O’ZBEKISTON RESPUBLIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI
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TOSHKENT DAVLAT TEXNIKA UNIVERSITETI

MATEMATIK FIZIKA TENGLAMALARI

Oliy texnika o’quv yurtlarining bakalavriat ta'lim yo'nalishi talabalari uchun o’quv qo’llanma

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O'quv qo'llanmaning tarkibiga matematik fizika tenglamalarining quyidagi bo'limlari kirgan: ikkinchi tartibli, xususiy hosilali tenglamalarni tiplarga ajratish, asosiy masalalarning qo'yilishi, tor tebranishi tenglamasi, issiqlik tarqalishi va diffuziya tenglamalari, Laplas tenglamasi.

Har bo'limda nazariy ma'lumotlar berilgan va tipik masalalar yechib ko'rsatilgan shuningdek mustaqil yechish uchun masalalar keltirilgan.

Qo'llanma oliy texnika o'quv yurtlarining bakalavriat - talabalari uchun mo'ljallangan bo'lib u O'zbekiston Respublikasi oliy va o'rta maxsus ta'lim vazirligi tasdiqlagan "Oliy matematika" fanidan namunaviy dastur asosida 10 soat ma'ruza, 10 soat amaliy mashg'ulot darslari hajmida yozilgan.

Abu Rayhon Beruniy nomidagi Toshkent davlat texnika universiteti ilmiy-uslubiy kengashi qaroriga asosan nashr etildi.

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Mundarija.

Kirish

Ushbu qo'llanmada matematik fizika tenglamalarining asosiy bayoni keltirilgan bo'lib, bu materiallar oliy texnika o'quv yurtlarining bakalavriat talabalariga mo'ljallangan. Mualliflar bu qo'llanmani yozishda o'quvchi "Oliy matematika" ning respublikamiz oliy o'quv yurtlarida o'qitiladigan kursi bilan tanishligini ko'zda tutishgan. Mualliflar shuni ham ko'zda tutishganki, o'quvchini bu qo'llanmada faqatgina o'z mutaxassisligiga aloqasi bo'lgan bo'limlar qiziqtirishi mumkin. Shunga ko'ra qo'llanma bayoni shunday qurilganki, o'quvchi o'zining mutaxassisligiga xos bo'lgan bo'limlarini alohida o'rganish imkoniga ega. Hususan, ko'pgina matematik fizika masalalarini yechish usuli, issiqlik tarqalish tenglamalari uchun ham, tor tebranish masalalari uchun ham bir xil darajada keltirilgan.

1-§ Ikkinchi tartibli xususiy hosilali differensial tenglamalarni tiplarga ajratish

Ikkinchi tartibli xususiy hosilali differensial tenglama deb ikki o'zgaruvchi x va y ga bog'liq u = u (x,y) funksiya va uning birinchi va ikkinchi tartibli xususiy hosilalari orasidagi quyidagi munosabatga aytiladi:

$$F(x, y, u, u'_x, u'_y, u''_{xx}, u''_{xy}, u''_{yy}) = 0$$

Agar tenglama quyidagicha yozilsa:

$$a_{11}u''_{xx} + 2a_{12}u''_{xy} + a_{22}u''_{yy} + F_1(x, y, u, u'_x, u'_y) = 0 \qquad (1)$$

u yuqori tartibli hosilalarga nisbatan chiziqli deyiladi.
Bu yerda a_{ij} koeffitsientlar x va y larning funksiyalari.

Agar tenglama barcha tartibli hosilalariga nisbatan va u(x,y) ga nisbatan chiziqli, ya'ni

$$a_{11}u''_{xx} + 2a_{12}u''_{xy} + a_{22}u''_{yy} + b_1u'_x + b_2u'_y + cu + f = 0 \qquad (2)$$

ko'rinishda bo'lsa, tenglama chiziqli deyiladi, Bunda a_{ij}, b_i, c va f lar faqat x va y larning funksiyalaridir. Agar ular x va y larga bog'liq bo'lmasalar, bu holda tenglama o'zgarmas koeffitsientli deyiladi. Agar f = 0 bo'lsa, u holda tenglama bir jinsli deyiladi. Teskarisi mavjud bo'lgan quyidagi

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y)$$

almashtirishlar orqali biz oldingiga ekvivalent yangi tenglama hosil qilamiz. Bunda quyidagi savol tug'iladi: ξ va η larni qanday tanlaganda bu o'zgaruvchilarga nisbatan tenglama yetarlicha soddaroq ko'rinishga ega bo'ladi?

(1) Tenglamaga nisbatan bu savolga javob qidiramiz: Hosilalarni yangi o'zgaruvchilarga o'tkazsak, quyidagilarni hosil qilamiz:

$$\begin{aligned}
u'_x &= u'_\xi \xi'_x + u'_\eta \eta'_x \\
u'_y &= u'_\xi \xi'_y + u'_\eta \eta'_y \\
u''_{xx} &= u''_{\xi\xi} \xi'^2_x + 2u''_{\xi\eta} \xi'_x \eta'_x + u''_{\eta\eta} \eta'^2_x + u'_\xi \xi''_{xx} + u'_\eta \eta''_{xx} \\
u''_{xy} &= u''_{\xi\xi} \xi'_x \xi'_y + u''_{\xi\eta} (\xi'_x \eta'_y + \xi'_y \eta'_x) + u''_{\eta\eta} \eta'_x \eta'_y + u'_\xi \xi''_{xy} + u'_\eta \xi''_{xy} \\
u''_{yy} &= u''_{\xi\xi} \xi'^2_y + 2u''_{\xi\eta} \xi'_y \eta'_y + u''_{\eta\eta} \eta'^2_y + u'_\xi \xi''_{yy} + u'_\eta \eta''_{yy}
\end{aligned} \tag{3}$$

(3) da hosilalarning qiymatlarini (1) tenglamaga qo'yib quyidagini olamiz:

$$\bar{a}_{11} u''_{\xi\xi} + 2\bar{a}_{12} u''_{\xi\eta} + \bar{a}_{22} u''_{\eta\eta} + \bar{F} = 0 \tag{4}$$

Bunda

$$\bar{a}_{11} = a_{11} \xi'^2_x + 2a_{12} \xi'_x \xi'_y + a_{22} \xi'^2_y$$

$$\bar{a}_{12} = a_{11} \xi'_x \eta'_x + a_{12} (\xi'_x \eta'_y + \eta'_x \xi'_y) + a_{22} \xi'_y \eta'_y$$

$$\bar{a}_{22} = a_{11} \eta'^2{}_x + 2a_{12} \eta'_x \eta'_y) + a_{22} \eta'^{\,2}_y$$

$\bar{F}$ esa ikkinchi tartibli hosilalarga bog'liq emas. ξ va η ni shunday tanlaymizki, $\bar{a}_{11} = 0$ bo'lsin. Shu munosabat bilan quyidagi birinchi tartibli xususiy hosilali differensial tenglamani ko'rib chiqamiz:

$$a_{11} z'^2_x + 2a_{12} z'_x z'_y + a_{22} z'^2_y = 0 \tag{5}$$

$z=\varphi(x,y)$ - bu tenglamaning birorta xususiy yechimi bo'lsin. Agar $\xi=\xi(x,y)$ deb olsak, albatta $\bar{a}_{11}$=0 bo'ladi. Shunday qilib yangi o'zgaruvchilarni tanlash masalasi (5) tenglamani yechish bilan bog'liq ekan.

Lemma:

1). Agar z=φ (x,y) funksiya

$$a_{11}z_x'^2+2a_{12}z_x'z_y'+a_{22}z_y'^2=0$$

tenglamaning xususiy yechimi bo'lsa, u holda φ (x,y)=C munosabat

$$a_{11}dy^2-2a_{12}dxdy+a_{22}dx^2=0 \qquad (6)$$

oddiy differensial tenglamaning umumiy integrali bo'ladi.

2). Agar φ (x,y)=C munosabat (6) tenglamaning integrali bo'lsa, u holda z=φ(x,y) funksiya (5) tenglamani qanoatlantiradi. (6) tenglama (1) tenglamaning xarakteristik tenglamasi deyiladi, uning integrallari uning xarakteristikalari deyiladi.

$\xi=\varphi$ (x,y) deb olib, (bu yerda φ (x,y)=C (6) tenglamaning umumiy integrali), biz (4) tenglamadagi $u''_{\xi\xi}$ oldidagi koeffitsientni nolga tenglaymiz. Agar Ψ (x,y)=C (6) tenglamaning boshqa φ (x,y) ga bog'liq bo'lmagan umumiy integrali bo'lsa, u holda $\eta=\Psi$ (x,y) deb olib, (4) dagi $u''_{\eta\eta}$ oldidagi koeffitsientni ham nolga tenglaymiz.

(6) tenglama quyidagi ikki tenglamaga ajraladi:

$$\frac{dy}{dx}=\frac{a_{12}+\sqrt{a_{12}^2-a_{11}a_{22}}}{a_{11}} \qquad (7)$$

$$\frac{dy}{dx}=\frac{a_{12}-\sqrt{a_{12}^2-a_{11}a_{22}}}{a_{11}} \qquad (8)$$

Ildiz ostidagi ifodaning ishorasi (1) - tenglamaning tipini

aniqlaydi;
M(x,y) nuqtada biz (1) tenglamani :

-agar M(x,y) nuqtada $a_{12}^2 - a_{11}, a_{22}$>0 bo'lsa, giperbolik tipdagi tenglama deymiz,

-agar M(x,y) nuqtada $a_{12}^2 - a_{11}, a_{22}$<0 bo'lsa, elliptik tipdagi tenglama deymiz,

-agar M(x,y) nuqtada $a_{12}^2 - a_{11}, a_{22}$=0 bo'lsa, parabolik tipdagi tenglama deymiz.

Tenglama yo'l qo'yadigan sohadagi turlicha nuqtalarda tenglama turlicha tipda bo'lishi mumkin. Barcha nuqtalarida tenglama bir xil tipda bo'lgan G sohani qaraymiz. G ning har bir nuqtasidan ikkita xarakteristika o'tadi; giperbolik tipdagi tenglama uchun xarakteristikalar haqiqiy va turlicha, elliptik tipdagi tenglamalar uchun kompleks ko'rinishda va turlicha, parabolik tipdagi tenglamalar uchun ikkala xarakteristikalar haqiqiy va bir xil.
Bu har bir holni alohida ko'rib chiqamiz:
(1) Giperbolik tip: $a_{12}^2 - a_{11}a_{22}$>0 va (7) va (8) ning o'ng tarafi haqiqiy va har xil. Ularning umumiy integrallari φ (x,y)=C va Ψ (x,y)=C lar haqiqiy xarakteristikalar oilasini aniqlaydilar. Agar $\xi=\varphi$ (x,y) va $\eta=\Psi$ (x,y) deb olsak, (4) dagi $\bar{a}_{11}$ va $\bar{a}_{22}$ koeffitsientlar nolga teng bo'ladi va $u''_{\xi\eta}$ oldidagi koeffitsientga bo'lish natijasida quyidagi tenglamaga kelib qolamiz:

$u''_{\xi\eta} = \phi$, bu yerda $\phi = -\dfrac{\overline{F}}{2\bar{a}_{12}}$;

Bu tenglama giperbolik tipdagi tenglamaning kanonik ko'rinishi deyiladi, ko'p hollarda kanonik ko'rinishning boshqa formasidan foydalaniladi, ya'ni $\xi = \alpha + \beta, \eta = \alpha - \beta$ deb olsak,

$\alpha=\dfrac{\xi+\eta}{2}$, $\beta=\dfrac{\xi-\eta}{2}$ bo'ladi (bu yerda α va β lar yangi o'zgaruvchilar). U holda:

$$u'_{\xi}=\frac{1}{2}(u'_{\alpha}+u'_{\beta}), \qquad u'_{\eta}=\frac{1}{2}(u'_{\alpha}-u'_{\beta}),$$

$$u''_{\xi\eta}=\frac{1}{4}(u''_{\alpha\alpha}-u''_{\beta\beta}).$$

Natijada $u''_{\alpha\alpha}-u''_{\beta\beta}=\phi_{1,}$ $\quad(\phi_1=4\phi)$.

2. Parabolik tip; $a_{12}^2-a_{11}a_{22}$=0, (7) va (8) tenglamalar bir xil va biz (6) tenglama uchun bitta umumiy φ (x,y) =C integralga ega bo'lamiz. Bu holda biz $\xi=\varphi(x,y),\eta=\eta(x,y)$ deb olamiz.

Bunda $\eta=\eta$(x,y) - ixtiyoriy φ (x,y) ga bog'liq bo'lmagan funksiya.

(ya'ni $J=\begin{vmatrix}\varphi'_x & \varphi'_y\\ \eta'_x & \eta'_y\end{vmatrix}\neq 0$)

O'zgaruvchilarni shu tariqa tanlaganimizda $\bar{a}_{11}$ koeffitsient uchun

$$\bar{a}_{11}=a_{11}\xi'^2_x+2a_{12}\xi'_x\xi'_y+a_{22}\xi'^2_y=(\sqrt{a_{11}}\xi'_x+\sqrt{a_{22}}\xi'_y)^2=0,$$

tenglik o'rinli bo'ladi, chunki $a_{12}=\sqrt{a_{11}}\sqrt{a_{22}}$. Bundan esa

$$\bar{a}_{12}=a_{11}\xi'_x\eta'_x+a_{12}(\xi'_x\eta'_y+\xi'_y\eta'_y)+a_{22}\xi'_y\eta'_y=$$

$$=(\sqrt{a_{11}}\xi'_x+\sqrt{a_{22}}\xi'_y)\cdot(\sqrt{a_{11}}\eta'_x+\sqrt{a_{22}}\eta'_y)=0.$$

(4) tenglamani $u_{\eta\eta}$ oldidagi koeffitsientga bo'lib, parabolik tipdagi tenglamaning kanonik ko'rinishiga kelamiz:

$$u''_{\eta\eta}=\phi, \qquad (\phi=-\frac{\bar{F}}{a_{22}}).$$

3. Elliptik tip: $a_{12}^2-a_{11}a_{22}$<0 va (7) , (8) larning o'ng tomon-

lari kompleks ko'rinishda. Demak, xarakteristikalar kompleks va qo'shma kompleks ko'rinishda $(\varphi(x,y)=\overline{\psi(x,y)})$ bo'ladi hamda

$$\xi=\frac{1}{2}(\varphi(x,y)+\psi(x,y))=\operatorname{Re}\varphi(x,y).$$

va

$$\eta=\frac{1}{2i}(\varphi(x,y)-\psi(x,y))=\operatorname{I\overline{m}}\varphi(x,y),$$

deb olamiz.

U holda (4) tenglamada $\bar{a}_{11}=\bar{a}_{22}$ va $\bar{a}_{12}=0$ ni hosil qilamiz. (4) ni $u''_{\xi\xi}$ oldidagi koeffitsientga bo'lib quyidagiga kelamiz:

$$u''_{\xi\xi}+u''_{\eta\eta}=\phi, \qquad (\phi=-\frac{\overline{F}}{\bar{a}_{22}}).$$

Shunday qilib $a_{12}^2-a_{11}a_{22}$ ifodaning ishorasiga qarab quyidagi kononik tenglamalarga ega bo'lamiz:

- $a_{12}^2-a_{11}a_{22}>0$ da $u''_{xx}-u''_{yy}=\phi$ yoki $u''_{xy}=\phi$ (giperbolik tip);
- $a_{12}^2-a_{11}a_{22}<0$ da $u''_{xx}+u''_{yy}=\phi$ (elliptik tip);
- $a_{12}^2-a_{11}a_{22}=0$ da $u''_{xx}=\phi$ yoki $u''_{yy}=\phi$ (parabolk tip);

Yuqorida ko'rilgan (1) tenglamani kanonik ko'rinishga keltirish metodi va uni yechish metodi xarakteristikalar metodi deyiladi.

2-§ Matematik fizika tenglamalarining tiplari

Ko'pgina fizika va texnika masalalarini yechish xususiy hosilali differensial tenglamalarga olib keladi. Bunda 2-tartibli xususiy hosilali differensial tenglamalar eng ko'p uchraydi.

u=u(x,y) ikki o'zgaruvchili funksiya uchun quyidagi misollar-ni keltiramiz.

1. **Torda to'lqin tarqalishi tenglamasi:**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Bu tenglamani tekshirishga torning ko'ndalang tebranishi jarayonini kuzatish, sterjenning bo'ylama tebranishi, simdagi elektr tebranishlari, gaz molekulalari tebranishlari va boshqalarni kuzatish davomida to'qnash kelamiz. Bu tenglamalar giperbolik tipdagi eng sodda tenglamalardir.

2. **Kesmada issiqlik tarqalishi tenglamasi:**

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

Bu tenglamani tekshirishga issiqlik tarqalishi jarayonini kuzatish, suyuqlik va gaz filtratsiyasi, ehtimollar nazariyasining ba'zi bir masalalari olib keladilar. Bu tenglama eng sodda ko'rinishdagi parabolik tip tenglamadir.

3. **Laplas tenglamasi:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3)$$

Bu tenglamani tekshirishga elektr va magnit maydoni haqidagi masalalarni kuzatish jarayonida, statsionar issiqlik holatini tekshirish, gidrodinamika masalalari, diffuziya va boshqa masalalarni yechishda duch kelinadi. Bu tenglama eng sodda elliptik tipdagi tenglamadir.

1-misol

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0, \qquad xy \neq 0$$

ko'rinshdagi tenglamani kanonik (sodda) ko'rinishiga keltiring.

Yechish.

Bu yerda

$a_{11} = x^2, a_{12} = 0, a_{22} = -y^2$ bo'lib, $a_{12}{}^2 - a_{11}a_{22} = x^2y^2 > 0$. Demak, tenglamamiz giperbolik tip ekan.

Uning xarakteristik tenglamasi $x^2dy^2 - y^2dx^2 = 0$ yoki $(xdy + ydx)(xdy - ydx) = 0$ bo'lib u ikkita differensial tenglamadan iborat, ularning o'zgaruvchilarini ajiratib, $\frac{dy}{y} + \frac{dx}{x} = 0, \frac{dy}{y} - \frac{dx}{x} = 0$ ko'rinishga ega bo'ladi, bularni integrallab $\ln y + \ln x = \ln c_1$ va $\ln y - \ln x = \ln c_2$ yoki $xy = c_1$ va $\frac{y}{x} = c_2$ ga ega bo'lib, va $\zeta = xy, \eta = \frac{y}{x}$ almashtirish bajarsak, eski o'zgaruvchilar bo'yicha olingan xususiy hosilalarni yangi o'zgaruvchilar orqali olamiz.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = y\frac{\partial u}{\partial \zeta} - \frac{y}{x^2}\frac{\partial u}{\partial \eta};$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = x\frac{\partial u}{\partial \zeta} - \frac{1}{x}\frac{\partial u}{\partial \eta}; \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(y\frac{\partial u}{\partial \zeta}) - \frac{\partial}{\partial x}(\frac{y}{x^2}\frac{\partial u}{\partial \eta}) =$$

$$y(\frac{\partial^2 u}{\partial \zeta^2}\frac{\partial \zeta}{\partial x} + \frac{\partial^2 u}{\partial \zeta \partial \eta}\frac{\partial \eta}{\partial x}) - \frac{y}{x^2}(\frac{\partial^2 u}{\partial \eta \partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}) +$$

$$\frac{2y}{x^3}\frac{\partial u}{\partial \eta} = y(y\frac{\partial^2 u}{\partial \zeta^2} - \frac{y}{x^2}\frac{\partial^2 u}{\partial \zeta \partial \eta}) - \frac{y}{x^2}(y\frac{\partial^2 u}{\partial \zeta \partial \eta} - \frac{y}{x^2}\frac{\partial^2 u}{\partial \eta^2}) + \frac{2y}{x^3}\frac{\partial u}{\partial \eta} =$$

$$= y^2\frac{\partial^2 u}{\partial \zeta^2} - 2\frac{y^2}{x^2}\frac{\partial^2 u}{\partial \zeta \partial \eta} + \frac{y^2}{x^4}\frac{\partial^2 u}{\partial \eta^2} + 2\frac{y}{x^3}\frac{\partial u}{\partial \zeta};$$

$$\frac{\partial^2 u}{\partial y^2} = x(\frac{\partial^2 u}{\partial \zeta^2}\frac{\partial \zeta}{\partial y} + \frac{\partial^2 u}{\partial \zeta \partial \eta}\frac{\partial \eta}{\partial y}) + \frac{1}{x}(\frac{\partial^2 u}{\partial \eta \partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{1}{x}\frac{\partial^2 u}{\partial y^2}) =$$

$$x^2\frac{\partial^2 u}{\partial \zeta^2} + 2\frac{\partial^2 u}{\partial \zeta \partial \eta} + \frac{1}{x^2}\frac{\partial^2 u}{\partial \eta^2}$$

$\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ - larning bu qiymatlarini berilgan differensial tenglamaga qo'yib va uni soddalashtirib, quyidagi kanonik tenglamaga ega bo'lamiz.

$$x^2(y^2\frac{\partial^2 u}{\partial \zeta^2}-2\frac{\partial^2 u}{\partial \zeta\partial \eta}+\frac{y^2}{x^4}\frac{\partial^2 u}{\partial \eta^2}+2\frac{y}{x^2}\frac{\partial u}{\partial \eta}-$$

$$y^2(x^2\frac{\partial^2 u}{\partial \zeta^2}+2\frac{\partial^2 u}{\partial \zeta\partial \eta}+\frac{1}{x^2}\frac{\partial^2 u}{\partial \eta^2})=0$$

yoki $-4y^2\frac{\partial^2 u}{\partial \zeta\partial \eta}+2\frac{y}{x}\frac{\partial u}{\partial \eta}=0$, yoki $\frac{\partial^2 u}{\partial \zeta\partial \eta}-\frac{1}{2}\frac{1}{xy}\frac{\partial u}{\partial \eta}=0$.

bundan $\frac{\partial^2 u}{\partial \zeta\partial \eta}-\frac{1}{2\zeta}\frac{\partial u}{\partial \eta}=0$

2-misol

$$\sin^2 x\cdot\frac{\partial^2 z}{\partial x^2}-2y\sin x\cdot\frac{\partial^2 z}{\partial x\partial y}+y^2\cdot\frac{\partial^2 z}{\partial y^2}=0.$$

ko'rinshdagi tenglamani kanonik (sodda) ko'rinishga keltiring.

Yechish:

Bu yerda $a_{11}=\sin^2 x$, $a_{12}=-y\sin x$, $a_{22}=y^2$ bo'lib, $a_{12}{}^2-a_{11}a_{22}=y^2\cdot\sin^2 x-y^2\cdot\sin^2 x=0.$ Demak, tenglama parabolik tip ekan. Xarakteristik tenglama quyidagi ko'rinishga ega:

$$\sin^2 x\cdot dy^2+2y\cdot\sin x\cdot dxdy+y^2\cdot dx^2=0,$$

yoki

$$(\sin x\cdot dy+ydx)^2=0.$$

Quyidagi

$$\sin x\cdot dy+ydx=0.$$

tenglamada o'zgaruvchilarni ajratib

$$\frac{dy}{y}+\frac{dx}{\sin x}=0.$$

tenglamani hosil qilamiz.
Uni integrallab

$$\ln y+\ln \operatorname{tg}\frac{x}{2}=\ln C,$$

yoki

$$y\cdot \operatorname{tg}\frac{x}{2}=C$$

yechimni olamiz.

Quyidagi almashtirishlarni bajaramiz: $\xi=y\cdot \operatorname{tg}\frac{x}{2}, \qquad \eta=y$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{1}{2}y\sec^2\frac{x}{2}\frac{\partial z}{\partial \xi};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}=\operatorname{tg}\frac{x}{2}\frac{\partial z}{\partial \xi}+\frac{\partial z}{\partial \eta};$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{2}y\sec^2\frac{x}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial x}+\frac{\partial^2 z}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial x}\right)+\frac{1}{2}y\sec^2\frac{x}{2}\cdot \operatorname{tg}\frac{x}{2}\frac{\partial z}{\partial \xi}\cdot=$$

$$=\frac{1}{4}y^2\sec^4\frac{x}{2}\frac{\partial^2 z}{\partial \xi^2}+\frac{1}{2}y\sec^2\frac{x}{2}\operatorname{tg}\frac{x}{2}\frac{\partial z}{\partial \xi};$$

$$\frac{\partial^2 z}{\partial y^2}=\operatorname{tg}\frac{x}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial y}\right)+\frac{\partial^2 z}{\partial \eta\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \eta^2}\frac{\partial \eta}{\partial y}=$$

$$=\operatorname{tg}^2\frac{x}{2}\frac{\partial^2 z}{\partial \xi^2}+2\operatorname{tg}\frac{x}{2}\frac{\partial^2 z}{\partial \xi\partial \eta}+\frac{\partial^2 z}{\partial \eta^2};$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{2}y\sec^2\frac{x}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial y}\right)+\frac{1}{2}\sec^2\frac{x}{2}\frac{\partial z}{\partial \xi}=$$

$$=\frac{1}{2}y\sec^2\frac{x}{2}\left(\operatorname{tg}\frac{x}{2}\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \xi\partial \eta}\right)+\frac{1}{2}\sec^2\frac{x}{2}\frac{\partial z}{\partial \xi}.$$

$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x\partial y}$ larning qiymatlarini differensial tenglamaga qo'yamiz:

$$\frac{1}{4}y^2\sec^4\frac{x}{2}\cdot\sin^2 x\frac{\partial^2 z}{\partial \xi^2}+\frac{1}{2}y\sec^2\frac{x}{2}\operatorname{tg}\frac{x}{2}\sin^2 x\frac{\partial z}{\partial \xi}-$$

$$-y^2\sec^2\frac{x}{2}\sin x\left(\operatorname{tg}\frac{x}{2}\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \xi\partial \eta}\right)-y\sec^2\frac{x}{2}\sin x\frac{\partial z}{\partial \xi}+$$

$$+y^2\left(\operatorname{tg}^2\frac{x}{2}\frac{\partial^2 z}{\partial \xi^2}+2\operatorname{tg}\frac{x}{2}\frac{\partial^2 z}{\partial \xi\partial \eta}+\frac{\partial^2 z}{\partial \eta^2}\right)=0.$$

Osongina ko'rsatish mumkinki, $\frac{\partial^2 z}{\partial \xi^2}$ va $\frac{\partial^2 z}{\partial \eta^2}$ -larni o'z ichiga olgan hadlar o'zaro qisqaradi va tenglama quyidagi ko'rinishga keladi:

$$\frac{1}{2}y\sec^2\frac{x}{2}\operatorname{tg}\frac{x}{2}\sin^2 x\frac{\partial z}{\partial \xi}+y^2\frac{\partial^2 z}{\partial \eta^2}-y\sec^2\frac{x}{2}\sin x\frac{\partial z}{\partial \xi}=0,$$

yoki

$$y\frac{\partial^2 z}{\partial \eta^2}=\sin x\frac{\partial z}{\partial \xi},$$

lekin

$$\sin x=\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}},\qquad \operatorname{tg}\frac{x}{2}=\frac{\xi}{\eta},$$

bo'lganligi uchun

$$\sin x = \frac{2\xi\eta}{\xi^2 + \eta^2}.$$

Natijada

$$\frac{\partial^2 z}{\partial \eta^2} = \frac{2\xi}{\xi^2 + \eta^2}\frac{\partial z}{\partial \xi}.$$

3-misol

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$$

ko'rinshdagi tenglamani kanonik (sodda) ko'rinishga keltiring.

Yechish:

Bu yerda

$a_{11} = 1, \quad a_{12} = -1, \quad a_{22} = 2,$ bo'lib, $a_{12}{}^2 - a_{11}a_{22} = -1 < 0.$

Demak tenglamamiz elliptik tip ekan. Xarakteristik tenglama

$dy^2 + 2\cdot dxdy + 2\cdot dx^2 = 0,$ yoki $y'^2 + 2y' + 2 = 0.$

Bundan $y' = -1 \pm i;$ Demak ikkita kompleks yechimlar oilasini hosil qilamiz: $y + x - ix = C_1$ i $y + x + ix = C_2$.

O'zgaruvchilarni almashtiramiz: $\xi = y + x, \qquad \eta = x;$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}; \quad \frac{dz}{dy} = \frac{dz}{d\xi}\cdot\frac{d\xi}{dy} + \frac{dz}{d\eta}\cdot\frac{d\eta}{dy} = \frac{dz}{d\xi};$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial^2 z}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right) =$$

$$= \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}; \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi \partial \eta};$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial^2 z}{\partial \xi^2}.$$

Differensial tenglamaga xususiy hosilalarning topilgan ifodalarini qo'yib, quyidagini hosil qilamiz:

$$\frac{\partial^2 z}{\partial \xi^2}+2\frac{\partial^2 z}{\partial \xi \partial \eta}+\frac{\partial^2 z}{\partial \eta^2}-2\frac{\partial^2 z}{\partial \xi^2}-2\frac{\partial^2 z}{\partial \xi \partial \eta}+2\frac{\partial^2 z}{\partial \xi^2}=0.$$

ya'ni

$$\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2}=0.$$

4-misol

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x \partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

ko'rinishdagi tenglamani kanonik (sodda) ko'rinishga keltiring.

Javob: $\frac{\partial^2 z}{\partial \eta^2}=0, \qquad \xi=\frac{y}{x}, \qquad \eta=y.$

5-misol

$$\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x \partial y}-3\frac{\partial^2 z}{\partial y^2}-2\frac{\partial z}{\partial x}+6\frac{\partial z}{\partial y}=0.$$

ko'rinishdagi tenglamani kanonik (sodda) ko'rinishga keltiring.

Javob: $\frac{\partial^2 z}{\partial \xi \partial \eta}-\frac{\partial z}{\partial \xi}=0, \qquad \xi=x+y, \qquad \eta=3x+y.$

6-misol

$$xy \neq 0 \text{ bo'lganda } \frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} = 0.$$

ko'rinishdagi tenglamani kanonik (sodda) ko'rinishga keltiring.

Javob:

$$\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{1}{2}\left(\frac{1}{\xi}\frac{\partial z}{\partial \xi} + \frac{1}{\eta}\frac{\partial z}{\partial \eta}\right) = 0, \qquad \xi = y^2, \qquad \eta = x^2.$$

3-§ Asosiy masalalarning qo'yilishi: Koshi masalasi, chegaraviy masalalar, aralash masalalar.

Fizik jarayonlarni matematik tilda yozish uchun avvalo masala qo'yilishi shart, ya'ni jarayonni bir qiymatli aniqlab beruvchi shartlar bayon qilinishi kerak. Oddiy differensial tenglamalar va ayniqsa xususiy hosilali differensial tenglamalar umuman olganda cheksiz ko'p yechimga ega.

Shuning uchun fizik masala xususiy hosilali differensial tenglamalarga olib kelinadigan holda, jarayonni bir qiymatli aniqlash uchun tenglamaga ba'zi bir qo'shimcha shartlar qo'yiladi. Ikkinchi tartibli differensial tenglamalar bo'lgan holda, yechim boshlang'ich shartlar orqali aniqlanishi mumkin, ya'ni argumentning boshlang'ich qiymatiga mos keluvchi funksiyaning va uning birinchi hosilasining qiymatlari beriladi (Koshi masalasi).

Xususiy hosilali differensial tenglamalar uchun ham turli ko'rinishdagi qo'shimcha shartlar qo'yilishi mumkin.

Eng avvalo quyidagi sodda masalani qaraymiz: uchlari mahkamlangan torning ko'ndalang tebranishi masalasi; Bu masalada $u(x,t)$ deb torning OX o'qidan chetlanishini qaraymiz. Agar torning uchlari $0 \leq x \leq \ell$ kesmaning uchlarida mahkamlangan bo'lsa, u holda quyidagi «chegaraviy shartlar» bajarilishi zarur

$$u(0,t)=0, u(\ell,t)=0 \qquad (1)$$

Tor tebranishi uning $t=t_0$ vaqtdagi boshlang'ich formasiga va tezlik taqsimlanishiga bog'liq bo'lganligi sababli quyidagi "boshlang'ich shart»lar berilishi zarur:

$$\begin{cases} u(x,t_0)=\varphi(x) \\ u'_t(x,t_0)=\psi(x) \end{cases} \qquad (2)$$

Shunday qilib, qo'shimcha shartlar «chegaraviy» va «boshlang'ich» shartlardan iborat bo'ladi, bunda φ (x) va ψ (x) lar berilgan funksiyalar. Bu shartlar tor tebranishi jarayonini, ya'ni

$$u''_{tt}=a^2u''_{xx} \qquad (3)$$

tenglama uchun (1), (2) chegaraviy masala yechimini to'liq aniqlab beradi. Boshqa ko'rinishdagi chegaraviy shartlar ham berilishi mumkin. (1) chegaraviy shartlar birinchi tur shartlar deb ataladi. u(x,t) funksiyaning x bo'yicha hosilasi $u'_x(x,t)$ga qo'yilgan chegaraviy shartlar

$$\begin{cases} u'_x(0,t)=0 \\ u'_x(\ell,t)=0 \end{cases} \qquad (4)$$

ikkinchi tur chegaraviy shartlar deb ataladi. Agar chegaraviy shartlar u(x,y) ning o'zining qiymatlariga va uning hosilasi $u'_x(x,t)$ ga qo'yilgan bo'lsa:

$$\begin{cases} \left(\frac{\partial u}{\partial x}+\delta u\right)_{x=0}=0 \\ \left(\frac{\partial u}{\partial x}+\delta u\right)_{x=\ell}=0 \end{cases} \qquad (5)$$

uchinchi tur chegaraviy shartlar deb ataladi. (1) - (5) shartlardan tashqari, bundan so'ng uchta asosiy tur chegaraviy shartlar haqida gapiramiz:

- birinchi tur chegaraviy shart: $u(0,t)=\mu(t)$ – berilgan rejim,

- ikkinchi tur chegaraviy shart: $u'_x(0,t)=\nu(t)-$berilgan kuch,

-uchinchi tur chegaraviy shart: $u'_x(0,t)=h[u(0,t)-\theta(t)]-$elastik mahkamlanish.

Xuddi shunday torning ikkinchi uchi $x=\ell$ ga ham chegaraviy shartlar qo'yiladi. Agar $\mu(t)$, $\nu(t)$, va $\theta(t)$ lar nolga teng bo'lsa, u holda bu shartlar bir jinsli deb ataladi. Endi $u''_{tt}=a^2u''_{xx}+f(x,t)$ tenglama uchun birinchi chegaraviy masalani keltiramiz: $0\le x\le\ell$ va t ≥ 0 sohada aniqlangan va

$$u''_{tt}=a^2u''_{xx}+f(x,t),\quad 0<x<\ell,\qquad t>0.$$

tenglamani,

$$\begin{cases} u(0,t)=\mu_1(t), \\ u(l,t)=\mu_2(t) \end{cases}\qquad t\ge 0$$

chegaraviy shartlarni va

$$\begin{cases} u(x,0)=\varphi(x) \\ u'_t(x,0)=\psi(x) \end{cases}\qquad 0\le x\le\ell.$$

boshlang'ich shartlarni qanoatlantiruvchi u(x,y) funksiyani topish masalasi.

Agar torning uchlari uchun ikkinchi yoki uchinchi chegaraviy shartlarni olsak, bularga mos keluvchi masalalar ikkinchi yoki uchinchi chegaraviy masalalar deb ataladi. Agar x=0 va x=ℓ lardagi chegaraviy shartlar turlicha tiplarga mansub bo'lsa, u holda aralash chegaraviy masalalarga kelamiz.

Agar bizni kichik vaqt oralig'idagi torning o'zgarishi qiziqtirsa, u holda chegaraning ta'siri kam bo'ladi va to'liq masala o'rniga cheksiz uzunlikdagi tor uchun boshlang'ich shartlar ostidagi chegaraviy masalalar qaraladi:

$$u''_{tt}=a^2u''_{xx}+f(x,t),\quad -\infty<x<\infty,\qquad t>0.$$

tenglamaning

$$\begin{cases} u(x,0) = \varphi(x) \\ u'_t(x,0) = \psi(x) \end{cases}, \qquad -\infty < x < \infty$$

boshlang'ich shartlarga bo'ysunuvchi yechimini topish masalasi. Bu masala Koshi masalasidir.

4-§ Matematik fizika masalalari qo'yilishining korrektligi.

Giperbolik va parabolik tipdagi tenglamalar ko'p hollarda vaqt o'tishiga bog'liq bo'ladigan jarayonlarni o'rganishda uchraydi. Bular tebranish tenglamalari, elektr to'lqinlarining tarqalishi tenglamalari va diffuziya tenglamalaridir. Bir o'lchamli holda doimo bitta koordinata x va vaqt t ishtirok etadi.

Shuning uchun bu tiplarning kanonik tenglamalari odatda quyidagicha yoziladi.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + cu = g(x,t) \quad \text{(giperbolik tip)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = g(x,t) \quad \text{(parabolik tip)}$$

Bunday tenglamalarga keltiruvchi masalalar uchun qo'shimcha shartlar boshlang'ich va chegaraviy shartlarga ajratiladi.

Boshlang'ich shartlar, biz ko'rdikki, izlanayotgan funksiya u – ning va uning hosilasining (giperbolik holda) t=0 dagi qiymatini berishdan iborat. Bu masalalarning chegaraviy shartlari u(x,t) noma'lum funksiyaning koordinatalari o'zgaradigan interval chegaralaridagi qiymatlarini berishdan iborat.

Agar kechayotgan jarayon tenglamasida x koordinata cheksiz oraliqda o'zgarsa, u holda chegaraviy shart bo'lmaydi va faqatgina boshlang'ich shartlarga ega masala hosil bo'lib, bu masalani Koshi masalasi deyiladi.

Agar masala chekli oraliqda qo'yilgan bo'lsa, boshlang'ich va chegaraviy shartlar beriladi va bu holda aralash masala haqida gap yuritiladi. Elliptik tipdagi tenglamalar statsionar jarayonlarni o'rganishda yuzaga keladi.

t vaqt bu tenglamalarga kirmaydi va ikkala bog'liqsiz o'zgaruvchilar nuqtaning koordinatalaridan iborat bo'ladi.

Bu tipdagi masalalar uchun faqatgina chegaraviy shartlar qo'yiladi, ya'ni sohaning chegarasida noma'lum funksiyaning holati ko'rsatiladi. Bular noma'lum funksiyalarning qiymati berilgan Dirixle masalasi, noma'lum funksiyaning normal bo'yicha hosilasining qiymati berilgan Neyman masalasi, va nihoyat soha chegarasida funksiyaning chiziqli kombinatsiyasi va uning normal bo'yicha hosilasi berilgan masalalardir.

Matematik fizika masalalarida o'rganilayotgan jarayon xarakteriga javob beruvchi yagona yechimini topish uchun, u yoki bu konkret masalalarda qanday qo'shimcha shartlar qo'yilishi lozimligini, aynan fizik mushohadalar ko'rsatadi. Bundan tashqari, shuni ko'zda tutish lozimki, tenglama jarayonning muhimroq tomonlarini aks ettiradi.

Fizik masalalarda boshlang'ich va chegaraviy shartlarga kiruvchi funksiyalar tajriba natijalariga ko'ra topiladi. Shunday qilib, yechim qaralayotgan barcha sohalarda masala shartlarining yetarlicha kichik o'zgarishlari uning yechimlarining ham kichik o'zgarishiga olib kelishi muhimdir. Bu holda masalaning boshlang'ich va chegaraviy shartlarga nisbatan turg'unligi to'g'risida gapiriladi.

Barcha biz ko'rgan masalalar yagona yechimga ega bo'lib, boshlang'ich ma'lumotlarga ko'ra turg'un masalalar tipiga kiradi. Aytish mumkinki, bu masalalar korrekt qo'yilgan. Yani, 1) yechim mavjud, 2) yechim yagona, 3) yechim turg'undir.

5-§ To'lqin va tor tebranishi tenglamalarini yechish uchun tarqalayotgan to'lqinlar usuli. Dalamber yechimi.

Giperbolik tipdagi tenglamalar uchun chegaraviy masalalarning yechimlarini topish masalasini biz chegaralanmagan tor uchun boshlang'ich shartlarni qanoatlantiruvchi masalani yechish usulini o'rganishdan boshlaymiz:

$$u''_{tt} - a^2 u''_{xx} = 0 \quad -\infty < x < \infty, \qquad t > 0 \qquad (1)$$

$$\begin{cases} u(x,0) = \varphi(x) \\ u'_t(x,0) = \psi(x) \end{cases} \qquad (2)$$

Avvalo bu tenglamani biz $u''_{xt} = 0$ ko'rinishga keltiramiz.

(1) ning $dx^2 - a^2 dt^2 = 0$ xarakteristik tenglamasi ikkita tenglamaga ajraladi: $dx - a \cdot dt = 0$, $dx + a \cdot dt = 0$,
Bularning integrallari: $x - a \cdot t = c_1$, $x + a \cdot t = c_2$
$\xi = x + a \cdot t, \eta = x - a \cdot t$ deb olsak, (1) ga asosan

$$u''_{\xi\eta} = 0 \qquad (3)$$

ga kelamiz. (3) ni integrallasak (ξ bo'yicha)
$u'_\eta(\xi, \eta) = f^*(\eta)$ ni hosil qilamiz. Bunda $f^*(\eta)$ - faqat η ga bog'liq funksiya. Endi oxirgi tenglamani η bo'yicha integrallasak, quyidagiga kelamiz.

$$u(\xi, \eta) = \int f^*(\eta) d\eta + f_1(\xi) = f_1(\xi) + f_2(\eta) \qquad (4)$$

f_1 - ξ ning, f_2 - esa η ning uzluksiz funksiyasi.
Aksincha, f_1 va f_2 lar qanday bo'lishidan qat'iy nazar ikki marta diferensiallanuvchi funksiya bo'lsalar, u holda (4) bilan aniqlangan u(ξ,η) funksiya (3) tenglamaning yechimi bo'ladi.
Demak,

$$u(x, t) = f_1(x + at) + f_2(x - at) \qquad (5)$$

funksiya (1) tenglamaning umumiy yechimi bo'ladi.
Faraz qilamiz, qaralayotgan masalaning yechimi mavjud bo'lsin, u holda bu yechim (5) orqali beriladi. Endi f_1 va f_2 larni shunday tanlab olamizki, ular boshlang'ich shartlarni

qanoatlantirsin, ya'ni:

$$u(x,0) = f_1(x) + f_2(x) = \varphi(x) \qquad (6)$$

$$u'_t(x,0) = a \cdot f'_1(x) - a \cdot f'_2(x) = \psi(x) \qquad (7)$$

(7) ni integrallab,

$$f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(\alpha)d\alpha + c,$$

x_0, c - lar o'zgarmaslar.

$$\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ f_1(x) - f_2(x) = \dfrac{1}{a}\displaystyle\int_{x_0}^{x} \psi(\alpha)d\alpha + c \end{cases}$$

dan

$$\begin{cases} f_1(x) = \dfrac{1}{2}\varphi(x) + \dfrac{1}{2a}\displaystyle\int_{x_0}^{x} \psi(\alpha)d\alpha + \dfrac{c}{2} \\ f_2(x) = \dfrac{1}{2}\varphi(x) - \dfrac{1}{2a}\displaystyle\int_{x_0}^{x} \psi(\alpha)d\alpha - \dfrac{c}{2} \end{cases} \qquad (8)$$

ni topamiz. Shunday qilib biz f_1 va f_2 larni φ va ψ orqali ifodaladik, va bu (8) tenglik ixtiyoriy argumentlar uchun o'rinli. (5) ga f_1 va f_2 ning qiymatlarini (8) dan qo'yib, quyidagini hosil qilamiz.

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\left(\int_{x_0}^{x+at} \psi(\alpha)d\alpha - \int_{x_0}^{x-at} \varphi(\alpha)d\alpha \right)$$

yoki

$$u(x,t) = \frac{\varphi(x+at) - \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha \qquad (9)$$

(9) formula Dalamber formulasi deb ataladi, biz uni berilgan masalaning yechimi mavjud degan farazda topdik. (9)

formula yechimning yagonaligini isbot kiladi.

7-misol

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

ko'rinishdagi differensial tenglamaning $u\Big|_{t=0}=x^2,\quad \frac{\partial u}{\partial t}\Big|_{t=o}=0$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

Yechish:

$$a=1,\qquad \psi(x)=0\qquad \varphi(x)=x^2$$

ekanligidan

$$u=\frac{\varphi(x-at)+\varphi(x+at)}{2}=\frac{(x-t)^2+(x+t)^2}{2}=$$

$$\frac{x^2-2xt+t^2+x^2+2xt+t^2}{2}=\frac{2x^2+2t^2}{2}=x^2+t^2$$

8-misol

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$$

tenglamaning $u\Big|_{t=0}=0,\quad \frac{\partial u}{\partial t}\Big|_{t=o}=x$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

Yechish:

Bu yerda $a=2, \varphi(x)=0, \psi(x)=x$, demak,

$$u=\frac{1}{4}\int_{x-2t}^{x+2t} z\cdot dz=\frac{1}{8}z^2\Big|_{x-2t}^{x+2t}=\frac{1}{8}\left[(x+2t)^2-(x-2t)^2\right]=$$

$$\frac{1}{8}(x^2+4xt+4t^2-x^2+4xt-4t^2)=\frac{1}{8}\cdot 8xt=xt$$

9-misol

$$\frac{\partial^2 u}{\partial t^2}=a^2\cdot\frac{\partial^2 u}{\partial x^2}$$

tenglamaning $u\Big|_{t=0} = \sin x, \quad \dfrac{\partial u}{\partial t}\Big|_{t=o} = 1$ boshlang'ich shartlarni qanoatlantiruvchi va $t = \dfrac{\pi}{2a}$ momentdagi yechimini toping,

Yechish: $u = \dfrac{\sin(x+at)+\sin(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} dz$, ya'ni

$$u = \sin x \cos at + \frac{1}{2a} \cdot z\Big|_{x-at}^{x+at},$$

Bundan esa

$$u = \sin x \cos at + t$$

ni hosil qilamiz. Agar $t = \dfrac{\pi}{2a}$ bo'lsa, $u = \dfrac{\pi}{2a}$, ya'ni tor OX o'qiga parallel.

10-misol Berilgan tenglamaning yechimini toping:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

Bu yerda

$$u\Big|_{t=0} = x, \qquad \frac{\partial u}{\partial t}\Big|_{t=o} = -x.$$

Javob. $u = x(1-t)$.

11-misol Berilgan tenglamaning yechimini toping

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},$$

Bu yerda

$$u\Big|_{t=0} = 0, \qquad \frac{\partial u}{\partial t}\Big|_{t=o} = \cos x.$$

Javob. $u = \dfrac{\cos x \cdot \sin at}{a}.$

12-misol

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

tenglama bilan berilgan torning $t = \pi$ bo'lgandagi

$$u\Big|_{t=0} = \sin x, \left.\frac{\partial u}{\partial t}\right|_{t=o} = \cos x.$$

boshlang'ich shartlarni qanoatlantiruvchi formulasini toping.

Javob. $u = -\sin x$

6-§ To'lqin tenglamasi uchun o'zgartiruvchilarni ajratish usuli

$0 < x < \ell, \qquad t > 0$ sohada berilgan

$$u''_{tt} = a^2 u''_{xx} \qquad (1)$$

tenglamaning bir jinsli chegaraviy

$$u(0,t) = 0, u(\ell,t) = 0 \qquad (2)$$

va boshlangich

$$\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \qquad (3)$$

shartlarni qanoatlantiruvchi yechimini izlaymiz. (1) tenglama bir jinsli va chiziqli, shuning uchun ham xususiy yechimlarning yig'indisi ham bu tenglamaning yechimi bo'ladi. Yetarli ko'p xususiy yechimlarni topgandan so'ng, bu yechimlarni ba'zi koeffitsientlarga ko'paytirib yig'indi olsak, yechim topilishi mumkin. Buning uchun avvalo asosiy yordamchi masala qo'yiladi:

$$u''_{tt} = a^2 u''_{xx}$$

tenglamaning aynan nolga teng bo'lmagan va bir jinsli

$$\begin{cases} u(0,t) = 0 \\ u(\ell,t) = 0 \end{cases} \qquad (4)$$

chegaraviy shartlarni qanoatlantiruvchi

$$u(x,t) = X(x)\cdot T(t) \qquad (5)$$

ko'rinishdagi yechim topilsin. Bu yerda X(x) - faqat x, T(t) – esa faqat t o'zgaruvchiga bog'liq funksiyalar. Agar (5) ni (1) ga qo'ysak,

$$X''(x)T(t) = \frac{1}{a^2}T''(t)X(x)$$

tenglamani hosil qilamiz. Bu tenglikning ikkala tomonini aynan nolga teng bo'lmagan X(x)T(t) ifodaga bo'lib, quyidagi tenglikka kelamiz:

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} \qquad (6)$$

(5) –funksiya (1) ning yechimi bo'lishi uchun (6) tenglik $0<x<\ell$, t>0 larda aynan bajarilishi kerak. (6) ning o'ng tomoni faqat t- ning, chap tomoni esa faqat x ning funksiyasidir. Agar biz t ni fiksirlab, x ni o'zgartirsak, chap tomondagi ifoda ixtiyoriy x - argumentning qiymatlarida o'zgarmas qiymat qabul qilishi ko'rinib qoladi va aksincha, ya'ni

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = -\lambda \qquad (7)$$

Bu yerda λ- o'zgarmas va uning oldidagi minus ishora keyingi hisoblashlar o'ng'ay bo'lishi uchun qo'yildi. Lekin λ - ixtiyoriy ishorada bo'lishi mumkin. (7) munosabatdan quyidagi X(x) va T(t) – funksiyalarni aniqlovchi oddiy differensial tenglamalarni hosil qilamiz:

$$X''(x) + \lambda\cdot X(x) = 0, \qquad X(x) \not\equiv 0 \qquad (8)$$

$$T''(t) + a^2\cdot\lambda\cdot T(t) = 0, \qquad T(t) \not\equiv 0 \qquad (9)$$

(4) chegaraviy shartlar quyidagicha bo'ladi:

$$u(0,t) = X(0)\cdot T(t) = 0$$

$$u(\ell,t) = X(\ell)\cdot T(t) = 0$$

Bulardan kelib chiqadiki, X(x) funksiya quyidagi tenglikni qanoatlantirishi kerak:

$$X(0) = X(\ell) = 0, \qquad (10)$$

aks holda biz

$$T(t) \equiv 0 \quad \text{va} \quad u(x,t) \equiv 0$$

yechimlarga kelib qolamiz. Bu esa noldan farqli yechimlar topilsin degan shartni qanoatlantirmaydi. T(t) funksiya uchun asosiy yordamchi masalada hech qanday qo'shimcha shart yo'q. Shunday qilib X(x) ni topish uchun quyidagi oddiy masala xos qiymatlarni topish masalasiga kelib qolamiz:

- λ parametrning shunday qiymatlari aniqlansinki, quyidagi

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases} \qquad (11)$$

masalaning noldan farqli yechimlari mavjud bo'lsin va bu yechimlar topilsin. λ ning shu xususiyatiga ega bo'lgan qiymatlari xos qiymatlar, bu qiymatlarga mos keluvchi (11) ning noldan farqli yechimlari (11) masalaning xos funksiyalari deyiladi.

Shu tariqa keltirilgan masala Shturm – Liuvill masalasi deb ataladi.

Endi biz λ ning musbat, manfiy va nolga teng qiymatlarini alohida ko'rib chiqamiz.

1. $\lambda<0$ bo'lgan holda masalaning noldan farqli yechimlari

mavjud emas. Haqiqatan, (8) ning umumiy yechimi

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

ko'rinishda bo'lib, chegaraviy shartlardan

$$X(0) = c_1 + c_2 = 0$$

$X(\ell) = c_1 e^{\alpha} + c_2 e^{-\alpha} = 0,$ $(\alpha = \ell \cdot \sqrt{-\lambda})$ larni hosil qilamiz, bulardan esa $c_1 = -c_2$ va $c_1 \cdot (e^{\alpha} - e^{-\alpha}) = 0$ larni hosil qilamiz.

Lekin ko'rilayotgan holda α - haqiqiy va musbat son bo'lib, bundan $e^{\alpha} - e^{-\alpha} \neq 0$ ligi kelib chiqadi.

Shuning uchun $c_1 = 0,$ $c_2 = 0$. Bundan esa $X(x) \equiv 0$ kelib chiqadi.

2. λ=0 bo'lgan holda ham, aynan nolga teng bo'lmagan yechimlar mavjud emas. Haqiqatan bu holda (8) tenglamaning umumiy yechimi $X(x) = c_1 x + c_2$ ko'rinishda bo'lib, chegaraviy shartlar quyidagilarni beradi:

$$X(0) = [c_1 x + c_2]_{x=0} = c_2 = 0$$

$$X(\ell) = c_1 \cdot \ell = 0$$

ya'ni $c_1 = 0,$ $c_2 = 0$; Bulardan esa $X(x) \equiv 0$ kelib chiqadi.

3. λ>0 bo'lgan holda (8) tenglamaning umumiy yechimini quyidagicha yozish mumkin:

$$X(x) = D_1 \cos\sqrt{\lambda}x + D_2 \sin\sqrt{\lambda}x$$

Chegaraviy shartlar quyidagilarni beradi:

$$X(0) = D_1 = 0$$

$$X(l) = D_2 \sin\sqrt{\lambda}\ell = 0$$

Agar X(x)-aynan nolga teng bo'lmasa, u holda $D_2 \neq 0$, shuning uchun

$$\sin\sqrt{\lambda}\ell = 0 \text{ yoki } \sqrt{\lambda} = \frac{\pi \cdot n}{\ell}; \qquad (12)$$

bu yerda n- ixtiyoriy butun son.

Shularga ko'ra, (11) masalaning aynan nolga teng bo'lmagan yechimlari faqatgina

$$\lambda = \lambda_n = \left(\frac{\pi n}{\ell}\right)^2$$

bo'lgan hollarda mavjuddir. Bu xos qiymatlarga quyidagi xos funksiyalar mos keladi:

$$X_n(x) = D_n \sin\frac{\pi n}{l}x\,,$$

bu yerda D_n - ixtiyoriy o'zgarmas. Shunday qilib λ parametrning faqatgina

$$\lambda_n = \left(\frac{\pi n}{\ell}\right)^2 \qquad (13)$$

qiymatlaridagina (11) masalaning aynan noldan farqli yechimlari mavjud bo'lib, uning ko'rinishi ixtiyoriy o'zgarmas ko'paytma aniqligida quyidagicha bo'ladi:

$$X_n(x) = \sin\frac{\pi n}{\ell}x \qquad (14)$$

(biz $D_n = 1$ deb oldik)

$\lambda = \lambda_n$ ning qiymatlarida (9) tenglamaning quyidagi yechimlari mos keladi:

$$T_n(t) = A_n \cos\frac{\pi n}{\ell}at + B_n \sin\frac{\pi n}{\ell}at \qquad (15)$$

bu yerda A_n va B_n -lar ixtiyoriy o'zgarmaslar. (1) - (3) masalaga qaytib, shuni xulosa qilamizki, quyidagi

$$\begin{aligned} u_n(x,t) = x_n(x)T_n(t) = (A_n \cos\frac{\pi n}{\ell}at + \\ + B_n \sin\frac{\pi n}{\ell}at)\sin\frac{\pi n}{\ell}x \end{aligned} \qquad (16)$$

funksiyalar (1) tenglamaning (4) chegaraviy shartlarni qanoatlantiruvchi va biri faqat x ning, ikkinchisi faqat t ning funksiyasi bo'lgan ikki funksiya ko'paytmasi (5) ko'rinishida yozish mumkin bo'lgan xususiy yechimlari bo'ladilar.

Bu yechimlar dastlabki qo'yilgan masalaning (3) boshlang'ich shartlarini, ulardan boshlang'ich funksiyalar φ (x) va ψ (x) larning faqatgina xususiy hollarida qanoatlantirishi mumkin.

Endi biz (1)-(3) masalaning umumiy yechimini topamiz.
(1) tenglamaning bir jinsliligi va chiziqliligiga ko'ra xususiy yechimlar yig'indisi:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}(A_n\cos\frac{\pi n}{\ell}at+B_n\sin\frac{\pi n}{\ell}at)\cdot\sin\frac{\pi n}{\ell}x \quad (17)$$

ham (1) tenglamani va (2) chegaraviy shartlarni qanoatlantiradi. Boshlang'ich shartlar A_n va B_n larni aniqlashga imkon beradi. (17) funksiya (3) shartlarni qanoatlantirishini talab qilamiz:

$$\begin{cases} u(x,0)=\varphi(x)=\sum\limits_{n=1}^{\infty}u_n(x,0)=\sum\limits_{n=1}^{\infty}A_n\sin\frac{\pi n}{\ell}x \\ u'_t(x,0)=\psi(x)=\sum\limits_{n=1}^{\infty}\frac{\partial u_n}{\partial t}(x,0)=\sum\limits_{n=1}^{\infty}\frac{\pi n}{\ell}aB_n\sin\frac{\pi n}{\ell}x \end{cases} \quad (18)$$

Furye qatorlari nazariyasidan ma'lumki, ixtiyoriy $0\le x\le\ell$ da aniqlangan bo'lakli-uzluksiz va bo'lakli-differensiyallanuvchi funksiya f(x) - Furye qatoriga yoyiladi:

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{\pi n}{\ell}x \quad (19)$$

bu yerda

$$b_n=\frac{2}{\ell}\int_0^{\ell}f(\xi)\sin\frac{\pi n}{\ell}\xi d\xi \quad (20)$$

Agar φ (x) va ψ (x) lar Furye qatoriga yoyilish shartlarini bajarsalar, u holda

$$\begin{cases} \varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin\frac{\pi n}{\ell} x \\ \varphi_n = \frac{2}{\ell}\int_0^{\ell} \varphi(\xi)\sin\frac{\pi n}{\ell}\xi d\xi \end{cases} \qquad (21)$$

$$\begin{cases} \psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\frac{\pi n}{\ell} x \\ \psi_n = \frac{2}{1}\int_0^{\ell} \psi(\xi)\sin\frac{\pi n}{\ell}\xi d\xi \end{cases} \qquad (22)$$

Bu qatorlarni (18) formula bilan solishtirish shuni ko'rsatadiki, boshlang'ich shartlar bajarilishi uchun (18) da

$$A_n = \varphi_n, \qquad B_n = \frac{1}{\pi n a}\psi_n \qquad (23)$$

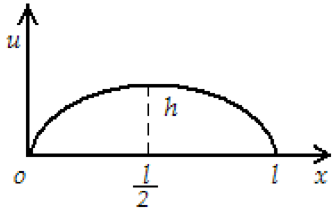
deb olish kerak ekan.

Shunday qilib o'rganilayotgan masalaning yechimi bo'lgan (17) funksiya to'liq aniqlandi.

Biz yechimni (17) cheksiz qator ko'rinishida aniqladik. Agar (17) qator uzoqlashsa yoki bu qator bilan aniqlangan funksiya differensiallanuvchi bo'lmasa, u holda bu qator berilgan differensial tenglamaning yechimi bo'la olmaydi.

13-misol

$x = 0, \qquad x = \ell$ uchlari mahkamlangan torning boshlang'ich momentdagi formasi $u = \frac{4h}{\ell^2}\cdot x(\ell - x)$ parabola ko'rinishida berilgan bo'lsin (1-rasm). Boshlang'ich tezlik nolga teng bo'lganda, tor nuqtalarining abssissa o'qidan chetlanishini toping.



1- rasm

Yechish:
Bu yerda

$$\varphi(x)=\frac{4h}{\ell^2}\cdot x(\ell-x),\qquad \psi(x)=0\,.$$

Qatorning koeffitsientlarini topamiz.

$$a_\kappa=\frac{2}{\ell}\int_0^\ell \varphi(x)\sin\frac{\kappa\pi x}{\ell}dx=\frac{8h}{\ell^3}\int_0^\ell(\ell x-x^2)\sin\frac{\kappa\pi x}{\ell}dx,\qquad b_\kappa=0$$

$$u_1=\ell x-x^2,\qquad d\upsilon_1=\sin\frac{k\pi x}{\ell}dx\,,\qquad du_1=(\ell-2x)dx\,,$$

$$\upsilon_1=-\frac{l}{k\pi}\cdot\cos\frac{k\pi x}{\ell}$$

$$a_k=-\frac{8h}{\ell^3}(lx-x^2)\frac{\ell}{k\pi}\cos\frac{k\pi x}{\ell}\bigg|_0^\ell+\frac{8h}{k\pi\ell^2}\int_0^\ell(\ell-2x)\cos\frac{k\pi x}{\ell}dx$$

ya'ni

$$a_k=\frac{8h}{k\pi\ell^2}\int_0^\ell(\ell-2x)\cos\frac{k\pi x}{\ell}dx\,,$$

$$u_2=\ell-2x\,,\quad d\upsilon_2=\cos\frac{k\pi x}{\ell}dx\,,\quad du_2=-2dx\,,\quad \upsilon_2=\frac{\ell}{k\pi}\cdot\sin\frac{k\pi x}{\ell}\,;$$

$$a_k=-\frac{8h}{k^2\pi^2\ell}(\ell-2x)\cdot\sin\frac{k\pi x}{\ell}\bigg|_0^\ell+\frac{16h}{k^2\pi^2\ell}\cdot\int_0^\ell\sin\frac{k\pi x}{\ell}dx=$$

$$=-\frac{16h}{k^3\pi^3}\cdot(\cos k\pi-1)=\frac{16h}{k^3\pi^3}\cdot[1-(-1)^k].$$

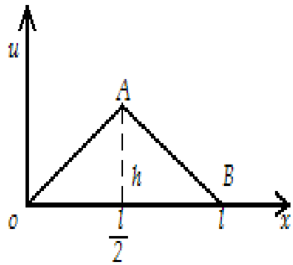
Demak, $u(x,t)=\sum_{k=1}^{\infty}\frac{16h}{k^3\pi^3}\cdot[1-(-1)^k]\cdot\cos\frac{k\pi at}{\ell}\cdot\sin\frac{k\pi x}{\ell}$

Lekin $k=2n$ da $1-(-1)^k=0$, $k=2n+1$ da esa $1-(-1)^k=2$, shuning uchun

$$u(x,t)=\frac{32h}{\pi^3}\cdot\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\cdot\cos\frac{(2n+1)\pi at}{\ell}\cdot\sin\frac{(2n+1)\pi x}{\ell}.$$

14-misol

Chetlari $x=0, x=\ell$ nuqtalar bilan mahkamlangan tor berilgan. Torning boshlang'ich momenti OAB siniq chiziqda berilgan (2-rasm). Agar boshlang'ich tezligi nolga teng bo'lsa, ixtiyoriy t-momentdagi tor tebranishi formulasini toping.



2-rasm

Yechish:

OA to'g'ri chiziqning burchak koeffitsienti $\frac{h}{\ell/2}=\frac{2h}{\ell}$ ga teng.

U holda $u=\frac{2h}{\ell}\cdot x$. AB to'g'ri chiziq esa koordinata o'qida ℓ va $2h$ kesmalarni kesadi, u holda uning tenglamasi $\frac{x}{\ell}+\frac{u}{2h}=\ell$, yoki $u=\frac{2h}{\ell}(\ell-x)$.

Shunday qilib, $\varphi(x)=\begin{cases}\frac{2h}{\ell}x, & 0\le x\le\frac{\ell}{2};\\ \frac{2h}{\ell}(\ell-x), & \frac{\ell}{2}\le x\le\ell,\end{cases}$, $\psi(x)=0$

Demak,

$$a_k=\frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\frac{k\pi x}{\ell}dx=$$

$$=\frac{4h}{\ell^2}\int_0^{\frac{\ell}{2}}x\sin\frac{k\pi x}{\ell}dx+\frac{4h}{\ell^2}\int_{\frac{1}{2}}^{\ell}(\ell-x)\sin\frac{k\pi x}{\ell}dx,\qquad b_k=0$$

buni bo'laklab integrallaymiz.

$$a_k=-\frac{4h}{k\pi\ell}x\cos\frac{k\pi x}{\ell}\Big|_0^{\frac{\ell}{2}}-\frac{4h}{k\pi\ell}\int_0^{\frac{\ell}{2}}\cos\frac{k\pi x}{\ell}dx-\frac{4h}{k\pi\ell}(\ell-x)\cos\frac{k\pi x}{\ell}\Big|_{\frac{\ell}{2}}^{\ell}$$

$$-\frac{4h}{k\pi\ell}\int_{\frac{\ell}{2}}^{\ell}\cos\frac{k\pi x}{\ell}dx=-\frac{2h}{k\pi}\cos\frac{k\pi}{2}+\frac{4h}{k^2\pi^2}\cdot\sin\frac{k\pi x}{\ell}\cdot\Big|_0^{\frac{\ell}{2}}+\frac{2h}{k\pi}\cdot\cos\frac{k\pi}{2}-$$

$$\frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{\ell}\Big|_{\frac{\ell}{2}}^{\ell}=\frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2}+\frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2}=\frac{8h}{k^2\pi^2}\sin\frac{k\pi}{2}$$

natijada, $u(x,t)=\frac{8h}{\pi^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sin\frac{k\pi}{2}\cdot\sin\frac{k\pi x}{\ell}\cdot\cos\frac{k\pi at}{\ell}$;

qatorning bir nechta hadlarini yozsak

$$u(x,t)=\frac{8h}{\pi^2}(\sin\frac{\pi x}{\ell}\cdot\cos\frac{\pi at}{\ell}-\frac{1}{3^2}\cdot\sin\frac{3\pi x}{\ell}\cdot\cos\frac{3\pi at}{\ell}+$$

$$+\frac{1}{5^2}\cdot\sin\frac{5\pi x}{\ell}\cdot\cos\frac{5\pi at}{\ell}-\frac{1}{7^2}\cdot\sin\frac{7\pi x}{\ell}\cdot\cos\frac{7\pi at}{\ell}+...)$$

ko'rinishga ega bo'lamiz.

15-misol

$x=0, x=\ell$ nuqtalar bilan mahkamlangan torning boshlang'ich og'ishi nolga teng, boshlang'ich tezligi esa, quyidagi formula bilan berilgan

$$\frac{\partial u}{\partial t}=\begin{cases} v_0, & \left|x-\frac{\ell}{2}\right|<\frac{h}{2} \\ 0, & \left|x-\frac{\ell}{2}\right|>\frac{h}{2}. \end{cases}$$

Torning t -momentidagi holatini toping.

Yechish:

Bu yerda $\varphi(x)=0,$ $\psi(x)$ esa $(\frac{\ell-h}{2},\frac{\ell+h}{2})$ intervalda v_0 ga teng bo'lib, bu intervaldan tashqarida 0ga teng.

Natijada: $a_k=0$

$$b_k=\frac{2}{k\pi a}\int_{\frac{1}{2}(l-n)}^{\frac{1}{2}(\ell+n)}\upsilon_0\sin\frac{k\pi x}{\ell}dx=-\frac{2\upsilon_0}{k\pi a}\frac{\ell}{k\pi}\cos\frac{k\pi x}{\ell}\Bigg|_{\frac{1}{2}(\ell-n)}^{\frac{1}{2}(\ell+n)}=$$

$$=\frac{2\upsilon_0\ell}{k^2\pi^2 a}\left[\cos\frac{k\pi(\ell-h)}{2\ell}-\cos\frac{k\pi(\ell+h)}{2\ell}\right]=\frac{4\upsilon_0\ell}{k^2\pi^2 a}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2\ell}$$

$$u(x,t)=\frac{4\upsilon_0\ell}{\pi^2 a}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2\ell}\sin\frac{k\pi at}{\ell}\sin\frac{k\pi x}{\ell}$$

yoki

$$u(x,t)=\frac{4\upsilon_0\ell}{\pi^2 a}(\sin\frac{\pi h}{2\ell}\sin\frac{\pi at}{\ell}\sin\frac{\pi x}{\ell}-$$

$$-\frac{1}{3^2}\sin\frac{3\pi h}{2\ell}\sin\frac{3\pi at}{\ell}\sin\frac{3\pi x}{\ell}+\frac{1}{5^2}\sin\frac{5\pi h}{2\ell}\sin\frac{5\pi at}{\ell}\sin\frac{5\pi x}{\ell}-...)$$

16-misol $x=0, x=\ell$ nuqtalar bilan mahkamlangan torning boshlang'ich og'ishi quyidagi formaga ega: $u=h(x^4-2x^3+x)$. Agar boshlang'ich tezlik nolga teng bo'lsa, torning ixtiyoriy t vaqtdagi og'ishi holatini toping.

Javob: $u=\frac{96\cdot h}{\pi^5}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^5}\cdot\cos(2k+1)\cdot\pi\cdot a\cdot t\cdot\sin(2k+1)\cdot\pi\cdot x.$

17-misol Tor $x=0$, $x=\ell$ nuqtalar bilan mahkamlangan. Torning boshlang'ich og'ishi nolga teng bo'lib, boshlang'ich tezligi esa, quyidagi formula bilan berilgan:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} \cos\dfrac{\pi\left(x-\dfrac{\ell}{2}\right)}{h}, & \left|x-\dfrac{\ell}{2}\right| < \dfrac{h}{2}. \\ 0, & \left|x-\dfrac{\ell}{2}\right| \end{cases}$$

Torning ixtiyoriy t vaqtdagi holatini toping.

Javob: $$u(x,t)=\frac{4h\ell^2}{\pi^2 a}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\sin\dfrac{\pi k}{2}\cdot\cos\dfrac{k\pi h}{\ell}}{\ell^2-k^2h^2}\sin\frac{k\pi x}{\ell}\sin\frac{k\pi at}{\ell}.$$

7-§ Kesmada issiqlik tarqalish tenglamalari uchun o'zgaruvchilarni ajratish usuli

$$u'_t = a^2 u''_{xx} \qquad 0<x<\ell, \qquad 0<t\le T \tag{1}$$

tenglamaning

$$u(x,0)=\varphi(x), \qquad 0\le x\le \ell \tag{2}$$

boshlang'ich va

$$u(0,t)=0, \qquad u(\ell,t)=0, \qquad 0\le t\le T \tag{3}$$

chegaraviy shartlarini qanoatlantiruvchi yechimini qidiramiz. Bu masalani yechishda biz o'zgaruvchilarni ajratish usulida qabul qilinganiga ko'ra avval asosiy yordamchi masalani yechamiz:

$$u'_t = a^2 u''_{xx}$$

tenglamaning aynan nolga teng bo'lmagan

$$u(0,t)=0, \qquad u(\ell,t)=0 \tag{3'}$$

bir jinsli chegaraviy shartga bo'ysunuvchi va

$$u(x,t)=X(x)\cdot T(t) \tag{4}$$

(bu yerda X(x)- funksiya faqat x ning, T(t)- esa faqat t ning funksiyasi) ko'rinishidagi yechimini axtaramiz. (4) da u(x,t) ning ko'rinishini (1) tenglamaga qo'yib va hosil bo'lgan tenglikning ikkala tomonini $a^2 X \cdot T$ ga bo'lib, quyidagini hosil qilamiz.

$$\frac{1}{a^2}\frac{T'}{T}=\frac{X''}{X}=-\lambda \qquad (5)$$

bu yerda λ=const, chunki tenglikning o'ng tomoni faqat-gina x ga, chap tomoni esa faqatgina t ga bog'liqdir. Bun-dan esa

$$X''+\lambda\cdot X=0 \qquad (6)$$

$$T'+a^2\lambda\cdot T=0 \qquad (7)$$

tengliklar hosil bo'ladi.

(3) chegaraviy shartlar esa quyidagilarni beradi:

$$X(0)=0, X(\ell)=0 \qquad (8)$$

Shunday qilib, X(x) funksiyani aniqlash uchun biz tebranish-lar tenglamasini yechishda o'rganilgan

$$X^{//}+\lambda\cdot X=0, \qquad X(0)=0, \qquad X(\ell)=0 \qquad (9)$$

xos qiymatlarni topish masalasiga (Shturm-Liuvill - masa-lasiga) keldik. Unda biz shuni ko'rsatgan edikki, λ ning fa-qatgina

$$\lambda_n=\left(\frac{\pi\cdot n}{\ell}\right)^2, \qquad n=1,2,\ldots \qquad (10)$$

qiymatlari uchungina (6) tenglamaning trivial bo'lmagan yechimlari mavjud bo'lib, ularning ko'rinishi quyidagicha edi:

$$X_n(x)=\sin\frac{\pi\cdot n}{\ell}x \qquad (11)$$

λ_n - ning bu qiymatlariga (7) tenglamaning quyidagi yechimlari mos keladi:

$$T_n(t) = C_n e^{-a^2\lambda_n t} \qquad (12)$$

bu yerda C_n -koeffitsientlar hozircha aniqlanmagan.
Agar asosiy masalaga qaytsak, ko'ramizki ,

$$u_n(x,t) = X_n(x) \cdot T_n(t) = C_n e^{-a^2\lambda_n t} \sin\frac{\pi \cdot n}{\ell} x \qquad (13)$$

funksiya (1)-tenglamaning (3') chegaraviy shartlarini bajaruvchi xususiy yechimlaridir.
Quyidagi qatorni tuzamiz:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{\pi n}{\ell}\right)^2 a^2 t} \sin\frac{\pi \cdot n}{\ell} x \qquad (14)$$

Bu qatorning har bir hadi chegaraviy shartlarni bajargani uchun $u(x,t)$- funksiyaning o'zi ham chegaraviy shartlarni bajaradi. Boshlang'ich shartlarning bajarilishini talab qilib quyidagini olamiz:

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{\pi \cdot n}{\ell} x \qquad (15)$$

ya'ni C_n lar φ(x) funksiyaning (0, ℓ) intervalda sinuslar bo'yicha qatorga yoyilmasidagi Furye koeffitsientlaridir:

$$C_n = \varphi_n = \frac{2}{\ell}\int_0^l \varphi(\xi)\sin\frac{\pi \cdot n}{\ell}\xi d\xi \qquad (16)$$

Endi (14) qator (1)-(3) masalaning barcha shartlarini bajaradigan bo'lishi uchun φ (x) funksiyaga qo'yiladigan shartlarni keltiramiz.

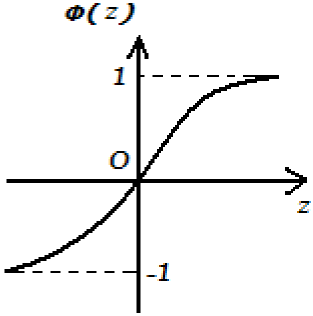
(14) qator t ≥ 0 da uzluksiz funksiyani aniqlashi uchun φ (x) funksiya uzluksiz, bo'lakli uzluksiz hosilaga ega va φ (0)=0, φ (ℓ)=0 shartlarni bajarishi kerak.

18-misol.

Torda issiqlik tarqalishining boshlang'ich taqsimoti

$$u(x,t)\big|_{t=o} = f(x) = \begin{cases} u_0, & x_1 < x < x_2 \\ 0, & x < x_1, \text{yoki} \quad x > x_2 \end{cases}$$

tenglik bilan berilsa, $\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$ tenglamaning yechimini toping.



3- rasm

Yechish:
Chegaralanmagan sterjen berilgan bo'lib, uning yechimini Puasson integrali ko'rinishida yozib olamiz: $u(x,t) = \frac{1}{2a\sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(\xi) \cdot e^{-\frac{(\xi - x)^2}{4a^2 t}} d\xi$

$f(x)$ funksiya (x_1, x_2) intervalda o'zgarmas u_0 haroratga, intervaldan tashqarida harorat nolga teng bo'lgani uchun, yechim

$u(x,t) = \frac{u_0}{2a\sqrt{\pi t}} \cdot \int_{x_1}^{x_2} e^{-\frac{(\xi - x)^2}{4a^2 t}} d\xi$ ko'rinishni oladi. Bu integralni quyidagi ehtimollar integrali bilan almashtirish mumkin : $\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\mu^2} d\mu$

Haqiqatan $\frac{x - \xi}{2a\sqrt{t}} = \mu$, $d\xi = -2a\sqrt{t} \cdot d\mu$ deb quyidagiga ega bo'lamiz

$$u(x,t) = -\frac{u_0}{\sqrt{\pi}} \cdot \int_{\frac{x - x_1}{2a\sqrt{t}}}^{\frac{x - x_2}{2a\sqrt{t}}} e^{-\mu^2} d\mu = \frac{u_0}{\sqrt{\pi}} \cdot \int_0^{\frac{x - x_1}{2a\sqrt{t}}} e^{-\mu^2} d\mu - \frac{u_0}{\sqrt{\pi}} \cdot \int_0^{\frac{x - x_2}{2a\sqrt{t}}} e^{-\mu^2} d\mu..$$

Shunday qilib, yechim quyidagi formula orqali ifodalanadi:

$$u(x,t)=\frac{u_0}{2}\cdot[\Phi(\frac{x-x_1}{2a\sqrt{t}})-\Phi(\frac{x-x_2}{2a\sqrt{t}})]$$

$\Phi(z)$ funksiyaning grafigi 3-rasmda berilgan. $\Phi(z)$ funksiya uchun maxsus jadvallar mavjud.

19-misol.

$0<x<\infty$ da $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u\big|_{t=0}=f(x)=u_0$ boshlang'ich va $u\big|_{x=0}=0$ chegaraviy shartlarni qanoatlantiruvchi yechimini toping.

Yechish.

Bu yerda bizga yarim chegaralangan tordagi issiqlik tarqalishining differensial tenglamasi berilgan.

Berilgan shartlarni qanoatlantiruvchi yechim ko'rinishi quyidagicha bo'ladi: $u(x,t)=\frac{1}{2\sqrt{\pi t}}\cdot\int_0^\infty u_0\cdot[e^{-\frac{(\xi-x)^2}{4t}}-e^{-\frac{(\xi+x)^2}{4t}}]d\xi$

yoki

$u(x,t)=\frac{u_0}{2\sqrt{\pi t}}\cdot\int_0^\infty [e^{-\frac{(\xi-x)^2}{4t}}-e^{-\frac{(\xi+x)^2}{4t}}]d\xi$ dan

a) $\frac{x-\xi}{2\sqrt{t}}=\mu$, $d\xi=-2\sqrt{t}\cdot d\mu$ deb olib, ehtimollar integrali yordamida birinchi integralni quyidagicha o'zgartiramiz:

$$\frac{u_0}{2\sqrt{\pi t}}\cdot\int_0^\infty e^{-\frac{(\xi-x)^2}{4t}}d\xi=\frac{u_0}{\sqrt{\pi}}\cdot\int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\mu^2}d\mu=\frac{u_0}{2}\cdot[1+\Phi(\frac{x}{2\sqrt{t}})]$$

b) $\frac{x+\xi}{2\sqrt{t}}=\mu$, $d\xi=2\sqrt{t}\cdot d\mu$ deb olib, ehtimollar integrali yordamida ikkinchi integralni quyidagicha o'zgartiramiz:

$$\frac{u_0}{2\sqrt{\pi t}}\cdot\int_0^\infty e^{-\frac{(\xi+x)^2}{4t}}d\xi=\frac{u_0}{\sqrt{\pi}}\cdot\int_{\frac{x}{2\sqrt{t}}}^{+\infty} e^{-\mu^2}d\mu=\frac{u_0}{2}\cdot[1-\Phi(\frac{x}{2\sqrt{t}})]$$

Shunday qilib yechim

$$u(x,t) = u_0 \cdot \Phi\left(\frac{x}{2\sqrt{t}}\right)$$

ko'rinishga keladi.

20-misol.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

tenglamaning $(0 < x < l), t > 0$ quyidagi

$$u\big|_{t=0} = f(x) = \begin{cases} x, & \text{agar} \quad 0 < x \le \dfrac{\ell}{2}; \\ \ell - x, & \text{agar} \quad \dfrac{\ell}{2} \le x < \ell. \end{cases}$$

boshlang'ich va $u\big|_{x=0} = u\big|_{x=\ell} \equiv 0$ chegaraviy shartlarni qanoatlantiruvchi yechimini toping.

Yechish:

Berilgan chegaraviy shartlarni qanoatlantiruvchi Koshi masalasining yechimini

$$u(x,t) = \sum_{k=1}^{\infty} b_k \cdot e^{-(\frac{k\pi}{\ell})^2 \cdot t} \cdot \sin\frac{k\pi x}{\ell}$$

ko'rinishda izlaymiz.

Bu yerda

$$b_k = \frac{2}{\ell}\int_0^{\frac{\ell}{2}} x \cdot \sin\frac{k\pi x}{\ell}\,dx = \frac{2}{\ell}\int_0^{\frac{\ell}{2}} x \cdot \sin\frac{k\pi x}{\ell}\,dx + \frac{2}{\ell}\int_{\frac{\ell}{2}}^{\ell} (\ell - x) \cdot \sin\frac{k\pi x}{\ell}\,dx .$$

$u = x, d\upsilon = \sin\frac{k\pi x}{\ell}dx, du = dx, \upsilon = -\frac{\ell}{k\pi} \cdot \cos\frac{k\pi x}{\ell}$ deb olib, bo’laklab integrallaymiz

$$b_k = \frac{2}{\ell}\left(-\frac{\ell x}{k\pi}\cdot\cos\frac{k\pi x}{\ell} + \frac{\ell^2}{k^2\pi^2}\cdot\sin\frac{k\pi x}{\ell}\right)\Bigg|_0^{\frac{\ell}{2}} +$$

$$+\frac{2}{\ell}\left(-\frac{\ell^2}{k\pi}\cdot\cos\frac{k\pi x}{\ell} + \frac{\ell x}{k\pi}\cdot\cos\frac{k\pi x}{\ell} - \frac{\ell^2}{k^2\pi^2}\cdot\sin\frac{k\pi x}{\ell}\right)\Bigg|_{\frac{\ell}{2}}^{\ell} = \frac{4\ell}{k^2\pi^2}\cdot\sin\frac{k\pi}{2}$$

Natijada yechim

$$u(x,t) = \frac{4\cdot\ell}{\pi^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sin\frac{k\pi}{2}\cdot e^{-\frac{k^2\pi^2 t}{\ell^2}}\cdot\sin\frac{k\pi x}{\ell}$$

yoki

$$u(x,t) = \frac{4\cdot\ell}{\pi^2}\cdot\sum_{n=1}^{\infty}(-1)^2\,\frac{1}{2n+1}\cdot e^{-\frac{(2n+1)^2\pi^2 t}{\ell^2}}\cdot\sin\frac{(2n+1)\pi x}{\ell}$$

bo’ladi. .

21-misol.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

tenglamaning quyidagi boshlang’ich shartlarni qanoatlantiruvchi yechimini toping.

$$u(x,t)\big|_{t=0} = f(x) = \begin{cases} 1-\frac{1}{\ell},\ 0\le x\le \ell; \\ 1+\frac{x}{\ell}, -\ell\le x\le 0; \\ 0 \quad ,\ x\ge \ell \text{ и } x\le -\ell \end{cases}$$

Ko’rsatma:

Yechim quyidagicha izlanadi.

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\cdot\int_{-\ell}^{0}(1+\frac{\xi}{\ell})\cdot e^{-\frac{(x-\xi)^2}{4t}}d\xi + \frac{1}{2\sqrt{\pi t}}\cdot\int_{0}^{\ell}(1-\frac{\xi}{\ell})\cdot e^{-\frac{(x-\xi)^2}{4t}}d\xi,$$

$\frac{x-\xi}{2\sqrt{t}} = \mu$ almashtirish bajarib javobni soddalashtiring.

Javob:

$$u(x,t)=\frac{1}{2}[(1+\frac{x}{\ell})\Phi(\frac{x+\ell}{2\sqrt{t}})-2\cdot\frac{x}{\ell}\cdot\Phi(\frac{x}{2\sqrt{t}})-(1-\frac{x}{\ell})\Phi(\frac{x-\ell}{2\sqrt{t}})]+$$

$$+\frac{1}{\ell}\sqrt{\frac{\ell}{\pi}}\cdot[e^{-\frac{(x+l)^2}{4t}}-2e^{-\frac{x^2}{4t}}+e^{-\frac{(x-\ell)^2}{4t}}]$$

22-misol. Yarim cheksiz sterjenning chapdagi oxiri issiqlik oʼtkazmaydigan boʼlsa va issiqlik tarqalishining boshlangʼich tezligi quyidagi

$$u\big|_{t=0}=f(x)=\begin{cases}0, & x<0;\\ u_0, & 0<x<\ell;\\ 0, & \ell<x.\end{cases}$$

formula bilan berilgan boʼlsa, issiqlik tarqalish tenglamasining yechimini toping.

Javob:

$$u(x,t)=\frac{u_0}{2}\cdot\left[\Phi\left(\frac{x+l}{2a\sqrt{t}}\right)-\Phi\left(\frac{x-l}{2a\sqrt{t}}\right)\right].$$

23-misol.

Yupqa bir jinsli, uzunligi ℓ ga teng boʼlgan sterjen berilgan. U tashqi muhitdan ajratilgan boʼlib, boshlangʼich harorati $f(x)=\frac{cx(\ell-x)}{\ell^2}$ ga teng. Sterjenning oxirlarida nolga teng boʼlgan harorat saqlanadi. Sterjenning ixtiyoriy $t>0$ vaqtdagi harorati aniqlansin.

Koʼrsatma. Sterjen haroratining tarqalish qonuni quyidagi tenglama bilan beriladi:

$$\frac{\partial u}{\partial t}=a^2\cdot\frac{\partial^2 u}{\partial x^2}.$$

Boshlang'ich sharti $u\big|_{t=0} = f(x) = \dfrac{cx(\ell - x)}{\ell^2}$ va chegaraviy shartlari $u\big|_{x=0} = u\big|_{x=\ell} = 0$ tengliklar bilan beriladi.

Javob:

$$u(x,t) = \frac{8c}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{(2n+1)^3} \cdot e^{-\frac{(2n+1)^2 \pi^2 a^2 t}{\ell^2}} \cdot \sin \frac{(2n+1)\pi x}{\ell} .$$

8-§ Elliptik tipdagi tenglama. Chegaraviy masalalarning qo'yilishi.

Bu tipdagi eng sodda tenglama quyidagi Laplas tenglamasidir:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0 \qquad (1)$$

Agar u(x,y) funksiya T-sohada o'zining ikkinchi tartibli hosilalari bilan birga uzluksiz bo'lsa va Laplas tenglamasini qanoatlantirsa, u holda u T-sohada garmonik deyiladi.

(1) tenglama t-vaqt bo'yicha statsionar jarayonni ifodalaydi, shuning uchun t parametr qatnashmaydi. Issiqlik manbalari mavjud bo'lganda quyidagi tenglamaga ega bo'lamiz:

$$\Delta u = -f, \qquad f = \frac{F}{K} . \qquad (2)$$

bu yerda F - issiqlik manbalarining zichligi, K - issiqlik o'tkazuvchanlik koeffisienti. (2) Puasson tenglamasi deb ataladi.

Σ - chegara bilan cheklangan biror T – soha ichida u(x,y) – issiqlikning statsionar taqsimoti to'g'risidagi masalani quyidagi holda keltiramiz:

- T soha ichida

$$\Delta u = -f(x, y) \qquad (3)$$

tenglamani va quyidagi ko'rinishlardan birortasidan iborat bo'lgan

I. $u = f_1 \quad \sum$ - da (1- chegaraviy masala)

II. $\dfrac{\partial u}{\partial n} = f_2 \quad \sum$ - da (2- chegaraviy masala)

III. $\dfrac{\partial u}{\partial n} + h(u - f_3) = 0 \quad \sum$ - da (3- chegaraviy masala)

chegaraviy shartlarni qanoatlantiruvchi u(x,y) funksiyani topish. Bu yerda f_1, f_2, f_3, h - berilgan funksiyalar, $\dfrac{\partial u}{\partial n}$ - $\sum$ chegaraga o'tkazilgan tashqi normal bo'yicha hosila. Ko'p hollarda Laplas tenglamasi uchun birinchi chegaraviy masalani Dirixle masalasi, ikkinchi chegaraviy masalani – Neyman masalasi deb atashadi.

Agar $\sum$ chegaraga nisbatan ichki (yoki tashqi) T_0 - sohada yechim qidirilsa, u holda bunga mos masalani ichki (yoki tashqi) chegaraviy masala deb atashadi.

9-§ Ikki o'lchamli Laplas tenglamasi va doira uchun Dirixle masalasi. O'zgaruvchilarni ajratish usuli.

Bu bo'limda biz T – soha doira bo'lgan, ya'ni doira uchun Dirixle masalasini o'zgaruvchilarni ajratish usulida yechamiz. Doiraning radiusini r bilan belgilab markazini koordinatalar boshida joylashtiramiz. Masalani qutb koordinatalar sistemasiga o'tib yechish maqsadga muvofiqdir.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

almashtirishlar yordamida masala quyidagicha qo'yiladi:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial \varphi})+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=0 \qquad (1)$$

Laplas tenglamasining $r<R$ uchun $u=u(r,\varphi)$ yechimi izlanadi, yechim $r=R$ da

$$u\big|_{r=R}=\tilde{u}(\varphi) \qquad (2)$$

berilgan qiymat qabul qiladi. O'zgaruvchilarni ajratish usuli shundan iboratki, avvalo (1) tenglamaning yechimini $u=V(r)\Phi(\varphi)$ ko'rinishda izlaymiz. Bunda $V(r)$ va $\Phi(\varphi)$ lar faqat r va φ larga mos ravishda bog'liqdir. Bu holda bu noma'lum funksiyalar uchun (1) tenglamadan quyidagilarni hosil qilamiz:

$$\frac{1}{r}\frac{d}{dr}(r\frac{dV}{dr})\Phi=-\frac{1}{r^2}\frac{d^2\Phi}{d\varphi^2}V$$

yoki

$$\frac{r}{V}\frac{d}{dr}(r\frac{dV}{dr})=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} \qquad (3)$$

(3) tenglamaning chap tomoni φ ga, o'ng tomoni esa r ga bog'liq bo'lmagani sababli bu tengliklar o'zgarmasdirlar:

$$\frac{r}{V}\frac{d}{dr}(r\frac{dV}{dr})=\lambda, \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-\lambda \qquad (4)$$

Bu yerda λ - o'zgaruvchilarni ajratishning o'zgarmasidir. λ - o'zgarmas musbat bo'lishi kerak. (4) tenglamalarning ikkinchisi $\frac{d^2\Phi}{d\varphi^2}=-\lambda\Phi$ quyidagi umumiy yechimga ega:

$$\Phi(\varphi)=A\cos\sqrt{\lambda}\varphi+B\sin\sqrt{\lambda}\varphi \qquad (5)$$

A va B lar ixtiyoriy o'zgarmaslar. Endi λ ixtiyoriy qiymat qabul qila olmasligini ko'rsatamiz. Bu shundan kelib chiqadiki, λ ning qiymatini 2π ga oshirish (r,φ) nuqtani oldingi holatga qaytaradi; demak biz qarayotgan barcha φ ga bog'liq funksiyalar φ bo'yicha 2π-davr bilan davriy bo'lishlari lozim. Shunday qilib $\Phi(\varphi+2\pi)=\Phi(\varphi)$, bu esa (5) formulaga asosan shuni bildiradiki, $\sqrt{\lambda}$ qabul qiladigan qiymatlari n=0,1,2,... butun sonlarga teng bo'lishi kerak. n ning manfiy qiymatlarini biz tashlab yuborishimiz mumkin, chunki n ning ishorasi faqat o'zgarmas B ning ishorasiga ta'sir ko'rsatadi. Shunday qilib $\lambda = n^2$ (xos sonlar) va

$$\Phi(\varphi) = A\cos n\varphi + B\sin n\varphi . \tag{6}$$

Endi biz (4) tenglamalarning birinchisiga qaytamiz va λ ning o'rniga n^2 ni qo'yamiz:

$$\frac{r}{V}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = n^2$$

Bundan esa quyidagini hosil qilamiz:

$$r^2\frac{d^2V}{dr^2} + r\frac{dV}{dr} - n^2V = 0$$

Bu tenglama yechimini $V = r^\alpha$ ko'rinishida izlaymiz: Ko'rsatkich α uchun osongina quyidagi tenglamani olamiz:

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0$$

yoki

$$\alpha(\alpha-1) + \alpha - n^2 = 0, \quad \alpha^2 = n^2, \quad \alpha = \pm n .$$

Demak, $V(r) = r^n$, yoki $V(r) = r^{-n}$

Bu yechimlardan ikkinchisini tashlab yuboramiz, chunki n>0 uchun r=0 doiraning markazida uning qiymati cheksizga aylanadi.

Natijada

$$u_n(r,\varphi) = r^n(A_n \cos n\varphi + B_n \sin n\varphi)$$

Biz u=V(r)Φ(φ) ko'rinishda izlagan (1) Laplas tenglamasining hususiy yechimini u_n orqali belgiladik, chunki u n ga bogliqdir, ixtiyoriy A_n va B_n o'zgarmaslar ham n ga bogliq.

Laplas tenglamasining chiziqliligi va bir jinsligi uchun xususiy yechimlar yig'indisi ham yechim bo'ladi:

$$u(r,\varphi)=\sum_{n=0}^{\infty}u_n(r,\varphi)=\sum_{n=0}^{\infty}r^n(A_n\cos n\varphi+B_n\sin n\varphi) \qquad (7)$$

Bu tenglama ikkita no'malum A_n va B_n koeffitsientlarni o'z ichiga oladi.

(7) ko'rinishidagi yechimga Furye qatori ko'rinishini berish uchun

$$A_0=\frac{a_0}{2},A_n=a_n,B_n=b_n,(n=1,2,.....)$$

deb olamiz. U holda u(r,φ) ni quyidagi ko'rinishda yozish mumkin.

$$u(r,\varphi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos n\varphi+b_n\sin n\varphi) \qquad (8)$$

Noma'lum koeffitsientlar a_0,a_n,b_n larni (2) chegaraviy shartdan aniqlaymiz, bundagi $\tilde{u}(\varphi)$ funksiya ham 2π davr bilan davriy bo'lishi kerak. (8) yechimda r = R deb quyidagi ko'rinishdagi chegaraviy shartni olamiz:

$$\tilde{u}(\varphi)=\frac{a_0}{2}+\sum_{n=0}^{\infty}(R^na_n\cos n\varphi+R^nb_n\sin n\varphi)$$

bu esa $\tilde{u}(\varphi)$ ning Furye qatoriga yoyilmasini beradi. Furye koeffitsientlari uchun ma'lum formulalarga asosan biz quyidagilarni hosil qilamiz:

$$R^na_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{u}(\psi)\cos n\psi d\psi$$

va
$$R^nb_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{u}(\psi)\sin n\psi d\psi$$

ya'ni
$$\begin{cases} a_n = \dfrac{1}{\pi R^n}\displaystyle\int_0^{2\pi}\tilde{u}(\psi)\cos n\psi d\psi \\ b_n = \dfrac{1}{\pi R^n}\displaystyle\int_0^{2\pi}\tilde{u}(\psi)\sin n\psi d\psi \end{cases} \qquad (9)$$

Doira uchun Dirixle masalasining yechimini olish uchun (8) ga (9) koeffisientlarning ifodalarini qo'yish yetarli.

10-§ Sferik koordinatalarda berilgan uch o'lchovli Laplas tenglamasi uchun o'zgaruvchilarni ajratish.

Endi biz quyidagi sferik koordinatalarda berilgan uch o'lchovli Laplas tenglamasi uchun o'zgaruvchilarni ajratish usulini qo'llaymiz:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0$$

Biz bu yerda u funksiya φ ga bog'liq bo'lmagan eng oddiy va muhim holni qaraymiz: ya'ni $u=u(r,\theta)$. Bu - u funksiya kenglikning har bir doirasida o'zgarmas bo'lgan ya'ni, $r=const$, $\theta=const$, $0\le\varphi\le\pi$ bo'lgan holdir.

U holda Laplas tenglamasi quyidagi ko'rinishni oladi:

$$\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) = 0 \qquad (1)$$

Bu tenglamaning

$$u=V(r)\Phi(\theta) \qquad (2)$$

ko'rinishdagi, markazi koordinata boshida va radiusi R ga teng bo'lgan sharda chegaralangan, yechimi qanday bo'lishi mumkinligini ko'raylik. (2) ni (1) ga qo'yib topamiz:

$$\frac{1}{V}\frac{d}{dr}(r^2\frac{dV}{dr}) = -\frac{1}{\Phi\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Phi}{d\theta}) \qquad (3)$$

O'zgaruvchilarni ajratish usuli uchun standart bo'lgan mulohazalarga asosan (3) ning ikkala tomoni ham o'zgarmas bo'lishi kerak:

a) $\frac{1}{V}\frac{d}{dr}(r^2\frac{dV}{dr}) = \lambda$; b) $\frac{1}{\Phi\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Phi}{d\theta}) = -\lambda$ (4)

Bu yerda λ ni yana manfiy bo'lmagan o'zgarmas son deb hisoblaymiz. Hisoblashlarni yengillashtirish maqsadida ν - ixtiyoriy o'zgarmas uchun $\lambda=\nu(\nu+1)$ deb olamiz.

Bu holda a) (4) tenglikdan:

$$r^2\frac{d^2V}{dr^2} + 2r\frac{dV}{dr} - \nu(\nu+1)V = 0$$

ni hosil qilamiz. $V = r^\alpha$ deb olib α ko'rsatkich uchun quyidagi tenglamani hosil qilamiz:

$\alpha(\alpha-1) + 2\alpha - \nu(\nu+1) = 0$ yoki $\alpha(\alpha+1) = \nu(\nu+1)$.

Bu tenglamaning yechimi:

$$\alpha=\nu, \qquad \alpha=-\nu-1.$$

$\nu\geq 0$ deb hisoblab, biz ikkinchi ildizni tashlab yuborishimiz mumkin, chunki r=0 da $V(r) = r^{-\nu-1}$ funksiya cheksizga aylanadi. (eslatamizki, bizning maqsadimiz chegaralangan yechimni topish).

Shunday qilib,

$$V(r) = r^\nu, \qquad \nu \geq 0$$

yechim qoladi.

Endi b) holga qaytamiz:

(4) tenglikdan hosil qilamiz:

$$\frac{1}{\Phi\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Phi}{d\theta}) = -\nu(\nu+1) \qquad (5)$$

$0\leq\theta\leq\pi$ bo'lganligi uchun biz (5) tenglamaning barcha θ lar uchun cheklangan yechimini topishimiz kerak.

(5) da θ- o'zgaruvchi o'rniga yangi bog'liqsiz o'zgaruvchi

$$x=\cos\theta \qquad (6)$$

ga o'tamiz. Bu yerda $0 \leq \theta \leq \pi$ dan $-1 \leq x \leq 1$ kelib chiqadi.

(6) almashtirishga ko'ra:

$$\frac{d\Phi}{d\theta} = \frac{d\Phi}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Phi}{dx},$$

$$\frac{d}{d\theta}(\sin\theta\frac{d\Phi}{d\theta}) = \sin\theta\frac{d}{dx}(\sin^2\theta\frac{d\Phi}{dx}),$$

va (5) tenglama quyidagi ko'rinishni oladi:

$$\frac{d}{dx}(\sin^2\theta\frac{d\Phi}{dx}) = -\nu(\nu+1)\Phi$$

yoki

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Phi}{dx}\right] + \nu(\nu+1)\Phi = 0$$

Bundan tashqari biz Ф bog'liq o'zgaruvchini x=cosθ ning funksiyasi sifatida y orqali belgilasak

$$\Phi(\theta) = \Phi(\arccos x) = y(x),$$

u holda y(x) izlanayotgan funksiyani aniqlash uchun quyidagi tenglamani olamiz

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \nu(\nu+1)y = 0$$

Bu tenglama Lejandr tenglamasi deb ataladi. Bu tenglamaning yechimi ko'pgina tatbiqiy masalalar uchun muhim rol o'ynaydi. x=±1 nuqtalar (7) differensial tenglama uchun muhim nuqta bo'lganliklari uchun Lejandr tenglamasi chekli yechimga ega bo'lishi uchun ν=n (n ≥ 0) butun son bo'lishi shart. Bunday ν lar uchun (7) Lejandr tenglamasining chekli yechimlari Lejandr ko'phadlari bo'ladi. Bu ko'phadlarning ko'rinishi yetarlicha sodda yopiq ifodaga ega:

$$y = P_n(x) = \frac{1}{2^2 n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right] \qquad (8)$$

Shunday qilib, biz shuni aniqladikki, Ф(θ) sifatida $P_n(x) = P_n(\cos\theta)$ ni olish lozim va shuning uchun (1) Laplas tenglamasining V(r)Ф(θ) ko'rinishidagi chekli yechimlari

$$u_n(r,\theta) = V_n(r)\Phi_n(\theta) = r^n P_n(\cos\theta) \qquad (9)$$

funksiyalar bo'ladi.

Laplas tenglamasining chiziqli va bir jinsliligidan uning yechimi sifatida

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta) \qquad (10)$$

funksiyani olish mumkin.

Bu yerda c_n – ixtiyoriy o'zgarmas koeffitsientlar bo'lib, biz ularni (10) funksiya

$$u(r,\theta)\big|_{r=R} = \tilde{u}(\theta) \qquad (11)$$

chegaraviy shartni bajaradigan qilib aniqlaymiz.

Buning uchun $0 \le \theta \le \pi$ bo'yicha

$$\tilde{u}(\theta) = \sum_{n=0}^{\infty} c_n R^n P_n(\cos\theta) \qquad (12)$$

munosabat aynan bajarilishini talab etish lozim. (12) - ayniyat bajariladi degan farazda biz bu koeffitsientlarni Lejandr ko'phadlarining ortogonalligi xossalaridan foydalanib topishimiz mumkin (Lejandr ko'phadlari (-1,1) intervalda ortogonaldir, ya'ni $n \ne m$ da

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0$$

(12) ayniyatni $P_m(\cos\theta)$ ga ko'paytiramiz va R radiusli sfera bo'yicha integrallaymiz: ($m \ge 0$-fiksirlangan butun son)

$$\oiint \tilde{u}(\theta)P_m(\cos\theta)d\delta = \sum_{n=0}^{\infty} C_n R^n \oiint P_n(\cos\theta)P_m(\cos\theta)d\delta \qquad (13)$$

$d\delta = R^2 \sin\theta d\theta d\varphi$ sfera sirti yuzasining elementi va sfera bo'yicha integrallash, θ bo'yicha 0 dan π gacha va φ bo'yicha 0 dan 2π gacha integrallashni bildiradi.

Shuning uchun

$$\oiint P_n(\cos\theta)P_m(\cos\theta)d\delta = 2\pi R^2\int_{-1}^{1} P_n(x)P_m(x)dx =$$

$$= \begin{cases} 0, n \neq m \\ 2\pi R^2 \dfrac{2}{2m+1}, n = m \end{cases}$$

chunki $\int_{-1}^{1} P_n^2(x)dx = \dfrac{2}{2n+1}$, $n = 0,1,2,.....)$

Shunday qilib, (13) tenglikning o'ng tomonidagi hadlarning n=m bo'lgan hadidan tashqari barchasi 0 ga teng. Binobarin, (13) tenglikning o'ng tomoni

$$C_m R^m \frac{4\pi R^2}{2m+1}$$

ga teng; bunga ko'ra

$$C_m = \frac{1}{R^m}(2m+1)\frac{1}{4\pi R^2}\oiint \tilde{u}(\theta)P_m(\cos\theta)d\delta \qquad (14)$$

Bu yerda m ning o'rniga n qo'yib va (14) ni (10) ga qo'yib, R-radiusli shar uchun Dirixle masalasining yechimini hosil qilamiz.

24-misol. Yon sirti issiqlik o'tkazmaydigan yupqa sterjenda issiqlikning statsionar tarqalishi qonunini toping. Chegaraviy shartlar quyidagicha:

$$u|_{x=0} = u_0, u|_{x=l} = u_1 .$$

Yechish:

Bir o'lchamli holda Dirixle masalasi, Laplas tenglamasi $\dfrac{d^2u}{dx^2} = 0$ ning quyidagi

$$u|_{x=0} = u_0, u|_{x=l} = u_1$$

chegaraviy shartlarni qanoatlantiruvchi u yechimini topishdan iboratdir. Bu tenglamaning umumiy yechimi $u = Ax + B$ ko'rinishda bo'lib, chegaraviy shartlarni hisobga olib quyidagi tenglikka kelamiz:

$$u = \frac{u_l - u_0}{l} x + u_0,$$

ya'ni yon sirti issiqlik o'tkazmaydigan yupqa sterjenda issiqlikning statsionar tarqalishi qonunini chiziqli funksiyadan iborat ekan.

25-misol. Umumiy OZ o'qqa ega bo'lgan ikki silindr orasidagi fazoda issiqlikning statsionar tarqalishi qonunini toping. Silindrlar sirtlarida harorat o'zgarmas.

Ko'rsatma. u funksiya θ va z lardan bog'liq emas deb faraz qilib, silindrik koordinatalar sistemasiga o'ting.

Javob: $u = u_a + (u_b - u_a) \cdot \dfrac{\ln \frac{r}{a}}{\ln \frac{b}{a}} = u_a + (u_b - u_a) \dfrac{\ln r - \ln a}{\ln b - \ln a}$.

26-misol.

R radiusli bir jinsli yupqa doirasimon plastinkaning yuqori yarmida 1 gradus haroratda, pastki yarmida esa 0 gradus haroratda saqlangan sharoitda, bu plastinkada issiqlikning statsionar tarqalishi qonunini toping.

Yechish:

$-\pi < \tau < 0$ da $f(\tau) = 0$ va $.0 < \tau < \pi$ da $f(\tau) = 1$.

Issiqlik tarqalish quyidagi integral orqali ifodalanadi:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\tau - \theta) + r^2} d\tau$$

(τ, θ) nuqta yuqori yarim doirada joylashgan, ya'ni $0 < \theta < \pi$,

u holda $\tau - \theta$ ning qiymati $-\theta$ dan $\tau - \theta$ gacha o'zgaradi va bu uzunligi π ga teng interval $\pm\pi$ nuqtani o'z ichiga olmaydi.

Shuning uchun quyidagi almashtirishlarni bajaramiz:

$$\operatorname{tg}\frac{\tau - \theta}{2} = t, \qquad \cos(\tau - \theta) = \frac{1 - t^2}{1 + t^2}, \qquad d\tau = \frac{2dt}{1 + t^2}.$$

Bundan quyidagini olamiz:

$$u(r,\theta)=\frac{1}{\pi}\int\limits_{-tg\frac{\theta}{2}}^{ctg\frac{\theta}{2}}\frac{R^2-r^2}{(R-r)^2-(R+r)^2\cdot t^2}dt=\frac{1}{\pi}\cdot arctg\left(\frac{R+r}{R-r}\cdot t\right)\Bigg|_{-tg\frac{\theta}{2}}^{ctg\frac{\theta}{2}}=$$

$$\frac{1}{\pi}\left\{arctg\left(\frac{R+r}{R-r}\cdot ctg\frac{\theta}{2}\right)+arctg\left(\frac{R+r}{R-r}\cdot tg\frac{\theta}{2}\right)\right\}=\frac{1}{\pi}arctg\frac{\frac{R+r}{R-r}\left(ctg\frac{\theta}{2}+tg\frac{\theta}{2}\right)}{1-\left(\frac{R+r}{R-r}\right)^2}=$$

$$=-\frac{1}{\pi}\cdot arctg\frac{R^2-r^2}{2Rr\sin\theta}$$

yoki

$$tg(u\pi)=-\frac{R^2-r^2}{2Rr\sin\theta}.$$

O'ng tomoni manfiy bo'lganligi uchun u funksiya $0<\theta<\pi$ bo'lganda $\frac{1}{2}<u<1$ tengsizlikni bajaradi. Bu hol uchun quyidagi yechimni olamiz:

$$tg(\pi-u\pi)=-\frac{R^2-r^2}{2Rr\sin\theta},$$

yoki

$$u=1-\frac{1}{\pi}arctg\frac{R^2-r^2}{2Rr\sin\theta}\quad(0<\theta<\pi).$$

Agar nuqta pastki $\pi<\theta<2\pi$ yarim doirada joylashgan bo'lsa, u holda $\tau-\theta$ ning o'zgarish intervali $(-\theta,\pi-\theta)$ $-\pi$ nuqtani o'z ichiga oladi, lekin 0 ni o'z ichiga olmaydi. Biz quyidagi almashtirishlarni bajarishimiz mumkin:

$$ctg\frac{\tau-\theta}{2}=t,\cos(\tau-\theta)=\frac{t^2-1}{t^2+1},d\tau=-\frac{2dt}{1+t^2}.$$

U holda θ ning bu qiymatlari uchun

$$u(r,\theta)=-\frac{1}{\pi}\int_{-ctg\frac{\theta}{2}}^{tg\frac{\theta}{2}}\frac{R^2-r^2}{(R+r)^2+(R-r)^2\cdot t^2}dt=-\frac{1}{\pi}\left\{arctg(\frac{R-r}{R+r}\cdot tg\frac{\theta}{2})+\right.$$

$$\left.+arctg(\frac{R-r}{R+r}\cdot ctg\frac{\theta}{2})\right\}$$

Shunga oʼxshash almashtirishlarni bajarib, quyidagini topamiz:

$$u=-\frac{1}{\pi}arctg\frac{R^2-r^2}{2Rr\sin\theta}\quad (\pi<\theta<2\pi).$$

Oʼng tomoni endi musbat boʼlganligi uchun $0<u<\frac{1}{2}$.

27-misol. Quyidagi Laplas tenglamasining

$$u\big|_{r=1}=0, u\big|_{r=2}=y.$$

chegaraviy shartlarni qanoatlantiruvchi yechimini $1\le r\le 2$ halqaning ichki qismi uchun toping.

Koʼrsatma. Qutb koordinatalar sistemasini kiriting.

Javob: $u(r,\theta)=\frac{8}{3}sh(\ln r)\cdot\sin\theta.$

Ilova:

Matematik fizika masalalarining qoʼyilishi va tenglamalarni keltirib chiqarish

Matemtika, fizika va keng ko'lamdagi texnika masalalari matematik fizika tenglamalarining xususiy holi bo'lgan ikkinchi darajali xususiy hosilali differensial tenglamalarni tekshirishga olib keladi. Bu tenglamalarni keltirib chiqarish mexanika va fizikaning qonunlariga suyanadi. Fizik va mexanik jarayonlarning matematik modeli bo'lgan bu tenglamalarni keltirib chiqarish usullaridan birini ko'rib chiqaylik.

1-misol.

Izotrop qattiq jismda issiqlik tarqalish tenglamasini keltirib chiqazing.

Yechilishi:

Qattiq jismning $M(x,y,z)$ nuqtasida t vaqtdagi haroratini $u(x,y,z,t)$ bilan belgilaylik. Ma'lumki, jismdagi issiqlik harakati issiqlik ko'proq qismidan issiqlik kamroq qismiga qarab yo'nalgan bo'ladi. Issiqlik tarqalish nazariyasida shu narsa qabul qilinganki, jism ichida joylashgan yuza elementi $\Delta\sigma$ orqali o'tuvchi ΔQ issiqlik miqdori $\Delta\Pi\cdot\Delta t$ ga proporsionaldir, bu yerda $\Delta\Pi$, $\Delta\sigma$-element orqali o'tuvchi $\operatorname{grad}(u)$ vektor oqimidir, ya'ni

$$\Delta Q = k\cdot\Delta\Pi\cdot\Delta t \qquad (1)$$

Bunda $k=k(x,y,z)$ - jismning issiqlik o'tkazuvchanlik koeffitsientidir.

Jism ichida Σ - silliq, yopiq yuza bilan chegaralangan ixtiyoriy V hajmni ajratamiz va bu hajm uchun issiqlik balansi tenglamasini tuzamiz. Q_1 deb $(t;t+\Delta t)$ vaqt oralig'ida Σ yuza orqali V hajmga kiruvchi issiqlik miqdorini belgilaymiz.

U holda (1) dan quyidagini olamiz:

$$Q_1 = \oiint_{\Sigma} k(M)\cdot(\operatorname{grad}(u), d\sigma)\cdot\Delta t$$

Q_2 deb esa, hajmda issiqlik tarqalish manbai (yoki issiqlik yutilish manbai) bo'lishligi natijasida $(t;t+\Delta t)$ vaqt oralig'ida

V hajmda issiqlik hosil bo'lishi yoki issiqlik yutilishi miqdorini belgilaymiz. Birlik vaqt va birlik hajmda tarqalayotgan yoki yutilayotgan issiqlik miqdorini ya'ni issiqlik tarqalish yoki yutilish manbalarining zichligini $F(x,y,z,t)$ bilan belgilasak,

$$Q_2 = \iiint_V F(x,y,z,t)dv\Delta t$$

bo'ladi. Bu holda Gauss-Ostrogradskiy formulasini $(t;t+\Delta t)$ vaqt oralig'ida V hajmga kelayotgan umumiy issiqlik miqdoriga qo'llasak, quyidagi tenglikka kelamiz:

$$Q_1 + Q_2 = \iiint_V \operatorname{div}(k\cdot \operatorname{grad}(u))dv\Delta t + \iiint_V F(x,y,z,t)dv\Delta t\,. \qquad (2)$$

Ikkinchi tomondan $(t;t+\Delta t)$ vaqt oralig'ida V jism hajmida issiqlikning $\Delta_t u = u(x,y,z,t+\Delta t) - u(x,y,z,t) \approx \frac{\partial u(M,t)}{\partial t}\Delta t$ kattalikka o'zgarishi uchun quyidagi miqdordagi issiqlik sarf qilinishi kerak:

$$Q_3 = \iiint_V [u(x,y,z,t+\Delta t) - u(x,y,z,t)]\cdot\gamma(x,y,z)\cdot\rho(x,y,z)dv \approx$$
$$\approx \iiint_V \gamma\cdot\rho\cdot\frac{\partial u}{\partial t}\cdot dv\cdot\Delta t \qquad (3)$$

bu yerda $\gamma = \gamma(M)$ jism yasalgan moddaning issiqlik sig'imi, $\rho = \rho(M)$ esa uning zichligidir. Lekin $Q_1 + Q_2 = Q_3$ bo'lganligi uchun (2) va (3) tengliklardan

$$\iiint_V [\gamma\cdot\rho\cdot\frac{\partial u}{\partial t} - \operatorname{div}(k\cdot \operatorname{grad}(u)) - F(x,y,z,t)]dv = 0$$

V hajm ixtiyoriy va integral ostidagi funksiya uzluksiz bo'lgani uchun, ixtiyoriy vaqt t da quyidagi munosbat o'rinli bo'lishi kerak:

$$\gamma\cdot\rho\cdot\frac{\partial u}{\partial t} = \operatorname{div}(k\cdot \operatorname{grad}(u)) + F(M,t) \qquad (4)$$

Bu tenglik bir jinsli bo'lmagan izotrop jismda issiqlik tarqalishi tenglamasidir. Agar jism bir jinsli bo'lsa, u holda γ,ρ

va k lar o'zgarmas bo'lib, (4) tenglik quyidagicha yoziladi:

$$\frac{\partial u}{\partial t} = a^2 \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f(x,y,z,t) \qquad (5)$$

Bu yerda $a = \sqrt{\frac{k}{\gamma\rho}}, \qquad f(x,y,z,t) = \frac{1}{\gamma\rho} F(x,y,z,t)$.

Ixtiyoriy t vaqt momentida jismning ixtiyoriy nuqtasidagi u(x,y,z,t) – haroratini hisoblash uchun (4) va (5) tenglamalarni yechish yetarli emas. Fizik munosabatlardan kelib chiqadiki, bulardan tashqari boshlang'ich vaqt momentida jism ichidagi issiqlik taqsimotini (boshlang'ich shart) va jism chegarasidagi xarorat rejimini (chegaraviy shart) ham bilish shartdir. Huddi shunday matematik fizikaning boshqa masalalarini yechish uchun ham boshlang'ich (agar jarayon nostatsionar bo'lsa) va chegaraviy shartlarni bilish talab qilinadi. Shuning uchun masalaning qo'yilishi deganda bundan keyin tekshirilayotgan fizik jarayonni xarakterlovchi funksiyani tanlash, bu jarayonga mos keluvchi tenglamani keltirib chiqarish (yoki tanlash), chegaraviy shartlarni topish va boshlang'ich sartlarni keltirish tushuniladi.

2-misol.

Agar yon sirti issiqlikdan izolyatsialangan bir jinsli izotrop $0 \le x \le \ell$ sterjenning boshlang'ich harorati x funksiya orqali aniqlanib, sterjen uchlariga tashqaridan issiqlik oqimi ta'sir qilganda sterjen harorati aniqlansin.

Yechilishi:

Sterjen harorati faqatgina nuqta koordinatalari $x \in (0,\ell)$ va vaqt t ga bog'liq bo'ladi, ya'ni $u = u(x,t)$. Sterjen ichida issiqlik manbalari yo'q, ya'ni $F(x,y,z,t) \equiv 0$. Shuning uchun (5) tenglama quyidagi ko'rinishda yoziladi:

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \;;$$

Bu yerda $a^2 = \frac{k}{\gamma\rho}$.

Boshlang'ich shart quyidagicha yoziladi:

$u(x,0) = \varphi(x), \quad 0 \le x \le \ell, \quad \varphi(x)$ - berilgan funksiya.

Chegaraviy shart quyidagi ko'rinishda bo'ladi:

$$u'_x(0,t) = -\frac{1}{k\sigma} q_1(t)$$

$$u'_x(\ell,t) = \frac{1}{k\sigma} q_2(t) \qquad 0 < t < \infty,$$

bu yerda σ - sterjenning ko'ndalang kesimi yuzasi, $q_1(t)$ va $q_2(t)$ lar sterjenning uchlari orqali unga kirayotgan issiqlik miqdori.

Shuday qilib quyidagi masalani qo'yish mumkin:

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad a^2 = \frac{k}{\gamma\rho}, \tag{6}$$

$$0 \le x \le \ell, \qquad 0 < t < \infty$$

tenglamaning

$$u(x,0) = \varphi(x)$$

$$\begin{cases} u'_x(0,t) = -\dfrac{1}{k\sigma} q_1(t) \\ u'_x(\ell,t) = \dfrac{1}{k\sigma} q_2(t) \end{cases} \tag{7}$$

boshlang'ich va chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Bu masala aralash masala deb ataladi, chunki bunda boshlang'ich va chegaraviy shartlar birgalikda qatnashadilar. $u'_x(x,t)$ - hosilaga qo'yilgan (7) –

chegaraviy sartlar II tur shartlar deb ataladi. Bundan tashqari $u(x,t)$ funksiya qiymatiga qo'yiladigan quyidagi I tur

$$\begin{aligned} u(0,t) &= \varphi_1(t) \\ u(\ell,t) &= \varphi_2(t) \end{aligned} \qquad (8)$$

shartlar va $u(x,t)$ funksiya qiymatiga qo'yiladigan shartlar bilan birgalikda bu funksiyaning $u'_x(x,t)$ hosilasiga ham qo'yiladigan
III tur

$$\begin{cases} \left(\dfrac{\partial u}{\partial x} + \delta u\right)\Big|_{x=0} = \Psi_1(t) \\ \left(\dfrac{\partial u}{\partial x} + \delta u\right)\Big|_{x=\ell} = \Psi_2(t) \end{cases} \qquad (9)$$

shartlar qaraladi. (9) shart $x = 0$ va $x = \ell$ nuqtalardagi elastik mahkamlanishni bildiradi.

Aralash masalalardan tashqari, ko'p hollarda quyidagi Koshi masalasi ham qaraladi: berilgan tenglamaning $-\infty < x < \infty$ va $0 < t < \infty$ sohadagi faqat-gina boshlang'ich shartlarga bo'ysunuvchi $u(x,t)$ yechimi topilsin.

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