vaqt momentida qandaydir M sonidan oshib ketmasa, u holda jism ichida harorat manbasi bolmasa M dan katta temperatura paydo bolishi mumkin emas.

Maksimal qiymat uchun teoremaning isbotini korib otamiz.

Teorema isbotini qarama-qarshidan boshlaymiz u(x,t) funktsiyasining t=0 ($0 \le x \le l$)yoki x=0, x=l da maksimal qiymatini M orqali elgilaymiz, va qandaydir (x_0, t_0) ($0 < x_0 < l$, $0 < t_0 < T$) nuqtada u(x, t) funktsiyasi ozining

$$u(x_0, t_0) = M + \varepsilon$$

Maksimal qiymatiga erishsin. (x_0, t_0) nuqtada (13) tenglamaning chap va ong taraflaridagi belgilarini tekshiramiz. (x_0, t_0) nuqtada funktsiya uzining maksimal qiymaticha erishadigan bolsa u holda

$$\frac{\partial u}{\partial x}(x_0, t_0) = 0 \text{ ва } \frac{\partial^2 u}{\partial x^2}(x_0, t_0) \le 0$$

Bulishi shart song $u(x_0, t_0)$ da $t = t_0$ maksimal qiymatga erishadigan bolsa

$$\frac{\partial u}{\partial t}(x_0, t_0) \ge 0$$

(*) ning ong va chap taraflaridan belgilarini solishtirib ularning har xil ekanligini koramiz. Bu hali teorema isbotlandi degani emas. Toliq isbotlash uchun (x_1, t_1) nuqtani topamiz u $\frac{\partial^2 u}{\partial x^2}(x_0, t_0) \geq 0$ va $\frac{\partial u}{\partial t} > 0$ shartini qanoatlanlantirsin. Uning uchun qoshimcha

$$\vartheta(x,t)=u(x,t)+k(t_0-t) \qquad (**)$$

Erdamga funktsiya kiritamiz k -qandaydir son

$$\vartheta(x_0, t_0)=u(x_0, t_0)=M+\varepsilon$$

va $\qquad k(t_0-t)\leq kT$

k>0 sonini $\quad kT<\frac{\varepsilon}{2}$ eki $k < \frac{\varepsilon}{2T}$ buladigandek etib saylab olamiz.

U holda $\vartheta(x,t)$ ning $t = 0$ yaki $x = 0, x = l$ diga maksimal qiymati $M + \frac{\varepsilon}{2}$ dan ortib ketmaydi ya`ni

$$\vartheta(x,t) \leq M + \frac{\varepsilon}{2} \qquad (t=0,\ x=0,\ x=l)$$

(**) formulaning birinchi qoshiluvchisi M dan ikkinchisi ega - $\frac{\varepsilon}{2}$ dan ortib ketmaydi. Agar u(x,t) ni chegaralangan $0\leq x\leq l$, $0\leq t\leq T$ sohada uzliksiz deb olmasak u holda u(x,t) hech bir nuqtada ozining maksimum nuqtasiga ega bola olmaydi. Har qanday uzluksiz funktsiya uzining maksimal qiymatiga tuyuq sohada erishadi degan teorema kuchiga kora

1) u(x,t) ozining maksimal qiymatiga tog`riburchakning pastin yoki yon tarafida erishadi va M ga teng boladi.
2) Agar u(x,t) ning qiymati kamida bitta nuqtada M dan katta bolsa, u holda
3) (x_0 ,t_0) nuqta bor bolib u(x,t) M dan ortib ketmaydigan maksimal qiymatga ega boladi

$$u(x_0, t_0)=M+\varepsilon, \quad (\varepsilon>0)$$

Bu erda $0<x_0<\ell$, $0<t_0 \leq T$ haqiqattan xam analizdan biz bilamiz f(x) funktsiya x_0 nuqtada (0,) interval ichida min ga ega bolishi uchun

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = 0, \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} > 0$$ sharti orinli bolishi zarur.

Shunday etib x_0 nuqtada f(x) funktsiya max ga ega bolishi uchun

1) $f`(x_0)=0$ 2) $f``(x_0)\leq 0$ bolishi kerak.

Bundan $t_0<T$ bolganda $\frac{\partial u}{\partial \tau} = 0$ agar $t0 = T$ bwlsa $\frac{\partial u}{\partial \tau} \geq 0$

v(x,t) ning uzluksizligidan u qandaydir (x_1, t_1) nuqtada ozining maksimal qiymatiga erishadi. Haqiyqattan ham

$$\vartheta(x1,t1)\geq\vartheta(x_0, t_0)=M+\varepsilon,$$

Shuning uchun $t_1>0$ va $0<x_1<\ell$ $t=0$ yoki $x=0, x=\ell$ da (17) formula orinli boladi (x_1, t_1) nuqtada (14) va (15) ga oxshash $v_{xx}(x_1, t_1)\leq 0$, $v_t(x_1, t_1)\geq 0$ bolishi shart (16)ni hisobiga olib

$$u_{xx}(x_1, t_1)=\vartheta_{xx}(x_1, t_1)\leq 0,$$

$$u_t(x_1, t_1)=\vartheta_t(x_1, t_1)+k\geq k>0$$

Bundan $u_t(x_1, t_1)-a^2 u_{xx}(x_1, t_1)\geq k>0$ ekanligi kelib chiqadi. Demak $u_t=a^2 u_{xx}$ tenglama (x_1, t_1) ichki nuqtada qanoatlandirilmaydi. Shuningdek (13) tenglamaning u(x,t) echimi soxa ichidagi qiymati u(x,t) ning chegaradigi qiymatidan oshib ketmas ekan ($t=0$, $x=0$, $x=\ell$) Endi birinchi chegaraviy shart uchun birdan birlik teoremasini isbotlaymiz. Birdan birlik teoremasi

$0\leq x\leq\ell$, $0\leq t\leq T$ soxada aniqlangan va uzluksiz $u_1(x,t)$ va $u_2(x,t)$ ikkita funktsiya

$$u_t=a^2 u_{xx}+f(x,t) \quad (0<x<\ell,\ t>0)$$

Harorat otkazgichlik tenglamasini qanoatlantirsa va bir hil boshlangich va chegaraviy shartlarini qanoatlandirsa

$u_1(x,0)=u_2(x,0)=\varphi(x)$

$u_1(0,t)=u_2(0,t)=\mu_1\ (t)$

$u_1(\ell,t)=u2(\ell,t)=\ )=\mu_2\ (t)$

U holda $u_1(x,t)=u_2(x,t)$

Bu teoremani isbotlash uchun $\vartheta(x,t)=u_2(x,t)-u_1(x,t)$

funktsiya kiritamiz. $u_1(x,t)$ va $u_2(x,t)$ funktsiyalari $0\le x\le \iota$ va

ℓ va $0\le t\le T$ da uzluksiz bwlgani uchun ushbu soxada $\vartheta(x,t)$ham uzluksizdir.

$0 < x < \ell, t > 0$ soxada ikkita echimning ayirmasi sifatida $\vartheta(x,t)$ushbu sohada harorat utkazuvchanlik tenglamasining echimi ham bwladi. Shunday qilib maksimum qiymat printsipi ushbu funktsiyaga ham tegishli ya`ni u uzining maksimal yoki minimal qiymatiga $t = 0$ yoki $x = 0$ yoki $x = \ell$ chegarada erishadi. Lekin shart boyicha

$$\vartheta(x,0)=0, \quad \vartheta(0,t)=0, \quad \vartheta(\ell,t)=0$$

Shuning uchun

$$\vartheta(x,t)\equiv 0,$$

ya`ni $u_1(x,t)\equiv u_2(x,t)$.

Bundan birinchi chegaraviy masalaning echimi yagona.

Xossalari.

1. Harorat otkazuvchanlikning ikkita echimi $u_1(x,t)$ va $u_2(x,t)$ lar

   $u_1(x,0)\le u_2(x,0)$

   $u_1(0,t)\le u_2(0,t)$, $u_1(\ell,t)\le u2(\ell,t)$

shartlarini qanoatlantirsa u holda barcha $0\le x\le \ell$ va $0\le t\le T$ lar uchun

$u_1(x,t)\le u_2(x,t)$ boladi.

Haqiqattan ham $\vartheta(x,t)=u_2(x,t)-u_1(x,t)$ ayirma maksimum qiymat printsipi shartlarini qanoatlandirsa va undan tashqari

$\vartheta(x,0)\ge 0, \quad \vartheta(0,t)\ge 0, \quad \vartheta(\ell,t)\ge 0$ shuning uchun $\vartheta(x,t)\ge 0,$

boladi $0<x<\ell$ va $0<t\leq T$. Teskari holda $\vartheta(x,t)$ funktsiya $0<x<\ell$ va $0<t\leq T$ sohada manfiy minimal qiymatni qabul qilar edi.

2. Harorat utkazuvchanlikning tenglamasining uchta echimi $\underline{u}(x,t),\ u(x,t),\ \overline{u}(x,t)$ лар $\underline{u}(x,t)\leq u(x,t)\leq \overline{u}(x,t)$

3. shartni t=0, x=0, va

$x=\ell$ da qanoatlantirsa u holda bu tengsizlik barcha x, va t lar uchun $0\leq x\leq$

va $0\leq t\leq T$ sohada orinli boladi.

Harorat otkazuvchanlikning tenglamasining $u_1(x,t)$ va $u_2(x,t)$ echimlari uchun $|u_1(x,t)-u_2(x,t)|\leq \varepsilon$,

$t=0,\ x=0, x=\ell$ da tengsizligi bajarilsa u xolda $0\leq x\leq \ell$ va $0\leq t\leq T$ shartini qanoatlantiruvchi barcha x va t lar uchun $|u_1(x,t)-u_2(x,t)|\leq \varepsilon$

sharti bajariladi.

§3. Uch qatlamli iteratsion sxemalar.

3.1 Zeydel metodi.

Noaniq sxemalar, aniq sxemalarga solishtirganda doimiy buladi. Oddiy noaniq sxemaga Zeydel metodi kiradi. Quydagi chiziqli algebraik tenglamalar sistemasini qaraymiz.

$$Au=f \qquad (26)$$

yaki $$\sum_{j=1}^{N} a_{ij} u_j = f_i, \quad i = 1,2,\dots,N \quad .$$

A matritsasining diogonal elementlari $A=(a_{ij})$ noldan farqli dep $a_{ij}\neq 0$, quydagi iteratsion metodni yozamiz.

$$\sum_{j=1}^{i} a_{ij} y_j^{k+1} + \sum_{j=i+1}^{N} a_{ij} y_j^{k} = f_i, \quad a_{ij} \neq 0 \qquad (27)$$

Bu erda y^k_j k-nomerdagi iteratsiya (k+1)- iteratsiyani i=1 dan boshlaymiz.

$$a_{11} y_1^{k+1} + \sum_{j=2}^{N} a_{1j} y_j^{k} = f_1, \quad a_{11} \neq 0$$

Bundan y_1^{k+1} ni topamiz. i=2 uchun

$$a_{21} y_1^{k+1} + a_{22} y_2^{k+1} + \sum_{j=3}^{N} a_{2j} y_j^{k} = f_2, \quad a_{22} \neq 0$$

Bizga y_1^{k+1} ning qiymati belgili bolganlikdan va $a_{22}\neq 0$, bulgani uchun y_2^{k+1} ni topamiz va h.z. A matritsasini quydagicha yig`indi korinishida yozamiz.

$$A= A^{-}+A^{+}+D \qquad (28)$$

bu erda $A^{-}=(a^{-}_{ij})$, $a^{-}_{ij} = a_{ij}$ agar va j<i va $a^{-}_{ij}=0$ agar j≥i bosh diogonaldagi quydagi uchburchakli nollik matritsa $A^{+}=(a^{+}_{ij})$,

$a^+_{ij}= a_{ij}$ va $j>i$ va $a^+_{ij}=0$ agar $j\le i$ bosh diogonaldagi yuqorgi uchburchakli nollik matritsa. $D=(a_{ii}\ \delta_{ij})$, $\delta_{ij}=1$ agar $j=i$ $\delta_{ij}=0$ agar $j\ne i$ diogonallik matritsa Ushbu belgilashlardan foydalanib Zeydel metodini quydagicha yozamiz

$$(A^-+D)y^{k+1}+A^+y^k=f, \qquad y=(y_1,y_2,\dots y_N) \qquad (29)$$

Bu ikki qatlamli sxemani kanonik turga keltiramiz.

$$(A+\overline{D})(y^{k+1}-y^k)+(A+\overline{D}+A^+)y^k=f$$

yoki
$$(A^-+D)(y^{k+1}-y^k)+Ay^k=f \qquad (30)$$

uni kanonik forma bilan solishtirib

$$B\frac{y^{k+1}-y^k}{\tau_{k+1}}+Ay^k=f \qquad k=0,1,..n-1 \qquad (31)$$

hohlagan $y_0\in H$ uchun $B=A^-+D$, $\tau_k=1$ ekanligini topamiz. Sxema noaniq, V-matritsasi uchburchakli va simmetriyali emas. ($B\ne B^*$-uz-uziga tuginlash emas).

$$\wedge u=-\varphi,\quad x\in\omega_h \qquad u|_{\gamma h}=0$$

Zeydel metodi quydagicha buladi.

$$y^{k+1}_{i_1-1}+y^{k+1}_{i_2-1}-4y^{k+1}+y^k_{i_1+1}+y^k_{i_2+1}=-h^2\varphi,$$

$$\text{демек}\quad y^{k+1}=\frac{y^{k+1}_{i_1-1}+y^{k+1}_{i_2-1}+y^k_{i_1+1}+y^k_{i_2+1}+h^2\varphi}{4} \qquad (32)$$

Hisoblash $i_1=1$, $i_2=1$ tuginlardan boshlanadi. (0,1) va (1,0) tuginlari chegarada etganlikdan, $y^{k+1}_{i_1-1}$ *ва* $y^{k+1}_{i_2-1}$ qiymatlari belgili boladi va (32) ning ong tarafining barchasi belgili boladi. y^{k+1} ning qiymati $i_1=1$, $i_2=1$ tuginda topiladi.

Undan song $i_2=2,3,\ldots$ larni $i_1=1$ deb olib y^{k+1} ni quydagi qatorda topamiz. Song $i_1=2,3,\ldots$ larga otamiz. Natijada y^{k+1} turning barcha tuginlarida topiladi.

Agar $A=A^*>0$ operatori oz-oziga tuginlash va ong aniqlangan bolsa (ong aniqlanadi, sababi $A\geq\gamma_1 E$, $\gamma_1=\min\lambda_k(A)>0$) Zeydel metodi yaqinlashuvchi. Zeydel metodining yaqinlashuv tezligini baholash uchun har hil fikr yuritamiz. Masalan agar

$$\sum_{j\neq i}|a_{ij}|\leq q|a_{ii}|, \qquad i=1,2,\ldots.N. \qquad (33)$$

Bolsa, unda q<1 bolganda Zeydel metodi geometrik progressiya tezligi bilan yaqinlashadi haqiqattan ham

$$a_{ii}z_i^{k+1}=-\sum_{j<i}a_{ij}z_j^{k+1}-\sum_{j>i}a_{ij}z_j^k$$

Formuladan

$z^{k+1}=y^{k+1}-u$ hatolik uchun

$$|a_{ii}||z_i^{k+1}|\leq\sum_{j<i}|a_{ij}||z_j^{k+1}|+\sum_{j>i}|a_{ij}|\|z_j^k\|_c$$

bu erda $\|z\|_c=\max\limits_{1\leq i\leq N}|z_i|$ Mayli $\max\limits_i|z_i^{k+1}|$, qandaydir $i=i_0$ da erishgan u holda $\|z^{k+1}\|_c=|z_{i0}^{k+1}|$ va

$$\|z^{k+1}\|_c\leq\left[\sum_{j>j_0}|a_{i_0 j}|/\left(|a_{i_0 i_0}|-\sum_{j<j_0}|a_{i_0 j}|\right)\right]\|z^k\|_c$$

(33) shart boyicha (a_{ij}) matritsasi dioganal bolib topiladi. Bu shart laplasning chegirmali operatori uchun orinli emas. Lekin Zeydel metodining yaqinlashuvchiligi bu holda oz-oziga tuginlachlikdan va A operatorining ong aniqlanganligidan kelib chiqadi.

Yuqori relaksatsiya metodi.

Iteratsion protsessni tezlatish uchun Zeydel metodini turlandiramiz yani (30) formulaga ω-parametrini shunday qilib kiritamiz, na`tiyjada

$$(A+\frac{1}{\omega}D)(y^{k+1}-y)+Ay^{k}=f \qquad (34)$$

Formulaga ega bolamiz. Bu metod relaksatsiya metodi deb nomlanadi. ω=1 bolganda Zeydel metodi kelib chiqadi. Agar parametr ω>1 bolsa u holda (34) iteratsion protsess yoqori relaksatsiya metodi deb nomlanadi.

(34) ni (31) bilan solishtirib $B=(A+\frac{1}{\omega}D)$, $\tau_k=1$ yoki

$V=\omega A+\bar{D}$, $\tau_k=\omega$ ga ega bolamiz.

V-operatori oz-oziga tuginlash emas. y^{k+1} ni hisoblash algoritmi uchburchakli matritsani turlandirishga olib keladi. Agar Zeydel metodi barcha $A=A^{9}>0$ lar uchun orinli bolsa, relaksatsiya metodining yaqilashuvchi bolishi uchun qoshimcha $0<\omega<2$ shartini talab etamiz. Yaqinlashish tezligi ω parametrdan g`arazli boladi. ω uchun nazariy baho va yaqinlashish tezligi urinli, lekin ularni qollanish $D^{-1}(A+A^{+})$ operatorining spektrini bilishni talab etadi. Uni topish hamma vaqt mumkin emas. Shuning uchun ω-parametrni amaliyotta iteratsiya sonini kamaytish uchun qolayli etib saylab olamiz. Bir xil tiptagi masalalar sinfini echishda bu metod juda qolayli.

Yuqori relaksatsiya metodini iteratsiya soni buyicha Chebishevning aniq metodi bilan solishtirish mumkin.

$$n\geq n_0(\varepsilon), \quad n_0(\varepsilon)=0\left(\frac{1}{h}\ln\frac{1}{\varepsilon}\right)$$

3.2. Noaniq iteratsion sxemalar.

Biz hohlagan $y_0 \in H$ uchun

$$\frac{y^{k+1}-y^k}{\tau_{k+1}}+Ay^k = f \qquad (35)$$

Korinishdagi aniq iteratsion sxemani tadqiq qildik.

Bu erda $y_{k+1}=y_k-\tau_{k+1}(Ay_k-f)$ formulasi buyicha hisoblandi. Yuqori relaksatsiya va Zeydel metodi $V\neq E$ noaniq sxemaga (31) misol buladi. Noaniq metodni qollashga toza y_{k+1} iteratsiyani aniqlashda quydagi tenglamani echish kerak.

$$By_{k+1}=F_k\,,\quad F_k=By_k-\tau_{k+1}(Ay_k-f) \qquad (36)$$

F_k- oldindan berilgan. Zeydel metodi holinda $V=\bar{A}+D$ uchburchakli matritsa, yuqori relaksatsiya metodinda $B=(\bar{A}+\frac{1}{\omega}D)$ uchburchakli matritsa.Shuning uchun y_{k+1} ni aniqlash minimal sondagi harakatni, modellik masalada esa y_{k+1} ni aniqlash uchun amallar soni turning tuginlariga proportsional boladi. V-operatoriga qoyiladigan talab quydagicha

1) iteratsiyaning minimum soni.

2) V-operatorining tejamkorligi: yani ikkinchi tartibli chegirmali elliptik tenglamalar uchun $By_{k+1}=F_k$

tenglamasining echimi turning tugunlar soniga proportsional arifmetik amalda topilishi kerak. $\{\tau_k\}$ parametrini optimal saylab olish $V\neq E$ bulgan holda noaniq sxemaga utkazilishi mumkin.

$$B\frac{y^{k+1}-y^k}{\tau_{k+1}}+Ay^k = f\,, \quad k=0,1,2,\dots n-1. \ \forall y_0\in H \qquad (37)$$

$$B=B^*>0, \quad A=A^*>0$$

$$\gamma_1 B \le A \le \gamma_2 B, \qquad \gamma_1 > 0 \qquad (38)$$

deb olamiz. Bu shartlar (31) iteratsion metodning semeystvosini aniqlaydi. Zeydel va yuqori relaksatsiya metodlari uchun $V \ne B^*$ oz-oziga tuginlash emas operator, unda ular (31) va (38) semeystvoga kirmaydi. $r_k = Ay_k - f$ uchun quydagi bir jinsli tenglamaga ega bo`lamiz. $r_0 = Ay_0 - f \in H$ berilgan

$$B\frac{r_{k+1} - r_k}{\tau_{k+1}} + Ar_k = 0, \qquad k = 0,1,2,\dots n-1 \qquad (39)$$

Bu sxema quydagi aniq sxemaga ekvivalent

$$\frac{x_{k+1} - x_k}{\tau_{k+1}} + cx_k = 0, \qquad k = 0,1,2,\dots n-1 \qquad (40)$$

$$x_0 \in H$$

Bu erda $x_k = B^{1/2} r_k$, $c = B^{-1/2} A B^{-1/2}$ haqiqattan ham V oz-oziga tuginlash ong operator bo`lganlikdan $B = B^* > 0$, u holda V-operatorining koreni $B^{1/2}$ bor bolib

$$(B^{1/2})^* = B^{1/2} > 0$$

(39) tenglamaga $B^{-1/2}$ operatori erdamida tasir qildirib va $r_k = B^{-1/2} r_k$ ni almashtirib (40) tenglamani olamiz.

Lemma 1. Mayli

$$A = A^* > 0, \; B = B^* > 0, \quad c = B^{-1/2} A B^{-1/2}$$

operatorlari berilsin. Unda quydagi operatorlik tengsizliklar

$$\gamma_1 B \le A \le \gamma_2 B, \qquad \gamma_1 > 0$$

$$\gamma_1 E \le C \le \gamma_2 E, \qquad (41)$$

ekvivalent.

Isboti.

$J = ((A - \gamma B)y, y) = (Ay, y) - \gamma(By, y) = (AB^{-1/2}(B^{1/2}y), B^{-1/2}(B^{1/2}y)) -$

$-\gamma(B^{1/2}y,B^{1/2}y)=(cx,x)-\gamma(x,x)=((c-\gamma E)x,x)$ bu erda $x=B^{1/2}y$.

y (shuningdek-x), N dan olingan erkli vektor bulganliktan

$$J=((A-\gamma B)y,y)=((c-\gamma E)x,x) \qquad (42)$$

deb $A-\gamma B$ va $c-\gamma E$ operatorlari birdek belgiga ega ekanligi kelib chiqadi.

Mayli $A-\gamma_1 B\geq 0$ unda (42) da $\gamma=\gamma_1$ dep olib

$J=((c-\gamma_1 E)x,x)\geq 0$ ega bo`lamiz $c\geq\gamma_1 E$ lemma isbotlandi.

Bu muzokaraning natijasi quydagicha Au=f tenglamasini echish uchun (31) noaniq sxemani qollash, $cv=\varphi$ tenglamasini

$$\frac{x_{k+1}-x_k}{\tau_{k+1}}+cx_k=\varphi, \qquad k=0,1,2,....n \qquad x_0\in H \qquad (43)$$

aniq sxema bilan echish ekvivalent agar

$$S=B^{-1/2}AB^{-1/2},\ \varphi=B^{-1/2}f \qquad \text{bo`lsa.}$$

Shuning uchun aniq sxema bilan olingan barcha natijalarni, noaniq sxemaga o`tgazishimiz mumkin.

Uch qatlamli iteratsion sxemalar.

Biz ilgari Au=f operatorlik tenglamani (A tuginlash operator bilan) γ_1 va γ_2 A operatorining chegaralari N_β da (bunda $B=B^* >0$) echish uchun ikki qatlamli sxemalarni ko`rib chiqqan edik.

Endi uch qatlamli iteratsion sxemalardan ko`rib o`tamiz. Mayli

$$Au=f \quad A:H\rightarrow H \qquad (44)$$

Tenglamasini oz-oziga tuginlash, musbat aniqlangan operatorda echish talab etilsin. Uning spektorining chegaralari belgili bo`lsin

$$A=A^*, \qquad \gamma_1 E\leq A_0\leq\gamma_2 E, \qquad \gamma_1>0 \qquad (45)$$

uch qatlamli iteratsion sxema uch iteratsiyani y_{k-1}, y_k va y_{k+1} ni bog`laydi. y_{k+1} iteratsiya y_{k-1}, y_k iteratsiyalar orqali anglatiladi. Aniq sxema odatta quydagicha yoziladi.

$$y_{k+1}=(1+\alpha)Sy_k-\alpha y_{k-1}+(1+\alpha)\tau_0 f \qquad (46)$$

$k=1,2\ldots.$ $y_k=Sy_0+\tau_0 f$, hohlagan $u_0\in H$ uchun bunda $S=E+\tau_0 A$ ikki qatlamli oddiy iteratsion sxema uchun τ_0 optimal parametr bilan otish operatori.

$$\tau_0=\frac{2}{\gamma_1+\gamma_2},\quad \rho_1=\frac{1-\xi}{1+\xi},\qquad \xi=\frac{\gamma_1}{\gamma_2} \qquad (47)$$

y_1-dastlabki iteratsiyani ikki qatlamli oddiy itertsiyadan topamiz. (46) sxemani kuydagisha usulda olamiz.

(44) tenglamani taer holida yozib olib τ parametrni $\|S\|$ minimal` bo`lgandek etib saylab olamiz.

$$u=u-\tau Au+\tau f=S(\tau)u+\tau f,\quad S(\tau)=E-\tau A,$$

Uning uchun $\tau=\tau_0$ deb olib

$$u=S(\tau_0)u+\tau_0 f \qquad (48)$$

ga ega bo`lamiz. Bu tenglamani quydagicha yozish mu`mkin. $(1+\alpha)u=(1+\alpha)Su+(1+\alpha)\tau_0 f$, va songi tenglamaga $(1+\alpha)Su$ di $(1+\alpha)Sy_k$ ga, αu di αy_k ga almashtirib noaniq sxemani qo`llaymiz. $\alpha$-parametr iteratsion minimumlik shartidan olinadi. Biz (46) sxemaning yaqinlashish tezligiga va $\alpha$-parametrni saylab olishga to`qtalib o`tirmaymiz. Faqat ohirgi natijani keltiramiz. (46) tenglamaga A operatorini qo`llaymiz, u $r_k=(Ay_k-f)$ birjinsli tenglamani qanoatlandiradi.

$$r_{k+1}=(1+\alpha)S\, r_k+\alpha\, r_{k-1},\quad k=1,2\ldots \qquad (49)$$

$r_1=Sr_0$ bunda $r_0=Ay_0-f\in H$ Bu masala uchun $\alpha=\rho_1^2$ deb olsak

$$\| Ay_n-f \|\le q_n\| Ay_0-f \| \qquad (50)$$

baholash urinli. Bunda

$$q_n = \rho_1^n\left(1+\frac{1-\rho_1^2}{1+\rho_1^2}\cdot n\right) \qquad (51)$$

iteratsiya soni uchun quydagicha baholash o`rinli.

$$n \ge \frac{\ln\frac{1}{\varepsilon}+\ln\left(1+\frac{1-\rho_1^2}{1+\rho_1^2}\cdot n\right)}{\ln\frac{1}{\rho_1}}, \quad \text{ол} \qquad n \ge \frac{\ln\frac{1}{\varepsilon}+\ln\left(1+\frac{2\sqrt{\xi}}{1+\xi}\cdot n\right)}{2\sqrt{\xi}}$$

da o`rinli. Uch qatlamli sxemada iteratsiya soni birqancha ko`b bo`ladi va ulkan xotirani talab etadi ($y_{k+1}$ ni aniqlashda $y_k$ va $y_{k-1}$ vektorlarni xotirada saqlash kerak) $\gamma_1$, $\gamma_2$ larning aniq berilishiga ko`proq g`arazli. Shuning uchun $\gamma_1$ va $\gamma_2$ berilsa amalda uch qatlamli sxemadan ko`ra ikki qatlamli sxemadan foydalangan avzal. Eslatma. Aniq sxemadan noaniq uch qatlamli sxemaga o`tish (46) tenglamada A ni va $B^{-1}A$ va f va $B^{-1}f$ ni almashtirishga olib keladi.

Demak

$$y_{k+1}=(1+\alpha)(E-\tau_0B^{-1}A)\, y_k -\alpha\, y_{k-1}+(1+\alpha)\tau_0B^{-1}Af \qquad \text{ya`ni}$$

$$By_{k+1}=(1+\alpha)(B-\tau_0A)\, y_k -\alpha B\, y_{k-1}+(1+\alpha)\tau_0f \qquad (52)$$

$By_1=B\ y_0-\tau_0A\ y_0\ +\tau_0f$ k=1,2,…, $y_0\in H$ berilgan. Agar (48) tenglamaning o`rniga

$$Bu=(1+\alpha)(Bu-\tau_0Au) -\alpha Bu+(1+\alpha)\tau_0f$$

tenglikni qo`ysaq (52) ni olish mu`mkin. τ_0, α- lar uchun (47) formula o`z ku`chida qoladi, lekin A-operatorining chegarasi N ta emas N_B da bo`ladi.

$$\gamma_1B \le A\le\gamma_2B, \qquad \gamma_1>0\ ,\ B=B^*>0 \qquad (53)$$

Unda (52) masalani echish uchun (50) tengsizlik o`rniga

$$\|\, Ay_n-f\, \|_{B^{-1}} \le q_n \|\, Ay_0-f\, \|_{B^{-1}} \qquad (54)$$

baholash o`rinli. Bu joyda q_n avvalgidek (51) formuladan aniqlanadi.

Eng tez tushish usuli.

Aniq sxema uchun eng tez tushish usuli

$y_{k+1}=y_k-\tau_{k+1}(Ay_k-f),\ \ k=1,2,\dots,\ \ \forall y_0\in H$ ko`rinishida bo`ladi. τ_{k+1} parametr quydagi formuladan aniqlanadi.

$$\tau_{k+1}=\frac{(r_k,r_k)}{(Ar_k,r_k)},\qquad r_k=Ay_k-f,\qquad k=0,1,2,\dots. \tag{55}$$

Bu formula N_A da $z_k=y_k-u$ qoldiqni minimizatsiyalash shartidan olinishi mu`mkin. ya`ni

$$\max_{\{\tau_{k+1}\}}\|z_{k+1}\|_A;\qquad \|z\|_A=\sqrt{(Az,z)}\qquad z_k=y_k-u$$

qoldiq uchun $z_{k+1}=z_k-\tau_{k+1}Az_k$ tenglamaga ega bulamiz. $v_k=A^{1/2}z_k$ deb olib

$$v_{k+1}=v_k-\tau_{k+1}Av_k \tag{56}$$

ga ega bo`lamiz. Normaning kvadratini hisoblab topamiz.

$$\|v_{k+1}\|^2=\|v_k\|^2-2\tau_{k+1}(v_k,Av_k)+\tau^2_{k+1}\|Av_k\|^2 \tag{57}$$

$\max\limits_{\{\tau_{k+1}\}}\|z_{k+1}\|^2$ shartidan

$$\tau_{k+1}=\frac{(Av_k,v_k)}{\|Av_k\|^2}, \tag{58}$$

ga ega bo`lamiz. Quydagi baholash o`rinli.

$$\|v_n\|\le\rho^n_0\|v_0\| \tag{59}$$

Endi v_k dan $z_k=A^{-1/2}v_k$ ga o`tish qoldi.

$Az_k=A(y_k-u)=Ay_k-f=r_k$, $(Av_k,v_k)=\|Az_k\|^2=\|r_k\|^2$, $\|Av_k\|^2=(A r_k, r_k)$ ekanligini hisobga olib (58) ni (55) ko`rinishiga olib kelamiz. (59) tengsizlikdan

$$\|y_k-u\|_A \le \rho^n_0 \| y_0-u\|_A \qquad (60)$$

sababi $\|v_n\|^2=(v_n ,v_n)=(A z_n, z_n)=\|z_n\|^2_A$ Shunday qilib N_A da eng tez tushish usuli.,

$$B\frac{y_{k+1}-y_k}{\tau_{k+1}}+Ay_k = f, \qquad k=0,1,2,.... \qquad \forall y_0\in H \qquad (61)$$

ko`rinishda bo`ladi. Yuqoridagi usul bo`yicha

$$\tau_{k+1}=\frac{(r_k, w_k)}{(Aw_k, w_k)}, \qquad w_k = B^{-1}r_k, \qquad r_k = Ay_k - f \qquad (62)$$

ega bo`lamiz. Agar $\gamma_1 B \le A \le \gamma_2 B$, $\gamma_1>0$ , sharti o`rinlansa

$\| Ay_n-f \|_{B^{-1}} \le q_n \| Ay_0-f \|_{B^{-1}}$ baholash o`rinli bo`ladi.

O`z-o`ziga tuginlash emas operator yordamida tenglamani eshish.

Au=f, $A:H\to H$ tenglamasini ko`rib o`tamiz. Bu joyda A-musbat aniqlangan o`z-o`ziga tuginlash emas chiziqli operator. Dastlabki ikki qatlamli doimiy parametrga ega sxemadan foydalanamiz.

$$\frac{y_{k+1}-y_k}{\tau}+Ay_k = f, \qquad k=0,1,2,.... \qquad \forall y_0\in H \qquad (63)$$

Iteratsiyaning yaqinlashish tezligini baholash uchun $z_{k+1}=Sz_k$, $S=E-\tau A$, $k=0,1,2,\dots$, $z_0\in H$ birjinsli tenglamani ko`rib o`tamiz. Bu joyda qoldiq $z_k=y_k-u$ bundan $\|z_{k+1}\|\le\|S\| \|z_k\|$

τ-parametrini $\max\limits_{\tau} \| S(\tau) \|$ shartidan saylab olamiz. Mayli A va A^{-1} operatorlarning quydagi chegaralari berilsin.

$A \le \gamma_1 E$ ya`ni $(Ay,y)\ge \gamma_1\| y\|^2$ $\gamma_1>0$

$A^{-1} \ge \frac{1}{\gamma_2}E$ ya`ni $\|Ay\|^2 \le\gamma_2 \|y\|^2$, $\gamma_2>0$ (64)

$B=B^*$ bo`lganda ikkinchi shart $A \le \gamma_2 E$ ge teng kuchli $2-\tau\gamma_2 \ge 0$ deb olib.

$\|Sy\|^2=\|y-\tau Ay\|^2=\|y\|^2-2\tau(Ay,y)+ \tau^2\|Ay\|^2 \le \|y\|^2-2(Ay,y)+\tau^2\gamma_2(Ay,y)=$

$= \|y\|^2-\tau(2-\tau\gamma_2)(Ay,y) \le \|y\|^2-\tau(2-\tau\gamma_2)\ \gamma_1\|y\|^2=(1-2\tau\gamma_1+\tau^2\ \gamma_1\gamma_2)\|y\|^2$

ya`ni $\|S\|^2 \le 1-2\tau\gamma_1+\tau^2\ \gamma_1\gamma_2$ uchxadning minimumlik shartidan τ ni topamiz.

$$\tau=\frac{1}{\gamma_2},\quad \|S\|^2 \le (1-\gamma_1/\gamma_2),\qquad \|S\| \le \sqrt{1-\xi},\ \tau=\frac{1}{\gamma_2},\ \xi=\frac{\gamma_1}{\gamma_2} \tag{65}$$

γ_1 va γ_2 parametr o`rniga $\gamma_1, \gamma_2, \gamma_3$ uch parametr berilgan holatni ko`ramiz. Endi A ni A_0 simmetrik operatorlar yig`indisi ko`rinishida va A_1 kosimmetrik operatorlarning yig`indisi ko`rinishida ko`ramiz.

$$A=A_0+A_1,\ A_0=1/2(A+A^*),\ A_1=1/2(A-A^*) \tag{66}$$

$A_0^*=A_0$, $A_1^*=-A_1$, $(A_1x,x)=-(x,A_1x)=0$ ya`ni $(Ax,x)=(A_0x,x)$

A-operatori

$$\gamma_1E \le A_0 \le \gamma_2E,\ \|A_1\| \le \gamma_3 \tag{67}$$

shartin qanoatlantiradi. Bunda $\gamma_2>\gamma_1>0$ va $\gamma_3 \ge 0$ berilgan sonlar. $z_{k+1}=(E-\tau A)z_k$ tenglamasini $z_{k+1}=y_{k+1}-u$ qoldiq uchun.

$$z_{k+1}=(E-\tau A_0-\tau A_1)z_k=(\theta E-\tau A_0)\ z_k+[(1-\theta)E-\tau A_1]z_k \tag{68}$$

bu joyda $0<\theta<1$ hohlagan son. τ va θ ni shunday etib saylab olamiz, ya`ni $\|S\|=\|E-\tau(A_0-A_1)\|$ minimal bo`lsin. Uchburchak tezligidan

$$\|z_{k+1}\| \le \theta\|E-\tau/\theta\ A_0\|\ \|z_k\|+\|(1-\theta)z_k-\tau A_1)z_k\| \tag{69}$$

A_0-operatori o`z-o`ziga tuginlash va $\gamma_1E \le A_0 \le \gamma_2E$ shuning uchun

$$\max_{\frac{\tau}{\theta}} \| E - \frac{\tau}{\theta} A_0 \| = \rho_{0,} \qquad \text{агар} \qquad \frac{\tau}{\theta} = \tau_0 = \frac{2}{\gamma_1 + \gamma_2} \tag{70}$$

Bu joyda $\rho_0 = \frac{1-\xi}{1+\xi}$, $\xi = \frac{\gamma_1}{\gamma_2}$, unda $\tau = \tau_0\theta$ (69) ning ung tomanidagi ikkinchi qushiluvchini ko`rib o`tamiz.

$\|(1-\theta)y-\tau A_1y\|^2=(1-\theta)^2\|y\|^2-2\tau(1-\theta)(A_1y,y)+\tau^2\|A_1y\|^2=(1-\theta)^2\|y\|^2+$
$\tau^2\|A_1y\|^2\leq [(1-\theta)^2+\tau^2\gamma_3{}^2]\,\|y\|^2= [(1-\theta)^2+\tau^2\theta^2\gamma_3{}^2]\,\|y\|^2$

Shunday qilib $\tau = \tau_0\theta$ bo`lganda

$$\|z_{k+1}\|\leq\|S\|\,\|z_k|, \quad \|S\|\leq f(\theta)$$

$$f(\theta)=\theta\rho_0+\sqrt{(1-\theta)^2+\theta^2a^2}, \qquad a^2=\tau_0^2\cdot\gamma_3^2 \tag{71}$$

tengsizlik o`rinli. Endi $f(\theta)$ funktsiyasining minimumini topamiz.

$$f'(\theta)=\rho_0-\frac{1-\theta-a^2\theta}{\sqrt{(1-\theta)^2+a^2\theta^2}}=\rho_0-\frac{\alpha-a^2}{\sqrt{\alpha^2+a^2}}, \qquad \alpha=\frac{1-\theta}{\theta}$$

$f'(\theta)=0$ sharti $\rho_0\sqrt{\alpha^2+a^2}=\alpha-a^2$ tenglikni beradi. Undan α-uchun kvadrat tenglama hosil qilamiz. $(1-\rho_0^2)\alpha^2-2a^2\alpha-a^2(a^2-\rho_0^2)=0$ uni echib

$$\alpha=a\frac{a+\rho_0\sqrt{1-\rho_0^2+a^2}}{1-\rho_0^2}$$

(ikkinchi koren a va ρ_0 parametrlarining qandaydir qiymatlarida manfiy bo`lishi mumkin)

$$\aleph=\frac{\gamma_3}{\sqrt{\gamma_1\gamma_2+\gamma_3^2}}$$ belgilash kiritamiz

$$\gamma_3^2=\frac{\aleph^2}{1-\aleph^2}\gamma_1\gamma_2, \quad a^2=\tau_0^2\cdot\gamma_3^2=\frac{4\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\cdot\frac{\aleph^2}{1-\aleph^2}=\frac{(1-\rho_0^2)\aleph^2}{1-\aleph^2}\cdot\frac{a^2}{\aleph^2}=\frac{1-\rho_0^2}{1-\aleph^2}$$

Ildiz osti ifodalanib

$$1-\rho_0^2+a^2=1-\rho_0^2+\frac{\aleph^2(1-\rho_0^2)}{1-\aleph^2}=\frac{1-\rho_0^2}{1-\aleph^2}=\frac{a^2}{\aleph^2}$$ bunnan

$$\alpha = \frac{a^2(\aleph - \rho_0)}{\aleph(1-\rho_0^2)} = \frac{\aleph(\aleph+\rho_0)}{1-\aleph^2}, \qquad 1+\alpha = \frac{1+\aleph\rho_0}{1-\aleph^2}, \quad \theta = \frac{1}{1+\alpha} = \frac{1-\aleph^2}{1+\aleph\rho_0}.$$

Endi

$$f(\theta) = \frac{1}{1+\alpha}\rho_0 + \frac{1}{1+\alpha}\sqrt{\alpha^2+a^2} = \frac{\rho_0^2+\alpha-a^2}{(1+\alpha)\rho_0}.$$

$$\rho_0^2+\alpha-a^2 = (1+\alpha)-(1-\rho_0^2+a^2) = 1+\alpha-\frac{a^2}{\aleph^2} = \frac{1+\aleph\rho_0}{1-\aleph^2} - \frac{1-\rho_0^2}{1-\aleph^2} = \rho_0\frac{\rho_0+\aleph}{1-\aleph^2}$$

ekanligini hisobga olib

$$\| S \| \le \frac{\aleph+\rho_0}{1+\aleph\rho_0}, \quad агар \quad \tau = \tau_0\frac{1-\aleph^2}{1+\aleph\rho_0}$$

bo`lganda. Shunday qilib (63) masalani echish uchun agar A operatori (67) shartni qanoatlantirsa $\|y_k-u\| \le \rho^n \| y_k-u\|$ baho o`rinli, bunda

$$\rho = \frac{\aleph+\rho_0}{1+\aleph\rho_0}, \qquad \aleph = \frac{\gamma_3}{\sqrt{\gamma_1\gamma_2+\gamma_3^2}}, \qquad \tau = \bar{\tau} = \frac{\tau_0(1-\aleph^2)}{1+\aleph\rho_0} \qquad (72)$$

iteratsiya soni $n \ge \dfrac{\ln\frac{1}{\varepsilon}}{\ln\frac{1}{\rho}}$. Endi o`z-o`ziga tuginlash $B=B^*>0$

operatori bilan birga quydagicha noaniq sxemani ko`rib o`taylik.

$$B\frac{y_{k+1}-y_k}{\tau_{k+1}} + Ay_k = f, \qquad k = 0,1,2,.... \quad \forall y_0 \in H \qquad (73)$$

Bu xolda quydagi aniq sxemaga o`tamiz.

$x_{k+1}=x_k-\tau(cx_k-\varphi)$

$x_k=B^{1/2}y_k$, $c=B^{-1/2}AB^{-1/2}$, $\varphi= B^{-1/2}f$ va (67) shart S uchun quydagicha yoziladi.

$$\gamma_1 B \le A_0 \le \gamma_2 B$$

$$(B^{-1}A_1y, A_1y) \le \gamma_3^2(By,y) \qquad (74)$$

ya`ni $\|A_1y\|_B^{-1} \le \gamma_3\|y\|_B$ Unda (72) tenglamaning o`rniga

$\|y_k-u\|_B \le \rho^n \| y_k-u\|_B$ ga ega bo`lamiz. Au=f tenglamasin o`z-o`ziga tuginlash emas A operatori bilan echish uchun eng kam tuzatishlar usulidan foydalansa ham bo`ladi.

$\tau = \bar{\tau}$ da u va (63) sxemaning tezligidek tezlik bilan yaqinlashuvchi bo`ladi. Aniq sxema uchun

$$\frac{y_{k+1}-y_k}{\tau_{k+1}} + Ay_k = f, \qquad k = 0,1,2,.... \quad \forall y_0 \in H \tag{75}$$

τ_{k+1} parametr

$$\tau_{k+1} = \frac{(Av_k, v_k)}{\| Av_k \|^2}, \qquad r_k = Ay_k - f, \qquad k = 0,1,2,.... \tag{76}$$

$\|r_{k+1}\|^2$ ing minimumlik shartidan formula bo`yicha xisoblanadi va bunda A-ning o`z-o`ziga tuginlashligi hech bir joyda qo`llanilmaydi. Eng kam tuzatishlar usulida (67) shartning o`rinlanishida (75) va (76) formulalar uchun $\|Ay_k-f\| \le \rho^n \|Ay_0-f\|$ baholash o`rinli. Bu joyda y_k- (75) masalaning echimi, ρ- (72) formula bo`yicha aniqlanadi. Eng tez tushish usuli bu xolda qo`llanilmaydi. Sababi A-o`z-o`ziga tuginlash operator. Noaniq eng kam tuzatishlar metodi uchun $\|A_1y-f\|_{B^{-1}} \le \rho^n \|y\|_{B^{-1}}$ baho o`rinli.

3.3 Matritsaviy haydash usulining ayrim turlari.

1. Ku`chli o`zgarivchili koefitsentlardagi chegirmalik tenglamani echishning oqim usuli.

Haydash usulining ku`chli o`zgaruvchi koefitsentlarida qo`llanilib echiladigan variantini ko`rib o`tamiz. Bunga harorat o`tkazuvchilik, gidrodinamika va magnitlik gidrodinamika masalalari kiradi. Bu joyda harorat o`tkazuvchilik va elektr o`tkazuvchilik koefitsentlari atrofning termodinamik parametridan g`arazli harorat masalalarida izotermik uchastkalar bo`lishi mu`mkin. U joyda harorat o`tkazuvchilik bo`lmasligi mu`mkin va izotermik uchastkalar bo`lishi mu`mkin, u joyda harorat o`tkazgichlik cheksiz katta. Ikkinchi tartibli chegirmali tenglamalarni echishda, bu masalalarni oddiy haydash usuli bilan approksimatsiyalaganda aytarlikdek darajada aniqlik yo`qoladi. Bunday holda haydash usulini oqim usulidan foydalanib qutilish mumkin. Haydash usulining bu varianti oddiy haydash usulini turlandirish natijasida olish mumkin. Shunday qilib quydagi chegirmali masalani ko`ramiz.

$$a_i y_{i-1} - c_i y_i + a_{i+1} y_{i+1} = -f_i \qquad i=1,2,..,N-1 \qquad (76)$$

$$y_0 = \aleph_1 y_1 + \gamma_1 \;, \qquad y_N = \aleph_2 y_{N-1} + \gamma_2 \qquad (77)$$

bunda $c_i = a_{i+1} + a_i + d_i$, $d_i > 0$, $0 \le a_i \le \infty$ (78)

$1 \ge \aleph_1$, $\aleph_2 \ge 0$, $\aleph_1 + \aleph_2 < 2$ Oddiy haydash formulalari (76)- (77) masala uchun (78) ni xisobga olsak

$$y_i = \alpha_{i+1} y_{i+1} + \beta_{i+1} \;, \quad i=0,1,2,\ldots N-1,$$

$$\alpha_{i+1} = \frac{a_{i+1}}{a_{i+1} + a_i (1-\alpha_i) + d_i} \qquad (79)$$

$$\beta_{i+1} = \frac{a_{i+1}}{a_{i+1}} (a_i \beta_i + f_i), \qquad i = 1,2,\ldots N-1$$

Yangi nomalum chegirmali funktsiyani kiritamiz.

$$\omega_i = a_i (y_{i-1} - y_i) \qquad (80)$$

va (76) tenglamani (77)-chegaralik shartini

$$\omega_i - \omega_{i+1} - d_i y_i = -f_i \;, \qquad i=1,2,\ldots,N-1 \qquad (81)$$

$$a_1 (1-\aleph_1) y_1 + \omega_1 = a_1 v_1, \quad a_N (1-\aleph_2) y_N - \aleph_2 \omega_N = a_N v_2 \qquad (82)$$

(79) ning birinchi formulasiga (80) ni y_i ning qiymatiga qo`yamiz. $y_i = y_{i+1} + \frac{\omega_{i+1}}{a_{i+1}}$ u xolda quydagiga ega bo`lamiz.

$$a_{i+1}(1-\alpha_{i+1})y_{i+1}+\omega_{i+1}=a_{i+1}\beta_{i+1} \qquad (83)$$

$\alpha_i=a_i(1-\alpha_i)$, $\gamma_i= a_i\beta_i$ belgilash kiritib (83)ni quydagi ko`rinishda yozamiz.

$$\alpha_i y_i+\omega_i=\gamma_i \qquad (84)$$

(81) dan va (84) dan y_i ni yo`qotib

$$\omega_i = \frac{\alpha_i}{\alpha_i + d_i}\omega_{i+1} + \frac{d_i\gamma_i - \alpha_i f_i}{\alpha_i + d_i} \qquad (85)$$

α_I va γ_i larni aniqlash uchun rekkurent formulalarni yozamiz.

$$\alpha_{i+1} = a_{i+1}(1-\alpha_{i+1}) = \frac{a_{i+1}\left[a_i(1-\alpha_i)+d_i\right]}{a_{i+1}+a_i(1-\alpha_i)+d_i}$$ ya`ni

$$\alpha_{i+1} = \frac{\alpha_i + d_i}{1+(\alpha_i + d_i)/a_{i+1}} \qquad (86)$$

$$\gamma_{i+1} = a_{i+1}\beta_{i+1} = \alpha_{i+1}(a_i\beta_i + f_i) = \alpha_{i+1}(\gamma_i + f_i) = \frac{a_{i+1}(\gamma_i + f_i)}{a_{i+1}+a_i(1-\alpha_i)+d_i}$$

ya`ni

$$\gamma_{i+1} = \frac{\gamma_i + f_i}{1+(\alpha_i + d_i)/a_{i+1}} \qquad (87)$$

i=1 bo`lganda (82) chegaralik shartning birinchisini (84) bilan solishtirib

$$\alpha_1=a_1(1-\aleph_1), \qquad \gamma_1=a_1\gamma_1$$

(86) va (87) formulalar $a_i\geq1$ bo`lganda qo`layli. Agar $a_i\leq1$ bo`lganda (86) va (87) formulalarni quydagi ko`rinishda qollagan maqul.

$$\alpha_{i+1} = \frac{a_{i+1}(\alpha_i + d_i)}{a_{i+1}+(\alpha_i + d_i)} \qquad (86)^l$$

$$\gamma_{i+1} = \frac{a_{i+1}(\gamma_i + f_i)}{a_{i+1}+(\alpha_i + d_i)} \qquad (87)^l$$

(78) shart urinlanganda (86) va $(86)^l$ formulalardan $\alpha_1 \geq 0$ ekanligi kelib chiqadi. Shunda (85) formuladagi $\frac{\alpha_i}{\alpha_i + d_i}$ koeffitsient hamisha kichkina va ω_i ni xisoblashda echimning o`rinliligini ta`minlaydi. y_i ni aniqlash uchun quydagi formulalardan foydalanish mu`mkin. $a_i \geq 1$ bo`lganda

$$y_i = \alpha_{i+1} y_{i+1} + \beta_{i+1} = \left(1 - \frac{\alpha_{i+1}}{a_{i+1}}\right) y_{i+1} + \frac{\gamma_{i+1}}{a_{i+1}} \qquad (88)$$

va $a_i \leq 1$ bo`lganda

$$y_i = \frac{a_{i+1}}{a_{i+1} + \alpha_i + d_i} y_{i+1} + \frac{\gamma_i + f_i}{a_{i+1} + \alpha_i + d_i} \qquad (89)$$

(88) va (89) dan ko`rinib turganidek haydash doimiy. (85) va (88) (89) formula bo`yicha xisoblash uchun ω_N, y_N larni xisoblash kerak. Ular (82) chegaraviy shartning ikkinchi tenglamasidan i=N bo`lganda (84) sharttan aniqlanadi.

$$y_N = \frac{a_N \nu_2 + \nu_N \mathrm{N}_2}{a_N (1 - \mathrm{N}_2) + N_2 d_N}, \qquad \omega_n = \frac{\gamma_N a_N (1 - \mathrm{N}_2) + \alpha_N a_N \nu^2}{a_N (1 - \mathrm{N}_2) + N_2 d_N}$$

(78) dan bu ikki ifodaning bo`limlari hamisha no`ldan katta. Endi oqimlik haydashning asosiy formulasini keltiramiz.

1) $\alpha_1 = a_1(1-\aleph_1)$, $\gamma_1 = a_1 \nu_1$ ni xisoblaymiz

2) $\alpha_{i+1} = \frac{\alpha_i + d_i}{1 + (\alpha_i + d_i)/a_{i+1}}$ agar $a_{i+1} \geq 1$ larning qiymatini i=1,2,…,N-1 uchun ketma-ket xisoblab topamiz. Ya`ni

$a_{i+1} < 1$ bo`lganda $\alpha_{i+1} = \dfrac{a_{i+1}(\alpha_i + d_i)}{a_{i+1} + (\alpha_i + d_i)}$,

$a_{i+1} \geq 1$ bo`lganda $\gamma_{i+1} = \dfrac{\gamma_i + f_i}{1 + (\alpha_i + d_i)/a_{i+1}}$

$a_{i+1} < 1$ bo`lganda $\gamma_{i+1} = \dfrac{a_{i+1}(\gamma_i + f_i)}{a_{i+1} + (\alpha_i + d_i)}$

3) $a_N \geq 1$ bo`lganda $y_N = \dfrac{\nu_2 + \gamma_N \aleph_2 / a_N}{1 - \aleph_2 + \aleph_2 \alpha_N / a_N}$,

agar $a_N < 1$ bwlsa $y_N = \dfrac{a_N \nu_2 + \gamma_N \aleph_2}{a_N (1 - \aleph_2) + \aleph_2 \alpha_N}$,

agar $a_N \geq 1$ bo`lsa $\omega_n = \dfrac{\gamma_N (1 - \aleph_2) + \alpha_N \nu_2}{1 - \aleph_2 + \aleph \alpha_N / a_N}$

agar $a_N < 1$ bo`lsa $\omega_n = \dfrac{\gamma_N (1 - \aleph_2) a_N + \alpha_N \nu_2 a_N}{a_N (1 - \aleph_2) + \aleph_2 \alpha_N}$

4) i=N-1, N-2,…0 lar uchun $\omega_i = \dfrac{\alpha_i}{\alpha_i + d_i} \omega_{i+1} + \dfrac{d_i \nu_i - \alpha_i f_i}{\alpha_i + d_i}$

agar $a_{i+1} \geq 1$ bo`lsa $y_i = \left(1 - \dfrac{\alpha_{i+1}}{a_{i+1}}\right) y_{i+1} + \dfrac{\gamma_{i+1}}{a_{i+1}}$,

$a_{i+1} < 1$ bo`lganda $y_i = \dfrac{a_{i+1}}{a_{i+1} + \alpha_i + d_i} y_{i+1} + \dfrac{\gamma_i + f_i}{a_{i+1} + \alpha_i + d_i}$ qiymatlarni xisoblaymiz.

Eslatma 1.

Yuqorida faqatgina y_i funktsiyalarini aniqlash uchun gina emas ω_i oqimni xisoblash uchun ham formulalar keltirildi. a_i – koeffitsentining katta qiymatlarida oqimni $\omega_i = a_i(y_{i-1} - y_i)$ formula buyicha xisoblash aniqlikni yo`qotishga olib keladi. Shuning uchun ham qo`shimcha izlanuvchi funktsiya sifatida ω_i oqim kiritildi va u (85) rekkurent formula buyicha xisoblanadi.

2. Tsikllik haydash usuli.

Bu usul chegirmali tenglamaning davriy echimlarini topish uchun qo`llaniladi. Bunday masalalar xususiy xosilali tenglamalarni tsilindrlik va sferik koordinatalar sistemasida taqribiy echishda hosil bo`ladi.

$$\begin{aligned} &a_1y_N-c_1y_1+b_1y_2=-f_1 \\ &a_iy_{i-1}-c_iy_i+a_{i+1}y_{i+1}=-f_i, \quad i=0,1,2,\ldots N-1, \\ &a_Ny_{N-1}-c_Ny_N+b_Ny_1=-f_N \end{aligned} \qquad (90)$$

tenglamalar sistemasini ko`rib o`taylik. Bunday algebraik masala $a_iy_{i-1}-c_iy_i+a_{i+1}y_{i+1}=-f_i$, uch xadli tenglamani $a_{i+N}=a_i$, $b_{i+N}=b_i$, $c_{i+N}=c_i$, $f_{i+N}=f_i$, shartlarda $y_{i+N}=y_i$, davriy echimni izlashda hosil bo`ladi.

(76) koeffitsientlari bo`yicha $a_i>0$, $b_i>0$, $c_i> a_i+b_i$ o`rinli.

(90) masalani echish formulasini ya`ni tsikllik haydash formulasini keltirimiz.

$$\alpha_{i+1}=\frac{b_i}{c_i-a_i\alpha_i}, \qquad \beta_{i+1}=\frac{f_i+a_i\beta_i}{c_i-a_i\alpha_i}, \qquad \gamma_{i+1}=\frac{a_i\gamma_i}{c_i-a_i\alpha_i} \qquad (91)$$

$$i=0,1,2,\ldots N,$$

$$\alpha_2=\frac{b_2}{c_1}, \qquad \beta_2=\frac{\beta_2}{c_1}, \qquad \gamma_2=\frac{a_1}{c_1}$$

$$\rho_i=\alpha_{i+1}\rho_{i+1}+\beta_{i+1}, \qquad q_i=\alpha_{i+1}q_{i+1}+\gamma_{i+1}$$

$$i=N-2,\ldots,1. \qquad \rho_{N-1}=\beta_N \;, \quad q_{N-1}=\alpha_N+\gamma_N \qquad (92)$$

$$y_N=\frac{\beta_{N+1}+\alpha_{N+1}\rho_1}{1-\alpha_{N+1}q_1-\gamma_{N+1}}, \qquad y_i=\rho_i+y_Nq_i \qquad (93)$$

$$i=0,1,2,\ldots N-1,$$

tsikllik haydash usuli o`rinli, sababi (93) masalaning echimi (91) shartlarning bajarilishida o`rniqli, mahraji, $1-\alpha_{N+1}q_1-\gamma_{N+1}$ ifoda y_N uchun nolga aylanmaydi.

Haqiqattan ham (91),(92) dan ko`rinib turganidek

$\alpha_i<1$, $\gamma_i>0$, $\alpha_2+\gamma_2<1$. $\alpha_i+\gamma_i<1$ deb olib

$$\alpha_{i+1}+\gamma_{i+1}=\frac{b_i+a_i\gamma_i}{c_i-a_i\alpha_i}<\frac{b_i+a_i-a_i\alpha_i}{c_i-a_i\alpha_i}<1 \qquad (94)$$

(93) va (94) ni xisobga olib $q_{N-1}<1$, $q_i<1$ ekanligini topamiz. Barcha aytilganlardan $1-\alpha_{N+1}q_1-\gamma_{N+1}>0$ ekanligi kelib chiqadi

Chegirmali tenglamani faktorizatsiyalash usuli.

Chegirmalik chegaraviy masala uchun haydash formulasi

$$Ly_k=A_ky_{k-1}-C_ky_k+B_ky_{k+1}=-F_k, \qquad k=1,2,\ldots N-1,$$

$$y_0=N_1y_1+v_1,\ y_N=N_2y_{N-1}+v_2 \qquad (95)$$

chegirmalik tenglamani faktorizatsiyalash natijasida olinishi mumkin. Siljitish operatori T ni $Ty_k=y_{k+1}$ deb olib, (95) tenglamaning chap tomonini kupaytma kurinishida hosil qilamiz.

$(b_kT-A_k)(\alpha_kT-E)\,y_{k-1}=(b_kT-A_k)\,(\alpha_ky_k-y_{k-1})=$

$=b_k\alpha_ky_{k-1}-(A_k\alpha_k-b_k)\,y_k+A_k\,y_{k-1}$

bu joyda E-birlik operator, $E\,y_k=y_k$. Bu ifodani (95) bilan solishtirib $\alpha_{k+1}\,b_k=B_k$, $A_k\,\alpha_k+b_k=C_k$ ni olamiz. Bundan

$b_k=C_k-A_k\,\alpha_k$ ni yo`qotib

$$\alpha_{k+1}=\frac{B_k}{C_k-A_k\alpha_k} \qquad (96)$$

ga ega bo`lamiz. Faktorizatsiyalangan $(b_kT-A_k)\,(\alpha_ky_k-y_{k-1})=-F_k$ tenglamani echish quydagicha bo`ladi. Dastlabki

$$(b_kT-A_k)\beta_k=b_k\,\beta_{k+1}-A_k\,\beta_k=F_k$$

tenglamaning ya`ni

$$\beta_{k+1} = \frac{A_k \beta_k + F_k}{C_k - A_k \alpha_k} \quad k=1,2,\dots N-1 \qquad (97)$$

deb β_k -funktsiyasi aniqlanadi. So`ng

$\alpha_k y_k - y_{k-1} = -\beta_k$ ya`ni

$$y_k=\alpha_{k+1}y_{k+1}+\beta_{k+1} \quad k=0,1,2,\dots N-1 \qquad (98)$$

formulasidan y_k aniqlanadi.

Natijada oddiy haydash formulasini xosil qilamiz. (97)-(98) formulalarga

$$\alpha_1=N_1,\ \beta_1=\gamma_1, \qquad y_N=\frac{N_2\beta_N+\nu_2}{1-N_2\alpha_N} \qquad (99)$$

boshlang`ish shartlarni qo`shish kerak. Agar $\Delta v_k=v_{k+1}-v_k$, o`ng va $\nabla v_k= v_k-v_{k-1}$ chap chegirmali operatorlarini kiritsak, unda faktorizatsiya operatorini $Ly_k=A_k y_{k-1}-C_k y_k+B_k y_{k+1}$, $L=L_1L_2$ deb $L_1=b_k\ \Delta+v_k$, $L_2=\nabla+(\alpha_k-1)$ turlandirish mu`mkin. $L_1L_2y_k= Ly_k$ tengligi urinli bo`ladi. Agar $v_k=b_k-A_k$, $\alpha_{k+1}b_k=B_k$, $A_k\alpha_k+ B_k/\alpha_{k+1}=C_k$ deb olsak. $v_k$ va $b_k$ koeffitsentlarni yuqotgandan so`ng yana (98)-(99) haydash formulalariga aylanib kelamiz.

Xulosa.

Yuqori o`quv muassosalarida talabalar va magistrantlar «Differentsial tenglamalar», «Matematik fizika tenglamalari» fanini o`rganishda ikkinchi tartibli xususiy xosilali differentsial tenglamalar ya`ni parabolik, giperbolik va elliptik tenglamalar, ularga qo`yilgan chegaraviy masalalar bilan ko`b uchiraydi. Echimni ko`b hollarda analitik usulda olish judayam qiyin shuning uchun biz sonli usullardan foydalanamiz.

Bu bitiruv malakaviy ishda to`r tenglamalarini matritsaviy haydash usulidan foydalanib echish yo`li ko`rsatilgan.

Bu usul uch diagonalli matritsalarga ega sistemalar uchun qo`llaniladi. (Ular ikkinchi tartibli differentsial tenglamalarni chegaraviy masalalar bilan echishda uchraydi.)

Bunday sistemalar kanonik ko`rinishda quydagicha yoziladi.

$$A_k y_{k-1} - C_k y_k + B_k y_{k+1} = -F_k, \qquad k=1,2,\ldots N-1,$$

$$y_0 = N_1 y_1 + \mu_1, \qquad y_N = N_2 y_{N-1} + \mu_2$$

barcha $k=1,2,\ldots N-1$, lar uchun $A_i \neq 0$, $B_i \neq 0$ bu ikkinchi tartibli chegirmali tenglama ya`ni uch nuqtali tenglama deb nomlanadi. Matritsaviy haydash usuli chegaraviy tenglamalarni soha murakkab bo`lganda tadqiq qiladi. Lekin matritsaviy haydash usuli $Q=O(N^k)$ arifmetik amalda bajariladi va katta xotirani talab qiladi. Shuning bilan birga chegaraviy shartlar bilan haydash usulidan foydalanib masalani echkanda haydash matritsasini $O(N^3)$ ga cha qisqartish mu`mkin. Xulosa qilib aytganda matritsaviy haydash usulining boshqa xisoblash usullariga ko`ra ancha yutiqga ega ekanligi misollar yordamida ko`rsatildi.

Foydalanilgan adabiyotlar.

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