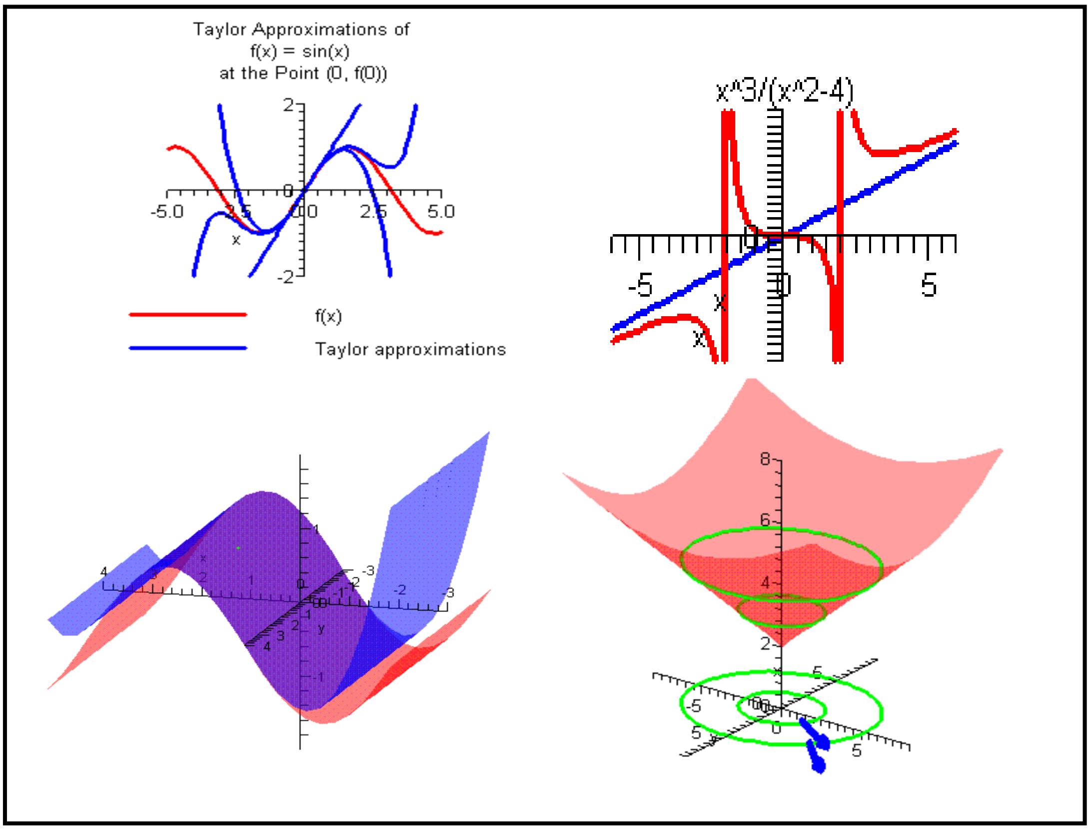
O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI
FARG'ONA POLITEXNIKA INSTITUTI

OLIY MATEMATIKA
kafedrasi

E.M.MIRZAKARIMOV

M A P L E 7
d a s t u r i yo r d a m i d a
OLIY MATEMATIKA
msalalarini echish

USLUBIY QO'LLANMA
(2- qism)



Farg'ona-2010

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Institut ilmiy-uslubiy kengashi
tomonidan tasdiqlangan
bayon №_3_”_26_”_02_2010yil

Farg‘ona-2010

Ushbu uslubiy qo‘llanma, barcha oliy o‘quv yurtidagi texnika yo‘nalishlari bo‘yicha ta’lim olayotgan talabalarning, 1-semestrdagi Oliy matematika fanidan, mustaqil ishlarni MAPLE7 dasturidan foydalanib bajarishlari uchun mo‘ljallangan.

Uslubiy qo‘llanma Oliy matematika fanidan bir va bir necha o‘zgaruvchili funktsiyalarning limiti, hosilalari va ularning tadbiqlariga bag‘ishlamgan. Har bir bo‘limda nazariya bo‘yicha zaruriy ma’lumotlar keltirilgan, ularga mos namunaviy masalalarning yechimlari berilgan va bu yechimlarni MAPLE7 dasturi yordamida bajarish yo‘llari ko‘rsatilgan, shuningdek mustaqil ishlash uchun variantlar tuzilgan.

Ushbu uslubiy qo‘llanma “Oliy matematika” kafedrasining uslubiy semenarida muxokama qilingan (№ 6, 18. 01. 2010)

Iqtisod va menejment fakulg‘tetining uslubiy komissiyasi tomonidan maqullangan. (№ 4 , 24 . 01. 2010)

Tuzuvchi katta o‘qituvchi **E.M.MIRZAKARIMOV**

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6. Funktsiyalar. Limitlar. Funktsiyaning uzluksizligi

Barcha ratsional (Q) va irratsional (I) sonlar to‘plami birgalikda haqiqiy sonlar to‘plamini tashkil qiladi. Haqiqiy sonlar to‘plami R harfi bilan belgilanadi.

To‘g’ri chiziqdagi nuqtalar to‘plami bilan R to‘plam orasida har doim o‘zaro bir qiymatli moslik o‘rnatish mumkin. Agar bu moslik o‘rnatilgan bo‘lsa, u holda to‘g‘ri chiziq *sonlar o‘qi* deyiladi. Quyidagi $a<x<b$ ($a<x<b$) tengsizlikni qanoatlantiruvchi hamma *X* sonlar to‘plami ochiq(yopiq) oraliq yoki interval (kesma, segment) deyiladi va (*a;b*) yoki]*a;b*[([*a;b*]) ko‘rinishlarning biri bilan belgilanadi, $a\leq x<b$ ([*a;b*)) va $a<x<b$ {(*a;b*)} lar esa yarim ochiq intervallar deyiladi.

Haqiqiy son *a* ning *absolyut qiymati (moduli)* quyidagicha aniqlanadi:

$$a=\begin{cases}-a, & agar\ a<0\ bo'lsa\\ a, & agar\ a>0\ bo'lsa\end{cases}$$

Ixtiyoriy ikkita *a* va *b* haqiqiy son uchun

$$1)\ |a+b|\leq|a\backslash+|b|,\qquad 2)\ |a|-|b|\leq|a-b|,\qquad 3)\ \frac{|a|}{|b|}=\left|\frac{a}{b}\right|(b\neq 0)$$

munosabatlar har doim to‘g’ridir.

Ta’rif. *D va E* to‘plamlar berilgan bo‘lsin. Agar *D* to‘plamdagi har bir *x* songa biror qoida yoki qonunga ko‘ra $E=\{f(x),x\in D\}$ to‘plamdan bitta *y* son mos qo‘yilsa, *D* to‘plamda *funktsiya* berilgan (aniqlangan) deb ataladi va u $y=f(x)$ kabi belgilanadi, bunda *x erkli o‘zgaruvchi* yoki *argument* deyiladi.

Bu ta’rifdagi *D* va *E* lar orasidagi bog’lanish *funktsional bog’lanish* deyiladi.

D to‘plam funktsiyaning *aniqlanish sohasi* deyiladi. *E* to‘plam, ya’ni *D* ning har bir *x* elementiga moe kelgan f(x) elementlar to‘plami funktsiyaning *o‘zgarish sohasi* deyiladi.

D va *E* to‘plamlar [*a;b*] kesma, (*a;b*) interval, (*a;b*] yoki [*a;b*), yarim intervallar, son o‘qining biror nuqtasi, butun son $(-\infty, +\infty)$ bo‘lishi mumkin.

Funktsiyalar *jadval, grafik analitik* usullarda berilishi mumkin: $y = f(x)$ funktsiya analitik usulda berilganda uning *D* va *E* sohalari berilmagan bo‘lishi mumkin, ammo ularni *f(x)* funktsiyaning xossalaridan foydalanib aniqlanadi.

M i s o l . $y = \lg(4-3x-x^2)$ funktsiyaning aniqlanish va o‘zgarish sohalarini toping.

YE ch i sh. Logarifmik funktsiya

$$4-3x-x^2>0$$

bo‘lganda mahnoga ega bo‘ladi. Kvadrat uchhadning ildizlari

$$x_1=-4,\ x_2=1$$

bo‘lgani uchun yusjridagi tengeizlikni

$$-(x+4)(x-1)>0$$

ko‘rinishda yozib olamiz. Bundan x>-4 va x<1 kelib chikadi.

Demak funktsiyaning aniqlanish sohasi, ya’ni *x* ning berilgan funktsiya mahnoga ega bo‘ladigan qiymatlari to‘plami *(D)* (- 4;1) intervaldan iborat. *D* sohada funktsiyaning qiymatlari

$$0<4-3x-x^2<25/4$$

oraliqda o‘zgargani uchun (bunda $y_0=25/4$ berilgan funktsiya aniqlaydigan parabola uchining ordinatasi)

$$(\infty;\ \lg(25/4))$$

interval . E sohadan iborat bo‘lib, u funktsiyaning qiymatlari to‘plami bo‘ladi.

Agar *D* sohani *E* sohaga akslantirganda o‘zaro bir qiymatli moslik ya’ni $y=f(x)$ funktsiya bajarilsa, u holda *x* ni *y* orqali $x=g(y)$ kabi ifodalash mumkin. Oxirgi funktsiya $y=f(x)$ funktsiyaga *teseskari funktsiya* deyiladi. $x=g(y)$ funktsiya uchun *E* — aniqlanish sohasi *D* esa funktsiyaning o‘zgarish sohasi bo‘ladi. $g(f(x))=x$ va $f(g(y))=y$ bo‘lgani uchun $y=f(x)$ va $x=g(y)$ funktsiyalar *o‘zaro teskari funktsiyalar* bo‘ladi.

Odatda *x=g(y)* teskari funktsiyadagi *x* ni *y* bilan, *y* ni *x* bilan almashtirib *y=g(x)* ko‘rinishda yoziladi.

Masalan, $y=x^3$ va $y=\sqrt[3]{x}$, $y=2^x$ va $y=log_2 x$, *y=sinx* va *y=arcsinx* har bir juft funktsiyalar o‘zaro teskari funktsiyalar. Ularning aniqlanish sohasi moe ravishda quyidagicha:

$x\in(-\infty, +\infty)$ va $x\in(-\infty, +\infty)$, $x\in(-\infty, +\infty)$ va $x\in(0; +\infty)$, $x\in(-\infty, +\infty)$ va $x\in[-1;+1]$.

Agar *u=φ(x)* funktsiya *D* sohada aniqlangan, *G* uning o‘zgarish sohasi, *y=f (u)* funktsiya *G* sohada aniqlangan bo‘lsa, u holda *u =f[φ(x)] = F(x)* funktsiya *murakkab funktsiya* deyiladi.

y=f(*φ(x)* murakkab funktsiya *y* =f*(u)* va *u = φ(x)* funktsiyalarning *kompozitsiyasi* deyiladi.

Funktsiyalarning kompozitsiyasi bir nechta funktsiyalardan iborat bo‘lishi mumkin. Masalan,

$$y=\sin(x^2+1)$$

funktsiya $y=\sin u$ va $u=x^2+1$,

ya’ni ikkita funktsiyaning kompozitsiyasi bo‘lsa,

$$y=\log(\cos 2^{x^2})$$

funktsiya esa uchta:.

$$y=lgu,\ u=\cos v,\ v=2^W,\ w=x^2$$

funktsiyalarning kompozitsiyasidan iboratdir *u, v, w* o‘zgaruvchilar *oraliq argumentlari* deyiladi.

Agar biror *(a;b)* oralikda aniqlangan *y =f*(x) funktsiya *F(x,y)* = 0 tenglamani qanoatlantirsa, u holda *y*=f(x) funktsiya *F(x,y)* = 0 tenglik bilan aniqlangan *oshkormas funktsiya* deyiladi. Funktsiyani oshkor ko‘rinishga keltirish uchun *F(x,y)*=0 tenglikdan *y* ni topish kerak. *y* ni har doim xam topib bo‘lavermaydi, bahzan esa umuman topib bulmaydi. Masalan, $y+x3^y=1$ tenglamani *y* ga nisbatan umuman echib bo‘lmaydi.

Tekislikning (*x; f*(*x*)) kabi aniqlangan nuqtalaridan iborat ushbu

$$\{(x; f(x))\} = \{(x; y) : x\in X,\ y=f(x) \in Y\}$$

to‘plam *y*=f(x) funktsiyaning grafigi deb ataladi. Murakkab funktsiyalarning grafigi ularning ordinatalari ustida (qo‘shish, ayirish, ko‘paytirish, bo‘lish, darajaga ko‘tarish, ildiz chiqarish, logarifmlash va x.k.) amallar bajarish yordamida takriban chiziladi.

Darajali, ko‘rsatkichli, logarifmik trigonometrik va teskari trigonometrik funktsiyalar *asosiy elementar funktsiyalar* deyiladi.

Asosiy elementar funktsiyalarning aniqlanish va o‘zgarish sohalari

№	Funktsiya	Funktsiyaning aniqlanish sohasi	Funktsiyaning o‘zgarish sohasi
1.	*y =x"*, *p*-natural son	$(-\infty; +\infty)$	*p* juft bo‘lganda: $[0;+\infty)$,
2.	$y=\sqrt[2n]{x}$	$[0;+\infty)$	$[0;+\infty)$
3.	$y=\sqrt[2n+1]{x}$	$(-\infty; +\infty)$	$(-\infty; +\infty)$
4.	$y=a^x$	$(-\infty; +\infty)$	(0;+»)
5.	y = lgx	$(0;+\infty)$	$(-\infty; +\infty)$
6.	$y = \sin x$	$(-\infty; +\infty)$	[-1, 1]
7.	$y = \cos x$	$(-\infty; +\infty)$	[-1, 1]
8.	*y =tgx*	$\left((2n+1)\frac{\pi}{2},(2n-1)\frac{\pi}{2}\right)$ n = 0,±1,±2,…	$(-\infty; +\infty)$

9.	y = ctg x	(nπ; (n+1) π), n=0,±1,±2>…	(- ∞; + ∞)
10	y = arcsin x	[-1;+1]	$[-\frac{\pi}{2},\frac{\pi}{2}]$
P.	y = arccos x	1-1;+1]	[0;π]
12	y = arctg x	(— ∞; +∞)	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
13	y = arcctg x	(— ∞;-+ ∞)	[0; π 1

6.2. Ketma-ketlik va funktsiyaning limiti. Funktsiya uzluksizligi

1-ta’rif. Agar har kanday $\varepsilon > 0$ son uchun shunday $n_0 = n_0(\varepsilon) > 0$ son topilsaki (bunda n_0 — butun son), barcha $n>n_0$ lar uchun $|x- A| < \varepsilon$ tengsizlik bajarilsa, u xrlda A son {x} sonli ketma-ketlikning *limiti* deiiladi va bu

$$\lim x_n = A$$

ko‘rinishda yoziladi.

Limitga ega bo‘lgan ketma-ketlik *yatsinlashuvchi,* aks holla esa *uzotslashuvchi* ketma-ketlik deyiladi.

$y = f(x)$ funktsiya biror x_0 nuqtaning atrofida aniq.-langan bo‘lsin.

2-ta’rif. Agar xrqanday $\varepsilon > 0$ son uchun shunday $\delta > 0$ ($\delta = \delta(\varepsilon)$) son topilsaki, argument x ning $0<|x-x_0|<\delta$ qiymatlari uchun $|f(x)-A| < \varepsilon$ tengsizlik bajarilsa, A son $y= f(x)$ funktsiyaning $x\to x_0$ dagi limiti deyiladi va bu quyidagicha yoziladi:

$$\lim_{x\to x_0} f(x) = A$$

3-ta’rif. Agar har kanday $\varepsilon > 0$ son uchun shunday $\delta = \delta(\varepsilon, x_0)$ son topilsaki, x argumentning $x_a<x<x_0+\delta$ ($x_0-\delta<x<x_0$) tengsizliklarni qanoatlantiruvchi barcha $x\epsilon X$ qiymatlarida $|f(x)-A|<\varepsilon$ tengsizlik bajarilsa, A son f(x) funktsiyaning x_0 nuqtadagi *ung (chap) limiti* deb ataladi va quyidagicha yoziladi:

$$\lim_{x\to +0} f(x) = A \quad \text{yoki} \quad f(x_0 + 0) = A,$$

$$\lim_{x\to -0} f(x) = A \quad \text{yoki} \quad f(x_0 - 0) = A$$

Agar $\lim_{x\to\infty} f(x)$ limit mavjud bo‘lsa, *A son y* =f(x) funktsiyaning $x\to\infty$ dagi limiti deyiladi.

Funktsiyaning chap va ung limitlari uning *bir tomonlama limitlari* deyiladi.

Agar ikkala bir tomonlama limit mavjud bo‘lib, ular o‘zaro teng $f(x_0 - 0) = f(x_0 + 0)$ bo‘lsa, f(x) funktsiya $x\to x_0$ da limitga ega deyiladi.

Teorema. Agar $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} f(x) = B$ bo‘lib, A va B lar chekli sonlar bo‘lganda:

$$\lim_{x\to a}[\ f(x)\pm\varphi(x)\ ] = A\pm B,$$

$$\lim_{x\to a} f(x)\cdot\varphi(x) = A\cdot B$$

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \frac{A}{B},\ (B\neq 0)$$

tengliklar o‘rinli bo‘ladi.

Agar $f(x)$ va $\varphi(x)$ funktsiyalar cheksiz kichik eki cheksiz katta ($x\to a$ da) bo‘lsa, $\frac{f(x)}{\varphi(x)}$ nisbat $\frac{0}{0}$ *ёки* $\frac{\infty}{\infty}$ aniqmaslikni ifodalaydi. Shuningdek limitlarni hisoblashda $:0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0, 1^\infty$

ko‘rinishdagi aniqmasliklar ham uchraydi. Limitlarni hisoblashda birinchi va ikkinchi ajoyib limitlar ahamiyatlidir.

Birinchi ajoyib limit ($\frac{0}{0}$ aniqmaslik)

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Bu limitdan $x\to 0$ da y(x) $\to 0$ bo‘lganda,

$$\lim_{x\to 0}\frac{\sin x}{u(x)}=1,\quad lim_{x\to 0}\frac{\sin mx}{nx}=\frac{m}{n}$$

Ikkinchi ajoyib limit (1^{∞} aniqmaslik)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\ell,\quad \ell=2{,}7182....,$$

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=\ell^k,\quad \frac{1}{x}=y\ \ddot{a}\grave{a}\quad lim_{x\to\infty}(1+ky)^{\frac{1}{y}}=\ell^k$$

Bu limitlarni hisablashda x va y larni bir xil ifoda bo‘lish ahamiyatlidir.

6.1 – misol. $\lim_{x\to\infty}\frac{8x^{10}+11x^9-1}{3x^{10}-x^7+x^6}$ limitni hisablang.

Bu berilgan limit $\frac{\infty}{\infty}$ aniqmaslikka ega, uni ochish uchun kasirning surat va maxrajini *x* ning eng yuqori darajaligisi x^{10} ga bo‘lamiz.

$$\lim_{x\to\infty}\frac{8+\frac{11}{x}-\frac{1}{x^{10}}}{3-\frac{1}{x^3}+\frac{1}{x^4}}=\left(x\to\infty\ da\ \frac{11}{x},\frac{1}{x^{10}},\frac{1}{x^3},\frac{1}{x^4}\ lar\to 0\right)=\frac{8}{3}$$

```
> Limit((8*x^10+11*x^9-1)/(3*x^10-x^7+x^5), x=infinity)=
limit((8*x^10+11*x^9-1)/(3*x^10-x^7+x^5), x=infinity);
```

$$\lim_{x\to\infty}\left(\frac{8x^{10}+11x^9-1}{3x^{10}-x^7+x^5}\right)=\frac{8}{3}$$

6.2 – misol. $\lim_{x\to 1}\frac{\sqrt{x+15}-\sqrt{17-x}}{x^2+5x-6}$ limitni hisablang.

Bu limit $x\to 1$ da surat va maxraji 0 ga intilgani uchun $\frac{0}{0}$ aniqmaslika ega. Bu aniqmaslikni ochish uchun kasirni surat va maxrajini suratning qo‘shmasi $\sqrt{x+15}+\sqrt{17-x}$ ga ko‘paytiramiz.

$$\lim_{x\to 1}\frac{x+15-17+x}{(x^2+5x-6)\left(\sqrt{x+15}+\sqrt{17+x}\right)}=\lim_{x\to 1}\frac{2(x-1)}{(x-1)(x+6)\left(\sqrt{x+15}+\sqrt{17+x}\right)}=$$

$$=\lim_{x\to 1}\frac{2}{(x+6)\left(\sqrt{x+15}+\sqrt{17+x}\right)}=\frac{1}{28}$$

```
> Limit((sqrt(x+15)-sqrt(17-x))/(x^2+5*x-6), x=1)=
limit((sqrt(x+15)-sqrt(17-x))/(x^2+5*x-6), x=1);
```

$$\lim_{x\to 1}\left(\frac{\sqrt{x+15}-\sqrt{17-x}}{x^2+5x-6}\right)=\frac{1}{28}$$

6.3 – misol. $\lim\limits_{x\to 0}\frac{tgx-\sin x}{x^3}$, $\left(\frac{0}{0}\right)$ limitni hisablang.

Bu limitni hisablashda funktsiyada shunday shakil almashtirish kerakki, unga barincha ajoyib limitni kullash mumkin bo‘lsin.

$$\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=\lim_{x\to 0}\frac{1}{\cos x}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}=$$

$$=\lim_{x\to 0}\frac{1}{\cos x}\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=1\cdot 1\cdot\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\cdot 1=\frac{1}{2}$$

```
> Limit((tan(x)-sin(x))/(x^3), x=0)=
limit((tan(x)-sin(x))/(x^3), x=0);
```

$$\lim_{x\to 0}\left(\frac{\tan(x)-\sin(x)}{x^3}\right)=\frac{1}{2}$$

6.4 - misol. $\lim\limits_{x\to\infty}(5x-1)[\ln(x-4)-\ln x]$, $(\infty-\infty)$ limitni hisablang.

Lagorifmlash qoidalari asosida:

$$\lim_{x\to\infty}(5x-1)\ln\frac{x-4}{x}=\lim_{x\to\infty}\ln\left(\frac{x-4}{x}\right)^{5x-1}=\lim_{x\to\infty}\ln\left[1+\left(-\frac{4}{x}\right)\right]^{5x-1}$$

$y=lnx$, $x>0$ funktsiya uzuliksiz ekanidan $\lim\limits_{x\to\infty}\ln x=\ln(\lim\limits_{x\to\infty}x)$ o‘rinli. Bundan:

$$\lim_{x\to\infty}\ln\left[1+\left(-\frac{4}{x}\right)\right]^{5x-1}=\ln\lim_{x\to\infty}\left[\left(1+\frac{-4}{x}\right)^x\right]^5\lim_{x\to\infty}\left[\left(1+\frac{-4}{x}\right)\right]^{-1}=$$

$$=\ln[(e^{-4})^5\cdot 1^{-1}]=\ln e^{-20}=-20;$$

```
> Limit((5*x-1)*(ln(x-4)-ln(x)),x=infinity)=limit((5*x-1)*(ln(x-4)-ln(x)),x=infinity);
```

$$\lim_{x\to\infty}((5x-1)(\ln(x-4)-\ln(x)))=-20$$

Qo‘shimch misollar:

6.5-misol.
```
> exr := ((5*x+3)/(x-2))^x;
```

$$expr:=\left(\frac{5x+3}{x-2}\right)^x$$

```
> Limit( exr, x=infinity )=limit( exr, x=infinity );
```

$$\lim_{x\to\infty}\left(\frac{5x+3}{x-2}\right)^x=\infty$$

6.6-misol.
```
> Limit( (5*x^3+6*x+1)/(6*x^3+4*x+2),x=infinity)=
limit( (5*x^3+6*x+1)/(6*x^3+4*x+2),x=infinity );
```

$$\lim_{x\to\infty}\left(\frac{5x^3+6x+1}{6x^3+4x+2}\right)=\frac{5}{6}$$

6.7-misol.
```
> Limit( (x^3/(x^3+9)^1/3), x=infinity )=
limit( (x^3/(x^3+9)^1/3), x=infinity );
```

$$\lim_{x\to\infty}\left(\frac{1}{3}\frac{x^3}{x^3+9}\right)=\frac{1}{3}$$

6.8-misol.
```
> Limit(((x^2-5*x+4)/(x^2-6*x+3)),x=4)=
```

```
limit(((x^2-5*x+4)/(x^2-6*x+3)),x=4 );
```

$$\lim_{x\to 4}\left(\frac{x^2-5x+4}{x^2-6x+3}\right)=0$$

6.9-misol.

```
> Limit(sqrt(1-cos(x))/x*sqrt(2),x=2)=
limit(sqrt(1-cos(x))/x*sqrt(2),x=2);
```

$$\lim_{x\to 2}\left(\frac{\sqrt{1-\cos(x)}\,\sqrt{2}}{x}\right)=\frac{1}{2}\sqrt{1-\cos(2)}\,\sqrt{2}$$

Funktsiyalarni uzliksizligini aniqlash

Funktsiya uzluksizligi tushunchasi muxim tushunchalardan xisoblanadi.

1) *f(x)* funktsiya $x = x_0$ nuqta va uning atrofida aniqlangan;
2) x_0 nuqtada *f*(x) funktsiya limitga ega;
3) funktsiyaning $x\to x_0$ dagi limiti funktsiyaning x_0 nuqtadagi qiymatiga teng, ya'ni

$$\lim_{x\to x_0} f(x)=f(x_0) \quad (1.2)$$

bo'lsa, $y=f(x)$ funktsiya $x = x_0$ nuqtada *uzluksiz* deiiladi. Agar *x* o'rniga $x = x_0+\Delta x$ ni kuysaq (1.2) uzluksizlik shartiga teng kuchli bo'lgan shartga ega bo'lamiz:

$$\lim_{\Delta x\to 0}\Delta f(x)=\lim_{\Delta x\to 0}\left[f(x_0+\Delta x)-f(x_0)\right]=0$$

ya'ni funktsiya argumentining x_0 nuqtadagi cheksiz kichik orttirmasi Δx ga funktsiyaning cheksiz kichik orttirmasi $\Delta f(x)$ moe kelganda va fakdt shunda $y = f(x)$ funktsiya shu nuqtada uzluksiz bo'ladi.

Biror sohaning xamma nuqtalarida uzluksiz bo'lgan $u = f(x)$ funktsiya shu sohada uzluksiz deyiladi.

Agar x_0 nuqtada funktsiya uzluksizligining yuqoridagi uchta shartidan birortasi bajarilmasa, x_0 nuqta *uzilish nutstasi* deyiladi.

Agar x nuqtada

$$\lim_{x\to x_0-0} f(x)=f(x_0-0) \text{ va } \lim_{x\to x_0+0} f(x)=f(x_0+0)$$

limitlar mavjud bo'lib, $f(x_0-0)\neq f(x_0+0)$ bo'lsa, x_Q nuqta 1- *tur uzilish nuqtasi* deyiladi.

Agar $\lim_{x\to x_0-0} f(x)$, $\lim_{x\to x_0+0} f(x)$ limitlar yoki ularning birortasi mavjud bo'lmasa yoki ∞ ga teng bo'lsa, x_0 nuqta *2-tur uzilish nuqtasi* deyiladi.

Agar x_0 nuqtada bir tomonlama limitlar mavjud va $f(x_0—0) = f(x_0+0)=fix_o)$ bo'lsa, u holda x_0 nuqta *bartaraf qilinish mumkin bo'lgan uzilish nuqtasi* deyiladi.

Funktsiyalarni aniqlanish sohasni va uzliksizligini aniqlashga doir misollar.

6.10-misol. $ó=\dfrac{x^2+1}{2x-1}$ **funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.**

Yechish. Bu kasir funktsiya mavjud bo'lishi uchun uning maxraji noldan farqli bo'lishi kerak. Yani $x\neq\frac{1}{2}$ bundan funktsiyani aniqlanish sohasi : $(-\infty,\frac{1}{2})$ va $(\frac{1}{2},+\infty,)$. $x=\frac{1}{2}$ funktsiyaning uzilish nuqtasi.

```
> restart;
```

$$f:=x\to\frac{x^2+1}{2x-1}$$

```
> f:=x->(x^2+1)/(2*x-1);
```

Funktsiyaning toqligini tekshirish:

```
>type(f(x),oddfunc(x));
```

false

Funktsiyaning juftlgini tekshirish:

```
> type(f(x),evenfunc(x));
```

false

Ko'rsatilgan oraliqdagi uzulik sizligi:

`> iscont(f(x),x=-2..2);` *false*

`> iscont(f(x),x=-2..0);` *true*

`> iscont(f(x),x=1..2);` *true*

Uzlisl nuqtaqsi:

`> discont(f(x),x);` $\left\{\frac{1}{2}\right\}$

Funktsiyaning aniqlanish sohasi:

`> solve(2*x-1<0,x);` $RealRange\left(-\infty, Open\left(\frac{1}{2}\right)\right)$

`> solve(2*x-1>0,x);` $RealRange\left(Open\left(\frac{1}{2}\right), \infty\right)$

6.11-misol. $ó=\sqrt{2x-3}$ **funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.**

Yechish. Bu kasir funktsiya mavjud bo'lishi uchun uning ilsiz ostisagi ifodasi manfiy bo'lmasligi kerak. Yani $2x-3\geq 0$ bundan $x\geq\frac{3}{2}$. Funktsiyani aniqlanish sohasi : $[\frac{3}{2}, +\infty,)$.

$x=\frac{3}{2}$ funktsiyaning kritik nuqtasi.

`> restart;`

`> f:=x->sqrt(2*x-3);` $f := x \to \sqrt{2x-3}$

Funktsiyaning toqligini tekshirish:

`> type(f(x),oddfunc(x));` *false*

Funktsiyaning juftlgini tekshirish:

`> type(f(x),evenfunc(x));` *false*

Kritik nuqta:

`> discont(f(x),x);` $\left\{\frac{3}{2}\right\}$

Ko'rsatilgan oraliqdagi uzulik sizligi:

`> iscont(f(x),x=-2..2);` *false*

`> iscont(f(x),x=2..4);` *true*

Funktsiyaning aniqlanish sohasi:

`> solve(2*x-3>=0,x);` $RealRange\left(\frac{3}{2}, \infty\right)$

6.12-misol. $ó=\sqrt{x-2}+\sqrt{4-x}$ **funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.**

Yechish. Bu funktsiyani aniqlanish sohasi uning ildiz ostidagi ifodalari manfiy bo'lmasligi bilan aniqlanadi.

Yani $x-2\geq 0$ va $4-x\geq 0$ bundan $x\geq 2,\ x\leq 4$. Funktsiyani aniqlanish sohasi : $2\leq x\leq 4$ yoki $[2, 4]$. $x=2,\ x=4$ funktsiyaning kritik nuqtasi.

`> restart;`

`> f:=x->sqrt(x-2)+sqrt(4-x);` $f := x \to \sqrt{x-2}+\sqrt{4-x}$

Funktsiyaning toqligini tekshirish:

```
> type(f(x),oddfunc(x));
```
false

Funktsiyaning juftlgini tekshirish:

```
> type(f(x),evenfunc(x));
```
false

Ko'rsatilgan oraliqdagi uzuliksizligi:

```
> iscont(f(x),x=0..4);
```
false

```
> iscont(f(x),x=3..4);
```
true

Uzlisl nuqtaqsi:

```
> discont(f(x),x);
```
$\{2, 4\}$

Funktsiyaning aniqlanish sohasi:

```
> solve({x-2>=0,4-x>=0},x);
```
$\{2 \% x, x \% 4\}$

6.13-misol. $ó = \arcsin\left(\frac{2x}{1+x}\right)$ funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.

Yechish. Bu funktsiya xossasiga asosan uning argumenti uchun $\left|\frac{2x}{1+x}\right| \le 1$ va $x \ne -1$ sartni bajarilishi kerak. Bundan

$$4x^2 \le 1 + 2x + x^2 \quad \text{yoki} \quad 3x^2 - 2x - 1 \le 0 \ \text{dan} \ -\frac{1}{3} \le x \le 1$$

Bu tengsizlikning eychimidan funktsiyani aniqlanish sohasi : $[-\frac{1}{3}, 1]$ buda $x = -1$ funktsiyaning kritik nuqtasi.

```
> restart;
> f:=x->arcsin(2*x/(1+x));
```
$f := x / \ \arcsin 0 \frac{2\,x}{1 \text{ C } x} 1$

Funktsiyaning toqligini tekshirish:

```
> type(f(x),oddfunc(x));
```
false

Funktsiyaning juftlgini tekshirish:

```
> type(f(x),evenfunc(x));
```
false

Ko'rsatilgan oraliqdagi uzulik sizligi:

```
> iscont(f(x),x=-2..2);
```
false

```
> iscont(f(x),x=-1..0);
```
true

Uzlisl nuqtaqsi:

```
> discont(f(x),x);
```
$\{\text{K } 1\}$

Funktsiyaning aniqlanish sohasi:

```
> solve(abs(2*x/(1+x))<1,x);
```
$RealRange\,0\, Open\,0 \frac{\text{K } 1}{3} 1, Open(1)\,1$

6.14-misol. $ó = \frac{tgx}{\sqrt{x^2 - x + 1}}$ funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.

Yechish. Bu funktsiya maxrajidagi ifoda argumentning barch qiymatlarida aniqlangan bo'lganligi uchun funktsiya suratidagi tgx ning uzulish nuqtalarini aniqlaymiz:

$$x = \frac{\pi}{2} + n\pi,\ n = 0, \pm 1, \pm 2,,,,$$

Funktsiyani aniqlanish sohasi : $-\frac{\pi}{2} < x < \frac{\pi}{2} + n\pi,\ n = 0, \pm 1, \pm 2,,,,$

```
> restart;
> f:=x->tan(x)/sqrt(x^2-x+1);
```

$f := x/\ \frac{\tan(x)}{\sqrt{x^2 \text{K}\ x \text{C}\ 1}}$

Funktsiyaning toqligini tekshirish:

```
> type(f(x),oddfunc(x));
```

false

Funktsiyaning juftlgini tekshirish:

```
> type(f(x),evenfunc(x));
```

false

tan(x) ni Uzlisl nuqtaqsi:

```
> discont(tan(x),x);
```

$\text{4}\ \_Z1\sim \text{C}\ \frac{1}{2}\ \text{p}\ \text{5}$

Funktsiyaning uzlisl nuqtaqsi:

```
>  discont(f(x),x);
```

$\text{4}\ \_Z2\sim \text{C}\ \frac{1}{2}\ \text{p}\ \text{5}$

```
> solve({x^2-x+1>0},x);
```

$\{x = x\}$

6.15-misol. $ó = \arccos\sqrt{3-2x-x^2}$ **funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.**

Yechish. Bu funktsiya aniqlanish sohasi uning xossasiga asosan uning argumenti uchun $0 \le \sqrt{3-2x-x^2} \le 1$ shartning bajarilishidan aniqlanadi. Yani

$$0 \le 3-2x-x^2 \le 1$$

dan funktsiyani aniqlanish sohasi :

$$-3 \le x \le 1-\sqrt{3} \text{ va } \sqrt{3}-1 \le x \le 1$$

bu erda $x = -3,\ x = 1$ funktsiyaning kritik nuqtasi.

```
> restart;
> f:=x->arccos(sqrt(3-2*x-x^2));
```

$f := x/\ \arccos \text{0}\sqrt{3 \text{K}\ 2\,x \text{K}\ x^2}\text{1}$

Funktsiyaning toqligini tekshirish:

```
> type(f(x),oddfunc(x));
```

false

Funktsiyaning juftlgini tekshirish:

```
> type(f(x),evenfunc(x));
```

false

Ko'rsatilgan oraliqdagi uzulik sizligi:

```
> iscont(f(x),x=-2..2);
```

false

```
> iscont(f(x),x=-2..0);
```

true

Funktsiyaning uzlisl nuqtaqsi:

```
> discont(f(x),x);
```

{K 3, 1, *RealRange*(*Open*(K 3), *Open*(1))}

Funktsiyaning aniqlanish sohasi:

```
> solve(0<3-2*x-x^2,x);
```

RealRange(*Open*(K 3), *Open*(1))

```
> solve(3-2*x-x^2<=1,x);
```

RealRange(K N, K 1 K $\sqrt{3}$), *RealRange*(K 1 C $\sqrt{3}$, N)

```
> evalf(solve({3-2*x-x^2>=0,3-2*x-x^2<=1},x));
```

{K 3.%x, x%K 2.732050808 }, {0.7320508076 %x, x%1. }

6.16-misol. $ó = \begin{cases} x^2, & x \le 0 \\ (x-1)^2, & 0 < x \le 2 \\ 5-x, & x < 2 \end{cases}$ **funktsiyani aniqlanish sohasi va uzluksizligi aniqlansin.**

```
> restart;
> f:=piecewise(x<=0, x^2, x>0 and x<2, (x-1)^2, x>2,5-x);
```

$$f := \begin{cases} x^2 & x \le 0 \\ (x-1)^2 & 0 < x \textbf{ and } x < 2 \\ 5-x & 2 < x \end{cases}$$

```
> simplify(%);
```

$$\begin{cases} x^2 & x \le 0 \\ (x-1)^2 & x < 2 \\ 0 & x = 2 \\ 5-x & 2 < x \end{cases}$$

Ko'rsatilgan oraliqdagi uzulik sizligi:

```
> iscont(f,x=0..2);
```

true

```
> discont(f,x);
```

{0, 2}

```
> eval( f, x=0. );
```

0

```
> eval( f, x= 2. );
```

0

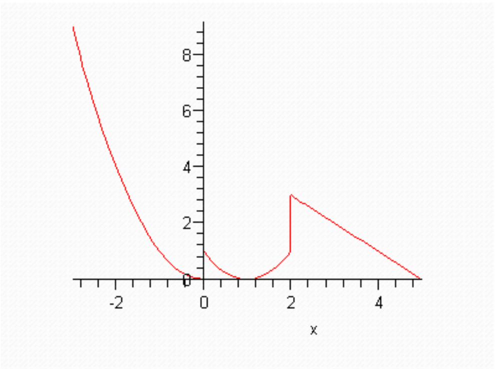
```
> limit(f, x=-0);limit(f, x=+0);
```

undefined *undefined*

```
> limit(f, x=2-0);limit(f, x=2+0);
```

undefined *undefined*

```
> plot(f, x=-3..6);
```



Nazorat ishi uchun variant namunalari.

1 – variant

1. $\lim\limits_{x\to 5}\dfrac{3x^2-17x+10}{3x^2-16x+5}$, 2. $\lim\limits_{x\to 5}\dfrac{5-x}{3-\sqrt{2x-1}}$;

3. $\lim\limits_{x\to 8}\dfrac{2x^2+x+1}{3x^3+x^2+1}$, 4. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-\sin x}}{\sin x}$;

5. $\lim\limits_{x\to\infty}\left(1-\dfrac{2}{x}\right)^x$;

2 – variant

1. $\lim\limits_{x\to 1}\dfrac{4x^2-7x+3}{3x^2-2x-1}$, 2. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{3+x}-\sqrt{3-x}}$;

3. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+tgx}-1}{3tgx}$, 4. $\lim\limits_{x\to\infty}\dfrac{5x^4-x^3+2x}{x^4-8x^3+1}$;

5. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{-x}$; 6. $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{2x-1}\right)^{5x}$.

6-MUSTAQIL ISHLASH UCHUN TOPSHIRIQLAR

Lopital qoidasidan foydalanmasdan quyidagi limitlarni hisoblang.

1. a) $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2+4}{4x^3+3x^2+1}$; b) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}$; v) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{5x^2}$; g) $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-2}\right)^x$;

2. a) $\lim\limits_{x\to\infty}\dfrac{3x^2-2x+1}{x^3-4x^2+5}$; b) $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2+4}{4x^3+3x^2+1}$; v) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{5x^2}$; g) $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x-3}\right)^x$

3. a) $\lim\limits_{x\to\infty}\dfrac{3x^4-2}{\sqrt{x^8+3x+4}}$; b) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$ v) $\lim\limits_{x\to 0}\dfrac{tg2x}{\sin 3x}$; g) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2}\right)^{x^2+1}$

4. a) $\lim\limits_{x\to 0}\dfrac{5x^3-2x}{4x^3-2x^2+2}$; b) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x}-\sqrt{1-2x}}{x+x^2}$ v) $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{5x}$; g) $\lim\limits_{x\to 0}(1+\sin x)^{\cos x}$

5. a) $\lim\limits_{x\to\infty}\dfrac{x^4+x+1}{5x^4+7x^2+x}$; b) $\lim\limits_{x\to -3}\dfrac{9-x}{1-\sqrt{4+x}}$ v) $\lim\limits_{x\to 0}\dfrac{8x}{\sin^2 5x}$; g) $\lim\limits_{x\to -1}2x\ln(1+\dfrac{1}{x})$.

6. a) $\lim\limits_{x\to\infty}\dfrac{3x^4-2x-9}{9x^2+3x+5}$; b) $\lim\limits_{x\to -1}\dfrac{\sqrt{2x+3}-1}{\sqrt{5+x}-2}$ v) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^5 x}{x^2}$; g) $\lim\limits_{x\to\infty}(x+2)[\ln(2x+1)-\ln(2x-1)]$

7. a) $\lim\limits_{x\to 3}\dfrac{(x+1)^3}{-5x^9+1}$; b) $\lim\limits_{x\to\infty}(x+2)[\ln(2x+1)-\ln(2x-1)]$, v) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{\sqrt{x-2}-1}$; g) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

8. a) $\lim\limits_{x\to\infty}\dfrac{6x^5-11x^2+3}{5x-3x^5}$; b) $\lim\limits_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)$ v) $\lim\limits_{x\to 0}\dfrac{\cos 5x-1}{xtg2x}$; g) $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x+8}\right)^{2x+3}$

9. a) $\lim\limits_{x\to\infty}\dfrac{8x^7-6x^6+5}{x-3x^2}$; b) $\lim\limits_{x\to\infty}x\left(\sqrt{x^2+1}-x\right)$ v) $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{3x}$ g) $\lim\limits_{x\to\infty}x[\ln(x+1)-\ln x]$

10. a) $\lim\limits_{x\to\infty}\dfrac{5x^2-8x+1}{x^3-1}$; b) $\lim\limits_{x\to -4}\dfrac{2-\sqrt{8+x}}{4+x}$; v) $\lim\limits_{x\to 0}(1+3x)^{\frac{2x+1}{3x}}$; g) $\lim\limits_{x\to 0}\dfrac{arctg3x}{x}$

11. a) $\lim\limits_{x\to\infty}\dfrac{x-2x^2+5x^4}{2+3x^2+x^4}$; b) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-2}{\sqrt{2x+5}-3}$; v) $\lim\limits_{x\to 0}\dfrac{x^2 ctg2x}{\sin 3x}$; g) $\lim\limits_{x\to\infty}(2x+3)[\ln(x+1)-\ln x]$.

12. a) $\lim\limits_{x\to\infty}\dfrac{2+x^2-3x^3}{1-3x+6x^3}$; b) $\lim\limits_{x\to 4}\dfrac{x^2-16}{\sqrt{x}-2}$; v) $\lim\limits_{x\to 0}\dfrac{tgx-\sin x}{x\sin x}$; g) $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x+1}\right)^{x+1}$

13. a) $\lim\limits_{x\to\infty}\dfrac{x^2+x-2}{x-1}$; b) $\lim\limits_{x\to\infty}(\sqrt{x^2-3x}-x)$; v) $\lim\limits_{x\to 0.5}\dfrac{\arcsin(1-2x)}{4x^2-1}$; g) $\lim\limits_{x\to\infty}x[\ln x-\ln(x+2)]$.

14. a) $\lim\limits_{x\to\infty}\dfrac{3x^3-7x+5}{4x^3+8x-3}$; b) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x^2}-1}{x^2+x^3}$ v) $\lim\limits_{x\to a}\dfrac{\cos x-\cos a}{x-a}$; g) $\lim\limits_{x\to\infty}(1-\dfrac{2}{x})^{5x}$.

15 1) $\lim\limits_{x\to 1}\dfrac{2x^2+5x-10}{x^3-1}$; 3) $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{\sqrt{x-2}-\sqrt{4-x}}$; 4) $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{5x}$; 6) $\lim\limits_{x\to\infty}\left(\dfrac{x-7}{x+1}\right)^{4x-2}$.

16 1) $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^2-4x+3}$; 2) $\lim\limits_{x\to -4}\dfrac{\sqrt{x+12}-\sqrt{4-x}}{x^2+2x-8}$; 3) $\lim\limits_{x\to 0}\dfrac{tg2x-\sin 2x}{x^2}$; 4) $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{2x-1}\right)^{5x}$.

17. 1) $\lim\limits_{x\to 2}\dfrac{3x^2+2x+1}{x^3-8}$; 2) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{\sqrt{x^2+16}-4}$; 3) $\lim\limits_{x\to 0}\dfrac{\sin 7x+\sin 3x}{x\sin x}$; 4) $\lim\limits_{x\to+\infty}\left(\dfrac{6x+5}{x-10}\right)^{5x}$;.

18. 1) $\lim\limits_{x\to 1}\dfrac{2x^2-3x-1}{x^4-1}$; 2) $\lim\limits_{x\to 0}\dfrac{3ч}{\sqrt{1+x}-\sqrt{1-x}}$; 3) $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos 4x}{arctg3x^2}$; 4) $\lim\limits_{x\to\infty}\left(\dfrac{2x-3}{x+4}\right)^{6x+1}$

19. 1) $\lim\limits_{x\to -4}\dfrac{2x^2+7x-4}{x^3+64}$; 2) $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{\sqrt{x-1}-2}$; 3) $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos 4x}{tg3x^2}$; 4) $\lim\limits_{x\to+\infty}\left(\dfrac{x-1}{4x+5}\right)^{5x}$

20. 1) $\lim\limits_{x\to 1}\dfrac{x^3+x-2}{x^3-x^2-x+1}$; 2) $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$; 3) $\lim\limits_{x\to 0}\dfrac{\cos^2 x-\cos^2 2x}{\arcsin x^2}$; 4) $\lim\limits_{x\to-\infty}\left(\dfrac{3x+7}{x+4}\right)^{5x}$

21. 1) $\lim\limits_{x\to -4}\dfrac{4x^3-2x^2+5x}{3x^2+7x}$; 2) $\lim\limits_{x\to 9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$; 3) $\lim\limits_{x\to 0}\dfrac{\arcsin 7x}{\sin x+\sin 7x}$; 4) $\lim\limits_{x\to+\infty}\left(\dfrac{x-2}{3x+10}\right)^{5x}$.

22. 1) $\lim\limits_{x\to -5}\dfrac{x^2-x-30}{x^3+125}$; 2) $\lim\limits_{x\to 4}\dfrac{2-\sqrt{x}}{\sqrt{6x+1}-5}$; 3) $\lim\limits_{x\to 0}\dfrac{\arcsin 5x}{2x^2-x}$; 4) $\lim\limits_{x\to+\infty}\left(\dfrac{4x+3}{x+5}\right)^{7x}$.

23 1) $\lim\limits_{x\to -6}\dfrac{x^2+2x-24}{2x^3+15x+18}$; 2) $\lim\limits_{x\to 7}\dfrac{\sqrt{x-3}-2}{\sqrt{x+2}-3}$; 3) $\lim\limits_{x\to 0}\dfrac{1-\cos 8x}{3arctg^2 x}$; 4) $\lim\limits_{x\to+\infty}\left(\dfrac{1-x}{2-10x}\right)^{15x+3}$

24. 1) $\lim\limits_{x\to 2}\dfrac{x^3-2x-40}{x^2-11x+18}$; 2) $\lim\limits_{x\to 2}\dfrac{\sqrt{4x+1}-3}{x^3-8}$; 3) $\lim\limits_{x\to 0}\dfrac{\sin^2 3x-\sin^2 x}{tgx\cdot arctgx}$; 4) $\lim\limits_{x\to-\infty}\left(\dfrac{x+5}{4x-2}\right)^{3x+7}$

Bir o'zgaruvchili funktsiyaning differentsial hisobi
7. Hosila

7.1. Hosilaning ta'rifi va differentsiallash qoidalari

<u>Ta'rif.</u> $y=f(x)$ ***funktsiyaning hosilasi deb, funktsiya ortirmasi Δy ni argumentning mos ortirmasi Δx ga nisbatining $\Delta x\to 0$ dagi limitiga aytiladi.***

Berilgan funktsiyadan hosila topish amali shu funktsiyani *differentsiallash* deyiladi. Uni quyidagicha belgilanadi:

$$y',\quad y'_x,\quad \frac{dy}{dx}\quad \text{eki}\quad f'(x),\quad \frac{df(x)}{dx}$$

$y = f(x)$ funktsiyaning hosilasini hisoblash quyidagi to‘rt bosqichda bajariladi.

I. x, Δx ; x + Δx da: $y + \Delta y = f(x + \Delta x)$

II. $\Delta y = f(x + \Delta x) - f(x)$

III. $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x-\Delta x)}{\Delta x}$

IV. $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x-\Delta x)}{\Delta x} = f'_x(x)$

Differentsiallash qoidalari:

1. $(U \pm V)'_x = U'_x \pm V'_x$

> Diff(u+v, x)=diff(u(x)+v(x), x);

$$\frac{\partial}{\partial x}(u+v) = \frac{d}{dx}u(x) + \frac{d}{dx}v(x)$$

2. $(U \cdot V)'_x = U' \cdot V + V' \cdot U \ , \ (CU)' = C \cdot U'$

> Diff(u*v, x)=diff(u(x)*v(x), x);

$$\frac{\partial}{\partial x}(u\,v) = \left(\frac{d}{dx}u(x)\right)v(x) + u(x)\left(\frac{d}{dx}v(x)\right)$$

3. $(U \cdot V \cdot W)'_x = U' \cdot V \cdot W + U \cdot V' \cdot W + U \cdot V \cdot W'$

> Diff(u*v*w, x)=diff(u(x)*v(x)*w(x), x);

$$\frac{\partial}{\partial x}(u\,v\,w) = \left(\frac{d}{dx}u(x)\right)v(x)\,w(x) + u(x)\left(\frac{d}{dx}v(x)\right)w(x) + u(x)\,v(x)\left(\frac{d}{dx}w(x)\right)$$

> Diff(u(x)*v(x), x$3)=diff(u(x)*v(x), x$3);

$$\frac{d^3}{dx^3}(u(x)\,v(x)) = \left(\frac{d^3}{dx^3}u(x)\right)v(x) + 3\left(\frac{d^2}{dx^2}u(x)\right)\left(\frac{d}{dx}v(x)\right) + 3\left(\frac{d}{dx}u(x)\right)\left(\frac{d^2}{dx^2}v(x)\right) + u(x)\left(\frac{d^3}{dx^3}v(x)\right)$$

4. $\left(\dfrac{U}{V}\right)' = \dfrac{U' \cdot V - V' \cdot U}{V^2}, \quad \left(\dfrac{C}{V}\right)' = -\dfrac{C}{V^2}V'$

> Diff(u/v, x)=diff(u(x)/v(x), x);

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{\frac{d}{dx}u(x)}{v(x)} - \frac{u(x)\left(\frac{d}{dx}v(x)\right)}{v(x)^2}$$

5. y = f(U), U = φ(x) bo‘lsa, $y'_x = f'_U \cdot f'_x$

> Diff(F(u(x)), x)=diff(F(u(x)), x);

$$\frac{d}{dx}F(u(x)) = (D(F))(u(x))\left(\frac{d}{dx}u(x)\right)$$

6. $y=[u(x)]^{V(x)}$, $y'_x=[u(x)]^{v(x)}\left[\frac{dv(x)}{dx}\ln(u(x))+\frac{v(x)}{u(x)}\frac{du(x)}{dx}\right]$

> Diff(u(x)^v(x), x)=diff(u(x)^v(x), x);

$$\frac{d}{dx}u(x)^{v(x)} = u(x)^{v(x)}\left(\left(\frac{d}{dx}v(x)\right)\ln(u(x)) + \frac{v(x)\left(\frac{d}{dx}u(x)\right)}{u(x)}\right)$$

7. x=φ(t), y=ψ(t) ; $y'_x=\frac{y'_t}{x'_t}$, $\frac{dy}{dx}=\frac{[\psi(t)]'_t}{[\varphi(t)]'_t}$

> x:=phi(t); y:=psi(t); dy/dx=diff(y,t)/diff(x,t);

$$x := \phi(t) \quad y := \psi(t) \quad \frac{dy}{dx} = \frac{\frac{d}{dt}\psi(t)}{\frac{d}{dt}\phi(t)}$$

> dy/dx =diff(si(t),t)/diff(hi (t),t); $\frac{dy}{dx} = \frac{\frac{d}{dt}\psi(t)}{\frac{d}{dt}\phi(t)}$

8. $F(x,\ y(x))=0$, $F'_x+F'_y y'_x=0$, $y'_x=-\frac{F'_x}{F'_y}$

> diff(F(x,y(x)), x)=0;

$$(D_1(F))(x, y(x)) + (D_2(F))(x, y(x))\left(\frac{d}{dx}y(x)\right) = 0$$

```
> d:=solve((D[1](F))(x,y(x))+(D[2](F))(x,y(x))*d, d);
```

$$d := -\frac{(D_1(F))(x, y(x))}{(D_2(F))(x, y(x))}$$

HOSILALAR JADVALI

1. $(C)'=0$ — $U=U(x)$ bo‘lganda,

2. $(X)'=1$ — $y=f(U)$ hosilasi:

3. $(x^n)'=nx^{n-1}$ — $(U^n)'=nU^{n-1}*U'_x$.

$$4.\ \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \qquad \left(\frac{1}{U}\right)' = -\frac{1}{U^2}$$

$$5.\ \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}} \qquad \left(\sqrt{U}\right)' = \frac{1}{2\sqrt{U}}U'$$

$$6.\ \left(a^x\right)' = a^x \ln a \qquad \left(a^u\right)' = a^u U' \ln a$$

$$7.\ \left(e^x\right)' = e^x \qquad \left(e^u\right)' = e^u U'$$

$$8.\ (\log_a x)' = \frac{1}{x}\log_a e \qquad (\log_a U)' = \frac{U'}{U}\log_a e$$

$$9.\ (\ln x)' = \frac{1}{x} \qquad (\ln U)' = \frac{U'}{U}$$

$$10.\ (\sin x)' = \cos x \qquad (\sin u)' = (\cos u)u'$$

$$11.\ (\cos x)' = -\sin x \qquad (\cos u)' = (-\sin u)u'$$

$$12.\ (tgx)' = \frac{1}{\cos^2 x} \qquad (tgu)' = \frac{1}{\cos^2 u}u'$$

$$13.\ (ctgx)' = -\frac{1}{\sin^2 x} \qquad (ctgu)' = -\frac{1}{\sin^2 u}u'$$

$$14.\ \left.\begin{matrix}(\arcsin x)' \\ (\arccos x)'\end{matrix}\right\} = \pm\frac{1}{\sqrt{1-x^2}} \qquad \left.\begin{matrix}(\arcsin u)' \\ (\arccos u)'\end{matrix}\right\} = \pm\frac{u'}{\sqrt{1-u^2}}$$

$$15.\ \left.\begin{matrix}(arctgx)' \\ (arcctgx)'\end{matrix}\right\} = \pm\frac{1}{1+x^2} \qquad \left.\begin{matrix}(arctgu)' \\ (arcctgu)'\end{matrix}\right\} = \pm\frac{u'}{1+u^2}$$

7.1-misol. Quyidagi funktsiyalarning hosilasini toping.

$$1.\ y = \ln(1-\sin x) \qquad 2.\ y = 2^{\arcsin(x^2)}$$

$$3.\ y = (tgx)^{\frac{1}{x}} \qquad 4.\ \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

$$5.\ \cos(xy) = x \qquad 6.\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Yechish.

1). $y = \ln(1-\sin x)$ - murakkab funktsiya bo‘lganligi uchun U(x)=1-sinx deb, y = ln(U(x)) funktsiya hosilasini topamiz. Murakkab funktsiya hosilasini topish qoidasiga asosan:

$$y' = \frac{1}{U(x)}U'_x = \frac{1}{1-\sin x}(1-\sin x)' = \frac{-\cos x}{1-\sin x}.$$

> Diff(ln(1-sin(x)), x)=diff(ln(1-sin(x)), x);

$$\frac{d}{dx}\ln(1\ \mathrm{K}\ \sin(x)) = \mathrm{K}\ \frac{\cos(x)}{1\ \mathrm{K}\ \sin(x)}$$

2). $y = 2^{\arcsin(x^2)}$ - murakkab funktsiya. Agar *arcsin* $(x^2) = U$ kabibelgilasak, bu funktsiya ham murakkab funktsiya bo‘ladi, bu holda:

$$y' = 2^u \ln 2 \cdot U'_x \quad ; \quad U'_x = \frac{1}{\sqrt{1-(x^2)^2}}(x^2)' = \frac{2x}{\sqrt{1-x^4}}$$

demak, $y = 2^{\arcsin(x^2)} \cdot \frac{2x}{\sqrt{1-x^4}} \ln 2$

> **Diff(2^(arcsin(x^2)), x)=diff(2^(arcsin(x^2)), x);**

$$\frac{d}{dx} 2^{\arcsin(x^2)} = \frac{2\, 2^{\arcsin(x^2)}\, x \ln(2)}{\sqrt{1 - x^4}}$$

3). $y = (tgx)^{\frac{1}{x}}$ funktsiya , $y = [U(x)]^{V(x)}$ ko‘rinishdagi funktsiya bo‘lib, uni daraja ko‘rsatkichli funktsiya deyiladi. Bu funktsiya hosilasini topish uchun uni logarifmlash bilan, xosil bo‘lgan lny murakkab funktsiya hosilasini topamiz :

$$\ln y = \frac{1}{x} \ln tgx \; ; \qquad (\ln y)'_x = (\frac{1}{x} \ln tgx)'_x$$

$$\frac{1}{y} y'_x = -\frac{1}{x^2} \ln tgx + \frac{1}{x} \frac{1}{tgx} \frac{1}{\cos^2 x}$$

$$\frac{1}{y} y'_x = \frac{1}{x} (\frac{1}{\sin x} \frac{1}{\cos x} - \frac{\ln tgx}{x})$$

$$y'_x = \frac{1}{x} (\frac{2}{\sin 2x} - \frac{\ln tgx}{x})$$

$$y'_x = (tgx)^{\frac{1}{x}} \frac{1}{x} (\frac{2}{\sin 2x} - \frac{\ln tgx}{x})$$

> **Diff(tan(x)^(1/x), x)=diff(tan(x)^(1/x), x);**

4) $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ Parametrik funktsiya hosilasini topish qoidasiga binoan:

$$y'_t = (a\sin t)'_t = a\cos t \; , \quad x'_t = (a\cos t)'_t = -a\sin t$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{a\cos t}{-a\sin t} = -ctgt$$

> **x:=a*cos(t); y:=b*sin(t); dy/dx=diff(y,t)/diff(x,t);**

$$x := a\cos(t) \quad y := b\sin(t)$$

$$\frac{dy}{dx} = -\frac{b\cos(t)}{a\sin(t)}$$

> **dy/dx=diff(b*sin(t),t)/diff(a*cos(t),t);**

$$\frac{dy}{dx} = -\frac{b\cos(t)}{a\sin(t)}$$

5) cos(x y) = x funktsiya *y* ga nisbatan echilmaganligi uchun u oshkormas deyiladi. Bunday funktsiya hosilasini topishda y = y(x) ekanligini e’tiborga olib, hosila topiladi:

$$[\cos(xy)]'_x = (x)'_x \qquad \sin(xy)(xy)'_x = 1 \qquad (1y + xy')\sin(xy) = 1$$

$$y + xy'_x = \frac{1}{\sin(xy)}, \qquad xy'_x = \frac{1 - y\sin(xy)}{\sin(xy)} \qquad y' = \frac{1}{x} \cdot \frac{1 - y\sin(xy)}{\sin(xy)} = \frac{1 - y\sin(xy)}{x\sin(xy)}$$

> **restart;**
> **diff(cos(x*y(x))-x, x)=0;**

$$-\sin(x\,y(x))\left(y(x)+x\left(\frac{d}{dx}\,y(x)\right)\right)-1=0$$

> d1:=solve(-sin(x*y(x))*(y(x)+x*d1)-1= 0, d1);

$$d1:=-\frac{\sin(x\,y(x))\,y(x)+1}{\sin(x\,y(x))\,x}$$

6) Shuningdek, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning 1-tartibli hosshdasini topish:

> restart;
> diff(x^2/a^2+y(x)^2/b^2-1, x)=0;

$$\frac{2\,x}{a^2}+\frac{2\,y(x)\left(\frac{d}{dx}\,y(x)\right)}{b^2}=0$$

> d:=solve(2*x/a^2+2*y(x)*d/b^2 = 0, d);

$$d:=-\frac{x\,b^2}{y(x)\,a^2}$$

7.2. Yuqori tartibli hosilalar va differentsiallar

Yuqorida funktsiyaning hosilasi argumentning ixtiyoriy qiymatida (aniqlanish soasiga tegishli) mavjud bo‘lsa, hosila ham funktsiyadan iborat ekanligini ko‘rdik.

Agar funktsiyaning *birinchi tartibli hosilasi* ham hosilaga ega bo‘lsa, *hosiladan olingan hosilani ikkinchi tartibli hosila* deb yuritiladi.

Funktsiyaning hosilasini uning *birinchi tartibli hosilasi* deb qabul qilsak, umumiy holda quyidagi ta’rifni berish mumkin.

Ta’rif. *Agar funktsiyaning (n-1) tartibli hosilasi differentsialanuvchi bo‘lsa, uning hosilasini funktsiyaning n-tartibli hosilasi* deyiladi va $y^{(n)}$, $\frac{d^n y}{dx^n}$, $f^{(n)}(x)$, $\frac{d^n f(x)}{dx^n}$ kabi belgilanadi. Bu holda *funktsiya n marta differentsiallanuvchi* deyiladi.

Demak,

$$y^{(n)}=\left(y^{(n-1)}\right)', \qquad n=1,\ 2,\ \ldots$$

bu erda funktsiyaning nolinchi tartibli hosilasi sifatida uning o‘zini qabul q ilish tabiiydir, ya’ni

$$y^{(0)}=y.$$

Eslatma. *Yuqori tartibli hosilani belgilashda hosila belgisini kerakli marta takrorlash usuli am o‘llaniladi. Masalan, y'' - ikkinchi, y''' - uchinchi va okazo tartibli hosilalardir. Shuningdek, bahzan Rim raqamlari ham qo‘llaniladi, masalan, y^{IV} - to‘rtinchi, y^{V} – beshinchi va hokazo tartibli hosilalardir.*

Quyidagi misollarni keltiramiz:

7.2-misol. $y=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ bo‘lsa, funktsiyaning n-inchi tartibli hosilasi:

$y'=na_0x^{n-1}+(n-1)a_nx^{n-2}+\ldots+a_{n-1}$,
$y''=n(n-1)a_0x^{n-2}+(n-1)(n-2)a_nx^{n-3}+\ldots+a_{n-1}$,
$y'''=n(n-1)(n-2)a_0x^{n-3}+(n-1)(n-2)(n-3)a_nx^{n-4}+\ldots+a_{n-1}$,
$y^{IV}=n(n-1)(n-2(n-3)a_0x^{n-4}+(n-1)(n-2)(n-3)(n-4)a_nx^{n-5}+\ldots+a_{n-1}$,

- - - - - - - - - - - - - - - - - - -

$y^{(n)}=n\cdot(n-1)\cdot\ldots\cdot2\cdot1\cdot a_0=a_0\,n!$,

$y^{(n+1)}=y^{(n+2)}=\ldots=0$.

```
> diff( a0*x^m,x$n);
```

$$\mathrm{pochhammer}(m-n+1,n)\,a0\,x^{(m-n)}$$

```
> _EnvFallingNotation := GAMMA;
```

$$\_EnvFallingNotation := \Gamma$$

```
> diff(x^m,x$n);
```

$$\frac{\Gamma(m+1)\,x^{(m-n)}}{\Gamma(m-n+1)}$$

Demak, *n* – darajali ko‘phadning *n* – tartibli hosilasi o‘zgarmas son bo‘lib, (*n+1*)- tartibli hosilasidan boshlab yuqori tartibli hosilalarining barchasi nolga teng bo‘lar ekan.

7.3-misol. $f(x)=e^{kx}$, k – o‘zgarmas $(k\neq0)$. funktsiyaning n-tartibli hosilasi

$f'(x)=e^{kx}(kx)'=ke^{kx}$;

$f''(x)=(f'(x))'=(ke^{kx})'=k(e^{kx})'=k\cdot ke^{kx}=k^2e^{kx}$

……………………….

$f^{(n)}(x)=k^ne^{kx}$

Demak,

$$(e^{kx})^{(n)}=k^n e^{kx},\ n\in N$$

```
> Diff( e^k*x, x$n )=diff( exp(k*x), x$n );
```

$$\frac{\partial^n}{\partial x^n}(e^k x)=k^n e^{(kx)}$$

7.4-misol. *f(x)=sinx*. funktsiyaning n-tartibli hosilasini topish.

$$f'(x)=\cos x=\sin(x+\frac{\pi}{2}),$$

$$f''(x)=(f'(x))'=(\sin(x+\frac{\pi}{2}))'=\cos(x+\frac{\pi}{2})\cdot 1=\sin(x),$$

- -

$$f^{(n)}(x)=\sin(x+n\cdot\frac{\pi}{2}),$$

ya'ni $(\sin x)^{(n)}=\sin(x+n\cdot\frac{\pi}{2}),\ \ n\in N$

```
> Diff(sin(x), x$n);
```

$$\frac{d^n}{dx^n}\sin(x)$$

```
> value(%);
```

$$\sin\left(x+\frac{1}{2}n\pi\right)$$

7.5-misol. *f(x)=cosx*. funktsiyaning n-tartibli hosilasini topish.
Yuqoridagiga o'xshash,

$$(\cos x)^{(n)}=\cos(x+n\cdot\frac{\pi}{2}),\ \ n\in N$$

ni olish mumkin.

```
> cn := diff( cos(x), x$n );
```

$$cn:=\cos\left(x+\frac{1}{2}n\pi\right)$$

7.6-misol. *f(x)=uv*, bu erda *u* va *v* lar ixtiyoriy tartibli hosilalari mavjud funktsiyalardir.

$$(uv)'=u'v+uv'$$

$$(uv)''=((uv)')'=(u'v+uv')'=u''v+u'v'+u'v'+uv''=u''v+2u'v'+uv''$$

va hokazo.

$$(u\cdot v)^{(n)}=\sum_{k=0}^{n}C_n^k\,u^{(n-k)}\cdot v^{(k)}$$

ni olish mumkin. Bu ***Leybnits formulasi*** deb yuritiladi. Bu yerda nolinchi tartibli hosila funktsiyaning o'zi ekanligini eslash lozim.

```
> Diff(u(x)*v(x), x$n);
```

$$\frac{d^n}{dx^n}(u(x)\,v(x))$$

```
> value(%);
```

$$\sum_{\_k1=0}^{n} \operatorname{binomial}(n, \_k1)\left(\frac{d^{\_k1}}{dx^{\_k1}} u(x)\right)\left(\frac{d^{n-\_k1}}{dx^{n-\_k1}} v(x)\right)$$

7.7- misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning x bo'yicha 2-tartibli hosilasini topish:

1) hosilasini topish qoidasiga asosan:

```
1)> restart;
> diff(x^2/a^2+y(x)^2/b^2-1, x$2)=0;
```

$$\frac{2}{a^2}+\frac{2\left(\frac{d}{dx} y(x)\right)^2}{b^2}+\frac{2\, y(x)\left(\frac{d^2}{dx^2} y(x)\right)}{b^2}=0$$

```
> d2:=solve(2/a^2+2*(diff(y(x),x))^2/b^2+2*y(x)*d2/b^2 =
0,d2);
```

$$d2 := -\frac{b^2+\left(\frac{d}{dx} y(x)\right)^2 a^2}{y(x)\, a^2}$$

2) oshkormas funktsiya hoslasini topish

```
> with(algcurves):
> F:=x^2/a^2+y^2/b^2-1=0;
```

$$F := \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

```
> yx:=implicitdiff(F,y,x);
```

$$yx := -\frac{x\, b^2}{y\, a^2}$$

```
> yxx:=implicitdiff(F,y,x$2);
```

$$yxx := -\frac{b^2\left(a^2 y^2+x^2 b^2\right)}{y^3 a^4}$$

Parametrik funktsiyaning 2- tartibli hosilasi:

```
> x:=phi(t); y:=psi(t);
d(dy/dx)/dx=diff(diff(y,t)/diff(x,t),t)/diff(x,t);
```

$$x := \phi(t) \qquad y := \psi(t)$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dx}=\frac{\frac{\frac{d^2}{dt^2}\psi(t)}{\frac{d}{dt}\phi(t)}-\frac{\left(\frac{d}{dt}\psi(t)\right)\left(\frac{d^2}{dt^2}\phi(t)\right)}{\left(\frac{d}{dt}\phi(t)\right)^2}}{\frac{d}{dt}\phi(t)}$$

7.8- misol. $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ Parametrik funktsiyani 2- tartibli hosilasini topish.

```
> x:=a*cos(t); y:=b*sin(t);
d(dy/dx)/dx=diff(diff(y,t)/diff(x,t),t)/diff(x,t);
```

$$x := a\cos(t) \qquad y := b\sin(t)$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = -\frac{\frac{b}{a}+\frac{b\cos(t)^2}{a\sin(t)^2}}{a\sin(t)}$$

Endi, yuqori tartibli differentsial tushunchasini kiritamiz. Buning uchun funktsiya differentsialini uning birinchi tartibli differentsiali argument orttirmasini o‘zgarmas deb qabul qilgan holda (*n–1*) – tartibli differentsialning differentsialini *n-tartibli differentsial deb ataymiz* va uning uchun $d^n y$, $d^n f(x)$ kabi belgilashlarni qo‘llaymiz.

Demak, Ta’rif bo‘yicha $d^n y=d(d^{n-1}y)$ekan. Oxirgi formula asosida

$$d^2y=d(dy)=d[f'(x)dx]=(f''(x)dx)dx=f''(x)dx^2$$

va hokazo,

$$d^n y=f^{(n)}(x)dx^n$$

formulani olamiz.

Bu yerda ikkinchi va undan yuqori tartibli differentsiallar birinchi tartibli differentsialning invariantlik xossasiga ega emasligini ammo, oraliq o‘zgaruvchi bo‘lgan murakkab funktsiya argumenti (erkli o‘zgaruvchi)ning chiziqli funktsiyasi bo‘lgan holda bu xossa saqlanishini aytamiz.

Yuqori tartibli hosila ma’nolariga kelsak, agar moddiy nuta $S=S(t)$ qonun bo‘yicha to‘g’ri chiziq bo‘ylab harakatlanayotgan bo‘lsa, undan (yo‘l funktsiyasidan) olingan birinchi tartibli hosila moddiy nuqtaning tezligi $\vartheta=\vartheta(t)$ ekanligi bizga ma’lum, ya’ni

$$\vartheta=\frac{dS}{dt}\ .$$

Agar tezlanishni qaralsa,

$$a=\lim_{\Delta t\to 0}\frac{\Delta\vartheta}{\Delta t}=\frac{d\vartheta}{dt}$$

ekanligini chiqarish qiyin emas:

$$a=\frac{d\vartheta}{dt}=\frac{d}{dt}\left(\frac{dS}{dt}\right)=\frac{d^2S}{dt^2}\ .$$

Demak, *to‘g’ri chiziqli harakatda bo‘lgan moddiy nuqtaning tezlanishi uning yo‘l funktsiyasidan olingan ikkinchi tartibli hosilaga teng* ekan. Bu ikkinchi tartibli hosilaning fizik mahnosidir.

7- MUSTAQIL ISHLASH UCHUN TOPSHIRIQLAR

Quyidagi funktsiyalarni hosilasini toping.

1. a) $y=\ln(x+\sqrt{x^2+1})$, b) $y=\left(\frac{x-1}{x+2}\right)^4$ v) $tg\frac{y}{x}=5x$, g) $\begin{cases}x=\cos\frac{t}{2}\\ y=t-\sin t\end{cases}$

2. a) $y=x^2\sqrt{1-x^2}$, b) $y=\frac{\sin x}{\cos^2 x}$ v) $x-y+arctgy=0$, g) $\begin{cases}x=t^3+8t\\ y=t^5+2t\end{cases}$

3. a) $y=tg^6x$, b) $y=\frac{e^{2x}}{\sqrt{3x-1}}$ v) $x-y+arcctgy=0$, g) $\begin{cases}x=2\sin^3 t\\ y=\cos^2 t\end{cases}$

4. a) $y=\sin^3\frac{x}{3}$, b) $y=x\sqrt{1+x^2}$ v) $y\sin x=\cos(x-y)$ g) $\begin{cases}x=\ln t\\ y=\frac{t}{2}+\frac{1}{t}\end{cases}$

5. a) $y=\ln tg\frac{t}{2}$, b) $y=\arcsin\sqrt{1-3x}$ v) $\frac{y}{x}=arctg\frac{x}{y}$, g) $\begin{cases}x=t+\ln\cos t\\ y=t-\ln\sin t\end{cases}$

6. a) $y=\ln(x+\sqrt{x^2+1})$, b) $y=\frac{1}{tg^2 2x}$ v) $y^2x=e^{\frac{y}{x}}$, g) $\begin{cases}x=2t-t^3\\ y=2t^2\end{cases}$

7. a) $y=\ln\sqrt{2\sin x+1}$, b) $y=3^{arctgx^2}$ v) $x-y+\arcsin y=0$, g) $\begin{cases}x=3\cos t\\ y=4\sin^2 t\end{cases}$

8. a) $y=\arcsin\frac{2x^2}{1+x^4}$, b) $y=x^2\sqrt{1-x^2}$ v) $x-y+a\sin y=0$, g) $\begin{cases}x=3\cos t\\ y=4t^2\end{cases}$

9. a) $y=arctg\frac{\ln x}{3}$, b) $y=\frac{4\sin x}{\cos^2 x}$ v) $x-y+a\cos y=0$, g) $\begin{cases}x=3\cos^2 t\\ y=2\sin^3 t\end{cases}$

10. a) $y=e^x arctge^x$, b) $y=\sin x-\frac{x}{\cos x}$ v) $x-y+e^y arctgx=0$, g) $\begin{cases}x=e^{2t}\\ y=\cos t\end{cases}$

11. a) $y=\frac{1}{2}tg^2\sqrt{x}$, b) $y=\frac{\ln x}{\sqrt{x^2+1}}$ v) $\ln y=arctg\frac{x}{y}$, g) $\begin{cases}x=t-\sin t\\ y=t-\cos t\end{cases}$

12. a) $y=\ln\cos\sqrt{x}$, b) $y=\frac{11}{2(x-2)^2}$ v) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, g) $\begin{cases}x=t^3+8t\\ y=t^5+2t\end{cases}$

13. a) $y=\ln\cos\sqrt{x}$, b) $y=(\frac{x}{x+1})^2$ v) $x^3+y^3+2xy=a^3$, g) $\begin{cases}x=\cos\frac{t}{2}\\ y=t-\sin t\end{cases}$

14. 1) $y=\sqrt[3]{4x^2-3x-4}-\frac{2}{(x-3)^5}$; 2) $y=\sqrt{\arccos 2x}\cdot 3^{-x}$;
3) $y=(\cos 5x)^{\sqrt{x}}$; 4) $y-4x=e^y$; .

15. 1) $y=\frac{7}{(x-1)^3}+\sqrt{8x-3+x^2}$; 2) $y=arctg5x\cdot ln(x^2+x-1)$;
3) $y=\left(\sqrt{3x+2}\right)^{3x}$; 4) $\frac{3x}{sin\,y}=5y$;

16 1) $y=\sqrt[5]{3x^2+4x-5}+\frac{4}{(x-4)^6}$; 2) $y=5^{-x^2}arcsin3x^3$;
3) $y=(2x-10)^{sin\sqrt{x}}$; 4) $xy=ctgy$;

17. 1) $y=\frac{3}{(x+2)^4}-\sqrt[7]{5x-7x^2-3}$; 2) $y=log_3(2x+5)\cdot\sqrt{arcctg6x}$;
3) $y=(7x+4)^{\sqrt{3x-5}}$; 4) $tgy=3x+6y$;

18 1) $y=\sqrt[4]{(x-1)^5}-\frac{4}{7x^2-3x+2}$; 2) $y=e^{-cos\,x}\cdot arcctg2x^3$;
3) $y=(log_2(5x+7))^{2x}$; 4) $y^2+x^2=sin\,y$;

19 1) $y=\sqrt[5]{(x-2)^6}-\frac{3}{(x^2-4)^2}$; 2) $y=ctg4x\cdot arcsin5x^6$;
3) $y=(lg(8x+3))^{\sqrt{2x}}$; 4) $4sin^2(x+y)=x$;

20 1) $y=\frac{3}{(4x-3x^2+1)^2}-\sqrt{(x+1)^5}$; 2) $y=2^{sin\,x}\cdot\sqrt[3]{arccos^2 3x}$;
3) $y=\left(lg\sqrt{2x+3}\right)^{(3x+8)}$; 4) $y=7x-ctgy$;

21 1) $y=\sqrt[3]{(x-8)^4}-\frac{2}{(1+3x-4x^2)^2}$; 2) $y=\sqrt{5x+1}\cdot arctg^3 4x$;
3) $y=(sin6x)^{\sqrt{10x+3}}$; 4) $x-6=cos\,y$;

22 $1)\ y=\dfrac{4}{(3x-7)^5}+\sqrt[3]{(3x^2-x+1)^4};$ $\quad 2)\ y=\sqrt{tg\,2x}\cdot lg(6x+1);$

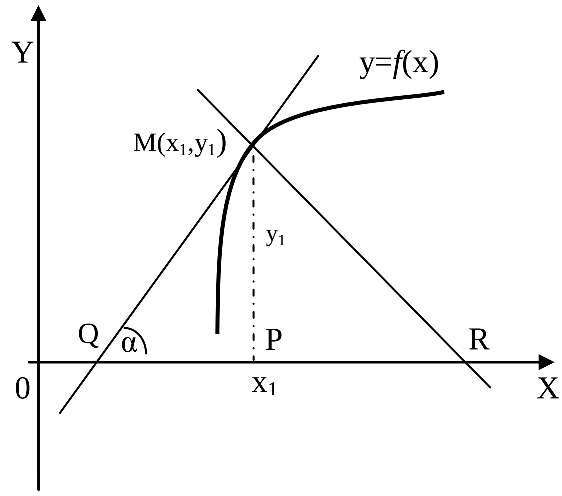
$3)\ y=(cos(4x+5))^{3x};$ $\quad 4)\ x^2y^2+x=5y;$

23 $1)\ y=\sqrt[7]{(2x+1)^5}+\dfrac{5}{(2x^2-4x+3)^6};$ $\quad 2)\ y=log_5(7x-2)\cdot arcsin\,x^3;$

$3)\ y=(lg\,7x)^{3^{\sqrt{x+2}}};$ $\quad 4)\ 3y-7=xy^3;$.

7.3. Egri chiziqning urinma va normal tenglamalari

1.Agar $y=f(x)$ funksiya x_1 nuqtada differensiallanuvchi bo‘lsa, bu nuqtaning qandaydir atrofida aniqlangan ekanligidan shu atrofdagi uning grafigini quramiz.



Funksiya grafigining $M(x_1;y_1)$ nuqtasidagi urinmaning burchak koeffitsienti funksiya hosilasining $x=x_0$ nuqtadagi qiymatiga teng bo‘ladi.

$$k_t=f'(x_1)$$

Bu hosilaning geometrik ma’nosidir. Bundan foydalanib, funksiya grafigining $M(x_1;y_1)$ nuqtasiga o‘tkazilgan urinma tenglamasini quyidagich yozamiz:

$$y-f(x_1)=f'(x_1)(x-x_1).$$

Xuddi shuningdek, qaralayotgan nuqtadagi normal (urinmaga erendikulyar to‘g‘ri chiziq) tenglamasi erendirulyarlik sharti

$$k_n=-\frac{1}{k_t}=-\frac{1}{f'(ő_1)}$$

ga asosan

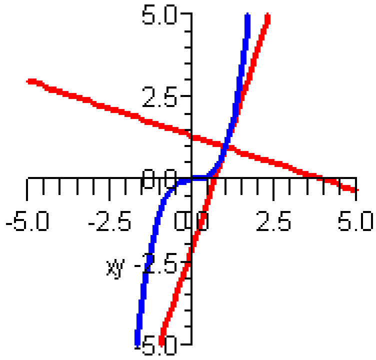
$$ó-f(x_0)=-\frac{1}{f'(ő_1)}(ő-ő_1)$$

bo‘ladi(bu yerda $f'(x_0)\neq 0$ deb faraz qilindi).

7.9-misol. $y=x^3$ funktsiya grafigining $x=1$ ga mos keluvch nuqtasidagi urinma va normalini toping.

Urinma, normal tenglamasi

```
> with(plots):
> f:=x->x^3:a:=1;   a := 1
> D(f);D(f)(a);   x/  3 x^2
                  3
> T:=x->f(a)+(x-a)*D(f)(a);   T := x/  f(a) C (x K a) (D(f))(a)
> N:=x->f(a)-(x-a)/D(f)(a);   N := x/  f(a) K  x K a / (D(f))(a)
> plot({f(x),N(x),T(x)},x=-5..5,y=-5..5,discont=true,
scaling=constrained, color=[red,red,blue]);
```



7.10-misol. $y=\frac{3}{2}\sqrt{4-x^2}$ funktsiya grafigining $x=\frac{1}{\sqrt{2}}$ ga mos keluvch nuqtasidagi urinma va normalini toping.

> **with(plots):**

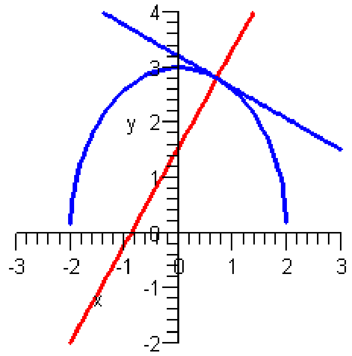
> **f:=x->3*sqrt(2^2-x^2)/2:a:=1/sqrt(2);** $a:=\frac{1}{2}\sqrt{2}$

> **D(f);D(f)(a);** $x \to -\frac{3}{2}\frac{x}{\sqrt{4-x^2}}$ $-\frac{3}{28}\sqrt{2}\sqrt{14}$

> **T:=x->f(a)+(x-a)*D(f)(a);** $T:=x \to f(a)+(x-a)\,(D(f))(a)$

> **N:=x->f(a)-(x-a)/D(f)(a);** $N:=x \to f(a)-\frac{x-a}{(D(f))(a)}$

> **plot({f(x),N(x),T(x)},x=-3..3,y=-2..4, color=[red,blue,blue],discont=true, scaling=constrained);**



2. Urinmaning egri chiziqdagi urinish nuqtasi bilan Ox o‘q orasidagi QM kesmaning T uzunligi *urinmaning uzunligi* deyiladi. Bu kesmaning Ox o‘qdagi proektsiyasi Q kesma urinma osti deyiladi va S_T kabi belgilanadi. MR normal uzunligini N , R proektsiyasini S_N bilan belgilaymiz.

y=f(x) funksiya egri chizig’ining *M(x₁;y₁)* dagi T , S_T, N, S_N miqdorlarni topamiz.

Shakildan: Q=|y₁ctgα|= $\left|\frac{y_1}{tg\alpha}\right|=\left|\frac{y_1}{y_1'}\right|$, shuning uchun

$$S_T=\left|\frac{y_1}{y_1'}\right|,\quad T=\sqrt{y_1^2+\frac{y_1^2}{y_1'^2}}=\left|\frac{y_1}{y_1'}\sqrt{y_1'^2+1}\right|$$

Yana shu shakildan R= $|y_1 tg\alpha|=|y_1 y_1'|$ bundan

$$S_N=\left|y_1 y_1'\right|, \quad N= N=\sqrt{y_1^2+(y_1 y_1')^2}=\left|y_1\sqrt{y_1'^2+1}\right|$$

Bu formulalar $y_1>0$, $y_1'>0$ degan farazda chiqarildi. Ammo umumiy holda ham saqlanadi.

7.11-misol. ***x=acost, y=bsint*** ellips uchun $t=\pi/4$ bo'lgan $M(x_1;\ y_1)$ nuqtada urinma va normal tenglamalari, urinma va urinma osti uzunliklari, normal va normal osti uzunliklari topilsin.

Echish. Quyidagilarni topamiz.

$$\frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=b\cos t\ ,\quad \frac{dy}{dx}=-\frac{b}{a}ctgt,\quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}}=-\frac{b}{a}$$

Urinish nuqtasi M ning koordinatalarini topamiz.

$$x_1=(x)_{t=\pi/4}=\frac{a}{\sqrt{2}},\quad y_1=(y)_{t=\pi/4}=\frac{b}{\sqrt{2}}$$

Urinma tenglamasi: $y-\frac{b}{\sqrt{2}}=-\frac{b}{a}(x-\frac{a}{\sqrt{2}}),\quad bx+ay-ab\sqrt{2}=0$

Normal tenglamasi: $y-\frac{b}{\sqrt{2}}=\frac{a}{b}(x-\frac{a}{\sqrt{2}}),\quad (ax+by)\sqrt{2}-a^2+b^2=0$

Urinma osti va normal osti uzunliklari:

$$S_T=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\right|=\frac{a}{\sqrt{2}},\quad S_N=\left|\frac{b}{\sqrt{2}}(-\frac{b}{a})\right|=\frac{b^2}{a\sqrt{2}}$$

Urinma va normal uzunliklari

$$T=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\sqrt{\left(-\frac{b}{a}\right)^2+1}\right|=\frac{1}{\sqrt{2}}\sqrt{a^2+b^2}\ ,\quad N=\left|\frac{b}{\sqrt{2}}\sqrt{\left(-\frac{b}{a}\right)^2+1}\right|=\frac{b}{a\sqrt{2}}\sqrt{a^2+b^2}$$

Yuqoridagi masalani Male7 dasturida bajarilishini ko'rsatamiz. Buning uchun ellips tenglamasini oshkor holdagi $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$ ko'rinishidan $y=+\frac{b}{a}\sqrt{a^2-x^2}$ (a>0,b>0) ni va tanlaymiz chunki, urinma uning yuqori qismidagi M nuqtaga o'tkaziladi deb hisoblaymiz. Quyidagicha masalani echamiz:

1) ellipsning berilgan nuqtasidagi urinma va normal tenglamalari, urinma va urinma osti uzunliklari, normal va normal osti uzunliklari topilsin.

2) topilganlar asosida ellips, uning urinma va normalning grafigini quring.

3) katta va kichik o'qlari a=3, b=2 bo'lgan ellipsning abtsissalari x=-2, x=2 bo'lgan, nuqtalariga o'tkazilgan urinmalar orasidagi burchakni toping.

1) **> restart;**

> f:=x->b*sqrt(a^2-x^2)/a; $f:=x\to\frac{b\sqrt{a^2-x^2}}{a}$

> x1:=a/sqrt(2);y1:=b/sqrt(2); $x1:=\frac{1}{2}a\sqrt{2}\qquad y1:=\frac{1}{2}b\sqrt{2}$

> diff(f(x), x);
$$-\frac{b\,x}{\sqrt{a^2-x^2}\;a}$$

> kt :=eval(diff(f(x), x), x=a/sqrt(2));df1:=kt;
$$kt := -\frac{b}{\sqrt{a^2}} \qquad df1 := -\frac{b}{\sqrt{a^2}}$$

> kn:=-1/kt;
$$kn := \frac{\sqrt{a^2}}{b}$$

> ur:=y-y1=kt*(x-x1);nr:=y-y1=kn*(x-x1)
$$ur := y-\frac{1}{2}\,b\,\sqrt{2} = -\frac{b\left(x-\frac{1}{2}\,a\,\sqrt{2}\right)}{\sqrt{a^2}} \qquad nr := y-\frac{1}{2}\,b\,\sqrt{2} = \frac{\sqrt{a^2}\left(x-\frac{1}{2}\,a\,\sqrt{2}\right)}{b}$$

> St:=abs(y1/df1);
$$St := \frac{1}{2}\,\sqrt{2}\,|a|$$

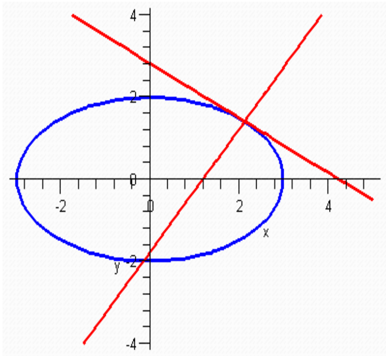
> T:=abs(y1*sqrt(1+df1^2)/df1);
$$T := \frac{1}{2}\,\sqrt{2}\;\;a\,\sqrt{1+\frac{b^2}{a^2}}$$

> Sn:=abs(y1*df1);
$$Sn := \frac{1}{2}\,\sqrt{2}\;\;\frac{b^2}{a}$$

> N:=abs(y1*sqrt(1+df1^2));
$$N := \frac{1}{2}\,\sqrt{2}\;\;\frac{b\sqrt{a^2}\,\sqrt{1+\frac{b^2}{a^2}}}{a}$$

2) **> a:=3;b:=2;el:=x^2/a^2+y^2/b^2=1;**
$$a := 3 \qquad b := 2 \qquad el := \frac{1}{9}\,x^2+\frac{1}{4}\,y^2 = 1$$

> with(plots):
imlicitplot([x^2/a^2+y^2/b^2=1,y-1/2*b*sqrt(2) = -b*(x-1/2*a*sqrt(2))/sqrt(a^2),
y-1/2*b*sqrt(2) = sqrt(a^2)*(x-1/2*a*sqrt(2))/b], x=-5..5,y=-4..4,
color=[blue,red,red],thickness=2);



3)
```
> restart;
```

```
> f:=x->b*sqrt(a^2-x^2)/a;
```
$$f := x l \frac{b\sqrt{a^2 K\ x^2}}{a}$$

```
> a:=3;b:=2;
```
$a := 3$ $b := 2$

```
> x1:=-2; x2:=2; y1:=f(x1);y2:=f(x2);
```
$$x1 := K\ 2 \quad x2 := 2 \quad y1 := \frac{2}{3}\sqrt{5} \quad y2 := \frac{2}{3}\sqrt{5}$$

```
> diff(f(x), x);
```
$$K\ \frac{2}{3}\frac{x}{\sqrt{9 K\ x^2}}$$

```
> k1t :=eval(diff(f(x), x), x=x1);
```
$$k1t := \frac{4}{15}\sqrt{5}$$

```
> k2t:=eval(diff(f(x), x), x=x2);
```
$$k2t := K\ \frac{4}{15}\sqrt{5}$$

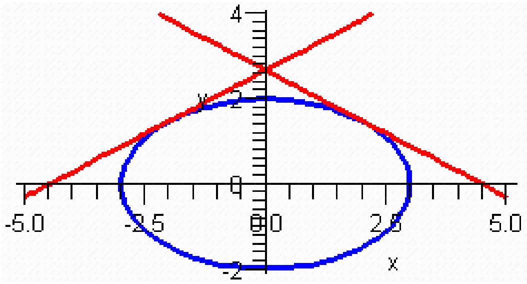
```
> ur1:=y-y1=k1t*(x-x1);ur2:=y-y2=k2t*(x-x2);
```
$$ur1 := y K\ \frac{2}{3}\sqrt{5} = \frac{4}{15}\sqrt{5}\ (x C\ 2) \quad ur2 := y K\ \frac{2}{3}\sqrt{5} = K\ \frac{4}{15}\sqrt{5}\ (x K\ 2)$$

```
> tg(phi):=(k2t-k1t)/(1-k1t*k2t);
```
$$tg(f\ ) := K\ \frac{24}{61}\sqrt{5}$$

```
> phi:=arctan((k2t-k1t)/(1-k1t*k2t));
```
$$f\ := K\ \arctan 0 \frac{24}{61}\sqrt{5}\ 1$$

```
> evalf(phi);
```
- 0.7215220863

```
> with(plots):
implicitplot([x^2/3^2+y^2/2^2=1,
y-2/3*sqrt(5)= 4/15*sqrt(5)*(x+2),
y-2/3*sqrt(5)=-4/15*sqrt(5)*(x-2)], x=-5..5,
y=-4..4,color=[blue,red,red],thickness=2);
Warning, the name changecoords has been redefined
```



3. *y=f₁(x)* va *y= f₂(x)* funktsiyalarning kesishish burchagini ularning kesishish nuqtasi $M(x_0;y_0)$ dagi urinmalari orasidagi burchak bilan aniqlaymiz.

$$tgw=\frac{k_2-k_1}{1+k_1\cdot k_2}=\frac{f_2'(x_0)-f_1'(x_0)}{1+f_1'(x_0)\cdot f_2'(x_0)}$$

7.12-misol. **$y_1= f_1(x)=x^2+y^2=1$ va $y_2= f_2(x)=x^2$ funktsiyalarning kesishish burchagini toping.**

1) funktsiyalarning kesishish nuqtalarini topamiz. Egri chiziq tenglamalarini sistema qilib echamiz:

$$\begin{cases} x^2+y^2=6 \\ y=x^2 \end{cases}$$ bundan, $x^4+x^2-6=0$, $x=\sqrt{2}$, $x=-\sqrt{2}$

2) bu nuqtalardagi funktsiyalarning hosilalarini topamiz.

$$y=\sqrt{6-x^2},\ y'=f_1'(x)=\frac{-x}{\sqrt{6-x^2}}\Big|_{x=\sqrt{2}}=-\frac{\sqrt{2}}{2}=k_1$$

$$y=x^2\quad y'=f_2'(x)=2x\big|_{x=\sqrt{2}}=2\sqrt{2}=k_2$$

3) $$tgw=\frac{k_2-k_1}{1+k_1\cdot k_2}=\frac{f_2'(x_0)-f_1'(x_0)}{1+f_1'(x_0)\cdot f_2'(x_0)}=\frac{2\sqrt{2}+\frac{\sqrt{2}}{2}}{1-2\sqrt{2}\cdot\frac{\sqrt{2}}{2}}=-\frac{5\sqrt{2}}{2}=-3.5355$$

```
> restart;
> with(plots):
> f1:=x->sqrt(6-x^2):f2:=x->x^2:a:=sqrt(2);
```

$a:=\sqrt{2}$

```
> f1(a);f2(a);
```

2 2

```
> D(f1); d1:=D(f1)(a); D(f2); d2:=D(f2)(a);
```

$x \to -\frac{x}{\sqrt{6-x^2}}\quad d1:=-\frac{1}{2}\sqrt{2}\qquad x\to 2x\quad d2:=2\sqrt{2}$

```
> T1:=x->f1(a)+(x-a)*D(f1)(a); yt1:=f1(a)+(x-a)*D(f1)(a);
```

$T1:=x\to f1(a)+(x-a)\,(D(f1))(a)\quad yt1:=2-\frac{1}{2}(x-\sqrt{2})\sqrt{2}$

```
> T2:=x->f2(a)+(x-a)*D(f2)(a); yt2:=f2(a)+(x-a)*D(f2)(a);
```

$T2:=x\to f2(a)+(x-a)\,(D(f2))(a)\quad yt2:=2+2(x-\sqrt{2})\sqrt{2}$

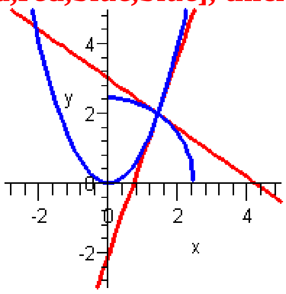
```
> N1:=x->f1(a)-(x-a)/D(f1)(a);yn1:=f1(a)+(x-a)/D(f1)(a);
```

$$N1 := x \to f1(a) - \frac{x - a}{(\mathrm{D}(f1))(a)} \qquad yn1 := 2 - (x - \sqrt{2})\sqrt{2}$$

> N2:=x->f2(a)-(x-a)/D(f2)(a); yn2:=f2(a)+(x-a)/D(f2)(a);

$$N2 := x \to f2(a) - \frac{x - a}{(\mathrm{D}(f2))(a)} \qquad yn2 := 2 + \frac{1}{4}(x - \sqrt{2})\sqrt{2}$$

> plot({f1(x),T1(x),f2(x),T2(x)},x=-3..5,y=-3..5, discont=true, scaling=constrained,color=[red,red,blue,blue], thickness=2);



7.4. Teylor formulasi

Berilgan *f(x)* funktsiyani x=*a* nuqta atrofida ko‘hadga yoyishning Teylor formulasi :

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$R_n(x)$ yoyilmaning *qoldiq hadi.*

Agar Teylor formulasini a=0 nuqta atrofida yozilsa,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

ga ega bo‘lamiz. Buni *Makloren formulasi* deb ham yuritiladi.

4.13-misol. $f(x)=e^x$ ni Makloren formulasiga yoying va uning yordamida $\sqrt{e}$ ni 0,001 aniqlikda hisoblang.

Yechish. Ma’lumki, $f^{(k)}(x)=(e^x)^{(k)}=e^x$, $k \in n$.

$$f(0) = e^0 = 1, \quad f^{(k)}(0) = e^0 = 1.$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}$$

Demak, $$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

bu yerda $c = \theta x$, $0 < \theta < 1$ dir.

> taylor(exp(x), x=0, 7);

$$1\,C\; x\,C\;\frac{1}{2}x^2\,C\;\frac{1}{6}x^3\,C\;\frac{1}{24}x^4\,C\;\frac{1}{120}x^5\,C\;\frac{1}{720}x^6\,C\;O(x^7)$$

Endi, $\sqrt{e}=e^{0,5}$ ni 0,001 aniqlikda taqribiy hisoblaylik. Buning uchun e^x ning yuqoridagi yoyilmasida qoldiq hadni tashlab yuborsak,

$$e^x \approx 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}$$

ni olamiz. Shart bo‘yicha $|Rn(x)| < 0,001$ bo‘lishi kerak.

$$\left|R\left(\frac{1}{2}\right)\right|=\frac{e^c}{(n+1)!}\cdot\left(\frac{1}{2}\right)^{n+1}=\frac{e^c}{2^{n+1}(n+1)!}<0,001, \qquad 0<c<\frac{1}{2}.$$

Ma’lumki, 2<e<3. Buni hisobga olsak,

$$\left|R_n\left(\frac{1}{2}\right)\right|<\frac{3}{2^{n+1}(n+1)!}<0,001.$$

Bu tengsizlik n≥4 lar uchun o‘rinlidir. Demak, n=4 deb olish yetarli ekan.

$$\sqrt{l}\approx 1+0,5+\frac{0,5^2}{2!}+\frac{0,5^3}{3!}+\frac{0,5^4}{4!}=1+0,5+0,125+0,02083+0,00260=1,64843=1,648.$$

$$\sqrt{l}\approx 1,648.$$

> taylor(exp(x),x=0,4); $1\,C\; x\,C\;\frac{1}{2}x^2\,C\;\frac{1}{6}x^3\,C\;O(x^4)$

> taylor(exp(0.5), x=0, 4); 1.648721271

Yoyilmadagi yaqinlashmalar:

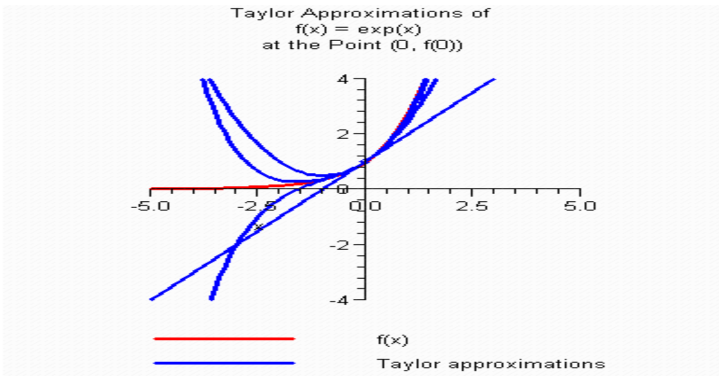
> TaylorApproximation(exp(x) , x = 0, order = 1..4);

$$1\,C\; x,\; 1\,C\; x\,C\;\frac{1}{2}x^2,\; 1\,C\; x\,C\;\frac{1}{2}x^2\,C\;\frac{1}{6}x^3$$

x=0 nuqtadagi yoyilma bo‘yicha yaqinlashmalarni ko‘rsatuvchi grafik:

> with(Student[Calculus1]):

> TaylorApproximation(ex(x), 0, outut = plot, order=1..4, view = [-5..5, -4..4],thickness=2);



7.14-misol. *f(x)=sinx* ni Makloren formulasiga yoying.

Yechish. $f^{(k)}(x)=(\sin x)^{(k)}=\sin\left(x+k\cdot\frac{\pi}{2}\right);\quad f(0)=\sin 0=0,$

$$f^{(k)}(0)=\sin\frac{k\pi}{2}=\begin{cases}0, & k=2m,\\ (-1)^{m+1}, & k=2m-1,\end{cases}\qquad m\in N$$

$$\sin x = ő-\frac{ő^3}{3!}+\frac{ő^5}{5!}-\frac{ő^7}{7!}+\ldots+(-1)^{m+1}\frac{ő^{2m-1}}{(2m-1)!}+R_{2m+1}(ő),$$

bu yerda qoldiq hadning Lagranj ko‘rinishi

$$R_{2m+1}(ő)=(-1)^m\frac{\cos c}{(2m+1)!}ő^{2m+1}$$

bo‘lib, $c=\theta x$, $0<\theta<1$ dir.

> taylor(sin(x), x=0 ,8);

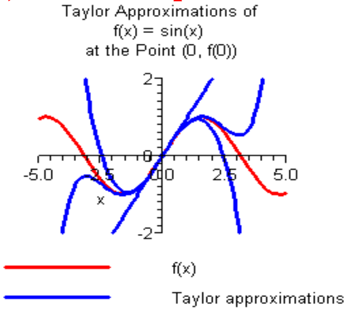
$$x\ \mathrm{K}\ \frac{1}{6}x^3\ \mathrm{C}\ \frac{1}{120}x^5\ \mathrm{K}\ \frac{1}{5040}x^7\ \mathrm{C}\ \mathrm{O}(x^8)$$

> TaylorApproximation(sin(i/9.),x=0,order=5); 0.3420201433

x=0 nuqtadagi yoyilma bo‘yicha yaqinlashmalarni ko‘rsatuvchi grafik:

> with(Student[Calculus1]):

> TaylorApproximation(sin(x), 0, outut = plot, order=1..5, view = [-5..5, -2..2]);



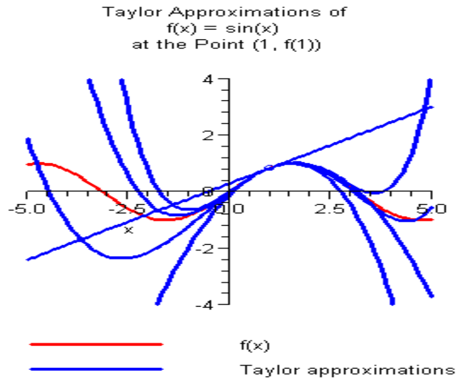
x=1 nuqtadagi yoyilma bo'yicha yaqinlashmalar:

```
> with(Student[Calculus1]):
> TaylorApproximation(sin(x) , x = 1., order = 1..5);
```

$0.3011686789\text{C }0.5403023059x,\ 0.3011686789$
$\text{C }0.5403023059x\text{ K }0.4207354924\,(x\text{ K }1.)^2,\ 0.3011686789$
$\text{C }0.5403023059x\text{ K }0.4207354924\,(x\text{ K }1.)^2$
$\text{K }0.09005038432\,(x\text{ K }1.)^3,\ 0.3011686789$
$\text{C }0.5403023059x\text{ K }0.4207354924\,(x\text{ K }1.)^2$
$\text{K }0.09005038432\,(x\text{ K }1.)^3$
$\text{C }0.03506129103\,(x\text{ K }1.)^4,\ 0.3011686789$
$\text{C }0.5403023059x\text{ K }0.4207354924\,(x\text{ K }1.)^2$
$\text{K }0.09005038432\,(x\text{ K }1.)^3\text{C }0.03506129103\,(x\text{ K }1.)^4$
$\text{C }0.004502519216\,(x\text{ K }1.)^5$

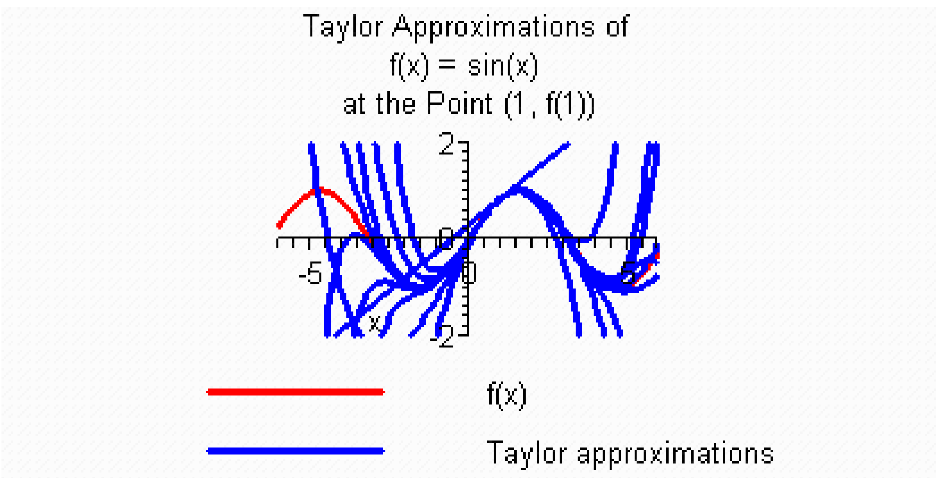
x=1 nuqtadagi yoyilma bo'yicha yaqinlashmalarni ko'rsatuvchi grafik:

```
> with(Student[Calculus1]):
> TaylorApproximation(sin(x), 1, outut = plot, order=1..5,   view = [-6..6, -
4..4],thickness=2);
```



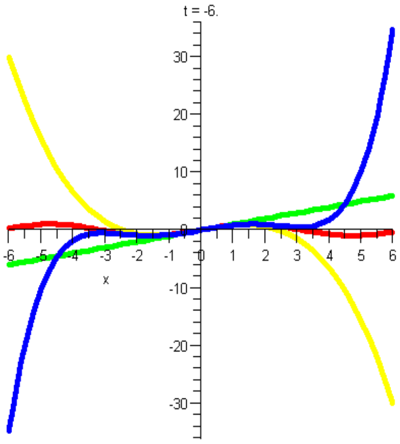
Yoyilma funktsiyalar bo'yicha yaqinlashmalarni animatsiya(ketma-ketnamoyish qilish) bilan ko'rsatuvchi grafik:

1)
```
> with(Student[Calculus1]):
> TaylorApproximation(sin(x),1, order=1..10, view = [-6..6, -2..2],
outut =animation,thickness=2);
```



2)
```
> with(plots):with(Student[Calculus1]):
```

> animate(plot, [[sin(x),x,x-1/6*x^3,x-1/6*x^3+1/120*x^5], x=-t..t], t=-6..6, frames=50,thickness=3);



7.15-misol. *f(x)=cosx* ni Makloren formulasiga yoying.

Mustaqil yeching. $cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m+2}(x),$

bu yerda $$R_{2m+2}(x) = (-1)^{m+1} \frac{\cos c}{(2m+2)!} x^{2m+2}$$

bo‘lib, $c = \theta x,\ 0 < \theta < 1$ dir.

> taylor(cos(x), x=0 ,8);

$$1 \mathrm{K} \frac{1}{2} x^2 \mathrm{C} \frac{1}{24} x^4 \mathrm{K} \frac{1}{720} x^6 \mathrm{C} \mathrm{O}(x^8)$$

Yoyilmadagi yaqinlashmalar:

> with(Student[Calculus1]):

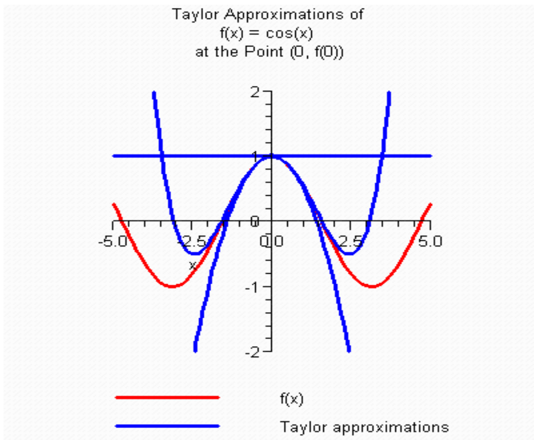
> TaylorApproximation(cos(x) , x = 0, order = 1..5);

$$1,\ 1 \mathrm{K} \frac{1}{2} x^2,\ 1 \mathrm{K} \frac{1}{2} x^2,\ 1 \mathrm{K} \frac{1}{2} x^2 \mathrm{C} \frac{1}{24} x^4,\ 1 \mathrm{K} \frac{1}{2} x^2 \mathrm{C} \frac{1}{24} x^4$$

x=0 nuqtadagi yoyilma bo‘yicha yaqinlashmalarni ko‘rsatuvchi grafik:

> with(Student[Calculus1]):

> TaylorApproximation(cos(x),0,outut=plot,order=1..5, view = [-5..5, -2..2], thickness=2);



Funktsiyalarning yoyilma funktsiyalari bo'yicha yaqinlashmalar grafigini animatsion jarayonni muloqat oynasida ko'rsatish:

```
> with(Student[Calculus1]):
TaylorApproximationTutor();
TaylorApproximationTutor(sin(x), 0);
TaylorApproximationTutor(cos(x), 0);
TaylorApproximationTutor(ex(x), 0);
```

8 – MUSTAQIL ISHLASH UCHUN TOPSHIRIQLAR

1.Quyidagi tenglamalar bilan berilgan egri chiziqlarning ko'rsatilgan nuqtasidagi urinma, urinma osti, normali va normal osti uzunliklarini hisoblang	2. Quyidagi tenglamalar bilan berilgan chiziqlarning kesishish burchagini toping.	3. Quyidagi funktsiyalarni ko'rsatilgan nuqtadagi Teylor formulasi bo'yicha yoyilmasini toping
1. y=tg2x, O(0;0) 2. $y=\arcsin\frac{x-1}{2}$, y=0 3. y=arccos3x, x=0 4, y=lnx, y=0 5. $x=\frac{1+t}{t^2}$, $y=\frac{3}{2t^2}+\frac{1}{2t}$, (2;2) 6. x=tcost, y=tsint, t=π/4 7. $x^3+y^2+2x-6=0$, y=3 8, $x^5+y^5-2xy=0$, (1;1) 9. y=(x-1)(x-2)(x-3), y=0 10. $y^4= 4x^4+8xy$, (1;2)	1. $y=(x-2)^2$, $y=-4+6x-x^2$ 2. $y=3x^3-4x+2$, y=2-x 3. $x^2+y^2=2$, x+2y=2 4. $x^2+y^2=2$, $y=x^2$ 5. $x^2+y^2=2$, $x^2-y^2=1$ 6. $x^2/4+y^2/9=1$, $y^2=4-x$ 7, $x^2/9+y^2/4=1$, $y=x^2$ 8. $x^2/9+y^2/4=1$, $y=x^2$ 9. $x^2+y^2=2$, $(x-1)^2+y^2=1$ 10. $y=x^2$, $y=-x^2+1$	1. y=lnx, x=1 2. y=1/x, x=1 3. $y=1/(x^2)$, x=-1 4. $y=1/(x^2+3x+2)$, x=-4 5. $y=1/(x^2+4x+7)$, x=-2 6. $y=e^x$, x=-2 7. $y=\sqrt{x}$, x=4 8. $y=\frac{1}{\sqrt{x}}$, x=1 9. $y=\frac{x}{\sqrt{x+1}}$, x=0 10. $y=\frac{2}{\sqrt{x+1}}$, x=0

.7. 5. Funktsiyani tekshirish va grafigini qurish

Funktsiyani tekshirish va grafigini qurishni umumiy sxemasi.

1.Funktsiyaning aniqlanish sohasini topish.

Buning uchun funktsiyaning asoslaridan va ko'rinishiga qarab aniqlanish sohasini topamiz:

M: $y=\sqrt{1-x^2}$, funktsiya xossasiga asosan $1-x^2\geq 0$, $(1-x)(1+x)\geq 0$ bundan $-1\leq x\leq 1$, da aniqlangan.

M: $y=\frac{3x}{x-1}$, funktsiya kfsir funktsiya bo'lgani uchun, $x-1\neq 0$ bo'ladi, eki $x\neq 1$ aniqlanish sohasi $(-\infty, 1)\cap(1, +\infty)$

2.Funktsiyani grafigini koordinata o'qlari bilan kesishish nuqtalarini topish uchun.

1) x = 0 da y = f(0) tenglamani echib 0Y o'qi bilan kesishish nuqtalarini topamiz.
2) y=0 da 0 = f(x) tenglamani echib 0X o'qi bilan kesishish nuqtalarini topamiz.

3.**Funktsiyaning davriyligi, juft va tokligi.**

Buning uchun quyidagi shartlarni tekshiramiz:

1) $f(x+T)=f(x)$, T-davrni funktsiya

2) f(-x)=f(x), juft funktsiya
3) f(-x)=-f(x), tok funktsiya

4. Funktsiya uzliksizligi, uzilish nuqtalarini topish va turini aniqlash.

Buning uchun funktsiya aniqlanmagan x_0 nuqta va uning atrofida uzuliksizligi shartlarini tekshiramiz:

$$f(x_0-0)=\lim_{x\to x_0-0} f(x) \text{ chap tomonli limit}$$
$$f(x_0+0)=\lim_{x\to x_0+0} f(x) \text{ o‘ng tomonli limit}$$

Agar: 1) $f(x_0)=\pm\infty$, $f(x_0+0)=\pm\infty$ bo‘lsa, funktsiya ikkinchi tur uzulishga ega.
2) $f(x_0-0)=A\neq f(x_0+0)=V$ bo‘lsa, funktsiya birinchi tur chekli uzulishga ega.

5. Funktsiya ekstremullarini aniqlash va monogonlik oraliklarini topish.

Buning uchun:
1) $f'(x)$ – hosilani topamiz.
2) $f'(x)=0$ tenglamani echib, $x=x_1$, $x=x_2$ yuritish nuqtalarni topamiz.
3) $x=x_1$ va $x=x_2$ nuqtalar atrofida $f'(x)$ hosilani orasini tekshiramiz:

agar: $x=x_1$ da $f'(x)$ chapda «-» unga «+»bo‘lsa, $u_{max}=f'(x_1)$
$x=x_2$ da $f'(x)$ chapda «-», unga «+» ishorali bo‘lsa, $u_{max}=f'(x_2)$

Monotonli oraliqini topish uchun x_1 va x_2 kritik nuqtalar atrofidagi hosila ishorasidan va uzulish nuqta atrofidagi $f'(x_0-0)$, $f'(x_0+0)$ limitlardan foydalanamiz.

6. Funktsiya burilish nuqtasi va botik, kavarikli oraliklarni topish.

Buning uchun:
1) $f''(x)=0$ tenglamani echib, burilish nuqtasining abstsissasi x = a nuqtani topamiz va $y_{bn}=f'(a)$ hisoblaymiz.
2) $f''(x)<0$ tengsizlikni echib kabariklik, $f''(x)>0$ tengsizlikni echib, botiklik orolig’ini aniqlaymiz, yoki x = a nuqta atrofida $f''(x)$ ning ishorasini aniqlab topaimz.

7. Funktsiya grafigining asitotasini topamiz.
1) $y=kx+b$ – og’ma asimtota.

Bunda $k=\lim\limits_{x\to\infty}\frac{s(x)}{x}$, $\quad b=\lim\limits_{x\to\infty}[f(x)-k\cdot x]$

2) $\lim\limits_{x\to\infty} f(x)=b$, - gorizontal asimtota.
3) $\lim\limits_{x\to\infty} f(x)=\pm\infty$, bo‘lsa x=a vertikal asimptota.

8.Yuqorida aniqlanganlarni koordinatalar sistemasida tasvirlaymiz.

7.16-misol. $y=\frac{3x^2-1}{x3}=\frac{3}{x}-\frac{1}{x^3}$ **funktsiyani tekshirish va grafigini quring**.

Echish. Yuqoridagi umumiy sxema asosida berilgan funktsiyani tekshiramiz.

$Faqat \; f(x) \; dan \; foydalaniib \; tekshirish:$

```
> restart;
> f:=x->(3*x^2-1)/x^3;
```

$$f := x \mapsto \frac{3x^2-1}{x^3}$$

1. Funktsiyaning aniqligi sohasi:

$$y=\frac{3}{x}-\frac{1}{x^3} \text{ da } x\neq 0 \text{ yoki } (-\infty, 0)\cap(0, +\infty)$$

Funktsiyaning uzilish nuqtasi: `> discont(f(x),x);` $\{0\}$

2. $y=\frac{3}{x}-\frac{1}{x^3}$ da x≠0 bo‘ladi. Bundan funktsiya grafigi OY o‘qi bilan kesishmasligi kelib chiqadi.
y=0 da OX o‘qi bilan kesish nuqtalarini topamiz.

$$\frac{3x^2-1}{x3}=0,\quad 3x^2-1=0,\quad x=\pm\frac{1}{\sqrt{3}},\qquad A\left(-\frac{1}{\sqrt{3}};0\right),\ B\left(\frac{1}{\sqrt{3}};0\right)$$

> solve(f(x),{x}); $\{x=\frac{1}{3}\sqrt{3}\},\{x=-\frac{1}{3}\sqrt{3}\}$

3. $f(x)=\frac{3x^2-1}{x^3}$ óáóí: $f(-x)=\frac{3(-x)^2-1}{(-)^3}=-\frac{3x^2-1}{x^3}=-f(x)$ bundan funktsiyaning toq ekanligi kelib chiqadi.

Funktsiyaning toq ekanligi tekshirish:

```
> type((3*x^2-1)/x^3,oddfunc(x));
```
true

4. Funktsiya *x=0* da mahnoga ega bo‘lmaganligi uchun uning bu nuqta atrofida chap va o‘ng limitlarini tekshiramiz.

$$f(-0)=\lim_{x\to-0}\frac{3x^2-1}{x^3}=\frac{-1}{-0}=+\infty,\qquad f(+0)=\lim_{x\to+0}\frac{3x^2-1}{x^3}=\frac{-1}{+0}=-\infty$$

Bundan *x=0* da funktsiya ikkinchi tur uzilishga ega ekanligi kelib chiqadi .

> limit(f(x), x=0,right); $-\infty$

> limit(f(x), x=0,left); $+\infty$

5. Ekstemumlarini topamiz.

1) $f'(x)=-\frac{3(x^2-1)}{x^4}$

2) $f'(x)=0,\ 3(x^2-1)=0,\ x_1=-1,\ x_2=1$

3) $x_1=-1$ atrofida hosila qiymatining ishorasini tekshiramiz.

$$f'(-2)=-\frac{3\left[(-2)^2-1\right]}{(-2)^4}=-\frac{9}{16},\ "-"\ \Rightarrow\quad f'(-\frac{1}{2})=-\frac{3\left[(-\frac{1}{2})^2-1\right]}{(-\frac{1}{2})^4}=-12,\ "+"$$

demak, $y_{min}=f(-1)=-\frac{3(-1)^2-1}{(-1)^4}=-2$

4) $x_2=1$ nuqta atrofida $f'(x)$ ishorasi:

$$f'(-\frac{1}{2})=12,\ "+"\Rightarrow f'(2)=-\frac{9}{16},\ "-"\ \ \text{äŀėŕę}\ ,\ Y\max=f(1)=-\frac{3\cdot1^2-1}{1^3}=2$$

Funktsiyaning 1- *tartibli hosilasi*

> df1:=diff(f(x),x); $df1:=\frac{6}{x^2}-\frac{3\,(3\,x^2-1)}{x^4}$

Funktsiyaning kritik nuqtasini birinch hosila yordamida aniqlash

> solve(diff(f(x),x)=0,{x}); $\{x=-1\},\ \{x=1\}$

Dastur yordamida aniqlash:

```
> with(Student[Calculus1]):
> CriticalPoints((3*x^2-1)/x^3);
```

[K 1, 0, 1]

Funktsiyaning ekstrimal qiymatlari :

> **yminmax:=extrema(f(x),{},x);** *ymaxmin* := {K 2, 2}

5) x_1=-1, x=0, x_2=1 nuqtalar atrofidagi hosila ishorasiga asosan funktsiyaning o‘sish va kamayish oraliqini topamiz.

Demak funktsiya : (-∞, -1) da kamayuvchi, (-1, 0) da o‘suvchi, (0, 1) da o‘suvchi, (1, +∞) da kamayuvchi.

6. Burilish nuqtasini topamiz. $y'' = \frac{6(x^2-2)}{x^5} = 0, \quad x = \pm\sqrt{2}$

1) $y_1 = f(-\sqrt{2}) = \frac{3(-\sqrt{2})^2 - 1}{(\sqrt{2})^3} = -\frac{5}{2\sqrt{2}}$ $\quad y_2 = f(+\sqrt{2}) = \frac{3(\sqrt{2})^2 - 1}{(\sqrt{2})^3} = -\frac{5}{2\sqrt{2}}$;

2) x= $-\sqrt{2}$ ni atrofida *f*”(x) ni ishorasi: *f*”(-2)=-$\frac{3}{8}$, «-»⇒*f*”(-1)=6, «+» bundan burilish nuqtasida funktsiya grafigi qabariqlikdan botiqlikka o‘tadi.

3) x= $\sqrt{2}$ ni atrofida *f*”(x) ni ishorasi: *f*”(1)=6, «+»⇒*f*”(2)=- $\frac{3}{8}$, «-»

bundan burilish nuqtasida funktsiya grafigi botiqlikdan qabariqlikka o‘tadi.

Demak, funktsiya (-∞, -1) da kamayuvchi, (-1, 0) da o‘suvchi, (0, 1) da o‘suvchi, (1, +∞) da kamayuvchi

Funktsiyaning burilish nuqtasining aniqlash:

> **xbn:=solve(df2,{x});** *xbn* := $\{x = \sqrt{2}\}, \{x = K\sqrt{2}\}$

```
> with(Student[Calculus1]):
InflectionPoints( (3*x^2-1)/x^3 );
```

$[K\sqrt{2}, 0, \sqrt{2}]$

> **x:=sqrt(2); ybn1:=f(x);** $x := \sqrt{2}$ $\quad ybn1 := \frac{5}{4}\sqrt{2}$

> **x:= -sqrt(2);ybn2:=f(x);** $x := K\sqrt{2}$ $\quad ybn2 := K\frac{5}{4}\sqrt{2}$

7. 1) *y*=*kx*+*b* og’ma asimtotasini topamiz:

$$k = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{3x^2-1}{x4} = \lim_{x\to\infty}\frac{3}{x^2} - \frac{1}{x^4} = 0$$

$$b = \lim_{x\to\infty}[f(x) - kx] = \lim_{x\to\infty}\left[\frac{3x^2-1}{x^4} - 0\cdot x\right] = \lim_{x\to\infty}\left(\frac{3}{x} - \frac{1}{x^3}\right) = 0$$

og’ma asimtotasi yo‘q.. y=0 gorizontal asimtota bo‘ladi.

y = kx C b assimptotasini aniqlash:

> **restart**

> k:=limit((3*x^2-1)/x^4, x=infinity); $k := 0$

> b:=limit((3*x^2-1)/x^4-k*x, x=infinity); $b := 0$

y=kx+b=0x+0=0, y=0

Ox o‘qi gorizontal assimptota

2) x=0 da ikkinchi tur uzulish bo‘lganligi uchun, x=0 vertikal asimtota.

3) Gorizontal asimtotasini topamiz:

a) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left(\frac{3}{x}-\frac{1}{x^3}\right) = 3\left(-\frac{1}{\infty}\right)-\left(-\frac{1}{\infty}\right) = -0$

b) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(\frac{3}{x}-\frac{1}{x^3}\right) = 3\cdot\left(\frac{1}{\infty}\right)-\frac{1}{\infty} = +0$

Bundan $x\to-\infty$ da egri chiziq y=0 (OX o‘qi) asimtotaga pastdan yaqinlashadi, $x\to+\infty$ da yuqoridan yaqinlashadi.

Gorizontal asimptota :

> limit(f(x), x=infinity); **0**

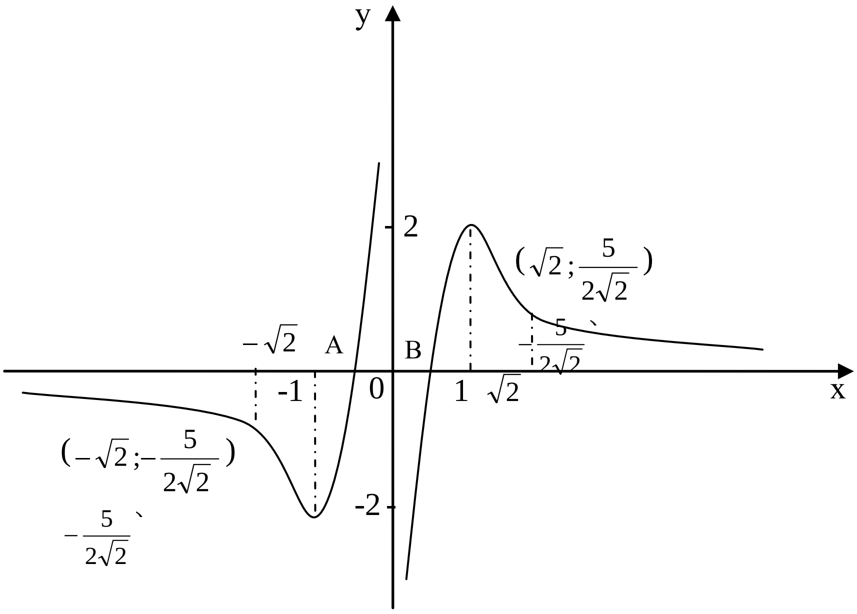
> limit(f(x), x=-infinity); **0**

8. Topilganlarni jadvalini tuzish.

Y \ X	$(-\infty,-\sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2},-1)$	-1	$(-1,-\frac{1}{\sqrt{3}})$	$-\frac{1}{\sqrt{3}}$	$(-\frac{1}{\sqrt{3}},0)$	0
y y′ y″	- - -	$-\frac{5}{2\sqrt{2}}$ $-\frac{3}{4}$ 0	- - +	-2 0 -6	- + +	0 18 $9\sqrt{3}$	+ + +	∞ ∞ ∞
Xolati	Kamayuvchi, kaborik	Bur. nuqtasi	Kamayuvchi, botik	min	O’suvchi, botik	Ox o‘q bilan kes ishishi	O’suvchi, botik	Cheksiz Uzun

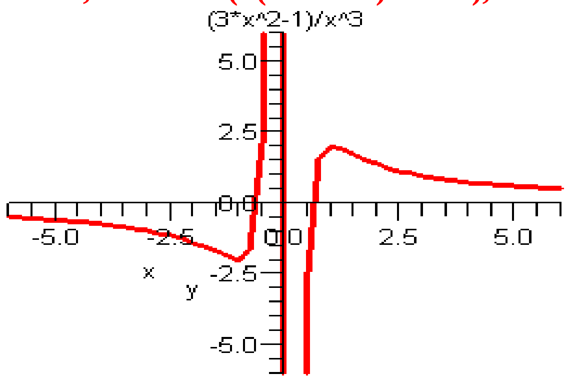
Y \ X	$(0,\frac{1}{\sqrt{3}})$	$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}},1)$	1	$(-1,\sqrt{2})$	$\sqrt{2}$	$(\sqrt{2},+\infty)$
y y′ y″	- + -	0 18 $9\sqrt{3}$	+ + -	2 0 -6	+ - -	$\frac{5}{2\sqrt{2}}$ $-\frac{3}{4}$ 0	+ - +
Xolati	O’suvchi, qaboriq	Ox o‘q bilan kes ishishi	O’suvchi, qaboriq	max	Kamayuvchi, botik	Bur. nuqtasi	Kamayuvchi, Botik

9. funktsiya grafigini koordinatalar sistemasida quramiz.



> restart; with(plots): f:=x->(3*x^2-1)/x^3; $f := x \to \frac{3x^2 - 1}{x^3}$

> plot(f(x), x=-6..6, y=-6..6, title = cat("(3*x^2-1)/x^3"), color=red);



Yuqoridagidek tizim bo'yicha quyidagi funktsiyalarni Mtple7 dasturi yordamida tekshirish va grafigini qurish.

7.17-misol. $y = \frac{x^3}{x^2 - 4}$ **funktsiyani tekshirish va grafigini quring**.

```
> restart;
```

```
> f:=x->x^3/(x^2-4);
```
$f := x \to \frac{x^3}{x^2 - 4}$

Toq funktsiya

```
> type(x^3/(x^2-4),oddfunc(x));
```
true

Judt funktsiya

```
> type(x^3/(x^2-4),evenfunc(x));
```
false

Uzilish nuqtalari:

```
> discont(f(x),x);
```
{K 2, 2}

```
> fdiscont(f(x),x=-5..5,0.01);
```
[K 2.00600319534110527..K 1.99302749523774536,
1.99526100311751132..2.00641059483572715]

Uzilish nuqtasining chap va ong tomondagi holati:

```
> limit(f(x), x=-2,right);
```
∞

```
> limit(f(x), x=-2,left);
```
$-\infty$

Kritik nuqta

```
> with(Student[Calculus1]):
> CriticalPoints(x^3/(x^2-4));
```
[K $2\sqrt{3}$, K 2, 0, 2, $2\sqrt{3}$]

Ekstremum nuqtasi va qiymati

```
> with(Student[Calculus1]):
> ExtremePoints(x^3/(x^2-4),x);
```
[K $2\sqrt{3}$, $2\sqrt{3}$]

```
> ymaxmin:=extrema(x^3/(x^2-4),{},x );
```
ymaxmin := {$3\sqrt{3}$, K $3\sqrt{3}$}

Burilish nuqtasining abtsisasi

```
> with(Student[Calculus1]):
> InflectionPoints( x^3/(x^2-4), x );
```
[K 2, 0, 2]

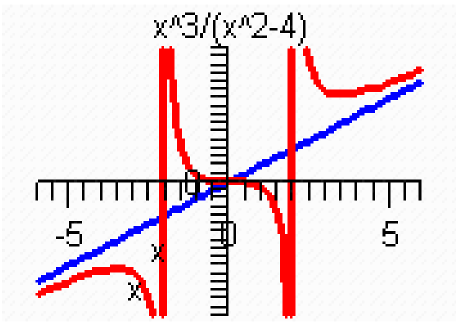
$y = kx$ C b *assimptotasini aniqlash*:

```
> k:=limit(f(x)/x, x=infinity);
```
$k := 1$

```
> b:=limit(f(x)-k*x, x=infinity);
```
$b := 0$

```
> y:=k*x+b;
```
$y := x$

```
> plot({f(x),y}, x=-6..6, y=-8..8,color=[blue,red],
thickness=2, title = cat( "x^3/(x^2-4)" ) );
```

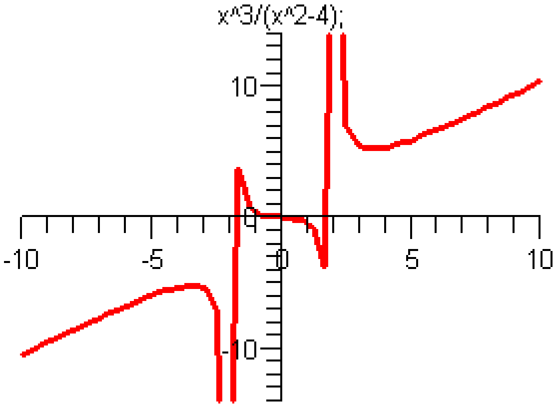


Funktsiya grafigini animatsiya yordamida qurish.

```
> restart;   with(plots): f:=x->x^3/(x^2-4);
```

$$f := x \to \frac{x^3}{x^2 - 4}$$

```
>animatecurve([x,f(x),x=-10..10],view=[-10..10,-14..14],
frames=100 ,thickness=2,title = cat( "x^3/(x^2-4);" ));
```



7.18-misol. $y = \arcsin\left(\frac{2x}{1+x^2}\right)$ **funktsiyani tekshirish va grafigini qurish.**

```
> restart;
> f:=x->arcsin(2*x/(1.+x^2));
```

$$f := x \to \arcsin\left(\frac{2x}{1. + x^2}\right)$$

Uzilish nuqtasi:

```
> discont(f(x),x);
```

$$\{\}$$

```
> fdiscont(f(x),x=-7..7,0.01);
```

$$[-1.00430138557337822..-.996600193794028422, 0.997172432673257392..1.00378407531488301]$$

Maxsus nuqtalar:

```
> singular(f(x));
```

$$\{x = 1.\, I\}, \{x = -1.\, I\}$$

OX oqi bilan kesishish nuqtalari:

```
> solve(f(x),{x});
```

$$\{x = 0.\}$$

Uzilish nuqtasining chap va ong tomondagi holati:

```
> limit(f(x), x=1,right);
```

$$1.570796327$$

```
> limit(f(x), x=1,left);
```
$$1.570796327$$

Gorizontal asimptota :

```
> limit(f(x), x=infinity);
```
$$0$$
```
> limit(f(x), x=-infinity);
```
$$0$$

Funktsiyaning 1 K *rinch tartibli hosilasi* :

```
> df1:=diff(f(x),x);
```

$$df1 := \frac{\frac{2}{1.\,\mathrm{C}\;x^2} \,\mathrm{K}\, \frac{4\,x^2}{(1.\,\mathrm{C}\;x^2)^2}}{\sqrt{1\,\mathrm{K}\,\frac{4\,x^2}{(1.\,\mathrm{C}\;x^2)^2}}}$$

Funktsiyaning kritik nuqtasini aniqlash :

```
> solve(diff(f(x),x)=0,x);
> solve(df1=0,{x});
```

Funktsiyaning ekstrimal qiymatlari :

```
> extrema(f(x),{},x );
```

Funktsiyaning 2 K *rinch tartibli hosilasi* :

```
> df2:=diff(f(x),x$2);
```

$$df2 := \frac{\mathrm{K}\,\frac{12\,x}{(1.\,\mathrm{C}\;x^2)^2} \,\mathrm{C}\, \frac{16\,x^3}{(1.\,\mathrm{C}\;x^2)^3}}{\sqrt{1\,\mathrm{K}\,\frac{4\,x^2}{(1.\,\mathrm{C}\;x^2)^2}}} \,\mathrm{K}\, \frac{1}{2} \frac{\left(\frac{2}{1.\,\mathrm{C}\;x^2} \,\mathrm{K}\, \frac{4\,x^2}{(1.\,\mathrm{C}\;x^2)^2}\right)\left(\mathrm{K}\,\frac{8\,x}{(1.\,\mathrm{C}\;x^2)^2} \,\mathrm{C}\, \frac{16\,x^3}{(1.\,\mathrm{C}\;x^2)^3}\right)}{\left(1\,\mathrm{K}\,\frac{4\,x^2}{(1.\,\mathrm{C}\;x^2)^2}\right)^{(3/2)}}$$

Funktsiyaning burilish nuqtasining aniqlash :

```
> xbn:=solve(df2,{x});
Warning, solutions may have been lost
```

$$xbn :=$$

```
> ybn1:=(f(x),{xbn});
```

$$ybn1 := \arcsin\left(\frac{2\,x}{1.\,\mathrm{C}\;x^2}\right), \{\,\}$$

```
> x:=0 ;ybn2:=f(x);
```
$x := 0 \quad ybn2 := 0.$

```
> restart; f:=x->arcsin(2*x/(1.+x^2));
```

$$f := x \to \arcsin\left(\frac{2\,x}{1. + x^2}\right)$$

Funktsiyaning maxsus nuqtasidagi ekstremum qiymati :

```
> minima,points:= minimize(f(x), x=-4..16,'location');
```

$minima, points :=$
$-1.570796327, \{[\{x = -1.\}, -1.570796327]\}$

```
> maxima,points:= maximize(f(x), x=-4..16,'location');
```

$maxima, points := 1.570796327, \{[\{x = 1.\}, 1.570796327]\}$

$y = kx + b$ *assimptotasini aniqlash* :

```
> k:=limit(f(x), x=infinity);
```
$k := 0.$

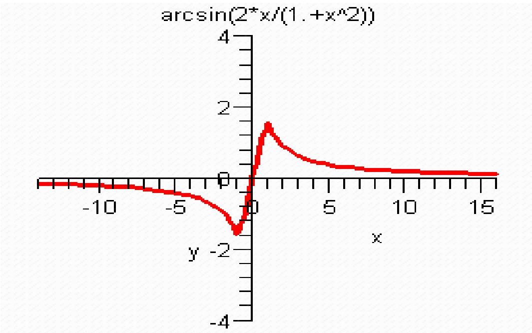
```
> b:=limit(f(x)-k*x, x=infinity);
```
$b := 0.$

funktsiya grafigini qurish:

```
> restart;   with(plots): f:=x->arcsin(2*x/(1.+x^2));
```

$$f := x \to \arcsin\left(\frac{2\,x}{1. + x^2}\right)$$

```
> plot(f(x), x=-14..16, y=-4..4,thickness=2, title =
cat( "arcsin(2*x/(1.+x^2))" ));
```

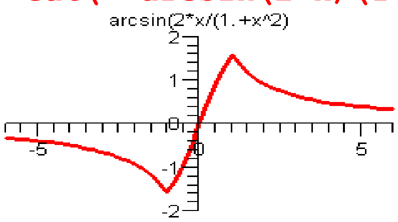


Funktsiya grafigini animatsiya yordamida qurish.

```
> restart;   with(plots):  f:=x->arcsin(2*x/(1.+x^2));
```

$$f := x \to \arcsin\left(\frac{2\,x}{1. + x^2}\right)$$

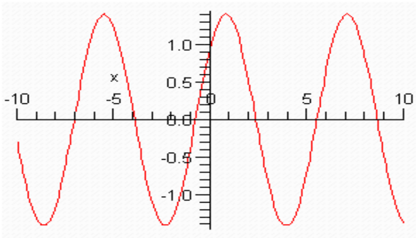
```
> animatecurve([x,f(x),x=-6..6],view=[-6..6,-2..2],frames=100,
thickness=2,title = cat( "arcsin(2*x/(1.+x^2)" ));
```



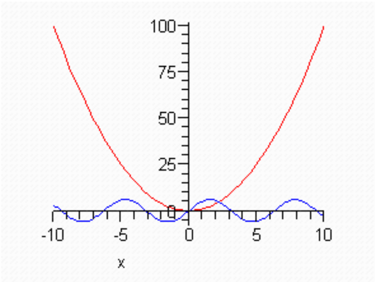
Qo'shimch masalalar

1. *Birnecha funktsiyalarni birgalikda grafigini qurish:*

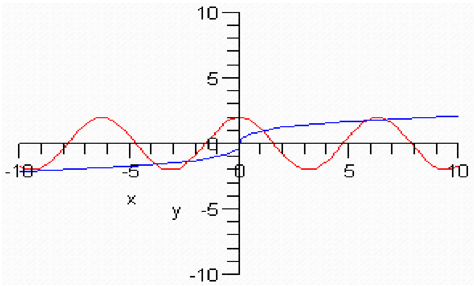
```
> with(plots):
> smartplot(cos(x) + sin(x));
```



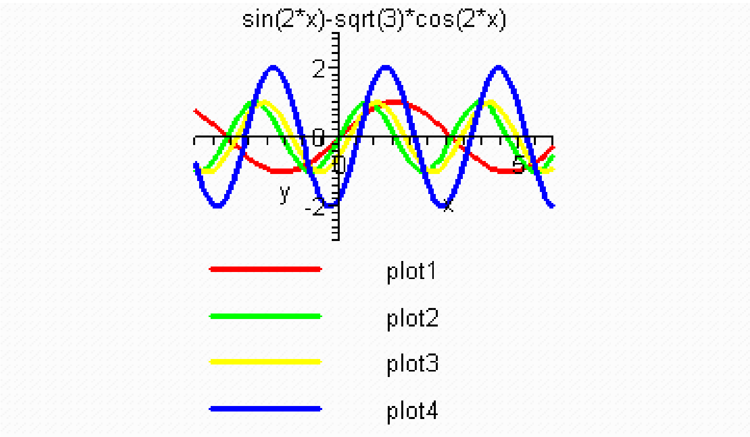

```
> with(plots):
>smartplot((x^2) , 6*sin(x));
```



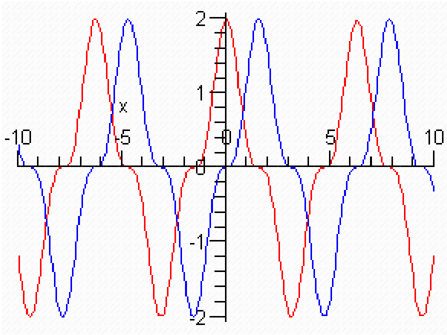

```
> with(plots):
>smartplot(2*cos(x) , (y^3));
```




```
> with(plots):
>plot([sin(x),sin(2*x),sin(2*x-Pi/6),sin(2*x)-
sqrt(3)*cos(2*x)], x=-4..6, y=-3..3,thickness=2,title =
cat("sin(2*x)-sqrt(3)*cos(2*x)"),
legend=[plot1,plot2,plot3,plot4]);
```



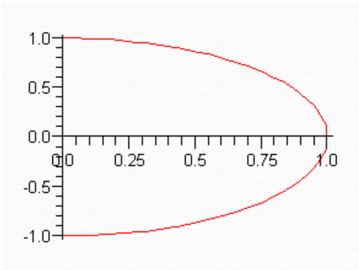
```
> with(plots):
> smartplot(2*cos(x)^3 , 2*sin(x)^3);
```



2.**Qutb koordinata sistemasida funktsiyalarni birgalikda grafigini qurish.**

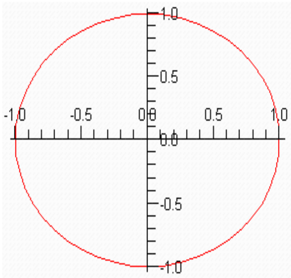
-Parabola

```
> with(plots):
> plot([x^2,x,x=0..3*i],-8..8,-10..10,coords=polar);
```



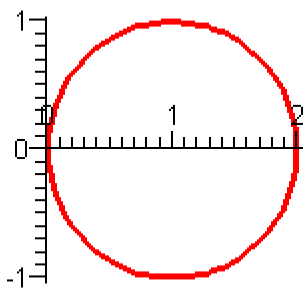
Aylana

```
> with(plots):  polarplot(1);
```



Markazi Ox o‘qida yotgan alana

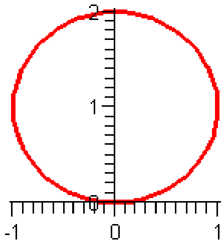
```
> with(plots):
> implicitplot({r=2*sin(phi)}, r=-14..14, phi=0..1*Pi,
coords=polar,thickness=2);
```



Markazi Oy o‘qida yotgan alana

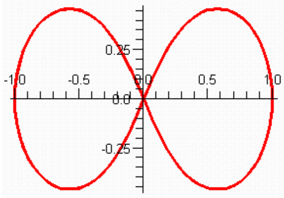
```
> with(plots):
```

```
> implicitplot({r=2*sin(phi)}, r=-14..14, phi=0..1*Pi,
coords=polar,thickness=2);
```

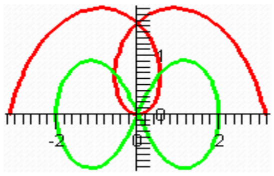


Bernuli lyumniskatasi

```
> with(plots):
polarplot([1*cos(t),1*sin(t),t=0..4*Pi],thickness=2,color=red);
```

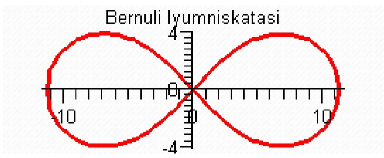



```
>with(plots):
polarplot({t,[2*cos(t),sin(t),t=-Pi..Pi]},t=-Pi..Pi,
thickness=2, numpoints=50);
```



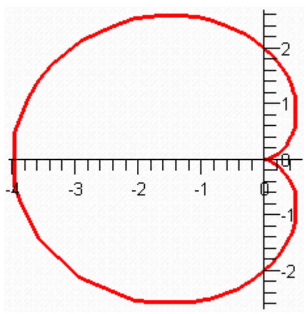
Bernuli lyumniskatasi

```
> with(plots):
> implicitplot({r^2=2*8^2*cos(2*theta)}, r=-16..16,
theta=0..1*Pi, coords=polar,thickness=2,
title=`Bernuli lyumniskatasi`);
```

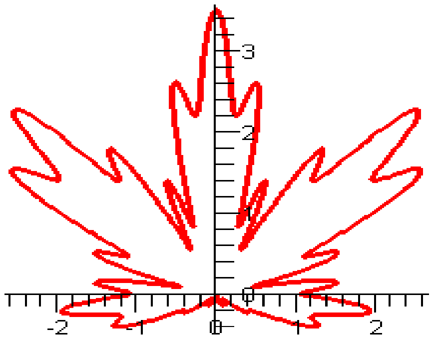


Kardoida

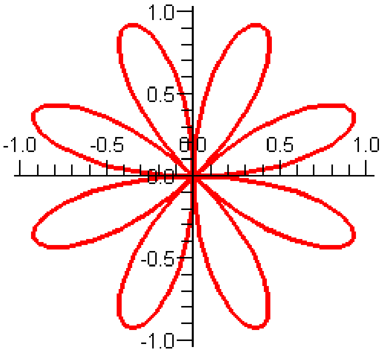
```
> with(plots):
> implicitplot(r=2*1*(1-cos(theta)), r=-6..6, theta=0..2*Pi,
coords=polar,thickness=2);
```



```
> with(plots):
S := t->100/(100+(t-Pi/2)^8): R := t -> S(t)*(2-sin(7*t)-
cos(30*t)/2):
polarplot([R,t->t,-Pi/2..3/2*Pi], thickness=2,
numpoints=2000,axes=NONE);
```

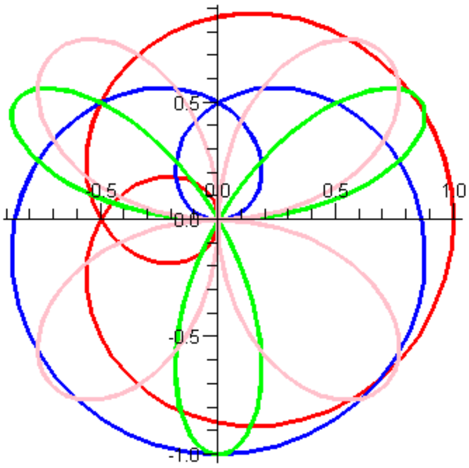


```
> with(plots):
> animate([sin(x*t),x,x=-4..4],t=1..4,coords=polar,
numpoints=150, frames=150,thickness=2);
```



Bitta grafikda bir necha functsiyani ifodalash.

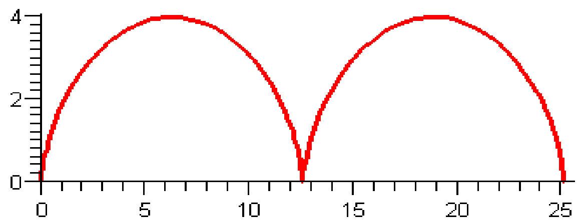
```
> with(plots):
polarplot([cos(t/3),sin(t/3),sin(3*t),sin(2*t)],t=0..4*Pi,
color=[red,blue,green,pink],thickness=2);
```



Parametric funktsiyalarning grafiklarini qurish

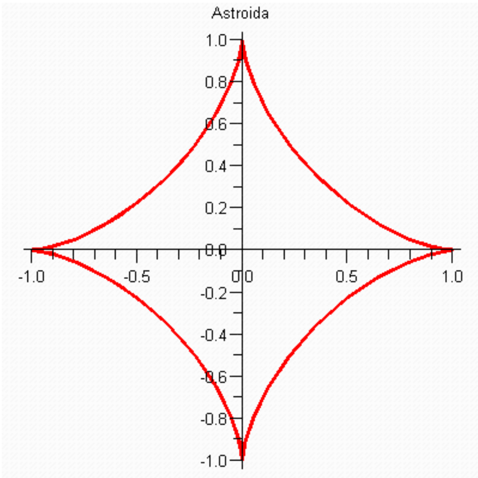
Tsikloida

```
> with(plots):
> plot([2*(t-sin(t)), 2*(1-cos(t)), t=0..4*Pi],thickness=2);
```



Astroida

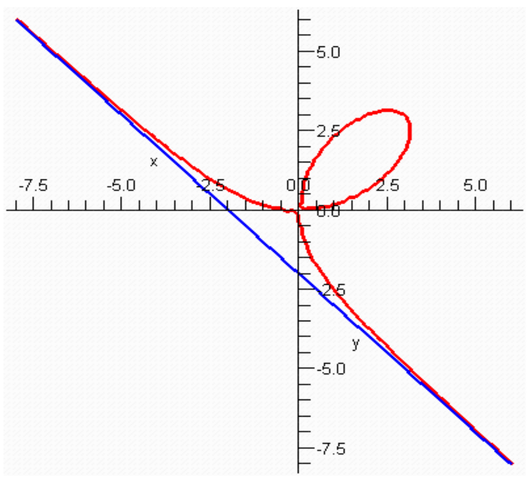
```
> with(plots):
> plot([1*cos(t)^3, 1*sin(t)^3, t=0..4*Pi],thickness=2,
title=`Astroida`);
```



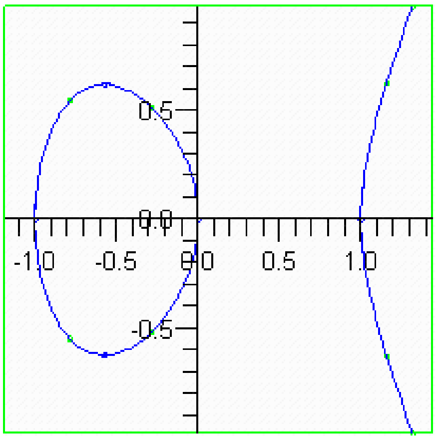
Oshkormas funktsiya grafigini qurish.

Dekart yarog'i

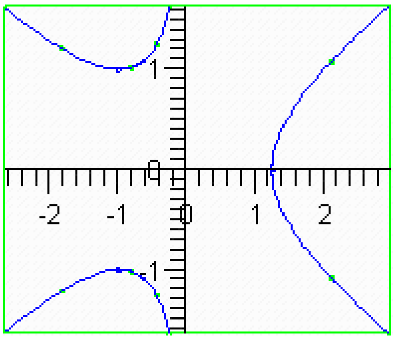
```
>with(plots):
> imlicitplot([x^3+y^3-3*2*x*y,x+y+2], x=-8..8, y=-8..8,numoints=2000, thickness=2,
color=[red,blue]);
```




```
> with(algcurves):
f:=y^2-x*(x^2-1):
plot_real_curve(f,x,y,showArrows=true);
```



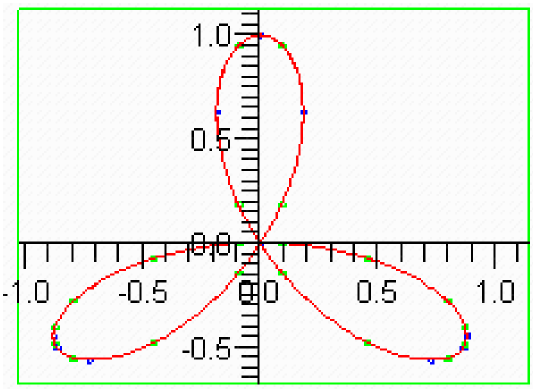

```
> with(algcurves):
f:=3*x*y^2-x^3+2:
plot_real_curve(f,x,y,showArrows=true);
```



Petalled roses (3- petalled rose, [0,0] is an ordinary triple point):

```
> f:=(x^2+y^2)^2 + 3*x^2*y - y^3;
plot_real_curve(f,x,y, colorOfCurve=red);
```

$$f := (x^2 \mathrm{C}\ y^2)^2 \mathrm{C}\ 3\,x^2\,y \mathrm{K}\ y^3$$



9 – MUSTAQIL ISHLASH UCHUN TOPSHIRIQLAR

Quydagi funktsiyalarni tekshiring va grafigini quring.

1) $y = \dfrac{x}{x^2 - 1}$, 2) $y = \dfrac{1}{1 + x^2}$, 3) $y = \dfrac{x^2}{x - 1}$ 4) $y = \dfrac{4x^3 + 5}{x}$

5) $y=\frac{x^2}{x-2}$ 6) $y=x\ln x$, 7) $y=\frac{x^2-5}{x-3}$ 8) $y=\frac{x^2-5}{x-2}$

9) $y=\frac{1}{x^2+3}$ 10) $y=\frac{1}{(x-2)^2}$, 11) $y=\frac{\ln x}{x}$ 12) $y=\frac{x}{x^2-4}$

13) $y=\frac{6x^2-x^4}{x}$, 14) $y=\frac{x^4-3}{x}$, 15) $y=\frac{8}{x^2-4}$, 16) $y=x^2+\frac{1}{x}$

17) $y=e^{2x-x^2}$ 18) $y=\frac{x(x+1)^2}{x-2}$. 19) $y=x\ln^2 x$. 20) $y=xe^{1/x}$.

21) $y=xe^x$. 22) $y=\frac{x^2}{(x+2)^2}$. 23) $y=\frac{x^4}{x^3-1}$. 24) $y=\ln(x^2-2x+6)$.

8. Bir necha o‘zgaruvchining funksiya

8.1. Bir necha o‘zgaruvchili(argumentli) funksiyaning ta’rifi

Aytaylik, R^n da n o‘lchovli D to‘lam berilgan bo‘lsin. Agar $M\in D$ nuqta bo‘lsa, uning n ta koordinatalari borligi, ya’ni $M(x_1;x_2;...;x_n)$ ekanligini eslatamiz.

8.1.1-ta’rif. Agar R^n da berilgan n o‘lchovli D to‘lamning har bir M nuqtasiga u sonli o‘zgaruvchining aniq bitta qiymati mos qo‘yilgan bo‘lsa, *u ni n o‘lchovli D to‘lamda aniqlangan funksiya* deyiladi va

$$u=f(M) \qquad (8.1.1)$$

kabi belgilanadi. D to‘lam funksiyaning *aniqlanish sohasi*, u ning qabo‘l qilishi mumkin bo‘lgan qiymatlari to‘lami esa, *o‘zgarish sohasi* deb yuritiladi.

(8.1) ga e’tibor bersak, u n o‘lchovli D to‘lamdan olingan M nuqtaning funksiyasi sifatida yozilgandir. Agar n o‘lchovli to‘lam nuqtasi $M(x_1;x_2;...;x_n)$ ekanligini hisobga olsak, (8.1.1) ni

$$u=f(x_1;x_2;...;x_n) \qquad (8.1.2)$$

ko‘rinishda ham yozish mumkin. Bu yozuvda u funksiya nechta o‘zgaruvchiga bog‘liq ekanligi yaqqol ko‘rinadi. Shu sababli, (8.1.2) ko‘rinishda yozilgan holda *bu funksiyani n o‘zgaruvchili (argumentli) funksiya deyilib,* $x_1;x_2;...;x_n$ *lar uning argumentlari* deb yuritiladi.

Biz, asosan, ikki va uch argumentli funksiyalar bilan ish yuritamiz. Shu sababli, an’anaviy belgilashlarni saqlab qolamiz:

ikki o‘zgaruvchili (argumentli) funksiya uchun

$$z=f(x;y) \qquad (8.1.3)$$

bu yerda x va y lar argumentlar, z esa funksiyadir;

uch o‘zgaruvchili (argumentli) funksiya uchun

$$u=f(x;y;z) \qquad (8.1.4)$$

bu yerda x , y va z lar argumentlar, u esa funksiyadir.

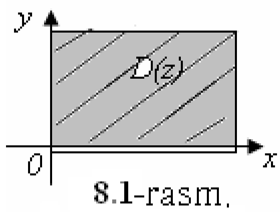
Masalan, to‘g‘ri burchakli uchburchak katetlarini x va y , giotenuzasini esa z bilan belgilasak, ifagor teoremasiga asosan

$$z=\sqrt{x^2+y^2} \qquad (8.1.5)$$

ya’ni (8.1.3) ko‘rinishdagi ikki argumentli; agar to‘g‘ri burchakli araleleiedning uch o‘lchovini x,y,z bilan, diagonalini esa u bilan belgilasak,

$$u=\sqrt{x^2+y^2+z^2} \qquad (8.1.6)$$

ya’ni (8.1.4) ko‘rinishdagi uch argumentli funksiyaga ega bo‘lamiz.



8.1-rasm.

Yuqorida keltirilgan misollarda, ularning geometrik ma’npolaridan kelib chiqib, aniqlanish va o‘zgarish sohalarini topish mumkin. (8.1.5) funksiya uchun *x* va *y* lar to‘g‘ri burchakli uchburchakning katetlari, *z* esa uning giotenuzasi ekanligidan $x > 0$, $y > 0$, $z > 0$ bo‘lishi kerakdir. Demak, (8.1.5) funksiyani anganiqlanish sohasi $D(z) = \{(x;y): x > 0, y > 0\}$ to‘lamdan, ya’ni 8.1-rasmda ko‘rsatilgan koordinatalar tekisligining birinchi choragi ichki nuqtalaridan iboratdir. O‘zgarish sohasi esa $E(z) = R^+$ ekanligi (8.1.5) ning o‘zidan kelib chiqadi.

Xuddi shunga o‘xshash, (8.6) funksiya uchun *x,y,z* to‘g‘ri burchakli paraleleiedning uch o‘lchovi va *u* uning diagonali ekanligi sababli, aniqlanish sohasi $D(u) = \{(x;y;z): x > 0, y > 0, z > 0\}$ to‘lam, ya’ni koordinatalar fazosi birinchi oktantasining ichki nuqtalari, o‘zgarish sohasi esa $E(u) = R^+$ dir.

Eslatma. Bu yerda funksiyaning aniqlanish va o‘zgarish sohalari uchun an’anaviy belgilashlarini saqlab qoldik.

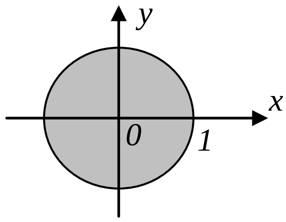
Agar (8.1.5) va (8.1.6) funksiyalar yozilgan bo‘lib, ularga hech qanday boshqa, masalan, geometrik, ma’no berilmagan bo‘lsa, uni aniqlovchi amallar bo‘yicha (8.1.5) uchun $D(z) = R^2$, $E(z) = [0;+\infty)$; (8.1.6) uchun esa, $D(u) = R^3$, $E(u) = [0;+\infty)$ larni topish mumkin.

Yana bir necha misollar keltiraylik.

8.1.1-misol. $z = \sqrt{1 - x^2 - y^2}$ funksiyaning aniqlanish va o‘zgarish sohalari topilsin.

Yechish. Bu funksiyaning aniqlanish sohasi $1 - x^2 - y^2 \geq 0$ tengsizlik yechimidir:

$$x^2 + y^2 \leq 1.$$



8.2-rasm.

Bu tengsizlikni qanoatlantiruvchi nuqtalar markazi koordinatalar boshida radiusi 1 ga teng bo‘lgan aylanada va uning ichida yotishi ravshandir (8.2-rasm).

Demak, $D(z) = \{(x;y): x^2 + y^2 \leq 1\}$.

O‘zgarish sohasi

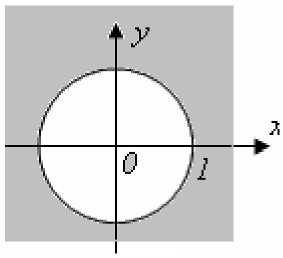
$$(x;y) \in D(z),\ \ 0 \leq x^2 + y^2 \leq 1 \Leftrightarrow 0 \leq 1 - (x^2 + y^2) \leq 1 \Leftrightarrow$$

$$\Leftrightarrow 0 \leq \sqrt{1 - (x^2 + y^2)} \leq 1 \Rightarrow 0 \leq z \leq 1 \Rightarrow E(z) = [0;1].$$

8.1.2-misol. $z = \lg\sqrt{x^2 + y^2 - 4}$ funksiyaning aniqlanish va o‘zgarish sohalari topilsin.

Yechish. Aniqlanish sohasini toaylik:

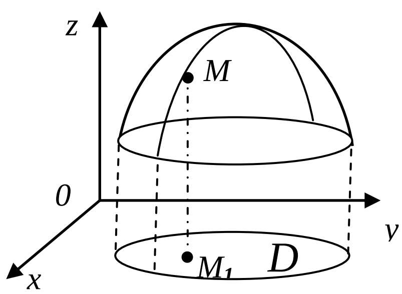
$$\sqrt{x^2 + y^2 - 4} > 0 \Leftrightarrow x^2 + y^2 - 4 > 0 \Leftrightarrow x^2 + y^2 > 4.$$



8.3 -rasm

Bu tengsizlikni koordinatalar tekisligining markazi koordinatalar boshida radiusi 2 ga teng bo‘lgan aylanadan tashqarida yotgan nuqtalari qanoatlantiradi, ya’ni $D(z) = \{(x;y): x^2 + y^2 > 4\}$ (8.3-rasm).

O‘zgarish sohasi $E(z) = R$ bo‘lishini ko‘rish qiyin emas. Agar $z = f(x;y)$ ikki o‘zgaruvchili funksiya biror *D* sohada aniqlangan bo‘lsa, fazoda *Oxyz* Dekart koordinatalar sistemasini kiritib, $(x;y) \in D$ bo‘lganda $M(x;y;f(x;y))$ nuqtani qurib, $M_0(x;y)$ ni *D* soha bo‘ylab harakatlantirsak, $M(x;y;z)$ $(z = f(x;y))$ nuqta fazoda qandaydir sirtni nuqtalarini aniqlaydi. Bu sirt *ikki o‘zgaruvchili funksiyaning grafigi* (geometrik tasviri) dir (8.4-rasm).



8.4-rasm.

Kezi kelganda, $z = f(x;y)$ funksiyani $z - f(x;y) = 0$ uch o‘zgaruvchili tenglama ko‘rinishida ifodalash mumkinligini hisobga olsak, $F(x;y;z) = 0$ uch o‘zgaruvchili tenglama fazoda biror sirtni aniqlashini eslatamiz (buni 6,7- boblarda ko‘rgan edik).

8.2. Ko‘p o‘zgaruvchili funksiyaning limiti

Agar $R^n \supset D$ n o‘lchovli to‘lamda $u = f(M)$ funksiya berilgan bo‘lib, M_0 nuqta D ning quyuqlik nuqtasi bo‘lsa, $f(M)$ funksiyaning bu nuqtadagi limiti tushunchasini quyidagicha kiritiladi.

8.2.1-ta’rif. *Agar M_0 nuqta D to‘lamning quyuqlik nuqtasi bo‘lib, $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ son mavjud bo‘lsaki, M_0 nuqtaning δ yaqin atrofidan olingan $\forall M \in D$ nuqtalarda, ya’ni $\{M : M \in D, 0 < \rho(M; M_0) < \delta\}$ to‘lamda*

$$|f(M) - b| < \varepsilon \qquad (8.2.1)$$

o‘rinli bo‘lsa, b son f(M) funksiyaning $M \to M_0$ dagi yoki M_0 nuqtadagi limiti deyiladi, hamda

$$\lim_{M \to M_0} f(M) = b \qquad (8.2.2)$$

kabi yoziladi.

Ko‘p o‘zgaruvchili funksiyaning limitining ta’rifi bir o‘zgaruvchili funksiyaning limitining ta’rifi bilan aynan bir xilligidan limitlar haqidagi teoremalar va xossalar hamda asosiy formulalar bu yerda ham o‘rinlidir.

Masalan,

$$\lim_{\begin{cases} x \to 0 \\ y \to 0 \end{cases}} \frac{\sin(xy)}{xy} = 1$$

ekanligini ko‘rish qiyin emas, chunki, $M(x;y) \to O(0;0) \Rightarrow x \cdot y \to 0$ va limit ostidagi funksiya $O(0;0)$ nuqta yaqin atrofida aniqlangan bo‘lib, birinchi ajoyib limitga ko‘ra yuqoridagi tenglik kelib chiqadi.

Bu yerda funksiyaning limiti tushunchasidan tashqari uning takroriy limitlari tushunchasi ham mavjud ekanini aytamiz. Bu tushunchani ikki o‘zgaruvchili funksiya uchun keltiramiz (soddalik uchun).

$$\lim_{\substack{x \to x_0 \\ y \to y}} f(x;y)$$

$- M_0(x_0; y_0)$ nuqtadagi funksiya limiti;

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x;y) \quad \text{va} \quad \lim_{y \to y_0} \lim_{x \to x_0} f(x;y)$$

- lar esa takroriy limitlardir.

Masalan, $\lim\limits_{x \to x_0} \lim\limits_{y \to y_0} f(x;y)$ takroriy limitni topishda, avval, x tayinlangan (o‘zgarmas) va $x \neq x_0$ deb faraz qilib, $\lim\limits_{y \to y_0} f(x;y) = \varphi(x)$ topiladi (agar mavjud bo‘lsa), so‘ngra, $\lim\limits_{x \to x_0} \varphi(x)$ limit topiladi (mavjud bo‘lsa). $\lim\limits_{y \to y_0} \lim\limits_{x \to x_0} f(x;y)$ da esa, avval, $\lim\limits_{x \to x_0} f(x;y) = \psi(y)$, keyin $\lim\limits_{y \to y_0} \psi(y)$ topiladi.

Bu yerda shuni ham takidlaymizki, $\lim\limits_{M \to M_0} f(M)$ ning mavjudligidan takroriy limitlarning mavjud ekanligi hamma vaqt ham kelib chiqavermaydi. Buning aksinchasi ham hamma vaqt o‘rinli bo‘lavermaydi.

Masalan: 1) $f(x;y) = \dfrac{xy}{x^2 + y^2}$ funksiya koordinatalar boshida aniqlanmagan bo‘lib, tekislikning boshqa barcha nuqtalarida mavjuddir. Uning O(0;0) nuqtadagi limitlarini qaraylik.

$$\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2}=\lim_{x\to 0}0=0;\quad \lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2}=\lim_{y\to 0}0=0.$$

Ammo, $\lim\limits_{\begin{cases}x\to 0\\ y\to 0\end{cases}}\frac{xy}{x^2+y^2}$ - mavjud emas. Buni ko‘rsatish uchun $M(x;y)\to O(0;0)$ intilish $y=kx$ to‘g‘ri chiziq bo‘ylab deb faraz qilsak, u vaqtda,

$$\lim_{\begin{cases}x\to 0\\ y=kx\end{cases}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{x\cdot kx}{x^2+(kx)^2}=\lim_{x\to 0}\frac{k}{1+k^2}=\frac{k}{1+k^2}$$

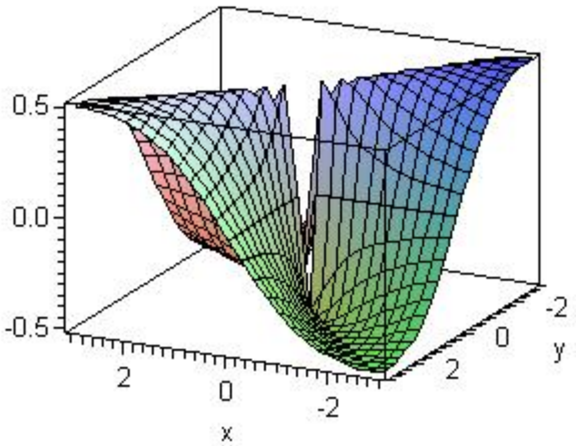
bo‘lib, turli k larda turli qiymatlar kelib chiqadi. Ya’ni limit M nuqtaning O(0; 0) nuqtaga intilish chizig‘iga bog‘liqdir. Agar $\lim\limits_{\begin{cases}x\to 0\\ y\to 0\end{cases}}\frac{xy}{x^2+y^2}$ mavjud bo‘lganda $M\to O(0;0)$ intilish qanday bo‘lishiga bog‘liq bo‘lmasligi kerak edi. Demak, $\lim\limits_{\begin{cases}x\to 0\\ y\to 0\end{cases}}\frac{xy}{x^2+y^2}$ limit mavjud emas ekan.

$f(x;y)=\dfrac{xy}{x^2+y^2}$ *funksiya grafigi:*

> restart;

> f:=(x,y)->x*y/(x^2+y^2); $f:=(x,y)\to\dfrac{x\,y}{x^2+y^2}$

> plot3d(f(x,y),x=-3..3,y=-3..3,axes=BOXED);



2) $f(x;y)=x\cdot\sin\dfrac{1}{y}$ funksiyani olsak, u $D=\{(x;y):\ y\neq 0,\ x\in R, y\in R\}$ sohada aniqlangandir. Bu funksiyaning $O(0;0)$ nuqtadagi limitini va takroriy limitlarini tekshirib ko‘raylik:

a) $\lim\limits_{\begin{cases}x\to 0\\ y\to 0\end{cases}}\left(x\cdot\sin\dfrac{1}{y}\right)=0$ - limit mavjud, chunki, $\left(x\to 0, y\to 0;\ \left|\sin\dfrac{1}{y}\right|\le 1\right)$;

b) $\lim\limits_{y\to 0}\lim\limits_{x\to 0}\left(x\cdot\sin\dfrac{1}{y}\right)=\lim\limits_{y\to 0}0=0;$

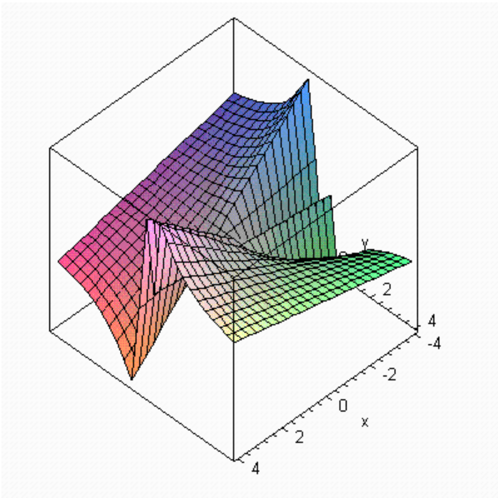
c) $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\left(x\cdot\sin\dfrac{1}{y}\right)$ - mavjud emas, chunki, $x\neq 0$ bo‘lganda, tayinlangan x uchun $\lim\limits_{y\to 0}\left(x\cdot\sin\dfrac{1}{y}\right)$ - mavjud emasdir.

$f(x;y)=x\cdot\sin\dfrac{1}{y}$ *funksiya grafigi:*

> restart;

> f:=(x,y)->x*sin(1/y); $f:=(x,y)\to x\sin\left(\frac{1}{y}\right)$

> plot3d(f(x,y),x=-4..4,y=-4..4,axes=BOXED);



Funksiyaning limiti va uning takroriy limitlarining orasidagi munosabatni aniqlovchi quyidagi teorema isbotlangandir (soddalik uchun uni ikki o‘zgaruvchili funksiya uchun keltiramiz).

8.2.1-teorema. Agar

1) chekli yoki cheksiz

$$A=\lim_{\begin{cases}x\to a\\ y\to b\end{cases}} f(x;y)$$

limit mavjud va

2) *y* argumentning *b* ga yetarlicha yaqin qiymatlari uchun

$$\varphi(y)=\lim_{x\to a} f(x;y)$$

limit mavjud bo‘lsa, u vaqtda, takroriy

$$\lim_{y\to b}\varphi(y)=\lim_{y\to b}\lim_{x\to a} f(x;y)$$

limit ham mavjud va *A* ga teng bo‘ladi.

Aytaylik, $f(x;y)$ funksiyaning aniqlanish sohasi $D(z)$ ning $M(x;y)$ nuqtasining birinchi koordinatasi *x* absolyut qiymat jixatdan xoxlagancha katta qiymat qabo‘l qila olsin, ikkinchisi *y* esa *b* songa yaqinlashsin. U vaqtda, $M_0(\infty;b)$ $D(z)$ ning quyuqlik nuqtasi deyishga shartlashib olgan holda quyidagi ta’rifni berish mumkin.

Agar $M_0(\infty;b)$ $z=f(x;y)$ *funksiyaning aniqlanish sohasi* $D(z)$ *ning quyuqlik nuqtasi bo‘lib,* $\forall\varepsilon>0$ *uchun shunday* $\delta_1>0,\delta_2>0$ *sonlar topilsaki,* $|x|>\delta_1, 0<|y-b|<\delta_2$ *tengsizliklarni qanoatlantiruvchi* $M(x;y)\in D(z)$ *nuqta uchun* $|f(x;y)-A|<\varepsilon$ *o‘rinli bo‘lsa, A son* $D(z)$ *ning* $M_0(\infty;b)$ *quyuqlik nuqtasidagi* $f(x;y)$ *funksiyaning limiti deyiladi va*

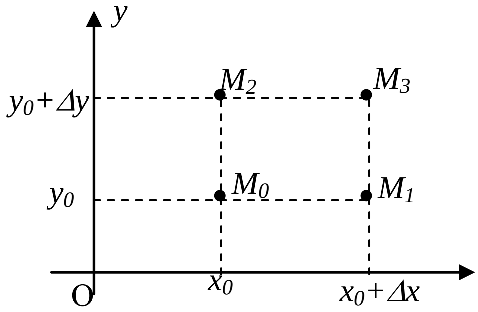
$$\lim_{\begin{cases}x\to\infty\\ y\to b\end{cases}} f(x;y)=A$$

kabi belgilanadi.

Agar bu ta'rifda $|x| > \delta_1$ o'rnida $x > \delta_1 \left(x < -\delta_1\right)$ tengsizlik ishlatilsa, $\lim\limits_{\substack{x\to+\infty \\ y\to b}} f(x;y) = A \left(\lim\limits_{\substack{x\to-\infty \\ y\to b}} f(x;y) = A \right)$ va nihoyat, $|f(x;y) - A| < \varepsilon$ o'rnida $|f(x;y)| > \varepsilon \;\; \left(f(x;y) > \varepsilon \; \left(f(x;y) < -\varepsilon\right)\right)$ tengsizlik ishlatilsa, A o'rnida $\infty\left(+\infty\left(-\infty\right)\right)$ yoziladi.

Xuddi shunga o'xshash, $D(z)$ning $M_0(a;\infty)$ yoki $M_0(\infty;\infty)$ quyuqlik nuqtasidagi funksiyaning limiti tushunchasi ham kiritiladi.

8.3. Ko'p o'zgaruvchili funksiyaning xususiy hosilalari



8.5-rasm.

Umumiylikka ta'sir qilmagan holda bu tushunchalarni ikki o'zgaruvchili funksiya uchun keltiraylik. Ya'ni $z = f(x;y)$ ikki o'zgaruvchili funksiya $M_0(x_0;y_0)$ nuqtada va uning qandaydir atrofida aniqlangan bo'lsa, M_0 nuqtani 8.5-rasmdagidek *Ox* o'qiga va *Oy* o'qiga arallel ko'chirish yo'li bilan $M_1(x_0+\Delta x; y_0)$ va $M_2(x_0;\, y_0+\Delta y)$ nuqtalarni hosil qilamiz. Undan tashqari, $M_3(x_0+\Delta y;\, y_0+\Delta y)$ nuqtani ham qaraymiz. U holda, $z = f(x;y)$ funksiyaning $M_0(x_0;y_0)$ nuqtadagi x hamda y argumentlari bo'yicha xususiy orttirmalari mos ravishda quyidagilardir:

$$\Delta_x z = f(x_0+\Delta x; y_0) - f(x_0;y_0),$$

$$\Delta_y z = f(x_0; y_0+\Delta y) - f(x_0;y_0).$$

Shu bilan birga, $M_0(x_0;y_0)$ nuqtani $M_3(x_0+\Delta y;\, y_0+\Delta y)$ nuqtaga ko'chirish natijasidagi funksiya orttirmasi

$$\Delta z = f(x_0+\Delta x; y_0+\Delta y) - f(x_0;y_0).$$

dan iboratdir. Bu $(\Delta x \neq 0; \Delta y \neq 0)$ funksiyaning $M_0(x_0;y_0)$ nuqtadagi ***to'liq orttirmasi*** deyiladi.

8.3.1-ta'rif. Agar $u = f(x_1;x_2;...;x_n)$ funksiya $M_0\left(x_1^{(0)};x_2^{(0)};...;x_n^{(0)}\right)$ nuqtaning qandaydir atrofida aniqlangan bo'lib, bu nuqtadagi x_i argument bo'yicha xususiy orttirmasini mos argument orttirmasiga nisbatining argument orttirmasi nolga intilgandagi limiti mavjud bo'lsa, bu limit funksiyaning x_i argument bo'yicha xususiy hosilasi deyiladi va

$$f'_{x_i}\left(x_1^{(0)};x_2^{(0)};...;x_n^{(0)}\right),\; u'_{x_i},\; \frac{\partial u}{\partial x_i},\; \frac{\partial f\left(x_1^{(0)};x_2^{(0)};...;x_n^{(0)}\right)}{\partial x_i}$$

kabi belgilashlardan biri ishlatiladi. Demak, ta'rif bo'yicha

$$\frac{\partial u}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\Delta_{x_i} u}{\Delta x_i}.$$

Xususiy hosila ta'rifidan uni topish qoidasi ham kelib chiqadi. Chunki, bu ta'rifga ko'ra qaysi argument bo'yicha xususiy hosila olish kerak bo'lsa, shu argumentga orttirma beriladi, qolganlari esa, M_0 nuqta mos koordinatasidan iborat o'zgarmas bo'lib qoladi, ya'ni, aslida, bir o'zgaruvchili funksiya bilan ish ko'riladi. Demak, *biror argument bo'yicha xususiy hosilani topish uchun* *funksiyaning shu argumentini o'zgaruvchi, qolganlarini esa, o'zgarmas deb faraz qilib, hosila olish kifoyadir.*

```
> restart;
> diff(f(x,y),x);diff(f(x,y),y);
```

$$\frac{\partial}{\partial x} f(x,y) \qquad \frac{\partial}{\partial y} f(x,y)$$

8.3.1-misol. $z = x^2 + xy - y^2 + 4x - 2y + 5$ funksiyaning xususiy hosilalarini topilsin.

Yechish.

$$\frac{\partial z}{\partial x} = (x^2)'_x + (xy)'_x - (y^2)'_x + (4x)'_x - (2y)'_x + 5'_x = 2x + y - 0 + 4 - 0 + 0 = 2x + y + 4;$$

$$\frac{\partial z}{\partial y} = (x^2)'_y + (xy)'_y - (y^2)'_y + (4x)'_y - (2y)'_y + 5'_y = 0 + x - 2y + 0 - 2 + 0 = x - 2y - 2.$$

```
> restart;
> u:=(x,y)->x^2+x*y-y^2+4*x-2*y+5;
```

$$u := (x,y) \to x^2 + x\,y - y^2 + 4\,x - 2\,y + 5$$

```
> Diff(u, x)=diff(u(x,y),x); Diff(u, y)=diff(u(x,y),y);
```

$$\frac{\partial}{\partial x} u = 2\,x + y + 4 \qquad \frac{\partial}{\partial y} u = x - 2\,y - 2$$

8.3.2-misol. $z = arctg\,\frac{x}{y}$ ning xususiy hosilalari topilsin.

Yechish. $y \neq 0$ deb faraz qilamiz.

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{x}{y}\right)'_x = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2};$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{x}{y}\right)'_y = \frac{y^2}{x^2+y^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}.$$

```
> restart;
> u:=(x,y)->arctan(x/y);
```

$$u := (x,y) \to \arctan\left(\frac{x}{y}\right)$$

```
> Diff(u, x)=diff(u(x,y),x); Diff(u, y)=diff(u(x,y),y);
```

$$\frac{\partial}{\partial x} u = \frac{1}{y\left(1 + \frac{x^2}{y^2}\right)} \qquad \frac{\partial}{\partial y} u = -\frac{x}{y^2\left(1 + \frac{x^2}{y^2}\right)}$$

8.3.3-misol.. $u = \sqrt{x^2 + y^2 + z^2}$ funktsiyaning x, y, z uzgaruvchilarga nisbatan xususiy hosilalarining (2; -2;1) nuqtadagi qiymatini hisoblang.

YEchish. Xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}},\quad \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}},\quad \frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

```
> u:=(x,y,z)->sqrt(x^2+y^2+z^2);
```

$$u := (x, y, z) \to \sqrt{x^2 + y^2 + z^2}$$

```
> Diff(u,x)=diff(u(x,y,z),x);
```

$$\frac{\partial}{\partial x} u = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

```
> Diff(u,y)=diff(u(x,y,z),y);
```

$$\frac{\partial}{\partial y} u = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

```
> Diff(u,z)=diff(u(x,y,z),z);
```

$$\frac{\partial}{\partial z} u = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Bu ifodalarga berilgan nuqtaning koordinatalarini qo‘yamiz:

$$\left.\frac{\partial u}{\partial x}\right|_P=\frac{2}{\sqrt{2^2+(-2)^2+1^2}}=\frac{2}{3},\qquad \left.\frac{\partial u}{\partial x}\right|_P=\frac{-2}{\sqrt{2^2+(-2)^2+1^2}}=-\frac{2}{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_P=\frac{1}{\sqrt{2^2+(-2)^2+1^2}}=\frac{1}{3}$$

```
> fx:=eval(diff(u(x,y,z),x),[x=2,y=-2,z=1]);
```

$$fx := \frac{2}{9}\sqrt{9}$$

```
> fy:=eval(diff(u(x,y,z),y),[x=2,y=-2,z=1]);
```

$$fy := -\frac{2}{9}\sqrt{9}$$

```
> fz:=eval(diff(u(x,y,z),z),[x=2,y=-2,z=1]);
```

$$fz := \frac{1}{9}\sqrt{9}$$

8.4. Ko‘ o‘zgaruvchili funksiyaning xususiy va to‘liq differensiallari

8.4.1-ta’rif. *Agar* $u=f(x_1;x_2;...;x_n)$ *ko‘ o‘zgaruvchili funksiya* $M_0\left(x_1^{(0)};x_2^{(0)};...;x_n^{(0)}\right)$ *nuqtaning qandaydir atrofida aniqlangan bo‘lib, bu nuqtadagi* x_i *argumenti bo‘yicha xususiy orttirmasining argument orttirmasi* Δx_i *ga nisbatan chiziqli bosh qismi mavjud bo‘lsa, bu chiziqli bosh qism funksiyaning* x_i *argumenti bo‘yicha xususiy differensiali deyiladi va* $d_{x_i}u$ *bilan belgilanadi.*

Xuddi bir o‘zgaruvchili funksiyalardagi kabi bu xususiy differensiallar uchun

$$d_{x_i}u=\frac{\partial u}{\partial x_i}dx_i \quad \left(i=\overline{1;n}\right) \tag{8.4.1}$$

formulani keltirib chiqarish mumkin bo‘lib, bu yerda $dx_i=\Delta x_i$ dir. Demak, xususiy differensial mavjud bo‘lishi uchun mos argument bo‘yicha xususiy hosila mavjud bo‘lishi yetarli ekan.

8.4.1-misol. $u=(xy^2)^{z^3}$ funktsiyaning xususiy diffe-rentsiallaryni toping.

Echish.

$$. \; d_x u = z^3 (xy^2)^{z^3-1} y^2 dx$$

$$d_y u = z^3 (xy^2)^{z^3-1} 2xydy$$

$$d_z u = (xy^2)^{z^3} \ln(xy^2) 3z^2 dz$$

```
> restart;
> u:=(x,y,z)->(x*y^2)^(z^3);
```

$$u := (x, y, z) \to (x\,y^2)^{(z^3)}$$

```
> Diff(u,x)=diff(u(x,y,z),x); Diff(u,y)=diff(u(x,y,z),y);
Diff(u,z)=diff(u(x,y,z),z);
```

$$\frac{\partial}{\partial x} u = \frac{(x\,y^2)^{(z^3)}\, z^3}{x} \qquad \frac{\partial}{\partial y} u = \frac{2\,(x\,y^2)^{(z^3)}\, z^3}{y}$$

$$\frac{\partial}{\partial z} u = 3\,(x\,y^2)^{(z^3)}\, z^2 \ln(x\,y^2)$$

```
> dux:=diff(u(x,y,z),x)*dx;duy:=diff(u(x,y,z),y)*dy;
duz:=diff(u(x,y,z),z)*dz;
```

$$dux := \frac{(x\,y^2)^{(z^3)}\, z^3\, dx}{x} \qquad duy := \frac{2\,(x\,y^2)^{(z^3)}\, z^3\, dy}{y}$$

$$duz := 3\,(x\,y^2)^{(z^3)}\, z^2 \ln(x\,y^2)\, dz$$

8.4.2-ta'rif. *Agar* $u = f(x_1; x_2; ...; x_n)$ *ko' o'zgaruvchili funksiya* $M_0\left(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)}\right)$ *nuqtaning qandaydir atrofida aniqlangan bo'lib, bu nuqtadagi funksiyaning to'liq orttirmasi argumentlar orttirmalari* Δx_i $\left(i = \overline{1;n}\right)$ *larga nisbatan chiziqli bosh qismga ega bo'lsa, bu chiziqli bosh qism funksiyaning to'liq differensiali deyiladi va* du *bilan belgilanadi. Bu holda funksiya differensiallanuvchi deb ataladi.*

To'liq differensial formulasini, umumiylikka ta'sir qilmagan holda, ikki o'zgaruvchili funksiya uchun keltirib chiqaramiz.

Aytaylik, $z = f(x; y)$ funksiya $M_0(x_0; y_0)$ nuqtaning qandaydir atrofida aniqlangan bo'lib, bu atrofda uning har ikkala argumenti bo'yicha uzluksiz xususiy hosilalari mavjud bo'lsin. U holda,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \qquad (8.4.2)$$

formula o'rinlidir, bu yerda $dx = \Delta x,\ dy = \Delta y$.

Umumiy holda n o'zgaruvchili $u = f(x_1; x_2; ...; x_n)$ funksiya $M_0\left(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)}\right)$ nuqta atrofida barcha argumentlari bo'yicha uzluksiz xususiy hosilalarga ega bo'lsa, uning to'liq differensiali uchun

$$du = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx_i \qquad (8.4.4)$$

formula o'rinlidir.

8.4.2-misol. $u = \ln(x^2 + y^2 - z^2)$

$$\frac{\partial u}{\partial x}=2\ln(x^2+y^2-z^2)\frac{2x}{x^2+y^2-z^2}=\frac{4x\ln(x^2+y^2-z^2)}{x^2+y^2-z^2}$$

$$\frac{\partial u}{\partial x}=2\ln(x^2+y^2-z^2)\frac{2y}{x^2+y^2-z^2}=\frac{4y\ln(x^2+y^2-z^2)}{x^2+y^2-z^2}$$

$$\frac{\partial u}{\partial x}=2\ln(x^2+y^2-z^2)\frac{-2z}{x^2+y^2-z^2}=\frac{-4z\ln(x^2+y^2-z^2)}{x^2+y^2-z^2}$$

$$.\ du=\frac{4\ln(x^2+y^2-z^2)}{x^2+y^2-z^2}(xdx+ydx-zdz)$$

```
> restart;
> u:=(x,y,z)->ln(x^2+y^2-z^2);
```

$$u := (x, y, z) / \quad \ln(x^2 \mathrm{C}\ y^2 \mathrm{K}\ z^2)$$

```
> Diff(u, x)=diff(u(x,y,z),x); Diff(u,
y)=diff(u(x,y,z),y);Diff(u, z)=diff(u(x,y,z),z);
```

$$\frac{\mathrm{V}}{\mathrm{V}x}u=\frac{2x}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2} \qquad \frac{\mathrm{V}}{\mathrm{V}y}u=\frac{2y}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2} \qquad \frac{\mathrm{V}}{\mathrm{V}z}u=\mathrm{K}\ \frac{2z}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2}$$

```
> dux:=diff(u(x,y,z),x)*dx;duy:=diff(u(x,y,z),y)*dy;
duz:=diff(u(x,y,z),z)*dz;
```

$$dux:=\frac{2\,x\,dx}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2} \qquad duy:=\frac{2\,y\,dy}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2} \qquad duz:=\mathrm{K}\ \frac{2\,z\,dz}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2}$$

```
> du:=diff(u(x,y,z),x)*dx+diff(u(x,y,z),y)*dy+
diff(u(x,y,z),z)*dz;
```

$$du:=\frac{2\,x\,dx}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2}\ \mathrm{C}\ \frac{2\,y\,dy}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2}\ \mathrm{K}\ \frac{2\,z\,dz}{x^2\mathrm{C}\ y^2\mathrm{K}\ z^2}$$

8.5. Ko‘p o‘zgaruvchili murakkab funksiyaning xususiy hosilalari
To‘liq hosila formulasi

Umumiylikka halal bermagan holda, mulohazalarimiz soddaroq bo‘lishi uchun, ikki o‘zgaruvchili murakkab funksiya ustida ish ko‘ramiz.

Aytaylik, $z=f(u;v)$ funksiya biror D sohada, $u=\varphi(x;y)$, $v=\psi(x;y)$ funksiyalar esa, biror D_0 sohada aniqlangan bo‘lib, ularning qiymatlaridan tuzilgan nuqta $(u;v)\in D$ bo‘lsin. U vaqtda, biz D_0 sohada aniqlangan $z=f[\varphi(x;y),\psi(x;y)]$ ikki o‘zgaruvchili murakkab funksiyaga ega bo‘lamiz. Bu yerda asosiy argumentlar (erkli o‘zgaruvchilar) x va y lar bo‘lib, u va v oraliq argumentlardir.

D_0 sohaga tegishli $(x_0;y_0)$ nuqtada $u=\varphi(x;y)$, $v=\psi(x;y)$ funksiyalar differensiallanuvchi bo‘lib, $(u_0;v_0)\in D$ nuqtada $z=f(u;v)$ ham differensiallanuvchi bo‘lsin, bu yerda $u_0=\varphi(x_0;y_0)$, $v_0=\psi(x_0;y_0)$.

Endi, $(x_0;y_0)$ nuqtani $(x_0+\Delta x;y_0)\in D_0$ nuqtaga qo‘zg‘ataylik, ya’ni y_0 ni o‘zgartirmagan holda x_0 ga $\Delta x\neq 0$ orttirma beraylik. Buning natijasida

$$\Delta_x u=\varphi(x_0+\Delta x;y_0)-\varphi(x_0;y_0),\ \Delta_x v=\psi(x_0+\Delta x;y_0)-\psi(x_0;y_0)$$

xususiy orttirmalarga va ularning ta’sirida

$$\Delta_x z = f(u_0 + \Delta_x u;\, v_0 + \Delta_x v) - f(u_0;\, v_0)$$

funksiya orttirmasiga ega bo'lamiz. Oxirgi olingan munosabatdan ko'rinadiki, natija funksiyaning to'liq orttirmasiga o'xshasada, aslida, u x argument bo'yicha xususiy orttirmadir.

φ, ψ va f funksiyalar differensiallanuvchi ekanligidan

$$\Delta_x u = (\varphi'_x(x_0\,;y_0) + \alpha)\Delta x,$$
$$\Delta_x v = (\psi'_x(x_0\,;y_0) + \beta)\Delta x,$$
$$\Delta_x z = (f'_u(u_0\,;v_0 + \Delta_x v)\Delta_x u + f'_v(u_0\,;v_0)\Delta_x v) + \rho\gamma$$

munosabatlar o'rinlidir. Bu yerda $\rho = \sqrt{\Delta_x u^2 + \Delta_x v^2}$,

$$\rho \to 0 \Rightarrow \alpha \to 0, \beta \to 0, \Delta_x u \to 0, \Delta_x v \to 0 \;.$$

Bo'larni hisobga olib,

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = f'_u(u_0\,;v_0)\cdot \varphi'_x(x_0\,;y_0) + f'_v(u_0\,;v_0)\cdot \psi'_x(x_0\,;y_0).$$

Ya'ni,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad (8.5.1)$$

formulani olamiz.

Xuddi shunga o'xshash

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \qquad (8.5.2)$$

formula ham olinadi.

8.5.1-misol.. Agar *z= cos(uv)* funktsiyada $u = 2x+3y$, $v = xy$ bo'lsa, uning xususiy hosilalarini toping.

Echish. (6.3) formulaga asosan:

$$\frac{\partial z}{\partial x} = -v\cos(uv)\cdot 2 - u\cos(uv)y = -(4xy + 3y^2)\cos(2x^2y + 3xy^2),$$
$$\frac{\partial z}{\partial y} = -v\cos(uv)\cdot 3 - u\cos(uv)\cdot x = -(6xy + 2y^2)\cos(2x^2y + 3xy^2)$$

```
> restart;
> F:=(u,v)->cos(u*v);  u:=(x,y)->2*x+3*y;v:=(x,y)->x*y;
```

F := (u, v)/ cos(u v)

u := (x, y)/ 2 x C 3 y

v := (x, y)/ x y

```
> Diff(F,x)=diff(F(u,v),u)*diff(u(x,y),x)+
diff(F(u,v),v)*diff(v(x,y),x);
Diff(F,y)=diff(F(u,v),u)*diff(u(x,y),y)+
diff(F(u,v),v)*diff(v(x,y),y);
```

V/Vx F = K 2 sin(u v) v K sin(u v) u y

V/Vy F = K 3 sin(u v) v K sin(u v) u x

Umumiy holda, *m* o'zgaruvchili differensiallanuvchi

$$u = f(v_1; v_2; ...; v_m)$$

funksiyaning argumentlari o‘z navbatida

$$v_i = \varphi_i(x_1; x_2; ...; x_n) \quad i = \overline{1;m}$$

n ta x_j $\left(j = \overline{1;n}\right)$ argumentlarning (erkli o‘zgaruvchilarning) differensiallanuvchi funksiyalari bo‘lsa, bu holda

$$\frac{\partial u}{\partial x_j} = \sum_{i=1}^{m} \frac{\partial u}{\partial v_i} \cdot \frac{\partial v_i}{\partial x_j}, \quad j = \overline{1;n} \qquad (8.5.3)$$

formula o‘rinlidir.

8.5.2-misol. $u = v^2 + w^2 \qquad v = x\cos(y+z), \ w = y\cos(x+z)$

$$\frac{\partial u}{\partial x} = 2v\cos(y+z) - 2wy\sin(x+z),$$

$$\frac{\partial u}{\partial y} = -2vx\sin(y+z) + 2w\cos(x+z),$$

$$\frac{\partial u}{\partial z} = -2vx\sin(y+z) - 2w\sin(x+z),$$

Endi, oraliq argumentlar bitta o‘zgaruvchining funksiyalari bo‘lgan holni qaraylik, ya’ni

$$v_i = \varphi_i(t) \quad i = \overline{1;m}$$

bo‘lsin, u holda $u = f(v_1; v_2; ...; v_m)$ bir o‘zgaruvchili ya’ni faqat t ning funksiyasi bo‘lib qoladi va uning hosilasi uchun (8.5.3) dan

$$\frac{du}{dt} = \sum_{i=1}^{m} \frac{\partial u}{\partial v_i} \cdot \frac{dv_i}{dt} \qquad (8.5.4)$$

ni olamiz. Bu *to‘liq hosila formulasi* deb yuritiladi.

8.5.3-misol.. $z = x^2 + xy + y^2$ da $x = t^2, \ y = t^3$ bo‘lsa, t bo‘yich to‘la hosilani toping.

1) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt} = (2x+y)2t + (x+2y)3t = 4t^3 + 5t^4 + 6t^5$

2) $z = t^4 + t^5 + t^6$ bo‘lsa, $\dfrac{dz}{dt} = 4t^3 + 5t^4 + 6t^5$

```
> restart;
> u:=(x,y)->x^2+x*y+y^2; x:=t->t^2;y:=t->t^3;
```

$$u := (x, y) \to x^2 + x\,y + y^2 \quad x := t \to t^2 \quad y := t \to t^3$$

```
> Diff(u(x,y),x)=diff(u(x,y),x);
Diff(x(t),t)=diff(x(t),t);Diff(u(x,y),y)=diff(u(x,y),y);
Diff(y(t),t)=diff(y(t),t);
```

$$\frac{\partial}{\partial x}(x^2 + x\,y + y^2) = 2\,x + y \qquad \frac{d}{dt}(t^2) = 2\,t$$

$$\frac{\partial}{\partial y}(x^2 + x\,y + y^2) = x + 2\,y \qquad \frac{d}{dt}(t^3) = 3\,t^2$$

```
> dut:=Diff(u,t)=diff(u(x,y),x)*diff(x(t),t)+
diff(u(x,y),y)*diff(y(t),t);
```

$$dut := \frac{\partial}{\partial t}\,u = 2\,(2\,x + y)\,t + 3\,(x + 2\,y)\,t^2$$

```
> eval(dut,[x=t^2,y=t^3]);
```

$$\frac{\partial}{\partial t}u = 2\,(2\,t^2 + t^3)\,t + 3\,(t^2 + 2\,t^3)\,t^2$$

Xususiy holda, erkli o‘zgaruvchi t sifatida oraliq o‘zgaruvchilardan birortasi olingan bo‘lsa, masalan, $u = f(x; v_2; v_3; ...; v_m)$ funksiyada $v_i = \varphi_i(x)$ $i = \overline{2;m}$ bo‘lsa,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \sum_{i=2}^{m} \frac{\partial u}{\partial v_i} \cdot \frac{dv_i}{dx} \qquad (8.5.5)$$

formulani olamiz.

$z = f(x; y)$ ikki o‘zgaruvchili funksiyada $y = \varphi(x)$ bo‘lsa,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad (8.5.6)$$

formulaga ega bo‘lamiz.

8.5.4-misol. $z = \arcsin\frac{x}{y}$ da $y = \sqrt{1+x^2}$ bo‘lsa, x bo‘yich $\frac{\partial z}{\partial x}$ va $\frac{dz}{dx}$ to‘la hosilani toping.

1) $$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{x}{y}\right)'_x = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$

2) $$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} = \frac{1}{\sqrt{y^2-x^2}} + \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right)\frac{1}{2\sqrt{1+x^2}}2x =$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{1}{y\sqrt{y^2-x^2}}\frac{x^2}{\sqrt{1+x^2}}$$

```
> restart;
> u:=(x,y)->arctan(x/y);y:=x->sqrt(1+x^2);
```

$$u := (x, y) \to \arctan\left(\frac{x}{y}\right) \qquad y := x \to \sqrt{1 + x^2}$$

```
> Diff(u, x)=diff(u(x,y),x); Diff(u, y)=diff(u(x,y),y);
Diff(y,x)=diff(y(x),x);
```

$$\frac{\partial}{\partial x}u = \frac{1}{y\left(1 + \frac{x^2}{y^2}\right)} \qquad \frac{\partial}{\partial y}u = -\frac{x}{y^2\left(1 + \frac{x^2}{y^2}\right)}$$

$$\frac{\partial}{\partial x}y = \frac{x}{\sqrt{1 + x^2}}$$

```
> dux:=Diff(u,x)=diff(u(x,y),x)+diff(u(x,y),y)*diff(y(x),x);
```

$$dux := \frac{\partial}{\partial x}u = \frac{1}{y\left(1 + \frac{x^2}{y^2}\right)} - \frac{x^2}{y^2\left(1 + \frac{x^2}{y^2}\right)\sqrt{1 + x^2}}$$

$y = \sqrt{1+x^2}$, $y^2 - x^2 = 1$ ekanligini etiborga olsak,

$$\frac{dz}{dx} = 1 - \frac{x^2}{\sqrt{1+x^2}\sqrt{1+x^2}} = 1 - \frac{x^2}{1+x^2} = \frac{1}{1+x^2}$$

```
> ux:=eval(dux,[y=sqrt(1+x^2)]);
```

$$ux := \frac{\partial}{\partial x} u = \frac{1}{\sqrt{1+x^2}\left(1+\frac{x^2}{1+x^2}\right)} - \frac{x^2}{(1+x^2)^{(3/2)}\left(1+\frac{x^2}{1+x^2}\right)}$$

8.6. Funksiyaning berilgan yo‘nalish bo‘yicha hosilasi va gradienti

Bu yerda ham soddalik uchun uch o‘lchovli D sohada anqlangan $u = f(x;y;z)$ uch o‘zgaruvchili funksiyani qaraymiz. Shu bilan birga qaralayotgan uch o‘lchovli fazoda biror $\vec{l}$ yo‘nalish berilgan deb faraz qilamiz. Agar D sohaning tayinlangan $M_0(x_0;y_0;z_0)$ nuqtasini $\vec{l}$ yo‘nalish bo‘yicha $M(x;y;z)$ nuqtaga qo‘zg‘atsak va bunda M_0M kesma D sohada to‘lig‘icha yotadi deb faraz qilsak, shu yo‘nalish bo‘yicha funksiya orttirmasiga ega bo‘lamiz. Endi M nuqta M_0 ga $\vec{l}$ yo‘nalishni saqlagan holda cheksiz yaqinlasha borsin. Agar

$$\lim_{\begin{cases} M \to M_0 \\ \overrightarrow{M_0M} \uparrow\uparrow \vec{l} \end{cases}} \frac{f(M) - f(M_0)}{\rho(M_0;M)}$$

limit mavjud bo‘lsa, bu limit $f(M)$ ning M_0 nuqtadagi $\vec{l}$ yo‘nalish bo‘yicha hosilasi deb ataladi va

$$\frac{\partial f(M_0)}{\partial l} \text{ yoki } \frac{\partial f(x_0;y_0;z_0)}{\partial l}$$

kabi belgilanadi. Bu yo‘nalish bo‘yicha hosila berilgan yo‘nalish bo‘yicha funksiyaning «o‘zgarish tezligi» dan iborat ekanligini ayqash qiyin emasdir.

Bu o‘rinda $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ xususiy hosilalarni mos ravishda Ox, Oy, Oz o‘qlarining yo‘nalishi bo‘yicha hosilalar ekanligini aytamiz.

Endi $\vec{l}$ yo‘nalish koordinatalar o‘qlari bilan

$$\alpha = (\vec{l};\hat{}\, Ox),\ \beta = (\vec{l};\hat{}\, Oy),\ \gamma = (\vec{l};\hat{}\, Oz)$$

burchaklar tashkil qiladi deylik (8.6.1-rasm), hamda $u = f(x;y;z)$ funksiya M_0 nuqtada barcha argumentlari bo‘yicha xususiy hosilaga ega bo‘lsin. U holda

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma \qquad (8.6.1)$$

o‘rinlidir.

$z = f(x,y)$ bo‘lganda $\alpha = (\vec{l};\hat{}\, Ox)$ burchak ostidagi $\vec{l} = \{\cos\alpha,\ \sin\alpha\}$ yo‘nalish bo‘yicha xosila:

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha \qquad (8.6.2)$$

bo‘ladi.

8.6.1-misol.. $Z=2x^2-3y^2$ funktsiyning M(1;0) huqtadagi Ox o‘qi bilan 120^o burchak hosil qiluvchi yo‘nalish bo‘yich hosilani toping.

Yechish. M(1;0) huqtadagi funktsiyni xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = 4x, \left(\frac{\partial u}{\partial x}\right)_M = 4\cdot 1 = 4, \quad \frac{\partial u}{\partial y} = -6y, \left(\frac{\partial u}{\partial y}\right)_M = -6\cdot 0 = 0$$

$$\cos\alpha = \cos 120^0 = -\frac{1}{2}, \quad \sin\alpha = \sin 120^0 = \frac{\sqrt{3}}{2},$$

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha = 4\left(-\frac{1}{2}\right) + 0\cdot\frac{\sqrt{3}}{2} = -2$$

Bundan, hosilaning manfiy ligidan bu yo‘nalish yo‘nalish funktsiyaning kamayuvch ekanligi lelib chiqadi.

8.6.2-misol. Z=x^2+y^2 funktsiyning M(1;2) huqtadagi $\vec{l} = \{3,\ 4\}$ yo‘nalish bo‘yich hosilani toping.

Yechish. M(1;2) huqtadagi funktsiyni xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = 2x, \left(\frac{\partial u}{\partial x}\right)_M = 2\cdot 1 = 2, \quad \frac{\partial u}{\partial y} = 2y, \left(\frac{\partial u}{\partial y}\right)_M = 2\cdot 2 = 4$$

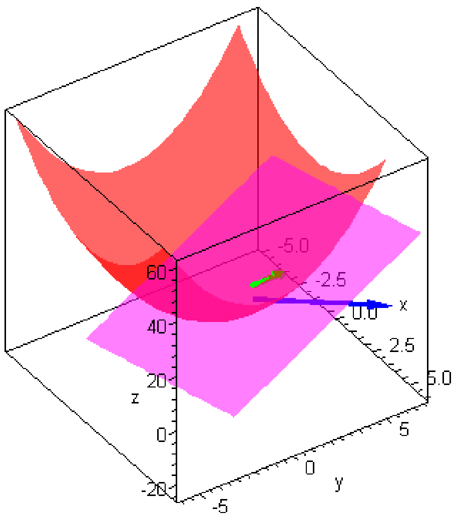
$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\cos\beta =$$

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\frac{l_1}{\sqrt{(l_1)^2 + (l_2)^2}} + \frac{\partial z}{\partial y}\frac{l_2}{\sqrt{(l_1)^2 + (l_2)^2}} == \frac{2\cdot 3}{\sqrt{3^2 + 4^2}} + \frac{4\cdot 4}{\sqrt{3^2 + 4^2}} = \frac{22}{5}$$

```
> with(Student[MultivariateCalculus]):
> DirectionalDerivative(x^2+y^2,[x,y]=[1,2],[3,4]);
```

$$\frac{22}{5}$$

```
> DirectionalDerivativeTutor(x^2+y^2, [x,y] = [1,2], [2,3], x=-
5..5, y=-6..6, z=0..6);
```

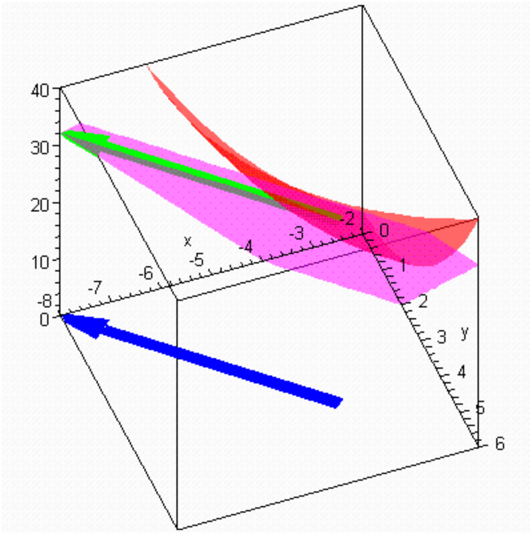


8.6.3-misol.. Z=x^2+y^2 funktsiyning M(-4;4) huqtadagi $\vec{l} = \{-6\ ,-6\}$ yo‘nalish bo‘yich hosilani toping.

```
> with(Student[MultivariateCalculus]):
> DirectionalDerivative(x^2+y^2,[x,y]=[-4,-4],[-6,-6]);
```

$$8\sqrt{2}$$

```
> DirectionalDerivative(x^2+y^2, [x,y]=[-4,4], [-6,-6],
x=-8..-2, y=0..6, z=0..40, output = plot);
```




```
> restart;
> Yo'nalish bo'yich hosila
> with(VectorCalculus):
> DirectionalDiff( f(x,y,z), <a,b,c>, [x,y,z] );
```

$$\frac{\left(\frac{\partial}{\partial x} f(x, y, z)\right) a + \left(\frac{\partial}{\partial y} f(x, y, z)\right) b + \left(\frac{\partial}{\partial z} f(x, y, z)\right) c}{\sqrt{a^2 + b^2 + c^2}}$$

8.6.4-misol.. $u=x^2+y^2+z^2$ funktsiyning M(1;1;1) huqtadagi $\vec{l}=\{2,1,3\}$ yo‘nalish bo‘yich hosilani toping.

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\frac{a}{\sqrt{a^2+b^2+c^2}} + \frac{\partial u}{\partial y}\frac{b}{\sqrt{a^2+b^2+c^2}} + \frac{\partial u}{\partial z}\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Formulaga asosan: x=1, y=1, z=1, a=2, b=1,c=3

$$\frac{\partial u}{\partial x} = 2x, \left(\frac{\partial u}{\partial x}\right)_M = 2, \quad \frac{\partial u}{\partial y} = 2y, \left(\frac{\partial u}{\partial y}\right)_M = 2, \quad \frac{\partial u}{\partial z} = 2z, \left(\frac{\partial u}{\partial z}\right)_M = 2,,$$

$$l_1 = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{14}}, \quad l_2 = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{14}}, \quad l_3 = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{14}}$$

$$\frac{\partial u}{\partial l} = \frac{12}{\sqrt{14}}$$

```
>restart;
>u:=(x,y,z)->x^2+y^2+z^2;
```

$$u := (x, y, z) \to x^2 + y^2 + z^2$$

```
> with(VectorCalculus):with(Student[MultivariateCalculus]):
> DirectionalDiff( f(x,y,z), <a,b,c>, [x,y,z] );
```

$$\frac{\left(\frac{\partial}{\partial x} f(x, y, z)\right) a + \left(\frac{\partial}{\partial y} f(x, y, z)\right) b + \left(\frac{\partial}{\partial z} f(x, y, z)\right) c}{\sqrt{a^2 + b^2 + c^2}}$$

```
> DirectionalDiff( u(x,y,z), <a,b,c>, [x,y,z] );
```

$$\frac{2\,x\,a + 2\,y\,b + 2\,z\,c}{\sqrt{a^2 + b^2 + c^2}}$$

```
> dl:=DirectionalDiff(u(x,y,z),<2,1,3>,[x,y,z]);
```

$$dl := \frac{1}{14}\sqrt{14}\,(4\,x + 2\,y + 6\,z)$$

```
> x:=1;y:=1;z:=1;dl :=1/14*sqrt(14)*(4*x+2*y+6*z);
```

$$x := 1 \quad y := 1 \quad z := 1 \quad dl := \frac{6}{7}\sqrt{14}$$

Endi yo‘nalish bo‘yicha hosila funksiyaning shu berilgan yo‘nalish bo‘yicha «o‘zgarish tezligi» ekanligini e’tiborga olgan holda berilgan nuqtada funksiya qaysi yo‘nalish bo‘yicha eng tez o‘sadi, degan savolni qo‘yaylik. Bu savol qaralayotgan nuqtada funksiyaning barcha xususiy hosilalari nolga teng bo‘lgan hol uchun o‘z ma’nosini yo‘qotadi. Chunki bunday nuqtada istalgan yo‘nalish bo‘yicha hosilalar nolga tengdir. Demak, xususiy hosilalardan aqalli bittasi noldan farqli bo‘lgan holni qarash lozim bo‘ladi.

$$\frac{\partial f(x_0;y_0;z_0)}{\partial x} = a, \quad \frac{\partial f(x_0;y_0;z_0)}{\partial y} = b, \quad \frac{\partial f(x_0;y_0;z_0)}{\partial z} = c$$

belgilashlarini kiritsak, $|a|+|b|+|c|>0$ shart bajarilishini talab qilamiz.

$\vec{N}(a;b;c)$ vektorni kiritaylik. Agar $\vec{l}$ yo‘nalishni birlik vektor deb qabo‘l qilsak, u holda (8.6.1) ni

$$\frac{\partial f(x_0;y_0;z_0)}{\partial l} = \vec{N}\cdot\vec{l}$$

ko‘rinishda yozish mumkinligi va skalyar ko‘aytma xossasiga asosan oxirgidan

$$\frac{\partial f(x_0;y_0;z_0)}{\partial l} = \Pi p_{\vec{l}}\,\vec{N} \qquad (8.6.3)$$

o‘rinli ekanligini olamiz. (8.6.3) dan agar $\vec{l} \uparrow\uparrow \vec{N}$ bo‘lsa, uning o‘ng tomonidagi roeksiya eng katta qiymatga erishishi kelib chiqadi. Demak, funksiyaning $\vec{N}(a;b;c)$ yo‘nalish bo‘yicha hosilasi eng kata bo‘lib, bu yo‘nalish bo‘yicha funksiya eng tez o‘sib borar ekan. Kiritilgan $\vec{N}(a;b;c)$ vektorni funksiyaning qaralayotgan nuqtadagi *gradienti* deb qabo‘l qilinadi.

Funksiyaning berilgan nuqtadagi gradienti deb, koordinatalari shu nuqtadagi funksiyaning mos xususiy hosilalaridan iborat vektorga aytiladi va $grad\ f(x_0;y_0;z_0)$ *kabi belgilanadi.*

Demak,

$$grad\ f(x_0;y_0;z_0) = \vec{N}\left(\frac{\partial f(x_0;y_0;z_0)}{\partial x};\ \frac{\partial f(x_0;y_0;z_0)}{\partial y};\ \frac{\partial f(x_0;y_0;z_0)}{\partial z}\right)$$

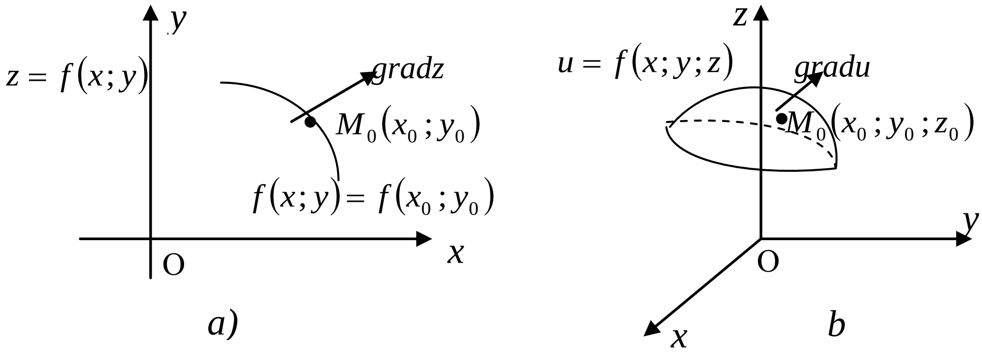
bo‘lib, bu vektor berilgan nuqtada funksiyaning eng tez o‘sish yo‘nalishidan iboratdir. Bu o‘rinda funksiya gradientiga qarama-qarshi yo‘nalish bo‘yicha hosila uning eng tez kamayish yo‘nalishi bo‘lishini ham aytamiz.

Funksiya gradientini koordinalar o‘qlarining ortlari bo‘yicha

$$grad u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}$$

ko‘rinishda yoziladi.

Uch o‘zgaruvchili funksiyaning qaralayotgan M_0 nuqtadagi gradienti (u mavjud va noldan farqli degan faraz asosida) bu funksiyaning M_0 nuqtaga mos satx sirtiga shu nuqtada normal vektordan iborat bo‘lishi, ya’ni sirtga tik yo‘nalgan bo‘lishi isbotlangandir ($8.6.2_b$-rasm). Ikki o‘zgaouvchili funksiya uchun uning gradienti satx chizig‘iga tikdir ($8.6.2_a$-rasm).



8.6.2-rasm

8.6.5-misol.. $u=x^2+y^2+z^2$ funktsiyning M(1;1;1) huqtadagi gradientini toping.

1) huqtadagi gradient: $gradu = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

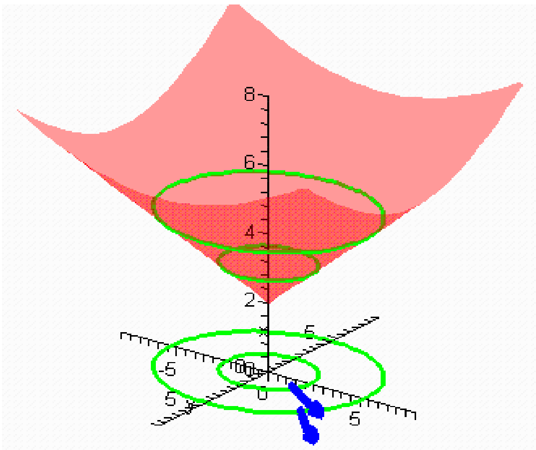
2) gradient yo‘nalishi bo‘yicha hosila:

$(gradu)_M = 2\vec{i} + 2\vec{j} + 2\vec{k}$, $\frac{\partial u}{\partial s} = |gradu|_M = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$

```
> restart;
> with(Student[MultivariateCalculus]):
> Gradient(x^2+y^2+z^2,[x,y,z]=[1,1,1]);
```
$[2 \quad 2 \quad 2]$

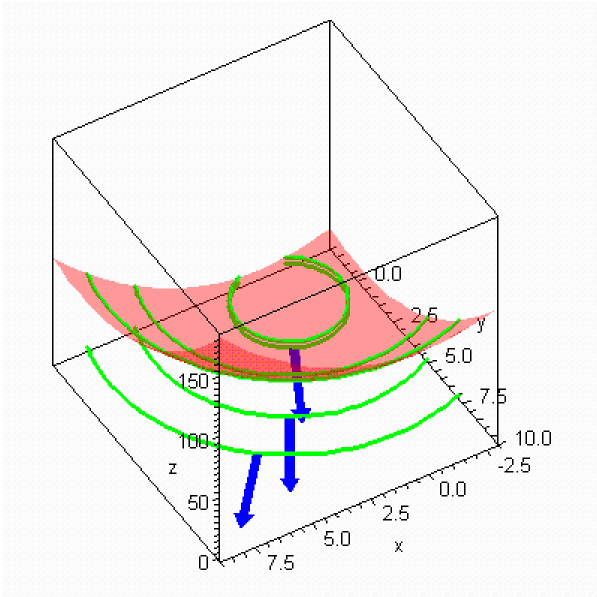
8.6.6-misol.. $x^2+y^2=(z-2)^2$ konuaning $M_1(1;2)$, $M_2(3;4)$ huqtalardagi gradientlarini toping.

```
> with(Student[MultivariateCalculus]):
> Gradient(sqrt(x^2/3+y^2/4)+2,[x,y]=[[1,2],[3,4]],
x=-8..8, y=-8..8, z=0..8, output=plot);
```



8.6.7-misol. $x^2+y^2=z$ paraboloidning $M_1(1;2)$, $M_2(3;5)$, $M_2(5;6)$ huqtalardagi gradientlarini toping.

```
> GradientTutor(x^2+y^2, [x,y] = [[1,2],[3,5],[5,6]]);
```



8.7. Egri chiziqqa urinma to‘g‘ri chiziq va normal tekislik

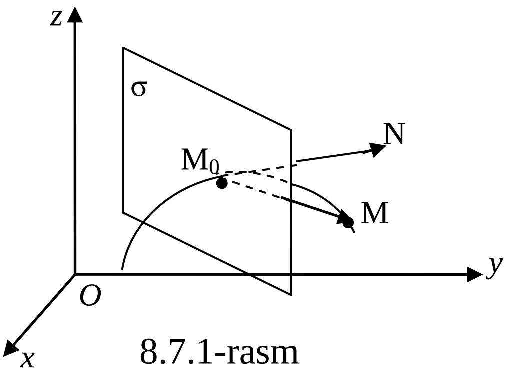
Aytaylik, fazoda egri chiziq o‘zining parametrik

$$x = x(t), y = y(t), z = z(t), t \in [\alpha, \beta] \qquad (8.7.1)$$

tenglamalari bilan berilgan bo‘lsin. arametrning $t_0 \in (\alpha, \beta)$ qiymatiga egri chiziqning $M_0(x_0, y_0, z_0)$ nuqtasi mos kelsin. Agar arametrga Δt orttirma bersak, bu nuqta egri chiziqning qandaydir $M(x_0+\Delta x; y_0+\Delta y; z_0+\Delta z)$ nuqtasiga qo‘zg‘aladi, bu yerda $\Delta x, \Delta y, \Delta z$ (8.7.1) parametrik tenglamalarga mos funksiyalarning orttirmalaridir. M_0M kesuvchini qarasak, bu to‘g‘ri chiziqning yo‘nalishi deb, $\overrightarrow{M_0M}$ vektorni olish mumkin (8.7.1-rasmga qarang).Bu vektorning koordinatalari $\overrightarrow{M_0M} = \{\Delta x, \Delta y, \Delta z\}$ bo‘lishi ravshandir. M_0M kesuvchining yo‘nalishi deb

$$\vec{m} = \left\{\frac{\Delta x}{\Delta t}; \frac{\Delta y}{\Delta t}; \frac{\Delta z}{\Delta t}\right\} \qquad (8.7.2)$$

ni qabo‘l qilish ham mumkin, chunki $\vec{m} \parallel \overrightarrow{M_0M}$ dir.



8.7.1-rasm

Agar (8.9.1) parametrik tenglamalar t_o nuqtada differensiallanuvchi va

$$|x'(t_0)| + |y'(t_0)| + |z'(t_0)| > 0$$

deb faraz qilinsa,

$\Delta t \to 0 \Rightarrow (\Delta x \to 0, \Delta y \to 0, \Delta z \to 0) \Rightarrow M \to M_0$ va

buning natijasida M_0M kesuvchi M_0N limitik holatga intlishi kelib chiqadi. Ana shu M_0N to‘g‘ri chiziq egri chiziqqa M_o nuqtada o‘tkazilgan urinma deb qabo‘l qilinadi. Bu urinmaning yo‘nalishining koordinatalarini $\vec{S}_0 = \{x'(t_0); y'(t_0); z'(t_0)\}$ deb qabo‘l qilish mumkin, chunki (8.7.2) da $\Delta t \to 0$ dagi limitga o‘tilsa,

$$\vec{S}_0 = \lim_{\Delta t \to 0} \vec{m} = \{x'(t_0), y'(t_0), z'(t_0)\}$$

bo‘ladi. Albatta, M_0N urinma yo‘nalishini $\vec{S}_0$ ga kollinear bo‘lgan ixtiyoriy vektor bilan almashtirish ham mumkin,masalan,

$$\vec{d} = \vec{S}_0 dt = \{dx, dy, dz\}$$

bilan. Endi urinma tenglamasini yozish qiyin emas

$$\frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} = \frac{z-z_0}{z'(t_0)} \quad \text{yoki} \quad \frac{x-x_0}{dx} = \frac{y-y_0}{dy} = \frac{z-z_0}{dz}$$

Agar yuqorida aytilgan ma’noda egri chiziqqa M_o nuqtasidagi urinma mavjud bo‘lsa, bu urinmaga urinish nuqtasi orqali erendikulyar qilib o‘tkazilgan tekislik egri chiziqqa M_o nuqtasidagi normal tekislik deb ataladi (8.9.1-rasmda σ tekislik).

Normal tekislik tenglamasi

$$x'(t_0)(x-x_0) + y'(t_0)(y-y_0) + z'(t_0)(z-z_0) = 0$$

bo‘lishini ko‘rish qiyin emas.

8.7.1-misol.. Uch o‘lchovli fazoda $x=t, y=t^2, z=t^3$ tenglamalar bilan berilgan egri chiziqning $t=1$ bo‘lgandagi urinma to‘g‘ri chiziq va normal tekislik tenglamalarini toping.

Yechish. $x=t, y=t^2, z=t^3$ da $t=t_0=1$ uchun: $x_0=1, y_0=1, z_0=1$

larning xusysiy hosilalari:

$$m_1 = x_t' = x'(t_0) = 1, \quad m_2 = y_t' = y'(t_0) = 2*1 = 2, \quad m_3 = z_t' = z'(t_0) = 3*1^2 = 3$$

urinma to‘g‘ri chiziq: $\dfrac{x-x_0}{m_1} = \dfrac{y-y_0}{m_2} = \dfrac{z-z_0}{m_3}, \qquad \dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-1}{3}$

$$x = 1+t, \; y = 1+2t, \; z = 1+3t$$

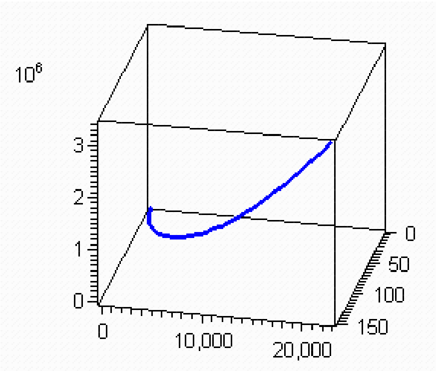
normal tekislik: $m_1 \cdot (x-x_0) + m_2 \cdot (y - í_0) + m_3 \cdot (z-z_0) = 0$

$$1\cdot(x-1) + 2\cdot(y-1) + 3\cdot(z-1) = 0 \quad \text{yoki} \quad x+2y+3z-6=0$$

```
> with(VectorCalculus):
> TangentLine( <t,t^2,t^3>, t=1 );
```

$$(1 C\ t)e_x C\ (1 C\ 2t)e_y C\ (1 C\ 3t)e_z$$

```
>with(plots):
spacecurve([t,t^2,t^3],t=0..8*Pi,thickness=2,color=BLUE);
```



8.8. Sirtning urinma tekisligi va normali

Bu yerda o‘zining $F(x,y,z)=0$ ko‘rnishdagi tenglamasi bilan berilgan sirtni qaraymiz va uning $M_0(x_0;y_0;z_0)$ nuqtasida (ya’ni $F(x_0;y_0;z_0)=0$) $F'_x(x_0;y_0;z_0)$, $F'_y(x_0;y_0;z_0)$, $F'_z(x_0;y_0;z_0)$ xususiy hosilalar mavjud hamda ulardan aqalli bittasi noldan farqli deb faraz qilamiz.

Sirtning $M_0(x_0;y_0;z_0)$ nuqtasidagi urinma tekislik tenglamasi:

$$F'_x(x_0;y_0;z_0)(x-x_0)+F'_y(x_0;y_0;z_0)(y-y_0)+F'_z(x_0;y_0;z_0)(z-z_0)=0 \qquad (8.8.2)$$

ko‘rinishda bo‘ladi.

Xususiy holda, sirt $z=f(x,y)$ tenglama orqali berilgan bo‘lsa, uning tenglamasini $z-f(x,y)=0$ ko‘rinishda yozsak, (8.8.2) urinma tekislik tenglamasini

$$z-z_0=f'_x(x_0;y_0)+f'_y(x_0;y_0)(y-y_0)$$

ko‘rinishda olish mumkin bo‘lib, bu yerda $z_0=f(x_0,y_0)$ dir.

Yuqoridagi M_0 nuqtadan o‘tub urinma tekislikka perpendikulyar to‘g‘ri chiziqni - sirtning M_0 nuqtasidagi *normali* deyiladi.

Normal tenglamasi.

$$\frac{x-x_0}{F'_x(x_0;y_0;z_0)}=\frac{y-y_0}{F'_y(x_0;y_0;z_0)}=\frac{z-z_0}{F'_z(x_0;y_0;z_0)}$$

ekanligiga ishonch hosil qilish qiyin emasdir.

8.8.1-misol.. $Z=x^2+y^2$ sirtning M(2;1;5) nuqtasidagi urinma tekislik va normal to‘g‘ri chiziq tenglamalarini toping.

```
> restart;
> f:=(x,y)->x^2+y^2;
```

$f:=(x,y)\to x^2+y^2$

Birinch tartibli xususiy hosilalar :

```
> fx:=diff(f(x,y),x);fy:=diff(f(x,y),y);
```

$fx:=2x \quad fy:=2y$

```
> x:=2;y:=1;z:=5; A:=fx;B:=fy;
```

$x:=2 \quad y:=1 \quad z:=5 \quad A:=4 \quad B:=2$

Nuqtasidagi urinma tekislik tenglamasi:

```
> a:=2;b:=1;c:=5; Z=c+A*(X-a)+B*(Y-b);
```

$a:=2 \quad b:=1 \quad c:=5 \quad Z=-5+4X+2Y$

Nuqtasidagi to‘g‘ri chiziq tenglamasi:

```
> (X-a)/A,"=",(Y-b)/B,"=",(Z-c)/1;
```

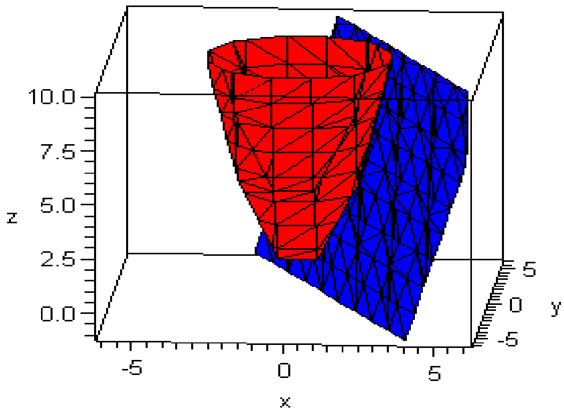
$\frac{1}{4}X-\frac{1}{2}, "=", \frac{1}{2}Y-\frac{1}{2}, "=", Z-5$

```
> X:=a+A*t; Y:=b+B*t; Z:=c+1*t;
```

$X:=2+4t \quad Y:=1+2t \quad Z:=5+t$

Sirt va urinma grafigi:

```
> restart;
> with(plots):
> implicitplot3d([x^2+y^2=z,-5+4*x+2*y=z],x=-6..6,y=-6..6,
z=-1..10,color=[red,blue], scaling=constrained,axes=BOXED);
```



8.8.2-misol. **xyz-z³=1** sirtning M(*0;1;-1*) nuqtasidagi urinma tekislik va normal to'g'ri chiziq tenglamalarini toping.

```
> restart;
>f:=(x,y,z)->3*x*y*z-z^3-1;
```

$f := (x, y, z) \rightarrow 3\,x\,y\,z - z^3 - 1$

Birinch tartibli xususiy hosilalar :

```
> fx:=diff(f(x,y,z),x);fy:=diff(f(x,y,z),y);
fz:=diff(f(x,y,z),z);
```

$fx := 3\,y\,z \quad fy := 3\,x\,z \quad fz := 3\,x\,y - 3\,z^2$

```
> x:=0;y:=1;z:=-1; A:=fx;B:=fy;C:=fz;
```

$x := 0 \quad y := 1 \quad z := -1 \quad A := -3 \quad B := 0 \quad C := -3$

Nuqtasidagi urinma tekislik tenglamasi:

```
> x0:=0;y0:=1;z0:=-1; A*(X-x0)+B*(Y-y0)+C*(Z-z0)=0;
```

$x0 := 0 \quad y0 := 1 \quad z0 := -1 \quad -3\,X - 3 - 3\,Z = 0$

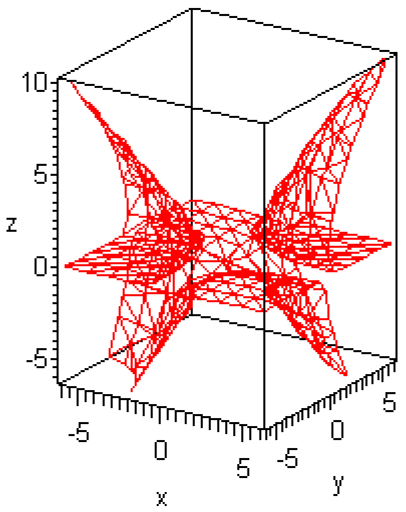
Nuqtasidagi to'g'ri chiziq tenglamasi:

```
> (X-x0)/A,"=",(Y-y0)/B,"=",(Z-z0)/C;
Error, numeric exception: division by zero
> X:=x0+A*t; Y:=y0+B*t; Z:=z0+C*t;
```

$X := -3\,t \quad Y := 1 \quad Z := -1 - 3\,t$

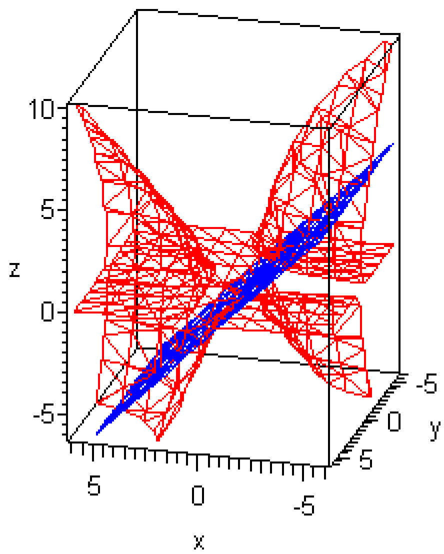
Sirt grafigi:

```
> restart;
> with(plots):
> implicitplot3d([3*x*y*z-z^3=1],x=-6..6,y=-6..6,
z=-6..10,color=red, scaling=constrained,axes=BOXED);
```



Sirt va urinma grafigi:

```
> implicitplot3d([3*x*y*z-z^3=1,x+z=-1],x=-6..6,y=-6..6,
z=-6..10,color=[red,blue],  scaling=constrained,axes=BOXED);
```



8.9. Ko‘p o‘zgaruvchili funksiya birinchi tartibli to‘liq differensiali shaklining invariantligi

Aytaylik, $u = f(x; y; z)$ funksiya uzluksiz xususiy hosilalarga ega va $x; y; z$ lar o‘z navbatida t va v o‘zgaruvchilarning funksiyalari

$$x = \varphi(t; v), \quad y = \psi(t; v), \quad z = \chi(t; v)$$

bo‘lib, $\varphi; \psi; \chi$ funksiyalar ham uzluksiz xususiy hosilalarga ega bo‘lsin. Bu holda u murakkab funksiyaning t va v bo‘yicha xususiy hosilalari mavjud va ular uzluksiz bo‘ladi.

Agar $x; y; z$ erkli o‘zgaruvchilar bo‘lsa, u funksiyaning to‘liq differensiali

$$du = u'_x dx + u'_y dy + u'_z dz \qquad (8.9.1)$$

bo‘lar edi. Qaralayotgan holda esa u funksiya $x; y; z$ lar vositasidagi t va v ga bog‘liqdir. Demak, bu o‘zgaruvchilarga nisbatan

$$du = u'_t dt + u'_v dv \qquad (8.9.2)$$

bo‘lishi kerak. Murakkab ko‘ o‘zgaruvchili funksiya xususiy hosilalarining formulasiga binoan

$$u'_t = u'_x x'_t + u'_y y'_t + u'_z z'_t ,$$
$$u'_v = u'_x x'_v + u'_y y'_v + u'_z z'_v$$

larga ega bo‘lamiz. Bo‘larni (8.9.2) ga qo‘yib,

$$du = \left(u'_x x'_t + u'_y y'_t + u'_z z'_t\right)dt + \left(u'_x x'_v + u'_y y'_v + u'_z z'_v\right)dv =$$
$$= u'_x\left(x'_t dt + x'_v dv\right) + u'_y\left(y'_t dt + y'_v dv\right) + u'_z\left(z'_t dt + z'_v dv\right)$$

ni olamiz. Qavslar ichidagi ifodalar t va v ning funksiyalari deb qaralgan $x;y;z$ larning to‘liq differensiallaridir, demak,

$$du = u'_x dx + u'_y dy + u'_z dz$$

ni, ya’ni (8.9.1) ko‘rinishni olamiz. Biroq bu olingan ifodada $dx;dy;dz$ lar oraliq funksiyalarning to‘liq differensiallaridan iborat bo‘lib, (8.9.1) da esa ular argumentlarining $\Delta x;\Delta y;\Delta z$ orttirmalaridan iborat edi, ya’ni ular bir xil ma’noga ega emas. Shunday qilib, bir o‘zgaruvchili funksiyadagi kabi, ko‘ o‘zgaruvchili funksiyaning ham birinchi tartibli to‘liq differensiali invariantlik xossasiga egadir.

8.10. Oshkormas funktsiyalar

Ikki *x* va *u* o‘zgaruvchini boglovchi ushbu

$$F(x,y) = 0 \qquad (8.10.1)$$

tenglamani qaraymiz. Agar x ning biror to‘lamdagi har bir qiymatiga x bilan birga (8.10.1) tenglamani qanoatlantiruvchi yagona u qiymat mos kelsa, (8.10.1) tenglama bu to‘lamda u =φ(x) oshkormas funktsiyani aniq.laydi.

Shunday qilib, x ning u oshkormas funktsiyasi x ga nisbatan echilmagan tenglama bilan aniqlanadi. Oshkormas funktsiyadan farqli ularoq x ga nisbatan echilgan y=f(x) tenglama bilan berilgan funktsiya oshkor funktsiya deb ataladi.

Masalan,

$$e^{2y}-x^2-1=0$$

tenglama barcha x>-1 lar uchun x ga nisbatan u funktsiyani oshkormas aniqlaydi. Uni u ga nisbatan echib, quyidagini hosil qlamiz:

$$y=\frac{1}{2}\ln(x^2+1)$$

Bu formula bizga *y* ni *x* ga nisbatan oshkor funktsiya sifatida beradi.

Birok, oshkormas ko‘rinishda berilgan har qanday funktsiyani ham oshkor ko‘rinishda tasvirlab bo‘lavermaydi. Masalan, ushbu

$$y - x - \frac{1}{2}\sin y = 0,$$

$$x - x^3 y - \ln y = 0$$

tenglamalar bilan berilgan funktsiyalarni y ga nisbatan echib bo‘lmaydi, to‘g’ri, ulardan birinchisini x ga nisbatan echish mumkin. Shuning uchun

$$y - x - \frac{1}{2}\sin y = 0,$$

tenglama u ga nisbatan x funktsiyani oshkormas aniqlaydi.

Bahzi hollarda F(*x*,*y*)=0 ko‘rinishdagi tenglama oshkormas funktsiyani umuman aniqlamasligi mumkin. Masalan, ushbu

$$x^2 + y^2 + R^2 = 0$$

tenglama hech kanday *x* va *y* haqiqiy sonlarda bajarilmaydi, demak, u hech qanday funktsiyani aniqlamaydi.

8.11. Oshkormas funktsiyaning hosilasi.

Oshkormas funktsiya

$$F(x, y) = 0$$

berilgan bo‘lsin va bu tenglama *y* ni biror *y=φ(x)* funktsiya sifatida aniqlasin. F(*x*,*y*) funktsiya oshkormas funktsiyaning mavjudlik teoremasi shartlarini kanoatlantirsin. Agar tenglamada u o‘rniga φ(x) funktsiyani qo‘ysak, u holda ushbu

$$F(x, \varphi(x)) = 0$$

ayniyatni hosil qilamiz.

Demak, *x* bo‘yicha F(*x*,*y*) funktsiyadan hosila ham (bu erda *y=φ(x)*) nolga teng bo‘lishi kerak. Murakkab funktsiyani differentsiallash qoidasi ((6.9) formula) bo‘yicha differentsiallab, quyidagini topamiz:

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}=0\,,\ \frac{\partial F}{\partial y}\neq 0$$

$$F_x'(x,y)+F_y'(x,y)y_x'=0$$

$$y_x'=-\frac{F_x'(x,y)}{F_y'(x,y)}\,,\qquad \frac{dy}{dx}=-\frac{F_x'(x;y)}{F_y'(x;y)} \qquad (*)$$

ni olamiz. Bu (*) tenglama vositasida berilgan bir o‘zgaruvchili oshkormas funksiyaning hosilasi formulasidir.

Olingan (*) formulaning har ikki tomonini differensiallab,

$$\frac{d^2y}{dx^2}=-\frac{\left(F''_{xx}(x;y)+F''_{xy}(x;y)\cdot\dfrac{dy}{dx}\right)F_y'(x;y)-F_x'(x;y)\left(F''_{yx}(x;y)+F''_{yy}(x;y)\cdot\dfrac{dy}{dx}\right)}{\left(F_y'(x;y)\right)^2}=$$

$$=\frac{\left(F''_{xx}(x;y)F_y'(x;y)-F''_{xy}(x;y)F_x'(x;y)\right)F_y'(x;y)-F_x'(x;y)\left(F''_{yx}(x;y)F_y'(x;y)+F''_{yy}(x;y)\cdot F_x'(x;y)\right)}{\left(F_y'(x;y)\right)^3}=$$

$$=\frac{2F_x'(x;y)F_y'(x;y)F''_{xy}(x;y)-\left(F_y'(x;y)\right)^2F''_{xx}(x;y)-\left(F_x'(x;y)\right)^2F''_{yy}(x;y)}{\left(F_y'(x;y)\right)^3}$$

va hokazo formulalarni olish mumkin.

Albatta buning uchun F dan kerakli tartibli uzluksiz xususiy hosilalar mavjudligini talab qilish kerak bo‘ladi.

Endi, $F(x_1;x_2;...;x_n;u)=0$ da $x_i\ \left(i=\overline{1;n}\right)$ lar argumentlar *u* funksiya deb faraz qilinsa, *n* o‘zgaruvchili oshkormas funksiya tenglamasiga ega bo‘lamiz. Bu yerda ham *F* funksiya barcha argumentlari bo‘yicha xususiy hosilalarga ega va $F_u'\neq 0$ deb faraz qilib, oshkormas funksiyaning xususiy hosilalari uchun

$$\frac{\partial u}{\partial x_i}=-\frac{F_{x_i}'}{F_u'}\,,\ i=\overline{1;n}$$

formulani olish qiyin emas.

8.11.1-misol. Ushbu $x^3 + y^3 = 3axy$ tenglama bilan berilgan oshkormas funktsiyaning xosilasini toping.

Yechish. $F(x,y) = x^3 + y^3 - 3axy$

funktsiya uchun bajarilishini tekshiramiz.

Funktsiya butun tekislikda anshugangan va uzluksizdir. Uning xususiy xrsilalari

$$F_x(x,y)=3x^2-3ay= 3(x^2-ay),\ F_y(x,y)=3y^2-3ax=3(y^2-ax)$$

ham butun tekislikda uzluksiz. Shuning uchun $F_y(x,y)=3(y^2-ax)\neq 0$ bo‘ladigan nuqtalarda *x* ga nisbatan ***y*** funktsiya y'_x hosilasi:

$$y'_x = -\frac{F'_x(x,y)}{F'_y(x,y)} = -\frac{x^2 - ay}{y^2 - ax}$$

```
> f := x^3+y^3-3*a*x*y;
```

$$f := x^3 C\ y^3 K\ 3\,a\,x\,y$$

```
> implicitdiff(f,y,x);
```

$$\frac{x^2 K\ a\,y}{K\ y^2 C\ a\,x}$$

Endi uchta o‘zgaruvchini boglaydigan

$$F\ (x, y, z) = 0$$

tenglamani qaraymiz. Bunda z ni biror z = φ(x,y) funktsiya sifatida aniqlaydi deb faraz qilaylik. Birinchi holda (*) formu lani keltirib chikarishda qanday yo‘l tutgan bo‘lsak, shunday yo‘l tutamiz. Tenglamada z ning urniga ($f(x,y)$) funktsiyani quyamiz va quyidagi ayniyatni xosil kilamiz:

$$F(x, y, f(x,y)) = 0.$$

Demak, F(x, y, z) funktsiyadan x va y buyicha xususiy hosilalar ham (bunda $z=f(x,y)$) nolga teng bo‘lishi kerak. Differentsiallab topamiz:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0,$$

$$F'_x(x,y,z) + F'_z(x,y,z)z'_x = 0$$

$$F'_y(x,y,z) + F'_z(x,y,z)z'_y = 0$$

$$z'_x = -\frac{F'_x(x,y,z)}{F'_z(x,y,z)},\ z'_y = -\frac{F'_y(x,y,z)}{F'_z(x,y,z)} \qquad (**)$$

8.11.2-misol. Ushbu

$$e^z - xyz = 0$$

tenglama bilan berilgan oshkormas funktsiyaning xususiy hosilalarini toping.

Yechish. $F(x, y, z) = e^z - xyz$ funktsiya hamma yerda aniqlangan va uzluksizdir. Uning xususiy hosilalari

$$F_x\ (x,y,z) = -yz,$$

$$F_y\ (x,y,z) = -xz.$$

$$F'_z\ (x,y,z) = e^z - xy$$

xamma erda uzluksiz funktsiyalar bo‘ladi. Shuning uchun $F_z\ (x,y,z) = e^z - xy \neq 0$ bo‘ladigan nuqtalarda z funktsiya (**) formula bilan xisoblaniladigan z'_x va z'_y xususiy hosilalarga ega bo‘ladi:

$$z'_x = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy},\ z'_x = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}$$

Ixtiyorny sondagi o‘zgaruvchilarning oshkormas funktsiyasi shunga O‘xshash aniqlanadi va ularning xususiy xrsilalari topiladi.

```
> f := exp(z)-x*y*z;
```

$$f := e^z K\ x\,y\,z$$

```
> Diff(z,x)=implicitdiff(f,z(x,y),x);
```

$$\frac{\vee}{\vee x} z = K\ \frac{y\,z}{K\ e^z C\ x\,y}$$

```
> Diff(z,y)=implicitdiff(f,z(x,y),y);
```

$$\frac{\vee}{\vee y} z = K\ \frac{x\,z}{K\ e^z C\ x\,y}$$

10. MUSTAQIL ISHLASH BAJARISH NAMUNASI VA TOPSHIRIQLARI

Mustaqil uy ishining har bir variantida oltita misol bo‘lib, ularning sharti quyidagicha.
1-misolda: berilgan funktsiyaning aniqlanish soxasini topish kerak.
2-misolda: berilgan funktsiyaning xususiy hosilasini va xususiy differentsialini topish kerak.
3-misolda: berilgan funktsiyaning $M_0(x_0;y_0;z_0)$ nuqtadagi $f'_x(M_0)$,$f'_y(M_0)$,$f'_z(M_0)$ xususiy hosilalarini qiymatlarini verguldan keyin ikkita raqamgacha aniqlik bilan hisoblash kerak.
4-misolda: berilgan funktsiyaning tola differentsialini topish kerak.
5-misolda: $u=u(x,y)$ (bunda $x=x\{t)$, $y=y(t)$) murakkab funktsiya hosilasining $t=t_0$ dagi qushmatini verguldan keyin ikkita raqamgacha aniqlik bilan hisoblash kerak.
6-misolda: $z(x,y)$ oshkormas ko‘rinishda berilgan funktsiya xususiy hosilasining $M_0(x_0;y_0;z_0)$ nuqtadagi qiymatini verguldan keyin ikkita raqamgacha aniqlik bilan hisoblash kerak.

1-misol. $z=\ln(x^2-3y+6)$ **funktsiyaning aniqlanish soxasini toping.**
Ychesh. Logarifm funktsiya argumenti musbat bo‘lish shartidan

$$x^2-3y+6>0,\quad y<\frac{1}{3}x^2+2$$

funktsiyaning aniqlanish soxasining parabolaning tashqarisida yotgan nuqtalardan iborat boladi.

2-misol. $z=e^{-\sqrt[3]{x^2+5y^2}}$ funktsiyaning xususiy hosilasini va xususiy differentsialini toping.

Yechish. 1) $$\frac{\partial z}{\partial x}=e^{-\sqrt[3]{x^2+5y^2}}\left(-\frac{1}{3}(x^2+5y^2)^{-\frac{2}{3}}\cdot 2x\right)=-\frac{2x}{3}e^{-\sqrt[3]{x^2+5y^2}}\frac{1}{\sqrt[3]{(x^2+5y^2)^2}}$$

$$\frac{\partial z}{\partial y}=e^{-\sqrt[3]{x^2+5y^2}}\left(-\frac{1}{3}(x^2+5y^2)^{-\frac{2}{3}}\cdot 10y\right)=-\frac{10y}{3}e^{-\sqrt[3]{x^2+5y^2}}\frac{1}{\sqrt[3]{(x^2+5y^2)^2}}$$

2) xususiy differentsial $d_x z=\frac{\partial z}{\partial x}dx,\ d_y z=\frac{\partial z}{\partial y}dy$ bu erdagi xususiy hosilasilalarning o‘rniga topilganlarni qo‘yish kifoya.

3-misol. $f(x,y,z)=\sqrt{xy}\cos z$ funktsiya xususiy hosilasilalar $M_0(1;1;\pi/3)$ nuqtadagi qiymatlari $f_x(1,1,\pi/3)$, $f_y(1,1,\pi/3)$, $f_z(1,1,\pi/3)$ ni 0.01 aniqlikda hisoblang.

Yechish. 1) $f'_x(x,y,z)=\frac{y}{\sqrt{xy}}\cos z$, $f'_x(1,1,\pi/3)=\frac{1}{\sqrt{1\cdot 1}}\cos(\pi/3)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}=0.25,$

2) $f'_y(x,y,z)=\frac{x}{\sqrt{xy}}\cos z$, $f'_y(1,1,\pi/3)=\frac{1}{\sqrt{1\cdot 1}}\cos(\pi/3)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}=0.25,$

3) $f'_z(x,y,z)=-\sqrt{xy}\sin z$, $f'_z(1,1,\pi/3)=-\sqrt{1\cdot 1}\sin(\pi/3)=-1\cdot\frac{\sqrt{3}}{2}=-0.86,$

4-misol. $z=arctg\sqrt{\frac{x}{y}}$ funktsiyaning tola differentsialini topish

Yechish. 1) $$\frac{\partial z}{\partial x}=\frac{1}{1+\frac{x}{y}}\frac{1}{2\sqrt{\frac{x}{y}}}\frac{1}{y}=\frac{y}{x+y}\frac{\sqrt{y}}{2\sqrt{x}}\frac{1}{y}=\frac{\sqrt{y}}{2(x+y)\sqrt{x}}$$

2) $\frac{\partial z}{\partial y}=\frac{1}{1+\frac{x}{y}}\frac{1}{2\sqrt{\frac{x}{y}}}(-\frac{x}{y^2})=\frac{y}{x+y}\frac{\sqrt{y}}{2\sqrt{x}}(-\frac{x}{y^2})=-\frac{\sqrt{x}}{2(x+y)\sqrt{y}}$

3) $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{\sqrt{y}}{2(x+y)\sqrt{x}}dx-\frac{\sqrt{x}}{2(x+y)\sqrt{y}}dy=\frac{ydx-xdy}{2(x+y)\sqrt{xy}}$

5-misol. $z=arcsin\frac{x^2}{y}$ bunda $x=1+lnt$, $y=-2e^{-t2+1}$, murakkab funktsiya hosilasining $t=1$ dagi qiymatini verguldan keyin ikkita raqamgacha aniqlik bilan hisoblash

Yechish. 1) $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{1}{\sqrt{1-\frac{x^4}{y^2}}}\frac{2x}{y}\frac{1}{t}-\frac{1}{\sqrt{1-\frac{x^4}{y^2}}}(-\frac{x^2}{y^2})(-2e^{-t^2+1})(-2t)$

2) $t=1$ da $x=1+ln1=1$, $y=-2e^{-1+1}=-2$ bo‘lgani uchun $\frac{dz}{dt}\Big|_{t=1}=\frac{4}{\sqrt{3}}$

6-misol. $4x^3-3y^2+2xyz-4xz=3-z^2$ oshkormas ko‘rinishda berilgan funktsiya xususiy hosilasining $M_0(0;1;-1)$ nuqtadagi qiymatini verguldan keyin ikkita raqamgacha aniqlik bilan hisoblash

Yechish. 1) $F(x,y,z)=4x^3-3y^3+2xyz-4xz+z^2-3$

$F'_x=12x^2+2yz-4z$, $F'_y=-9y^2+2xz$, $F'_z=42xy-4x+2z$

$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}=-\frac{12x^2+2yz-4z}{42xy-4x+2z}$, $\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}=-\frac{-9y^2+2xz}{42xy-4x+2z}$

2) $M_0(0;1;-1)$ da x=0,y=1,z=-1, $\frac{\partial z(M_0)}{\partial x}=1$, $\frac{\partial z(M_0)}{\partial y}=-4.5$

1-вариант

1. $z=\sqrt{x^2+y^2-5}$
2. z = tg(x^3+y^2).
3. f(x,y,z) = ln cos(x^2 y^2+z), $M_0(0;0;1)$.
3. z = cos(x^2 -y^2) + x^3.
4. u = ln(e^x+e^y), x= t^2, $y=t^3$, t_0 =1
5. $z^3+3xyz+3y=7$, $M_0(1;1;1)$.

2-вариант

1. Z = arccos(x + y).
2. $\cdot = ctg\sqrt{xy^3}$
3. /(x, y, z) = 27 $\sqrt[3]{x^2+y^2+z^2}$, M_0 (3; 4; 2).
4. z = 1п(3x^2-2 y^2).
5. $u=x^y$, $x=e'$, y = lnt, t_0 = 1.
6. $\cos^2x+\cos^2y+\cos^2z=3/2$, $M_0(\pi/4, 3\pi/4, \pi/4)$.

3-вариант

1. $z=3x+\frac{y}{2-x+y}$
2. $z=e^{-x^2+y^2}$
3. f(x,y,z) = arctg(xy^2 + z), M_0 (2;1;0).
4. z= 5xy^2 - 3x^3y^4.

4-вариант

1. $z=\sqrt{9-x^2-y^2}$.
2. z = 1п(3x^2-y^4).
3. f(x,y,z) = arcsin(x^2/y-z), $M_0(2;0;4)$.
4. z = arcsin(x +y)
5. z = x^2e^{-y}, x = sint, y = $\sin^2$t $t_o=\pi/2$

5. $z = e^{y-2x}$, $x = sint, y = t^3$, $t_0 = 0$.
6. e^{z-1} = cosx cosy + 1, $M_0(0; \pi/2;1)$.

6. $x^2 + y^2 + z^2 - 6x = 0$, $M_0(1;2;1)$.

5-вариант

1. $z = \ln(x^2 + y^3 - 3)$.
2. z = arccos (y/x).
3. f(x, *y, z)* = $\sqrt{z}$ sin(y/x) , $M_0(2;0;4)$.
4. z = arctg(2x - *y).*
5. z= 1п($e^{-x}+e^{y}$), x = t^2, $y = t^3$, t_0 = - l
6. xy = z^2 - l , $M_0(0;1;-1)$.

6-вариант

1. $z = \sqrt{2x^2 - y^2}$.
2. z=arctg(xy^2)
3. f(x, *y, z)* = $\dfrac{y}{\sqrt{x^2+z^2}}$, $M_0(-1;1;0)$
4. Z = $7x^3y - \sqrt{xy}$
5. $z = e^{y-2x}$, , $x = cost, y = \sin t$, $t_0 = \pi/2$
6. $x^2 + 2y^2 + 3z^2 - yz + y = 2$, $M_0(l;l;l)$.

7-вариант

1. $z = \dfrac{4xy}{2-3y+1}$
2. $Z = \cos\sqrt{x^2+y^2}$.
3. *f(x,y,z) = aictg*(xz/y)., $M_0(2;l;l)$.
4. $Z = y\sqrt{x^2+y^2} - 2xy$.
5. *u* = arcsin (xz/y), x =sin t, *y* = cos *t*, $t_0 = \pi$.
6. $x^2 + y^2 + z^2 + 2x$ ^ = 5, $M_0(0;2;l)$.

8-вариант

1. z = $\dfrac{\sqrt{xy}}{x^2+z^2}$.
2. z = $sin\sqrt{x - y^3}$.
3. f(x,y,z) = lnsin(x-2y +z/4), $M_0(l;1/2; \pi)$.
4. z= e^{x+y-4}
5. *u* = arccos (2x/y), x = sint, y = cos *t*, $t_0 = \pi$.
6. xcos*y* + *ycosz* + zcosx = π /2, M_0 (0; π/2 ; π).

9-вариант

1. *z* = arcsin(x/y)
2. z = tg(x^3y4).
3. f(x,y,z) = $\dfrac{y}{x} + \dfrac{z}{y} + \dfrac{x}{z}$,, $M_0(l;l;2)$.
4. *z* = cos(3x + *y)* - x^2.
5. u = $\dfrac{x^2}{1+y}$, x = 1 - 2t, y = arctgt, $t_0 = 0$.
6 $x^2y^2 + 2xyz^2 - 2x^3z + 4y^3z = 4$, M_0 (2; 1; 2).

10-вариант

1. z = $\ln(y^2 - x^2)$.
2. z= ctg(3x - 2y).
3. *f(x,y,z)*= $\dfrac{1}{\sqrt{x^2+y^2-z^2}}$, $M_0(1;2;2)$.
4. z= $\dfrac{tg(x+y)}{x-y}$
5. $u = \dfrac{x}{y}$, x = e^t, *y* = 2 -e^{2t}, $t_0 = 0$.
6. $x^2 - 2y^2 + z^2 - 4x + 2z + 2 = 0$, M_0 (1; 1; 1).

8.12. Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differensiallari

Bu yerda ham soddalik uchun funksiya ikki o‘zgaruvchili bo‘lgan holni qaraymiz. Aytaylik, $z = f(x;\, y)$ funksiya biror (ochiq) D sohada argumentlaridan birortasi bo‘yicha xususiy hosilaga ega bo‘lsa, u o‘z navbatida $x,\, y$ ning funksiyasi sifatida, biror $(x_0\,;y_0)$ nuqtada xususiy hosilalarga ega bo‘lishi mumkin. Dastlabki funksiya uchun bu xusususiy hosilalar *ikkinchi tartibli xususiy hosilalar* yoki *ikkinchi xususiy hosila* deb yuritiladi. Bu holda xususiy hosilani birinchi xususiy hosila deb qabo‘l qilamiz.

Masalan, birinchi xususiy hosila x bo‘yicha olingan bo‘lsa, uning x,y bo‘yicha olingan xususiy hosilalari ya’ni ikkinchi xususiy hosilalar quyidagicha belgilanadi:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

yoki

$$z''_{xx} = f''_{xx}(x_0\,;y_0),\; z''_{xy} = f''_{xy}(x_0\,;y_0).$$

Xuddi shunga o‘xshash, birinchi xususiy hosila y bo‘yicha olingan bo‘lsa, undan x va y bo‘yicha xususiy hosilalar olib,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

yoki

$$z''_{yx} = f''_{yx}(x_0\,;y_0),\; z''_{yy} = f''_{yy}(x_0\,;y_0).$$

Umuman, agar funksiyaning *(n-1)* – tartibli xususiy hosilalari olingan bo‘lsa, undan olingan xususiy hosilalar *n –tartibli xususiy hosilalar* deb ataladi.

8.12.1-misol. $z = x^4 y^3$ bo‘lsin, bu holda:

$$z'_x = 4x^3y^3\,,\; z'_y = 3x^4y^2\,;$$

$$z''_{xx} = 12x^2y^3\,,\; z''_{xy} = 12x^3y^2\,,\; z''_{yx} = 12x^3y^2\,,\; z''_{yy} = 6x^4y;$$

$$z'''_{xxx} = 24xy^3\,,\; z'''_{xxy} = 36x^2y^2\,,\; z'''_{xyx} = 36x^2y^2\,,\; z'''_{xyy} = 24x^3y,$$

$$z'''_{yxx} = 36x^2y^2\,,\; z'''_{yxy} = 24x^3y,\; z'''_{yyx} = 24x^3y,\; z'''_{yyy} = 6x^4\,.$$

va hokazo.

```
> restart;
> u:=(x,y)->x^4*y^3;
```

$$u := (x, y) \to x^4\, y^3$$

```
> Diff(u,x)=diff(u(x,y),x); Diff(u, y)=diff(u(x,y),y);
```

$$\frac{\partial}{\partial x} u = 4\, x^3\, y^3 \qquad \frac{\partial}{\partial y} u = 3\, x^4\, y^2$$

```
> Diff(u,x$2)=diff(u(x,y),x$2); Diff(u,y$2)=diff(u(x,y),y$2);
```

$$\frac{\partial^2}{\partial x^2} u = 12\, x^2\, y^3 \qquad \frac{\partial^2}{\partial y^2} u = 6\, x^4\, y$$

```
> Diff(u,x,y)=diff(u(x,y),x,y); Diff(u,y,x)=diff(u(x,y),y,x);
```

$$\frac{\partial^2}{\partial y\,\partial x}u = 12\,x^3\,y^2 \qquad \frac{\partial^2}{\partial x\,\partial y}u = 12\,x^3\,y^2$$

> Diff(u,x$3)=diff(u(x,y),x$3); Diff(u,y$3)=diff(u(x,y),y$3);

$$\frac{\partial^3}{\partial x^3}u = 24\,x\,y^3 \qquad \frac{\partial^3}{\partial y^3}u = 6\,x^4$$

8.12.2-misol. $z = arctg\,\frac{x}{y}$ ni olsak, oldinroq **8.5.4-misolda**

$$z'_x = \frac{y}{x^2+y^2},\quad z'_y = -\frac{x}{x^2+y^2}$$

ekanligini olgan edik. Bundan 2-tartibli xususiy hosilalarni topamiz:

$$z''_{xx} = \left(\frac{y}{x^2+y^2}\right)'_x = -\frac{y}{\left(x^2+y^2\right)^2}\cdot\left(x^2+y^2\right)'_x = -\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$z''_{xy} = \left(\frac{y}{x^2+y^2}\right)'_y = \frac{1\left(x^2+y^2\right)-y\cdot 2y}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2},$$

$$z''_{yx} = \left(-\frac{x}{x^2+y^2}\right)'_x = -\frac{1\left(x^2+y^2\right)-x\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2},$$

$$z''_{yy} = \left(-\frac{x}{x^2+y^2}\right)'_y = -\left(-\frac{x}{\left(x^2+y^2\right)^2}\cdot 2y\right) = \frac{2xy}{\left(x^2+y^2\right)^2}.$$

> restart;

> u:=(x,y)->arctan(x/y); $u := (x, y) \to \arctan\left(\frac{x}{y}\right)$

> Diff(u, x)=diff(u(x,y),x); Diff(u, y)=diff(u(x,y),y);

$$\frac{\partial}{\partial x}u = \frac{1}{y(1+\frac{x^2}{y^2})} \qquad \frac{\partial}{\partial y}u = -\frac{x}{y^2(1+\frac{x^2}{y^2})}$$

> Diff(u,x$2)=diff(u(x,y),x$2); Diff(u,y$2)=diff(u(x,y),y$2);

$$\frac{\partial^2}{\partial x^2}u = \frac{2x}{y^3(1+\frac{x^2}{y^2})^2} \qquad \frac{\partial^2}{\partial y^2}u = \frac{2x}{y^3(1+\frac{x^2}{y^2})} - \frac{2x^3}{y^5(1+\frac{x^2}{y^2})^2}$$

> Diff(u,x,y)=diff(u(x,y),x,y); Diff(u,y,x)=diff(u(x,y),y,x);

$$\frac{\partial^2}{\partial y\,\partial x}u = \frac{1}{y^2(1+\frac{x^2}{y^2})} + \frac{2x^2}{y^4(1+\frac{x^2}{y^2})^2}$$

$$\frac{\partial^2}{\partial x\partial y}u=\frac{1}{y^2(1+\frac{x^2}{y^2})}+\frac{2x^2}{y^4(1+\frac{x^2}{y^2})^2}$$

8.12.3-misol. Faraz qilaylik, x va y ning oshkormas z funksiyasi

$$\frac{x^2}{a^2}+\frac{y^2}{в^2}+\frac{z^2}{c^2}=1$$

tenglamadan aniqlansin (ellipsoid). Xususiy hosilalarini toping.

Yechish. $F'_x=\frac{2x}{a^2},\ F'_y=\frac{2y}{в^2},\ F'_z=\frac{2z}{c^2}$

$$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}=-\frac{c^2x}{a^2z},\ \frac{\partial z}{\partial x}=-\frac{c^2y}{в^2z}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(-\frac{c^2x}{a^2z}\right)=-\frac{c^2}{a^2}\left(\frac{x}{z}\right)'_x=$$

$$=-\frac{c^2}{a^2}\cdot\frac{1\cdot z-x\cdot z'_x}{z^2}=-\frac{c^2}{a^2}\cdot\frac{z+\frac{c^2}{a^2}\frac{x^2}{x}}{z^2}=$$

$$=-\frac{c^2}{a^2z^3}\left(z^2+\frac{c^2x^2}{a^2}\right)=-\frac{c^4}{a^2z^3}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}\right)=-\frac{c^4}{a^2z^3}\left(1-\frac{y^2}{в^2}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{c^2}{a^2}\left(\frac{x}{z}\right)'_y=\frac{c^2}{a^2}\cdot\frac{xz'_y}{z^2}=$$

$$\frac{c^2x}{a^2}\cdot\frac{\left(-\frac{c^2}{в^2}\frac{y}{z}\right)}{z^2}=-\frac{c^4xy}{a^2в^2z^3}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=-\frac{c^4}{в^2z^3}\left(\frac{y^2}{в^2}+\frac{z^2}{c^2}\right)=-\frac{c^4}{в^2z^3}\left(1-\frac{x^2}{a^2}\right).$$

Va hokazo, shu yo‘sinda ixtiyoriy tartibli xususiy hosilalarni topish mumkin.

8.12.4-misol. $z=f(x+at)+\varphi(x-at)$ berilgan bo‘lib, $a=const$, $f(u)$, $\varphi(u)$ lar ikkinchi tartibgacha hosilalari mavjud bo‘lgan funksiyalar bo‘lsin. U holda, z

$$\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$$

tenglamani qanoatlantirishini ko‘rsataylik.

$$\frac{\partial z}{\partial x}=f'(x+at)+\varphi'(x-at),\ \frac{\partial^2 z}{\partial x^2}=f''(x+at)+\varphi''(x-at);$$

$$\frac{\partial z}{\partial t}=f'(x+at)\cdot a+\varphi'(x-at)\cdot(-a)=a(f'(x+at)-\varphi'(x-at));$$

$$\frac{\partial^2 z}{\partial t^2}=a(f''(x+at)\cdot a-\varphi''(x-at)(-a))=a^2(f''(x+at)+\varphi''(x-at))=a^2\frac{\partial^2 z}{\partial x^2}.$$

```
> restart;
> z:=(x,y)->f(x+a*t)-hi(x-a*t);
```

$$z:=(x,y)/\ \ f(x\,C\ a\,t)\,K\ f\ (x\,K\ a\,t)$$

> Diff(z,x)=diff(z(x,y),x); Diff(z,t)=diff(z(x,y),t);

$$\frac{V}{Vx} z = (D(f))(x\,C\,a\,t)\,K\,(D(f\,))(x\,K\,a\,t)$$

$$\frac{V}{Vt} z = (D(f))(x\,C\,a\,t)\,a\,C\,(D(f\,))(x\,K\,a\,t)\,a$$

> Diff(z,x$2)=diff(z(x,y),x$2); Diff(z,t$2)=diff(z(x,y),t$2);

$$\frac{V^2}{Vx^2} z = ((D^{(2)})(f))(x\,C\,a\,t)\,K\,((D^{(2)})(f\,))(x\,K\,a\,t)$$

$$\frac{V^2}{Vt^2} z = ((D^{(2)})(f))(x\,C\,a\,t)\,a^2\,K\,((D^{(2)})(f\,))(x\,K\,a\,t)\,a^2$$

Yuqorida keltirilgan misollardan birinchisida

$$z''_{xy} = z''_{yx}\,,\quad z'''_{xxy} = z'''_{xyx} = z'''_{yxx},\quad z'''_{xyy} = z'''_{yxy} = z'''_{yyx}\,;$$

ikkinchisida esa,

$$z''_{xy} = z''_{yx}$$

ekanligini ko‘rish mumkin. Ularni *aralash xususiy hosilalar* deb ataladi. Bu misollarda bir xil tartibli aralash xususiy hosilalarni olish argumentlarining tarkibi bir xil bo‘lganda ular tengdir. Bu hol tasodifiy bo‘lmay umumiydir. Ya’ni *bir xil tartibli xususiy hosilalarni olish bo‘yicha argumentlar tarkibi bir xil bo‘lgan barcha aralash xususiy hosilalar qaralayotgan nuqtada mavjud va uzluksiz bo‘lsa, ular teng bo‘lishi isbotlangandir.*

Yuqori tartibli differensiallar tushunchasi bu yerda ham bir o‘zgaruvchili funksiyadagi kabi kiritiladi masalan, yuqori tartibli xususiy differensiallar uchun formulalar bir o‘zgaruvchili funksiyadagi bilan aynan bir xildir. Buni ikki o‘zgaruvchili $z = f(x;y)$ funksiya uchun qarasak, n – tartibli xususiy differensiallar deb

$$d_x^n z = d_x\left(d_x^{n-1} z\right),\ d_y^n z = d_y\left(d_y^{n-1} z\right)$$

qabo‘l qilinadi ($n=2;3;\ldots;\ d^1=d$). Shu sababli ularga to‘xtab o‘tirmaymiz.

n –tartibli to‘liq differensial deb, ta’rif bo‘yicha

$$d^n z = d\,\left(d^{n-1} z\right),\ \left(n = 2;3;\ldots,\ d^1 z = dz\right)$$

qabo‘l qilinadi.

Bu yerda $z = f(x;y)$ ikki o‘zgaruvchili funksiyani qaraymiz. Uni berilgan nuqtada yetarlicha tartibli uzluksiz xususiy hosilalari mavjud va $dx = \Delta x$, $dy = \Delta y$ lar o‘zgarmas deb faraz qilamiz. Endi, ikkinchi tartibli to‘liq differensial uchun formula chiqaraylik:

$$d^2 z = d(dz) = d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dy =$$

$$= \frac{\partial^2 z}{\partial x^2}dx^2 + \frac{\partial^2 z}{\partial y \partial x}dydx + \frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2\,;$$

$$d^2 z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2\,.$$

Bu ikkinchi tartibli to‘liq differensial formulasi bo‘lib, uni shartli ravishda

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z$$

ko‘rinishda yozish mumkin bo‘lib, bu yerda

$$\frac{\partial^2}{\partial x^2}z = \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2}{\partial x\partial y}z = \frac{\partial^2 z}{\partial x\partial y},\ \frac{\partial^2}{\partial y^2}z = \frac{\partial^2 z}{\partial y^2}$$

deb qabo‘l qilamiz.

Yuqoridagiga o‘xshash,

$$d^3z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^3 z$$

va nihoyat, n –tartibli to‘liq differensial uchun

$$d^nz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z$$

«simvolik» formulani olish mumkin. Bu formulada

$$\frac{\partial^n}{\partial x^m \partial y^{n-m}}z = \frac{\partial^n z}{\partial x^m \partial y^{n-m}}$$

deb qabo‘l qilish kerak bo‘ladi ($m \in Z_0$, $0 \le m \le n$).

Agar m o‘zgaruvchili $u = f(x_1; x_2; ...; x_m)$ funksiyani olsak, uning n –tartibli to‘liq differensiali uchun ham

$$d^nu = \left(\sum_{i=1}^{m}\frac{\partial}{\partial x_i}dx_i\right)^n u$$

«simvolik» formula o‘rinlidir. Bu yerda

$$\frac{\partial^n}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_m^{k_m}}u = \frac{\partial^n u}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_m^{k_m}}$$

deb qabo‘l qilamiz ($\sum_{i=1}^{m}k_i = n;\quad k_i \in Z_0,\ \ 0 \le k_i \le n$).

Eslatma. Ikkinchi va undan yuqori tartibli to‘liq differensiallar uchun umumiy holda invariantlik xossasi o‘rinli emas. Ammo, oraliq argumentlar yakuniy argumentlarning (erkli o‘zgaruvchilarning) chiziqli funksiyalari bo‘lgan holda bu invariantlik xossasi saqlanadi.

8.13. Ko‘ o‘zgaruvchili funksiya uchun Teylor formulasi

Agar bir o‘zgaruvchili $F(t)$ funksiya t_0 nuqtada $n+1$- tartibligacha uzluksiz hosilalari mavjud bo‘lsa, uning Teylor formulasi bo‘yicha yoyilmasi

$$F(t) = F(t_0) + F'(t_0)(t-t_0) + \frac{1}{2!}F''(t_0)(t-t_0)^2 + ... + \frac{1}{n!}F^{(n)}(t_0)(t-t_0)^n +$$

$$+\frac{1}{(n+1)!}F^{(n+1)}(t_0 + \theta(t-t_0))(t-t_0)^{n+1},$$

(0<θ<1) ekanligini bilamiz. Endi, $t - t_0 = \Delta t$, $F(t) - F(t_0) = \Delta F(t_0)$ faraz qilib, bu formulani

$$\Delta F(t_0)=dF(t_0)+\frac{1}{2!}d^2F(t_0)+...+\frac{1}{n!}d^nF(t_0)+\frac{1}{(n+1)!}d^{n+1}F(t_0+\theta\Delta t)$$

ko‘rinishda yozish mumkin (bu yerda $dt=\Delta t$ dir).

Ikki argumentli $f(x;y)$ funksiyani qaraylik, u tayinlangan $(x_0;y_0)$ nuqta atrofida $(n+1)$-tartibgacha uzluksiz xususiy hosilalarga ega deb faraz qilaylik. Δx va Δy orttirmalarni shunday beraylikki, $(x_0;y_0)$ va $(x_0+\Delta x;y_0+\Delta y)$ nuqtalarni tutashtiruvchi kesma $(x_0;y_0)$ nuqtaning ko‘rilayotgan atrofidan chiqmasin.

$$\Delta f(x_0;y_0)=f(x_0+\Delta x;y_0+\Delta y)-f(x_0;y_0)=$$
$$=df(x_0;y_0)+\frac{1}{2!}d^2f(x_0;y_0)+...+\frac{1}{n!}d^nf(x_0;y_0)+ \qquad (8.13.1)$$
$$+\frac{1}{(n+1)!}d^{n+1}f(x_0+\theta\Delta x;y_0+\theta\Delta y),\quad (0<\theta<1)$$

tenglikning to‘g‘ri ekanligini isbotlaylik (bu yerda $dx=\Delta x$, $dy=\Delta y$ dir).

Buning uchun yangi t o‘zgaruvchini

$$x=x_0+t\Delta x,\ y=y_0+t\Delta y \qquad (0\le t\le 1) \qquad (8.13.2)$$

tenglamalar yordamida kiritamiz. (8.13.2) ni $f(x;y)$ ga qo‘yib, bitta t o‘zgaruvchining murakkab

$$F(t)=f(x_0+t\Delta x;y_0+t\Delta y)$$

funksiyani hosil qilamiz.

Kiritilgan (8.13.2) tenglamalar geometrik jihatdan $M_0(x_0;y_0)$ va $M_1(x_0+\Delta x;y_0+\Delta y)$ nuqtalarni tutashtiruvchi kesmani ifoda qilishi bizga ma’lum.

Bo‘lar asosida

$$\Delta f(x_0;y_0)=\Delta F(0)=F(1)-F(0)$$

ekanligini tushunish qiyin emas.

Endi, $F(t)$ ga yuqoridagi Teylor formulasini qo‘llasak,

$$\Delta F(0)=dF(0)+\frac{1}{2!}d^2F(0)+...+\frac{1}{n!}d^nF(0)+\frac{1}{(n+1)!}d^{n+1}F(\theta)\qquad (0<\theta<1)\ (8.13.3)$$

ga ega bo‘lamiz (bu formulaning o‘ng tomoniga kirgan dt turli darajalari bilan birga $\Delta t=1-0=1$ ga tengdir).

Endi, o‘zgaruvchilarni chiziqli almashtirish natijasida hatto yuqori tartibli differensiallar ham shaklini saqlab qolishidan (invariantlik xossasi) foydalanib, ushbuni yoza olamiz:

$$dF(0)=f'_x(x_0;y_0)dx+f'_y(x_0;y_0)dy=df(x_0;y_0)$$

$$d^2F(0)=f''_{xx}(x_0;y_0)dx^2+2f''_{xy}(x_0;y_0)dxdy+f''_{yy}(x_0;y_0)dy^2=d^2f(x_0;y_0)$$

va hokazo. Nihoyat, *(n+1)*-differensial uchun

$$d^{n+1}F(\theta)=d^{n+1}f(x_0+\theta\Delta x;\ y_0+\theta\Delta y).$$

bo‘larni (8.13.3) ga qo‘yib, (8.13.1) ni olamiz.

Shunday qilib, differensial shaklda yozilgan Teylor formulasi ko‘ o‘zgaruvchili funksiya uchun ham bir o‘zgaruvchili funksiyadek kabi sodda ko‘rinishga ega ekan. Ammo, uni argumentlar bo‘yicha yozilgan ko‘rinishi ancha murakkabdir. Masalan, ikki o‘zgaruvchili funksiya uchun uning birinchi uchta hadi quyidagichadir:

$$f(x_0+\Delta x;y_0+\Delta y)-f(x_0;y_0)=\left[f'_x(x_0;y_0)\Delta x+f'_y(x_0;y_0)\Delta y\right]+ \\ +\frac{1}{2!}\left[f''_{x^2}(x_0;y_0)\Delta x^2+2f''_{xy}(x_0;y_0)\Delta x\Delta y+f''_{y^2}(x_0;y_0)\Delta y^2\right]+ \\ +\frac{1}{3!}\left[f'''_{x^3}(x_0;y_0)\Delta x^3+3f'''_{x^2y}(x_0;y_0)\Delta x^2\Delta y+3f'''_{xy^2}(x_0;y_0)\Delta x\Delta y^2+f'''_{y^3}(x_0;y_0)\Delta y^3\right]+\dots \quad (8.13.4)$$

Bu formulani ($x_0;y_0$) nuqta atrofidagi yoyilmasinin quyidagicha yozamiz:

$$f(x,y)=f(x_0;y_0)+[f'_x(x_0;y_0)(x-x_0)+f'_y(x_0;y_0)(y-y_0)]+ \\ +\frac{1}{2!}[f''_{x^2}(x_0;y_0)(x-x_0)^2+2f''_{xy}(x_0;y_0)(x-x_0)(y-y_0)+f''_{y^2}(x_0;y_0)(y-y_0)^2]+ \\ +\frac{1}{3!}\{f'''_{x^3}(x_0;y_0)(x-x_0)^3+3f'''_{x^2y}(x_0;y_0)(x-x_0)^2(y-y_0)+ \\ 3f'''_{xy^2}(x_0;y_0)(x-x_0)(y-y_0)^2+f'''_{y^3}(x_0;y_0)(y-y_0)^3\}+\dots \quad (8.13.5)$$

Xuddi shunga o‘xshash ishni n o‘lchovli R^n fazoga tegishli M_0 nuqtaning atrofida $(m+1)$-tartibgacha xususiy hosilalari uzluksiz bo‘lgan $f(M)$ funksiya uchun bajarib, quyidagicha Teylor formulasiga ega bo‘lamiz:

$$\Delta f(M_0)=\sum_{k=1}^{m}\frac{1}{k!}d^k f(M_0)+\frac{1}{(m+1)!}d^{m+1}f(M_\theta) \quad (8.13.6)$$

bu yerda

$$M_0\left(x_1^{(0)};x_2^{(0)};\dots;x_n^{(0)}\right),\ M_\theta\left(x_1^{(0)}+\theta\Delta x_1;x_2^{(0)}+\theta\Delta x_2;\dots;x_n^{(0)}+\theta\Delta x_n\right),$$
$$0<\theta<1,\ \Delta f(M_0)=f(M)-f(M_0),\ M\left(x_1^{(0)}+\Delta x_1;x_2^{(0)}+\Delta x_2;\dots;x_n^{(0)}+\Delta x_n\right).$$

Eslatma. (8.13.6) formula m=0 bo‘lganda ham o‘rinli bo‘lib, bu holda (8.13.1) chekli orttirmalar formulasiga ega bo‘lamiz.

8.13.1-misol. $f(x,y)=x^y$ funksiyатш M(1;1) nuqta atrofida Teylor formulasi bo‘yicha yoying.

Yechish. (8.13.3) Teylor formulasini quyidagicha yozamiz:

$$f(x,y)=f(1,1)+\frac{df(1,1)}{1!}+\frac{d^2f(1,1)}{2!}+\frac{d^3f(1,1)}{3!}+R_3 \quad (*)$$

3-tartibligach hususiy hosilalarni topamiz.

$f'_x=yx^{y-1},\ f'_x(1,1)=1,\quad f''_{xx}=y(y-1)x^{y-2},\ f''_{xx}(1,1)=0,\ f'''_{xxx}=y(y-1)(y-2)x^{y-3},\ f'''_{xxx}(1,1)=0,$

$f'_y=x^y\ln x,\ f'_y(1,1)=0,\quad f''_{yy}=x^y(\ln x)^2,\ f''_{yy}(1,1)=0,\ f'''_{yy}=x^y(\ln x)^3,\ f'''_{yyy}(1,1)=0,$

$f''_{xy}=x^{y-1}+yx^{y-1}\ln x,\ f''_{xy}(1,1)=1,\quad f'''_{xyy}=2x^{y-1}\ln x+yx^{y-1}(\ln x)^2,\ f'''_{xyy}(1,1)=0,$

$f''_{xxy}=(2y-1)x^{y-2}+y(y-1)x^{y-2}\ln x,\ f'''_{xxy}(1,1)=1$

3-tartibligach differentsiallarni topamiz.

$$df(1,1)=f'_x(1,1)\Delta x+f'_y(1,1)\Delta y=\Delta x$$
$$d^2f(1,1)=f''_{xx}(1,1)\Delta x^2+2f''_{xy}(1,1)\Delta x\Delta y+f''_{yy}(1,1)\Delta y^2=2\Delta x\Delta y$$
$$d^3f(1,1)=f'''_{xxx}(1,1)\Delta x^3+3f'''_{xxy}(1,1)\Delta x^2\Delta y+3f'''_{xyy}(1,1)\Delta x\Delta y^2+f'''_{yyy}(1,1)\Delta y^3=3\Delta x^2\Delta y$$

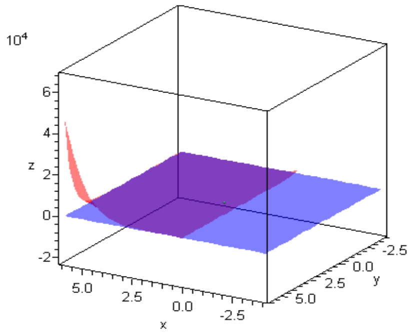
Topilganlarni (*)formulaga qo‘ysak:

$$x^y=1+\Delta x+\Delta x\Delta y+\frac{1}{2}\Delta x^2\Delta y+R_3=1+(x-1)+(x-1)(y-1)+\frac{1}{2}(x-1)^2(y-1)+R_3$$

```
> mtaylor(x^y, [x=1,y=1], 6, [2,1]);
```

$$x+(y-1)(x-1)+\frac{1}{2}(y-1)(x-1)^2$$

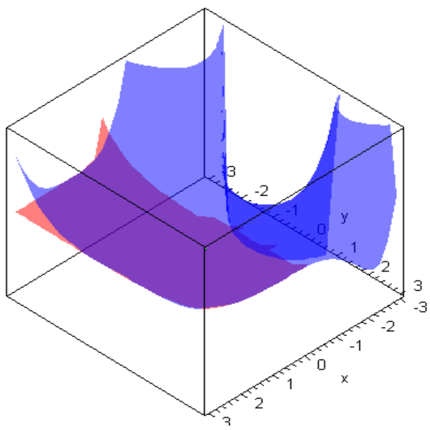
```
> TaylorApproximationTutor(x^y, [x,y] = [1,1], 3, x=-3..3, y=-3..3,
z=-12..12);
```




```
> mtaylor(x^y, [x=1,y=1], 8);
```

$$x + (y-1)(x-1) + \frac{1}{2}(y-1)(x-1)^2 - \frac{1}{6}(y-1)(x-1)^3 + \frac{1}{2}(y-1)^2(x-1)^2 + \frac{1}{12}(y-1)(x-1)^4$$

```
> TaylorApproximationTutor(x^y, [x,y] = [1,1], 8, x=-3..3, y=-3..3,
z=-12..12);
```



8.13.2-misol. Teylorning (8.13.5) formulasi asosida z(1;1)=1 bo'lganda, z^3-2xz+y=0 oshkormas funksiyani (x-1) va (y+1) ning darajalari bo'yicha yoying.

Yechish. z^3-2xz+y=0 ning xususiy hosilalarining x=1,y=1,z=1 nuqtadagi qiymatlarin topamiz va ularni (8.13.5) Teylorning formulaga qo'yib yoyilmani topamiz::

```
> restart;
> f :=z^3-2*x*z+y;
```

$$f := z^3 - 2\,x\,z + y$$

```
> Ax:=implicitdiff(f,z,x);A1:=eval(Ax,[x=1,y=1,z=1]);
```

$$Ax := -\frac{2\,z}{-3\,z^2 + 2\,x} \qquad A1 := 2$$

```
> Axx:=implicitdiff(f,z(x,y),x$2);A11:=eval(Axx,[x=1,y=1,z=1]);
```

$$Axx := \frac{16\,z\,x}{-27\,z^6 + 54\,z^4\,x - 36\,z^2\,x^2 + 8\,x^3} \qquad A11 := -16$$

```
> Ay:=implicitdiff(f,z(x,y),y);A2:=eval(Ay,[x=1,y=1,z=1]);
```

$$Ay := \frac{1}{\mathrm{K}\ 3\,z^2\ \mathrm{C}\ 2\,x} \qquad A2 := \mathrm{K}\ 1$$

```
> Ayy:=implicitdiff(f,z(x,y),y$2);A22:=eval(Ayy,[x=1,y=1,z=1]);
```

$$Ayy := \frac{6\,z}{\mathrm{K}\ 27\,z^6\ \mathrm{C}\ 54\,z^4\,x\ \mathrm{K}\ 36\,z^2\,x^2\ \mathrm{C}\ 8\,x^3} \qquad A22 := \mathrm{K}\ 6$$

```
> Axy:=implicitdiff(f,z(x,y),x,y);A12:=eval(Axy,[x=1,y=1,z=1]);
```

$$Axy := \mathrm{K}\ \frac{2\,(3\,z^2\ \mathrm{C}\ 2\,x)}{\mathrm{K}\ 27\,z^6\ \mathrm{C}\ 54\,z^4\,x\ \mathrm{K}\ 36\,z^2\,x^2\ \mathrm{C}\ 8\,x^3} \qquad A12 := 10$$

```
> a:=1;b:=-1;c:=1;
```

$a := 1 \quad b := \mathrm{K}\ 1 \quad c := 1$

```
> T:=z+[A1*(x-a)+A2*(y-b)]+[A11*(x-a)^2+2*A12*(x-a)*(y-b)+
A22*(y-b)^2]/2;
```

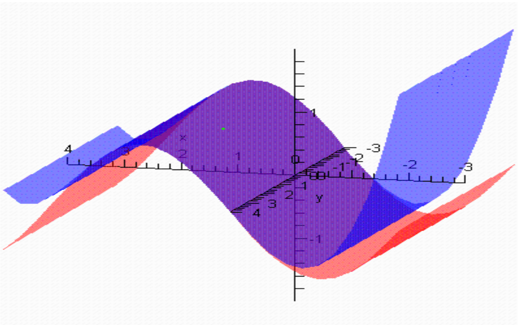
$$T := z\ \mathrm{C}\ [\mathrm{K}\ 8\,(x\ \mathrm{K}\ 1)^2\ \mathrm{C}\ 10\,(x\ \mathrm{K}\ 1)\,(y\ \mathrm{C}\ 1)\ \mathrm{K}\ 3\,(y\ \mathrm{C}\ 1)^2$$
$$\mathrm{C}\ 2\,x\ \mathrm{K}\ 3\ \mathrm{K}\ y]$$

Qo'shimch masalalar.

```
> with(Student[MultivariateCalculus]):
> mtaylor(sin(x), [x,y], 8);
```

$$x\ \mathrm{K}\ \frac{1}{6}\,x^3\ \mathrm{C}\ \frac{1}{120}\,x^5\ \mathrm{K}\ \frac{1}{5040}\,x^7$$

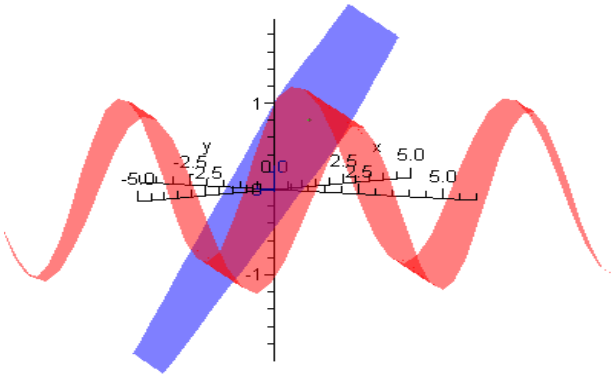
```
> TaylorApproximation(sin(x),[x,y]=[1,1],5,x=-3..4,
y=-3..4,output=plot);
```



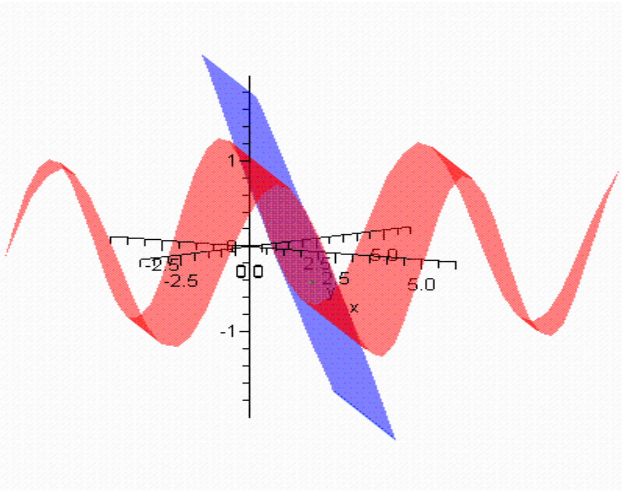
```
> TaylorApproximation(sin(x+y),[x,y]=[1,0],3);
```

$$\mathrm{K}\ \frac{1}{6}\cos(1)\,y^3\ \mathrm{K}\ \frac{1}{2}\cos(1)\,(x\ \mathrm{K}\ 1)\,y^2\ \mathrm{K}\ \frac{1}{2}\sin(1)\,y^2$$
$$\mathrm{C}\ \cos(1)\,y\ \mathrm{K}\ \frac{1}{2}\cos(1)\,(x\ \mathrm{K}\ 1)^2\,y\ \mathrm{K}\ \sin(1)\,(x\ \mathrm{K}\ 1)\,y$$
$$\mathrm{K}\ \frac{1}{6}\cos(1)\,(x\ \mathrm{K}\ 1)^3\ \mathrm{K}\ \frac{1}{2}\sin(1)\,(x\ \mathrm{K}\ 1)^2\ \mathrm{C}\ \sin(1)$$
$$\mathrm{C}\ \cos(1)\,(x\ \mathrm{K}\ 1)$$

```
> with(Student[MultivariateCalculus]):
> TaylorApproximation(sin(x+y),[x,y]=[1,0],5,
output=animation);
```



```
> with(Student[MultivariateCalculus]):
> TaylorApproximation(cos(x+y),[x,y]=[1,1],3,
output=animation);
```



8.14. Ko‘ o‘zgaruvchili funksiyaning ekstremumlari

Faraz qilaylik, $u = f(M)$ funksiya *n* o‘lchovli *D* sohada aniqlangan bo‘lib, M_0 bu sohaning ichki nuqtasi bo‘lgan holda uning shunday atrofi mavjud bo‘lsinki, unga tegishli barcha nuqtalar uchun $f(M) \le f(M_0)$ $\left(f(M) \ge f(M_0)\right)$ tengsizlik o‘rinli bo‘lsa, M_0 nuqta *funksiyaning maksimum (minimum) nuqtasi,* $f(M_0)$ esa uning *maksimumi (minimumi)* deyiladi.

Agar M_0 nuqta $f(M)$ funksiyaning maksimum (minimum) nuqtasi bo‘lib, bu nuqtaning shunday yaqin atrofi mavjud bo‘lsaki, unga tegishli barcha M nuqtalar uchun $f(M) < f(M_0)$ $\left(f(M) > f(M_0)\right)$ tengsizlik bajarilsa, M_0 funksiyaning *sof maksimum (sof minimum) nuqtasi* $f(M_0)$ esa uning *sof maksimumi (sof minimumi)* deyiladi.

Funksiyaning maksimum va minimumlari bitta nom bilan uning *ekstremumlari* deb ataladi.

Agar $u = f(x_1; x_2; ...; x_n)$ funksiya $\left(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)}\right)$ nuqtada ekstremumga ega bo‘lgan holda, bu nuqtada uning biror argumenti bo‘yicha chekli xususiy hosilasi mavjud bo‘lsa, u nolga teng bo‘ladi. Masalan, f'_{x_1} mavjud va chekli bo‘lsin, u vaqtda $f\left(x_1; x_2^{(0)}; ...; x_n^{(0)}\right)$ funksiyani qaralsa, u bir o‘zgaruvchili bo‘lib, $x_1^{(0)}$ nuqtada ekstremumga ega ekanligi va uning chekli hosilasi mavjudligidan bu nuqtada $f'_{x_1}\left(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)}\right) = 0$ kelib chiqadi.

Bu yerda shuni ham aytamizki, ko‘ o‘zgaruvchili funksiyaning barcha xususiy hosilalari nolga aylanadigan nuqtalar uning *statsionar nuqtalari* deb yuritiladi. Demak, statsionar nuqtalar ekstremum uchun «gumonli» nuqtalar hisoblanadi. Ya’ni ularning ba’zilari (yoki barchasi) ekstremum nuqtasi bo‘lishi ham, bo‘lmasligi ham mumkin.

$u = f(x_1; x_2; ...; x_n)$ *n* o‘zgaruvchili funksiyaning statsionar nuqtalarini topish

$$f'_{x_i}(x_1; x_2; ...; x_n) = 0, \; i = \overline{1;n} \qquad (8.14.1)$$

tenglamalar sistemasini yechish orqali amalga oshiriladi.

(8.14.1) o‘rniga

$$df(x_1; x_2; ...; x_n) = 0 \qquad (8.14.2)$$

ni qarash mumkin. Yana ekstremum nuqtasida ba’zi xususiy hosilalar mavjud bo‘lmay qolishi yoki cheksizga aylanishi ham mumkin. Demak, ekstremumga «gumonli» nuqtalar qatoriga, statsionar nuqtalardan tashqari, funksiya xususiy hosilalarining barchasi mavjud bo‘lmaydigan yoki ulardan bir qismi nolga teng bo‘lib, qolganlari mavjud bo‘lmaydigan nuqtalarini ham qo‘shish kerak bo‘ladi.

Yuqorida aytilganlar ekstremumning *zaruriy shartlarni* bildiradi, lekin u yetarli emas.

Endi, ko‘ o‘zgaruvchili funksiya ekstremumining yetarli shartlarini ko‘ramiz. Biz, ikki o‘zgaruvchili $z = f(x; y)$ funksiya uchun qaraymiz. Aytaylik, $(x_0; y_0)$ qaralayotgan funksiyaning statsionar nuqtasi, ya’ni

$$f'_x(x_0; y_0) = 0, \; f'_y(x_0; y_0) = 0$$

bo‘lib, bu nuqtada funksiyaning ikkinchi tartibli uzluksiz xususiy hosilalari mavjud bo‘lsin. U holda funksiyaning ekstremumi etarli shartini aniqlovchi quyidagi teoremani ko‘ramiz.

Teorema. *Agar z=f(x,y) funksiya (x_0;y_0) kritik nuqtada va uning boror atrofida ikkinchi tartibli xususiy hosilalariga ega bo‘lib, bundan tashqari, bu nuqtadagi xususiy hosilalar no‘lga teng bo‘lsa:*

$$f'_x(x_0; y_0) = 0, \; f'_y(x_0; y_0) = 0$$

U holda (x_0;y_0) nuqtada

$A = f'_{xx}{}''(x_0; y_0) = 0, \; B = f_{xy}{}''_y(x_0; y_0) = 0, \; C = f_{yy}{}''(x_0; y_0) = 0$ *bo‘lib,*

$\Delta = AC - B^2$ *ifodaga asosan:*

1)agar $\Delta > 0$ *bo‘lsa, ekstremum mavjud, bunda:*

a) $A > 0$ *bo‘lsa, minimum;*

b) $A < 0$ *bo‘lsa, maksimim;*

2) agar $\Delta < 0$ *bo‘lsa, ekstremum mavjud emas;*

3) agar $\Delta = 0$ *bo‘lsa, ekstremum bo‘lishi ham va bo‘lmasligi ham mumkin.*

8.14.1-misol. $z = x^3 + y^3 - 3xy$ funktsiyaning ekstremumini tekshiring.

Yech i sh. Berilgan funktsiya uchun z'_x va z'_y har doim mavjud va bu xususiy hosilalarni topamiz

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \qquad \frac{\partial z}{\partial y} = 3y^2 - 3x$$

Endi quyidagi sistemani tuzamiz:

$$\begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases}$$

bundan x, = 0, x_2 = 1, $y_1 = 0$, $y_2 = 1$. Shunday qilib, $M_1(0;0)$ va $M_2(1;1)$ ikkita statsionar nuqtaga ega bo‘ldik. Endi quyidagilarni topamiz:

$$A = \frac{\partial^2 z}{\partial x^2} = 6x, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -3, \; C = \frac{\partial^2 z}{\partial y^2} = 6y,$$

U holda

$$\Delta = AC - B^2 = 36xy - 9$$

$M_1(0;0)$ nuqtada $\Delta = -9 < 0$ bo‘lgani uchun bu nuqtada ekstremum yo‘q.

$M_2(1;1)$ nuqtada $\Delta = 27 > 0$ va $A = 6 > 0$ bo‘lgani uchun bu nuqtada berilgan funktsiya lokal minimumga erishadi: $z_{min}=-1$

Xususiy hosilalarni topish:

```
> restart;
> f5:=(x,y)->x^3+y^3-3*x*y;
```

$$f5 := (x, y) \to x^3 + y^3 - 3\,x\,y$$

```
> f5x:=diff(f5(x,y),x);
```

$$f5x := 3\,x^2 - 3\,y$$

```
> f5y:=diff(f5(x,y),y);
```

$$f5y := 3\,y^2 - 3\,x$$

```
> f5xx:=diff(f5(x,y),x$2);
```

$$f5xx := 6\,x$$

```
> f5yy:=diff(f5(x,y),y$2);
```

$$f5yy := 6\,y$$

```
> f5xy:=diff(f5(x,y),x,y);
```

$$f5xy := -3$$

Zaruriy shart

```
> solvefor({f5x,f5y});
Warning, solvefor is deprecated. Please use solve command.
```

$$\{x = 0, y = 0\},\ \{x = 1, y = 1\},\ \left\{x = -\frac{1}{2} + \frac{1}{2}\,I\sqrt{3},\ y = \frac{1}{4}\,(-1 + I\sqrt{3})^2\right\},\ \left\{x = -\frac{1}{2} - \frac{1}{2}\,I\sqrt{3},\ y = \frac{1}{4}\,(1 + I\sqrt{3})^2\right\}$$

Statsionar nuqtalarda ekstremumni tekshirish:

1) $M1\,(0;\,0)$ *nuqtada:*

```
> x:=0;y:=0;A5:=f5xx; B5:=evalf(f5xy);C5:=evalf(f5yy);
d5:=A5*C5-B5^2;
```

$$x := 0 \quad y := 0 \quad A5 := 0 \quad B5 := -3. \quad C5 := 0. \quad d5 := -9.$$

Ekstremim yo‘q, chunki A5<0, d5<0;

```
> x:=0; y:=0; f5M1:=x^3+y^3-3*x*y;
```

$$x := 0 \quad y := 0 \quad f5M1 := 0$$

1) $M2\,(1;\,1)$ *nuqtada:*

```
> x:=1;y:=1;A5:=f5xx; B5:=evalf(f5xy);C5:=evalf(f5yy);
d5:=A5*C5-B5^2;
```

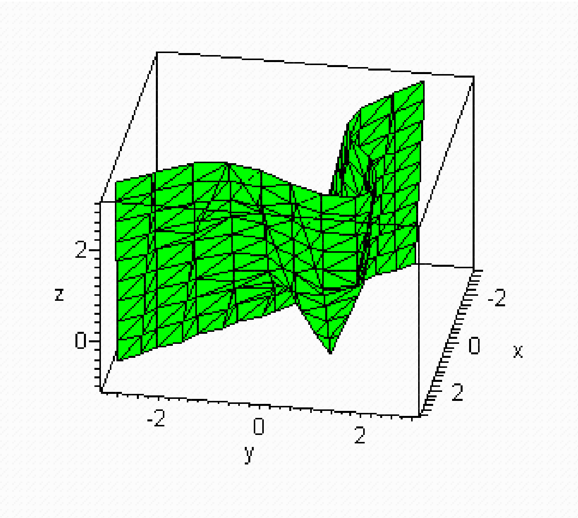
$$x := 1 \quad y := 1 \quad A5 := 6 \quad B5 := -3. \quad C5 := 6. \quad d5 := 27.$$

Funktsiya minimumga ega, chunki A5>0, d5>0;

```
> x:=1; y:=1; f5M2min:=x^3+y^3-3*x*y;
```

$$x := 1 \quad y := 1 \quad f5M2min := -1$$

```
Funktsiya grafigini qurish.
> restart;
> with(plots):
> implicitplot3d([x^3+y^3-3*x*y=z],x=-3..3,y=-3..3,
z=-1..3,color=green, scaling=constrained,axes=BOXED);
```



8.14.2-misol. $x^3 + y^3 - 3xy = 0$ tenglama bilan berilgan y oshkormas funksiyaning ekstremumini toping.

Yechish. Bu yerda $F(x;y) = x^3 + y^3 - 3xy$ desak,

$$F'_x(x;y) = 3x^2 - 3y = 3(x^2 - y), \quad F'_y(x;y) = 3y^2 - 3x = 3(y^2 - x).$$

bo‘lib, (13.18.9) formulaga ko‘ra

$$y' = -\frac{3(x^2 - y)}{3(y^2 - x)} = \frac{x^2 - y}{x - y^2}$$

Kritik nuqtani topamiz:

$$y' = 0 \Rightarrow F'_x(x;y) = 0 \Rightarrow x^2 - y = 0 \Rightarrow y = x^2$$

buni oshkormas funksiya tenglamasiga qo‘yib,

$$x^3 + x^6 - 3x^3 = 0 \Rightarrow x^3(x^3 - 2) = 0$$

tenglamani olamiz. Bundan $x_1 = 0$, $y_1 = 0$ va $x_2 = \sqrt[3]{2}$, $y_2 = \sqrt[3]{4}$ ya’ni $(0;0)$ va $(\sqrt[3]{2}, \sqrt[3]{4})$ kritik nuqtalarga ega bo‘lamiz. Endi, y'' uchun chiqarilgan formulada $F'_x(x;y) = 0$ ekanligini hisobga olsak, bu kritik nuqtalarda

$$y'' = -\frac{F''_{xx}(x;y)}{F'_y(x;y)}$$

ni olamiz. $F''_{xx}(x;y) = 3(x^2 - y)'_x = 6x$.

Endi, $(0;0)$ nuqtani qarasak, bu nuqtada $F'_y(x;y) = 0$ ham nolga aylanadi, shu sababli, oshkormas funksiya mavjudlik sharti buzilganligi uchun, bu nuqtani qaramaymiz. Ikkinchi $\sqrt[3]{2}$, $\sqrt[3]{4}$ nuqtada $F''_{xx}(\sqrt[3]{2};\sqrt[3]{4}) = 6\sqrt[3]{2}$; $F'_y(\sqrt[3]{2};\sqrt[3]{4}) = \sqrt[3]{2}$

$$y'' = -\frac{6\sqrt[3]{2}}{\sqrt[3]{2}} = -6 < 0$$

Demak, bu nuqtada maksimumga egamiz: $y_{max} = \sqrt[3]{4}$.

Oshkormas funktsiy ekstremumini aniqlash.

```
> restart;
> with(algcurves):
> F:=x^3+y^3-3*x*y=0;
```

$$F := x^3 \text{ C } y^3 \text{ K } 3\,x\,y = 0$$

Oshkormas funksiyan xususiy hosilasini topish:

```
> yx:=implicitdiff(F,y,x);
```

$$yx := \frac{x^2 \text{ K } y}{\text{K } y^2 \text{ C } x}$$

Zaruriy shartni tekshirish:

```
> solvefor(yx=0,F);evalf(solvefor(yx=0,F));
```

$$4\,x = 0\,\text{K}\,\frac{1}{2}\,2^{(1/3)}\ \text{C}\ \frac{1}{2}\,\text{I}\sqrt{3}\,2^{(1/3)}\,1^2,\ x = \text{K}\,\frac{1}{2}\,2^{(1/3)}$$

$$\text{C}\ \frac{1}{2}\,\text{I}\sqrt{3}\,2^{(1/3)}\,5\,\{x = 2^{(1/3)},\ y = 2^{(2/3)}\},\ 4$$

$$y = 0\,\text{K}\,\frac{1}{2}\,2^{(1/3)}\ \text{K}\ \frac{1}{2}\,\text{I}\sqrt{3}\,2^{(1/3)}\,1^2,$$

$$x = \text{K}\,\frac{1}{2}\,2^{(1/3)}\ \text{K}\ \frac{1}{2}\,\text{I}\sqrt{3}\,2^{(1/3)}\,5$$

$$[\{x = \text{K } 0.6299605250$$
$$\text{C } 1.091123636\text{I},\ y = \text{K } 0.7937005260\text{K } 1.374729637\text{I}\},\ \{$$
$$x = 1.259921050,\ y = 1.587401052\},\ \{y = \text{K } 0.7937005260$$
$$\text{C } 1.374729637\text{I},\ x = \text{K } 0.6299605250\text{K } 1.091123636\text{I}\}\,]$$

Ekstremumni topishda ikkinch tartibli hosiladan foydalanamiz:

```
> yxx:=implicitdiff(F,y,x$2);
```

$$yxx := \frac{2\,y\,x\,(y^3 \text{ K } 3\,x\,y \text{ C } x^3 \text{ C } 1)}{\text{K } y^6 \text{ C } 3\,x\,y^4 \text{ K } 3\,y^2\,x^2 \text{ C } x^3}$$

```
> x := 2^(1/3); y := 2^(2/3);Fxx:=yxx;
```

$$x := 2^{(1/3)} \quad y := 2^{(2/3)} \quad Fxx := \text{K } 2$$

Berilgan funksiya **Fxx=-2<0** *ekanidan maksimumga ega:* $y_{\max} = \sqrt[3]{4}$

8.15. Ko‘p o‘zgaruvchi funksiyaning eng kichik va eng katta qiymatlari

Faraz qilaylik, $u = f(x_1; x_2; ...; x_n)$ funksiya chegaralangan yoiq n o‘lchovli $\overline{D}$ sohada aniqlangan va uzluksiz bo‘lsin; bundan tashqari, u $\overline{D}$ sohaning ayrim nuqtalaridan boshqa barcha nuqtalarida chekli xususiy hosilalarga ega bo‘lsin. Veyershtrasning teoremasiga asosan bu sohada shunday $(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)})$ nuqta topiladiki, bu nuqtada $u = f(x_1; x_2; ...; x_n)$ funksiya o‘zining eng katta (yoki eng kichik) qiymatiga erishadi. Agar $(x_1^{(0)}; x_2^{(0)}; ...; x_n^{(0)})$ D sohaning ichki nuqtasi bo‘lsa, funksiya shu nuqtada maksimum (yoki minimumga) ega. demak, bu holda bizni qiziqtirgan nuqta ekstremumga “gumonli” nuqtalar orasida bo‘ladi. Biroq, u funksiya o‘zining eng katta (eng kichik) qiymatiga sohaning chegarasida ham erishishi mumkin. Shu sababli, $u = f(x_1; x_2; ...; x_n)$ funksiyaning D sohadagi ekstremumga “gumonli” barcha ichki nuqtalarini topib, funksiyaning shu nuqtalardagi qiymatlarini sohaning chegaraviy nuqtalardagi qiymatlari bilan solishtirish kerak; bu qiymatlarning eng kattasi (yoki eng kichigi) funksiyaning butun yoiq $\overline{D}$ sohadagi eng katta (yoki eng kichik) qiymati bo‘ladi.

Aytilgandagidek $\varphi(x, y) = 0$ funktsiya yordamida $z=f(x,y)$ funktsiyaning topilgan ekstremumini shartli ekstremum deyiladi. $\varphi(x, y) = 0$ tenglama bog‘lovchi tenglama deyiladi.

Geometrik masalalarda shartli ekstremumlarni aniqlash $z=f(x,y)$ sirtning $\varphi(x, y) = 0$ tsilindr bilan kesishishidan hosil bo‘lgan egri chizikding ekstremum nuk,talarini topishga keltiriladi.
Agar $\varphi(x, y) = 0$ boklovchi tenglamadan $y = y(x)$ ni topib (agar uni topish mumkin bo‘lsa), uni $z=f(x,y)$ funktsiyaga kuysak, shartli ekstremumni topish masalasi $z=f(x,y(x))$ bir o‘zgaruvchili funktsiyaning ekstremumini topishga keltiriladi.

8.15.1-misol. $z=x^2-y^2$ funktsiyaning $y=2x-6$ shart bo‘yicha ekstremumini toping.

YEchish. $y=2x-6$ ni berilgan funktsiyaga qo‘yib, x o‘zgaruvchiga nisbatan bir o‘zgaruvchili quyidagi funktsiyani hosil qilamiz:

$$z = x^2 - (2x - 6)^2, \qquad z = -3x^2+24x-36.$$

Uning hosilasini topamiz va uni nolga tenglaymiz:

$$z' = -6x+ 24; \quad z' = -6x+ 24= 0$$

bundan

$$x = 4, y = 2x-6 = 8\cdot 2-6 = 2.$$

Ikkinchi tartibli hosila $z''=\sim 6<0$
bo‘lgani uchun M(4;2) nuqtada berilgan funktsiya shartli maksimumga erishadi:

$$z_{max}= f(4,2)=12.$$

y=2x-6 shart bo‘yicha berilgan funktsiya ekstremumini topish:

```
> restart;
> with(Optimization);
> Minimize(x^2-y^2, {y=2*x-6});
```

$$[4.32000000000000, [x = 2.39999999999999990, y = -1.19999999999999928]]$$

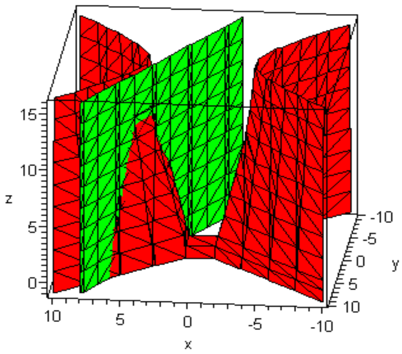
```
> Maximize(x^2-y^2, {y=2*x-6});
```

$$[12.0000000000000, [x = 3.99999999999999912, y = 2.]]$$

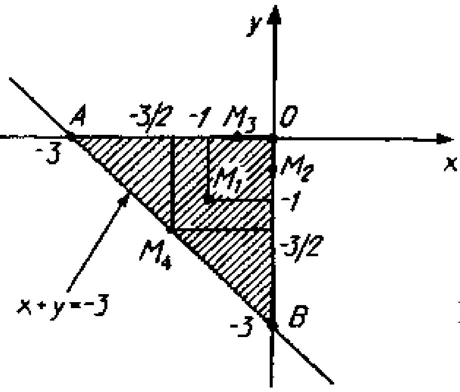
```
> extrema( x^2-y^2,{x,y});
```

$$\{0\}$$

```
> with(plots):
> implicitplot3d([x^2-y^2=z,y=2*x-6],x=-10..10,y=-10..10,
z=-1..16,color=[red,green], scaling=constrained,axes=BOXED);
```



Differentsiallanuvchi funktsiya chegaralangan yoiq. D sohada uzining eng katta (eng kichik) qiymatiga yoik. D soha ichida yotuvchi statsionar nuqtasida yoki shu sohaning chegarasida erishadi. z=f(x,y) funktsiyaning chegaralangan yoik. D «dadagi eng katta va eng kichik qiymatlarini topish uchun funktsiyaning bu sohaga tegishli kritik nuqtalardagi qiymatlarini hmda uning D sohaning chegarasidagi eng katta va eng kichik qiymatlar aniqlanadi. Bu qiymat-larning orasidagi eng kattasi va eng kichigi berilgan funktsiyaning D sohadagi mos ravishda eng katta va eng kichik qiymatlari bo‘ladi. Buni quyidagi misolda ko‘rsatamiz.



8.15.2-misol. $z = x^2 + y^2 - xy + x + y$ funktsiyaning $x = 0, y=0$, $x+ y= -3$ chizikdar bilan chegaralangan sohadagi eng katta va eng kichik qiymatlarini toping.

Yechish. O*xy* tekisligida D sohani chizib olamiz (65-chizma). D sohaga tegishli statsionar nuqtalarni aniqlaymiz:

$$\frac{\partial z}{\partial x} = 2x - y + 1 = 0, \quad \frac{\partial z}{\partial y} = 2y - x + 1 = 0$$

bundan $x=-1, y=-1$. M(- $1; - 1$) nuqtani hosil qildik, bu nuqtada $z_1=z(-l;-l)=-l$(65-chizma). Berilgan funktsiyani soha chegarasida tekshiramiz. OB tugri chiziqda (65-chizma) $x=0$ bo‘lib, $z=y^2+y$ tenglamaga ega bo‘lamiz va bu tenglama [$-3;0$] kesmada bir o‘zgaruvchili funktsiyaning eng katta va eng kichik qiymatini topish masalasiga keladi.

$$z' = 2y + 1$$

ni topib, uni nolga tenglaymiz:

$$2y + 1 = 0, \text{ bundan } y = -\frac{1}{2}, \quad z''_{yy} = 2$$

bo‘lgani uchun $M_2(0; -\frac{1}{2})$ shartli lokal minimum nuqtaga ega bo‘lamiz va unda

$$z_2 = z(0; -\frac{1}{2}) = -\frac{1}{4}$$

qiymatni xreil kilamiz. OB kesma uchlarida:

$$z_3=z(0,-3) = 6, \qquad z_4=z(0,0) = 0.$$

OA tug’ri chiziqda, y = 0 bo‘lib,

$$z=x^2 + x$$

ni xreil kilamiz.

$$z'_x= 2x+ 1 = 0, \; x = -\frac{1}{2}, \; z''_{xx} =2,$$

yani $M_3(-\frac{1}{2};0)$ lokal minimum nuqtasi bo‘lib, unda

$$z_5 = Z(-\frac{1}{2},0)= -\frac{1}{4}.$$

A nuqtada: $z_6 = z(-3,0) = 6.$

AB kesma tenglamasi $x + y = -3$ bo‘lib, undan

$$y = -x-3;$$

$$z=3x^2+9x+6,\ \ z'_x = 6x+9,\ x= -\frac{3}{2},\ \ M_4(-\frac{3}{2},-\frac{3}{2})$$

statsionar nuqtaga ega bo‘ldik:

$$z_7 = z(-\frac{3}{2},-\frac{3}{2}) = -\frac{3}{4}$$

Funktsiyaning AB kesma uchlardagi qiymatlari yuqrrida aniqdangan edi.

Berilgan z funktsiyaning topilgan barcha qiymatlarini solishtirib, funktsiya A (-3;0) va B(0;-3) nuqtalarda eng katta

$$z_{eng\ kat} = 6$$

va $M_1(-1;-1)$ statsionar nuqtada eng kichik

$$z_{eng\ kich} = -1$$

qiymatlarga erishishini aniqdaymiz.

```
> restart;
> with(Optimization);
> Minimize(x^2+y^2-x*y+x+y, {x=0,y=0,y+x=-3});
```

[0., [x = 0., y = 0.]]

```
> Minimize(x^2+y^2-x*y+x+y, {y+x=-3});
```

[
K .750000000000000, [x = K 1.49999999999999978
, y = K 1.49999999999999888]]

```
> solvefor({x^2+y^2-x*y+x+y,y=-x-3});
```

[{y = K 1, x = K 2}, {y = K 2, x = K 1}]

```
> minima, points := minimize( x^2+y^2-x*y+x+y, x=-4..4,
y=-2..2, 'location' );
```

minima, *points* := K 1, { [{y = K 1, x = K 1}, K 1] }

```
> minima, points := minimize( x^2+y^2-x*y+x+y, x=-2..0,
y=-2, 'location' );
```

minima, *points* := K 1/4, 4 4 x = K 3/2 5 K 1/4 5

```
> maxima, points := maximize( x^2+y^2-x*y+x+y, x=-3..0,
 y=0, 'location' );
```

maxima , *points* := 6, { [{x = K 3}, 6] }

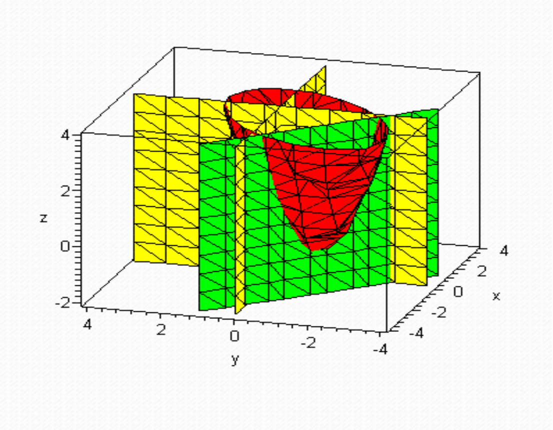
```
> maxima, points := maximize( x^2+y^2-x*y+x+y, x=0,
 y=-3..0, 'location' );
```

$maxima, points := 6, \{[\{y = K\ 3\}, 6]\}$

```
> extrema( x^2+y^2-x*y+x+y,{x,y});
```
$\{0\}$

$z = x^2 + y^2 - xy + x + y$ funktsiya grafigin qurish.

```
> with(plots):
> implicitplot3d([x^2+y^2-x*y+x+y=z,x=0,y=0,y+x=-3],
x=-4..4,y=-4..4,z=-2..4,color=[red,yellow,yellow,green],
scaling=constrained,axes=BOXED);
```



11. MUSTAQIL ISHLASH BAJARISH NAMUNASI VA TOPSHIRIQLARI

Mazkur mustakil uy ishining har bir variantida beshta misol bo‘lib, ularning sharti quyidagicha:

1-misolda: berilgan S sirtga $M_0(x_0;y_0;z_0)$ nuqtada o‘tkazilgan urinma va normal tekisliklar tenglamasini topish kerak.

2-misolda: berilgan funktsiyaning ikkinchi tartibli hosilalarini topish va $z''_{xy}=z''_{yx}$ tenglik tug’iligini tekshirish kerak.

3-misolda: berilgan U funktsiya berilgan tenglamani qanoatlantirishini tekshirish kerak.

4-misolda: funktsiyaning ekstremumini tekshirish kerak.

5-misolda: berilgan chiziklar bilan chegaralangan D soxada $z=z(x,y)$ funktsiyaning eng katta va eng kichik qiymatlarini topish kerak.

Quyida variant misollarini echish namunasini keltiramiz.

1 - m i s o l . *S:* $z = x^2 - y^2 + 3xy - 4x + 2y - 4$ sirtga $M_0(-1;0;1)$ nuqtada o’tkazilgan urinma va normal tekisliklar tenglamasini toping.

Echish. Xususiy xosilalarni topamiz:

$$\frac{\partial z}{\partial x} = 2x + 3y - 4, \qquad \frac{\partial z}{\partial y} = -2y + 3x + 2.$$

Xosil qilingan ifodalarga $M_0(-1;0;1)$ nuqtaning koordinatalarini qo’yamiz, natijada *S* sirtga perpendikulyar va berilgan nuqtadan o’tuvchi ***n*** vektorning koordinatalariga ega bo’lamiz:

$$A = \frac{\partial z}{\partial x}\Big|_{M_0} = 6, \qquad B = \frac{\partial z}{\partial y}\Big|_{M_0} = -1, \quad C=-1$$

(6.8) formulaga asosan urinma tekislik tenglamasi quyidagi ko’rinishda bo’ladi:

$$6(x + 1) - y - (z - 1) = 0 \quad \text{yoki} \quad 6x + y + z + 5 = 0.$$

(6.9) formulaga asosan normal tenglamasi:

$$\frac{x+1}{6} = \frac{y}{1} = \frac{z-1}{1}$$

2-misol. $z = \arccos\sqrt{\dfrac{x}{y}}$ funktsiyaning ikkinchi tartibli xususiy xosilalarini toping.

YE ch i sh . Dastlab berilgan funktsiyaning birinchi tartibli xususiy xosilalarini topamiz:

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x}{y}}}\frac{1}{\sqrt{\frac{x}{y}}}\frac{1}{y} = \frac{1}{2\sqrt{x}\sqrt{y-x}}\ , \quad \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x}{y}}}\frac{1}{\sqrt{\frac{x}{y}}}\left(-\frac{x}{y^2}\right) = -\frac{\sqrt{x}}{2y\sqrt{y-x}}$$

Bu xosilalarning xar birini x va u bo'yicha differentsiallab, berilgan funktsiyaning ikkinchi tartibli xususiy xosilalarini topamiz:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\frac{1}{2\sqrt{x}}\sqrt{y-x} - \frac{\sqrt{x}}{2\sqrt{y-x}}}{2x(y-x)} = \frac{y-x-x}{4\sqrt{x}(y-x)\sqrt{y-x}} = \frac{y-2x}{4\sqrt{x}(y-x)\sqrt{y-x}},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{x}}{2}\left(\frac{\sqrt{y-x} + \frac{y}{2\sqrt{y-x}}}{y^2(y-x)}\right) = \frac{\sqrt{x}(2x+2y)}{2y^2(y-x)},$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{2\sqrt{x}}\left(-\frac{1}{2}\right)(y-x)^{-\frac{3}{2}} = \frac{1}{4\sqrt{x}(y-x)\sqrt{y-x}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\frac{\sqrt{y-x}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{y-x}}}{y-x} = \frac{y-x+x}{4y(y-x)\sqrt{x}\sqrt{y-x}} = \frac{1}{4\sqrt{x}(y-x)\sqrt{y-x}}$$

Bulardan aralash xususiy xosilalar tengligi ($z''_{xy} = z''_{yx}$) ko'rinib turibdi.

3 - m i s o l. $u = \ln(x^2 + y^2)$ funktsiya

$$\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 \breve{a}}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{4y^2}{x^2+y^2}\frac{\partial u}{\partial x}$$

tenglamani qanoatlantirishini tekshiring.

YE ch i sh . Berilgan i funktsiyaning x va y o'zgaruvchilar bo'yicha birinchi va ikkinchi tartibli xususiy xosilala xosilalarini topamiz

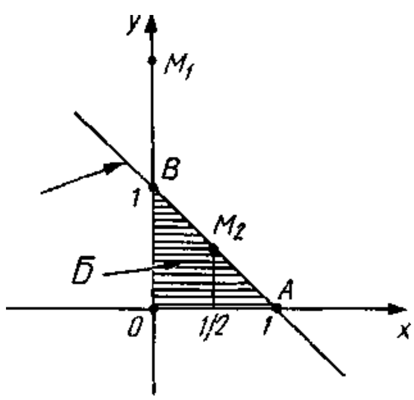
$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2},. \quad \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

Topilganlarni berilgan tenglamaning chap va ong tomoniga qo'yamiz.

$$\frac{8x^2y^2}{(x^2+y^2)^2} \neq \frac{8xy^2}{(x^2+y^2)^2}:$$

Xosil kilingan natijalardan berilgan funktsiya tenglamani qanoatlantirmasligi ko'rinib turibdi.



4-m i s o l. $z = xy\{x + y-2)$ funktsiyaning lokal ek stremumlarini tekshiring.

YE ch i sh. Berilgan funktsiyaning birinchi tartibli xususiy xosilalarini topamiz:

$$z'_x = 2xy + y^2 - 2y, \qquad z'_y = x^2 + 2xy - 2x.$$

Ularni nolga tenglab quyidagi tenglamalar sistemasini xosil kilamiz:

$$y(2x + y - 2) = 0,$$
$$x(x + 2y - 2) = 0$$

Bundan berilgan funktsiyaning $M_1(0;0)$, $M_2(2;0)$, $M_3(0;2)$, $M_4\left(\frac{2}{3};\frac{2}{3}\right)$ statsionar nuqtalarini aniqlaymiz. 2-teorema erdamida bu nuqtalarning kaysi birlari ekstremum nuqtalari ekanligini aniqlaymiz. Uning uchun berilgan funktsiyaning ikkinchi tartibli xususiy xosilalarini topamiz:

$$z''_{xx} = 2y, \quad z''_{xy} = 2x + 2y - 2, \quad z''_{yy} = 2x.$$

Xosil kilingan ifodalarga statsionar nuqtalarning koordinatalarini qo'yamiz va ekstremum mavjudligini zaruriy shartidan foydalanib quyidagiga ega bo'lamiz:

M_x nuqta uchun A = - 4 < O, ya'ni ekstremum yo'q.;

M_2 nuqta uchun A = - 4 < 0, ya'ni ekstremum yo'q;

M_h nuqta uchun A = - 4 < 0, ya'ni ekstremum yo'q.;

M4 nuqta uchun $\Delta = \frac{12}{9} > 0$, $\Delta = \frac{4}{3} > 0$, ya'ni ekstremum nuqta yo'q lekin berilgan funktsiya lokal minimum nuk.taga ega va unda

$$z_{\min} = z\left(\frac{2}{3};\frac{2}{3}\right) = -\frac{8}{27}$$

5 -misol. x = 0, $y = 0$, $x + y - 1 = 0$ chiziqlar bilan chegaralangan D soxada $z = xy - y^2 + 3x + 4y$ funktsiyaning eng katta va eng kichik qiymatini toping.

YEchish. Dastlab berilgan D sox.ani chizib olamiz (chizma). Berilgan D soxa, ya'ni OAV uchburchakning ichida yotuvchi statsionar nuqtalar bor yoki yukligini aniqlaymiz. Uning uchun berilgan funktsiyaning xususiy xrsi-lalarini topamiz:

$$z'_x = y + 3, \quad z'_y = x - 2y + 4.$$

Bundan

$$z'_x = y + 3 = 0, \quad z'_u = x - 2y + 4 = 0$$

yoki

$$y + 3 = 0, \quad x - 2y + 4 = 0.$$

Xosil qilingan sistemani echib M(-10;-3) statsionar nuqtani topamiz. Bu nuqta D soxa tashqarisida bulgani uchun uni masalani echishda xisobga olmaymiz. Funktsiyaning qiymatlarini D soxa chegarasida tekshiramiz.

OAB uchburchakning OA tomonida ($y = 0$, $0<x<1$) z funktsiya $z=3x$ ko'rinishda bo'ladi. OA kesmada statsionar nuqta yo'q chunki $z' = 3$. O *va* A nuqtalarda, moc ravishda $z(0,0)=0$, $z(1,0) = 3$. Uchburchakning OB tomonida (x = 0,) $0 < y < 1$) z funktsiya:

$$z' = y^2 + 4y, \quad z' = -2y + 4.$$

Statsionar nuqtani

$$-2y + 4 = 0$$

tenglamadan topamiz, ya'ni $y=2$. $M_1(0;2)$ nuqta D soxaga tegishli emas. V nuqtadagi funktsiyaning qiymati z(0,1)=3. Endi tenglamasi $x+y=1$ bo'lgan tomondagi eng katta va eng kichik qiymatini topamiz. Bunda

$$y = 1 - x, \quad z = -2x^2 + 2x + 3,$$

u holda $z' = -4x + 2$ va $z' = 0$ dan $x = \frac{1}{2}$ ga ega bo'lamiz va natijada *D* soxaga tegishli bulgan $M_2(\frac{1}{2};\frac{1}{2})$ statsionar nuqtaga ega bo'ldik. Bu nuqtada funktsiyaning qiymati: $z(\frac{1}{2};\frac{1}{2}) = 3.5$. Olingan funktsiyaning barcha qiymatlariga ko'ra

$$z_{eng\ kat} = z(\frac{1}{2};\frac{1}{2}) = 3.5,\quad z_{eng\ kich} = z(0;0) = 0$$

1-вариант

1. $S: x^2+y^2+z^2-6y+4z+4=0$, $M_0(2;1;-1)$.
2. $z = \operatorname{arctg}(x+y)$.
3. $x^2\frac{\partial^2 ă}{\partial x^2}+y^2\frac{\partial^2 ă}{\partial y^2}=0,\ u=e^{xy}$
4. $z = 2x^s + 2y^3 - 6xy + 5$.
5. $z = x^7 + y^2 - 2x - 2y + 8$,
   D : x = 0, y = 0, $x + y - 1 = 0$.

2-вариант

1. $S: x^2+z^2-5yz+3y=46$, $M_0(l;2;-3)$.
2. z = arccos(2x + y).
3. $x^2\frac{\partial^2 ă}{\partial x^2}-y^2\frac{\partial^2 ă}{\partial y^2}=0,\ u=y\sqrt{\frac{x}{y}}$
4. $z= 3x^3+3y^3-9xy+10$.
5. $z = 2x^e - xy^2 + y$, $D: x = 0, x = 1, y = 0, y = 6$.

3-вариант

1. $S: x^2 + y^2 - xz - yz = 0$, $M_0(0;2;2)$.
2. $z = \operatorname{arcctg}(x - 3y)$.
3. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,\quad u=\frac{1}{\sqrt{x^2+z^2+z^2}}$
4. z=x^2+xy+y^2+x~y+1
5. $z = 3x + 6y - x^2 - xy - y^2$,
   D : x = 0, x = 1, y = 0, y = 1.

4-вариант

1. $S: x^2+y^2+2yz-z^2+y-2z=2$, $M_0(1;1;1)$.
2. $z = \arcsin(x-y)$.
3. $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=0,\ u=\sin^2(x-ay)$
4. $z = 4(x - y) - x^2 - y^2$.
5. $z=x^2 - 2y^2 + 4xy - 6x - l$,
   $D: x = 0, y=0, x + y - 3 = 0$.

5-вариант

1. $S: y^2-z^2+x^2-2xz+2x=z$, $M_0(l;l;l)$.
2. $z = \ln(3x^2-2y^2)$.
3. $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=0,\quad u=e^{-\cos(x-ay)}$
4. $z= 6(x-y)-3x^2-3y^2$,
5. $z = x^2 + 2xy - 1=0$,
   $D: y = 0,\ y = x^2 - 4$.

6-вариант

1. $S: z = x^2+y^2-2xy+2x-y$, $M_0(-1;-1;-1)$.
2. $z = e^{2x^2+y^2}$
3. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0,\quad u=(x-y)(y-z)(z-x)$
4. $z = x^2 + xy + y^2 - 6x - 9y$
5. $z = xy-2y-y$, $D: x = 0, y = O, x = 3, y = 4$.

7-вариант

1. $S: z = y^2-x^2+2xy-3y$, $M_0(l;-l;l)$.
2. z = ctg(y/x)
3. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u,\quad u=x\ln\frac{y}{x}$
4. $z = (x-2)^2 + 2y^2 - 10$.

8-вариант

1. S: $z=x^2-y^2-2xy-x-2y$, $M_0(-1;1;1)$.
2. $z = tg\sqrt{xy}$
3. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0,\quad u=\ln(x^2+y^2)$.
4. $z = (x-5)^2 + y^2 + 1$.

5. $z = \frac{1}{2}x^2 - xy$, D: y = 8, $y = 2x^2$.

5. $z = 3x^2 + 3y^2 - 2x - 2y + 2$,

D: $x = 0, y = 0, x + y - 1 = 0$.

9-вариант

1. $S: x^2 - 2y^2 + z^2 + xz - 4y = 13$, $M_0(3;l;2)$.
2. $z = \cos(x^2y^2-5)$.
3. $x^2\frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} + y^2 = 0, \quad u = \frac{y^2}{3x}$
4. $z = x^3 + y^3 - 3xy$
5. $z = 2x^2 + 3y^2 + 1$,

D: $y = \sqrt{9 - \frac{9}{4}x^2}$, y = 0.

10-вариант

1. $S: 4x^2 – z^2 + 4xy - xz + 3z = 9$, M_0 (1;-2;1).
2. $z = \sin\sqrt{x^3y}$
3. $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0, \quad u = 0, u = e^{xy}$
4. $z = 2xy - 2x^2 - 4y1$.
5. $z=x^2- 2xy - y^2+4x+ l$,

D: x=-3, y = 0, x + y+ 1=0

11-вариант

1. *S:* $z = x^2 + y^2 – 3xy - x + y + 2$, $M_0(2;l;0)$.
2. $z = \arcsin(x - 2y)$.
3. $\frac{\partial^2 u}{\partial x \partial y} = 0, \quad u = arctg\frac{x+y}{1-xy}$
4. $z = x\sqrt{y} - x^2 - y + 6x + 3$.
5. $z = 3x^2 + 3y^2 - x - y + 1$,

$D: x = 5, y = 0, x - y - 1 = 0$.

12-вариант

1. S: $2x^2 - y^2 + 2z^2 + xy + xz = 3$, $M_0(1;2;1)$.
2. г = arccos (4x - *y)*.
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = arctg\frac{x+y}{1-xy}$
4. $z = 2xy - 5x^2 - 3y^2 + 2$.
5. $z= 2x^2 + 2xy^2 - \frac{1}{2}y^2 - 4x$,

D: y = 2x , y = 2, x = 0.

13-вариант

S: $x^2-y^2+z^2-4x + 2y =14$, M0(3;1;4).

z = arctg (5x + 2y).

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + u = 0, \quad u = \frac{2x+3y}{x^2+y^2}$

z = xy(12 - x - y).

5. $z = x^2 - 2xy + \frac{5}{2}y^2 - 2x$,

D : x = 0, x = 2, y = 0, y = 2

14-вариант

1. $5:x^2 + y^2-г^2+xг + y + 4$, $M_0(1;1;2)$.
2. z = arctg(2x-*y)*.
3. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1, \quad u = \sqrt{x^2 + z^2 + z^2}$
4. $z = xy – x^2 - y^2 + 9$
5. $z = xy - 3x – 2y$, $D:$ $x = 0$, $x = 4$, $y = 0$, $y = 4$.

Adabiyot

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