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“FUNKSIYALAR NAZARIYASI” KAFEDRASI

401 – MATEMATIKA GURUHI TALABASI
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Mavzu: α –potensiallar va sig‘im

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Klassik potensiallar nazariyasini garmonik va subgarmonik funksiyalar tashkil etadi. Ma'lumki, potensiallar nazariyasi asosiy tushunchalaridan bo'lgan garmonik funksiyalar, Grin funksiyasi, sig'im va ularning xossalai kompleks tekislikda aniqlangan analitik funksiyalarni tadqiq etishda keng qo'llanilib kelinadi. Kompleks fazolarda aniqlangan analitik funksiyalar uchun klassik potensiallar nazariyasi metodlarini qo'llab bo'lmaydi. Shu sababdan o'tgan asrning 80-yillariga kelib kompleks potensial-lar(plyuripotensiallar) nazariyasi yaratildi. Ushbu nazariya E.Bedford, B.A.Teylor, A.Sadullaev kabi yetuk olimlarning ko'p yillik samarali mexnati asosida shakllandi. Plyuripotensiallar nazariyasini asosini plyurisubgarmonik funksiyalar va qolaversa ushbu

$$(dd^c u)^n = 0$$

Kompleks Monge-Ampere tenglamasi tashkil etadi. Ushbu nazariyaga muvofiq sig'im tushunchasi ekstremal plyurisubgarmonik funksiyalar va kompleks Monge-Ampere operatori orqali aniqlanib unga $P-$sig'im deyiladi.

Yaqinda B.Abdullayev va A.Sadullayevlar tomonidan kompleks Gessianlar yordamida aniqlanuvchi $m-$ subgarmonik funksiyalar sinfiga asoslangan $m-$ potensiallar nazariyasi asoslari ishlab chiqildi. Ushbu nazariyada $m-$subgarmonik funksiyalar quyidagicha operator yordamida aniqlangan:

$$(dd^c)^k \wedge \beta^{n-k}$$

Bu yerda $\beta = dd^c \left|z\right|^2 -$ $\mathbb{C}^n -$ fazoning evklid metrikasiga mos standart Keli formasi. Biz ushbu bitiruv malakaviy ishida biz

$$\beta = dd^c \left|z\right|^2 = dz_1 \wedge d\overline{z}_1 + \ldots + dz_n \wedge d\overline{z}_n$$

formani $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}_+$ musbat haqiqiy koordinatali vektor uchun $\alpha = \alpha_1 dz_1 \wedge d\overline{z}_1 + \ldots + \alpha_n dz_n \wedge d\overline{z}_n$ forma bilan almashtirib quyidagi

$$dd^c u \wedge \alpha^{n-1} \geq 0$$

tengsizlikni qanoatlantiruvchi funksiyalar sinfini o'rganamiz. Bu funksiyalar sinfini $\alpha-$subgarmonik funksiyalar deb nomladik.

Ushbu bitiruv malakaviy ishining maqsadi $\alpha-$ subgarmonik funksiyalarga asoslangan $\alpha-$ potensiallar nazariyasi asoslarini ishlab chiqishdan iborat.

Bitiruv malakaviy ishi ikki bobdan tashkil topgan bo'lib, birinchi bobda klassik potensiallar nazariyasining asosiy tushunchalari o'rganilgan. Ikkinchi bobda $\alpha-$ subgarmonik funksiyalar, ularning xossalari, $\alpha-$ potensial tushunchasi va $\alpha-$sig'im tushunchalari o'rganiladi. Shuni ta'kidlab o'tish joizki ushbu tushunchalar mazkur bitiruv malakaviy ishida ilk bora kiritilmoqda. Shu sababdan ba'zi belgilashlarda kamchiliklar bo'lishi mumkin.

I-BOB. KLASSIK POTENSIALLAR NAZARIYASI

1.1. Ba'zi mihim funksiyalar sinfi

Yarim uzluksiz funksiyalar. Aytaylik, $f(x)$ funksiya $D \subset \mathbb{R}^n$ to'plamda berilgan bo'lib, $x^0 \in D$ shu to'plamning limit nuqtasi bo'lsin. Ma'lumki

$$\lim_{x \to x_0} f(x) = f(x^0) \qquad (x \in D)$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada *uzluksiz* deyiladi.

Ta'rif 1.1.1. Agar $f : D \to [-\infty, \infty)$ funksiyasi uchun

$$\overline{\lim_{x \to x_0}} f(x) \le f(x^0) \qquad (x \in D) \tag{1.1.1}$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada *yuqoridan yarim uzluksiz* deyiladi.

Quyidan yarim uzluksiz funksiya ham shunga o'xshash ta'riflanadi: $f : D \to (-\infty, \infty]$ va

$$\underline{\lim_{x \to x_0}} f(x) \ge f(x^0) \qquad (x \in D).$$

Agar $f(x)$ funksiya D to'plamning har bir nuqtasida yuqoridan yarim uzluksiz bo'lsa, funksiya D da *yuqoridan yarim uzluksiz* deyiladi. Agar $f(x)$ funksiya yuqoridan yarim uzluksiz bo'lsa, $-f(x)$ funksiya quyidan yarim uzluksiz bo'ladi. Agar $f(x)$ va $-f(x)$ funksiyalar yuqoridan yarim uzluksiz bo'lsalar, u holda $f(x)$ uzluksiz funksiya bo'ladi.

Yuqoridan yarim uzluksiz funksiyalar quyidagi xossalarga ega:

a) agar $f(x)$ funksiya K kompaktda yuqoridan yarim uzluksiz bo'lsa, u holda $f(x)$ yuqoridan chegaralangan va u o'zining eng katta qiymatiga erishadi:

$$\exists x^0 \in K : \qquad f(x) \le f(x^0) \qquad (x \in K);$$

b) agar yuqoridan yarim uzluksiz funksiyalar ketma – ketligi $\{ f_k(x) \}$ $k \to \infty$ da monoton kamayuvchi bo'lsa, u holda

$$\lim_{k \to \infty} f_k(x) = f(x)$$

ham yuqoridan yarim uzluksiz bo'ladi;

v) yuqoridan yarim uzluksiz va tekis yaqinlashuvchi funksiyalar ketma – ketligi limiti yuqoridan yarim uzluksizdir.

$F(D)$ **– sinf.** $\mathbb{R}^n$ fazoda D soha $(D \subset \mathbb{R}^n)$ olib, bu sohadagi C^∞ sinfga tegishli va *finit funksiyalar* to‘plami $F(D)$ deb belgilanadi. Demak, $\varphi \in F(D)$ bo‘lsa, u holda $\varphi \in C^\infty(D)$ bo‘lib, uning xavzasi

$$\mathrm{supp}\varphi = \overline{\{x \in D : \varphi(x) \neq 0\}} \subset\subset D$$

bo‘ladi.

Endi $F(D)$ da yaqinlashish tushunchasini kiritamiz.

Agar

$$\mathrm{supp}\varphi_k \subset D_0 \subset\subset D \qquad (k = 1,2,3,...)$$

bo‘lib bu $\varphi_k(x)$ ketma-ketlik $\varphi(x)$ ga o‘zining ixtiyoriy hususiy hosilasi bilan tekis yaqinlashsa, unda

$$k \to \infty \text{ da } \varphi_k \to \varphi$$

deyiladi. Bunday yaqinlashish ma’nosida $F(D)$ topologik chiziqli fazodir. $F(D)$ topologik chiziqli fazoga *Asosiy funksiyalar fazosi deyiladi.*

Umumlashgan funksiyalar. Chiziqli $F(D)$ fazoda ushbu

$$f : F(D) \to \mathbb{R}$$

akslantirish aniqlangan bo‘lib, bu akslantirish quyidagi shartlarni qanoatlantirsin:

a) $f -$ chiziqli, yani ixtiyoriy $\varphi, \psi \in F$ va ixtiyoriy $\lambda, \mu \in \mathbb{R}$ uchun

$$f(\lambda\varphi + \mu\psi) = \lambda f(\varphi) + \mu f(\psi);$$

b) $f -$uzluksiz, ya’ni ixtiyoriy $\varphi_k \in F$ va $\lim\limits_{k\to\infty} \varphi_k = \varphi$ ketma-ketlik uchun

$$\lim_{k\to\infty} f(\varphi_k) = f(\varphi).$$

U holda f akslantirish F fazoda aniqlangan *chiziqli uzluksiz funksional* deyiladi.

Aytaylik, f funksiya D sohada aniqlangan bo‘lib, u integrallanuvchi bo‘lsin: $f \in L^1(D)$. Bu funksiya $F(D)$ chiziqli fazoda uzluksiz va chiziqli bo‘lgan ushbu

$$f(\varphi) = \int_D f\varphi dV, \qquad \varphi \in F(D) \tag{1.1.2}$$

funksionalni aniqlaydi. Ayni payitda, $F(D)$ fazoda (1.1.2) ko‘rinishga ega bo‘lmagan chiziqli funksionallar ham mavjud.

Bunday funksionalga quyidagi

$$f(\varphi) = \varphi(x^0), \qquad \varphi \in F(D), \qquad x^0 \in D$$

funksional misol bo‘la oladi.

Integrallanuvchi funksiyalar sinfini kengaytirish maqsadida umumlashgan funksiyalar sinifini kiritamiz.

Ta’rif 1.1.2. $F(D)$ fazoda aniqlangan chiziqli va uzluksiz funksianal D sohada aniqlangan *umumlashgan funksiya* deyiladi.

Agar $\varphi(x) \geq 0$ uchun $f(\varphi) \geq 0$ bo‘lsa, umumlashgan f funksiya musbat funksiya deyiladi.

Masalan, ushbu

$$f(\varphi) = \int \varphi d\mu \tag{1.1.3}$$

funksional musbat funksional bo‘ladi, bunda μ D da aniqlangan ixtiyoriy Borel o‘lchovi. Aksincha, har qanday musbat funksional (1.1.3) ko‘rinishda ifodalanadi.

Aytaylik, $f(x)$ funksiya D da uzluksiz differensillanuvchi funksiya bo‘lsin: $f(x) \in C^1(D)$. Agar ushbu

$$f(\varphi) = \int f\varphi dV, \qquad \varphi \in F(D)$$

formula yordamida ifodalangan f ni funksional deb qaraydigan bo‘lsak, uning xususiy hosilasi quyidagi

$$\frac{\partial f}{\partial x_k}(\varphi) = \int \frac{\partial f}{\partial x_k}\varphi dV, \qquad (1 \leq k \leq n)$$

tenglik orqali boshqa bir funksionalni aniqlashni ko‘ramiz. Bu tenglikning o‘ng tomondagi integralni bo‘laklab integrallash bilan

$$\int \frac{\partial f}{\partial x_k}\varphi dV = -\int f\frac{\partial \varphi}{\partial x_k}dV$$

tenglikga kelamiz. Natijada

$$\frac{\partial f}{\partial x_k}(\varphi) = -\int f\frac{\partial \varphi}{\partial x_k}dV$$

bo‘ladi. Bundagi

$$-\int f \frac{\partial \varphi}{\partial x_k} dV$$

integral ixtiyoriy f funksional uchun aniqlangan bo'lib, bu holat D da aniqlangan ixtiyoriy f umumlashgan funksiyaning hosilasini quyidagicha aniqlanadi.

Ta'rif 1.1.3. Ushbu

$$F(\varphi) = -f\left(\frac{\partial \varphi}{\partial x_k}\right)$$

funksional f ning x_k bo'yicha hususiy hosilasi deyiladi va $\frac{\partial f}{\partial x_k}$ kabi belgilanadi.

Shuningdek, f ning yuqori tartibli hususiy hosilalariga ham tarif berish mumkin:

$$D^k f(\varphi) \equiv (-1)^{|k|} f(D^k \varphi),$$

bunda

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}, \qquad k = (k_1, \ldots, k_n).$$

Demak, ixtiyoriy umumlashgan funksiyaning istalgan tartibdagi xususiy hosilasi mavjud va u ham umumlashgan funksiya bo'ladi.

Garmonik funksiyalar. $D \subset \mathbb{R}^n$ sohada ushbu

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

tenglamani qanoatlantiruvchi $u \in C^2(D)$ funksiya *garmonik funksiya* deyiladi, bunda

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

Laplas operatori.

Barcha garmonik funksiyalar to'plami $h(D)$ kabi belgilanadi.

Teorema 1.1.1. Agar $u(x)$ garmonik funksiya bo'lsa, u holda ixtiyoriy $x^0 \in D$ nuqta va $S(x^0, r) \subset D$ sfera uchun

$$u(x^0) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^0,r)} u(x)d\sigma \qquad (1.1.4)$$

bo‘ladi, bunda σ_n – $\mathbb{R}^n$ fazodagi birlik sferaning yuzi.

Aksincha, agar D da olingan har bir $x^0 \in D$ nuqta va yetarlicha kichik $r\,(0 < r < r(x^0))$ lar uchun (1.1.4) formula o‘rinli bo‘lsa, u holda $u(x) \in h(D)$ bo‘ladi.

Endi garmonik funksiyalarning ba’zi bir xossalarini keltiramiz.

a) Agar $u, v \in h(D)$ bo‘lib, $\lambda, \mu \in R$ bo‘lsa, u holda

$$\lambda u + \mu v \in h(D)$$

bo‘ladi.

b) Agar umumlashgan funksiya f uchun $\Delta f = 0$, ya’ni ixtiyoriy $\varphi \in F(D)$ da $f(\Delta\varphi) = 0$ tenglik bajarilsa, u holda f garmonik funksiya bo‘ladi, aniqroq qilib aytganda, shunday $v(x) \in C^2(D)$, $\Delta v = 0$ bo‘lgan funksiya mavjudki,

$$f(\varphi) = v(\varphi) = \int v(x)\varphi(x)dV$$

bo‘ladi.

c) Agar $u(x)$ funksiya $B(x^0, r)$ sharda garmonik, uning yopig‘ida esa uzluksiz, ya’ni

$$u(x) \in h(B(x^0,r)) \cap C(\overline{B(x^0,r)})$$

bo‘lsa, u holda

$$u(x) = \int\limits_{S(x^0,r)} u(y)P(x,y)d\sigma(y)$$

formula o‘rinli bo‘ladi, bunda

$$P(x,y) = \frac{r^2 - \left|x - x^0\right|^2}{\sigma_n r \left|x - y\right|^n}, \qquad n \geq 2,$$

Puasson yadrosi.

Ikkinchi tomondan, agar $\varphi(y)$ funksiya $S(x^0, r)$ sferada uzluksiz bo‘lsa, u holda

$$u(x) = \int\limits_{S(x^0,r)} \varphi(y)P(x,y)d\sigma(y)$$

funksiya $B(x^0, r)$ sharda ushbu

$$\Delta u = 0, \qquad u\Big|_{S(x^0,r)} = \varphi(y)$$

Dirixle masalasining yechimi bo‘ladi.

Dirixle masalasining chegarasi silliq bo‘lgan ixtiyoriy $D \subset \mathbb{R}^n$ sohada ham qarash mumkin: $\varphi(y) \in C(\partial D)$ uchun yagona $u(x) \in C(\overline{D})$ mavjudki, $\Delta u = 0$, $u\,\big|_{\partial D} = \varphi\,(y)$ bajariladi;

d) kompleks tekislik $\mathbb{C} \approx \mathbb{R}^2$ da garmonik funksiyalar golomorf

funksiyalarga bevosita bog‘langandir.

Agar $z = x + iy$ deyilsa, unda

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}$$

bo‘ladi va har bir golomorf $f \in \mathcal{O}\big(D\big)$ funksiya uchun

$$\operatorname{Re} f = \frac{f + \overline{f}}{2}, \quad \operatorname{Im} f = \frac{f - \overline{f}}{2i}$$

tengliklardan

$$\Delta \operatorname{Re} f = 0 \ \ , \quad \Delta \operatorname{Im} f = 0 \ \ ,$$

ya’ni $\operatorname{Re} f$ va $\operatorname{Im} f$ larning garmonik funksiyalar ekanligi kelib chiqadi.

Shuningdek, agar $u(z) \in h\big(D\big)$ bo‘lsa, u holda har bir $z^o \in D$ nuqta va uning atrofi $B(z^o, r) \subset D$ uchun shunday $f \in \mathcal{O}\big(D\big)$ funksiya topiladiki, $\operatorname{Re} f(z) = u(z), \ \ z \in B$ bo‘ladi.

Haqiqatan ham, $z^0 = 0$ deb $B = B(0,r)$ doirada Puasson formulasini ushbu

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\xi)\frac{r^2 - |z|^2}{|\xi - z|^2}\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(\xi)\,\operatorname{Re}\frac{\xi + z}{\xi - z}\,dt\,,$$

bunda $\xi = re^{it}$, ko‘rinishda ifodalaymiz. Bu formuladan $B(0,r)$ da golomorf

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}u(\xi)\frac{\xi+z}{\xi-z}dt \tag{1.1.5}$$

funksiyasi uchun $\operatorname{Re} f(z)=u(z)$ ekanligi kelib chiqadi.

(1.1.5) formula golomorf funksiyani uning haqiqiy qismi orqali ifodalashda ham ishlatiladi: ixtiyoriy $f(z)\in\mathcal{O}(B)\cap C(\overline{B})$ uchun ushbu

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}u(\xi)\frac{\xi+z}{\xi-z}dt+i\operatorname{Im} f(0)$$

formula o‘rinlidir.

Subgarmonik funksiyalar. Faraz qilaylik, $D\subset\mathbb{R}^n$ sohada aniqlangan lokal integrallanuvchi

$$u:\ D\to[-\infty,+\infty)$$

funksiya berilgan bo‘lsin, $u\in L^1_{loc}(D)$. Agar bu funksiya quyidagi ikki shartni bajarsa:

1) $u(x)$ yuqoridan yarim uzluksiz:

$$\forall x^o\in D \quad \text{uchun} \quad \overline{\lim_{x\to x^o}}\, u(x)\le u(x^o)$$

(bundan $u(x)$ funksiyaning D ga tegishli ixtiyoriy kompaktda yuqoridan chegaralanganligi kelib chiqadi);

2) har bir $x^o\in D$ nuqta uchun shunday $r(x^o)>0$ topiladiki, $r\le r(x^o)$ uchun

$$u(x^o)\le\frac{1}{\sigma_n r^{n-1}}\int_{S(x^o,r)}u(x)d\sigma\ ; \tag{1.1.6}$$

u holda $u(x)$ funksiya D sohada *subgarmonik funksiya* deyiladi.

Subgarmonik funksiyalar sinfi $sh\left(D\right)$ kabi belgilanadi. Kelgusida qulaylik uchun biz trivial $u(x)\equiv-\infty$ funksiyasi ham D da $sh(D)$ ga tegishli deb qaraymiz. Endi subgarmonik funksiyalarning xossalarini keltiramiz.

a) subgarmonik funksiyalarning musbat koeffisentli chiziqli kombinatsiyasi subgarmonik funksiya bo‘ladi:

$$u_k(x)\in sh(D),\quad a_k\in R_+\quad (k=1,2,\dots,m)\Rightarrow$$
$$\Rightarrow a_1u_1(x)+a_2u_2(x)+\dots+a_mu_m(x)\in sh(D)\ ;$$

b) chekli sondagi subgarmonik funksiyalarning maksimumi subgarmonik funksiya bo‘ladi:

$$u_1(x),u_2(x),\dots,u_m(x)\in sh(D)\ \Rightarrow\ \max\left\{\ u_1(x),u_2(x),\dots,u_m(x)\ \right\}\in sh(D)\ ;$$

c) monoton kamayuvchi subgarmonik funksiyalar ketma – ketligining limiti subgarmonik funksiya bo‘ladi:

$$u_j(x)\in sh(D)\ \ ,\qquad u_j(x)\geq u_{j+1}(x)\quad (j=1,2,\dots)\Rightarrow\ \lim_{j\to\infty}u_j(x)\in sh(D)\ ;$$

d) tekis yaqinlashuvchi subgarmonik funksiyalar ketma – ketligi subgarmonik funksiyaga yaqinlashadi:

$$u_j(x)\in sh(D)\ \ (k=1,2,3,\dots),\ \ u_j(x)\overset{\rightarrow}{\rightarrow}u(x)\in sh(D)$$

a) – d) xossalarning isbot bevosita yuqorida keltirilgan ta’rifdan kelib chiqadi.

e) *Maksimumlar prinsipi.* Agar $u(x)\in sh(D)$ funksiya biror $x^o\in D$ nuqtada o‘zining maksimumiga erishsa, ya’ni

$$u(x^o)=\sup_{x\in D}u(x)\tag{1.1.7}$$

bo‘lsa, u holda $u(x) \equiv const$ bo‘ladi.

Ushbu

$$M = \left\{ x \in D: \; u(x) = u(x^o) \right\}$$

to‘plamni qaraylik. $u(x)$ funksiyaning yuqoridan yarim uzluksiz bo‘lganligidan va (1.1.7) shartdan M to‘plamning D da yopiqligi kelib chiqadi.

Ikkinchi tomondan, ixtiyoriy $p \in M$ uchun (1.1.6) formulaga ko‘ra

$$u(x^o) = u(p) \leq \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(p,r)} u(x) d\sigma \leq u(x^o) , \qquad r \leq r(p) ,$$

bo‘ladi. Bu munosabatdan $u\big|_{S(p,r)} \equiv u(x^o)$, ya’ni p ning $B(p, r(p))$ atrofida $u(x) = u(x^o)$ bo‘lishini ko‘ramiz. Bu esa M to‘plamning D da ochiq ekanligini ko‘rsatadi. Demak, $M = D$.

f) Agar $\vartheta(x) \in sh(D)$, $u(x) \in h(D)$ funksiyalar uchun $S(x^o, r) \subset\subset D$ sferada

$$\vartheta\big|_S \leq u\big|_S$$

bo‘lsa, u holda $B(x^o, r)$ sharda $\vartheta(x) \leq u(x)$ tengsizlik o‘rinli bo‘ladi.

g) – xossadan quyidagi muhim xulosaga kelamiz. Faraz qilaylik,

$$\vartheta(x) \in sh(D) \cap C(D)$$

berilgan bo‘lib, $B(x^o, r) \subset\subset D$ bo‘lsin. Ushbu

$$u(x) = \int\limits_{S(x^o, r)} \vartheta(y) P(x, y) d\sigma(y) , \qquad x \in B(x^o, r) ,$$

Puasson integralini qaraylik. $u(x) \in h(B)$, $u\big|_S = \vartheta\big|_S$ bo‘lib, $e)$ – xossaga ko‘ra $B = B(x^o, r)$ da $\vartheta(x) \leq u(x)$ bo‘ladi. Uzluksiz funksiyalar uchun isbotlangan bu xossa ixtiyoriy $\vartheta \in sh\left(D\right)$ uchun ham o'rinlidir: $\vartheta(x) \le u(x)$ va deyarli barcha $x \in S$ lar uchun $\vartheta(x) = u(x)$ o'rinlidir;

j) aytaylik, $u(x) \in sh(D)$ funksiya va $x^o \in D$ nuqta berilgan bo'lsin. Ushbu,

$$m(x^o, r) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^0, r)} u(x) d\sigma$$

va

$$n(x^o, r) = \frac{1}{V_n r^n} \int\limits_{B(x^o, r)} u(x) dV ,$$

bunda $V_n r^n$ miqdor $B(x^o, r)$ ning hajmi, integrallarni (o'rta qiymatlarni) qaraylik.

Teorema 1.1.2. Faraz qilaylik, $u(x) \in sh(D)$ va $u(x) \not\equiv -\infty$ bo'lsin. U holda ixtiyoriy $x^o \in D$ uchun

$$u(x^0) \le n(x^o, r) \le m(x^o, r) ,\ n(x^o, r) > -\infty ,\ 0 < r < \rho(x^o, \partial D) ,$$

tengsizliklar o'rinli bo'ladi. Bundan tashqari, agar r monoton kamayib nolga intilsa, unda $n(x^o, r)$ va $m(x^o, r)$ o'rta qiymatlar monoton kamayib, $u(x^o)$ ga intiladi.

Teoremadan ushbu muhim natijaga ega bo'lamiz: $D \subset \mathbb{R}^n$ sohada subgarmonik $u(x) \ne -\infty$ funksiya D da lokal integrallanuvchidir, ya'ni ixtiyoriy $B \subset\subset D$ uchun $\int\limits_B u(x) dV$ integral mavjuddir;

z) subgarmonik funksiya D sohaning har bir nuqtasida uzilishga ega bo‘lishi mumkin. Ayni paytda, quyidagi teorema o‘rinli bo‘ladi.

Teorema 1.1.3. Agar $u(z) \in sh(D)$ bo‘lsa, u holda shunday monoton to‘plamlar

$$D_j \subset D_{j+1} \subset D, \quad \bigcup_{j=1}^{\infty} D_j = D$$

ketma – ketligi va shunday monoton kamayuvchi funksiyalar

$$u_j(x) \in sh(D_j) \bigcap C^{\infty}(D_j), \qquad j = 1,2,3,...$$

ketma – ketligi mavjudki,

$$u(x) = \lim_{j\to\infty} u_j(x) \qquad (x \in D)$$

bo‘ladi.

Ushbu

$$K(x) = \begin{cases} ce^{-\frac{1}{1-|x|}}, & agar\ |x| \le 1\ bo'lsa, \\ 0, & agar\ |x| > 1\ bo'lsa. \end{cases}$$

funksiyani qaraylik. Bu funksiya $C^{\infty}(\mathbb{R}^n)$ sinfga tegishli bo‘lib, uning xavzasi $\mathrm{supp} K(x) = B(0,1)$ dir. Endi $K(x)$ ning ifodasidagi o‘zgarmas c ni shunday tanlab olamizki,

$$\int_{\mathbf{R}^n} K(x) dV = 1$$

bo‘lsin. $u(x) \not\equiv -\infty$ deb, bu yadro yordamida ushbu

$$u_\delta(x)=\frac{1}{\delta^n}\int\limits_{|y-x|\le\delta} u(y)K\left(\frac{y-x}{\delta}\right)dV\ , \quad \delta>0\ , \qquad (1.1.8)$$

integralni qaraylik. $u_\delta(x)$ funksiyasi

$$D_\delta=\{x\in D:\ \rho(x,\partial D)<\delta\}$$

ochiq to'plamda aniqlangan bo'lib, $x\in D_\delta$ da

$$u_\delta(x)=\frac{1}{\delta^n}\int\limits_{\mathbf{R}^n} u(y)\,K\left(\frac{y-x}{\delta}\right)dV=\int\limits_{\mathbf{R}^n} u(x+\delta y)\,K(y)dV$$

bo'ladi. Keyingi tenglikdagi birinchi integral $C^\infty(D_\delta)$ sinfga, ikkinchi integral esa $sh(D_\delta)$ sinfga tegishli bo'ladi. Binobarin,

$$u_\delta(x)\in sh(D_\delta)\bigcap C^\infty(D_\delta)\,.$$

$u(x)$ ning subgarmonik funksiya ekanligidan, u_δ ning $\delta\downarrow 0$ da, monoton kamayuvchi bo'lishligi va

$$\int\limits_{\mathbf{R}^n} K(x)dV=1$$

tenglikdan, $u_\delta(x)$ ning $u(x)$ ga intilishini topamiz.

i) Subgarmonik funksiyalar umumiy holda C^2 sinfga tegishli bo'lmasligidan, Δu - Laplas operatorini faqat umumlashgan funksiya sifatida qarash mumkin bo'ladi:

$$\Delta u(\varphi)=\int u\Delta\varphi, \qquad \varphi\in F(D).$$

Teorema 1.1.4. Har qanday subgarmonik funksiya

$u(x)\in sh(D)$, $u\not\equiv-\infty$, uchun umumlashgan ma'noda $\Delta u\ge 0$ bo'ladi, ya'ni ixtiyoriy $\varphi\in F(D)$, $\varphi\ge 0$ uchun $\int u\,\Delta\varphi\ge 0$ munosabat o'rinlidir.

Jumladan, $u(x)$ funksiya C^2 sinfga tegishli bo‘lib, u subgarmonik bo‘lsa, $u \in sh(D) \bigcap C^2(D)$, u holda $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_n^2} \geq 0$ dir.

Yuqorida keltirilgan teoremaning aksi ham o‘rinli: agar umumlashgan funksiya u uchun

$$\Delta u(\varphi) = \int u \Delta \varphi \geq 0 \ , \quad \varphi \in F(D) \ , \quad \varphi \geq 0 \ ,$$

munosabat bajarilsa, u holda u subgarmonik funksiya bo‘ladi;

k) F.Riss teoremasi. Har qanday $u(x) \in sh(D)$, $u \not\equiv -\infty$, funksiya olinganda ham D sohada shunday musbat o‘lcham μ mavjudki, ixtiyoriy $D^0 \subset\subset D$ soha uchun

$$u(x) = \int\limits_{D^0} K(x-y) d\mu_y + \Phi_{D^0}(x) \qquad (1.1.9)$$

bo‘ladi, bunda $\Phi_{D^0}(x) \in h(D^0)$ va

$$K(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, & agar \ \ n = 2 \quad bo'lsa, \\ -\frac{1}{(n-2)\sigma_n} \cdot \frac{1}{|x|^{n-2}}, & agar \ n > 2 \ bo'lsa. \end{cases}$$

Eslatib o‘tamiz, $K(x)$ funksiyasi Laplas tenglamasining fundamental yechimidir: $\Delta K(x) = \delta(x)$ bo‘lib, bu yerda $\delta(x)$ - Dirakning umumlashgan funksiyasi, $\delta \circ \varphi(x) = \varphi(0)$, $\varphi \in F(\mathbb{R}^n)$. δ ga massasi $x = 0$ da bo‘lgan birlik o‘lcham (zaryad) mos keladi.

F. Riss teoremasini isbotlashda Δu ning musbat funksionalligidan foydalaniladi. Ma'lumki, har bir musbat funksional biror μ o'lchamni aniqlaydi: $\Delta u \circ \varphi = \int \varphi \, d\mu$, $\varphi \in F(D)$. $\mu /_{D^0}$ chekli o'lcham bo'lib,

$$\vartheta(x) = \int_{D^0} K(x-y) d\mu$$

integral mavjuddir. Ayni paytda, $\Delta K(x-y) = \delta(x-y)$ ligidan, $\Delta \vartheta = \mu$, ya'ni

$$\int \vartheta \Delta \varphi \, dV = \int \varphi \, d\mu, \quad \forall \varphi \in F(D^0),$$

bo'lishini ko'rish qiyin emas. Bundan ushbu

$$\Phi_{D^0}(x) = u(x) - \vartheta(x)$$

ayirma uchun $\Delta \Phi_{D^0} = 0$ bo'lishi, ya'ni Φ_{D^0} ning D^0 sohada garmonik funksiya ekanligi kelib chiqadi.

l) Agar $u(x) \in sh(D)$ bo'lsa, u holda $e^{u(x)} \in sh(D)$ dir.

Isbot: i) – xossaga ko'ra, $u(x)$ funksiyasini *sh* va C^{∞} funksiyalar bilan yaqinlashtirish mumkin. Shundan kelib chiqib, qaralayotgan xossani $u(x) \in sh(D) \cap C^{\infty}(D)$ funksiyalar uchun isbotlash yetarlidir. Bu holda $D e^{u(x)} = e^{u(x)} \left[\Delta u + \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \right)^2 \right] \geq 0$ ekanligini ko'rish qiyin emas.

Xuddi shunday quyidagi xossa ham o'rinlidir.

m) Agar $u(x) \in sh(D)$, $u|_D < 0$ bo'lsa, u holda $-\ln[-u(x)] \in sh(D)$ dir.

1.2 Differensial formalar va oqimlar

$f \in C^1(G)$ funksiya $G \subset \mathbb{C}^n$ sohada berilgan bo'sin.

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$$

differensial 1-darajali differensial formaga misol bo'ladi. Umuman aytganda

$$w = \sum_{j=1}^{n} a_j(x) dx_j$$

ko'rinishdagi ifoda 1-darajali differensial forma deyiladi va $\deg w = 1$ yoziladi. Yuqori darajali differensial tarifini berish uchun $\wedge$ -tashqi ko'paytma tushunchasini kiritamiz. Bu ko'paytma quyidagi xossalarni qanoatlantiradi:

a) $a(x) \wedge dx_j = dx_j \wedge a(x) = a(x) dx_j$

b) $dx_k \wedge dx_j = -dx_j \wedge dx_k$, bu yerdan $dx_j \wedge dx_j = 0$ bo'lishi kelib chiqadi.

Ta'rif 1.2.1. Ushbu

$$w = \sum_{0 \le j_1 < \ldots < j_p \le n} a_{j_1 j_2 \ldots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p}$$

ifodaga $p-$ darajali differensial forma deyiladi, bu yerda $a_{j_1 j_2 \ldots j_p(x)} - G$ da aniqlangan funksiya va $\deg w = p$, $p-$darajali differensial formalar to'plami $\mathcal{F}^p(G)$ orqali belgilaymiz.

$w \in \mathcal{F}^p(G)$ differensial formani quyidagi ekvivalent formulada yozish qulay

$$w = \sum_{0 \le j_1 < \ldots < j_p \le n} a_{j_1 j_2 \ldots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p} = \sum_J a_J dx_J$$

bu yerda $dx_J = dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p}$, $J = (j_1, j_2, \ldots, j_p)$, $1 \le j_1 < \ldots < j_p \le n$ multindeks.

$0-$darajali differensial forma- bu $a(x)$ skalyar funksiya.

$n-$darajali differensial forma esa $w = a(x) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ ko'rinishga ega.

$p > n$ da $p-$darajali differensial forma nolga teng bo'ladi.

Differensial formalarning tashqi ko'paytmasi yig'indini ko'paytirish qoyidasiga bo'ysunadi

$$\left(\sum_J a_J dx_J \right) \wedge \left(\sum_I b_I dx_I \right) = \sum_{I,J} a_J b_I dx_J \wedge dx_I .$$

Shunday qilib, ikkita formaning $w_1 \wedge w_2$ ko'paytmasida, bu yerda

$\deg w_1 = n, \ \deg w_2 = p, \quad p+q > n$ bo‘lsa biz 0 ga ega bo‘lamiz.

Agar w differensial formani koefisientlari C^k sinfdan olingan bo‘lsa, u holda w forma C^k sinfga tegishli deyiladi, $w \in C^k$ kabi belgilanadi. Silliq formalar differensillashga ruxsat beradi. Agar

$$w = \sum_J a_J(x) dx_J$$

bo‘lsa, u holda

$$dw = \sum_J da_J(x) \wedge dx_J = \sum_J \left(\frac{\partial a_J}{\partial x_1} dx_1 + \frac{\partial a_J}{\partial x_2} dx_2 + \dots + \frac{\partial a_J}{\partial x_n} dx_n \right) \wedge dx_J =$$
$$= \sum_J \sum_{k=1}^n \frac{\partial a_J}{\partial x_k} dx_k \wedge dx_J.$$

Faqat bu yerda o‘xshash differensial formalarni tartibga keltirish kerak. Buning uchun $dx_j \wedge dx_j$ ko‘rinishdagi tashqi ko‘paytma qantashgan qo‘shiluvchilarni nolga tenglash kerak. Ta’rifdan ko‘rinadiki, $p-$darajali w differensial formulaning differensiali $p+1-$ darajali diferensial formani ifodalaydi:

$$d: \ \mathcal{F}^p(G) \to \mathcal{F}^{p+1}(G).$$

Differensiallash quyidagi xossalarni qanoatlantiradi:

a) $d(w_1 + w_2) = dw_1 + dw_2$

b) $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

c) $d(dw) = d^2 w = 0$

$\mathbb{C}^n \approx \mathbb{R}^{2n}$ kompleks fazoda differensial formalarni dz_j va $d\overline{z}_j$ dan foydalanib yozish mumkin. $z = (z_1, z_2, \dots, z_n)$ lar $\mathbb{C}^n \approx \mathbb{R}^{2n}$ dagi kompleks koordinatalar bo‘lsin, $z_j = x_j + ix_{j+n}, \quad j = 1, 2, \dots$

U holda

$$dx_j = \frac{dz_j + d\overline{z}_j}{2}, \quad dx_{j+n} = \frac{dz_j - d\overline{z}_j}{2i}$$

va

$$w = \sum_J a_J(x) dx_J$$

Differensial formani dz_j va $d\overline{z}_k$ larning chiziqli kombinatsiyasi ko‘rinishida yozish mumkin. Bunda har bir hadda dz_j va $d\overline{z}_k$ lar soni turlicha bo‘lishi mumkin. Ba'zan w da barcha qo'shiluvchilarda dz_j larning soni p ga, $d\overline{z}_k$ larning soni q ga teng bo'ladi. Bunday hollarda biz w differensial forma (p,q) bidarajaga ega deb aytamiz va $w \in \mathcal{F}^{(p,q)}$ deb yozamiz.

Agar

$$\partial a = \frac{\partial a}{\partial z_1} dz_1 + ... + \frac{\partial a}{\partial z_n} dz_n$$

va

$$\overline{\partial} a = \frac{\partial a}{\partial \overline{z}_1} d\overline{z}_1 + ... + \frac{\partial a}{\partial \overline{z}_n} d\overline{z}_n$$

deb belgilasak, u holda differensiallash operatori $d = \partial + \overline{\partial}$ ko'rinishga ega bo'ladi. Bu yerda shuni ta'kidlaymizki

$$d^2 w = ddw = (\partial + \overline{\partial})(\partial + \overline{\partial})w = d^2 w + \partial\overline{\partial} w + \overline{\partial}\partial w + \overline{\partial}^2 w = \partial^2 w + \overline{\partial}^2 w$$

bu yerda $d^2 w = 0$ ligidan $\partial^2 w + \overline{\partial}^2 w = 0$ tenglik istalgan w differensial forma uchun bajarilishi kelib chiqadi. Shu bilan birga $w \in \mathcal{F}^{(p,q)}$ bo'lsa, u holda $\partial^2 w \in \mathcal{F}^{p+2,q}$ va $\overline{\partial}^2 w \in \mathcal{F}^{p,q+2}$ va $\partial^2 w + \overline{\partial}^2 w = 0$ tenglikdan $\partial^2 w = 0$, $\overline{\partial}^2 w = 0$ ekani kelib chiqadi.

Quyidagi qo'shma differensial operatorini kiritamiz

$$d^c = \frac{\partial - \overline{\partial}}{4i} = \sum_{j=1}^{n} (-\frac{\partial}{\partial y_j} dx_j + \frac{\partial}{\partial x_j} dy_j)$$

u holda

$$dd^c = \frac{i}{2} \partial\overline{\partial}$$

va binobarin

$$dd^c \left| z \right|^2 = \frac{i}{2} (dz_1 \wedge d\overline{z}_1 + ... + dz_n \wedge d\overline{z}_n)$$

va

$$(dd^c \left| z \right|^2)^n = \underbrace{dd^c \left| z \right|^2 \wedge ... \wedge dd^c \left| z \right|^2}_{n\ marta} =$$

$$= n! \left(\frac{i}{2} \right)^n dz_1 \wedge d\overline{z}_1 \wedge ... \wedge dz_n \wedge d\overline{z}_n = n! dx_1 \wedge dx_2 \wedge ... \wedge dx_{2n} = dV$$

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$ dagi jismning hajmini ifodalaydi.

Yuqorida differensial forma tushunchasi $\mathbb{C}^n$ yevklid fazosida kiritildi.

$x = \varphi(y)$ o'zgaruvchi almashtirishda, bu yerda $\varphi(y) = (\varphi_1(y),...,\varphi_n(y))$, o'rniga qo'yish va differensiallash qonunlariga muvofiq hisoblanadi. Agar

$$w(x) = \sum_J a_J(x) dx_J = \sum_{1\le j_1 \le ... \le j_p \le n} a_J dx_{j_1} \wedge ... \wedge dx_{j_p}$$

bo'lsa, u holda yangi koordinatalarda

$$\tilde{w}(y) = \sum_{1\le j_1 < ... < j_p \le n} a_J(\varphi(y)) \left(\sum_{i=1}^n \frac{\partial \varphi_{j_1}}{\partial y_i} dy_i \right) \wedge ... \wedge \left(\sum_{i=1}^n \frac{\partial \varphi_{j_n}}{\partial y_i} dy_i \right)$$

ga ega bo'lamiz.

Musbat aniqlangan differensial formalar. ω differensial forma, $\deg\omega = p$ bo'lsin. Bu yerda biz faqat (p,p) bidarajali musbat differensial formalarni ko'rib chiqamiz. Shu bilan birga $\mathbb{C}^n$ fazodagi ixtiyoriy kompleks ko'pxillikda qiyinchiliksiz kiritiladigan mulohazalarni nazarda tutamiz.

Ta'rif 1.2.2. Ushbu

$$\omega = \mathop{\wedge}_{j=1}^{p} \frac{i}{2} dl_j \wedge d\bar{l}_j$$

ko'rinishidagi differensial formaga (p,p) bidarajali **bosh kuchli musbat differensial forma** deyiladi. Bu yerda $dl_j = a_{j1}dz_1 + ... + a_{jn}dz_n -$ ikkiyoqlama bazis bo'yicha chiziqli kombinatsiya $(dz_1,...,dz_n)$, $a_{jk} \in \mathbb{C}$, $j = 1,2,...,p$, $k = 1,2,...,n$. Bunday formalarning ixtiyoriy chiziqli kombinatsiyasi kuchli musbat differensial formalar deb ataladi. Yani **kuchli musbat differensial forma** bu

$$\omega = \sum_s \lambda_s(z) \mathop{\wedge}_{j=1}^{p} \frac{i}{2} dl_{js} \wedge d\bar{l}_{js},$$

ko'rinishda bo'ladi. Bu yerda $\lambda_s(z) -$ $G \subset \mathbb{C}^n$ sohada aniqlangan musbat funksiya.

Agar $z_j = x_j + ix_{n+j}$ kompleks koordinatalardan haqiqiy x_j, x_{n+j}, $j = 1,2,...,n$ koordinatalarga o'tsak, unda

$$\frac{i}{2}dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_p \wedge d\bar{z}_p = dx_1 \wedge dx_{n+1} \wedge ... \wedge dx_p \wedge dx_{n+p}$$

ifoda $\mathbb{R}^{2n}$ fazoda $2p$ darajali musbat forma deyiladi.

(n,n) bidarajali differensial forma (kuchli) musbat bo‘ladi faqat va faqat shu holdaki, qachonki u

$$\omega = \lambda(z) \mathop{\wedge}_{j=1}^{n} \frac{i}{2}dz_j \wedge d\bar{z}_j = \lambda(z) dx_1 \wedge dx_{n+1} \wedge ... \wedge dx_n \wedge dx_{2n}$$

ko‘rinishga ega bo‘lsa. Bu yerda $\lambda(z)-$ musbat funksiya. (n,n) bidarajali $\Omega = \mathop{\wedge}_{j=1}^{n} \frac{i}{2}dz_j \wedge d\bar{z}_j = dV$ differensial forma $\mathbb{C}^n$ da hajm formasi rolini o‘ynaydi. $p=1$ da $\omega = \frac{i}{2}\sum_{jk}\lambda_{jk}dz_j \wedge d\bar{z}_k -$ musbat bo‘ladi, qachonki, faqat va faqat shu holdaki, $\sum_{jk}\lambda_{jk}\xi_j\bar{\xi}_k -$ ermit kvadratik formasi musbat bo‘lsa. Binobarin, ixtiyoriy $z^0 \in G$ tayinlangan nuqtada uni unitar almashtirish yordamida $\omega = \frac{i}{2}\sum_{j=1}^{n}\mu_j dz_j \wedge d\bar{z}_j$ ko‘rinishda yozish mumkin, bu yerda $\mu_j \geq 0$.

Ta’rif 1.2.3. $\omega \in \mathcal{F}^{p,q}$ differensial forma **kuchsiz musbat** yoki **oddiy musbat** deyiladi, agar ixtiyoriy kuchli musbat $\nu \in \mathcal{F}^{n-p,n-p}$ forma uchun $\omega \wedge \nu \in \mathcal{F}^{n,n}$ musbat bo‘lsa.

$p = 0,\ 1,\ n-1,\ n$ da kuchli musbatlik kuchsiz musbatlik bilan ekvivalent. Ammo boshqa p lar uchun bu sinflar ustma-ust tushmaydi. Buni tushuntirish uchun kuchli musbat formani quyidagi ko‘rinishda qayta yozamiz:

$$\omega = \sum_s \lambda_s(z) \mathop{\wedge}_{j=1}^{p} \frac{i}{2} dl_{js} \wedge d\bar{l}_{js} = \frac{\mathrm{i}^{p^2}}{2^p}\sum_s \lambda_s(z) \mathop{\wedge}_{j=1}^{p} dl_{js} \mathop{\wedge}_{j=1}^{p} d\bar{l}_{js} =$$

$$= \frac{\mathrm{i}^{p^2}}{2^p}\sum_s \lambda_s(z) dl_s \wedge d\bar{l}_s. \tag{1.2.1}$$

Agar shunday $\theta = \sum_s \mu_s(z) dl_s$, bu yerda $\mu_s -$ kompleks qiymatli funksiya formani ko‘rib chiqsak, u holda $\frac{i^{p^2}}{2^p}\theta\wedge\overline{\theta}$ ko‘rinishidagi forma (kuchsiz) musbat bo‘ladi, biroq $2\leq p\leq n-2$ da u har doim ham (1.2.1) ko‘rinishda bo‘lmaydi.

Xulosa o‘rnida shuni aytish mumkinki, ta’rif 1.2.2 ikki yoqlama: $\nu\in\mathcal{F}^{p,p}$ forma kuchli musbat bo‘ladi, faqat shu holdaki $\omega\wedge\nu\in\mathcal{F}^{n,n}$ $-\nu\in\mathcal{F}^{n-p,n-p}$ musbat formalar uchun musbat bo‘lsa.

Oqim. Quyida biz faqat haqiqiy $\mathbb{R}^n$ fazoda p darajali oqimlar va $\mathbb{C}^n$ kompleks fazoda (p,p) bidarajali oqimlarning asosiy tushuncha va xossalarini ko‘rishimiz kerak bo‘ladi.

$G\subset\mathbb{R}^n$ tayinlangan soha uchun umumlashgan funksiyalar kabi **asosiy differensial formalar fazosi** ko‘rib chiqiladi.

$$F^p=F^p\left(G\right)=\left\{\omega:\omega\in\mathcal{F}^p:\left(G\right)\cap C^\infty\left(G\right):\operatorname{supp}\omega\subset\subset G\right\}$$

F^p dagi topologiya $F(G)$ fazo kabi aniqlanadi.

Ta’rif 1.2.4. F^p fazodagi chiziqli uzluksiz (kompleks qiymatli) T funksional $n-p=q$ darajali **oqim** deyiladi va $\boldsymbol{\mathcal{F}}^q$ orqali belgilanadi. $\boldsymbol{\mathcal{F}}^q$ oqimlar fazosi umumlashgan funksiyalarning barcha xossalariga ega bo‘ladi. Xususan, uning uchun kuchsiz yaqinlashish tushunchasini kiritish mumkin va $T_\delta\in\boldsymbol{\mathcal{F}}^q\cap C^\infty$ o‘ramani aniqlaydi. $\delta\to 0$ da oqimga yaqinlashadi ya’ni $T_\delta\circ\omega\to T\circ\omega,\forall\omega\in\mathcal{F}^p$.

II BOB. α –POTENSIALLAR VA SIG‘IM.

2.1. α –garmonik va α –subgarmonik funksiyalar

Quyidagi musbat aniqlangan formani qaraymiz

$$dd^c\left|z\right|^2 = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + ... + dz_n \wedge d\overline{z}_n)$$

bu yerda $\left|z\right|^2 = z_1\overline{z}_1 + z_2\overline{z}_2 + ... + z_n\overline{z}_n$ ga teng.

Biz shunday $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}_+^n$ vektor olib, ushbu funksiyani qaraymiz $w = \alpha_1 z_1\overline{z}_1 + \alpha_2 z_2\overline{z}_2 + ... + \alpha_n z_n\overline{z}_n$. Endi quyidagi differensial formani qaraymiz:

$$dd^c w = \frac{i}{2}\partial\overline{\partial} w = \frac{i}{2}\sum_{\substack{j=1,\\ k=1}}^{n} \frac{\partial^2 \alpha_j z_j \overline{z}_j}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k =$$

$$= \frac{i}{2}(\alpha_1 dz_1 \wedge d\overline{z}_1 + \alpha_2 dz_2 \wedge d\overline{z}_2 + ... + \alpha_n dz_n \wedge d\overline{z}_n)$$

Bu formani $n-$ darajasini hisoblaymiz:

$$(dd^c w)^n = \left(\frac{i}{2}\right)^n \cdot \alpha_1 \alpha_2 \cdot ... \cdot \alpha_n n! dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge ... \wedge dz_n \wedge d\overline{z}_n =$$

$$= \alpha_1 \alpha_2 \cdot ... \cdot \alpha_n n! dx_1 \wedge dx_2 \wedge ... \wedge dx_{2n} = \alpha_1 \alpha_2 \cdot ... \cdot \alpha_n n! dV$$

bu yerda dV shu fazodagi parallelopipedning hajmini bildiradi.

Izoh: $\frac{i}{2} dz_j \wedge d\overline{z}_j = dx_j \wedge dx_{j+n}$ ga teng.

Quyidagi operatorni qaraymiz

$$dd^c u \wedge (dd^c w)^{n-1} \tag{2.1.1}$$

bu yerda $dd^c w = \sum_{j=1}^{n} \alpha_j dz_j \wedge d\overline{z}_j$ ga teng. Endi $\left(dd^c w\right)^{n-1}$ ni hisoblaymiz:

$$\left(dd^c w\right)^{n-1} = (n-1)! \sum_{1 \le j_1 < j_2 < ... < j_{n-1} \le n} \alpha_{j_1} \alpha_{j_2} \cdot ... \cdot \alpha_{j_{n-1}} dz_{j_1} \wedge d\overline{z}_{j_1} \wedge ... \wedge dz_{j_{n-1}} \wedge d\overline{z}_{j_{n-1}}$$

$$(dd^c u) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k,$$

topilganlarni (2.1.1) tenglikga qo'yib,

$$dd^c u \wedge (dd^c w)^{n-1} = (n-1)! \sum_{\substack{1 \le j_1 < \ldots < j_{n-1} \le n \\ j \ne j_1, j_2, \ldots, j_{n-1}}} \alpha_{j_1} \alpha_{j_2} \ldots \alpha_{j_{n-1}} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV \ge 0$$

ifodaga ega bo'lamiz. Bu ifodani formal tarzda quyidagi shaklda ham yozish mumkin

$$dd^c u \wedge (dd^c w)^{n-1} = (n-1)! \sum_{j=1}^{n} \frac{\alpha_1 \alpha_2 \ldots \alpha_n}{\alpha_j} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV$$

u funksiya $D(D \subset \mathbb{C}^n)$ sohada aniqlangan bo'lsin. Ohirgi tenglikdan ko'rinadiki u funksiya uchun $dd^c u \wedge (dd^c w)^{n-1} = 0$ tenglikni bajarilishi

$$\sum_{j=1,n} \frac{\alpha_1 \alpha_2 \ldots \alpha_n}{\alpha_j} \cdot \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} = 0 \qquad (2.1.2)$$

tenglikni bajarilishi bilan ekvivalent ekan. Biz ana shu yuqoridagi (2.1.2) tenglamaning yechimlarini o'rganamiz.

Ta'rif 2.1.1. (2.1.2) tenglikni qanoatlantiruvchi $u \in C^2(D)$ funksiyaga $\alpha -$garmonik funksiya deyiladi.

$\alpha -$garmonik funksiyalar sinfi $\alpha - h(D)$ kabi belgilanadi.

Endi $\alpha -$garmonik funksiyalarning xossalarini ko'rib chiqamiz.

a) $\alpha -$ garmonik funksiyaning musbat koeffitsentli chiziqli kombinatsiyasi yana $\alpha -$garmonik funksiya bo'ladi ya'ni

$$u_k(z) \in \alpha - h(D),\ b_k \in R_+ \Rightarrow b_1 u_1(z) + b_2 u_2(z) + \ldots + b_m u_m(z) \in \alpha - h(D),$$

$$k = \overline{1, m}$$

Isbot: $u_1(z), u_2(z), \ldots, u_m(z)$ funksiyalar $\alpha -$ garmonik bo'ganligi uchun (2.1.2) tenglikni qanoatlantiradi, shunga asosan

$$b_1 \sum_{j=1,n} \frac{\alpha_1\alpha_2...\alpha_n}{\alpha_j} \cdot \frac{\partial^2 u_1}{\partial z_j \partial \overline{z}_j} + ... + b_m \sum_{j=1,n} \frac{\alpha_1\alpha_2...\alpha_n}{\alpha_j} \cdot \frac{\partial^2 u_m}{\partial z_j \partial \overline{z}_j} = 0$$

chunki $b_i \geq 0$, bundan tenlikni o'rinli ekanligi kelib chiqadi.

Teorema 2.1.1. Agar $u(z)$ funksiya D sohada α –garmonik bo'lsa, u holda $\forall z^0 \in D$ nuqta uchun $\exists \rho_0 > 0$ son topilib, $\forall\, 0 < \rho < \rho_0$ uchun

$$u(z^0) = \frac{1}{\tilde{V}(\rho)} \int\limits_{\alpha_1\left|z_1 - z_1^0\right|^2 + ... + \alpha_n\left|z_n - z_n^0\right|^2 \leq \rho^2} u(\tau) dV(\tau)$$

tenglik o'rinli bo'ladi. Bu yerda

$$\tilde{V}(\rho) - \left\{\alpha_1 \left|z_1 - z_1^0\right|^2 + ... + \alpha_n \left|z_n - z_n^0\right|^2 \leq \rho^2\right\}$$

ellipsoid bilan chegaralangan jism hajmi va $dV(\tau)$ –hajm elementi.

Isbot: Umumiylikni buzmagan holda $z^0 = (0,0,...,0) \in D$ bo'lsin deb olishimiz mumkin. Agar $u \in \alpha - h(D)$ bo'lsa, u holda

$$u\left(\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_1}} z_1, \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_2}} z_2, ..., \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_n}} z_n\right)$$

funksiya $z^0 \in D$ nuqtada garmonik funksiya bo'ladi. Demak, u funksiya garmonik bo'lgani uchun

$$\triangle u\left(\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_1}} z_1, \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_2}} z_2, ..., \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_n}} z_n\right) = \triangle_\alpha u(z) = 0$$

Garmonik funksiyalar uchun o'rta qiymat haqidagi teoremaga asosan bu funksiya uchun shunday r_0 mavjud bo'lib, $0 < r < r_0$ da

$$u(0) = \frac{1}{V(r)} \int\limits_{\left|\xi_1\right|^2 + \left|\xi_2\right|^2 + ... + \left|\xi_n\right|^2 \leq r^2} u\left(\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_1}} \xi_1, \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_2}} \xi_2, ..., \sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_n}} \xi_n\right) dV(\xi)$$

o‘rinli bo‘ladi. Endi quyidagicha almashtirish bajaramiz:

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_1}}\xi_1 = \tau_1 \Rightarrow \xi_1 = \sqrt{\frac{\alpha_1}{\alpha_1\alpha_2...\alpha_n}}\tau_1 \Rightarrow d\xi_1 = \sqrt{\frac{\alpha_1}{\alpha_1\alpha_2...\alpha_n}}d\tau_1,$$

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_2}}\xi_2 = \tau_2 \Rightarrow \xi_2 = \sqrt{\frac{\alpha_2}{\alpha_1\alpha_2...\alpha_n}}\tau_2 \Rightarrow d\xi_2 = \sqrt{\frac{\alpha_2}{\alpha_1\alpha_2...\alpha_n}}d\tau_2,$$

$$... \qquad ... \qquad ... \qquad ... \qquad ...$$

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_n}}\xi_n = \tau_n \Rightarrow \xi_n = \sqrt{\frac{\alpha_n}{\alpha_1\alpha_2...\alpha_n}}\tau_n \Rightarrow d\xi_n = \sqrt{\frac{\alpha_n}{\alpha_1\alpha_2...\alpha_n}}d\tau_n$$

va $\alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 = r^2\alpha_1...\alpha_1$ tenglikda $\rho^2 = r^2\alpha_1...\alpha_n$ deb belgilab olamiz va bundan $r = \frac{\rho}{\sqrt{\alpha_1...\alpha_n}}$ kelib chiqadi. Natijada r_0 songa mos ρ_0 soni ham topilib, $\rho > \rho_0$ da integral quyidagi ko‘rinishni oladi.

$$u(0) = \frac{1}{V(r)\sqrt{(\alpha_1...\alpha_n)^{n-1}}}\int\limits_{\alpha_1\left|\tau_1\right|^2+...+\alpha_n\left|\tau_n\right|^2\leq\rho^2} u(\tau_1,...,\tau_n)dV(\tau) =$$
$$= \frac{1}{\tilde{V}(\rho)}\int\limits_{\alpha_1\left|\tau_1\right|^2+...+\alpha_n\left|\tau_n\right|^2\leq\rho^2} u(\tau_1,...,\tau_n)dV(\tau)$$

bu yerda $\tilde{V}(\rho) - \alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 \leq \rho^2$ ellipsoidning hajmi. Demak, bu integral ellipsoid bilan chegaralangan soha bo‘yicha o‘rta qiymatni bildiradi.

Xuddi shunday quyidagi teoremani o’rinli ekaligini ko’rish mumkin**.**

Teorema 2.1.2. Agar $u(z)$ funksiya D sohada $\alpha-$garmonik bo‘lsa, u holda $\forall z^0 \in D$ nuqta uchun $\exists r_0 > 0$ son topilib, $\forall\, 0 < r < r_0$ uchun

$$u(z^0) = \frac{1}{S(r)}\int\limits_{\alpha_1\left|z_1-z_1^0\right|^2+...+\alpha_n\left|z_n-z_n^0\right|^2=r^2} u(\xi)d\sigma(\xi)$$

tengsizlik o‘rinli bo‘ladi. Bu yerda $S(r) - \alpha_1\left|z_1-z_1^0\right|^2 + ... + \alpha_n\left|z_n-z_n^0\right|^2 = r^2$ ellipsoid sirti va $d\sigma(\xi)$ - sirt elementi.

Bu teoremalar yordamida quyidagi xossalarning isboti kelib chiqadi.

b) Chekli sondagi $\alpha-$garmonik funksiyaning maksimimi ham $\alpha-$garmonik funksiya bo‘ladi ya’ni

$$u_1(z), u_2(z), ..., u_m(z) \in \alpha - h(D) \Rightarrow \max\{u_1(z), u_2(z), ..., u_m(z)\} \in \alpha - h(D).$$

c) Monoton kamayuvchi $\alpha -$ garmonik funksiyalar ketma-ketligining limiti $\alpha -$garmonik funksiya bo‘ladi ya’ni

$$u_j(z) \in \alpha - h(D),\ u_j(z) \geq u_{j+1}(z) \Rightarrow \lim_{j \to +\infty} u_j(z) \in \alpha - h(D),\ \ j = 1,2,...,n.$$

d) Tekis yaqinlashuvchi α-garmonik funksiyalar ketma-ketligi yana α-garmonik funksiyaga yaqinlashadi ya’ni

$$u_j(z) \in \alpha - h(D),\ \ u_j(z) \rightrightarrows u(z) \Rightarrow u(z) \in \alpha - h(D),\ (k = 1,2,3...)$$

e) *Maksimum prinspi.* Agar $u(z) \in \alpha - h(D)$ funksiya biror $z^0 \in D$ nuqtada o‘zining maksimumiga erishsa, ya’ni

$$u(z^0) = \sup_{x \in D} u(z)$$

bo‘lsa, u holda $u(z) = const$ bo‘ladi.

α – subgarmonik funksiyalar.

Ta’rif 2.1.2. $D \subset \mathbb{C}^n$ soha berilgan bo‘lsin. Ushbu

$$\sum_{j=1,n} \frac{\alpha_1 \alpha_2 ... \alpha_n}{\alpha_j} \cdot \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} \geq 0 \tag{2.1.3}$$

tengsizlikni qanoatlantiruvchi $u \in C^2(D)$ funksiyaga $\alpha -$subgarmonik funksiya deyiladi.

D sohada $\alpha -$subgarmonik funksiyalar sinfini $\alpha - sh(D)$ kabi belgilaymiz.

Endi $\alpha -$subgarmonik funksiyani xossalarini ko‘rib chiqamiz.

a) $\alpha -$ subgarmonik funksiyaning musbat koeffitsentli chiziqli kombinatsiyasi yana α-subgarmonik funksiya bo‘ladi ya’ni

$$u_k(z) \in \alpha - sh(D),\ b_k \in R_+ \Rightarrow b_1 u_1(z) + b_2 u_2(z) + ... + b_m u_m(z) \in \alpha - sh(D)$$

$$k = \overline{1, m}.$$

Isbot: $u_1(z), u_2(z), ..., u_m(z)$ funksiyalar $\alpha -$ subgarmonik bo‘lganligi uchun (2.1.3) tensizlikni qanoatlantiradi, shunga asosan

$$b_1 \sum_{j=1,n} \frac{\alpha_1 \alpha_2 ... \alpha_n}{\alpha_j} \cdot \frac{\partial^2 u_1}{\partial z_j \partial \overline{z}_j} + ... + b_m \sum_{j=1,n} \frac{\alpha_1 \alpha_2 ... \alpha_n}{\alpha_j} \cdot \frac{\partial^2 u_m}{\partial z_j \partial \overline{z}_j} \geq 0$$

chunki $b_i \geq 0$ bundan tensizlikni o‘rinli ekanligi kelib chiqadi.

Bu yerda ham subgarmonik funksiyaning xossasi kabi xossa o‘rinli.

Teorema 2.1.3. Agar $u(z)$ funksiya D sohada α –subgarmonik bo‘lsa, u holda $\forall z^0 \in D$ nuqta uchun $\exists \rho_0 > 0$ son topilib, $\forall\, 0 < \rho < \rho_0$ uchun

$$u(z^0) \leq \frac{1}{\tilde{V}(\rho)} \int\limits_{\alpha_1\left|z_1 - z_1^0\right|^2 + \ldots + \alpha_n\left|z_n - z_n^0\right|^2 \leq \rho^2} u(\tau)dV(\tau)$$

tenglik o‘rinli bo‘ladi. Bu yerda

$$\tilde{V}(\rho) - \left\{\alpha_1\left|z_1 - z_1^0\right|^2 + \ldots + \alpha_n\left|z_n - z_n^0\right|^2 \leq \rho^2\right\}$$

ellipsoid bilan chegaralangan jism hajmi va $dV(\tau)$ –hajm elementi.

Isbot: Umumiylikni buzmagan holda $z^0 = (0,0,\ldots,0) \in D$ bo‘lsin olishimiz mumkin. Agar $u \in \alpha - sh(D)$ bo‘lsa, u holda

$$u\left(\sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_1}}z_1, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_2}}z_2, \ldots, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_n}}z_n\right)$$

funksiya $z^0 \in D$ nuqtada subgarmonik funksiya bo‘ladi. Demak, u funksiya subgarmonik bo‘lgani uchun

$$\Delta u\left(\sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_1}}z_1, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_2}}z_2, \ldots, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_n}}z_n\right) = \Delta_\alpha u(z) \geq 0$$

Bu funksiya uchun r_0 mavjud bo‘lib, $0 < r < r_0$ da

$$u(0) \leq \frac{1}{V(r)} \int\limits_{|\xi_1|^2 + |\xi_2|^2 + \ldots + |\xi_n|^2 \leq r^2} u\left(\sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_1}}\xi_1, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_2}}\xi_2, \ldots, \sqrt{\frac{\alpha_1\alpha_2\ldots\alpha_n}{\alpha_n}}\xi_n\right) dV(\xi)$$

o‘rinli bo‘ladi. Endi quyidagicha almashtirish bajaramiz:

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_1}}\xi_1 = \tau_1 \Rightarrow \xi_1 = \sqrt{\frac{\alpha_1}{\alpha_1\alpha_2...\alpha_n}}\tau_1 \Rightarrow \ d\xi_1 = \sqrt{\frac{\alpha_1}{\alpha_1\alpha_2...\alpha_n}}d\tau_1,$$

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_2}}\xi_2 = \tau_2 \Rightarrow \xi_2 = \sqrt{\frac{\alpha_2}{\alpha_1\alpha_2...\alpha_n}}\tau_2 \Rightarrow \ d\xi_2 = \sqrt{\frac{\alpha_2}{\alpha_1\alpha_2...\alpha_n}}d\tau_2,$$

$$... \qquad ... \qquad ... \qquad ... \qquad ...$$

$$\sqrt{\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_n}}\xi_n = \tau_n \Rightarrow \xi_n = \sqrt{\frac{\alpha_n}{\alpha_1\alpha_2...\alpha_n}}\tau_n \Rightarrow \ d\xi_n = \sqrt{\frac{\alpha_n}{\alpha_1\alpha_2...\alpha_n}}d\tau_n$$

va $\alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 = r^2\alpha_1...\alpha_1$ tenglikda $\rho^2 = r^2\alpha_1...\alpha_n$ deb belgilab olamiz va bundan $r = \frac{\rho}{\sqrt{\alpha_1...\alpha_n}}$. Natijada r_0 songa mos ρ_0 soni ham topilib, $\rho > \rho_0$ da integral quyidagi ko‘rinishni oladi.

$$u(0) \le \frac{1}{V(r)\sqrt{(\alpha_1...\alpha_1)^{n-1}}} \int\limits_{\alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 \le \rho^2} u(\tau_1,...,\tau_n)dV(\tau) =$$

$$= \frac{1}{\tilde{V}(\rho)} \int\limits_{\alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 \le \rho^2} u(\tau_1,...,\tau_n)dV(\tau)$$

bu yerda $\tilde{V}(\rho) - \alpha_1\left|\tau_1\right|^2 + ... + \alpha_n\left|\tau_n\right|^2 \le \rho^2$ ellipsoidning hajmi. Demak, bu integral ellipsoid bilan chegaralangan soha bo‘yicha o‘rta qiymatni bildiradi.

Teorema 2.1.4. Agar $u(z)$ funksiya D sohada $\alpha -$subgarmonik bo‘lsa, u holda $\forall z^0 \in D$ nuqta uchun $\exists r_0 > 0$ son topilib, $\forall\, 0 < r < r_0$ uchun

$$u(z^0) \le \frac{1}{S(r)} \int\limits_{\alpha_1\left|z_1 - z_1^0\right|^2 + ... + \alpha_n\left|z_n - z_n^0\right|^2 = r^2} u(\xi)d\sigma(\xi)$$

tengsizlik o‘rinli bo‘ladi. Bu yerda $\sigma(r)$ elliptik sferaning sirti.

Bu o‘rta qiymat haqidagi teoremalar yordamida $\alpha -$ subgarmonik funksiyalarning quyidagi xossalarini isboti kelib chiqadi.

b) Chekli sondagi $\alpha -$ subgarmonik funksiyaning maksimimi ham $\alpha -$subgarmonik funksiya bo‘ladi ya’ni

$$u_1(z), u_2(z), ..., u_m(z) \in \alpha - sh(D) \Rightarrow \max\{u_1(z), u_2(z), ..., u_m(z)\} \in \alpha - sh(D).$$

c) Monoton kamayuvchi $\alpha -$subgarmonik funksiyalar ketma-ketligining limiti $\alpha -$subgarmonik funksiya bo‘ladi ya’ni

$$u_j(z) \in \alpha - sh(D),\ u_j(z) \geq u_{j+1}(z) \Rightarrow \lim_{j\to+\infty} u_j(z) \in \alpha - sh(D),\ \ j = 1,2,...,n.$$

e) Tekis yaqinlashuvchi α-subgarmonik funksiyalar ketma-ketligi yana α-subgarmonik funksiyaga yaqinlashadi ya’ni

$$u_j(z) \in \alpha - sh(D),\ \ u_j(z) \rightrightarrows u(z) \Rightarrow u(z) \in \alpha - sh(D),\ \ (k = 1,2,3...)$$

e) *Maksimum prinsipi.* Agar $u(z) \in \alpha - sh(D)$ funksiya biror $z^0 \in D$ nuqtada o‘zining maksimumiga erishsa, ya’ni

$$u(z^0) = \sup_{x\in D} u(z)$$

bo‘lsa, u holda $u(z) = const$ bo‘ladi.

f) $\alpha -$ subgarmonik funksiyalar umumiy holda C^2 sinfga tegishli bo‘lmasligidan, $\Delta_\alpha u -$ Laplas operatorini umumlashgan funksiyasi sifatida qarash mumkin, ya’ni

$$\Delta_\alpha u(\varphi) = \int u \cdot \Delta_\alpha \varphi, \qquad \varphi \in F(D).$$

k) $\alpha -$subgarmonik funksiyalar fazosiada metrika quyidagi ko‘rinishga ega:

$$\rho(z,w) = \sqrt{\sum_{i=1}^{n} \alpha_i \left| z_i - w_i \right|^2} \tag{2.1.4}$$

$D \subset \mathbb{C}^n$ sohada aniqlangan

$$u : D \to [-\infty, \infty)$$

funksiya berilgan bo‘lsin.

Ta'rif 2.1.3. Quyidagi ikkita shart o'rinli bo'lsa:

1) $u(z)$ yuqoridan yarim uzluksiz, ya'ni

$$\forall z^0 \in D \text{ uchun } \varlimsup_{z \to z^0} u(z) \leq u(z^0)$$

2) har bir $z^0 \in D$ nuqta uchun shunday $\rho(z^0) > 0$ topiladiki, $\rho \leq \rho(z^0)$ uchun

$$u(z^0) \leq \frac{1}{\tilde{V}(\rho)} \int\limits_{\alpha_1|z_1 - z_1^0|^2 + \ldots + \alpha_n|z_n - z_n^0|^2 \leq \rho^2} u(\tau) dV(\tau)$$

shartni qanoatlantirsa, u holda $u(z)$ funksiya D sohada $\alpha -$ subgarmonik funksiya deyiladi.

$\alpha -$subgarmonik funksiya va subgarmonik funksiyalarni farqli jihatlarini ko'rsatuvchi misollarni ko'rib o'tamiz.

Misol: $\mathbb{C}^3$ fazoda bizga $\alpha = (2,1,2) \in \mathbb{R}_+^3$ vektor berilgan bo'lsin. $u = -|z_1|^2 + 1,5|z_2|^2 - |z_3|^2$ bu funksiya subgarmonik funksiya emas, lekin $\alpha -$subgarmonik funksiya bo'ladi.

Haqiqatan ham,

$$\Delta u = \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} + \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} = -1 + 1,5 - 1 = -0,5 \leq 0$$

bundan ko'rinib turibdi u funksiya subgarmonik funksiya emas ekan, lekin quyidagiga asosan

$$\Delta_\alpha u = 2\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + 4\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} + 2\frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} = -2 + 6 - 2 = 2 \geq 0$$

ya'ni u $\alpha -$subgarmonik funksiya ekan.

Teorema 2.1.5. Berilgan $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}_+^n$, $n \geq 2$ vektor uchun ushbu

$$K(z) = \frac{1}{(n-1)\cdot(\alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n)^{n-1}} \tag{2.1.5}$$

funksiya quyidagi

$$\sum_{j=1}^{n} \frac{\alpha_1 \alpha_2 ... \alpha_n}{\alpha_j} \cdot \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} = 0 \tag{2.1.6}$$

elliptik tenglamaning fundamental yechimi bo‘ladi.

Isbot: Bu yadrodan hususiy xosilalarni olamiz:

$$\frac{\partial K(z)}{\partial z_1} = \frac{1}{(n-1)} \cdot \frac{\alpha_1 \overline{z}_1 (n-1)}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} = \frac{\alpha_1 \overline{z}_1}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n},$$

$$\frac{\partial K(z)}{\partial z_2} = \frac{1}{(n-1)} \cdot \frac{\alpha_2 \overline{z}_2 (n-1)}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} = \frac{\alpha_2 \overline{z}_2}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n},$$

$$... \qquad ... \qquad ... \qquad ... \qquad ... \qquad ...$$

$$\frac{\partial K(z)}{\partial z_n} = \frac{1}{(n-1)} \cdot \frac{\alpha_n \overline{z}_n (n-1)}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} = \frac{\alpha_n \overline{z}_n}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n}$$

endi bu hususiy hosilada $\overline{z}_j$, $(j = 1,..,n)$ bo‘yicha hosila olamiz.

$$\frac{\partial^2 K(z)}{\partial z_1 \partial \overline{z}_1} = \frac{\alpha_1}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} - \frac{n {\alpha_1}^2 z_1 \overline{z}_1}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^{n+1}},$$

$$\frac{\partial^2 K(z)}{\partial z_2 \partial \overline{z}_2} = \frac{\alpha_2}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} - \frac{n {\alpha_2}^2 z_2 \overline{z}_2}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^{n+1}},$$

$$... \qquad ... \qquad ... \qquad ... \qquad ...$$

$$\frac{\partial^2 K(z)}{\partial z_n \partial \overline{z}_n} = \frac{\alpha_n}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^n} - \frac{n {\alpha_n}^2 z_n \overline{z}_n}{(\alpha_1 z_1 \overline{z}_1 + ... + \alpha_n z_n \overline{z}_n)^{n+1}}$$

bu qiymatlarni (2.1.6) tenglikga qo‘yamiz:

$$\sum_{j=1}^{n}\frac{\alpha_1\alpha_2...\alpha_n}{\alpha_j}\cdot\frac{\partial^2 u}{\partial z_j\partial\overline{z}_j}=\alpha_2\alpha_3...\alpha_n\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}+...+\alpha_1\alpha_3...\alpha_{n-1}\frac{\partial^2 u}{\partial z_n\partial\overline{z}_n}=$$

$$=\alpha_2\alpha_3...\alpha_n\left(\frac{\alpha_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_1}^2 z_1\overline{z}_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)+$$

$$+\alpha_1\alpha_3...\alpha_n\left(\frac{\alpha_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_2}^2 z_2\overline{z}_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)+...+$$

$$+\alpha_1\alpha_2...\alpha_{n-1}\left(\frac{\alpha_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_n}^2 z_n\overline{z}_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)=$$

$$=\left(\frac{\alpha_1\alpha_2\alpha_3...\alpha_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_1}^2\alpha_2\alpha_3...\alpha_n z_1\overline{z}_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)+$$

$$+\left(\frac{\alpha_1\alpha_2\alpha_3...\alpha_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n\alpha_1{\alpha_2}^2\alpha_3...\alpha_n z_2\overline{z}_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)+...+$$

$$+\left(\frac{\alpha_1\alpha_2\alpha_3...\alpha_{n-1}\alpha_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n\alpha_1\alpha_2...\alpha_{n-1}{\alpha_n}^2 z_n\overline{z}_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)=$$

$$=n\alpha_1\alpha_2...\alpha_{n-1}\alpha_n\left(\frac{1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}\right)=0$$

bundan esa (2.1.5) ning (2.1.6) ni fundamental yechimi ekanligi kelib chiqadi.

α **−polyar to‘plamlar.** Bizga D sohaning qismi bo‘lgan biror E to‘plam berilgan bo‘lsin.

Ta’rif 2.1.4. Shunday $u(z)\in\alpha-sh(D)$ va $u(z)\not\equiv-\infty$ funksiya mavjud bo‘lib, $\forall z\in E$ va $u(z)=-\infty$ tenglik bajarilsa, E to‘plam D sohada α −polyar to‘plam deyiladi.

a) ixtiyoriy $D\subset\mathbb{C}^n$ sohada α − polyar to‘plam butun $\mathbb{C}^n$ da ham α −polyar to‘plam bo‘ladi, ya’ni

$$u(z)\in\alpha-sh(D),\ \ u(z)\neq-\infty,\ \ u(z)\big|_E=-\infty$$

bo‘lsa, u holda shunday $v(z)\in\alpha-sh(\mathbb{C}^n)$ funksiya topiladiki,

$$v(z)\neq-\infty \text{ va } v\big|_E=-\infty$$

bo‘ladi.

b) sanoqli sondagi α − polyar to‘plamlar yig‘indisi polyar to‘plam bo‘ladi, ya’ni $E_1, E_2, \ldots$ $\alpha-$polyar to‘plam bo‘lsa, $E=\bigcup_{j=1}^{\infty} E_j$ ham $\alpha-$polyar to‘plam bo‘ladi.

c) agar E $\alpha-$ polyar to‘plam bo‘lsa, uning Xausdorf o‘lchovi $H_{2n-2+\varepsilon}(E)=0,\ \varepsilon>0$, bo‘ladi va aksincha agar $H_{2n-2+\varepsilon}(E)<\infty$ bo‘lsa, E $\alpha-$polyar to‘plam bo‘ladi.

d) agar har bir $z^0 \in D$ nuqta uchun shunday atrof $B(z^0, r) \subset\subset D$ va shunday M o‘zgarmas topilib, $\left\{u_\beta\right\} \subset \alpha-sh(D)$ sinfdan olingan har bir $u_\beta(z)$ funksiya ushbu

$$u_\beta(z) \leq M \qquad (z \in B(z^0, r))$$

2.2 $\alpha -$ potensiallar va uning xossalari

$B(D)$ orqali D ning barcha borel qism to'plamlarining $\sigma-$ algebrasini, ya'ni D dan olingan barcha kompaktlarni o'z ichiga oluvchi eng kichik $\sigma-$ algebrani belgilaymiz.

Ta'rif 2.2.1. D da μ o'lchov deb, $B(D)$ da aniqlangan va barcha $K \subset D$ kompaktlarda chekli nomanfiy σ – additiv to'plam funksiyasiga aytiladi.

Agar $\mu(D) < \infty$ bo'lsa, u holda μ o'lchov chekli deyiladi. D dagi barcha o'lchovlar to'plami $\mathfrak{R}_D^+$ orqali belgilaymiz.

$D_0 = D_0(\mu)$ to'plam $\mu(D_0) = 0$ bo'lgan eng katta ochiq to'plam bo'lsin (u bo'sh bo'lishi ham mumkin). U holda $S(\mu) = D \setminus D_0(\mu)$ yopiq to'plam (D ga nisbatan) μ o'lchovning xavzasi deyiladi. Agar $S(\mu)$ –kompakt bo'lsa, u holda μ o'lchov finit o'lchov deyiladi. μ o'lchov $E \subset B(D)$ to'plamda markazlashgan deyiladi, agar $\mu(D \setminus E) = 0$ bo'lsa.

Ta'rif 2.2.2. D sohada ν zaryad deb, $B(D)$ da aniqlangan va $K \subset D$ kompaktda chekli haqiqiy σ–additiv to'plam funksiyasini tushunamiz. Ikkita o'lchov ayirmasi zaryad bo'ladi.

D- $\mathbb{C}^n$ da aniqlangan soha bo'lib, $K(z,w)$ $D \times D$ aniqlangan funksiya hamda ν D ning xavzasidagi zaryad bo'lsin.

Ta'rif 2.2.3. Ushbu

$$U_K^{\ \nu} = \int_D K(z,w)d\nu_w \tag{2.2.1}$$

ifodaga ν zaryadning $K-$potensiali deyiladi. Agar D bilan $\mathbb{C}^n$ ustma-ust tushib qolsa ($n \geq 2$)

$$K(z,w)=K_\alpha(z-w)=\frac{1}{n-1}\frac{1}{(\alpha_1|z_1-w_1|^2+\alpha_2|z_2-w_2|^2+...+\alpha_n|z_n-w_n|^2)^{n-1}}\ ,\quad (\alpha=(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbb{R}^n_+)$$

va unga mos potensialni

$$U^\nu_\alpha(z)=\int_D K_\alpha(z-w)d\nu_w \tag{2.2.2}$$

qaraymiz. Bu potensialga ν_w zaryadning α - potensiali deyiladi.

$E\subset\mathbb{C}^n$ kompaktni qaraymiz va $\mathfrak{M}(E)$ orqali xavzasi E da bo‘lgan barcha o‘lchovlar to‘plamini belgilaymiz. To‘plamning o‘lchovi $\mu(E)=1$ shartni qanoatlantiruvchi barcha μ o‘lchovlardan tuzilgan to‘plamini $S(E)$ orqali belgilaymiz. Ushbu energiya integralini qaraymiz:

$$I_\alpha(\mu)=\iint_{E\times E} K_\alpha(z-w)d\mu(z)d\mu(w)$$

$$W_\alpha(E)=\inf I_\alpha(\mu)$$

qiymatni belgilaymiz, bu yerda aniq quyi chegara barcha $\mu\in S(E)$ o‘lchovlar bo‘yicha olinadi. $K_\alpha(z-w)\geq\dfrac{const}{d^{2n-2}}$ ligidan, bu yerda $d-E$ ning diametri,

$$0<W_\alpha(E)\leq\infty \tag{2.2.3}$$

bo‘ladi.

Ta’rif 2.2.4. E kompaktning sig‘imi (α–sig‘imi) deb $C_\alpha(E)=W_\alpha^{-1}(E)$ songa aytiladi.

(2.2.3) ga ko‘ra $C_\alpha(E)$ chekli va nomanfiy bo‘ladi.

Teorema 2.2.1. Agar $S(\mu)$ da $U_\alpha^\mu(z) \le M$ $(0 < \alpha < n)$ tengsizlik o'rinli bo'lsa, $\mathbb{C}^n$ fazoning barcha joyida

$$U_\alpha^\mu(z) \le 2^{2n-2} M$$

tengsizlik o'rinli.

Isbot: $z \notin S(\mu)$, $z^0 - S(\mu)$ ning z ga yaqin nuqtasi bo'lsin. $\alpha-$ subgarmonik funksiyalar fazosida metrika (2.1.4) ga asosan kiritilganligidan, istalgan $w \in S(\mu)$ uchun

$$\rho(z^0, w) \le \rho(z, w) + \rho(z, z^0) \le 2\rho(z, w)$$

ga ega bo'lamiz, bu yerdan

$$\rho^{2-2n}(z^0, w) \ge 2^{2-2n} \rho^{2-2n}(z, w)$$

va

$$U_\alpha^\mu(z) \le 2^{2n-2} U_\alpha^\mu(z^0) \le 2^{2n-2} M$$

tengsizlik o'rinli ekanligi kelib chiqadi.

Teorema 2.2.2. Agar μ o'lchovning $U_\alpha^\mu(x)$ potensiali $S(\mu)$ xavzasida uzluksiz bo'lsa, u holda u $\mathbb{C}^n$ da uzluksiz bo'ladi.

Isboti. $U_\alpha^\mu(x)$ ning uzluksizligini faqat $S(\mu)$ ning nuqtalarida o'rnatish zarur.

$z^0 \in S(\mu)$, $\mu_\varepsilon - \mu$ o'lchovning $\rho(z^0, w) < \varepsilon$ $(\varepsilon > 0)$ shartdagi qismi va $\mu_1 = \mu - \mu_\varepsilon$ bo'lsin. U holda $U_\alpha^\mu(z) = U_\alpha^{\mu_\varepsilon}(z) + U_\alpha^{\mu_1}(z)$ va

$$| U_\alpha^\mu(z^0) - U_\alpha^\mu(z) | \le U_\alpha^{\mu_\varepsilon}(z^0) + U_\alpha^{\mu_\varepsilon}(z) + | U_\alpha^{\mu_1}(z^0) - U_\alpha^{\mu_1}(z) |$$

bo‘ladi.

$U_\alpha^{\mu_1}(z)$ potensial z^0 nuqtada uzluksizligidan, $U_\alpha^{\mu_\varepsilon}(z)$ potensial $\varepsilon \to 0$ da nolga tekis yaqinlashadi.

$U_\alpha^{\mu_\varepsilon}(z)$ potensialning chekliligidan istalgan berilgan $\eta > 0$ da shunday $\varepsilon_1 > 0$ soni topish mumkinki,

$$U_\alpha^{\mu_{\varepsilon_1}}(z^0) < \frac{1}{2}\eta$$

bo‘ladi. Lekin $S(\mu)$ da $U_\alpha^{\mu_{\varepsilon_1}}(z)$ potensial z^0 nuqtada uzluksiz; shuning uchun $S(\mu)$ ning $\rho(z, z^0) < \varepsilon(\eta) < \varepsilon_1$ tengsizlik bajariladigan qiymatlarida $U_\alpha^{\mu_{\varepsilon 1}}(z) < \eta$ tengsizlik o‘rinli va $U_\alpha^{\mu_\varepsilon}(z) < \eta$ ham bajariladi. Bu $S(\mu_\varepsilon)$ ning barcha nuqtalarida o‘rinli ekanligidan $\mathbb{C}^n$ ning barcha joyida $U_\alpha^{\mu_\varepsilon}(z) < 2^{2n-2}\eta$ o‘rinli va teorema isbotlandi.

Teorema 2.2.3. $\mu -$ deyarli hamma joyda $U_\alpha^{\mu}(z)$ chekli potensialga ega bo‘lgan finit o‘lchov bo‘lsin. Istalgan $\varepsilon > 0$ da shunday $K \subset S(\mu)$ kompakt topish mumkinki, μ o‘lchovning $K \subset S(\mu)$ kompaktda quyidagi shartlarni qanoatlantiruvchi μ' qismi mavjud:

1) $U_\alpha^{\mu'}(x)$ $\mathbb{C}^n$ da uzluksiz.

2) $\mu(1) - \mu'(1) < \varepsilon$

Isboti. Luzin teoremasiga ko‘ra shunday $K \subset S(\mu)$ topiladiki, 2) va $U_\alpha^{\mu}(z)$ K da uzluksiz bo‘lsin. U holda $U_\alpha^{\mu_1}(z)$ K da uzluksiz bo‘ladi. Bunga ishonch hosil qilish uchun

$$U_\alpha^{\mu'}(z) = U_\alpha^{\mu}(z) - U_\alpha^{\mu-\mu'}(z)$$

tenglikni yozamiz, uning o‘ng qismi yuqoridan yarim uzluksizligi uchun chap qismining quyidan yarim uzluksizligi kelib chiqadi. Teorema 2.2.1 ga ko‘ra $U_\alpha^{\mu'}(z)$ $\mathbb{C}^n$ da uzluksiz bo‘ladi.

Teorema 2.2.4. μ – deyarli hamma joyda μ chekli potensialli o‘lchov bo‘lsin. U holda $\{\mu_n\}$ finit o‘lchovlarning monoton o‘suvchi ketma – ketlik mavjud, shu bilan birga

1) $U_\alpha^{\mu_n}(z)$ $\mathbb{C}^n$ da uzluksiz

2) $\lim\limits_{n\to\infty} U_\alpha^{\mu_n}(z) = U_\alpha^{\mu}(z)$ bo‘ladi.

Isbot. Avvalo μ o‘lchov finit bo‘lsin. Monoton kamayib nolga intiluvchi $\{\varepsilon_n\}$ sonlar ketma–ketligini olamiz. Teorema 2.2.1 ni μ va ε_1 ga qo‘llab, uzluksiz potensialli λ_1 o‘lchovni topamizki $\mu(1) - \lambda_1(1) < \varepsilon_1$ bo‘ladi.

Endi teorema 2.2.2. ni $\mu - \lambda_1$ o‘lchovga va ε_2 songa qo‘llaymiz. $\mu - \lambda_1$ ning qismini ifodalovchi , shu bilan birga $\mu(1) - \lambda_1(1) - \lambda_2(1) < \varepsilon_2$ bo‘lgan uzluksiz potensialli λ_2 o‘lchovni olamiz. Bu mulohazalarni davom qildirib va $\mu_n(1) = \sum\limits_{k=1}^{n} \lambda_k$ deb olib, uzluksiz potensialli monoton o‘suvchi $\{\mu_n\}$ ketma – ketlikni olamiz, shu bilan birga $\mu_n(1) \to \mu(1)$. Bu erdan μ_n μ kelib chiqadi va shuning uchun 2) o‘rinli. μ finit bo‘lmagan o‘lchov bo‘lgan holda uni biz $\mu = \sum\limits_{k=1}^{\infty} \mu_k$ ko‘rinishda tasvirlaymiz, bu yerda μ_k – μ ning $k - 1 \le |z| < k$ qatlamdagi qismi va keyin har bir μ_k o‘lchov uchun $S(\mu_{k_n}) < S(\mu_k)$

bajariladigan $\{\mu_{k_n}\}$ ketma–ketlikni quramiz.

$U_\alpha^{\mu_{k_n}}(z)$ uzluksiz va $\lim\limits_{n\to\infty} U_\alpha^{\mu_{k_n}}(z) = U_\alpha^{\mu_k}(z)$ $\lambda_n = \sum\limits_{k=1}^{n} \mu_{kn}$ deb olamiz.

$U_\alpha^{\lambda_n}(z)$ potensiallar $\mathbb{C}^n$ da uzluksiz, λ_n finit o‘lchovlar monoton o‘suvchi ketma –ketlikni ifodalaydi, shu bilan birga λ_n μ ga intiladi. Binobarin

$$U_\alpha^{\mu}(z) = \lim_{n\to\infty} U_\alpha^{\mu_n}(z)$$

va teorema isbotlandi.

2.3 $\alpha -$ sig‘im va uning xossalari

Teorema 2.3.1. $E-\mathbb{C}^n$ dagi kompakt to‘plam va $U_\alpha\left(E\right)>-\infty$ tengsizlik o‘rinli bo‘lsin. U holda E da μ birlik massaning shunday taqsimoti mavjudki,

$$I_\alpha\left(\mu\right)=U_\alpha\left(E\right)$$

bo‘ladi.

Isbot. $\mu_n - E$ dagi shunday birlik massa taqsimoti ketma – ketligi bo‘lsinki $I_\alpha\left(\mu_n\right)\to U_\alpha\left(E\right)$ bo‘lsin. Qism ketma-ketlikda limitga o‘tishdan foydalanib, μ_n ketma-ketlik E da qandaydir μ o‘lchovga sust yaqinlashadi deyishimiz mumkin.

E da $f=1$ deb olib,

$$\mu\left(E\right)=\int fd\mu=\lim_{n\to\infty}\int fd\mu_n=\lim_{n\to\infty}\mu_n\left(E\right)=1$$

ga ega bo‘lamiz. Shunday qilib μ birlik o‘lchov. Endi quyidagi muhim tasdiqni isbotlaymiz.

Lemma 2.3.1. agar fiksirlangan kompakt to‘plamda $\mu_n -$ o‘lchovlar ketma - ketligi μ o‘lchovga sust yaqinlashsa va $f(z)$ F da yuqoridan yarim uzluksiz bo‘lsa, u holda

$$\overline{\lim_{n\to\infty}}\int_F f(z)d\mu_n\left(z\right)\le\int_F f(z)d\mu\left(z\right)$$

tengsizlik o‘rinli bo‘ladi.

Isbot. $f(z)$ funksiya yuqoridan yarim uzluksiz funksiya bo‘lgani uchun shunday kamayuvchi $f_m\left(z\right)$ uzluksiz funksiyalar ketma-ketligi mavjudki $f_m\left(z\right)\to f\left(z\right),\quad m\to\infty$ o‘rinli. Shuning uchun $m\to\infty$ da $\int_F f_m\left(x\right)d\mu\left(x\right)\to\int_F f\left(x\right)d\mu\left(x\right)$ va demak, $\forall\varepsilon>0$ uchun shunday m nomer mavjudki $\int_F f_m\left(z\right)d\mu\left(z\right)<\int_F fzd\mu\left(z\right)+\varepsilon$ tengsizlik o‘rinli bo‘ladi.

Shunday qilib,

$$\overline{\lim_{n\to+\infty}}\int_F fd\mu_n\le\lim_{n\to\infty}\int_F f_m d\mu_n=\int_F f_m d\mu<\int_F fd\mu+\varepsilon$$

tengsizlik o‘rinli bo‘ladi.

Bu lemmani $E\times E$ to‘plamda berilgan $K_\alpha\left(z-w\right)$ funksiyaga tadbiq etib $I_\alpha\left(\mu\right) \geq \overline{\lim} I_\alpha\left(\mu_n\right) = U_\alpha\left(E\right)$ ekanligini olamiz. Teskari tengsizlik ta'rifdan kelib chiqadi.

Ta'rif 2.3.1. μ o'lchov E dagi sig'im (tekis taqsimot) taqsimoti deyiladi,

$$U_\alpha\left(w\right) = \int_E K_\alpha\left(z-w\right) d\mu\left(z\right)$$

funksiya esa E to'plamning sig'im (tekis) potensiali deyiladi.

Bu yerda nokompakt to'plamlarning sig'imini aniqlash qulay. Buni quyidagicha bajaramiz. E – ixtiyoriy to'plam bo'lsin. E dagi barcha F kompakt qism to'plamlar $C_\alpha(F)$ sig'imlarining aniq yuqori chegarasi $C_\alpha(E)$ ichki sig'im deyiladi. Shuningdek, tashqi sig'imni quyidagicha aniqlaymiz: $C_\alpha^*(E) = \inf C_\alpha(G)$ bunda quyi chegara E ni o'zida saqlovchi barcha G ochiq to'plamlar bo'yicha olinayapti.

Ravshanki, agar va to'plam - $E_1 \subset E_2$ shartni qanoatlantiruvchi ixtiyoriy to'plamlar bo'lsa, u holda $C_\alpha\left(E_1\right) \leq C_\alpha\left(E_2\right)$. Demak, agar E ixtiyoriy chegaralangan to'plam bo'lsa u holda hamma vaqt $C_\alpha\left(E\right) \leq C_\alpha^*\left(E\right)$ o'lchov. Bu tengsizlik tenglikka aylangan paytda E to'plam sig'im bo'yicha o'lchovli (yoki C_α - o'lchovli) deb ataymiz.

Aytish mumkinki, agar E_1 va E_2 to'plamlar uchun $E_1 \subset E_2$ bo'lsa, u holda ${C^*}_\alpha\left(E_1\right) \leq {C^*}_\alpha\left(E_2\right)$.

Ta'rifdan kelib chiqadiki, agar G ochiq to'plam bo'lsa, u holda $C_\alpha^*\left(G\right) = C_\alpha\left(G\right)$, demak barcha ochiq to'plamlar sig'im bo'yicha o'lchovli. Kompakt to'plamlar uchun mos natijani quyidagi teorema beradi.

Teorema 2.3.2. E kompakt to'plam va G_n lar quyidagi shartni qanoatlantiruvchi chegaralangan ochiq to'plamlar ketma - ketligi bo'lsin.

$$\overline{G}_{n+1} \subset G_n, n = 1,2,... \text{ va } \bigcap_{n=1}^{\infty} G_n = \mathrm{E}$$

u holda

$$C_\alpha\left(G_n\right) \to C_\alpha\left(E\right), n \to \infty$$

shunday qilib kompakt to'plamlar sig'im bo'yicha o'lchovli.

Isbot. Sig'imning monotonligidan kelib chiqadiki agar $C_n = C_\alpha\left(G_n\right)$ bo'lsa, u holda $C_n \geq C_{n+1}, n = 1,2,\ldots$. Har bir fiksirlangan n larda esa $C_\alpha\left(E\right) \leq C_n$. Demak, $C = \lim\limits_{n\to\infty} C_n$ limit mavjud va $C_\alpha\left(E\right) \leq C$ tengsizlik o'rinli.

$\mu_n - G_n$ kompakt to'plamdagi maksimal qiymati

$$U_n = I_\alpha\left(\mu_n\right) = \int\limits_{G_n \times G_n} K_\alpha\left(z-w\right)d\mu_n\left(z\right)d\mu_n\left(w\right)$$

bo'lgan birlik massa taqsimoti bo'lsin. U holda yuqoridagiga teoremaga ko'ra μ birlik o'lchovga sust yaqinlashuvchi μ_{n_p} qismiy ketma - ketlik topish mumkin.

Bu μ limit o'lchov $\overline{G}_{n_p}$ da har bir ρ uchun taqsimlangan va shuning uchun E to'plamda taqsimlangan. Agar kerak bo'lsa qism ketma – ketlikka o'tib umumiylikka zid qilmagan holda, μ_n ketma - ketlik μ ga sust yaqinlashadi deb hisoblashimiz mumkin. Lemmani qo'llab

$$\overline{\lim_{n\to\infty}} U_n = \overline{\lim_{n\to\infty}} \int\limits_{\overline{G}_n \times \overline{G}_n} K_\alpha(z-w)d\mu_n(z)d\mu_n(w) \leq U = \int\limits_{E\times E} K_\alpha(z-w)d\mu(z)d\mu(w)$$

munosabatni olamiz. Bundan sig'im ta'rifiga ko'ra $C = \lim\limits_{n\to\infty} C_\alpha(\overline{G}_n) \leq C_\alpha(E)$ ekanligi kelib chiqadi va bu teoremani isbotlaydi.

Teorema 2.3.3. E to'plam $C_\alpha(E) > 0$ bo'lgan kompakt to'plam bo'lsin, $\mu - E$ dagi mos tekis taqsimot va E^*-μ ning xavzasi bo'lsin. U holda

$$C_\alpha(E^*) = C_\alpha(E)$$

tenglik o'rinli bo'ladi.

Isbot: Agar $E_0 = E - E^*$ bo'lsa, u holda $\mu(E_0) = 0$ o'lchov. Xakikatan ham agar $z^0 - E_0$ ning ixtiyoriy nuqtasi bo'lsa, u holda markazi z^0 nuqtada bo‘lgan E^* bilan kesishmaydigan shunday D ochiq shar topiladiki, $\mu(D\ )=0$ bo‘ladi. U holda markazi ratsional koordinatali nuqta $z^0 \in D_0 \subset D$ da bo‘lgan ratsional radiusli shunday D_0 ochiq shar mavjudki $z^0 \in D_0 \subset D$. Bundan kelib chiqadiki, $\mu(D_0)=0$ bo‘ladi. Barcha bunday sharlar to‘plami sanoqli shuning uchun ularni $\mu\left(E_0\right) \le \sum\limits_i \mu\left(D_i\right) = 0$ quyidagicha D_i ketma–ketlik shaklida joylashtirish mumkin. Shunday qilib $\mu - E^*$ dagi birlik massa taqsimoti bo‘ladi va

$$U_\alpha(E^*) \ge \int\limits_{E\times E} K_\alpha(z-w)d\mu(z)d\mu(w) = \int\limits_{E\times E} K_\alpha(z-w)d\mu(z)d\mu(w) = U_\alpha(E)$$

Bundan quyidagi tengsizlik kelib chiqadi: $C_\alpha(E^*) \ge C_\alpha(E)$. Oldingi teoremaga asosan teskari tengsizlik $E^* \subset E$ ekanligidan oson kelib chiqadi.

2.4 α –subgarmonik funksiyalarning Riss ifodasi

Teorema 2.4.1. Har qanday $u(z) \in \alpha - sh(D)$ funksiya olinganda ham D sohada shunday musbat μ o‘lchov mavjudki, ixtiyoriy $D^0 \subset\subset D$ soha uchun

$$u(z) = \int_{D^0} K(z-w)d\mu_w + h(z) \tag{2.4.1}$$

bo‘ladi, bunda $h(z) \in \alpha - h(D)$ funksiya va

$$K(z) = \frac{1}{(n-1)\cdot(\alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n)^{n-1}}, \text{ agar } n > 2 \text{ bo‘lsa}$$

ga teng. $K(z) = \delta(z)$ bo‘lib, bu yerda $\delta(z)$ Dirakning umumlashgan funksiyasi, $\delta(\varphi(z)) = \varphi(0)$, $\varphi \in F(\mathbb{C}^n)$. δ ga massasi $z = 0$ da bo‘lgan birlik o‘lchov mos keladi.

Isbot: $u(z)$ α - subgarmonik funksiya bo‘lganligi uchun $\triangle_\alpha u$ musbat funksional bo‘ladi. Har bir musbat funksional biror μ o‘lchovni aniqlaydi: $\Delta_\alpha u(\varphi) = \int \varphi d\mu$, $\varphi \in F(D)$. $\mu /_{D^0}$ chekli o‘lchov bo‘lib,

$$v(z) = \int_{D^0} K(z-w)d\mu_w$$

integral mavjud bo‘ladi. $\Delta_\alpha K(z-w) = \delta(z-w)$ ekanligidan $\Delta_\alpha v = \mu$, ya’ni

$$\int v\Delta\varphi dV = \int \varphi d\mu_w, \qquad \forall\varphi \in F(D^0)$$

tenglik o‘rinli bo‘ladi. Bundan ushbu

$$h(z) = u(z) - v(x)$$

ayrma uchun $\Delta_\alpha h(z) = 0$ bo‘lishi, yani $h(z)$ ning D^0 sohada α – garmonik funksiya ekanligini bildiradi.

Xulosa

Ushbu bitiruv malakaviy ishida $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{R}_+$ musbat haqiqiy koordinatali vektor berilganda

$$\alpha = \alpha_1 dz_1 \wedge d\overline{z}_1 + ... + \alpha_n dz_n \wedge d\overline{z}_n$$

forma uchun $dd^c u \wedge \alpha^{n-1} \geq 0$ tengsizlikni qanoatlantiruvchi $\alpha -$subgarmonik funksiyalar sinfi va ularning xossalari o‘rganildi. Hususan oddiy subgarmonik funksiyalar uchun o‘rinli bo‘lgan barcha xossalar bu sinf funksiyalari uchun ham o‘rinli ekani ko‘rsatildi. Ba’zi alohida jihatlariga to‘xtaladigan bo‘lsak ushbu sinfda o‘rta qiymat xossasi shar bo‘yicha emas balki elliptik sharlar uchun bajarilishiga amin bo‘lamiz. Shuningdek ushbu sinfda $\alpha -$ potensiallar nazariyasi asoslari ishlab chiqildi. $\alpha -$ sig‘im tushunchasi kiritildi va xossalari isbot qilindi.

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