Urganch Davlat Universiteti Fizika-matematika fakulteti 403-"Amaliy matematika va informatika" guruhi talabasi Qodirova Reymajon Farhodovna ning « Borg yagonalik teoremasini L.A.Chudov usulida isbotlash» mavzusida yozgan bitiruv malakaviy ishiga

TAQRIZ

Matematika va fizikaning bir qator muhim yo'nalishlarining rivojlanishiga ushbu Shturm-Liuvill

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0, \pi]$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0 \end{cases}$$

chegaraviy masalasi spektral nazariyasining to'g'ri va teskari masalalari katta xissa qo'shdi. Berilgan potensial bo`yicha Shturm-Liuvill operatorining spektral xarakteristikalarni topish va ularning xossalarini o'rganish masalasiga spektral analizning to'g'ri masalasi va aksincha, spektral xarakteristikalar bo'yicha Shturm-Liuvill operatorini tiklash masalasiga esa spektral analizning teskari masalasi deyiladi. Turli xil chegaraviy shartlardagi spektrlar, spektral funksiya, sochilish nazariyasining berilganlari va shunga o'xshash xarakteristikalar spektral xarakteristikalar bo'la olishi mumkin.

Ushbu bitiruv malakaviy ishi chekli oraliqda berilgan Shturm-Liuvill operatori uchun qo'yilgan yagonalik teoremalarini o'rganishga bag'ishlangan bo'lib, bu ish kirish, 3 ta paragraf va adabiyotlar ro'yxatidan tashkil topgan. Birinchi paragrafda Shturm-Liuvill chegaraviy masalasining xos qiymatlari va xos funksiyalari xossalari o'rganilgan. Ikkinchi paragrafda Shturm-Liuvill chegaraviy masalasiga teskari masalaning qo'yilishi, bir nechta yagonalik teoremalari va ularning isbotlari o'rganilgan. Oxirgi paragrafda esa, Borg yagonalik teoremasini Chudov usulida isbotlash o'rganilgan.

Bitiruv malakaviy ishini yozishda Qodirova Reymajon o'z bilimini, intizomi va mehnatsevarligini namoyon qildi.

Mazkur bitiruv ishi O‘zbekiston Respublikasi Oliy va O‘rta Maxsus Ta’lim Vazirligi tomonidan bitiruv malakaviy ishlariga qo‘yilgan barcha talablarga javob beradi.

Ilmiy rahbar: **Yaxshimuratov A.B.**

Urganch Davlat Universiteti Fizika-matematika fakulteti 403-“Amaliy matematika va informatika” guruhi talabasi Qodirova Reymajon Farhodovna ning « Borg yagonalik teoremasini L.A.Chudov usulida isbotlash» mavzusida yozgan bitiruv malakaviy ishiga

TAQRIZ

Shturm-Liuvill chegaraviy masalasining $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o‘zgarmaslariga yoki spektral funksiyasiga spektral xarakteristikalar deyiladi. Spektral xarakteristikalarni topish va ularning xossalarini o‘rganish masalasiga spektral analizning to‘g‘ri masalasi deyiladi. Yoqoridagi spektral xarakteristikalar yordamida tenglama koeffitsiyentlarini va chegaraviy shartlarini aniqlash masalasiga esa spektral analizning teskari masalasi deyiladi.

Shturm-Liuvill operatoriga qo‘yilgan teskari masalalar V.A.Ambarsumian, G.Borg, A.N.Tixonov, V.A.Marchenko, M.G.Kreyn, I.M.Gelfand, B.M.Levitan va boshqa olimlar tomonidan yetarlicha o‘rganilgan. Teskari masalarni matematik fizikaning ayrim nochiziqli tenglamalarini yechishga qo‘llash bu yo‘nalishni o‘rganishga bo‘lgan qiziqishni yanada oshirdi.

Mazkur bitiruv malakaviy ishida Borg yagonalik teoremasini L.A.Chudov usulida isbotlash o‘rganilgan.

Ushbu bitiriv malakaviy ishi yuqori matematik saviyada yozilgan, keltirilgan fikrlar qat’iy matematik isbotlarga ega.

Mazkur bitiruv ishi O‘zbekiston Respublikasi Oliy va O‘rta Maxsus Ta’lim Vazirligi tomonidan bitiruv malakaviy ishlariga qo‘yilgan barcha talablarni qanoatlantiradi.

Qodirova Reymajonning « Borg yagonalik teoremasini L.A.Chudov usulida isbotlash» mavzusidagi bitiruv malakaviy ishi yuqori saviyada yozilgan bo‘lib, himoyadan keyin «a’lo» bahoga loyiq deb hisoblayman.

Taqrizchi: Boltayev J., f-m.f.n, TATU Urganch filiali qoshidagi 1-son akademik litsey direktori.

O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS
TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

FIZIKA-MATEMATIKA FAKULTETI

Qodirova Reymajon Farhodovnaning

5130200 – «Amaliy matematika va informatika»

ta’lim yo‘nalishi bo‘yicha

bakalavr darajasini olish uchun

BITIRUV MALAKAVIY ISHI

Mavzu:Borg yagonalik teoremasini L.A.Chudov usulida isbotlash

Ilmiy rahbar: Yaxshimuratov A.B.

Urganch
2015-yil

O‘ZBEKISTON RESPUBLIKASI

OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

Fizika-matematika fakulteti

Amaliy matematika va matematik fizika kafedrasi

Borg yagonalik teoremasini L.A.Chudov usulida isbotlash

Bajaruvchi: Qodirova R.F.

Ilmiy rahbar: Yaxshimuratov A.B.

Urganch shahri
2015-yil

Urganch davlat universiteti “Fizika-matematika” fakulteti

“Amaliy matematika va matematik fizika” kafedrasi

«Amaliy matematika va informatika» bakalavr ta’lim yo‘nalishi

Tasdiqlayman
Fakultet dekani
dots. Abdullayev B.I.
“10” 06 2015 y.

BITIRUV MALAKAVIY ISH BO‘YICHA TOPSHIRIQ

Talaba Qodirova Reymajon Farhodovna

1. Ishning mavzusi: “Borg yagonalik teoremasini L.A.Chudov usulida isbotlash” mavzusi 19.12.2014 yil universitet rektorining № 216-T &-1 sonli buyrug‘i bilan tasdiqlangan.

2. Ishni topshirish muddati: “10” iyun 2015 y.

3. Mavzu bo‘yicha dastlabki ma’lumotlar beruvchi adabiyotlar ro‘yxati

1. *Ambarzumian V.* Über eine Frage der Eigenwerttheorie // Zeitschrift für Physik. 1929. V. 53. P. 690-695.
2. *Borg G.* Eine Umkehrung der Shturm-Liouvillesschen Eigenwertaufgabe, Acta Mathematica, 78, No 2 (1945), 1-96.
3. Чудов Л.А. Математический сборник. 1949.
4. *Stone M.* Linear Transformations in Hilbert Space and their Application to Analysis, New York, 1932.
5. *Beiberbach L.* Lehrbuch der Funktionentheorie, II, Leipzig-Berlin, 1927.
6. *Hasanov A.B.* Shturm-Liuvill chegaraviy masalalari nazariyasiga kirish.I. Toshkent-2011.
7. www.math-net.ru

4. Ishning maqsadi: Borg yagonalik teoremasini L.A.Chudov usulida isbotlash.

5. Maslahatchilar: Yaxshimuratov A.B.

Bo'limlar	Maslahatchi F.I.SH.	Imzo, sana	
		Topshiriq berdi	Topshiriq qabul qildi
Kirish	Yaxshimuratov A.B.	19.12.2014	08.01.2015
Xos qiymatlarning va xos funksiya-larning sodda xossalari.	Yaxshimuratov A.B.	10.01.2015	22.02.2015
Teskari masala-ning qo'yilishi. Yagonalik teorema-lari.	Yaxshimuratov A.B.	24.02.2015	14.03.2015
Borg yagonalik teoremasini L.A Chudov usulida isbotlash.	Yaxshimuratov A.B.	16.03.2015	25.04.2015
Xulosa	Yaxshimuratov A.B.	11.05.2015	30.05.2015
Adabiyotlar	Yaxshimuratov A.B.	6.06.2015	15.06.2015

6. Ishga taqriz yozuvchining F.I.SH., ilmiy darajasi, unvoni: Boltayev J., f-m.f.n, TATU Urganch filiali qoshidagi 1-son akademik litsey direktori.

7. Ilmiy rahbar: Yaxshimuratov A.B.

BMI bajaruvchi talaba: Qodirova R.

Kafedra mudiri: Mamedov Q.A.

O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

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FIZIKA-MATEMATIKA FAKULTETI

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5130200 – «Amaliy matematika va informatika» yo‘nalishi talabasi

ta’lim yo‘nalishi bo‘yicha

bakalavr darajasini olish uchun

BITIRUV MALAKAVIY ISHI

Mavzu:Borg yagonalik teoremasini L.A.Chudov usulida isbotlash

Ilmiy rahbar: **Yaxshimuratov A.B.**

Urganch
2015-yil

O‘ZBEKISTON RESPUBLIKASI

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URGANCH DAVLAT UNIVERSITETI

Fizika-matematika fakulteti

Amaliy matematika va matematik fizika kafedrasi

Borg yagonalik teoremasini L.A.Chudov usulida isbotlash

Bajaruvchi: Qodirova R.F.

Ilmiy rahbar: Yaxshimuratov A.B.

Urganch shahri

2015-yil

URGANCH DAVLAT UNIVERSITETI

Fizika-matematika fakulteti

Amaliy matematika va matematik fizika kafedrasi

BITIRUV MALAKAVIY ISHNI BAJARISH BO‘YICHA
TOPSHIRIQLAR REJASI:

1. Talaba Qodirova Reymajon Farhodovna Universitet rektorining № 216-T &-1 sonli buyrug‘i bilan bitiruv malakaviy ish bajarish uchun “**Borg yagonalik teoremasini L.A.Chudov usulida isbotlash**” mavzusi tasdiqlangan.
2. Kafedra majlisining qaroriga binoan Yaxshimuratov A.B. bitiruv malakaviy ishini bajarishga rahbar qilib tayinlangan.
3. Bitiruv malakaviy ishining tarkibiy tuzilmasi: Bitiruv malakaviy ishi kirish, asosiy qism, xulosa va adabiyotlar bo‘limlaridan iborat.
4. Bitiruv malakaviy ish uchun ma’lumotlar mavzu bo‘yicha ma’lumotlar beruvchi adabiyotlar, mavzuga oid maqolalar va internet saytlaridan olinadi.
5. Bitiruv malakaviy ishga ilova qilinmaydi.

Urganch davlat universiteti “Fizika-matematika” fakulteti

“Amaliy matematika va matematik fizika” kafedrasi

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“___” ____________ 2015 y.

BITIRUV MALAKAVIY ISH BO‘YICHA TOPSHIRIQ

Talaba Qodirova Reymajon Farhodovna

1. Ishning mavzusi: “**Borg yagonalik teoremasini L.A.Chudov usulida isbotlash**” mavzusi 19.12.2014 yil universitet rektorining № 216-T &-1 sonli buyrug‘i bilan tasdiqlangan.

2. Ishni topshirish muddati: “ ” iyun 2015 y.

3. Mavzu bo‘yicha dastlabki ma’lumotlar beruvchi adabiyotlar ro‘yxati

1. *Ambarzumian V.* Über eine Frage der Eigenwerttheorie // Zeitschrift f□r Physik. 1929. V. 53. P. 690-695.

2. *Borg G.* Eine Umkehrung der Shturm-Liouvillesschen Eigenwertaufgabe, Acta Mathematica, 78, No 2 (1945), 1-96.

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4. *Stone M.* Linear Transformations in Hilbert Space and their Application to Analysis, New York, 1932.

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6. *Hasanov A.B.* Shturm-Liuvill chegaraviy masalalari nazariyasiga kirish.I. Toshkent-2011.

7. www.math-net.ru

4. Ishning maqsadi: Borg yagonalik teoremasini L.A.Chudov usulida isbotlash.

.

5. Maslahatchilar: Yaxshimuratov A.B.

Bo‘limlar	Maslahatchi F.I.SH.	Imzo, sana	
		Topshiriq berdi	Topshiriq qabul qildi
Kirish	Yaxshimuratov A.B.	19.12.2014	08.01.2015
Xos qiymatlarning va xos funksiya-larning sodda xossalari.	Yaxshimuratov A.B.	10.01.2015	22.02.2015
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Adabiyotlar	Yaxshimuratov A.B.	6.06.2015	15.06.2015

6. **Ishga taqriz yozuvchining F.I.SH., ilmiy darajasi, unvoni:** Boltayev J., f-m.f.n, TATU Urganch filiali qoshidagi 1-son akademik litsey direktori.

7. Ilmiy rahbar: Yaxshimuratov A.B.

BMI bajaruvchi talaba: Qodirova R.

Kafedra mudiri: Mamedov Q.A.

M U N D A R I J A

Kirish

Mavzuning dolzarbligi. Hozirgi zamon fizikasi va texnikasining ayrim sohalari uchun (nurlanish nazariyasi, kristallar optikasi, elastiklik nazariyasi) spektral analizning bitta muammosi katta qiziqish kasb etadi. Shturm-Liuvill chegaraviy masalasining $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o‘zgarmaslariga yoki spektral funksiyasiga spektral xarakteristikalar deyiladi. Spektral xarakteristikalarni topish va ularning xossalarini o‘rganish masalasiga spektral analizning to‘g‘ri masalasi deyiladi. Yuqoridagi spektral xarakteristikalar yordamida tenglama koeffitsiyentlarini va chegaraviy shartlarini aniqlash masalasiga esa spektral analizning teskari masalasi deyiladi. Teskari masalar haqidagi ilk natija 1929-yilda astranomiyaga tadbiq qilish natijasida mashhur astranom V.A.Ambarsumyan [1] tomonidan olingan.

Keyinchlik, Shturm-Liuvill operatoriga qo‘yilgan teskari masalalar G.Borg, A.N.Tixonov, V.A.Marchenko, M.G.Kreyn, I.M.Gelfand, B.M.Levitan va boshqa olimlar tomonidan atroflicha o‘rganilgan.

Hozirgi kunda kvant fizikasining bir qator masalalari Grafda berilgan Shturm-Liuvill operatorlarining o‘rganishni taqazo etadi. Bu soha teskari masalalari Brovn, Vekart, Pifarchik, Vassel, Yurkovlarning ilmiy ishlarida o‘rganilgan.

Tadqiqot maqsadi. Borg yagonalik teoremasini L.A.Chudov usulida isbotlashni o‘rganish.

Tadqiqot vazifasi. Borg yagonalik teoremasini L.A.Chudov usulida isbotlash usulini tahlil qilib uni Dirak sistemasiga tadbiq qilish.

Tadqiqot obyekti. Borg yagonalik teoremasini L.A.Chudov usulidagi isboti.

Tadqiqot predmeti. Teskari spektral masala usuli.

Tadqiqot usullari. Kompleks analiz usuli, Matematik fizika tenglamalari, Differensial operatorlarning spektral nazariyasi, Funksional analiz.

Tadqiqot natijalarining ilmiy jihatdan yangilik darajasi. Mazkur bitiruv malakaviy ishidagi 3-paragrafdagi teoremalar.

Tadqiqot natijalarining amaliy ahamiyati va tadbiqi. Olingan natijalar va qo‘llanilgan usullar nochiziqli tenglamalarni integrallashga tadbiq qilinadi.

Ish tuzilishi va tarkibi. Ushbu bitiruv malakaviy ishi kirish, 3 ta paragraf, xulosa va adabiyotlar ro‘yxatidan tashkil topgan va 36 betdan iborat.

Ushbu bitiruv malakaviy ishda Borg yagonalik teoremasini L.A.Chudov usulida isbotlash o‘rganiladi.

1-§. Xos qiymatlarning va xos funksiyalarning sodda xossalari

Quyidagi masalaga

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0, \pi], \tag{1.1}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.2}$$

Shturm-Liuvill chegaraviy masalasi deyiladi. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya bo‘lib, α va β berilgan haqiqiy sonlardir, λ esa kompleks parametr.

Agar (1.1) tenglamani $y(0) = 0,\ y(\pi) = 0$ chegaraviy shartlar bilan qarasak, hosil bo‘ladigan chegaraviy masalaga Dirixle masalasi deyiladi, agar $y'(0) = 0,\ y'(\pi) = 0$ chegaraviy shartlar bilan qarasak, hosil bo‘ladigan chegaraviy masalaga Neyman masalasi deyiladi.

(1.1) tenglamaning $q(x)$ koeffitsiyentiga (1.1)+(1.2) Shturm-Liuvill masalasining potensiali deyiladi.

Ta’rif 1.1. Agar λ parametrning biror $\lambda = \lambda_0$ qiymatida (1.1)+(1.2) chegaraviy masala noldan farqli $y(x, \lambda_0) \not\equiv 0$ yechimga ega bo‘lsa, λ_0 songa (1.1)+(1.2) chegaraviy masalaning xos qiymati deyiladi, $y(x, \lambda_0)$ yechimga esa λ_0 xos qiymatga mos keluvchi xos funksiyasi deyiladi.

(1.1)+(1.2) Shturm-Liuvill masalasining barcha xos qiymatlaridan tuzilgan to‘plamga uning spektri deyiladi.

1-xossa. $y_1(x, \lambda)$ va $y_2(x, \lambda)$ (1.1) tenglamaning ixtiyoriy yechimlari bo‘lsin. U holda ulardan tuzilgan

$$W\{y_1(x,\lambda), y_2(x,\lambda)\} = \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) \end{vmatrix},$$

Vronskiy determinanti x o‘zgaruvchiga bog‘liq bo‘lmaydi.

Isbot. Buning uchun ushbu

$$\frac{dW}{dx} \equiv 0,$$

tenglik bajarilishini ko‘rsatish yetarli.

$$\frac{dW}{dx} \equiv (y_1 y_2' - y_1' y_2)' = y_1 y_2'' - y_1'' y_2 = y_1[q(x)y_2 - \lambda\, y_2] - y_2[q(x)y_1 - \lambda\, y_1] = 0.$$

2-xossa. (1.1) tenglamaning ikki yechimi chiziqli bog‘liq bo‘lishi uchun ulardan tuzilgan Vronskiy determinanti nolga teng bo‘lishi zarur va etarli.

Isbot. Ushbu

$$\frac{d}{dx}\left\{\frac{y_1(x,\lambda)}{y_2(x,\lambda)}\right\} = \frac{y_1'(x,\lambda)y_2(x,\lambda) - y_1(x,\lambda)y_2'(x,\lambda)}{y_2^2(x,\lambda)} =$$

$$= -\frac{1}{y_2^2(x,\lambda)} W\{y_1(x,\lambda), y_2(x,\lambda)\}$$

ayniyatdan quyidagi

$$\frac{y_1(x,\lambda)}{y_2(x,\lambda)} = \text{const}$$

munosabatning bajarilishi uchun $W\{y_1(x,\lambda), y_2(x,\lambda)\} = 0$ bo‘lishi zarur va etarli ekani kelib chiqadi.

3-xossa. (*Grin ayniyati*). Ixtiyoriy $y(x), z(x) \in C^2[0,\pi]$ funksiyalar uchun ushbu

$$\int_0^\pi Ly \cdot \bar{z}\, dx = W_\pi\{y,\bar{z}\} - W_0\{y,\bar{z}\} + \int_0^\pi y \cdot \overline{Lz}\, dx,$$

ayniyat bajariladi.

Isbot. Quyidagi ayirmani hisoblaymiz:

$$\int_0^\pi (\bar{z}Ly - y\overline{Lz})dx = \int_0^\pi \{\bar{z}[-y'' + q(x)y] - y[-\bar{z}'' + q(x)\bar{z}]\}dx =$$

$$= \int_0^\pi (\bar{z}''y - y''\bar{z})dx = \int_0^\pi (\bar{z}'y - y'\bar{z})'dx = \left.\begin{vmatrix} y(x) & \bar{z}(x) \\ y'(x) & \bar{z}'(x) \end{vmatrix}\right|_0^\pi = W_\pi\{y,\bar{z}\} - W_0\{y,\bar{z}\}.$$

4-xossa. Ixtiyoriy $y(x), z(x) \in C^2[0,\pi]$ funksiyalar uchun ushbu

$$\int_0^\pi Ly \cdot \bar{z}\, dx = [y(0)\cos\alpha + y'(0)\sin\alpha]\cdot[\bar{z}(0)\sin\alpha - \bar{z}'(0)\cos\alpha] -$$

$$-[y(0)\sin\alpha - y'(0)\cos\alpha]\cdot[\bar{z}(0)\cos\alpha + \bar{z}'(0)\sin\alpha] +$$

$$+\left[y(\pi)\cos\beta + y'(\pi)\sin\beta\right]\cdot\left[-\bar{z}(\pi)\sin\beta + \bar{z}'(\pi)\cos\beta\right]+$$

$$+\left[y(\pi)\sin\beta - y'(\pi)\cos\beta\right]\cdot\left[\bar{z}(\pi)\cos\beta + \bar{z}'(\pi)\sin\beta\right]+\int_0^\pi y\cdot\overline{Lz}\,dx, \qquad (1.3)$$

tenglik bajariladi.

Isbot. Grin ayniyatidagi $W_\pi\{y,\bar{z}\}-W_0\{y,\bar{z}\}$ ifodani kerakli ko‘rinishda yozamiz. Buning uchun quyidagi sistemani tuzib olamiz:

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = U_1, \\ y(0)\sin\alpha - y'(0)\cos\alpha = U_2, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = U_3, \\ y(\pi)\sin\beta - y'(\pi)\cos\beta = U_4, \end{cases}$$

va undan ushbu

$$\begin{cases} y(0) = U_1\cos\alpha + U_2\sin\alpha, \\ y'(0) = U_1\sin\alpha - U_2\cos\alpha, \\ y(\pi) = U_3\cos\beta + U_4\sin\beta, \\ y'(\pi) = U_3\sin\beta - U_4\cos\beta, \end{cases}$$

tengliklarni hosil qilamiz. Bularni Grin ayniyatiga qo‘ysak, (1.3) tenglik hosil bo‘ladi.

Natija 1.1. Agar $y(x), z(x)\in C^2[0,\pi]$ bo‘lib, $y(x)$ funksiya (1.2) chegaraviy shartlarni qanoatlantirsa, u holda

$$\int_0^\pi Ly\cdot\bar{z}\,dx = \int_0^\pi y\cdot\overline{Lz}dx,$$

tenglik bajarilishi uchun $z(x)$ funksiya ham (1.2) chegaraviy shartlarni qanoatlantirishi zarur va yetarlidir.

Yuqoridagi natija, (1.1)+(1.2) chegaraviy masala yordamida aniqlangan L chiziqli operator $L^2(0,\pi)$ Gilbert fazosida o‘z-o‘ziga qo‘shma operatorni ifodalashini ko‘rsatadi.

5-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining xos qiymatlari haqiqiydir.

Isbot. $\lambda = u + iv$, $i=\sqrt{-1}$, ($v\neq 0$) son (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining xos qiymati bo‘lsin deb faraz qilaylik va unga mos keluvchi xos funksiyani $y(x)$ bilan belgilaylik. U holda $\overline{\lambda}=u-iv$ son ham shu chegaraviy masalasining xos qiymati bo‘ladi va unga $\overline{y}(x)$ xos funksiya mos keladi. Quyidagi

$$(\lambda-\overline{\lambda})\int_0^{\pi}|y(x)|^2dx=\int_0^{\pi}(\lambda-\overline{\lambda})y(x)\overline{y}(x)dx=\int_0^{\pi}[(\lambda y)\overline{y}-y(\overline{\lambda}\overline{y})]dx=$$

$$=\int_0^{\pi}\{\overline{y}[-y''+q(x)y]-y[-\overline{y}''+q(x)\overline{y}]\}dx=\int_0^{\pi}(\overline{y}''y-y''\overline{y})dx=\int_0^{\pi}(\overline{y}'y-y'\overline{y})'dx=$$

$$=\begin{vmatrix} y(\pi) & \overline{y}(\pi) \\ -y(\pi)ctg\beta & -\overline{y}(\pi)ctg\beta \end{vmatrix}-\begin{vmatrix} y(0) & \overline{y}(0) \\ -y(0)ctg\alpha & -\overline{y}(0)ctg\alpha \end{vmatrix}=0,$$

tenglikdan $\overline{\lambda}=\lambda$ ekanligi kelib chiqadi. Bu esa farazimizga zid.

Natija 1.2. Xos funksiyani haqiqiy qilib tanlash mumkin. Chunki xos qiymat haqiqiy ekanligidan qaralayotgan tenglamaning haqiqiyligi kelib chiqadi. Chegaraviy shartlar esa hamisha haqiqiy.

6-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining turli xos qiymatlariga mos keluvchi xos funksiyalari o‘zaro ortogonaldir, ya’ni $\lambda_1\neq\lambda_2$ xos qiymatlarga mos keluvchi $y_1(x)$, $y_2(x)$ xos funksiyalar uchun ushbu

$$\int_0^{\pi}y_1(x)\cdot y_2(x)\,dx=0, \tag{1.4}$$

tenglik o‘rinli bo‘ladi.

Isbot. Ushbu

$$(\lambda_1-\lambda_2)\int_0^{\pi}y_1(x)y_2(x)dx=\int_0^{\pi}[(\lambda_1y_1)y_2-y_1(\lambda_2y_2)]dx=$$

$$=\int_0^{\pi}\{y_2[-y_1''+q(x)y_1]-y_1[-y_2''+q(x)y_2]\}dx=$$

$$=\int_0^{\pi}(y_2''y_1-y_1''y_2)dx=\int_0^{\pi}(y_2'y_1-y_1'y_2)'dx=\left.\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right|_0^{\pi}=0,$$

ayniyatda $\lambda_1\neq\lambda_2$ bo‘lgani uchun (1.4) tenglik o‘rinli bo‘lishligi kelib chiqadi.

7-xossa. (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari oddiy (karrasiz), ya’ni bitta xos qiymatga mos keluvchi xos funksiyalar bir-biriga proporsionaldir.

Isbot. λ xos qiymatga $y_1(x)$, $y_2(x)$ chiziqli erkli xos funksiyalar mos keladi deb faraz qilaylik. U holda

$$W\{y_1,y_2\}=\lim_{x\to 0}W\{y_1,y_2\}=\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0)\end{vmatrix}=\begin{vmatrix} y_1(0) & y_2(0) \\ -y_1(0)ctg\alpha & -y_2(0)ctg\alpha\end{vmatrix}=0,$$

bo‘lgani uchun, $y_1(x)$, $y_2(x)$ chiziqli bog‘liq bo‘ladi. Bu esa farazimizga ziddir.

8-xossa. $y(x,\lambda)$ funksiya Shturm-Liuvill tenglamasining λ bo‘yicha uzluksiz differensiallanuvchi ixtiyoriy yechimi bo‘lsin. U holda

$$\int_0^\pi y^2(x,\lambda)dx=W_\pi\{\dot y,y\}-W_0\{\dot y,y\},$$

tenglik bajariladi. Bu yerda $\dot y=\dfrac{\partial y(x,\lambda)}{\partial\lambda}$.

Isbot. Ushbu

$$-y''+q(x)y=\lambda y, \tag{1.5}$$

ayniyatdan λ bo‘yicha hosila olsak,

$$-\dot y''+q(x)\dot y=y+\lambda\dot y, \tag{1.6}$$

tenglik kelib chiqadi. (1.5) va (1.6) tengliklarni mos ravishda $\dot y$ va y funksiyalarga ko‘paytirib, bir-biridan ayirsak, ushbu

$$\dot y''y-y''\dot y=-y^2,$$

ayniyat hosil bo‘ladi. Bu tenglikni $[0,\pi]$ kesmada integrallasak, ushbu

$$\int_0^\pi y^2(x,\lambda)dx=\int_0^\pi(y'\dot y-\dot y'y)'dx=W_\pi\{\dot y,y\}-W_0\{\dot y,y\},$$

formula kelib chiqadi.

9-xossa. Agar quyidagi chegaraviy masalaning

$$\begin{cases} -y''+q(x)y=\lambda y, \\ y(0)\cos\alpha+y'(0)\sin\alpha=0, \\ y(\pi)\cos\beta+y'(\pi)\sin\beta=0, \end{cases}$$

xos qiymatlari $\lambda_0,\lambda_1,\lambda_2,...$ va xos funksiyalari $y_0(x), y_1(x), y_2(x),...$ bo‘lsa, u holda ushbu

$$\begin{cases} -y'' + [q(x) + c]y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.7}$$

chegaraviy masalaning xos qiymatlari $\lambda_0 + c, \lambda_1 + c, \lambda_2 + c, ...$ va xos funksiyalari $y_0(x), y_1(x), y_2(x), ...$ bo‘ladi. Bu yerda c o‘zgarmas son.

Isbot. Ushbu

$$\begin{cases} -y'' + q(x)y = (\lambda - c)y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases}$$

chegaraviy masala noldan farqli yechimga ega bo‘lishi uchun $\lambda - c = \lambda_n$ bo‘lishi zarur va yetarli. Shartga ko‘ra, bu holda oxirgi chegaraviy masala $y_n(x) \not\equiv 0$ yechimga ega. Demak, (1.7) masalaning xos qiymatlari $\lambda_0 + c, \lambda_1 + c, \lambda_2 + c, ...$ va xos funksiyalari $y_0(x), y_1(x), y_2(x), ...$ bo‘ladi.

(1.1) differensial tenglamaning quyidagi

$$\varphi(0,\lambda) = -\sin\alpha, \;\; \varphi'(0,\lambda) = \cos\alpha$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini $\varphi(x,\lambda)$ orqali belgilaymiz. Xuddi shuningdek, (1.1) tenglamaning ushbu

$$\psi(\pi,\lambda) = -\sin\beta, \;\; \psi'(\pi,\lambda) = \cos\beta$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini $\psi(x,\lambda)$ orqali belgilab olamiz. Bu yerda $\varphi(x,\lambda)$ yechim (1.2) chegaraviy shartlardan birinchisini qanoatlantiradi, $\psi(x,\lambda)$ yechim esa (1.2) chegaraviy shartlardan ikkinchisini qanoatlantiradi. Bu $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ yechimlarni mos ravishda (1.2) chegaraviy shartlardan ikkinchisiga va birinchisiga qo‘yib, ushbu

$$\Delta(\lambda) \equiv \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta = 0\,,$$

$$\widetilde{\Delta}(\lambda) \equiv \psi(0,\lambda)\cos\alpha + \psi'(0,\lambda)\sin\alpha = 0\,,$$

tenglamalarni hosil qilamiz. Bu tenglamalarga (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining xarakteristik tenglamalari deyiladi. Shturm-Liuvill tenglamasining $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ yechimlaridan tuzilgan ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda), \psi(x,\lambda)\} \equiv \begin{vmatrix} \varphi(x,\lambda) & \psi(x,\lambda) \\ \varphi'(x,\lambda) & \psi'(x,\lambda) \end{vmatrix},$$

Vronskiy determinantini qaraymiz. Biz yuqorida bu determinant x o‘zgaruvchiga bog‘liq emasligini ko‘rsatgan edik. Shuning uchun ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda), \psi(x,\lambda)\} = \lim_{x\to 0} W\{\varphi(x,\lambda), \psi(x,\lambda)\} = \lim_{x\to \pi} W\{\varphi(x,\lambda), \psi(x,\lambda)\},$$

tengliklarni yozishimiz mumkin. Bu tengliklardan

$$\omega(\lambda) = \Delta(\lambda) = -\widetilde{\Delta}(\lambda),$$

kelib chiqadi. Bu yerdagi $\omega(\lambda)$, $\Delta(\lambda)$, $\widetilde{\Delta}(\lambda)$ funksiyalar λ o‘zgaruvchining butun funksiyalari bo‘lib, sanoqlita $\{\lambda_n\}_{n=0}^{\infty}$ nollarga ega ekanligini keyinchalik ko‘rsatamiz.

$\Delta(\lambda) = 0$ xarakteristik tenglamaning λ_n, $n = 0, 1, 2, ...$ ildizlari Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan iborat bo‘lib, $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar uning xos funksiyalari bo‘ladi va ushbu

$$\psi(x,\lambda_n) = c_n \varphi(x,\lambda_n), \quad c_n \neq 0 \tag{1.8}$$

tenglik bajariladi. Haqiqatan ham, $\lambda = \lambda_n$ son $\Delta(\lambda) = 0$ tenglamaning ildizi bo‘lsa, u holda $\omega(\lambda_n) = 0$ bo‘lgani uchun (1.8) tenglik o‘rinli bo‘ladi. $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar (1.2) chegaraviy shartlarni qanoatlantiradi, bundan esa $\lambda = \lambda_n$ son xos qiymat hamda $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar Shturm-Liuvill chegaraviy masalasining xos funksiyalari ekanligi kelib chiqadi.

Izoh 1.1. Odatda, agar (1.2) chegaraviy shartlardan birinchisi ushbu $y'(0) - hy(0) = 0$ ko‘rinishda bo‘lsa, u holda $\varphi(x,\lambda)$ yechim $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$ boshlang‘ich shartlarni qanoatlantiradigan qilib olinadi, agar (1.2) chegaraviy shartlardan birinchisi $y(0) = 0$ ko‘rinishda bo‘lsa, u holda $\varphi(x,\lambda)$ yechim $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$ boshlang‘ich shartlarni qanoatlantiradigan qilib olinadi.

Agar $\lambda = \lambda_0$ soni Shturm-Liuvill chegaraviy masalasining xos qiymati bo‘lib, $y(x,\lambda_0)$ unga mos keluvchi xos funksiya bo‘lsa, u holda $y(0,\lambda_0)$ va $y'(0,\lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo‘ladi, aks xolda yechimning yagonaligi xaqidagi Koshi teoremasidan $y(x,\lambda_0) \equiv 0$ ekanligi kelib chiqadi. Bu esa xos funksiya ta'rifiga ziddir. Xuddi shuningdek, $y(\pi,\lambda_0)$ va $y'(\pi,\lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo'lishi ko'rsatiladi.

Quyidagi

$$\alpha_n = \sqrt{\int_0^\pi \varphi^2(x,\lambda_n)dx}\,, \quad n = 0, 1, 2, \ldots,$$

sonlarga (1.1)+(1.2) chegaraviy masalaning normallovchi o'zgarmaslari deyiladi. (1.1)+(1.2) masalaning ortonormallangan xos funksiyalari quyidagi tengliklardan topiladi:

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x,\lambda_n), \; n = 0,1,2,\ldots.$$

Ta'rif 1.2. Ushbu $\{\lambda_n\}_{n=0}^\infty$, $\{\alpha_n\}_{n=0}^\infty$ sonli ketma-ketliklar juftligiga Shturm-Liuvill chegaraviy masalasining spektral berilganlari (spektral xarakteristikalari) deyiladi.

Ta'rif 1.3. Monoton o'suvchi, chapdan uzluksiz ushbu

$$\rho(\lambda) = \begin{cases} 0, & \lambda = 0, \\ -\sum\limits_{\lambda \le \lambda_n < 0} \dfrac{1}{\alpha_n^2}, & \lambda < 0, \\ \sum\limits_{0 < \lambda_n < \lambda} \dfrac{1}{\alpha_n^2}, & \lambda > 0, \end{cases} \tag{1.9}$$

funksiyaga Shturm-Liuvill chegaraviy masalasining spektral funksiyasi deyiladi.

10-xossa. Shturm-Liuvill chegaraviy masalasining normallovchi o'zgarmaslari uchun ushbu

$$\alpha_n^2 = \begin{cases} \varphi'(\pi,\lambda_n)\dot{\Delta}(\lambda_n), & \sin\beta = 0, \\ -\dfrac{\varphi(\pi,\lambda_n)}{\sin\beta}\dot{\Delta}(\lambda_n), & \sin\beta \ne 0, \end{cases} \tag{1.10}$$

tenglik o'rinli bo'ladi. Bu yerda

$$\Delta(\lambda) = \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta\,,$$

funksiya (1.1)+(1.2) Shturm-Liuvill masalasining xarakteristik funksiyasi, $\varphi(x,\lambda)$ funksiya esa (1.1) tenglamaning $\varphi(0,\lambda) = -\sin\alpha$, $\varphi'(0,\lambda) = \cos\alpha$ boshlang'ich shartlarni qanoatlantiruvchi yechimidir.

Isbot. 7-xossada $y(x,\lambda)$ yechim o‘rnida $\varphi(x,\lambda)$ yechimni olib, $\lambda=\lambda_n$ desak, ushbu

$$\alpha_n^2=\int_0^\pi \varphi^2(x,\lambda_n)dx=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix}-\begin{vmatrix}\dot\varphi(0,\lambda_n) & \varphi(0,\lambda_n)\\ \dot\varphi'(0,\lambda_n) & \varphi'(0,\lambda_n)\end{vmatrix}=$$

$$=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix}-\begin{vmatrix}0 & -\sin\alpha\\ 0 & \cos\alpha\end{vmatrix}=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n)\end{vmatrix},$$

tenglik hosil bo‘ladi.

Quyidagi ikkita holni ko‘rib chiqamiz.

1) $\sin\beta=0$ bo‘lsin. Bu holda $\Delta(\lambda)=\varphi(\pi,\lambda)$, $\varphi(\pi,\lambda_n)=0$ bo‘lgani uchun

$$\alpha_n^2=\varphi'(\pi,\lambda_n)\dot\Delta(\lambda_n)$$

tenglik o‘rinli.

2) $\sin\beta\neq 0$ bo‘lsin. Bu holda $\varphi'(\pi,\lambda_n)=-\varphi(\pi,\lambda_n)ctg\beta$ bo‘lgani uchun

$$\alpha_n^2=\begin{vmatrix}\dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n)\\ \dot\varphi'(\pi,\lambda_n) & -\varphi(\pi,\lambda_n)ctg\beta\end{vmatrix}=\varphi(\pi,\lambda_n)[-\dot\varphi(\pi,\lambda_n)ctg\beta-\dot\varphi'(\pi,\lambda_n)]=$$

$$=-\frac{\varphi(\pi,\lambda_n)}{\sin\beta}[\dot\varphi(\pi,\lambda_n)\cos\beta+\dot\varphi'(\pi,\lambda_n)\sin\beta]=-\frac{\varphi(\pi,\lambda_n)}{\sin\beta}\dot\Delta(\lambda_n)$$

tenglik bajariladi.

Natija 1.3. (1.8) tenglikdan foydalanib, (1.10) tenglikni quyidagi ko‘rinishda yozish mumkin:

$$\alpha_n^2\cdot c_n=\begin{cases}\cos\beta\cdot\dot\Delta(\lambda_n), & \sin\beta=0,\\ \dot\Delta(\lambda_n), & \sin\beta\neq 0.\end{cases}$$

Natija 1.4. (1.10) formuladan (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining

$$\Delta(\lambda)=\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta,$$

xarakteristik funksiyasi karrali ildizga ega emasligi kelib chiqadi.

2-§. Teskari masalaning qo'yilishi. Yagonalik teoremalari

Ushbu

$$-y''+q(x)y=\lambda y,\ y'(0)=0,\quad y'(\pi)=0, \tag{2.1}$$

Shturm-Liuvill chegaraviy masalasini qaraymiz. Bu yerda $q(x)\in C[0,\pi]$ haqiqiy uzluksiz funksiya.

Agar (2.1) chegaraviy masalada $q(x)\equiv 0$, $x\in[0,\pi]$ bo'lsa, u holda $\lambda_n=n^2$, $n\geq 0$ bo'lishi ravshan.

Teorema 2.1 *(V.A. Ambarsumyan).* Agar (2.1) chegaraviy masalaning xos qiymatlari uchun, ushbu

$$\lambda_n=n^2,\ n\geq 0, \tag{2.2}$$

tenglik bajarilsa, u holda $q(x)\equiv 0$, $x\in[0,\pi]$ bo'ladi.

Isbot. (2.1) chegaraviy masalaning xos qiymatlari uchun

$$\sqrt{\lambda_n}=n+\frac{c_0}{n}+\frac{\gamma_n}{n}\quad,\ n\to\infty,\ \{\gamma_n\}\in l_2, \tag{2.3}$$

asimptotik formula o'rinli. Bu yerda

$$c_0=\frac{1}{2\pi}\int_0^\pi q(x)dx. \tag{2.4}$$

(2.2) va (2.3) tengliklarni tenglashtirib,

$$c_0=0,\ \int_0^\pi q(x)dx=0, \tag{2.5}$$

topamiz. (2.1) chegaraviy masalaning $\lambda_0=0$ eng kichik xos qiymatiga mos keluvchi xos funksiya $y_0(x)$ bo'lsa u holda

$$y_0''+q(x)y_0=0,\quad y_0'(0)=0,\quad y_0'(\pi)=0, \tag{2.6}$$

bo'ladi. Ossilyatsiya teoremasiga asosan $y_0(x)\neq 0$, $x\in[0,\pi]$ bo'ladi, chunki $y_0(x)$ funksiya $[0,\pi]$ oraliqda nolga ega emas. Ushbu

$$0=\int_0^\pi q(x)dx=\int_0^\pi\frac{y_0''(x)}{y_0(x)}dx=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)'dx=\int_0^\pi\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\frac{y_0'(x)}{y_0(x)}\Bigg|_{x=0}^{x=\pi}=$$

$$=\int_0^{\pi}\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx+\frac{y_0'(\pi)}{y_0(\pi)}-\frac{y_0'(0)}{y_0(0)}=\int_0^{\pi}\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx,$$

tenglikdan

$$\int_0^{\pi}\left(\frac{y_0'(x)}{y_0(x)}\right)^2dx=0,$$

hosil bo‘ladi. Oxirgi tenglikdan $y_0'(x)=0$, ya’ni $y_0(x)=const$ kelib chiqadi. (2.6) tenglamadan

$$y_0''(x)-q(x)y_0(x)=0,$$

$q(x)=0$ ekanligi kelib chiqadi.

Umuman olganda $\lambda_0,\lambda_1,\lambda_2,...,\lambda_n,...$ xos qiymatlarning, ya’ni spektrning berilishi Shturm-Liuvill chegaraviy masalasini yagona aniqlamaydi. Chunki, ikkita har-xil Shturm-Liuvill chegaraviy masalasi bir xil spektrga ega bo‘lishi mumkin.

Quyidagi

$$-y''+q(x)y=\lambda y,\ x\in[0,\pi], \tag{2.7}$$

$$y'(0)-hy(0)=0, \tag{2.8}$$

$$y'(\pi)-Hy(\pi)=0, \tag{2.9}$$

Shturm-Liuvill chegaraviy masalasi berilgan bo‘lsin. Bu yerda $q(x)\in C[0,\pi]$ haqiqiy uzluksiz funksiya, h, H - chekli haqiqiy sonlar.

$\varphi(x,\lambda)$ orqali (2.7) tenglamaning

$$\varphi(0,\lambda)=1,\ \ \varphi'(0,\lambda)=h,$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(2.7)-(2.9) Shturm-Liuvill chegaraviy masalasining spektrini

$$\lambda_0,\lambda_1,\lambda_2,...,\lambda_n,...,$$

bilan, uning xos funksiyalari ketma-ketligini

$$\varphi(x,\lambda_0),\ \varphi(x,\lambda_1),\ \dots,\ \varphi(x,\lambda_n),\ \dots,$$

bilan belgilaylik. U holda

$$\alpha_n=\int_0^{\pi}\varphi^2(x,\lambda_n)dx,\ n\geq 0,$$

orqali normallovchi o‘zgarmaslar ketma-ketligini belgilaymiz.

Ta’rif 2.1. Ushbu $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ juftliklar ketma-ketligiga Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari yoki spektral berilganlari deyiladi.

Ta’rif 2.2. $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar yordamida $q(x)$ funksiyani va h, H sonlarni topish masalasiga teskari masala deyiladi.

Yuqoridagi masalani atroflicha o‘rganish maqsadida ikkinchi

$$-y'' + \tilde{q}(x)y = \lambda y,\ x \in [0,\pi], \tag{2.10}$$

$$y'(0) - \tilde{h}y(0) = 0, \tag{2.11}$$

$$y'(\pi) + \tilde{H}y(\pi) = 0, \tag{2.12}$$

Shturm-Liuvill chegaraviy masalasini qaraymiz. Bu yerda $\tilde{q}(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya, $\tilde{h}$, $\tilde{H}$ - chekli haqiqiy sonlar.

$\tilde{\varphi}(x,\lambda)$ orqali (2.10) tenglamaning

$$\tilde{\varphi}(0,\lambda) = 1,\ \tilde{\varphi}'(0,\lambda) = \tilde{h},$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

Endi (2.10)-(2.12) chegaraviy masalasining spektrini

$$\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_n, ...,$$

bilan, uning xos funksiyalari ketma-ketligini

$$\tilde{\varphi}(x,\tilde{\lambda}_0),\ \tilde{\varphi}(x,\tilde{\lambda}_1),\ \dots,\ \tilde{\varphi}(x,\tilde{\lambda}_n),\ \dots,$$

bilan belgilaylik. (2.10)-(2.12) chegaraviy masalalarning normallovchi o‘zgarmaslar ketma-ketligini

$$\tilde{\alpha}_n = \int_0^{\pi} \tilde{\varphi}^2(x,\tilde{\lambda}_n)dx,\ n \ge 0,$$

orqali belgilaymiz.

Teorema 2.2 *(V.A.Marchenko).* Agar $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n \ge 0$ bo‘lsa, u holda ushbu

$$h = \tilde{h},\ H = \tilde{H},\ q(x) = \tilde{q}(x),\ x \in [0,\pi],$$

tengliklar o‘rinli bo‘ladi.

Isbot. Almashtirish operatorlaridan foydalanib, $\varphi(x,\lambda)$ va $\widetilde{\varphi}(x,\widetilde{\lambda})$ yechimlarni quyidagi ko‘rinishda yozib olamiz:

$$\begin{aligned} \varphi(x,\lambda) &= \cos\sqrt{\lambda}x + \int_0^x G(x,t)\cos\sqrt{\lambda}t\,dt, \\ \widetilde{\varphi}(x,\lambda) &= \cos\sqrt{\lambda}x + \int_0^x \widetilde{G}(x,t)\cos\sqrt{\lambda}t\,dt. \end{aligned} \tag{2.13}$$

Bu yerda

$$\begin{aligned} G(x,x) &= h + \frac{1}{2}\int_0^x q(t)\,dt, \\ \widetilde{G}(x,x) &= \widetilde{h} + \frac{1}{2}\int_0^x \widetilde{q}(t)\,dt. \end{aligned} \tag{2.15}$$

Yuqoridagi tengliklarni (qisqacha) operator ko‘rinishda ham yozish mumkin:

$$\begin{aligned} \varphi(x,\lambda) &= (E+G)\cos\sqrt{\lambda}x, \\ \widetilde{\varphi}(x,\lambda) &= (E+\widetilde{G})\cos\sqrt{\lambda}x. \end{aligned} \tag{2.16}$$

Bu yerda E birlik operator bo‘lib,

$$(E+G)f(x) = f(x) + \int_0^x G(x,t)f(t)\,dt,$$

$$(E+\widetilde{G})f(x) = f(x) + \int_0^x \widetilde{G}(x,t)f(t)\,dt.$$

Ushbu

$$\widetilde{\varphi}(x,\lambda) = (E+\widetilde{G})\cos\sqrt{\lambda}x,$$

tenglama ikkinchi tur Volterra integral tenglamasi bo‘lgani uchun, uni $\cos\sqrt{\lambda}x$ ga nisbatan yechib,

$$\cos\sqrt{\lambda}x = \widetilde{\varphi}(x,\lambda) + \int_0^x \widetilde{H}(x,t)\widetilde{\varphi}(t,\lambda)\,dt, \tag{2.17}$$

topamiz. Bunda $\widetilde{H}(x,t)$-haqiqiy uzluksiz funksiya bo‘lib, teskari operatorning yadrosini ifodalaydi:

$$(E+\widetilde{G})^{-1} = (E+\widetilde{H}),$$

$$\widetilde{H}f(x)=\int_0^x \widetilde{H}(x,t)f(t)dt.$$

(2.17) tenglikni (2.13) ayniyatlarning birinchisiga qo‘yib, ushbu

$$\varphi(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x G(x,t)\cos\sqrt{\lambda}tdt=\widetilde{\varphi}(x,\lambda)+\int_0^x \widetilde{H}(x,t)\widetilde{\varphi}(t,\lambda)dt+$$

$$+\int_0^x G(x,t)\left\{\widetilde{\varphi}(t,\lambda)+\int_0^t \widetilde{H}(t,s)\widetilde{\varphi}(s,\lambda)ds\right\}dt=\widetilde{\varphi}(x,\lambda)+\int_0^x \widetilde{H}(x,t)\widetilde{\varphi}(t,\lambda)dt+$$

$$+\int_0^x G(x,t)\widetilde{\varphi}(t,\lambda)dt+\int_0^x G(x,t)\int_0^t \widetilde{H}(t,s)\widetilde{\varphi}(s,\lambda)dsdt=\widetilde{\varphi}(x,\lambda)+\int_0^x \widetilde{H}(x,t)\widetilde{\varphi}(t,\lambda)dt+$$

$$+\int_0^x G(x,t)\widetilde{\varphi}(t,\lambda)dt+\int_0^x\left[\int_t^x G(x,s)\widetilde{H}(t,s)ds\right]\widetilde{\varphi}(t,\lambda)dt=\widetilde{\varphi}(x,\lambda)+$$

$$+\int_0^x\left[G(x,t)+\widetilde{H}(x,t)+\int_t^x G(x,s)\widetilde{H}(t,s)ds\right]\widetilde{\varphi}(t,s)dt=\widetilde{\varphi}(x,\lambda)+\int_0^x Q(x,t)\widetilde{\varphi}(t,\lambda)dt,$$

$$\varphi(x,\lambda)=\widetilde{\varphi}(x,\lambda)+\int_0^x Q(x,t)\widetilde{\varphi}(t,\lambda)dt, \tag{2.18}$$

tenglikni hosil qilamiz. Bu yerda

$$Q(x,t)=G(x,t)+\widetilde{H}(x,t)+\int_t^x G(x,s)\widetilde{H}(t,s)ds.$$

Endi ixtiyoriy $f(x)\in L_2(0,\pi)$ funksiyaning (f,φ) Fure koeffitsiyentini (2.18) tenglikdan foydalanib hisoblaymiz:

$$\int_0^\pi f(x)\varphi(x,\lambda)dx=\int_0^\pi f(x)\left[\widetilde{\varphi}(x,\lambda)+\int_0^x Q(x,t)\widetilde{\varphi}(t,\lambda)dt\right]dx=$$

$$=\int_0^\pi f(x)\widetilde{\varphi}(x,\lambda)dx+\int_0^\pi f(x)\left[\int_0^x Q(x,t)\widetilde{\varphi}(t,\lambda)dt\right]dx=$$

$$=\int_0^\pi f(x)\widetilde{\varphi}(x,\lambda)dx+\int_0^\pi\left[\int_x^\pi Q(t,x)f(t)dt\right]\widetilde{\varphi}(x,\lambda)dx=$$

$$=\int_0^\pi\left[f(x)+\int_x^\pi Q(t,x)f(t)dt\right]\widetilde{\varphi}(x,\lambda)dx=\int_0^\pi g(x)\widetilde{\varphi}(x,\lambda)dx. \tag{2.19}$$

Bu yerda

$$g(x) = f(x) + \int_x^{\pi} Q(t,x) f(t) dt. \tag{2.19'}$$

(2.19) tenglikda $\lambda = \lambda_n$, $n = 0,1,2,...$ deb olsak, u holda ushbu

$$a_n = \int_0^{\pi} f(x)\varphi(x,\lambda_n)dx, \qquad \tilde{a}_n = \int_0^{\pi} g(x)\tilde{\varphi}(x,\lambda_n)dx,$$

Fure koeffitsiyentlari uchun

$$a_n = \tilde{a}_n, \; n = 0,1,2,...,$$

tenglik bajariladi. Parseval tengligiga asosan

$$\int_0^{\pi} |f(x)|^2 dx = \sum_{n=0}^{\infty} \frac{|a_n|^2}{\alpha_n^2} = \sum_{n=0}^{\infty} \frac{|\tilde{a}_n|^2}{\tilde{\alpha}_n^2} = \int_0^{\pi} |g(x)|^2 dx,$$

kelib chiqadi, ya'ni

$$\|f\|_{L_2} = \|g\|_{L_2}. \tag{2.20}$$

Ushbu

$$Af(x) = f(x) + \int_x^{\pi} Q(t,x) f(t) dt,$$

operatorni qaraylik. U holda (2.19') ga asosan

$$Af(x) = g(x),$$

bo‘ladi. (2.20) tenglikdan esa

$$\|Af\|_{L_2} = \|f\|_{L_2},$$

kelib chiqadi. Bu esa A operatorning $L_2(0,\pi)$ fazoda unitarligini bildiradi. Unitar operatorlar uchun

$$A^* = A^{-1}, \; \left(A^*\right)^{-1} = A,$$

tenglik o‘rinli bo‘ladi. Bu tenglik faqat $Q(x,t) \equiv 0$ bo‘lgan holda o‘rinli bo‘ladi. Shuning uchun (2.18) tenglikdan

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda),$$

kelib chiqadi. Boshlang‘ich shartlardan $h = \tilde{h}$ tenglik kelib chiqadi. $x = \pi$ nuqtadagi chegaraviy shartlardan $H = \tilde{H}$ kelib chiqadi.

Ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi,$$

$$-\varphi'' + \tilde{q}(x)\varphi = \lambda\varphi,$$

differensial tenglamalardan foydalanib,

$$[q(x) - \tilde{q}(x)]\varphi(x,\lambda) = 0,$$

topamiz. Bundan va $q(x)$, $\tilde{q}(x)$ funksiyalarning uzluksizligini hamda $\varphi(x,\lambda)$ funksiyaning nollarini ajralganligini e'tiborga olib, $q(x) = \tilde{q}(x)$ ayniyatni keltirib chiqaramiz.

Quyidagi chegaraviy masalalarni qaraylik:

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.21}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - h_1 y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.22}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \\ y'(0) - \tilde{h}y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0, \end{cases} \tag{2.23}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \\ y'(0) - \tilde{h}_1 y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0, \end{cases} \tag{2.24}$$

Bu yerda $q(x), \tilde{q}(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiyalar bo'lib, h, h_1, $\tilde{h}_1$, H va $\tilde{H}$ chekli haqiqiy sonlar.

(2.21), (2.22), (2.23) va (2.24) chegaraviy masalalarning xos qiymatlarini mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$, $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ va $\{\tilde{\mu}_n\}_{n=0}^{\infty}$ orqali belgilaylik.

$\varphi(x,\lambda)$ va $\tilde{\varphi}(x,\lambda)$ lar orqali (2.21) va (2.23) differensial tenglamalarining mos ravishda ushbu

$$\varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = h, \ \tilde{\varphi}(0,\lambda) = 1, \ \tilde{\varphi}'(0,\lambda) = \tilde{h},$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlari bo'lsin. Bu chegaraviy masalalarning normallovchi o'zgarmaslarini mos ravishda

$$\alpha_n = \sqrt{\int_0^{\pi} \varphi^2(x,\lambda_n)dx}, \ \tilde{\alpha}_n = \sqrt{\int_0^{\pi} \tilde{\varphi}^2(x,\lambda_n)dx},$$

orqali belgilaymiz.

Teorema 2.3. (*G.Borg*) Agar $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, $n = 0,1,2,...$ bo'lsa, u holda

$$q(x)=\tilde{q}(x),\ h=h_1,\ \tilde{h}=\tilde{h}_1,\ H=\tilde{H},$$

bo‘ladi.

Isbot. Ushbu

$$\alpha_n^2=\frac{\tilde{h}-h}{\tilde{\lambda}_n-\lambda_n}\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\lambda_k-\lambda_n}{\tilde{\lambda}_k-\lambda_n},\quad n=0,1,2,...$$

formuladan foydalanib, berilgan $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ spektrlar yordamida $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o‘zgarmaslar ketma-ketligini aniqlaymiz:

$$h_1-h=\lim_{n\to\infty}n(\sqrt{\mu_n}-\sqrt{\lambda_n}),\tag{2.25}$$

$$\alpha_n^2=\frac{h_1-h}{\mu_n-\lambda_n}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\lambda_k-\lambda_n}{\mu_k-\lambda_k},\ n=0,1,2,...,\tag{2.26}$$

Xuddi shuningdek $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ va $\{\tilde{\mu}_n\}_{n=0}^{\infty}$ spektrlar yordamida $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$ normallovchi o‘zgarmaslar ketma-ketligini topamiz:

$$\tilde{h}_1-\tilde{h}=\lim_{n\to\infty}n(\sqrt{\tilde{\mu}_n}-\sqrt{\tilde{\lambda}_n}),\tag{2.27}$$

$$\tilde{\alpha}_n^2=\frac{\tilde{h}_1-\tilde{h}}{\tilde{\mu}_n-\tilde{\lambda}_n}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\tilde{\lambda}_k-\tilde{\lambda}_n}{\tilde{\mu}_k-\tilde{\lambda}_k},\ n=0,1,2,...\tag{2.28}$$

Toerema shartlaridan foydalanib, ushbu

$$\tilde{h}_1-\tilde{h}=h_1-h,\tag{2.29}$$

$$\tilde{\alpha}_n^2=\alpha_n^2,\ n=0,1,2,...,\tag{2.30}$$

tengliklarni hosil qilamiz.

Shunday qilib, teoremaning shartlari bajarilganda $\{\lambda_n,\alpha_n\}_{n=0}^{\infty}$ va $\{\tilde{\lambda}_n,\tilde{\alpha}_n\}_{n=0}^{\infty}$ spektral xarakteristikalar ustma-ust tushar ekan, ya’ni

$$\tilde{\lambda}_n=\lambda_n,\ \tilde{\alpha}_n=\alpha_n,\ \ n=0,1,2,...,$$

V.A.Marchenko yagonalik teoremasiga asosan $\tilde{q}(x)=q(x)$, $\tilde{h}=h$, $\tilde{H}=H$ tengliklar bajariladi. (2.29) tenglikda $\tilde{h}=h$ ekanligini e’tiborga olsak, $\tilde{h}_1=h_1$ kelib chiqadi.

3-§. Borg yagonalik teoremasini L.A Chudov usulida isbotlash

Teorema 3.1. *(V.A. Ambarsumyan).* Agar ushbu

$$-y'' + q(x)y = \lambda y, \quad x \in [0,\pi]$$

$$y'(0) = 0, \; y'(\pi) = 0, \quad q(x) \in C[0,\pi]$$

chegaraviy masalaning xos qiymatlari

$$\lambda_n = n^2, \; (n = 0,1,2,...),$$

bo‘lsa, u holda $q(x) \equiv 0$, $x \in [0,\pi]$ bo‘ladi.

Bu masalaga 1946-yilda shved matematigi G.Borg e’tibor bergan. U [2] ishida quyidagi teoremani isbotlagan.

Teorema 3.2. (G.Borg,1946) Agar ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad x \in [0,\pi] \\ \alpha y(0) + \beta y'(0) = 0 \\ \gamma y(\pi) + \delta y'(\pi) = 0 \end{cases} \tag{3.1}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad x \in [0,\pi] \\ \alpha y(0) + \beta y'(0) = 0 \\ \gamma_1 y(\pi) + \delta_1 y'(\pi) = 0 \end{cases} \tag{3.2}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \quad x \in [0,\pi] \\ \alpha y(0) + \beta y'(0) = 0 \\ \gamma y(\pi) + \delta y'(\pi) = 0 \end{cases} \tag{3.3}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \quad x \in [0,\pi] \\ \alpha y(0) + \beta y'(0) = 0 \\ \gamma_1 y(\pi) + \delta_1 y'(\pi) = 0 \end{cases} \tag{3.4}$$

masalalarning xos qiymatlari mos ravishda $\{\lambda_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty}, \{\tilde{\lambda}_n\}_{n=0}^{\infty}, \{\tilde{\mu}_n\}_{n=0}^{\infty}$ bo‘lib $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, $n = 0,1,2,...$ bo‘lsa hamda quyidagi

$$\delta \cdot \delta_1 = 0, \; |\delta| + |\delta_1| > 0 \tag{3.5}$$

shartlar bajarilsa, u holda $q(x) \equiv \tilde{q}(x)$ bo‘ladi.

L.A.Chudov [3] tomonidan 1949-yilda G.Borg teoremasi uchun (3.5) shartlar ortiqcha ekanligi ko‘rsatildi, ya’ni L.A.Chudov yuqoridagi teoramani (3.5) shartlarsiz yangi usulda isbot qilgan.

Quyidagi tenglamani ko‘rib chiqamiz

$$-y''+q(x)y=F(x),\quad x\in[0,\pi] \tag{3.6}$$

Bu yerda $q(x),F(x)$ haqiqiy uzluksiz funksiyalar. $y_1(x)$ va $y_2(x)$ orqali (3.6) tenglamaga mos keluvchi ushbu

$$-y''+q(x)y=0 \tag{3.7}$$

bir jinsli tenglamaning chiziqli erkli yechimlarini belgilaymiz. Ma'lumki

$$W\{y_1,y_2\}=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'\neq 0$$

bo'ladi.

Teorema 3.3. (3.6) tenglamaning

$$y(0)=a_1,\ y'(0)=a_2 \tag{3.8}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi mavjud, yagona bo'lib, u quyidagi formula yordamida beriladi:

$$y(x)=c_1\cdot y_1(x)+c_2\cdot y_2(x)+\frac{1}{W(y_1,y_2)}\cdot\int_0^x[y_1(x)y_2(t)-y_1(t)y_2(x)]F(t)dt \tag{3.9}$$

bu yerdagi c_1 va c_2 sonlar ushbu

$$\begin{cases} c_1y_1(0)+c_2y_2(0)=a_1 \\ c_1y_1'(0)+c_2y_2'(0)=a_2 \end{cases} \tag{3.10}$$

sistemadan bir qiymatli ravishda topiladi.

Isbot: (3.9) formula bilan berilgan funksiya (3.6) tenglamaning (3.8) boshlang'ich shartlarni qanoatlantiruvchi yechimi bo'lishini to'g'ridan-to'g'ri hosila olish yordamida isbotlaymiz. Bunda biz ushbu

$$\left(\int_{a(x)}^{b(x)} f(t,x)dt\right)_x' = f(b(x),x)\cdot b'(x)-f(a(x),x)\cdot a'(x)+\int_{a(x)}^{b(x)} f_x'(t,x)dt$$

fomuladan foydalanamiz.

$$y'(x)=c_1\cdot y_1'(x)+c_2\cdot y_2'(x)+\frac{1}{W(y_1,y_2)}\cdot\left\{0+\int_0^x[y_1'(x)y_2(t)-y_1(t)y_2'(x)]F(t)dt\right\} \tag{3.11}$$

$$y''(x)=c_1\cdot y_1''(x)+c_2\cdot y_2''(x)+\frac{1}{W(y_1,y_2)}\cdot\{[y_1'(x)y_2(x)-y_1(x)y_2'(x)]F(x)+$$

$$+\int_0^x \left[y_1''(x)y_2(t)-y_1(t)y_2''(x)\right]F(t)dt\Bigg\}$$

$$y''(x)=c_1\cdot q(x)\cdot y_1(x)+c_2\cdot q(x)\cdot y_2(x)-F(x)+$$

$$+\frac{1}{W(y_1,y_2)}\int_0^x q(x)\left[y_1(x)y_2(t)-y_1(t)y_2(x)\right]F(t)dt,$$

$$y''(x)=q(x)\left\{c_1\cdot y_1(x)+c_2\cdot y_2(x)+\frac{1}{W(y_1,y_2)}\int_0^x \left[y_1(x)y_2(t)-y_1(t)y_2(x)\right]F(t)dt\right\}-F(x)$$

,

$$y''(x)=q(x)y(x)-F(x),$$

ya'ni (3.9) funksiya (3.6) tenglamani qanoatlantirar ekan. (3.9) da $x=0$ desak (3.10) ga ko'ra

$$y(0)=c_1y_1(0)+c_2y_2(0)=a_1$$

kelib chiqadi. (3.11) da $x=0$ desak, (3.10) ga ko'ra

$$y'(0)=c_1y_1'(0)+c_2y_2'(0)=a_2$$

bo'ladi, ya'ni $y(x)$ yechim (3.8) boshlang'ich shartlarni qanoatlantirar ekan. Bu yechim yagonaligi differensial tenglamalar kursidagi Koshi teoremasidan kelib chiqadi.

Teorema 3.4. $y(x,\lambda)$ funksiya ushbu

$$-y''+q(x)y=\lambda y \tag{3.12}$$

tenglamaning biror yechimi bo'lib $y(0,\lambda)$ va $y'(0,\lambda)$ lar λ ga bog'liq bo'lmasin. $f(x)$ funksiya quyidagi

$$-f''+q(x)f=0 \tag{3.13}$$

tenglamaning ushbu

$$W(y(x,0),f(x))=1 \tag{3.14}$$

shartni qanoatlantiruvchi yechimi bo'lsin hamda

$$K(x,t)=y(x,0)f(t)-y(t,0)f(x)$$

bo'lsin. U holda quyidagi yoyilmalar o'rinli bo'ladi:

$$y(x,\lambda)=y(x,0)+\lambda\cdot\int_0^x K(x,t)y(t,0)dt+\ldots+$$

$$+\lambda^{n+1}\int_0^x\int_0^{t_1}\ldots\int_0^{t_n}K(x,t_1)K(t_1,t_2)\ldots K(t_{n-1},t_n)y(t_n,0)dt_1\ldots dt_n+\ldots, \tag{3.15}$$

$$y(x,\lambda)=y'(x,0)+\lambda\cdot\int_0^x K'_x(x,t)y(t,0)dt+\ldots+$$

$$+\lambda^{n+1}\int_0^x\int_0^{t_1}\ldots\int_0^{t_n}K'_x(x,t_1)K(t_1,t_2)\ldots K(t_{n-1},t_n)y(t_n,0)dt_1\ldots dt_n+\ldots \tag{3.16}$$

Bunda (3.15) va (3.16) tengliklardagi qatorlar $[0,\pi]$ kesmada absalyut va tekis yaqinlashuvchi bo‘ladi hamda x ning tayinlangan qiymatlarida λ ga nisbatan $\frac{1}{2}$-tartibdagi butun funksiya bo‘ladi.

Isbot. (3.12) tenglamada $F(x)=\lambda\cdot y(x,\lambda)$ belgilash olamiz va 3.3 teoremani tadbiq qilamiz. Bunda $y_1(x)$ sifatida $y(x,0)$ ni, $y_2(x)$ sifatida $f(x)$ ni olamiz. Bu holda (3.9) formulaga ko‘ra

$$y(x)=c_1\cdot y(x,0)+c_2\cdot f(x)+\frac{1}{W(y(x,0),f(x))}\cdot\int_0^x[y(x,0)f(t)-y(t,0)f(x)]\lambda y(t,\lambda)dt$$

ushbu tenglik kelib chiqadi. (3.14) tenglikdan foydalansak,

$$y(x)=c_1\cdot y(x,0)+c_2\cdot f(x)+\lambda\int_0^x K(x,t)y(t,\lambda)dt \tag{3.17}$$

hosil bo‘ladi. Bu yerda $x=0$ desak,

$$y(0,\lambda)=c_1\cdot y(0,0)+c_2\cdot f(0) \tag{3.18}$$

kelib chiqadi. (3.17) dan hosila olib, $x=0$ desak ,

$$y'(x)=c_1\cdot y'(x,0)+c_2\cdot f'(x)+\lambda\int_0^x K'_x(x,t)y(t,\lambda)dt$$

$$y'(0,\lambda)=c_1\cdot y'(0,0)+c_2\cdot f'(0) \tag{3.19}$$

(3.18) va (3.19) ga ko‘ra

$$\Delta=\begin{vmatrix} y(0,0) & f(0)\\ y'(0,0) & f'(0)\end{vmatrix}=W(y(0,0),f(0))=1,$$

$$\Delta_1=\begin{vmatrix} y(0,\lambda) & f(0)\\ y'(0,\lambda) & f'(0)\end{vmatrix}=y(0,\lambda)\cdot f'(0)-y'(0,\lambda)\cdot f(0),$$

$$\Delta_2=\begin{vmatrix} y(0,0) & y(0,\lambda)\\ y'(0,0) & y'(0,\lambda)\end{vmatrix}=y(0,0)\cdot y'(0,\lambda)-y'(0,0)\cdot y(0,\lambda),$$

$$c_1 = y(0,\lambda)\cdot f'(0) - y'(0,\lambda)\cdot f(0), \tag{3.20}$$

$$c_2 = y(0,0)\cdot y'(0,\lambda) - y'(0,0)\cdot y(0,\lambda). \tag{3.21}$$

(3.20) va (3.21) da $\lambda = 0$ desak, $c_1 = 1$ va $c_2 = 0$ bo‘ladi.

$y(0,\lambda)$ va $y'(0,\lambda)$ lar λ ga bog‘liq bo‘lmagani uchun hamisha $c_1 = 1$ va $c_2 = 0$ bo‘ladi.

Bunga ko‘ra (3.17) formuladan ushbu

$$y(x) = y(x,0) + \lambda\int_0^x K(x,t)y(t,\lambda)dt \tag{3.22}$$

integral tenglama kelib chiqadi. Bu integral tenglama Volterraning 2-tur integral tenglamasidir. Bizga yaxshi ma’lumki, uning yechimi x bo‘yicha obsalyut va tekis yaqinlashuvchi (3.15) qator yig‘indisidan iborat bo‘ladi.

$K'_x(x,t)$ hosilaning uzluksizligini ishlatib (3.15) qatorni hadlab differensiallash orqali hosil qilingan (3.16) qator ham obsalyut va tekis yaqinlashuvchi ekanligi ko‘rsatiladi. (3.15) qator kompleks tekislikdagi ixtiyoriy $|\lambda| \le N$ doirasida tekis yaqinlashuvchi bo‘lgani uchun kompleks analizdagi Veyershtrass teoremasiga ko‘ra uning yig‘indisi $y(x,\lambda)$ ham $|\lambda| \le N$ golomorf bo‘ladi. Bu doira ixtiyoriy ekanligidan $y(x,\lambda)$ funksiya λ ga nisbatan butun funksiya bo‘lishi kelib chiqadi. Bu butun funksiyaning tartibi $\frac{1}{2}$ dan oshmasligini ko‘rsatamiz. Buning uchun (3.15) qator koeffitsiyentlari uchun baholash olamiz. (3.15) qatorni ushbu

$$y(x,\lambda) = \sum_{n=0}^{\infty} C_n(x)\lambda^n \tag{3.23}$$

ko‘rinishida yozib olsak, $C_0(x) = y(x,0)$ kelib chiqadi. Agar (3.23) funksiyani (3.12) differensial tenglamaga qo‘ysak,

$$-C''_{n+1}(x) + q(x)C_{n+1}(x) = C_n(x), \qquad n = \overline{1,\infty} \tag{3.24}$$

tengliklar hosil bo‘ladi. (3.23) da $x = 0$ desak va $y(0,\lambda) = y(0,0)$ ekanligidan foydalansak,

$$\sum_{n=0}^{\infty} C_n(0)\lambda^n = 0$$

kelib chiqadi.bunga ko‘ra

$$C_n(0)=0, \quad n=\overline{1,\infty} \tag{3.25}$$

bo‘ladi. (3.23) da x bo‘yicha hosila olib $x=0$ desak, va $y'(0,\lambda)=y'(0,0)$ dan foydalansak,

$$C'_n(0)=0, \quad n=\overline{1,\infty} \tag{3.26}$$

tengliklarni olamiz.

Agar (3.24) da $C_n(x)=F(x)$ deb olib uni $C_{n+1}(x)$ ga nisbatan tenglama deb qarasak, hamda $C_n(0)=0$, $C'_n(0)=0$ ekanligini hisobga olsak, 3.3 teoremadan

$$C_{n+1}(x)=\int_0^x K(x,t)C_n(t)dt, \; C_0(x)=y(x,0) \tag{3.27}$$

kelib chiqadi. 3.3 teoremaga ko‘ra (3.27) tenglik va (3.24), (3.25), va (3.26) tengliklarga ekvivalent bo‘ladi. Induksiya usulini qo‘llab, quyidagi baholash o‘rinli bo‘lishini ko‘rsatamiz.

$$\left|C_n(x)\right| \le C\frac{x^{2n}}{(2n)!}\cdot A^n \tag{3.28}$$

Bu yerda A va C biror musbat o‘zgarmas sonlar. $n=0$ da

$$\left|C_0(x)\right|=\left|y(x,0)\right| \le const = C.$$

(3.28) baholash $k \le n$ uchun to‘g‘ri deb olib, $k=n+1$ uchun to‘g‘ri ekanligini ko‘rsatamiz. Buning uchun ushbu

$$m=\max_{x,t\in[0,\pi]}\left|K(x,t)\right|, M=\int_0^\pi \left|q(x)\right|dx$$

(3.29) belgilashlarni kiritib olamiz. (3.24) da $x=t_2$ deb olamiz, u holda

$$C''_{n+1}(t_2)=q(t_2)C_{n+1}(t_2)-C_n(t_2).$$

Buni t_2 bo‘yicha $[0,t_1]$ kesmada integrallaymiz:

$$\int_0^{t_1} C''_{n+1}(t_2)dt_2=\int_0^{t_1} q(t_2)C_{n+1}(t_2)dt_2-\int_0^{t_1} C_n(t_2)dt_2 .$$

Nyuton-Leybnis formulasiga ko‘ra

$$C'_{n+1}(t_1)-C'_{n+1}(0)=\int_0^{t_1}q(t_2)C_{n+1}(t_2)dt_2-\int_0^{t_1}C_n(t_2)dt_2$$

(3.26) tenglikni hisobga olsak,

$$C'_{n+1}(t_1)=\int_0^{t_1}q(t_2)C_{n+1}(t_2)dt_2-\int_0^{t_1}C_n(t_2)dt_2\,. \tag{3.30}$$

(3.30) ayniyatni t_1 bo‘yicha $[0,x]$ kesmada integrallaymiz:

$$\int_0^x C'_{n+1}(t_1)dt_1=\int_0^x\left\{\int_0^{t_1}q(t_2)C_{n+1}(t_2)dt_2\right\}dt_1-\int_0^x\left\{\int_0^{t_1}C_n(t_2)dt_2\right\}dt_1\,.$$

$$C_{n+1}(x)-C_{n+1}(0)=\int_0^x\left\{\int_0^{t_1}q(t_2)C_{n+1}(t_2)dt_2\right\}dt_1-\int_0^x\left\{\int_0^{t_1}C_n(t_2)dt_2\right\}dt_1\,.$$

Bu yerda (3.25) ni hisobga olsak

$$C_{n+1}(x)=\int_0^x\left\{\int_0^{t_1}q(t_2)C_{n+1}(t_2)dt_2\right\}dt_1-\int_0^x\left\{\int_0^{t_1}C_n(t_2)dt_2\right\}dt_1 \tag{3.31}$$

kelib chiqadi.

(3.27) ga ko‘ra ushbu

$$C_{n+1}(t_2)=\int_0^{t_2}K(t_2,t_3)C_n(t_3)dt_3 \tag{3.32}$$

tenglik o‘rinli bo‘ladi. (3.31) tenglikka (3.32) ifodani qo‘yamiz:

$$C_{n+1}(x)=\int_0^x\int_0^{t_1}\int_0^{t_2}q(t_2)K(t_2,t_3)C_n(t_3)dt_3dt_2dt_1-\int_0^x\int_0^{t_1}C_n(t_2)dt_2dt_1\,. \tag{3.33}$$

Bunga ko‘ra

$$|C_{n+1}(x)|\le\int_0^x\int_0^{t_1}\int_0^{t_2}|q(t_2)||K(t_2,t_3)||C_n(t_3)|dt_3dt_2dt_1+\int_0^x\int_0^{t_1}|C_n(t_2)|dt_2dt_1 \tag{3.34}$$

(3.29) belgilashlarni va (3.28) baholashni hisobga olsak, (3.34) tengsizlikdan quyidagi tengsizlik kelib chiqadi:

$$|C_{n+1}(x)|=m\cdot C\cdot A^n\cdot\int_0^x\int_0^{t_1}\int_0^{t_2}|q(t_2)|\cdot\frac{t_3^{2n}}{(2n)!}dt_3dt_2dt_1+\int_0^x\int_0^{t_1}\frac{t_2^{2n}}{(2n)!}dt_2dt_1\,,$$

$$|C_{n+1}(x)| = m\cdot C\cdot A^n\cdot\int_0^x\int_0^{t_1}|q(t_2)|\cdot\frac{t_2^{2n+1}}{(2n+1)!}dt_2dt_1 + C\cdot A^n\cdot\int_0^x\frac{t_2^{2n+1}}{(2n+1)!}dt_1 \le$$

$$\le m\cdot C\cdot A^n\cdot\frac{x^{2n+2}}{(2n+2)!}\cdot M + C\cdot A^n\cdot\frac{x^{2n+2}}{(2n+2)!},$$

Demak,

$$|C_{n+1}(x)| \le C\cdot A^n\cdot\frac{x^{2n+2}}{(2n+2)!}\cdot[mM+1].$$

Agar biz (3.28) da $A \ge mM+1$ deb olsak, ushbu

$$|C_{n+1}(x)| \le C\cdot A^{n+1}\cdot\frac{x^{2(n+1)}}{(2(n+1))!}$$

tengsizlik hosil bo‘ladi. Bu esa (3.28) tengsizlik $k = n+1$ da ham o‘rinli ekanligini bildiradi. Shunday qilib (3.28) tengsizlik barcha $n \ge 0$ da to‘g‘ri ekan.

(3.23) va (3.28) ga ko‘ra

$$|y(x,\lambda)| \le \sum_{n=0}^{\infty}|C_n(x)|\cdot|\lambda|^n \le C\cdot\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\cdot A^n|\lambda|^n = C\cdot\sum_{n=0}^{\infty}\frac{\left(x\sqrt{A}\sqrt{|\lambda|}\right)^{2n}}{(2n)!} =$$

$$= C\cdot ch\left(x\sqrt{A}\sqrt{|\lambda|}\right) = C\cdot\frac{e^{\left(x\sqrt{A}\sqrt{|\lambda|}\right)} + e^{-\left(x\sqrt{A}\sqrt{|\lambda|}\right)}}{2} \le Ce^{\left(x\sqrt{A}\sqrt{|\lambda|}\right)}.$$

Demak,

$$|y(x,\lambda)| \le C\cdot e^{\left(x\sqrt{A}\sqrt{|\lambda|}\right)}.$$

(3.35)

Bu esa $y(x,\lambda)$ yechim har bir fikserlangan x da λ bo‘yicha tartibi $\le\frac{1}{2}$ ekanini bildiradi. Huddi shu tarzda $y'(x,\lambda)$ funksiyaning har bir fikserlangan x da λ bo‘yicha tartibi $\le\frac{1}{2}$ bo‘lishi ko‘rsatiladi.

Ikkinchi tomondan kompleks o‘zgaruvchili funksiyalar nazariyasida ([4], 241-bet,)quyidagi teorema isbotlangan: agar butun funksiyaning tartibi chekli bo‘lib, butun songa teng bo‘lmasa, u holda u quyidagi shartlarni qanoatlantiruvchi k sonlarning aniq quyi chegarasiga teng bo‘ladi:

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^k},\ \lambda_n \neq 0$$

Bu yerda λ_n orqali qaralayotgan butun funksiyaning nollari belgilanadi. Shturm-Liuvill chegaraviy masalasining xos qiymatlari uchun ushbu

$$\lambda_n = n^2 + C_0 + \frac{\gamma_n}{n},\ \ \gamma_n \in l_2$$

formula o‘rinli. Bundan foydalansak

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^k} \sim \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \tag{3.36}$$

kelib chiqadi. $2k = 1+\varepsilon$, $\varepsilon > 0$ bo‘lsa, (3.36) qator Koshining integral alomatiga ko‘ra yaqinlashadi. Demak, $k = \frac{1}{2} + \frac{\varepsilon}{2}$, $\varepsilon > 0$ ekan, bunga ko‘ra

$$\inf k = \inf_{\varepsilon>0}\left(\frac{1}{2} + \frac{\varepsilon}{2}\right) = \frac{1}{2}$$

bo‘ladi. Demak, har bir funksiyani x da $y(x,\lambda)$ va $y'(x,\lambda)$ funksiyalar λ bo‘yicha $\frac{1}{2}$ tartibli butun funksiyalar ekan.

Xulosa

Shturm-Liuvill operatori uchun qo‘yilgan to‘g‘ri va teskari spektral masalalar fizikaning turli sohalarida uchraydigan muhim tenglamalarni yechishga tadbiqi bo‘lgani uchun, bu soha o‘z dolzarbligini saqlab kelmoqda.

Ushbu bitiruv malakaviy ishi kirish, 3 ta paragraf va adabiyotlar ro‘yxatidan tashkil topgan. Birinchi paragrafda Shturm-Liuvill chegaraviy masalasining xos qiymatlari va xos funksiyalari xossalari o‘rganilgan. Ikkinchi paragrafda Shturm-Liuvill chegaraviy masalasiga teskari masalaning qo‘yilishi, bir nechta yagonalik teoremalari va ularning isbotlari o‘rganilgan. Oxirgi paragrafda esa, Borg yagonalik teoremasini Chudov usulida isbotlash o‘rganilgan.

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