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p-adic $(3, 2)$-rational dynamical systems with two fixed points
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Ikkita qo'zg'almas nuqtaga ega bo'lgan p-adik $(3, 2)$-*ratsional dinamik sistemalar*

Bu maqolada C_p - *p*-adik kompleks sonlar maydonida aniqlangan $(3, 2)$-ratsional funksiyaning dinamik sistemasi o'rganilgan. Bunda Zigel disklar va attraktorlar topilgan.

В данной работе изучены *p*-адические динамические системы (3,2)-рациональной функции, определенной на поле C_p - *p*-адических комплексных чисел. Найдены диски Зигеля и аттракторы этой системы.

1. Introduction

The *p*-adic numbers were first introduced by the German mathematician K.Hensel. For about a century after the discovery of *p*-adic numbers, they were mainly considered objects of pure mathematics. Beginning with 1980's various models described in the language of *p*-adic analysis have been actively studied. More precisely, models over the field of *p*-adic numbers have been considered which, based on the conception that *p*-adic numbers might provide a more exact and more adequate description of micro-world phenomena. Numerous applications of these numbers to theoretical physics have been proposed in papers [1], [3], [8], [12],[13] to quantum mechanics [5], to *p*-adic valued physical observable [5] and many others [6],[11].

p-adic numbers. Let Q be the field of rational numbers. The greatest common divisor of the positive integers n and m is denotes by (n, m). Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number.

The *p*-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

It has the following properties:

    1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,

    2) $|xy|_p = |x|_p|y|_p$,

    3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

    3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$,

    3.2) if $|x|_p = |y|_p$ then $|x + y|_p \leq |x|_p$,

    this is a non-Archimedean one.

The completion of Q with respect to p-adic norm defines the p-adic field which is denoted by Q_p.

The well-known Ostrovsky's theorem asserts that norms $|x| = |x|_\infty$ and $|x|_p$, $p = 2, 3, 5...$ exhaust all nonequivalent norms on Q (see [7]). Any p-adic number $x \neq 0$ can be uniquely represented in the canonical series:

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, ...$ (see more detail [4],[7]). Observe that in this case $|x|_p = p^{-\gamma(x)}$.

The algebraic completion of Q_p is denoted by C_p and it is called *complex p-adic numbers*. For any $a \in C_p$ and $r > 0$ denote

$$U_r(a) = \{x \in C_p : |x - a|_p \leq r\}, \quad V_r(a) = \{x \in C_p : |x - a|_p < r\},$$

$$S_r(a) = \{x \in C_p : |x - a|_p = r\}.$$

A function $f : U_r(a) \to C_p$ is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n(x - a)^n, \quad f_n \in C_p,$$

which converges uniformly on the ball $U_r(a)$.

Dynamical systems in C_p. In this section we recall some known facts concerning dynamical systems (f, U) in C_p, where $f : x \in U \to f(x) \in U$ is an analytic function and $U = U_r(a)$ or C_p.

Now let $f : U \to U$ be an analytic function. Denote $x_n = f^n(x_0)$, where $x_0 \in U$ and $f^n(x) = \underbrace{f \circ \cdots \circ f}_{n}(x)$.

Recall some the standard terminology of the theory of dynamical systems (see for example [9]). If $f(x_0) = x_0$ then x_0 is called a *fixed point*. The set

of all fixed points of f is denoted by $\text{Fix}(f)$. A fixed point x_0 is called an *attractor* if there exists a neighborhood $V(x_0)$ of x_0 such that for all points $y \in V(x_0)$ it holds $\lim_{n \to \infty} y_n = x_0$. If x_0 is an attractor then its *basin of attraction* is

$$A(x_0) = \{y \in C_p : y_n \to x_0, \ n \to \infty\}.$$

A fixed point x_0 is called *repeller* if there exists a neighborhood $V(x_0)$ of x_0 such that $|f(x) - x_0|_p > |x - x_0|_p$ for $x \in V(x_0)$, $x \neq x_0$. Let x_0 be a fixed point of a function $f(x)$. The ball $V_r(x_0)$ (contained in U) is said to be a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $x_n \in S_\rho(x_0)$ for all $n = 1, 2 \ldots$. The union of all Siegel desks with the center at x_0 is said to *a maximum Siegel disk* and is denoted by $SI(x_0)$.

Let x_0 be a fixed point of an analytic function $f(x)$. Put

$$\lambda = \frac{d}{dx}f(x_0).$$

The point x_0 is *attractive* if $0 \leq |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

We investigate the behavior of trajectory of an arbitrary $(3,2)$-rational p-adic dynamical system in complex p-adic filed C_p when the there are two fixed points. It is studied Siegel disks and attractors of such dynamical systems.

2. $(3,2)$-Rational p-adic dynamical systems

A function is called a (n,m)-rational function if and only if it can be written in the form $f(x) = \frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomial functions with degree n and m respectively, $Q_m(x)$ is not the zero polynomial (see [2]).

In this paper we consider the dynamical system associated with the $(3,2)$-rational function $f : C_p \to C_p$ defined by

$$f(x) = \frac{x^3 + ax^2 + bx + c}{x^2 + dx + e}, \quad a, b, c, d, e \in C_p, \tag{1}$$

where $x \neq \hat{x}_{1,2} = \frac{-d \pm \sqrt{d^2 - 4e}}{2}$.

In this paper we consider the case $a \neq d$. There are two fixed points:

$$x_{1,2} = \frac{e - b \pm \sqrt{(e-b)^2 + 4(d-a)c}}{2(a-d)}. \tag{2}$$

Denote
$$\mathcal{P} = \{x \in C_p : \exists n \in N \cup \{0\}, f^n(x) = \hat{x}_{1,2}\}.$$

For any $x \in C_p$, $x \neq \hat{x}_{1,2}$, by simple calculations we get

$$|f(x) - x_i|_p = |x - x_i|_p \cdot \frac{|[(x - x_i) + \alpha_1(x_i)]^2 + \alpha_2(x_i)|_p}{|[(x - x_i) + \beta_1(x_i)]^2 + e - \frac{d^2}{4}|_p}, \quad i = 1, 2, \quad (3)$$

where

$$\alpha_1(x) = x + \frac{a}{2}, \quad \alpha_2(x) = \frac{ex^2 + (ae - c)x + be - cd}{x^2 + dx + e} - \frac{a^2}{4}, \quad \beta_1(x) = x + \frac{d}{2}.$$

Denote

$$\alpha_{1,i} = |\alpha_1(x_i)|_p, \quad \alpha_{2,i} = |\alpha_2(x_i)|_p, \quad \beta_{1,i} = |\beta_1(x_i)|_p, \quad \beta_2 = |e - \frac{d^2}{4}|_p, \quad i = 1, 2,$$

and

$$\alpha_i = \max\{\alpha_{1,i}, \sqrt{\alpha_{2,i}}\}, \quad \beta_i = \max\{\beta_{1,i}, \sqrt{\beta_2}\}.$$

Let

$$\alpha_{1,i} \neq \sqrt{\alpha_{2,i}} \quad and \quad \beta_{1,i} \neq \sqrt{\beta_2}.$$

Consider the following functions:

For $0 \leq \alpha < \beta$ define the function $\varphi_{\alpha,\beta} : [0, +\infty) \to [0, +\infty)$ by

$$\varphi_{\alpha,\beta}(r) = \begin{cases} \frac{\alpha^2}{\beta^2} r, & \text{if } r < \alpha, \\ \alpha^*, & \text{if } r = \alpha, \\ \frac{r^3}{\beta^2}, & \text{if } \alpha < r < \beta, \\ \beta^*, & \text{if } r = \beta, \\ r, & \text{if } r > \beta, \end{cases}$$

where α^* and β^* are some given numbers with $\alpha^* \leq \frac{\alpha^3}{\beta^2}$, $\beta^* \geq \beta$.

For $0 \leq \beta < \alpha$ define the function $\phi_{\alpha,\beta} : [0, +\infty) \to [0, +\infty)$ by

$$\phi_{\alpha,\beta}(r) = \begin{cases} \frac{\alpha^2}{\beta^2} r, & \text{if } r < \beta, \\ \beta', & \text{if } r = \beta, \\ \frac{\alpha^2}{r}, & \text{if } \beta < r < \alpha, \\ \alpha', & \text{if } r = \alpha, \\ r, & \text{if } r > \alpha, \end{cases}$$

where α' and β' some positive numbers with $\alpha' \le \alpha$, $\beta' \ge \frac{\alpha^2}{\beta}$.

For $\alpha = \beta \ge 0$ we define the function $\psi_\alpha : [0, +\infty) \to [0, +\infty)$ by

$$
\psi_\alpha(r) = \begin{cases} r, & \text{if } r \ne \alpha, \\ \hat{\alpha}, & \text{if } r = \alpha, \end{cases}
$$

where $\hat{\alpha}$ is a given number.

Using the formula (3) we easily get the following:

Lemma 1. If $x \in S_r(x_i)$, then the following formula holds

$$
|f^n(x) - x_i|_p = \begin{cases} \varphi^n_{\alpha_i, \beta_i}(r), & \text{if } \alpha_i < \beta_i, \\ \phi^n_{\alpha_i, \beta_i}(r), & \text{if } \alpha_i > \beta_i, \quad n \ge 1, \quad i = 1, 2. \\ \psi^n_{\alpha_i}(r), & \text{if } \alpha_i = \beta_i. \end{cases}
$$

Thus the *p*-adic dynamical system $f^n(x), n \ge 1, x \in C_p \setminus \mathcal{P}$ is related to the real dynamical systems generated by $\varphi_{\alpha,\beta}$, $\phi_{\alpha,\beta}$ and ψ_α. Now we are going to study these (real) dynamical systems.

Lemma 2. The dynamical system generated by $\varphi_{\alpha,\beta}(r), \alpha < \beta$ has the following properties:

1. $\text{Fix}(\varphi_{\alpha,\beta}) = \{0\} \cup \{r : r > \beta\} \cup \{\beta^* : \text{if } \beta = \beta^*\}$

2. Independently on α, we have

$$
\lim_{n \to \infty} \varphi^n_{\alpha,\beta}(r) = \begin{cases} 0, & \text{for all } r < \beta, \\ r, & \text{for all } r > \beta, \quad ; \\ \beta^*, & \text{if } r = \beta \end{cases}
$$

proof. 1. This is the result of a simple analysis of the equation $\varphi_{\alpha,\beta}(r) = r$.

Proof of part 2 follow from the property that $\varphi_{\alpha,\beta}(r), r \ne \alpha, \beta$ is an increasing function.

Lemma 3. The dynamical system generated by $\phi_{\alpha,\beta}(r), \alpha > \beta$ has the following properties:

A. $\text{Fix}(\phi_{\alpha,\beta}) = \{0\} \cup \{r : r > \alpha\} \cup \{\alpha : \text{if } \alpha = \alpha'\}$

B. We have

$$\lim_{n\to\infty} \phi_{\alpha,\beta}^n(r) = \begin{cases} 0, & \text{if } r = 0, \\ \in B, & \text{for all } 0 < r \leq \alpha, \\ r, & \text{for all } r > \alpha, \end{cases} ;$$

where $B = (\alpha, \phi_{\alpha,\beta}(\beta))$.

proof. Since $\phi_{\alpha,\beta}(r)$ is a piecewise linear function the proof consists simple computations, using graph of the function and varying the parameters α, β.

The following lemma is obvious:

Lemma 4. The dynamical system generated by $\psi_\alpha(r), \alpha \geq 0$ has the following properties:

(i) $\text{Fix}(\psi_\alpha) = \{0\} \cup \{r : r \neq \alpha\} \cup \{\alpha : \text{if } \alpha = \hat{\alpha}\}$

(ii)

$$\lim_{n\to\infty} \psi_\alpha^n(r) = \begin{cases} r, & \text{for any } r \neq \alpha, \alpha \neq \hat{\alpha}, \\ \hat{\alpha}, & \text{for } r = \alpha, \alpha \neq \hat{\alpha}, \\ r, & \text{for any } r \geq 0, \alpha = \hat{\alpha}, \end{cases}$$

Now we shall apply these lemmas for study of the p-adic dynamical system generated by f.

For $x \in S_{\alpha_i}(x_i)$, we denote

$$\alpha_i^*(x) = \alpha_i \cdot \frac{|[(x - x_i) + \alpha_1(x_i)]^2 + \alpha_2(x_i)|_p}{|[(x - x_i) + \beta_1(x_i)]^2 + e - \frac{d^2}{4}|_p}, \quad i = 1, 2. \qquad (4)$$

For $x \in S_{\beta_i}(x_i)$, we denote

$$\beta_i^*(x) = \beta_i \cdot \frac{|[(x - x_i) + \alpha_1(x_i)]^2 + \alpha_2(x_i)|_p}{|[(x - x_i) + \beta_1(x_i)]^2 + e - \frac{d^2}{4}|_p}, \quad i = 1, 2. \qquad (5)$$

Using Lemma 1 and Lemma 2 we obtain the following

Theorem 5. If $\alpha_i < \beta_i$ and $x \in S_r(x_i)$, $i = 1, 2$ then the p-adic dynamical system generated by f has the following properties:

1. The following spheres are invariant with respect to f:

$$S_r(x_i), \text{ if } r > \beta_i,$$

2. We have

$$\lim_{n\to\infty} f^n(x) = x_i, \quad \text{for all} \quad r < \beta_i,$$

$$f\left(S_r(x_i) \setminus \mathcal{P}\right) \subset S_r(x_i), \quad \text{for all} \quad r > \beta_i,$$

$$\lim_{n\to\infty} f^n(x) \in S_{\beta_i^*(x)}(x_i), \quad \text{if} \quad r = \beta_i;$$

By Lemma 1 and Lemma 3 we obtain the following

Theorem 6. If $\alpha_i > \beta_i$, $i = 1, 2$ then the *p*-adic dynamical system generated by f has the following properties:

A. The following spheres are invariant:

$$S_r(x_i), \quad if \quad r > \alpha_i,$$

B. We have

$$\lim_{n\to\infty} f^n(x) \in S_\mu(x_i), \quad \mu \in B, \quad \text{for any} \quad 0 < r \leq \alpha_i,$$

$$f\left(S_r(x_i) \setminus \mathcal{P}\right) \subset S_r(x_i), \quad \text{for any} \quad r > \alpha_i;$$

By Lemma 1 and Lemma 4 we get

Theorem 7. If $\alpha_i = \beta_i$, and $x \in S_r(x_i)$, $i = 1, 2$ then the dynamical system generated by f has the following properties:

(i) For any $r \neq \alpha_i$ the sphere $S_r(x_i)$ is an invariant set.

(ii) (ii.a) If $\alpha_i \neq \alpha_i^*(x)$ then

$$f\left(S_r(x_i) \setminus \mathcal{P}\right) \subset S_r(x_i), \quad \text{for all} \quad r \neq \alpha_i;$$

$$f\left(S_r(x_i) \setminus \mathcal{P}\right) \subset S_{\alpha_i^*(x)}(x_i), \quad \text{if} \quad r = \alpha_i;$$

(ii.b) If $\alpha_i = \alpha_i^*(x)$ then

$$f\left(S_r(x_i) \setminus \mathcal{P}\right) \subset S_r(x_i), \quad \text{for all} \quad r \neq 0;$$

Remark 8. By above analysis it is clear that both fixed points x_1, x_2 have the same character. More precisely if x_1 is attractor (resp. repeller or indifferent) if and only if x_2 is attractor (resp. repeller or indifferent). If fixed points $x_i, i = 1, 2$ is attractor then its basin of attraction is $A(x_i) = V_{\beta_i}(x_i)$. We note that if fixed points are repeller then $|f(x) - x_i|_p > |x - x_i|_p$ for any

$x \in V_{\alpha_i}(x_i), x \neq x_i$. If fixed points are indifferent then they have the same Siegel disk and 1. $SI(x_i) = V_{\alpha_i}(x_i)$, *if* $\hat{\alpha} \neq \alpha_i$; 2. $SI(x_i) = C_p \setminus \mathcal{P}$, if $\hat{\alpha} = \alpha_i$.

References

1. I.Ya.Araf'eva, B.Dragovich, P.H.Frampton and I.V.Volovich, *Wave function of the universe and p-adic gravity*, Mod.Phys. Lett. **A6**(1991) 4341-4358.

2. S. Albeverio, U.A. Rozikov, I.A. Sattarov. p-adic $(2,1)$-rational dynamical systems. *Jour. Math. Anal. Appl.* **398**(2) (2013), 553–566.

3. P.G.O.Freund, and E.Witten, *Adelic string ampletudes*, Phys. Lett. **B199**(1987) 191-194.

4. F.Q.Gouvea, *p-adic numbers*, Springer, Berlin 1991.

5. A.Yu.Khrennikov, *p-adic quantum mechanics with p-adic valued functions*, J.Math.Phys. **32**(1991), 932-936.

6. A.Yu.Khrennikov, *p-adic Valued Distributions in Mathematical Physics* Kluwer, Netherlands, 1994.

7. N.Koblitz, *p-adic numbers, p-adic analysis and zeta-function* Springer, Berlin, 1977.

8. E.Marinari and G.Parisi, *On the p-adic five point function*, Phys.Lett. **203B**(1988) 52-56.

9. H.-O.Peitgen, H.Jungers and D.Saupe, *Chaos Fractals*, Springer, Heidelberg-New York, 1992.

10. J. Silverman. *The arithmetic of dynamical systems.* Number 241 in Graduate Texts in Mathematics. Springer-Verlag, New York, 2007.

11. V.S.Vladimirov, I.V.Volovich and E.I.Zelenov, *p-adic Analysis and Mathematical Physics*, World Scientific, Singapour, 1994.

12. I.V.Volovich, *Number theory as the ultimate physical theory*, Preprint, TH, 4781/87.

13. I.V.Volovich, *p-adic strings*, Class. Quantum Grav. **4**(1987) L83-L87.

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