Al-Xorazmiy nomli Urganch Davlat universiteti

Fizika-matematika fakulteti 201-amaliy matematika va informatika guruhi talabasi Safarboyeva Maftunaning differensial tenglamalar fanidan «Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi» mavzusidagi referatiga

TAQRIZ

Ushbu «Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi» mavzusidagi referatda chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi o'rganilgan. Shturm-Liuvill operatori uchun taskari masalalar B.A.Ambartsumyan, G.Borg, B.A.Marchenko, I.M.Gelfand, B.M.Levitan va boshqa olimlar tomonidan o'rganilgan bo'lib mazkur bitiruv malakaviy ishida G.Borg yagonalik teoremasini analogi, ikki spektr yordamida teskari masala haqidagi mavjudlik va yagonalik teoremasi isbotlangan hamda Shturm-Liuvill chegaraviy masalasini ikki spektr yordamida qurish algoritmi I.M.Gasimov va B.M.Levitan usuli o'rganilgan.

Shturm-Liuvill chegaraviy masalasining $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o'zgarmaslariga yoki spektral funksiyasiga spektral xarakteristikalar deyiladi. Spektral xarakteristikalarni topish va ularning xossalarini o'rganish masalasiga spektral analizning to'g'ri masalasi deyiladi. Spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasi koeffitsientlarini va chegaraviy shartlarni aniqlash masalasiga spektral analizning teskari masalasi deyiladi.

Talaba Safarboyeva Maftunaning «Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi» mavzusidagi referati Oliy va O`rta maxsus ta'lim Vazirligi tomonidan referatni bajarishga qo`yilgan talablarga to`la javob beradi.

Taqrizchi: **"Amaliy matematika" kafedrasi o'qituvchisi Matyakubov M.**

O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI AL-XORAZMIY NOMLI URGANCH DAVLAT UNIVERSITETI DEVONXONA

Al-Xorazmiy nomli Urganch Davlat universiteti
Fizika-matematika fakulteti 201-amaliy matematika va informatika guruhi talabasi Safarboyeva Maftunaning differensial tenglamalar fanidan «Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi» mavzusidagi referatiga

T A Q R I Z

Bizga ma'lumki matematik-fizikaning bir qancha yo'nalishlarining rivojlanishiga Shturm-Liuvill

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x \le \pi, \quad q(x) \in C[0,\pi] \\ y'(0) - hy(0) = 0, & \\ y'(\pi) + Hy(\pi) = 0, & \end{cases}$$

chegaraviy masalasining spektral nazariyasida qo'llaniladigan usullari katta xissa qo'shib kelmoqda. 1946 yilda G.Borg Shturm-Liuvill chegaraviy masalasi uchun teskari spektral masalani o'zgacha qo'yish mumkinligini tavsiya qildi. Jumladan, Shturm-Liuvill differentsial operatorini ikkita chegaraviy masalaning spektrlari, ya'ni umumiy differentsial tenglama va faqat bitta chegaraviy shart bilan farq qiluvchi ikkita Shturm-Liuvill chegaraviy masalasining spektrlari yordamida qurishning yagonaligini ko'rsatib berdi.

Talaba Safarboyeva Maftunaning «Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi» mavzusidagi referati Oliy va O`rta maxsus ta'lim Vazirligi tomonidan referatni bajarishga qo`yilgan talablarga to`la javob beradi.

Taqrizchi: **"Axborot ta'lim texnologiyalari" kafedrasi o'qituvchisi f.m.f.n Mamedov Q.**

IMZONI TASDIQLAYMAN

KADRLAR BO'LIMI BOSHLIG'I

O’ZBEKISTON RESPULIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

AL-XORAZIMIY NOMLI

URGANCH DAVLAT UNIVERSITETI

“FIZIKA-MATEMATIKA” FAKULTETI

201-AMALIY MATEMATIKA VA INFORMATIKA

GURUH TALABASI SAFARBOYEVA MAFTUNANING

DIFFERENSIAL TENGLAMALAR FANIDAN

REFERATI

Mavzu: Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi

Topshirdi: **Safarboyeva M.**

Qabul qildi: **Salayev S.**

Reja:

1. Xos qiymatlarning va xos funksiyalarning sodda xossalari

2. Shturm-Liuvill tenglamasi yechimining asimptotikasi

3. Chegaraviy shartlarida $y(0)=0$ yoki $y(\pi)=0$ bo‘lgan Shturm-Liuvill masalasining xos qiymatlari uchun asimptotik formulalar

Chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi

1-§. Xos qiymatlarning va xos funksiyalarning sodda xossalari

Quyidagi masalaga

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0,\pi], \tag{1.1.1}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.1.2}$$

Shturm-Liuvill chegaraviy masalasi deyiladi. Bu yerda $q(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya bo‘lib, α va β berilgan haqiqiy sonlardir, λ esa kompleks parametr.

Agar (1.1.1) tenglamani $y(0)=0,\ y(\pi)=0$ chegaraviy shartlar bilan qarasak, hosil bo‘ladigan chegaraviy masalaga Dirixle masalasi deyiladi, agar $y'(0)=0,\ y'(\pi)=0$ chegaraviy shartlar bilan qarasak, hosil bo‘ladigan chegaraviy masalaga Neyman masalasi deyiladi.

(1.1.1) tenglamaning $q(x)$ koeffitsiyentiga (1.1.1)+(1.1.2) Shturm-Liuvill masalasining potensiali deyiladi.

Ta’rif 1.1.1. Agar λ parametrning biror $\lambda=\lambda_0$ qiymatida (1.1.1)+(1.1.2) chegaraviy masala noldan farqli $y(x,\lambda_0)\not\equiv 0$ yechimga ega bo‘lsa, λ_0 songa (1.1.1)+(1.1.2) chegaraviy masalaning xos qiymati deyiladi, $y(x,\lambda_0)$ yechimga esa λ_0 xos qiymatga mos keluvchi xos funksiyasi deyiladi.

(1.1.1)+(1.1.2) Shturm-Liuvill masalasining barcha xos qiymatlaridan tuzilgan to‘plamga uning spektri deyiladi.

1-xossa. $y_1(x,\lambda)$ va $y_2(x,\lambda)$ (1.1.1) tenglamaning ixtiyoriy yechimlari bo‘lsin. U holda ulardan tuzilgan

$$W\{y_1(x,\lambda), y_2(x,\lambda)\} = \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) \end{vmatrix},$$

Vronskiy determinanti x o‘zgaruvchiga bog‘liq bo‘lmaydi.

Isbot. Buning uchun ushbu

$$\frac{dW}{dx} \equiv 0,$$

tenglik bajarilishini ko‘rsatish yetarli.

$$\frac{dW}{dx} \equiv (y_1 y_2' - y_1' y_2)' = y_1 y_2'' - y_1'' y_2 = y_1[q(x)y_2 - \lambda\, y_2] - y_2[q(x)y_1 - \lambda\, y_1] = 0.$$

2-xossa. Ushbu

$$\frac{d}{dx}\left\{\frac{y_1(x,\lambda)}{y_2(x,\lambda)}\right\} = \frac{y_1'(x,\lambda)y_2(x,\lambda) - y_1(x,\lambda)y_2'(x,\lambda)}{y_2^2(x,\lambda)} =$$

$$= -\frac{1}{y_2^2(x,\lambda)} W\{y_1(x,\lambda), y_2(x,\lambda)\}$$

ayniyatgan quyidagi

$$\frac{y_1(x,\lambda)}{y_2(x,\lambda)} = \text{const}$$

munosabatning bajarilishi uchun $W\{y_1(x,\lambda), y_2(x,\lambda)\} = 0$ bo‘lishi zarur va etarli ekani kelib chiqadi.

3-xossa. (*Grin ayniyati*). Ixtiyoriy $y(x), z(x) \in C^2[0,\pi]$ funksiyalar uchun ushbu

$$\int_0^\pi Ly \cdot \bar{z}\, dx = W_\pi\{y,\bar{z}\} - W_0\{y,\bar{z}\} + \int_0^\pi y \cdot \overline{Lz}\, dx,$$

ayniyat bajariladi.

Isbot. Quyidagi ayirmani hisoblaymiz:

$$\int_0^\pi (\bar{z}Ly - y\overline{Lz})dx = \int_0^\pi \{\bar{z}[-y'' + q(x)y] - y[-\bar{z}'' + q(x)\bar{z}]\}dx =$$

$$= \int_0^\pi (\bar{z}''y - y''\bar{z})dx = \int_0^\pi (\bar{z}'y - y'\bar{z})'dx = \left.\begin{vmatrix} y(x) & \bar{z}(x) \\ y'(x) & \bar{z}'(x) \end{vmatrix}\right|_0^\pi = W_\pi\{y,\bar{z}\} - W_0\{y,\bar{z}\}.$$

4-xossa. Ixtiyoriy $y(x), z(x) \in C^2[0,\pi]$ funksiyalar uchun ushbu

$$\int_0^\pi Ly \cdot \bar{z}\, dx = [y(0)\cos\alpha + y'(0)\sin\alpha] \cdot [\bar{z}(0)\sin\alpha - \bar{z}'(0)\cos\alpha] -$$

$$- [y(0)\sin\alpha - y'(0)\cos\alpha] \cdot [\bar{z}(0)\cos\alpha + \bar{z}'(0)\sin\alpha] +$$

$$+\left[y(\pi)\cos\beta+y'(\pi)\sin\beta\right]\cdot\left[-\bar{z}(\pi)\sin\beta+\bar{z}'(\pi)\cos\beta\right]+$$

$$+\left[y(\pi)\sin\beta-y'(\pi)\cos\beta\right]\cdot\left[\bar{z}(\pi)\cos\beta+\bar{z}'(\pi)\sin\beta\right]+\int_0^\pi y\cdot\overline{Lz}\,dx\,, \quad (1.1.3)$$

tenglik bajariladi.

Isbot. Grin ayniyatidagi $W_\pi\{y,\bar{z}\}-W_0\{y,\bar{z}\}$ ifodani kerakli ko'rinishda yozamiz. Buning uchun quyidagi sistemani tuzib olamiz:

$$\begin{cases} y(0)\cos\alpha+y'(0)\sin\alpha=U_1, \\ y(0)\sin\alpha-y'(0)\cos\alpha=U_2, \\ y(\pi)\cos\beta+y'(\pi)\sin\beta=U_3, \\ y(\pi)\sin\beta-y'(\pi)\cos\beta=U_4, \end{cases}$$

va undan ushbu

$$\begin{cases} y(0)=U_1\cos\alpha+U_2\sin\alpha, \\ y'(0)=U_1\sin\alpha-U_2\cos\alpha, \\ y(\pi)=U_3\cos\beta+U_4\sin\beta, \\ y'(\pi)=U_3\sin\beta-U_4\cos\beta, \end{cases}$$

tengliklarni hosil qilamiz. Bularni Grin ayniyatiga qo'ysak, (1.1.3) tenglik hosil bo'ladi.

Natija 1.1.1. Agar $y(x), z(x)\in C^2[0,\pi]$ bo'lib, $y(x)$ funksiya (1.1.2) chegaraviy shartlarni qanoatlantirsa, u holda

$$\int_0^\pi Ly\cdot\bar{z}\,dx=\int_0^\pi y\cdot\overline{Lz}dx\,,$$

tenglik bajarilishi uchun $z(x)$ funksiya ham (1.1.2) chegaraviy shartlarni qanoatlantirishi zarur va yetarlidir.

Yuqoridagi natija, (1.1.1)+(1.1.2) chegaraviy masala yordamida aniqlangan L chiziqli operator $L^2(0,\pi)$ Gilbert fazosida o'z-o'ziga qo'shma operatorni ifodalashini ko'rsatadi.

5-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill masalasining xos qiymatlari haqiqiydir.

Isbot. $\lambda=u+iv$, $i=\sqrt{-1}$, ($v\neq 0$) son (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xos qiymati bo'lsin deb faraz qilaylik va unga mos keluvchi xos funksiyani $y(x)$ bilan belgilaylik. U holda $\overline{\lambda} = u - iv$ son ham shu chegaraviy masalasining xos qiymati bo‘ladi va unga $\overline{y}(x)$ xos funksiya mos keladi. Quyidagi

$$(\lambda - \overline{\lambda})\int_0^{\pi} |y(x)|^2 dx = \int_0^{\pi} (\lambda - \overline{\lambda}) y(x)\overline{y}(x) dx = \int_0^{\pi} [(\lambda y)\overline{y} - y(\overline{\lambda}\overline{y})] dx =$$

$$= \int_0^{\pi} \{\overline{y}[-y'' + q(x)y] - y[-\overline{y}'' + q(x)\overline{y}]\} dx = \int_0^{\pi} (\overline{y}''y - y''\overline{y}) dx = \int_0^{\pi} (\overline{y}'y - y'\overline{y})' dx =$$

$$= \begin{vmatrix} y(\pi) & \overline{y}(\pi) \\ -y(\pi)ctg\beta & -\overline{y}(\pi)ctg\beta \end{vmatrix} - \begin{vmatrix} y(0) & \overline{y}(0) \\ -y(0)ctg\alpha & -\overline{y}(0)ctg\alpha \end{vmatrix} = 0,$$

tenglikdan $\overline{\lambda} = \lambda$ ekanligi kelib chiqadi. Bu esa farazimizga zid.

Natija 1.1.2. Xos funksiyani haqiqiy qilib tanlash mumkin. Chunki xos qiymat haqiqiy ekanligidan qaralayotgan tenglamaning haqiqiyligi kelib chiqadi. Chegaraviy shartlar esa hamisha haqiqiy.

6-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill masalasining turli xos qiymatlariga mos keluvchi xos funksiyalari o‘zaro ortogonaldir, ya’ni $\lambda_1 \neq \lambda_2$ xos qiymatlarga mos keluvchi $y_1(x)$, $y_2(x)$ xos funksiyalar uchun ushbu

$$\int_0^{\pi} y_1(x) \cdot y_2(x) dx = 0, \tag{1.1.4}$$

tenglik o‘rinli bo‘ladi.

Isbot. Ushbu

$$(\lambda_1 - \lambda_2)\int_0^{\pi} y_1(x) y_2(x) dx = \int_0^{\pi} [(\lambda_1 y_1) y_2 - y_1(\lambda_2 y_2)] dx =$$

$$= \int_0^{\pi} \{y_2[-y_1'' + q(x)y_1] - y_1[-y_2'' + q(x)y_2]\} dx =$$

$$= \int_0^{\pi} (y_2'' y_1 - y_1'' y_2) dx = \int_0^{\pi} (y_2' y_1 - y_1' y_2)' dx = \left. \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right|_0^{\pi} = 0,$$

ayniyatda $\lambda_1 \neq \lambda_2$ bo‘lgani uchun (1.1.4) tenglik o‘rinli bo‘lishligi kelib chiqadi.

7-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari oddiy (karrasiz), ya’ni bitta xos qiymatga mos keluvchi xos funksiyalar bir-biriga proporsionaldir.

Isbot. λ xos qiymatga $y_1(x)$, $y_2(x)$ chiziqli erkli xos funksiyalar mos keladi deb faraz qilaylik. U holda

$$W\{y_1, y_2\} = \lim_{x\to 0} W\{y_1, y_2\} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} y_1(0) & y_2(0) \\ -y_1(0)ctg\alpha & -y_2(0)ctg\alpha \end{vmatrix} = 0,$$

bo‘lgani uchun, $y_1(x)$, $y_2(x)$ chiziqli bog‘liq bo‘ladi. Bu esa farazimizga ziddir.

8-xossa. $y(x,\lambda)$ funksiya Shturm-Liuvill tenglamasining λ bo‘yicha uzluksiz differensiallanuvchi ixtiyoriy yechimi bo‘lsin. U holda

$$\int_0^\pi y^2(x,\lambda)dx = W_\pi\{\dot{y}, y\} - W_0\{\dot{y}, y\},$$

tenglik bajariladi. Bu yerda $\dot{y} = \dfrac{\partial y(x,\lambda)}{\partial \lambda}$.

Isbot. Ushbu

$$-y'' + q(x)y = \lambda y, \tag{1.1.5}$$

ayniyatdan λ bo‘yicha hosila olsak,

$$-\dot{y}'' + q(x)\dot{y} = y + \lambda\dot{y}, \tag{1.1.6}$$

tenglik kelib chiqadi. (1.1.5) va (1.1.6) tengliklarni mos ravishda $\dot{y}$ va y funksiyalarga ko‘paytirib, bir-biridan ayirsak, ushbu

$$\dot{y}''y - y''\dot{y} = -y^2,$$

ayniyat hosil bo‘ladi. Bu tenglikni $[0,\pi]$ kesmada integrallasak, ushbu

$$\int_0^\pi y^2(x,\lambda)dx = \int_0^\pi (y'\dot{y} - \dot{y}'y)'dx = W_\pi\{\dot{y}, y\} - W_0\{\dot{y}, y\},$$

formula kelib chiqadi.

9-xossa. Agar quyidagi chegaraviy masalaning

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases}$$

xos qiymatlari $\lambda_0, \lambda_1, \lambda_2, \ldots$ va xos funksiyalari $y_0(x), y_1(x), y_2(x), \ldots$ bo‘lsa, u holda ushbu

$$\begin{cases} -y'' + [q(x) + c]y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.1.7}$$

chegaraviy masalaning xos qiymatlari $\lambda_0 + c, \lambda_1 + c, \lambda_2 + c, \ldots$ va xos funksiyalari $y_0(x), y_1(x), y_2(x), \ldots$ bo‘ladi. Bu yerda c o‘zgarmas son.

Isbot. Ushbu

$$\begin{cases} -y'' + q(x)y = (\lambda - c)y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases}$$

chegaraviy masala noldan farqli yechimga ega bo‘lishi uchun $\lambda - c = \lambda_n$ bo‘lishi zarur va yetarli. Shartga ko‘ra, bu holda oxirgi chegaraviy masala $y_n(x) \not\equiv 0$ yechimga ega. Demak, (1.1.7) masalaning xos qiymatlari $\lambda_0 + c, \lambda_1 + c, \lambda_2 + c, \ldots$ va xos funksiyalari $y_0(x), y_1(x), y_2(x), \ldots$ bo‘ladi.

(1.1.1) differensial tenglamaning quyidagi

$$\varphi(0,\lambda) = -\sin\alpha, \quad \varphi'(0,\lambda) = \cos\alpha$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini $\varphi(x,\lambda)$ orqali belgilaymiz. Xuddi shuningdek, (1.1.1) tenglamaning ushbu

$$\psi(\pi,\lambda) = -\sin\beta, \quad \psi'(\pi,\lambda) = \cos\beta$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini $\psi(x,\lambda)$ orqali belgilab olamiz. Bu yerda $\varphi(x,\lambda)$ yechim (1.1.2) chegaraviy shartlardan birinchisini qanoatlantiradi, $\psi(x,\lambda)$ yechim esa (1.1.2) chegaraviy shartlardan ikkinchisini qanoatlantiradi. Bu $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ yechimlarni mos ravishda (1.1.2) chegaraviy shartlardan ikkinchisiga va birinchisiga qo‘yib, ushbu

$$\Delta(\lambda) \equiv \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta = 0,$$

$$\widetilde{\Delta}(\lambda) \equiv \psi(0,\lambda)\cos\alpha + \psi'(0,\lambda)\sin\alpha = 0,$$

tenglamalarni hosil qilamiz. Bu tenglamalarga (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xarakteristik tenglamalari deyiladi. Shturm-Liuvill tenglamasining $\varphi(x,\lambda)$ **va** $\psi(x,\lambda)$ yechimlaridan tuzilgan ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda), \psi(x,\lambda)\} \equiv \begin{vmatrix} \varphi(x,\lambda) & \psi(x,\lambda) \\ \varphi'(x,\lambda) & \psi'(x,\lambda) \end{vmatrix},$$

Vronskiy determinantini qaraymiz. Biz yuqorida bu determinant x o‘zgaruvchiga bog‘liq emasligini ko‘rsatgan edik. Shuning uchun ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda), \psi(x,\lambda)\} = \lim_{x\to 0} W\{\varphi(x,\lambda), \psi(x,\lambda)\} = \lim_{x\to \pi} W\{\varphi(x,\lambda), \psi(x,\lambda)\},$$

tengliklarni yozishimiz mumkin. Bu tengliklardan

$$\omega(\lambda) = \Delta(\lambda) = -\widetilde{\Delta}(\lambda),$$

kelib chiqadi. Bu yerdagi $\omega(\lambda)$, $\Delta(\lambda)$, $\widetilde{\Delta}(\lambda)$ funksiyalar λ o‘zgaruvchining butun funksiyalari bo‘lib, sanoqlita $\{\lambda_n\}_{n=0}^{\infty}$ nollarga ega ekanligini keyinchalik ko‘rsatamiz.

$\Delta(\lambda) = 0$ xarakteristik tenglamaning λ_n, $n = 0, 1, 2, \ldots$ ildizlari Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan iborat bo‘lib, $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar uning xos funksiyalari bo‘ladi va ushbu

$$\psi(x,\lambda_n) = c_n \varphi(x,\lambda_n), \quad c_n \neq 0 \tag{1.1.8}$$

tenglik bajariladi. Haqiqatan ham, $\lambda = \lambda_n$ son $\Delta(\lambda) = 0$ tenglamaning ildizi bo‘lsa, u holda $\omega(\lambda_n) = 0$ bo‘lgani uchun (1.1.8) tenglik o‘rinli bo‘ladi. $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar (1.1.2) chegaraviy shartlarni qanoatlantiradi, bundan esa $\lambda = \lambda_n$ son xos qiymat hamda $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar Shturm-Liuvill chegaraviy masalasining xos funksiyalari ekanligi kelib chiqadi.

Izoh 1.1.2. Odatda, agar (1.1.2) chegaraviy shartlardan birinchisi ushbu $y'(0) - hy(0) = 0$ ko‘rinishda bo‘lsa, u holda $\varphi(x,\lambda)$ yechim $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$ boshlang‘ich shartlarni qanoatlantiradigan qilib olinadi, agar (1.1.2) chegaraviy shartlardan birinchisi $y(0) = 0$ ko‘rinishda bo‘lsa, u holda $\varphi(x,\lambda)$ yechim $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$ boshlang‘ich shartlarni qanoatlantiradigan qilib olinadi.

Agar $\lambda = \lambda_0$ soni Shturm-Liuvill chegaraviy masalasining xos qiymati bo‘lib, $y(x,\lambda_0)$ unga mos keluvchi xos funksiya bo‘lsa, u holda $y(0,\lambda_0)$ va

$y'(0,\lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo‘ladi, aks xolda yechimning yagonaligi xaqidagi Koshi teoremasidan $y(x,\lambda_0)\equiv 0$ ekanligi kelib chiqadi. Bu esa xos funksiya ta’rifiga ziddir. Xuddi shuningdek, $y(\pi,\lambda_0)$ va $y'(\pi,\lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo‘lishi ko‘rsatiladi.

Quyidagi

$$\alpha_n = \sqrt{\int_0^\pi \varphi^2(x,\lambda_n)dx}\,, \quad n = 0,1,2,...,$$

sonlarga (1.1.1)+(1.1.2) chegaraviy masalaning normallovchi o‘zgarmaslari deyiladi. (1.1.1)+(1.1.2) masalaning ortonormallangan xos funksiyalari quyidagi tengliklardan topiladi:

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x,\lambda_n),\ n = 0,1,2,....$$

Ta’rif 1.1.2. Ushbu $\{\lambda_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ sonli ketma-ketliklar juftligiga Shturm-Liuvill chegaraviy masalasining spektral berilganlari (spektral xarakteristikalari) deyiladi.

Ta’rif 1.1.3. Monoton o‘suvchi, chapdan uzluksiz ushbu

$$\rho(\lambda) = \begin{cases} 0, & \lambda = 0, \\ -\sum\limits_{\lambda \le \lambda_n < 0} \dfrac{1}{\alpha_n^2}, & \lambda < 0, \\ \sum\limits_{0<\lambda_n<\lambda} \dfrac{1}{\alpha_n^2}, & \lambda > 0, \end{cases} \tag{1.1.9}$$

funksiyaga Shturm-Liuvill chegaraviy masalasining spektral funksiyasi deyiladi.

10-xossa. Shturm-Liuvill chegaraviy masalasining normallovchi o‘zgarmaslari uchun ushbu

$$\alpha_n^2 = \begin{cases} \varphi'(\pi,\lambda_n)\dot{\Delta}(\lambda_n), & \sin\beta = 0, \\ -\dfrac{\varphi(\pi,\lambda_n)}{\sin\beta}\dot{\Delta}(\lambda_n), & \sin\beta \ne 0, \end{cases} \tag{1.1.10}$$

tenglik o‘rinli bo‘ladi. Bu yerda

$$\Delta(\lambda) = \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta\,,$$

funksiya (1.1.1)+(1.1.2) Shturm-Liuvill masalasining xarakteristik funksiyasi, $\varphi(x,\lambda)$ funksiya esa (1.1.1) tenglamaning $\varphi(0,\lambda)=-\sin\alpha$, $\varphi'(0,\lambda)=\cos\alpha$ boshlang'ich shartlarni qanoatlantiruvchi yechimidir.

Isbot. 7-xossada $y(x,\lambda)$ yechim o'rnida $\varphi(x,\lambda)$ yechimni olib, $\lambda=\lambda_n$ desak, ushbu

$$\alpha_n^2=\int_0^\pi \varphi^2(x,\lambda_n)dx=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix}-\begin{vmatrix}\dot\varphi(0,\lambda_n) & \varphi(0,\lambda_n)\\ \dot\varphi'(0,\lambda_n) & \varphi'(0,\lambda_n)\end{vmatrix}=$$

$$=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix}-\begin{vmatrix}0 & -\sin\alpha\\ 0 & \cos\alpha\end{vmatrix}=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix},$$

tenglik hosil bo'ladi.

Quyidagi ikkita holni ko'rib chiqamiz.

1) $\sin\beta=0$ bo'lsin. Bu holda $\Delta(\lambda)=\varphi(\pi,\lambda)$, $\varphi(\pi,\lambda_n)=0$ bo'lgani uchun

$$\alpha_n^2=\varphi'(\pi,\lambda_n)\dot\Delta(\lambda_n)$$

tenglik o'rinli.

2) $\sin\beta\neq 0$ bo'lsin. Bu holda $\varphi'(\pi,\lambda_n)=-\varphi(\pi,\lambda_n)ctg\beta$ bo'lgani uchun

$$\alpha_n^2=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & -\varphi(\pi,\lambda_n)ctg\beta\end{vmatrix}=\varphi(\pi,\lambda_n)[-\dot\varphi(\pi,\lambda_n)ctg\beta-\dot\varphi'(\pi,\lambda_n)]=$$

$$=-\frac{\varphi(\pi,\lambda_n)}{\sin\beta}[\dot\varphi(\pi,\lambda_n)\cos\beta+\dot\varphi'(\pi,\lambda_n)\sin\beta]=-\frac{\varphi(\pi,\lambda_n)}{\sin\beta}\dot\Delta(\lambda_n)$$

tenglik bajariladi.

Natija 1.1.4. (1.1.8) tenglikdan foydalanib, (1.1.10) tenglikni quyidagi ko'rinishda yozish mumkin:

$$\alpha_n^2\cdot c_n=\begin{cases}\cos\beta\cdot\dot\Delta(\lambda_n), & \sin\beta=0,\\ \dot\Delta(\lambda_n), & \sin\beta\neq 0.\end{cases}$$

Natija 1.1.5. (1.1.10) formuladan (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining

$$\Delta(\lambda)=\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta\,,$$

xarakteristik funksiyasi karrali ildizga ega emasligi kelib chiqadi.

Misol. Ushbu

$$\begin{cases} -y'' = \lambda y, \quad 0 \le x \le \pi, \\ y'(0) - hy(0) = 0, \\ y'(\pi) - hy(\pi) = 0, \end{cases} \tag{1.1.11}$$

chegaraviy masalaning xos qiymatlarini, ortonormallangan xos funksiyalarini va spektral funksiyasini topamiz. Avvalo ushbu

$$y'' + \lambda y = 0,$$

tenglamaning umumiy yechimini topamiz:

$$y(x,\lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

So‘ngra, quyidagi

$$\varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = h,$$

boshlang‘ich shartlarni qanoatlantiruvchi $\varphi(x,\lambda)$ yechimni aniqlaymiz:

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + h\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Topilgan bu yechim chegaraviy shartlardan birinchisini qanoatlantirishi ravshan. Bu yechimni chegaraviy shartlardan ikkinchisiga qo‘yib, xarakteristik tenglamani keltirib chiqaramiz:

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + h\cos\sqrt{\lambda}x,$$

$$-\sqrt{\lambda}\sin\sqrt{\lambda}\pi + h\cos\sqrt{\lambda}\pi - h\cdot\left\{\cos\sqrt{\lambda}\pi + h\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\right\} = 0,$$

$$-\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}(\lambda + h^2) = 0.$$

Oxirgi xarakteristik tenglamadan xos qiymatlarni topamiz:

$$\lambda_0 = -h^2, \quad \lambda_n = n^2, \qquad n = 1, 2, \ldots \ .$$

Bu xos qiymatlarga quyidagi xos funksiyalar mos keladi:

$$\varphi(x,\lambda_0) = e^{hx},$$

$$\varphi(x,\lambda_n) = \cos nx + h\cdot\frac{\sin nx}{n}, \quad n = 1, 2, \ldots \ .$$

Endi esa, normallovchi o‘zgarmaslarni hisoblaymiz:

$$\alpha_0^2=\int_0^\pi \varphi^2(x,\lambda_0)dx=\int_0^\pi e^{2hx}dx=\frac{1}{2h}e^{2hx}\Big|_0^\pi=\frac{1}{2h}(e^{2h\pi}-1),$$

$$\alpha_n^2=\int_0^\pi \varphi^2(x,\lambda_n)dx=\int_0^\pi \left\{\cos nx+h\frac{\sin nx}{n}\right\}^2 dx=$$

$$=\int_0^\pi \left\{\cos^2 nx+\frac{h}{n}\sin 2nx+\frac{h^2}{n^2}\sin^2 nx\right\}dx=$$

$$=\int_0^\pi \left\{\frac{1}{2}+\frac{h^2}{2n^2}+\frac{h}{n}\sin 2nx+\frac{1}{2}\cos 2nx-\frac{h^2}{2n^2}\cos 2nx)\right\}dx=$$

$$=\frac{(n^2+h^2)\pi}{2n^2},\quad n=1,2,\dots\ .$$

Demak, (1.1.11) Shturm-Liuvill chegaraviy masalasining ortonormallangan xos funksiyalari quyidagi funksiyalardan iborat:

$$u_0(x)=\sqrt{\frac{2h}{e^{2h\pi}-1}}e^{hx},$$

$$u_n(x)=\sqrt{\frac{2}{(n^2+h^2)\pi}}\{n\cos nx+h\sin nx\},\quad n=1,2,\dots,$$

spektral funksiyasi esa ($h\neq 0$ bo‘lganida) ushbu

$$\rho(\lambda)=\begin{cases}-\dfrac{2h}{e^{2h\pi}-1}, & \lambda\le -h^2,\\ 0, & -h^2<\lambda\le 1,\\ \dfrac{2}{\pi}\sum\limits_{0<n<\sqrt{\lambda}}\dfrac{n^2}{n^2+h^2}, & \lambda>1,\end{cases}$$

formula bilan aniqlanadi.

Xususan, $h=0$ bo‘lganda, (1.1.11) Shturm-Liuvill chegaraviy masalasi quyidagi ko‘rinishni oladi:

$$\begin{cases}-y''=\lambda y, & 0\le x\le\pi,\\ y'(0)=0, \\ y'(\pi)=0.\end{cases}\tag{1.1.12}$$

Bu chegaraviy masalaning xos qiymatlari $\lambda_n=n^2,\ \ n=0,1,2,\dots$ bo‘lib, unga mos keluvchi ortonormallangan xos funksiyalari

$$u_0(x)=\sqrt{\frac{1}{\pi}},\ \ u_n(x)=\sqrt{\frac{2}{\pi}}\cos nx,\ \ n=1,2,...,$$

bo‘ladi. Normallovchi o‘zgarmaslar ketma-ketligi esa

$$\alpha_0=\sqrt{\pi},\ \ \alpha_n=\sqrt{\frac{\pi}{2}},\ \ n=1,2,...,$$

bo‘ladi. (1.1.9) formuladan foydalanib, (1.1.12) Neyman chegaraviy masalasining spektral funksiyasini topamiz:

$$\rho(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{\pi}, & 0<\lambda\le 1,\\ \dfrac{1}{\pi}+\dfrac{2}{\pi}\sum\limits_{0<n<\sqrt{\lambda}}1, & \lambda>1.\end{cases}$$

Buni quyidagicha tarzda yozish mumkin

$$\rho(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{\pi}, & 0<\lambda\le 1,\\ \dfrac{1}{\pi}+\dfrac{2}{\pi}\left[\sqrt{\lambda}\right], & \lambda>1.\end{cases}$$

Bu yerda $[a]$ belgilash a sonining butun qismini bildiradi.

2-§. Shturm-Liuvill tenglamasi yechimining asimptotikasi

Quyidagi Koshi masalasini ko‘rib chiqamiz

$$-y''+q(x)\,y=\lambda y,\ \ 0\le x\le\pi\,, \tag{1.3.1}$$

$$\begin{cases} y(0)=y_0,\\ y'(0)=y_1.\end{cases} \tag{1.3.2}$$

Bu yerda $q(x)\in C[0,\pi]$ haqiqiy funksiya bo‘lib, y_0, y_1 haqiqiy sonlar.

(1.3.1)+(1.3.2) Koshi masalasining $y(x,\lambda)$ yechimi mavjud, yagona va λ ga nisbatan butun funksiya bo‘lishini isbot qilgan edik.

Endi, $y(x,\lambda)$ yechimning $|\lambda| \to \infty$ bo‘lganda asimptotikasini o‘rganish maqsadida (1.3.1)+(1.3.2) Koshi masalasiga ekvivalent bo‘lgan integral tenglama tuzamiz. Buning uchun avvalo (1.3.1) tenglamani ushbu

$$y'' + \lambda y = f(x), \tag{1.3.3}$$

ko‘rinishda yozib olamiz. Bu yerda

$$f(x) = q(x)y(x,\lambda). \tag{1.3.4}$$

So‘ngra (1.3.3) tenglama uchun Koshi funksiyasini tuzamiz, ya’ni tarkibida t parametr qatnashgan ushbu

$$\begin{cases} y'' + \lambda y = 0, \\ y|_{x=t} = 0, \\ y'|_{x=t} = 1, \end{cases}$$

Koshi masalasining yechimini topamiz:

$$y(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \frac{\sin\sqrt{\lambda} x}{\sqrt{\lambda}},$$

$$y'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\cos\sqrt{\lambda} x,$$

$$\begin{cases} c_1\cos\sqrt{\lambda}\, t + c_2 \dfrac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} = 0, \\ -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, t + c_2\cos\sqrt{\lambda}\, t = 1, \end{cases}$$

$$\Delta = \begin{vmatrix} \cos\sqrt{\lambda}\, t & \dfrac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\, t & \cos\sqrt{\lambda}\, t \end{vmatrix} = 1,$$

$$\Delta_1 = \begin{vmatrix} 0 & \dfrac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} \\ 1 & \cos\sqrt{\lambda}\, t \end{vmatrix} = -\frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}}, \quad \Delta_2 = \begin{vmatrix} \cos\sqrt{\lambda}\, t & 0 \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\, t & 1 \end{vmatrix} = \cos\sqrt{\lambda}\, t,$$

$$c_1 = -\frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}}, \quad c_2 = \cos\sqrt{\lambda}\, t,$$

$$K(x,t) = -\frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}}\cos\sqrt{\lambda}\, x + \cos\sqrt{\lambda}\, t\frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,(x-t).$$

Koshi funksiyasi yordamida (1.3.3) tenglamaning umumiy yechimi quyidagicha ifodalanadi

$$y(x,\lambda) = A_0 \cos\sqrt{\lambda}\, x + A_1 \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x f(t)\cdot \sin\sqrt{\lambda}\ (x-t)\, dt. \qquad (1.3.5)$$

(1.3.4) belgilashni va boshlang‘ich shartlarni inobatga olsak, (1.3.5) tenglik ushbu

$$y(x,\lambda) = y_0 \cos\sqrt{\lambda}\, x + y_1 \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x q(t) y(t,\lambda) \sin\sqrt{\lambda}\ (x-t)\, dt, \qquad (1.3.6)$$

ko‘rinishni oladi. Bu tenglik biz izlagan integral tenglamadir. Bu tenglama Volterraning ikkinchi turdagi integral tenglamasidir, unga Liuvill integral tenglamasi deb ham aytiladi.

$c(x,\lambda)$ va $s(x,\lambda)$ orqali (1.3.1) tenglamaning quyidagi

$$\begin{cases} c(0,\lambda) = 1 \\ c'(0,\lambda) = 0 \end{cases} \quad \text{va} \quad \begin{cases} s(0,\lambda) = 0 \\ s'(0,\lambda) = 1 \end{cases}$$

boshlang‘ich shartlarini qanoatlantiruvchi yechimlarini belgilaymiz. Bu yechimlar uchun (1.3.6) integral tenglama ushbu

$$c(x,\lambda) = \cos\sqrt{\lambda}\, x + \frac{1}{\sqrt{\lambda}} \int_0^x q(t)\, c(t,\lambda) \sin\sqrt{\lambda}\ (x-t)\, dt, \qquad (1.3.7)$$

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x q(t)\, s(t,\lambda) \sin\sqrt{\lambda}(x-t)\, dt, \qquad (1.3.8)$$

ko‘rinishlarda bo‘ladi.

Lemma 1.3.1. Agar $x \in [0,\pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$, $|k| > 2\int_0^\pi |q(t)|\, dt$ bo‘lsa, u holda

$$|c(x,\lambda)| < 2e^{|\tau| x}, \qquad (1.3.9)$$

$$|s(x,\lambda)| < 2\frac{e^{|\tau| x}}{|k|}, \qquad (1.3.10)$$

baholashlar o‘rinli bo‘ladi.

Isbot. Quyidagi

$$F(x,\lambda)=\frac{c(x,\lambda)}{e^{|\tau|x}},$$

belgilashni kiritib olamiz. U holda

$$c(x,\lambda)=e^{|\tau|x}F(x,\lambda)$$

bo‘ladi. Buni (1.3.7) tenglamaga qo‘yamiz:

$$e^{|\tau|x}F(x,\lambda)=\cos kx+\frac{1}{k}\int_0^x q(t)e^{|\tau|t}F(t,\lambda)\sin k(x-t)dt,$$

$$F(x,\lambda)=\frac{\cos kx}{e^{|\tau|x}}+\frac{1}{k}\int_0^x q(t)F(t,\lambda)\frac{\sin k(x-t)}{e^{|\tau|(x-t)}}dt. \qquad (1.3.11)$$

Quyidagi baholashlarni bajaramiz:

$$\left|\frac{\cos kx}{e^{|\tau|x}}\right|=\frac{1}{2}\left|\frac{e^{ikx}+e^{-ikx}}{e^{|\tau|x}}\right|=\frac{1}{2}\left|\frac{e^{i\sigma x-\tau x}+e^{-i\sigma x+\tau x}}{e^{|\tau|x}}\right|\leq\frac{1}{2}(e^{-|\tau|x-\tau x}+e^{-|\tau|x+\tau x})\leq 1,$$

$$\left|\frac{\sin kx}{e^{|\tau|x}}\right|=\frac{1}{2}\left|\frac{e^{ikx}-e^{-ikx}}{e^{|\tau|x}}\right|=\frac{1}{2}\left|\frac{e^{i\sigma x-\tau x}-e^{-i\sigma x+\tau x}}{e^{|\tau|x}}\right|\leq\frac{1}{2}(e^{-|\tau|x-\tau x}+e^{-|\tau|x+|\tau|x})\leq 1.$$

$M(\lambda)=\max\limits_{0\leq x\leq\pi}|F(x,\lambda)|$ bo‘lsin. U holda yuqorida olingan baholashlarga ko‘ra (1.3.11) tenglikdan ushbu

$$|F(x,\lambda)|\leq 1+\frac{1}{|k|}\int_0^\pi |q(t)|M(\lambda)dt,$$

tengsizlik kelib chiqadi. Bu tengsizlikdan esa

$$M(\lambda)\leq 1+M(\lambda)\cdot\frac{1}{|k|}\int_0^\pi |q(t)|dt,$$

ya’ni

$$M(\lambda)\left(1-\frac{1}{|k|}\int_0^\pi |q(t)|dt\right)\leq 1, \qquad (1.3.12)$$

hosil bo‘ladi. Lemma shartiga ko‘ra

$$\frac{1}{|k|}\int_0^{\pi}|q(t)|dt<\frac{1}{2},$$

$$1-\frac{1}{|k|}\int_0^{\pi}|q(t)|dt>\frac{1}{2}, \qquad (1.3.13)$$

bo‘ladi. (1.3.12) va (1.3.13) tengsizliklardan ushbu $M(\lambda)<2$ baholash kelib chiqadi, ya’ni $|F(x,\lambda)|<2$ bo‘ladi. Shunday qilib (1.3.9) baholash isbot qilindi. (1.3.10) baholash ham shu tarzda isbot qilinadi.

Lemma 1.3.2. Agar $x\in[0,\pi]$, $\sqrt{\lambda}=k=\sigma+i\tau$, $|k|>2\int_0^{\pi}|q(t)|dt$ bo‘lsa, u holda

$$\left|c(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{2}{|k|}\int_0^x|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.14)$$

$$\left|s(x,\lambda)-\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right|\le\frac{2}{|k|^2}\int_0^x|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.15)$$

$$\left|c'(x,\lambda)+\sqrt{\lambda}\sin\sqrt{\lambda}x\right|\le 2\int_0^x|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.16)$$

$$\left|s'(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{2}{|k|}\int_0^x|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.17)$$

baholashlar o‘rinli bo‘ladi.

Isbot. (1.3.7) integral tenglama va (1.3.9) tengsizlikka ko‘ra

$$\left|c(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{1}{|k|}\int_0^x|q(t)|\cdot|c(t,\lambda)|\cdot|\sin k(x-t)|dt\le$$

$$\le\frac{1}{|k|}\int_0^x|q(t)|\cdot 2e^{|\tau|t}\cdot e^{|\tau|(x-t)}dt\le\frac{2}{|k|}\int_0^x|q(t)|dt\cdot e^{|\tau|x},$$

bo‘ladi. (1.3.8) integral tenglama va (1.3.10) baholashga ko‘ra

$$\left|s(x,\lambda)-\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right|\le\frac{1}{|k|}\int_0^x|q(t)|\cdot|s(t,\lambda)|\cdot|\sin k(x-t)|dt\le$$

$$\le\frac{1}{|k|}\int_0^x|q(t)|\cdot 2\frac{e^{|\tau|t}}{|k|}\cdot e^{|\tau|(x-t)}dt\le\frac{2}{|k|^2}\int_0^x|q(t)|dt\cdot e^{|\tau|x},$$

bo‘ladi. (1.3.14) va (1.3.15) baholashlar isbotlandi. (1.3.16) va (1.3.17) tengsizliklarni isbot qilish maqsadida, avvalo (1.3.7) va (1.3.8) tenglamalardan hosila olamiz:

$$c'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}\,x + \int_0^x q(t)\,c(t,\lambda)\cos\sqrt{\lambda}\,(x-t)\,dt\,,$$

$$s'(x,\lambda) = \cos\sqrt{\lambda}\,x + \int_0^x q(t)\,s(t,\lambda)\cos\sqrt{\lambda}(x-t)\,dt\,.$$

Bu tengliklardan hamda (1.3.9) va (1.3.10) tengsizliklardan foydalanib, ushbu

$$\left|c'(x,\lambda) + \sqrt{\lambda}\sin\sqrt{\lambda}x\right| \le \int_0^x |q(t)|\cdot|c(t,\lambda)|\cdot|\cos k(x-t)|dt \le$$

$$\le \int_0^x |q(t)|\cdot 2e^{|\tau|t}\cdot e^{|\tau|(x-t)}dt \le 2\int_0^x |q(t)|dt\cdot e^{|\tau|x}\,,$$

$$\left|s'(x,\lambda) - \cos\sqrt{\lambda}x\right| \le \int_0^x |q(t)|\cdot|s(t,\lambda)|\cdot|\cos k(x-t)|dt \le$$

$$\le \int_0^x |q(t)|\cdot 2\frac{e^{|\tau|t}}{|k|}\cdot e^{|\tau|(x-t)}dt \le \frac{2}{|k|}\int_0^x |q(t)|dt\cdot e^{|\tau|x}\,,$$

baholashlarni olamiz.

Natija 1.3.1. Agar $x\in[0,\pi]$, $\sqrt{\lambda}=k=\sigma+i\tau$, $|k|>2\int_0^\pi |q(t)|dt$ bo‘lsa, u holda quyidagi

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right),\quad s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right),$$

$$c'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}\left(e^{|\tau|x}\right),\quad s'(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right),$$

asimptotik formulalar o‘rinli bo‘ladi.

Natija 1.3.2. Agar $\varphi(x,\lambda)$ orqali (1.3.1) tenglamaning ushbu

$$\varphi(0,\lambda)=1\,,\ \varphi'(0,\lambda)=h$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilasak, u holda

$$\varphi(x,\lambda) = c(x,\lambda) + h\,s(x,\lambda)$$

bo‘ladi. Bundan $x \in [0,\pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$, $|k| > 2\int_0^{\pi} |q(t)| dt$ bo‘lganda ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right),$$

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}e\left(^{|\mathrm{Im}\sqrt{\lambda}|x}\right),$$

asimptotik formulalar o‘rinli bo‘lishi kelib chiqadi.

3-§. Chegaraviy shartlarida $y(0) = 0$ yoki $y(\pi) = 0$ bo‘lgan Shturm-Liuvill masalasining xos qiymatlari uchun asimptotik formulalar

1. ***Quyidagi***

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.1}$$

$$\begin{cases} y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.6.2}$$

Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz. Bu yerda $q(x) \in C[0,\pi]$ ***haqiqiy uzluksiz funksiya va*** H ***chekli haqiqiy son.***

$s(x,\lambda)$ orqali (1.6.1) tenglamaning ushbu

$$s(0,\lambda) = 0, \; s'(0,\lambda) = 1,$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. $s(x,\lambda)$ funksiya aniqlanishiga ko‘ra chegaraviy shartlardan birinchisini qanoatlantiradi. (1.6.1)+(1.6.2) masalaning xos qiymatlarini topish uchun bu funksiyani chegaraviy shartlarning ikkinchisiga qo‘yamiz:

$$s'(\pi,\lambda) + Hs(\pi,\lambda) = 0. \tag{1.6.3}$$

Hosil bo‘lgan tenglamaga (1.6.1)+(1.6.2) Shturm-Liuvill masalasining xarakteristik tenglamasi deyiladi. Uning ildizlari xos qiymatlardan iborat.

Teorema 1.6.1. (1.6.1)+(1.6.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to‘plam quyidan chegaralangan, ya’ni shunday son mavjudki, undan kichik xos qiymat yo‘q.

Isbot. Teskarisini faraz qilaylik, ya'ni Shturm-Liuvill masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralanmagan deb hisoblaylik. U holda, bu to'plamdan $-\infty$ ga intiladigan xos qiymatlardan tuzilgan qismiy ketma-ketlik ajratib olish mumkin bo'ladi. Aytaylik, $\{\lambda_n\}_{n=0}^{\infty}$ shu shartni qanoatlantiruvchi ketma-ketlik bo'lsin. $\mu_n = -\lambda_n > 0$ belgilash kiritamiz.

Agar ushbu

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im}\sqrt{\lambda}|x}}{\lambda}\right), \quad (|\lambda| \to \infty), \tag{1.6.4}$$

$$s'(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right), \quad (|\lambda| \to \infty), \tag{1.6.5}$$

asimptotik formulalarda, λ parametrning haqiqiy manfiy qiymatlarini qarasak va $\mu = -\lambda > 0$ belgilashni kiritsak, quyidagi formulalarni hosil qilamiz:

$$s(x,\lambda) = \frac{sh\sqrt{\mu}x}{\sqrt{\mu}} + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu}x}}{\mu}\right), \quad \mu \to +\infty, \tag{1.6.6}$$

$$s'(x,\lambda) = ch\sqrt{\mu}x + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu}x}}{\sqrt{\mu}}\right), \quad \mu \to +\infty. \tag{1.6.7}$$

(1.6.6) va (1.6.7) asimptotik formulalarni xarakteristik tenglamaga qo'yib,

$$ch\sqrt{\mu_n}\pi + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu_n}\pi}}{\sqrt{\mu_n}}\right) = 0, \qquad n \to \infty,$$

$$\frac{ch\sqrt{\mu_n}\pi}{e^{\sqrt{\mu_n}\pi}} + O\left(\frac{1}{\sqrt{\mu_n}}\right) = 0, \qquad n \to \infty,$$

bo'lishini ko'ramiz. $\mu_n \to \infty$ bo'lgani uchun oxirgi tenglikning chap tomoni $\frac{1}{2}$ ga intiladi, o'ng tomoni esa nolga teng, ziddiyat.

Natija 1.6.1. Butun funksiyaning nollar to'plami chekli limit nuqtaga ega bo'lmaganligi uchun (1.6.1)+(1.6.2) Shturm-Liuvill chegaraviy masalasi xos qiymatlarining to'plami chekli limit nuqtaga ega bo'lmaydi. Chunki xarakteristik tenglamaning chap tomoni butun funksiya. Xarakteristik tenglama faqat haqiqiy ildizlarga ega bo‘lgani uchun xos qiymatlar ketma-ketligi faqat $+\infty$ ga intilishi mumkin.

Teorema 1.6.2. (1.6.1)+(1.6.2) chegaraviy masalaning cheksizta xos qiymati mavjud. Bu xos qiymatlarni o‘sib borish tartibida $\lambda_n,\ n=0,1,2,...$ orqali belgilasak, ushbu

$$\sqrt{\lambda_n}=n+\frac{1}{2}+\underline{\underline{O}}\left(\frac{1}{n}\right),\quad n\to\infty, \tag{1.6.8}$$

asimptotik formula o‘rinli bo‘ladi.

Isbot. (1.6.1)+(1.6.2) chegaraviy masalaning xos qiymatlari orasida cheklitasi manfiy bo‘lishi mumkin. Demak, n ning biror qiymatidan boshlab barchasi musbat bo‘ladi. Biz λ_n ketma-ketlik uchun n ning yetarlicha katta qiymatlaridagi asimptotikasini keltirib chiqaramiz. Buning uchun ushbu

$$s(\pi,\lambda)=\frac{\sin k\pi}{k}+\underline{\underline{O}}\left(\frac{1}{k^2}\right),\quad k\to+\infty, \tag{1.6.9}$$

$$s'(\pi,\lambda)=\cos k\pi+\underline{\underline{O}}\left(\frac{1}{k}\right),\quad k\to+\infty, \tag{1.6.10}$$

asimptotik formulalardan foydalanamiz. Bu yerda $k=\sqrt{\lambda}>0$.

(1.6.9) va (1.6.10) ifodalarni xarakteristik tenglamaga qo‘yamiz:

$$\cos k\pi+\underline{\underline{O}}\left(\frac{1}{k}\right)=0,\quad k\to+\infty. \tag{1.6.11}$$

Bu tenglamadan

$$\cos k\pi=\underline{\underline{O}}\left(\frac{1}{k}\right),\quad k\to+\infty, \tag{1.6.12}$$

kelib chiqadi. (1.6.12) tenglamaning cheksizta ildizi bo‘lib, bu ildizlar $\cos k\pi=0$ tenglamaning ildizlari atrofida joylashgan bo‘ladi, aks holda (1.6.12) tenglama o‘ng tomoni nolga intiladi va chap tomoni nolga intilmaydi. Shuning uchun

$$k_n=m_n+\frac{1}{2}+\delta_n, \tag{1.6.13}$$

bo‘ladi. Bu yerda m_n butun son va

$$\delta_n \to 0, \quad (n \to \infty). \tag{1.6.14}$$

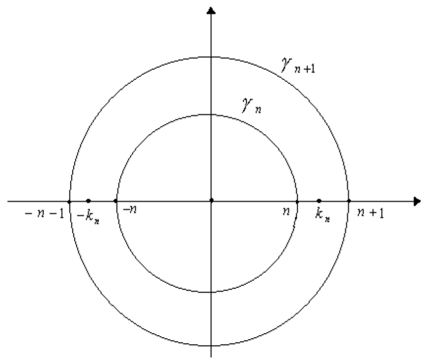
Rushe teoremasiga asoslanib, $m_n = n$ bo‘lishini ko‘rsatamiz. Buning uchun

$$f(k) = \cos k\pi \qquad \text{va} \qquad g(k) = s'(\pi, k^2) + Hs(\pi, k^2) - \cos k\pi$$

funksiyalarni ko‘rib chiqamiz. Bu funksiyalarning yig‘indisi (1.6.1)+(1.6.2) chegaraviy masalaning xarakteristik funksiyasini beradi. $g(k)$ funksiya uchun (1.6.4) va (1.6.5) asimptotik formulalarga ko‘ra

$$g(k) = \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im} k|\pi}}{k}\right), \quad (|k| \to \infty), \tag{1.6.15}$$

topamiz. γ_n orqali ushbu $k = ne^{i\alpha}$, $0 \le \alpha \le 2\pi$ aylanani belgilaymiz. Bu yerda n natural son.



2- rasm.

n natural sonning biror n_0 qiymatidan boshlab, γ_n aylana ustida $|f(k)| > |g(k)|$ bo‘ladi. Haqiqatan, ham

$$|f(k)| = |\cos k\pi| = \left|\cos\{\pi n e^{i\alpha}\}\right| = \underline{\underline{O}}\left(e^{\pi n|\sin\alpha|}\right),$$

$$|g(k)| = \underline{\underline{O}}\left(\frac{e^{\pi n|\sin\alpha|}}{n}\right),$$

bo‘lgani uchun

$$\frac{|f(k)|}{|g(k)|} = \underline{\underline{O}}(n) \to \infty,$$

bo‘ladi, xususan n natural sonning biror n_0 qiymatidan boshlab, ushbu $\dfrac{|f(k)|}{|g(k)|} > 1$ tengsizlik bajariladi. $f(k)$ funksiyaning ildizlari karrasiz bo‘lib, γ_n aylana ichidagi ildizlar $\pm\frac{1}{2},\ \pm\frac{3}{2},\ ...,\ \pm\left(n-\frac{1}{2}\right)$ sonlardan iborat. Ularning soni $2n$ ta. Rushe teoremasiga ko‘ra qaralayotgan aylana ichida $f(k)$ va $f(k)+g(k)$ funksiyalar bir xil sondagi ildizlarga ega bo‘ladi. Bunga asosan, $f(k)+g(k)$ funtsiya γ_n aylana ichidagi ildizlarining soni $2n$ ta. γ_{n+1} aylananing ichida esa $2(n+1)=2n+2$ ta ildizga ega, ya’ni, γ_n va γ_{n+1} aylanalar bilan chegaralangan halqada 2 ta ildiz joylashgan. $f(k)+g(k)=s'(\pi,k^2)+Hs(\pi,k^2)$ juft funksiya bo‘lgani uchun uning γ_{n+1} aylana ichidagi barcha ildizlari: $\pm k_0,\ \pm k_1,\ ...,\ \pm k_{n-1},\ \pm k_n$ bo‘ladi. $\pm k_n$ ildizlarning qaralayotgan halqada yotishini va haqiqiyligini e’tiborga olsak,

$$k_n\in(n,\ n+1),\quad n>n_0,\quad \text{ya'ni}\quad \left|k_n-\left(n+\frac{1}{2}\right)\right|<\frac{1}{2},\quad n>n_0,$$

bo‘lishi ko‘rinadi. Demak,

$$k_n=n+\frac{1}{2}+\delta_n,\quad |\delta_n|<\frac{1}{2},\qquad n>n_0. \tag{1.6.16}$$

δ_n ning asimptotikasini o‘rganish maqsadida (1.6.16) ifodani (1.6.12) tenglikka qo‘yamiz:

$$\cos\left(\frac{\pi}{2}+\pi n+\pi\delta_n\right)=\underline{\underline{O}}\left(\frac{1}{n+\frac{1}{2}+\delta_n}\right),\quad n\to\infty,$$

$$\sin(\pi\delta_n)=\underline{\underline{O}}\left(\frac{1}{n}\right),\quad n\to\infty,$$

$$\delta_n=\underline{\underline{O}}\left(\frac{1}{n}\right),\quad n\to\infty. \tag{1.6.17}$$

(1.6.17) ifodani (1.6.16) tenglikka qo‘ysak, ushbu

$$k_n=n+\frac{1}{2}+\underline{\underline{O}}\left(\frac{1}{n}\right),\quad n\to\infty, \tag{1.6.18}$$

formulaga ega bo‘lamiz. ■

Natija 1.6.2. (1.6.8) tenglikni kvadratga ko‘tarsak, ushbu

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c + \underline{\underline{O}}\left(\frac{1}{n^2}\right), \quad n \to \infty, \tag{1.6.19}$$

formula kelib chiqadi.

Teorema 1.6.3. Agar $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya bo‘lib, (1.6.1)+(1.6.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo‘lsa, u holda ushbu

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_1}{n + \frac{1}{2}} + \frac{\gamma_n}{n}, \quad n \to \infty, \tag{1.6.20}$$

asimptotik formula o‘rinli bo‘ladi. Bu yerda

$$c_1 = \frac{H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \ \{\gamma_n\} \in l_2, \sum_{n=0}^{\infty} |\gamma_n|^2 < \infty. \tag{1.6.21}$$

Isbot. Teoremaning isbotini o‘quvchiga havola qilamiz.

2. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.22}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.6.23}$$

Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy funksiya va h chekli haqiqiy son.

Teorema 1.6.4. (1.6.22)+(1.6.23) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to‘plam quyidan chegaralangan, ya’ni shunday son mavjudki, undan kichik xos qiymat yo‘q.

Natija 1.6.3. Butun funksiyaning nollar to‘plami chekli limit nuqtaga ega bo‘lmaganligi uchun Shturm-Liuvill chegaraviy masalasi xos qiymatlarining to‘plami chekli limit nuqtaga ega bo‘lmaydi. Chunki xarakteristik tenglamaning chap tomoni butun funksiya. Xarakteristik tenglama faqat haqiqiy ildizlarga ega bo‘lgani uchun xos qiymatlar ketma-ketligi faqat $+\infty$ ga intilishi mumkin.

Teorema 1.6.5. (1.6.22)+(1.6.23) chegaraviy masalaning cheksizta xos qiymati mavjud. Bu xos qiymatlarni o‘sib borish tartibida λ_n, $n = 0, 1, 2, \ldots$ orqali belgilasak, ushbu

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.6.24}$$

asimptotik formula o‘rinli bo‘ladi.

Teorema 1.6.6. Agar $q(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya bo‘lib, (1.6.22)+(1.6.23) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo‘lsa, u holda ushbu

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_2}{n + \frac{1}{2}} + \frac{\gamma_n}{n}, \quad n \to \infty, \qquad 1.6.25)$$

asimptotik formula o‘rinli bo‘ladi. Bu yerda

$$c_2 = \frac{h}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \ \{\gamma_n\} \in l_2, \sum_{n=0}^{\infty} |\gamma_n|^2 < \infty. \tag{1.6.26}$$

Isbot. (1.6.22)+(1.6.23) masalada $t = \pi - x$ almashtirish bajarsak va

$$q_1(t) = q(\pi - t), \ z(t) = y(\pi - t), \ H_1 = h$$

belgilashlarni kiritsak, ushbu

$$-z'' + q_1(t)z = \lambda z, \ 0 \le x \le \pi, \tag{1.6.27}$$

$$\begin{cases} z(0) = 0, \\ z'(\pi) + H_1 z(\pi) = 0, \end{cases} \tag{1.6.28}$$

chegaraviy masala hosil bo‘ladi. (1.6.27)+(1.6.28) chegaraviy masala xos qiymatlarining asimptotikasi o‘rganilgan edi. Bundan (1.6.25) va (1.6.26) kelib chiqadi.

3. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.29}$$

$$\begin{cases} y(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.6.30}$$

Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz. Bu yerda $q(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya.

Quyidagi keltirilgan teoremalar xuddi yuqoridagi usullar yordamida isbotlanadi.

Teorema 1.6.7. (1.6.29)+(1.6.30) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to‘plam quyidan chegaralangan, ya’ni shunday son mavjudki, undan kichik xos qiymat yo‘q.

Teorema 1.6.1.8. (1.6.29)+(1.6.30) Shturm-Liuvill chegaraviy masalasining cheksizta xos qiymati mavjud. Bu xos qiymatlarni o‘sib borish tartibida $\lambda_n,\ n = 1, 2, ...$ orqali belgilasak, ushbu

$$\sqrt{\lambda_n} = n + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.6.31}$$

asimptotik formula o‘rinli bo‘ladi.

Teorema 1.6.1.9. Agar $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya bo‘lib, (1.6.29)+(1.6.30) masalaning xos qiymatlari $\{\lambda_n\}_{n=1}^{\infty}$ bo‘lsa, u holda ushbu

$$\sqrt{\lambda_n} = n + \frac{c_3}{n} + \frac{\gamma_n}{n}, \quad n \to \infty \tag{1.6.32}$$

asimptotik formula o‘rinli bo‘ladi. Bu yerda

$$c_3 = \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\gamma_n\} \in l_2, \sum_{n=0}^{\infty}|\gamma_n|^2 < \infty. \tag{1.6.33}$$

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