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Qat'iy m - sh funksiyalar

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O‘ZBEKISTON RESPUBLIKASI
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Kafedra Funktsiyalar nazariyasi Ilmiy rahbar Sadullayev Azimboy

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(yo‘nalishlar bo‘yicha)

**“QAT’IY $m - sh$ FUNKSIYALAR ” MAVZUSIDAGI MAGISTRLIK
DISSERTATSIYASI ANNOTATSIYASI**

Mavzuning dolzarbligi. Plyuripotensiallar nazariyasi yoki kompleks potentsiallar nazariyasi $(dd^c u)^n$ – Monje-Ampere operatoriga asoslangan va zamonaviy matematikaning asosiy yo‘nalishlaridan biri hisoblanadi. $m - sh$ funksiyalar sinfida potentsiallar nazariyasi A.Sa’dullayev va B.Abdullayevlar tomonidan o‘rganilgan va hozirda matematikaning bir qator yo‘nalishlarida muvaffaqiyat bilan qo‘llanilmoqda. $m - sh$ funksiyalar sinfi subgarmonik va plyurisubgarmonik funksiyalar sinflari orasidagi sinf bo‘lgani bois bu soha kompleks potentsiallar nazariyasida alohida obyekt sifatida mutaxassislar e’tiborida bo‘lib kelmoqda.

Ishning maqsadi. $m - sh$ funksiyalarning ta’riflari va xossalariga asoslanib, qat’iy $m - sh$ funksiyalarni o‘rganish

Ishning vazifalari.

— $\forall u \in C^2(D) \ (D \subset \mathbb{C}^n)$ funksiya uchun
 $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} \geq 0 \ \forall k = \overline{1, m}$ tengsizlikning $k = m$ uchun
bajarilishidan qolgan barcha $\forall k = \overline{1, m-1}$ lar uchun bajarilishi yoki bajarilmasligini aniqlash

— m - sh funksiyalar sinfi $m = n$ bo'lgan holda plyurisubgarmonik funksiyalar sinfi bilan ustma-ust tushishini o'rganish.

— m - sh funksiyalarning qat'iy m - sh bo'lish shartlarini o'rganish

— **Tadqiqot obyekti.** m - sh funksiyalar, qat'iy m - sh funksiyalar.

Tadqiqot predmeti. $(dd^c u)^n$ – Monje –Ampere operatori hamda gessianlar, oqimlar to'g'risidagi masalalar.

Tadqiqot usullari. Ko'p kompleks o'zgaruvchining funksiyalar nazariyasi, kompleks potentsiallar nazariyasi va chiziqli algebra usullari.

Tadqiqot natijalarining ilmiy jihatdan yangilik darajasi. Mazkur ishning asosiy natijalari yangi.

Tadqiqot natijalarining amaliy ahamiyati va tadbiqu. Olingan natijalar va usullar funksiyalar nazariyasining keyingi rivojlanishida va kompleks analizning tadbirlarida qo'llanilishi mumkin.

Ish tuzilishi va tarkibi. Dissertatsiya ishi kirish, uchta bob hamda adabiyotlar ro'yhatidan iborat. Har bir bob bo'limlardan tuzilgan.

Bajarilgan ishning asosiy natijalari.

1) $\forall u \in C^2(D)$ ($D \subset \mathbb{C}^n$) funksiya uchun
 $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} \geq 0 \quad \forall k = \overline{1, m}$ tengsizlikning $k = m$ uchun
 bajarilishidan qolgan barcha $\forall k = \overline{1, m-1}$ lar uchun yuqoridagi
 tengsizlikning bajarilmasligi aniqlandi.

2) $D \subset \mathbb{C}^n$ sohada n - sh bo'lgan ixtiyoriy ikki marta silliq
 funksiya psh bo'lishi polinomlar nazariyasining asosiy teoremlaridan
 biri bo'lgan Dekart teoremasi yordamida isbotlandi.

3) Agar $H_1(\lambda) \geq 0, H_2(\lambda) \geq 0, \dots, H_{m-1}(\lambda) \geq 0, H_m(\lambda) > 0$ (bunda
 $H_m(\lambda) = \sum_{1 \leq j_1 < \dots < j_m \leq n} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_m}$ – gessian vektori) bo'lsa, u holda
 $H_{m-1}(\lambda) > 0$ bo'lishi isbotlandi.

Xulosa va takliflarning qisqacha umumlashtirilgan ifodasi.

Tadqiqot ishining asosiy natijalari bo'yicha quyidagi xulosalarga kelindi:

1. $m - sh$ funksiyalar va psh funksiyalar sinflari orasidagi munosabatlar aniqlandi..

2. $m - sh$ funksiyalarning qat'iy $m - sh$ bo'lish shartlari o'rganildi.

Ilmiy rahbar:



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Magistratura talabasi:



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Speciality mathematics (*to the
directions*)

ANNOTATION OF MASTER DEGREE DISSERTATION ON THE
THEME STRICTLY $m-sh$ FUNCTIONS

Actuality of research. The classical potential theory is based on subharmonic (sh) functions. It is related to the Laplace equation $\Delta u = 0$. The pluripotential theory constructed in the 1980s is based on plurisubharmonic (psh) functions and is related to the Monge–Ampere equation $(dd^c u)^n = 0$, where, as usual, $d = \partial + \bar{\partial}$ and $d^c = \frac{\partial - \bar{\partial}}{4i}$. This theory is well reflected in the literature, it is being successfully developed, and it has numerous applications. The goal of the present paper is to study the equation $(dd^c u)^m \wedge \beta^{n-m} = m!(n-m)!H_m(u)\beta^n$, $1 \leq m \leq n$, (1) and the related m -subharmonic functions. Here $\beta = dd^c |z|^2$ is the fundamental form in $\mathbb{C}^n$ and f is a positive function or measure in a domain $D \subset \mathbb{C}^n$ under consideration. Since $(dd^c u) \wedge \beta^{n-1} = \Delta u \beta^n$, equation (1) for $m=1$ reduces to the Poisson equation; for $m=n$, this gives the inhomogeneous Monge–Ampere equation. Equation (1) is called the complex Hessian equation. Such equations and the properties of their solutions have been systematically studied in the last decade.

The aim of the research. To study discussed $m-sh$ functions and

properties them also strictly $m-sh$ functions.

Tasks of research:

- to study In this dissertation, the weak solutions of $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k}$ operator will be dealt with. $m-sh$ functions will be discusted with the help of special detiations. In this way conchition can be drawn.
- $m-sh$ functions are discusted wether they are plurisubharmonics.
- $m-sh$ functions are discusted wether they are strictly $m-sh$ functions.

Object of research. $m-sh$ functions, strictly $m-sh$ functions.

Subject of research. operator $(dd^c u)^n$ – Monje –Ampere, hessians, currents.

Methods of research. It is uses methods of theory of functions of several complex variables, complex theory of potential and lines algebra.

The degree of novelty of the obtained results. The main results of this work are new.

The practical value of the research. Obtained results can be used in future researches in theory of functions and its applications.

The structure of the work. Dissertation consists of introduction part, three chapters, conclusion and bibliography.

Main results of the investigation:

1) the weak solutions of $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k}$ operator will be dealt with $m-sh$ functions will be discusted with the help of special detiations. In this way conchition can be drawn.

2) $m-sh$ functions are discusted wether they are plurisubharmonics.

3) $m-sh$ functions are discusted wether they are strictly $m-sh$ functions.

From research arguments we make the next conclusion:

1. $m-sh$ functions are discusted wether they are

plurisubharmonics.

2. $m-sh$ functions are discussed whether they are strictly $m-sh$ functions.

Supervisor:



(signature)

Master student:



(signature)

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KIRISH

1. Ishning umumiy tavsifi

1.1. Mavzuning dolzarbligi. Plyuripotensiallar nazariyasi yoki kompleks potentsiallar nazariyasi $(dd^c u)^n$ – Monje-Ampere operatoriga asoslangan va zamonaviy matematikaning asosiy yo'nalishlaridan biri hisoblanadi. $m-sh$ funksiyalar sinfida potentsiallar nazariyasi A.Sadullayev va B.Abdullayevlar tomonidan o'rganilgan va hozirda matematikaning bir qator yo'nalishlarida muvaffaqiyat bilan qo'llanilmoqda. $m-sh$ funksiyalar sinfi subgarmonik va plyurisubgarmonik funksiyalar sinflari orasidagi sinf bo'lgani bois bu soha kompleks potentsiallar nazariyasida alohida obyekt sifatida mutaxassislar e'tiborida bo'lib kelmoqda.

1.2. Ishning maqsadi. $m-sh$ funksiyalarning ta'riflari va xossalariga asoslanib qat'iy $m-sh$ funksiyalarni o'rganish.

1.3. Ishning vazifalari.

— $\forall u \in C^2(D) \ (D \subset \mathbb{C}^n)$ funksiya uchun

$(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} \geq 0 \ \forall k = \overline{1, m}$ tengsizlikning $k = m$ uchun bajarilishidan qolgan barcha $\forall k = \overline{1, m-1}$ lar uchun bajarilishi yoki bajarilmasligini aniqlash

— $m-sh$ funksiyalar sinfi $m = n$ bo'lgan holda plyurisubgarmonik funksiyalar sinfi bilan ustma-ust tushishini o'rganish.

— $m-sh$ funksiyalarning qat'iy $m-sh$ bo'lish shartlarini o'rganish

1.4.Tadqiqot obyekti. $m - sh$ funksiyalar, qat'iy $m - sh$ funksiyalar.

1.5.Tadqiqot predmeti. $(dd^c u)^n$ – Monje –Ampere operatori hamda gessianlar, oqimlar to'g'risidagi masalalar.

1.6.Tadqiqot usullari. Ko'p kompleks o'zgaruvchining funksiyalar nazariyasi, kompleks potentsiallar nazariyasi va chiziqli algebra usullari.

1.7.Tadqiqot natijalarining ilmiy jihatdan yangilik darajasi. Mazkur ishning asosiy natijalari yangi.

1.8.Tadqiqot natijalarining amaliy ahamiyati va tadbiqu. Olingan natijalar va usullar funksiyalar nazariyasining keyingi rivojlanishida va kompleks analizning tadbirlarida qo'llanilishi mumkin.

1.9.Ish tuzilishi va tarkibi. Dissertatsiya ishi kirish, uchta bob hamda adabiyotlar ro'yhatidan iborat. Har bir bob bo'limlardan tuzilgan.

2. Ishning mazmuni

Dissertatsiya ishi kirish, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yhatidan iborat. Birinchi bob uchta paragrafdan iborat bo'lib, unda garmonik, subgarmonik va plyurisubgarmonik funksiyalar haqida zarur ma'lumotlar hamda ularning xossalari keltirilgan.

Ikkinchi bob ikki paragrafdan iborat. Bu bobda musbat aniqlangan differensial forma va oqimlar hamda musbat gessianlar bilan differensial formalar orasidagi bog'lanishlar o'rganilgan.

Ikki marta silliq ixtiyoriy $u \in C^2(D)$ funksiya uchun uning 2-tartibli differensial

$$dd^c u = \frac{i}{2} \sum_{j,k} u_{j,k} dz_j \wedge d\bar{z}_k \quad (1)$$

(tayinlangan $z^0 \in D$ nuqtada) ermit kvadratlik formasidan iborat bo`ladi.

Bu yerda odatdagidek, $d = \partial + \bar{\partial}$, $d^c = \frac{\partial - \bar{\partial}}{4i}$ bo`lib,

$$du = \partial u + \bar{\partial} u = \sum_{k=1}^n \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^n \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_k$$

$$d^c u = \frac{1}{4i} \left(\sum_{k=1}^n \frac{\partial u}{\partial z_k} dz_k - \sum_{k=1}^n \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_k \right).$$

(1) tenglik unitar ([4] ga qarang) almashtirishlar yordamida quyidagi diagonal formaga keltiriladi:

$$dd^c u = \frac{i}{2} \left(\lambda_1 dz_1 \wedge d\bar{z}_1 + \dots + \lambda_n dz_n \wedge d\bar{z}_n \right) \quad (2)$$

Bunda $\lambda_1, \dots, \lambda_n$ lar ermit $(u_{j,k})$ matrisasining xos qiymatlari, bo`lib,

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ haqiqiy qiymatlardan iborat bo`ladi.

1-ta'rif. Ikki marta silliq $u \in C^2(D)$ ($D \subset \mathbb{C}^n$) funksiya uchun (tayinlangan $z^0 \in D$ nuqtada)

$$(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} \geq 0, \quad \forall k = 1, 2, \dots, m \quad (3)$$

tengsizlik bajarilsa, u holda bu funksiya D sohada m -sh ($1 \leq m \leq n$) funksiya deyiladi.

1-teorema. $D \subset \mathbb{C}^n$ sohada $n - sh$ bo'lgan ixtiyoriy funksiya psh dir.

2-ta'rif. Agar $u \in m - sh(D) \cap C^2(D), (D \subset \mathbb{C}^n)$ funksiya uchun $\varepsilon > 0$ son topilib, $u - \varepsilon \|z\|^2$ funksiya ham D sohada $m - sh$ funksiya bo'lsa, u holda $u(z)$ funksiya D sohada qat'iy $m - sh$ funksiya deyiladi,

Biz ikki marta silliq $u \in C^2(D)$ funksiya uchun qat'iy $m - sh$ funksiya ta'rifini boshqacha ko'rinishda ham ifodalashimiz mumkin:

3-ta'rif. Agar $u \in C^2(D), (D \subset \mathbb{C}^n)$ funksiya uchun ushbu

$$(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} > 0, \quad \forall k = 1, 2, \dots, m \quad (4)$$

qat'iy tengsizliklar o'rinli bo'lsa, u holda bu funksiya qat'iy $m - sh$ funksiya deyiladi. Bu ikki ta'rif bir-biriga ekvivalent.

1-lemma. Agar

$$\begin{cases} H_1(\lambda) = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n \geq 0 \\ H_2(\lambda) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \dots + \lambda_{n-1} \lambda_n \geq 0 \\ H_3(\lambda) = \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \dots + \lambda_{n-2} \lambda_{n-1} \lambda_n \geq 0 \\ \dots \\ H_{m-1}(\lambda) = \lambda_1 \dots \lambda_{m-1} + \dots + \lambda_{n-m+2} \dots \lambda_n \geq 0 \\ H_m(\lambda) = \lambda_1 \dots \lambda_m + \dots + \lambda_{n-m+1} \dots \lambda_n > 0. \end{cases} \quad (5)$$

bo'lsa, u holda $H_{m-1} > 0$ bo'ladi.

Yuqoridagi lemma asosida quyidagi teoremani ham isbotlash mumkin:

2-teorema. Agar $u \in m - sh(D) \cap C^2(D)$ bo'lib, (3) tengsizliklarda

$k = m$ uchun $(dd^c u)^m \wedge (dd^c |z|^2)^{n-m} > 0$ bo'lsa, u holda qolgan barcha

$k = \overline{1, m-1}$ lar uchun ham $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} > 0$ bo`ladi, ya'ni

$u \in m-sh(D) \cap C^2(D)$ funksiyaning qat'iy $m-sh$ bo`lishligi uchun

$(dd^c u)^m \wedge (dd^c |z|^2)^{n-m} > 0$ ning bajarilishi zarur va yetarlidir.

Muallif o'z ilmiy rahbari akademik A.S.Sadullayevga masalani qo'yishda, qo'yilgan masalalarni hal etishda ko`rsatgan doimiy e'tibori uchun chuqur minnatdorchiligini izhor etadi. Shuningdek, Fizika-matematika fanlari nomzodi dotsent B.I.Abdullayev va A.A.Atamuratovlarga hamda kafedraning barcha o'qituvchilariga dissertatsiya ishini tayyorlashda bergan qimmatli maslahatlari hamda dissertatsiya ishi masalalarini muhokama qilishda ko`rsatgan yordamlari uchun o'zining minnatdorchiligini bildiradi.

I BOB. FUNKSIYANING BA'ZI MUHIM SINFLARI

Subgarmonik va plyurisubgarmonik funksiyalar sinflari potentsiallar nazariyasining asosiy obyektlaridan bo'lib, bu nazariyaning barcha metodlari asosida ushbu tushunchalar yotadi. Masalan, Laplas tenglamasi uchun Dirixle masalasi, kompleks Monje - Ampere tenglamasi uchun Dirixle masalasi, ko'p argumentli golomorf funksiyalarni analitik davom ettirish masalalari, ko'phad va ratsional funksiyalar bilan yaqinlashtirishlar masalalarida ekstremal plyurisubgarmonik funksiyalar muhim ro'l o'ynaydi. ([2],[6],[7],[8],[13],[16] ga qarang)

1.1.Garmonik funksiyalar.

$D \subset \mathbf{R}^n$ sohada ushbu

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

tenglamani qanoatlantiruvchi $u \in C^2(D)$ funksiya *garmonik funksiya* deyiladi, bunda

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

laplas operatori.

Barcha garmonik funksiyalar to'plami $H(D)$ kabi belgilanadi.

1.1.1 – teorema. Agar $u(x)$ garmonik funksiya bo'lsa, u holda ixtiyoriy $x^0 \in D$ nuqta va $S(x^0, r) \subset D$ sfera uchun

$$u(x^o) = \frac{1}{\sigma_n r^{n-1}} \int_{S(x^o, r)} u(x) d\sigma \quad (1.1.1)$$

bo`ladi, bunda σ_n - birlik sferaning yuzi.

Aksincha, agar D dan olingan har bir $x^o \in D$ nuqta va yetarlicha kichik r ($0 < r \leq r_0(x^o)$) lar uchun (1.1.1) formula o`rinli bo`lsa, u holda $u(x) \in H(D)$ bo`ladi.

Endi garmonik funksiyalarning ba`zi bir xossalarini keltiramiz.

a) Agar $u, v \in H(D)$ bo`lib,  $\lambda, \mu \in R$  bo`lsa, u holda

$$\lambda u + \mu v \in H(D)$$

bo`ladi;

b) agar umumlashgan funksiya f uchun $\Delta f = 0$, ya`ni ixtiyoriy  $\varphi \in F(D)$  da  $f(\Delta\varphi) = 0$  tenglik bajarilsa, u holda  $f$  garmonik funksiya bo`ladi, aniqroq qilib aytganda, shunday $v(x) \in C^2(D)$, $\Delta v = 0$ bo`lgan funksiya mavjudki,

$$f(\varphi) = v(\varphi) = \int v(x) \varphi(x) dV$$

bo`ladi;

v) agar $u(x)$ funksiya $B(x_o, r)$ sharda garmonik, uning yopig`ida esa uzluksiz, ya`ni

$$u(x) \in H(B(x_o, r)) \cap \overline{C(B(x_o, r))}$$

bo`lsa, u holda

$$u(x) = \int_{S(x^o, r)} u(y) P(x, y) d\sigma(y)$$

formula o`rinli bo`ladi, bunda

$$P(x, y) = \frac{r^2 - |x - x_o|^2}{\sigma_n r |x - y|^n}, \quad n \geq 2, \quad ,$$

Puasson yadrosi.

Ikkinchi tomondan, agar $\varphi(y)$ funksiya $S(x^o, r)$ sferada uzluksiz bo'lsa, u holda

$$u(x) = \int_{S(x^o, r)} \varphi(y) P(x, y) d\sigma(y)$$

funksiya $B(x_o, r)$ sharda Dirixle masalasi

$$\Delta u = 0, \quad u|_{S(x^o, r)} = \varphi(y)$$

ning yechimi bo'ladi.

Dirixle masalasini chegarasi silliq bo'lgan ixtiyoriy $D \subset \mathbf{R}^n$ sohada ham qarash mumkin: $\varphi(y) \in C(\partial D)$ uchun yagona $u(x) \in C(\bar{D})$ mavjudki, $\Delta u = 0$, $u|_{\partial D} = \varphi(y)$ bajariladi;

g) kompleks tekislik $\mathbf{C} \approx \mathbf{R}^2$ da garmonik funksiyalar golomorf funksiyalarga bevosita bog'langandir.

Agar $z = x + iy$ deyilsa, unda

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

bo'ladi va har bir golomorf $f \in O(D)$ funksiya uchun

$$\operatorname{Re} f = \frac{f + \bar{f}}{2}, \quad \operatorname{Im} f = \frac{f - \bar{f}}{2i}$$

tengliklardan

$$\Delta \operatorname{Re} f = 0, \quad \Delta \operatorname{Im} f = 0,$$

ya'ni $\operatorname{Re} f$ va $\operatorname{Im} f$ larning garmonik funksiyalar ekanligi kelib chiqadi.

Shuningdek, agar $u(z) \in H(D)$ bo'lsa, u holda har bir $z^0 \in D$ nuqta va uning atrofi $B(z^0, r) \subset D$ uchun shunday $f \in O(D)$ funksiya topiladiki, $\operatorname{Re} f(z) = u(z)$, $z \in B$ bo'ladi.

Haqiqatan ham, $z^0 = 0$ deb $B = B(0, r)$ doirada Puasson formulasini ushbu

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{r^2 - |z|^2}{|\xi - z|^2} dt = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \operatorname{Re} \frac{\xi + z}{\xi - z} dt,$$

bunda $\xi = re^{it}$, ko'rinishda ifodalaymiz. Bu formuladan $B(0, r)$ da golomorf

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} dt \quad (1.1.2)$$

funksiyasi uchun $\operatorname{Re} f(z) = u(z)$ ekanligi kelib chiqadi.

(1.1.2) formula golomorf funksiyaning uning haqiqiy qismi orqali ifodalashda ham ishlatiladi: ixtiyoriy $f(z) \in O(B) \cap C(\overline{B})$ uchun ushbu

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} dt + i \operatorname{Im} f(0)$$

formula o'rinlidir.

1.2. Subgarmonik funksiyalar

Faraz qilaylik, $D \subset \mathbf{R}^n$ sohada aniqlangan lokal integrallanuvchi

$$u: D \rightarrow [-\infty, +\infty)$$

funksiya berilgan bo'lsin, $u \in L^1_{loc}(D)$.

Agar bu funksiya quyidagi ikki shartni bajarsa:

1) $u(x)$ yuqoridan yarim uzluksiz:

ya'ni $\forall x^o \in D$ uchun $\overline{\lim}_{x \rightarrow x^o} u(x) \leq u(x^o)$

(bundan $u(x)$ funksiyaning D ga tegishli ixtiyoriy kompaktda yuqoridan chegaralanganligi kelib chiqadi);

2) har bir $x^o \in D$ nuqta uchun shunday $r(x^o) > 0$ topiladiki, $r \leq r(x^o)$ uchun

$$u(x^o) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(x^o, r)} u(x) d\sigma ; \quad (1.2.1)$$

u holda $u(x)$ funksiya D sohada *subgarmonik funksiya* deyiladi.

Subgarmonik funksiyalar sinfi $Sh(D)$ kabi belgilanadi. Kelgusida qulaylik uchun biz trivial $u(x) \equiv -\infty$ funksiyasi ham D da $Sh(D)$ ga tegishli deb qaraymiz. Endi subgarmonik funksiyalarning xossalari keltiramiz.

a) subgarmonik funksiyalarning musbat koeffisientli chiziqli kombinativasi subgarmonik funksiya bo`ladi:

$$\begin{aligned} u_k(x) \in Sh(D), \quad a_k \in R_+ \quad (k=1,2,\dots,m) &\Rightarrow \\ \Rightarrow a_1 u_1(x) + a_2 u_2(x) + \dots + a_m u_m(x) &\in Sh(D) ; \end{aligned}$$

b) chekli sondagi subgarmonik funksiyalarning maksimumi subgarmonik funksiya bo`ladi:

$$\begin{aligned} u_1(x), u_2(x), \dots, u_m(x) \in Sh(D) &\Rightarrow \\ \Rightarrow \max\{u_1(x), u_2(x), \dots, u_m(x)\} &\in Sh(D) ; \end{aligned}$$

v) monoton kamayuvchi subgarmonik funksiyalar ketma – ketligining limiti subgarmonik funksiya bo`ladi:

$$\begin{aligned} u_j(x) \in Sh(D) \quad , \quad u_j(x) \geq u_{j+1}(x) \quad (j=1,2,\dots) &\Rightarrow \\ \Rightarrow \lim_{j \rightarrow \infty} u_j(x) &\in Sh(D) ; \end{aligned}$$

g) tekis yaqinlashuvchi subgarmonik funksiyalar ketma – ketligi subgarmonik

funksiyaga yaqinlashadi:

$$u_j(x) \in Sh(D) \quad (k=1,2,3,\dots) \quad , \quad u_j(x) \rightrightarrows u(x) \implies u(x) \in Sh(D) .$$

a) – g) xossalarning isbot bevosita yuqorida keltirilgan ta’rifdan kelib chiqadi.

d) *Maksimumlar prinsipi*. Agar $u(x) \in Sh(D)$ funksiya biror $x^o \in D$ nuqtada o’zining maksimumiga erishsa, ya’ni

$$u(x^o) = \sup_{x \in D} u(x) \quad (1.2.2)$$

bo’lsa, u holda $u(x) \equiv const$ bo’ladi.

Ushbu

$$M = \{x \in D: u(x) = u(x^o)\}$$

to’plamni qaraylik. $u(x)$ funksiyaning yuqoridan yarim uzluksiz bo’lganligidan va (1.2.2) shartdan M to’plamning D da yopiqligi kelib chiqadi.

Ikkinchi tomondan, ixtiyoriy $p \in M$ uchun (1.2.1) formulaga ko’ra

$$u(x^o) = u(p) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(p,r)} u(x) d\sigma \leq u(x^o) , \quad r \leq r(p) ,$$

bo’ladi. Bu munosabatdan $u|_{S(p,r)} \equiv u(x^o)$, ya’ni p ning $B(p, r(p))$ atrofida $u(x) = u(x^o)$ bo’lishini ko’ramiz. Bu esa M to’plamning D da ochiq ekanligini ko’rsatadi. Demak, $M = D$.

e) Agar $\vartheta(x) \in Sh(D)$, $u(x) \in H(D)$ funksiyalar uchun $S(x^o, r) \subset\subset D$ sferada

$$\vartheta|_S \leq u|_S$$

bo’lsa, u holda $B(x^o, r)$ sharda $\vartheta(x) \leq u(x)$ tengsizlik o’rinli bo’ladi.

Bu xossaning isboti yuqoridagi d) xossadan kelib chiqadi;

e) – xossadan quyidagi muxim xulosaga kelamiz. Faraz qilaylik,

$$\mathcal{G}(x) \in Sh(D) \cap C(D)$$

berilgan bo`lib,  $B(x^o, r) \subset\subset D$  bo`lsin. Ushbu

$$u(x) = \int_{S(x^o, r)} \mathcal{G}(y) P(x, y) d\sigma(y), \quad x \in B(x^o, r),$$

Puasson integralini qaraylik. $u(x) \in h(B)$, $u|_S = \mathcal{G}|_S$ bo`lib, e) – xossaga ko`ra $B = B(x^o, r)$ da $\mathcal{G}(x) \leq u(x)$ bo`ladi. Uzluksiz funksiyalar uchun isbotlangan bu xossa ixtiyoriy  $\mathcal{G} \in Sh(D)$  uchun ham o`rinlidir: $\mathcal{G}(x) \leq u(x)$ va deyarli barcha $x \in S$ lar uchun $\mathcal{G}(x) = u(x)$ o`rinlidir;

j) aytaylik, $u(x) \in Sh(D)$ funksiya va $x^o \in D$ nuqta berilgan bo`lsin.

Ushbu

$$m(x^o, r) = \frac{1}{\sigma_n r^{n-1}} \int_{S(x^o, r)} u(x) d\sigma$$

va

$$n(x^o, r) = \frac{1}{V_n r^n} \int_{B(x^o, r)} u(x) dV,$$

bunda $V_n r^n$ miqdor $B(x^o, r)$ ning hajmi, integrallarni (o`rta qiymatlarni) qaraylik.

1.2.1 – teorema. Faraz qilaylik, $u(x) \in Sh(D)$ va $u(x) \not\equiv -\infty$ bo`lsin. U holda ixtiyoriy $x^o \in D$ uchun

$$u(x^o) \leq n(x^o, r) \leq m(x^o, r), \quad n(x^o, r) > -\infty, \quad 0 < r < \rho(x^o, \partial D),$$

tengsizliklar o'rinli bo'ladi. Bundan tashqari, agar r monoton kamayib nolga intilsa, unda $n(x^o, r)$ va $m(x^o, r)$ o'rta qiymatlar monoton kamayib, $u(x^o)$ ga intiladi.

1.2.1 – teoremadan ushbu muhim natijaga ega bo'lamiz: $D \subset \mathbf{R}^n$ sohada subgarmonik $u(x) \neq -\infty$ funksiya D da lokal integrallanuvchidir, ya'ni ixtiyoriy

$B \subset\subset D$ uchun $\int_B u(x) dV$ integral mavjuddir;

z) subgarmonik funksiya D sohaning har bir nuqtasida uzilishga ega bo'lishi mumkin. Ayni paytda, quyidagi teorema o'rinli bo'ladi.

1.2.2 – t e o r e m a. Agar $u(x) \in Sh(D)$ bo'lsa, u holda shunday monoton to'plamlar

$$D_j \subset D_{j+1} \subset D, \quad \bigcup_{j=1}^{\infty} D_j = D$$

ketma – ketligi va shunday monoton kamayuvchi funksiyalar

$$u_j(x) \in Sh(D_j) \cap C^\infty(D_j), \quad j = 1, 2, 3, \dots$$

ketma – ketligi mavjudki,

$$u(x) = \lim_{j \rightarrow \infty} u_j(x) \quad (x \in D)$$

bo'ladi.

Isboti. Ushbu

$$K(x) = \begin{cases} ce^{-\frac{1}{1-|x|}}, & \text{agar } |x| \leq 1 \text{ bo'lsa,} \\ 0, & \text{agar } |x| > 1 \text{ bo'lsa.} \end{cases}$$

funksiyani qaraylik. Bu funksiya $C^\infty(\mathbf{R}^n)$ sinfga tegishli bo'lib, uning salmog'i $\text{supp}K(x) = B(0,1)$ dir. Endi $K(x)$ ning ifodasidagi o'zgarmas c ni shunday tanlab olamizki,

$$\int_{\mathbf{R}^n} K(x) dV = 1$$

bo'lsin. $u(x) \neq -\infty$ deb, bu yadro yordamida ushbu

$$u_\delta(x) = \frac{1}{\delta^n} \int_{|y-x| \leq \delta} u(y) K\left(\frac{y-x}{\delta}\right) dV, \quad \delta > 0, \quad (1.2.3)$$

integralni qaraylik. $u_\delta(x)$ funksiyasi

$$D_\delta = \{x \in D : \rho(x, \partial D) < \delta\}$$

ochiq to'plamda aniqlangan bo'lib, $x \in D_\delta$ da

$$u_\delta(x) = \frac{1}{\delta^n} \int_{\mathbf{R}^n} u(y) K\left(\frac{y-x}{\delta}\right) dV = \int_{\mathbf{R}^n} u(x + \delta y) K(y) dV$$

bo'ladi. Keyingi tenglikdagi birinchi integral $C^\infty(D_\delta)$ sinfga, ikkinchi integral esa $Sh(D_\delta)$ sinfga tegishli bo'ladi. Binobarin,

$$u_\delta(x) \in Sh(D_\delta) \cap C^\infty(D_\delta).$$

$u(x)$ ning subgarmonik funksiya ekanligidan, u_δ ning $\delta \downarrow 0$ da, monoton kamayuvchi bo'lishligi va

$$\int_{\mathbf{R}^n} K(x) dV = 1$$

tenglikdan, $u_\delta(x)$ ning $u(x)$ ga intilishini topamiz. Teorema isbotlandi.

i) Subgarmonik funksiyalar umumiy holda C^2 sinfga tegishli bo'lganligidan, Δu - Laplas operatorini faqat umumlashgan funksiya sifatida qarash mumkin bo'ladi:

$$\Delta u(\varphi) = \int u \Delta \varphi, \quad \varphi \in F(D).$$

1.2.3-teorema. Har qanday subgarmonik funksiya $u(x) \in Sh(D)$, $u \neq -\infty$, uchun umumlashgan ma'noda $\Delta u \geq 0$ bo'ladi, ya'ni ixtiyoriy $\varphi \in F(D)$, $\varphi \geq 0$ uchun $\int u \Delta \varphi \geq 0$ munosabat o'rinlidir.

Jumladan, $u(x)$ funksiya C^2 sinfga tegishli bo'lib, u subgarmonik bo'lsa,

$$u \in Sh(D) \cap C^2(D), \text{ u holda } \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} \geq 0 \text{ dir.}$$

Yuqorida keltirilgan teoremaning aksi ham o'rinli: agar umumlashgan funksiya u uchun

$$\Delta u(\varphi) = \int u \Delta \varphi \geq 0, \quad \varphi \in F(D), \quad \varphi \geq 0,$$

munosabat bajarilsa, u holda u subgarmonik funksiya bo'ladi;

1.3. Plyurisubgarmonik funksiyalar

1.3.1. Plyurigarmonik funksiyalar.

$\mathbf{C}^n$ kompleks fazodagi $D \subset \mathbf{C}^n$ sohada $u(z)$ funksiya berilgan bo'lib, $u(z) \in C^2(D)$ bo'lsin.

Agar ixtiyoriy $z^0 \in D$ va bu nuqtadan o'tuvchi ixtiyoriy kompleks to'g'ri

chiziq $l: z = z^0 + w\xi$, $w \in \mathbb{C}^n$, $\xi \in \mathbb{C}$ uchun ushbu $\varphi(\xi) = u/l = u(z^0 + w\xi)$ kesim – funksiya $\xi = 0$ nuqtada garmonik, ya'ni $\xi = 0$ da $\frac{\partial^2 \varphi}{\partial \xi \partial \bar{\xi}} = 0$ bo'lsa, $u(z)$ funksiya D da *plyurigarmonik funksiya* deyiladi.

Plyurigarmonik funksiyalar sinfi $Ph(D)$ kabi belgilanadi. Aytaylik, $u(z) \in Ph(D)$ bo'lsin. Unda

$$\frac{\partial^2 u(z^0 + w\xi)}{\partial \xi \partial \bar{\xi}} = 0$$

bo'lib, murakkab funksiyaning differensiallash qoidasidan

$$\sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k = 0, \quad w \in \mathbb{C}^n, \text{ ekanligini ko'ramiz. Bundan va } w \in \mathbb{C}^n \text{ vektorining}$$

ixtiyoriyligidan

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0 \quad (k, j = 1, 2, \dots, n) \quad (1.3.1)$$

bo'lishi kelib chiqadi.

Demak, plyurigarmonik funksiyalar sinfi (1.3.1) tenglamalar sistemasi bilan aniqlanadi va ba'zan bu sistema plyurigarmonik funksiyaning ta'rifi sifatida ham qaraladi.

1.3.1 –teorema. Agar $f(z) = u(z) + iv(z)$ funksiya $D \subset \mathbb{C}^n$ sohada golomorf bo'lsa, u holda

$$\operatorname{Re} f(z) = u(z), \quad \operatorname{Im} f(z) = v(z)$$

funksiyalar D sohada plyurigarmonik bo'ladi va, aksincha, agar $u(z)$ funksiya D sohada plyurigarmonik bo'lsa, u holda ixtiyoriy $B(z^0, r) \subset D$

atrofda $u(z)$ funksiya biror $f(z) \in O(B(z^0, r))$ holomorff funksiyaning haqiqiy qismi bo`ladi: $u(z) \equiv \operatorname{Re} f(z)$.

1.3.2. Plyurisubgarmonik funksiyalar.

Agar $D \subset \mathbb{C}^n$ sohada aniqlangan $u(z): D \rightarrow [-\infty, \infty)$ funksiya quyidagi ikki shartni bajarsa:

- 1) $u(z)$ yuqoridagi yarim uzluksiz;
- 2) ixtiyoriy l kompleks to'g'ri chiziq uchun $u|_l$ funksiya $l \cap D$ da subgarmonik bo`lsa, $u(z)$ funksiya D sohada *plyurisubgarmonik* funksiya deyiladi.

Plyurisubgarmonik funksiyalar to'plami $Psh(D)$ kabi belgilanadi. $D \subset \mathbb{C}^n$ sohada plyurisubgarmonik bo`lgan funksiya  $D \subset \mathbb{R}^{2n}$  da aniqlangan funksiya sifatida subgarmonik funksiya bo`ladi. Uni subgarmonik funksiyalarning 2 – shartini bajarishini ko`rish qiyin emas. Binobarin, subgarmonik funksiyalarning xossalari plyurisubgarmonik funksiyalar uchun ham saqlanadi.

Endi plyurisubgarmonik funksiyalarning xossalarini keltiramiz:

- a) agar $u(z) \in Psh(D)$ bo`lsa, u holda  $u_\delta(z)$  funksiya  $Psh(D_\delta) \cap C^\infty(D_\delta)$  sinfga tegishli bo`lib, $\delta \downarrow 0$ da $u_\delta(z) \downarrow u(z)$ bo`ladi;
- b) agar $u(z) \in Psh(D)$ funksiya $z^0 \in D$ nuqtada maksimumga erishsa, ya'ni $u(z^0) \geq u(z)$ ($z \in D$) bo`lsa, u holda  $u(z) \equiv \operatorname{const}$  bo`ladi;
- v) plyurisubgarmonik funksiyalarning musbat koeffitientli chiziqli

kombinativasi plyurisubgarmonik funksiya bo`ladi;

g) monoton kamayuvchi yoki tekis yaqinlashuvchi plyurisubgarmonik funksiyalar ketma – ketligining limiti plyurisubgarmonik funksiya bo`ladi;

d) D sohada plyurisubgarmonik funksiyalar sinfi $\{u_\alpha\}_{\alpha \in \Lambda}$ berilgan bo`lib,  $u = \sup_{\alpha \in \Lambda} u_\alpha$  yuqoridan yarim uzluksiz funksiya bo`lsin. U holda $u(z)$ plyurisubgarmonik funksiya bo`ladi,  $u \in Psh(D)$ . Jumladan,  $u_j(z) \in Psh(D)$  ( $j = 1, 2, \dots, k$ ) bo`lsa,

$$\sup\{u_1(z), u_2(z), \dots, u_k(z)\} \in Psh(D)$$

bo`ladi;

e) agar $f(z) \in O(D)$ bo`lsa,  $\alpha \cdot \ln|f(z)| \in Psh(D)$  bo`ladi, bunda $\alpha \geq 0$;

j) agar $u(z) \in Psh(D)$ bo`lsa, u holda  $e^{u(z)} \in Psh(D)$  dir, agar  $u(z) \in Psh(D)$ ,  $u|_D < 0$  bo`lsa, u holda $-\ln[-u(z)] \in Psh(D)$ dir.

Bu xossaning isboti subgarmonik funksiyalarning shu xossasidan bevosita kelib chiqadi;

z) D sohada aniqlangan $u(z) \in C^2(D)$ funksiyaning D da plyurisubgarmonik bo`lishi uchun ushbu

$$L(u, w) = \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \quad (1.3.2)$$

kvadratik formaning (Levi formasining) ixtiyoriy $z \in D$ da musbat aniqlangan bo`lishi zarur va yetarlidir. (1.3.2) munosabatda

$w = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n$ bo`lib, kvadratik formaning musbat aniqlanganligi $\forall w \in \mathbb{C}^n$

da $L(u, w) \geq 0$ bo'lishini anglatadi.

Agar $\forall w \in \mathbf{C}^n, w \neq 0$ da $L(u, w) > 0$ bo'lsa, $u(z)$ funksiya z nuqtada *qat'iy plyurisubgarmonik* funksiya deyiladi; agar $u(z)$ funksiya D sohaning har bir nuqtasida qat'iy plyurisubgarmonik bo'lsa, $u(z)$ *sohada qat'iy plyurisubgarmonik* funksiya deyiladi.

II BOB.MUSBAT GESSIANLAR BILAN DIFFERENSIAL FORMALAR ORASIDAGI BOG`LANISH

2.1.Musbat aniqlangan differensial forma va oqimlar.

2.1.1.Differensial formalar. $f \in C^1(G)$, $G \subset \mathbb{C}^n$ funksiyaning differensial

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$$

1-darajali differensial formaga misol bo`ladi. Umuman aytganda,

$\omega = \sum_{j=1}^n a_j(x) dx_j$ ko`rinishdagi ifodaga **1-darajali differensial forma** deyiladi va

$\deg \omega = 1$ kabi belgilanadi. Yuqori darajali differensial formalarni ta`riflash

uchun biz quyidagi shartlarni qanoatlantiruvchi $\ll \wedge \gg$ —tashqi ko`paytma

belgisini kiritamiz:

a) $a(x) \wedge dx_j = dx_j \wedge a(x) = a(x) dx_j$

b) $dx_k \wedge dx_j = -dx_j \wedge dx_k, \Rightarrow dx_j \wedge dx_j = 0$

2.1.1-ta`rif. Ushbu

$$\omega = \sum_{0 \leq j_1 < j_2 < \dots < j_p \leq n} a_{j_1 j_2 \dots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_p}$$

ko`rinishdagi ifodaga p-darajali differensial forma (bunda $a_{j_1 j_2 \dots j_p} - G$ sohada

aniqlangan funksiya) deyiladi va $\deg \omega = p$ kabi belgilanadi. p-darajali

differensial formalar to`plami $F^p(G)$ orqali belgilanadi. $\omega \in F^p(G)$ differensial

formani quyidagi qulay ko`rinishda ham yozishimiz mumkin:

$$\omega = \sum_{0 \leq j_1 < j_2 < \dots < j_p \leq n} a_{j_1 j_2 \dots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_p} = \sum_J a_J dx_J$$

bunda $dx_J = dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_p}$, $J = (j_1, j_2, \dots, j_p)$, $1 \leq j_1, j_2, \dots, j_p \leq n$ —

multiindekslar. Ravshanki, 0-darajali differensial forma bu- $a(x)$ skalyar funksiya.

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$ kompleks fazoda differensial formani dz_j va $d\bar{z}_j$ lardan foydalanib yozish mumkin. $z = (z_1, z_2, \dots, z_n)$ koordinatalar

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_j = x_j + ix_{n+j}$, $j = 1, 2, \dots$ dagi kompleks koordinatalar bo'lsin. Unda

$$dx_j = \frac{dz_j + d\bar{z}_j}{2}, \quad dx_{n+j} = \frac{dz_j - d\bar{z}_j}{2i}$$

bo'lganligidan $\omega = \sum_J a_J dx_J$ differensial formani dz_j va $d\bar{z}_k$ larning chiziqli

kombinatsiyasi ko'rinishida yozish mumkin. Har bir hadda dz_j va $d\bar{z}_k$ lar

turlicha bo'lishi mumkin. Masalan, $\mathbb{C}^2 \simeq \mathbb{R}^4$ da

$$dx_1 \wedge dx_2 \wedge dx_3 = \frac{i}{4} (-dz_1 \wedge dz_2 \wedge d\bar{z}_1 + dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2)$$

Ammo ba'zi hollarda ω dagi barcha qo'shiluvchilarda dz_j larning darajasi p ga, $d\bar{z}_k$ larniki esa q ga teng bo'lib qolishi mumkin. Bunday hollarda biz ω differensial formani (p, q) bidarajali differensial forma deb aytaamiz va $\omega \in F^{p,q}$ deb belgilaymiz.

2.1.2. Musbat aniqlangan differensial formalar.

2.1.2-ta'rif. Ushbu

$$\omega = \bigwedge_{j=1}^p \frac{i}{2} dl_j \wedge d\bar{l}_j$$

ko`rinishdagi differensial forma (p, p) bidarajali bosh kuchli musbat differensial forma deyiladi, bu yerda $l_j = a_{j_1} dz_1 + \dots + a_{j_n} dz_n$ - chiziqli kombinatsiya $(dz_1, \dots, dz_n)$ bazis bo'yicha $a_{jk} \in \mathbb{C}$, $j = 1, 2, \dots, p, k = 1, 2, \dots, n$. Bunday formalarning ixtiyoriy chiziqli kombinatsiyasi kuchli musbat differensial forma deb ataladi, ya'ni kuchli musbat differensial forma bu-

$$\omega = \sum_s \lambda_s(z) \wedge_{j=1}^p \frac{i}{2} dl_{j_s} \wedge d\bar{l}_{j_k}$$

ko`rinishdagi formadir, bu yerda  $\lambda_s(z)$  - qaralayotgan  $G \subset \mathbb{C}^n$  sohadagi manfiy bo`lmagan funksiya.

Agar $z_j = x_j + ix_{n+j}$ kompleks koordinatalardan x_j, x_{n+j} , $j = 1, 2, \dots, n$ haqiqiy koordinatalarga o'tsak, unda ushbu

$$\frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_p \wedge d\bar{z}_p = dx_1 \wedge dx_{n+1} \wedge \dots \wedge dx_p \wedge dx_{n+p}$$

ifoda $\mathbb{R}^{2n}$ fazoda $2p$ darajali musbat forma bo`ladi.

(n, n) bidarajali differensial forma (kuchli) musbat bo`ladi, faqat va faqat shu holdaki, qachonki u

$$\omega = \lambda(z) \wedge_{j=1}^n \frac{i}{2} dz_j \wedge d\bar{z}_j = \lambda(z) dx_1 \wedge dx_{n+1} \wedge \dots \wedge dx_n \wedge dx_{2n}$$

ko`rinishga ega bo`lsa, bu yerda $\lambda(z)$ - musbat funksiya.

(n, n) bidarajali $\Omega = \wedge_{j=1}^n \frac{i}{2} dz_j \wedge d\bar{z}_j = dV$ differensial forma $\mathbb{C}^n$ da hajm formasi rolini o'ynaydi.

$p=1$ da $\omega = \sum \lambda_{jk} dz_j \wedge d\bar{z}_k$ - differensial forma musbat bo`ladi, faqat va faqat

shu holdaki, qachonki $\sum_{jk} \lambda_{jk} \xi_j \xi_k$ – ermit kvadratik formasi musbat bo`lsa.

Shuningdek, ixtiyoriy $z^o \in G$ tayinlangan nuqtada unitar almashtirish yordamida uni

$$\omega = \frac{i}{2} \sum_{j=1}^n \mu_j dz_j \wedge d\bar{z}_j, \quad \mu_j \geq 0$$

ko`rinishda yozish mumkin.

2.1.3-ta'rif. $\omega \in F^{p,p}$ differensial forma kuchsiz musbat yoki musbat deyiladi, agar ixtiyoriy kuchli musbat $\nu \in F^{n-p,n-p}$ forma uchun $\omega \wedge \nu \in F^{n,n}$ forma musbat bo`lsa.

$p = 0, 1, n-1, n$ da kuchli musbatlik kuchsiz musbatlikka ekvivalent. Biroq boshqa p lar uchun bu sinflar ustma-ust tushmaydi. Buni aniq tushunish uchun kuchli musbat formani quyidagi ko`rinishda yozamiz:

$$\begin{aligned} \omega &= \sum_s \lambda_s(z) \wedge_{j=1}^p \frac{i}{2} dl_{j_s} \wedge d\bar{l}_{j_s} = \frac{i^{p^2}}{2^p} \sum_s \lambda_s(z) \wedge_{j=1}^p \frac{i}{2} dl_{j_s} \wedge_{j=1}^p d\bar{l}_{j_s} = \\ &= \frac{i^{p^2}}{2^p} \sum_s \lambda_s(z) dl_{j_s} \wedge d\bar{l}_{j_s} \end{aligned} \quad (*)$$

Agar $\theta = \sum_s \mu_s(z) dl_s$ formani qarasak, bunda μ_s – kompleks qiymatli

funksiya, unda $\frac{i^{p^2}}{2^p} \theta \wedge \bar{\theta}$ ko`rinishdagi forma (kuchsiz) musbat bo`ladi, biroq

$2 \leq p \leq n-2$ da u har doim ham $(*)$ ko`rinishga ega bo`lavermaydi.

2.1.3.Oqimlar. Qo`pol qilib aytganda,  $M$  ko`pxillikdagi oqim bu-

koefitsiyentlari umumlashgan funksiyalardan iborat differensial formadir.

Tayinlangan $G \subset \mathbb{R}^n$ soha uchun asosiy differensial formalar fazosi qaraladi:

$$F^p = F^p(G) = \{\omega \in F^p(G) \cap C^\infty(G) : \text{supp } \omega \subset\subset G\}$$

F^p da topologiya $F(G)$ fazodagi topologiyadan aniqlanadi.

2.1.4-ta'rif. F^p fazodagi uzluksiz chiziqli (kompleks qiymatli) T funksional $n - p = q$ darajali oqim deyiladi.

Barcha q – darajali oqimlar fazosini F^q orqali belgilaymiz. F^q oqimlar fazosi umumlashgan funksiyalarning barcha xossalarini qanoatlantiradi. Xususan, ular uchun kuchsiz yaqinlashish, kuchsiz chegaralanganlik tushunchalarini kiritish mumkin va $T_\delta \in F^q \cap C^\infty$

o 'ramani aniqlash mumkin, ya'ni $\delta \rightarrow 0$ da oqim sifatida

$$T_\delta \circ \omega \rightarrow T \circ \omega, \quad \forall \omega \in F^p$$

undan tashqari T oqim differensial formadek

$$\text{differensiallanadi: } dT \circ \omega = (-1)^p T \circ d\omega$$

2.1.5-ta'rif. Agar $F^p = \{\omega \in F^p(G) \cap C^\infty(G) : \text{supp } \omega \subset\subset G\}$ fazodan olingan T

oqim (funktional) $\{\omega \in F^p(G) \cap C^m(G) : \text{supp } \omega \subset\subset G\}$ fazoga uzluksiz davom

qilsa, unda T oqim m dan ko'p bo'lmagan singulyarlikka ega deyiladi, bunda

$$0 \leq m \leq \infty$$

Agar undan tashqari T oqim $\{\omega \in F^p(G) \cap C^{m-1}(G) : \text{supp } \omega \subset\subset G\}$ fazoga

davom qilmasa, unda T oqim aniq m singulyarlikka ega deyiladi.

$T = \sum_{|k|=q} \beta_k dx_k$ koefitsiyentlari umumlashgan funksiya bo`lgan differensial

forma oqim deyiladi.

2.1.1-teorema. Ixtiyoriy q – darajali oqim koefitsiyentlari umumlashgan funksiya bo`lgan differensial forma bo`ladi, shu bilan birga singulyarligi nolga teng bo`lgan oqim o`lchov tipidagi oqim bo`ladi va uni barcha koefitsiyentlari kompleks qiymatli o`lchov bo`ladi.

O`lchov tipidagi oqimlar fazosini  $F_0^q$  orqali belgilaymiz.  $F_0^q$  nafaqat  $G \subset \mathbb{R}^n$  soha uchun, balki ixtiyoriy  $E \subset \mathbb{R}^n$  borel to`plami uchun ham aniqlangan.

Xususan, $K \subset \mathbb{R}^n$ kompakt uchun u Banax fazosi bo`ladi, xuddi vektorlarni tekis normasi bilan differensial forma koefitsiyentlaridan tashkil topgan

$F_0^q(K) = F_0^q(K) \cap C(K)$ fazoga qo`shma fazo kabi.

$T \circ \omega = \mu(z) \Omega$ O`lchov tipidagi $T \in F_0^q$ oqim uchun

$$T \circ \omega = \int_G d\mu \text{ o`rinli.}$$

Bu yerda μ – kompleks qiymatli o`lchov bo`lib, u $\omega \in F_0^p$ ga bog`liq va

$$\Omega = dx_1 \wedge \dots \wedge dx_n$$

Xuddi o`lchovlar fazosi kabi quyidagi xossalar

o`rinli:

a) $\{T_j\} \subset F_0^q$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, ya`ni

$\lim_{j \rightarrow \infty} T_j \circ \omega = T \circ \omega, \forall \omega \in F_0^p$ faqat va faqat shu holdaki, qachon mos o`lchovlar

ketma-ketligini yaqinlashishi T_j ni koefitsiyentlaridan bo`lsa.

b) $\{T_j\} \subset F_0^q$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, faqat va faqat shu holdaki, qachon $T_j \left\| T_j \right\|_k \leq c, \forall K \subset\subset G, j=1,2,\dots$ norma bo'yicha chegaralangan bo'lsa va $j \rightarrow \infty$ da $T_j \circ \omega = T \circ \omega$ limit biror hamma yerda zich $E \subset F_0^p(K)$ to'plamdan olingan barcha ω lar uchun o'rinli bo'lsa, masalan barcha cheksiz differensiallanuvchi $\omega \in F^p \cap C^\infty(K)$ lar uchun d) ixtiyoriy kuchsiz chegaralangan $E \subset F_0^q(K)$ to'plam osti kuchsiz kompakt ya'ni uni ixtiyoriy ketma-ketligi kuchsiz yaqinlashuvchi qisman ketma-ketligini o'zida saqlaydi.

2.1.6-ta'rif. q – darajali differensial forma (oqim) yopiq deyiladi, agar uni differensial nolga teng bo'lsa;

U aniq deyiladi, agar u biror $q-1$ darajali formani (oqimni) differensial bo'lsa. Aniq formalar (oqimlar) yopiq bo'ladi, teskari tasdiq uchun G sohaga qo'shimcha geometrik shartlarni qo'yish zarur.

Endi $\mathbb{C}^n$ da asosiy differensial formalar fazosini qaraymiz:

$$F^{p,p} = F^{p,p}(G) \left\{ \omega \in F^{p,p}(G) \cap C^\infty(G) : \text{supp } \omega \subset\subset G \right\}$$

2.1.7-ta'rif. $F^{p,p}$ fazoda uzluksiz chiziqli (kompleks qiymatli) T funksionalga

$(n-p, n-p) = (q, q)$ bidarajali oqim deyiladi.

2.1.2-teorema. Musbat oqimlar o'lchov tipidagi oqimlar bo'ladi.

2.1.1-misol. Koeffitsiyentlari L'_{loc} sinfdan olingan ixtiyoriy $T \in F^{q,q}$ differensial forma (q, q) bidarajali oqim bo'ladi.

$T \circ \omega$ funksional $T \circ \omega = \int_G T \wedge \omega, \quad \forall \omega \in F^{p,p}$ kabi aniqlanadi. Bunday oqimlar regulyar oqim deyiladi.

2.1.2-misol. $A \subset \mathbb{C}^n - p$ o'lchamli analitik to'plam bo'lsin. U holda

$$[A] \circ \omega = \int_A \omega \text{ integral } (q, q) \text{ bidarajali oqimni aniqlaydi. Ko'rish qiyin}$$

emaski, $[A]$ musbat oqim bo'ladi.

2.2. Musbat gessianalar bilan differensial formalar orasidagi bog'lanish

2.2.1. Gessianlar. 2-marta silliq $u \in C^2(D)$ funksiya uchun uning ikkinchi tartibli differensial

$$dd^c u = \frac{i}{2} \sum_{j,k} u_{j\bar{k}} dz_j \wedge d\bar{z}_k \quad \left(\text{bunda } u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \quad (2.2.1)$$

(tayinlangan $z^0 \in D$ nuqtada) Ermit kvadratlik formasidan iborat bo'ladi.

$(dd^c u)^n$ operator Monje-Ampere operatori deyiladi. Bunda

$$d = \partial + \bar{\partial} \quad d^c = \frac{1}{4i}(\partial - \bar{\partial})$$

$$\begin{aligned} du &= \partial u + \bar{\partial} u = \sum_{k=1}^n \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^n \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_k \\ d^c u &= \frac{1}{4i} \left(\sum_{k=1}^n \frac{\partial u}{\partial z_k} dz_k - \sum_{k=1}^n \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_k \right). \end{aligned}$$

(2.2.1) tenglikni $n=1$ uchun ko'rib chiqamiz:

$$\begin{aligned} du &= \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z} \\ d^c u &= \frac{1}{4i} \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right) \end{aligned}$$

$$dd^c u = \frac{1}{4i} \left(\partial \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right) + \bar{\partial} \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right) = \right. \\ \left. \frac{1}{4i} \left(\frac{\partial^2 u}{\partial z^2} dz \wedge dz - \frac{\partial^2 u}{\partial z \partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial^2 u}{\partial \bar{z} \partial z} dz \wedge d\bar{z} - \frac{\partial^2 u}{\partial \bar{z}^2} d\bar{z} \wedge d\bar{z} \right) \right)$$

Tashqi ko`paytmada

$$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad d\bar{z} \wedge dz = -dz \wedge d\bar{z} \quad \text{bo`lishidan foydalansak,}$$

$$dd^c u = \frac{1}{2i} \frac{\partial^2 u}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$$

ifodaga ega bo`lamiz.

Demak, $u \in C^2(D), D \subset \mathbb{C}$ uchun $dd^c u = \frac{i}{2} \partial \bar{\partial} u$ bo`ladi.

(2.2.1) formulani $n=3$ uchun ham ko`rib chiqamiz.

$$du = \partial u + \bar{\partial} u = \frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2 + \frac{\partial u}{\partial z_3} dz_3 + \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2 + \frac{\partial u}{\partial \bar{z}_3} d\bar{z}_3 \\ d^c u = \frac{1}{4i} \left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2 + \frac{\partial u}{\partial z_3} dz_3 - \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 - \frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2 - \frac{\partial u}{\partial \bar{z}_3} d\bar{z}_3 \right) \\ dd^c u = \frac{1}{4i} \left[\partial \left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2 + \frac{\partial u}{\partial z_3} dz_3 - \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 - \frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2 - \frac{\partial u}{\partial \bar{z}_3} d\bar{z}_3 \right) + \right. \\ \left. + \bar{\partial} \left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2 + \frac{\partial u}{\partial z_3} dz_3 - \frac{\partial u}{\partial \bar{z}_1} d\bar{z}_1 - \frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2 - \frac{\partial u}{\partial \bar{z}_3} d\bar{z}_3 \right) \right] = \\ = \frac{1}{4i} \left[\left(\frac{\partial^2 u}{\partial z_2 \partial z_1} dz_1 \wedge dz_2 + \frac{\partial^2 u}{\partial z_3 \partial z_1} dz_1 \wedge dz_3 + \frac{\partial^2 u}{\partial z_1 \partial z_2} dz_2 \wedge dz_1 + \frac{\partial^2 u}{\partial z_3 \partial z_2} dz_2 \wedge dz_3 + \right. \right. \\ \left. \left. + \frac{\partial^2 u}{\partial z_1 \partial z_3} dz_3 \wedge dz_1 + \frac{\partial^2 u}{\partial z_2 \partial z_3} dz_3 \wedge dz_2 - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_1 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_2 - \right. \right. \\ \left. \left. - \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_3 - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_2 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_2 - \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_3 - \right. \right. \\ \left. \left. - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_3 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_3 - \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_3 \right) \right]$$

$$\begin{aligned}
& -\frac{\partial^2 u}{\partial z_1 \partial z_3} d\bar{z}_3 \wedge dz_1 - \frac{\partial^2 u}{\partial z_2 \partial z_3} d\bar{z}_3 \wedge dz_2 - \frac{\partial^2 u}{\partial z_3 \partial z_3} d\bar{z}_3 \wedge dz_3 \Big) + \frac{\partial^2 u}{\partial z_1 \partial z_1} dz_1 \wedge d\bar{z}_1 + \\
& + \frac{\partial^2 u}{\partial z_2 \partial z_1} dz_1 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial z_3 \partial z_1} dz_1 \wedge d\bar{z}_3 + \frac{\partial^2 u}{\partial z_1 \partial z_2} dz_2 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial z_2 \partial z_2} dz_2 \wedge d\bar{z}_2 + \\
& + \frac{\partial^2 u}{\partial z_3 \partial z_2} dz_2 \wedge d\bar{z}_3 + \frac{\partial^2 u}{\partial z_1 \partial z_3} dz_3 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial z_2 \partial z_3} dz_3 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial z_3 \partial z_3} dz_3 \wedge d\bar{z}_3 - \\
& - \frac{\partial^2 u}{\partial z_2 \partial z_1} d\bar{z}_1 \wedge dz_2 - \frac{\partial^2 u}{\partial z_3 \partial z_1} d\bar{z}_1 \wedge dz_3 - \frac{\partial^2 u}{\partial z_1 \partial z_1} d\bar{z}_2 \wedge dz_1 - \frac{\partial^2 u}{\partial z_3 \partial z_2} d\bar{z}_2 \wedge dz_3 - \\
& - \frac{\partial^2 u}{\partial z_1 \partial z_3} d\bar{z}_3 \wedge dz_1 - \frac{\partial^2 u}{\partial z_2 \partial z_3} d\bar{z}_3 \wedge dz_2 \Big) \Big] = \\
& = \frac{1}{4i} \left(2 \frac{\partial^2 u}{\partial z_1 \partial z_1} dz_1 \wedge d\bar{z}_1 + 2 \frac{\partial^2 u}{\partial z_2 \partial z_1} dz_1 \wedge d\bar{z}_2 + 2 \frac{\partial^2 u}{\partial z_3 \partial z_1} dz_1 \wedge d\bar{z}_3 + \right. \\
& + 2 \frac{\partial^2 u}{\partial z_2 \partial z_1} dz_1 \wedge d\bar{z}_2 + 2 \frac{\partial^2 u}{\partial z_2 \partial z_3} dz_3 \wedge d\bar{z}_2 + 2 \frac{\partial^2 u}{\partial z_3 \partial z_3} dz_3 \wedge d\bar{z}_3 + \\
& \left. 2 \frac{\partial^2 u}{\partial z_2 \partial z_1} d\bar{z}_1 \wedge dz_2 + 2 \frac{\partial^2 u}{\partial z_3 \partial z_1} dz_2 \wedge d\bar{z}_3 + 2 \frac{\partial^2 u}{\partial z_1 \partial z_3} dz_3 \wedge d\bar{z}_1 \right) = \\
& = \frac{1}{2i} \left(u_{11} dz_1 \wedge d\bar{z}_1 + u_{12} dz_1 \wedge d\bar{z}_2 + u_{13} dz_1 \wedge d\bar{z}_3 + u_{21} dz_2 \wedge d\bar{z}_1 + \right. \\
& + u_{22} dz_2 \wedge d\bar{z}_2 + u_{23} dz_2 \wedge d\bar{z}_3 + u_{31} dz_3 \wedge d\bar{z}_1 + u_{32} dz_3 \wedge d\bar{z}_2 + \\
& \left. + u_{33} dz_3 \wedge d\bar{z}_3 \right) = \frac{1}{2i} \sum_{j,k=1}^3 u_{j\bar{k}} dz_j \wedge d\bar{z}_k.
\end{aligned}$$

Demak, $n=3$ uchun ham $dd^c u = \frac{i}{2} \partial \bar{\partial} u$ tenglik o`rinli ekan.

Ermit kvadratlik formasi bu — $q(x) = \overline{\varphi(x, x)} = \varphi(x, x)$, ya`ni qo`shmasi o`ziga teng bo`lgan haqiqiy kvadratlik forma.

2.2.1-ta`rif. Agar elementlari $\mathbb{C}$ maydondan olingan $A = (a_{ik})$ matritsa

uchun $A^T = \bar{A}$ (ya`ni $\bar{a}_{ik} = a_{ki}$) tenglik bajarilsa, u ermit matritsasi deyiladi:

Unitar ([4] ga qarang) almashtirishlardan keyin u quyidagi diagonal formaga keltiriladi:

$$dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge \bar{d}z_1 + \dots + \lambda_n dz_n \wedge \bar{d}z_n]$$

bunda $\lambda_1(u), \dots, \lambda_n(u) = u_{j,\bar{k}}$ ermit matritsasini xos qiymatlari, ya'ni

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, haqiqiy qiymatlardan iborat.

Endi xos vektorlar va xos qiymat ta'riflarini ham eslatib o'tamiz:

2.2.2-ta'rif: Agar noldan farqli $x \in u$ vektor va λ son uchun $Ax = \lambda x$ tenglik o'rinli bo'lsa, u holda x vektorga A chiziqli almashtirishni xos vektori deyiladi, unga mos bo'lgan λ songa esa xos qiymat deyiladi.

$(dd^c u)^n = 0$ ko'rinishdagi tenglamaga Monje-Ampere tenglamasi deyiladi.

Bunda

$$(dd^c u)^n = dd^c u \wedge dd^c u \wedge \dots \wedge dd^c u$$

$n=1$ bo'lganda $dd^c u = \Delta u$ Laplas operatori bilan ustma-ust tushadi.

Endi bu tasdiqni isbotlashga harakat qilamiz

$$dd^c u = \frac{i}{2} \partial \bar{\partial} u.$$

Faraz qilaylik, $f(z)$ funksiya ($f(z) = u(x, y) + iv(x, y)$) $z_0 = x_0 + iy_0 \in D (D \in \mathbb{C})$

nuqtada $\mathbb{R}^2$ ma'noda differensiallanuvchi bo'lsin. Ushbu $du(x_0, y_0) + idv(x_0, y_0)$

ifoda $f(z)$ funksiyaning z_0 nuqtadagi differensial deyiladi va $df(z_0)$ kabi

belgilanadi:

$$df(z_0) = du(x_0, y_0) + idv(x_0, y_0)$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Shularni e'tiborga olib,

$$\begin{aligned} df &= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + i \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dx + \\ &+ \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \end{aligned} \quad (2.2.2)$$

$$z = x + iy \quad \bar{z} = x - iy \quad dx = \frac{1}{2}(dz + d\bar{z}), dy = \frac{1}{2i}(dz - d\bar{z}) \quad (2.2.3)$$

(2.2.2) va (2.2.3) dan

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial x} \cdot \frac{1}{2}(dz + d\bar{z}) + \frac{\partial f}{\partial y} \cdot \frac{1}{2i}(dz - d\bar{z}) = \\ &\frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) d\bar{z} \end{aligned}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (2.2.4)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \quad (2.2.5)$$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$\begin{aligned} dd^c u &= \frac{i}{2} \partial \bar{\partial} u = \frac{i}{2} \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2} + i \frac{\partial^2 f}{\partial x \partial y} - i \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \right) = 0 \Rightarrow \\ &\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \end{aligned}$$

$$\text{Ko`rish qiyin emaski, } (dd^c u)^m \wedge \beta^{n-m} = m!(n-m)! H_m(u) \beta^n \quad (2.2.6)$$

$$\text{Bunda} \quad H_m(u) = \sum_{1 \leq j_1 < \dots < j_m \leq n} \lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_m} \quad - \text{ gessian vektori}$$

Shunday qilib, 2 marta silliq u funksiya uchun $(dd^c u)^m \wedge \beta^{n-m}$ operator gessian vektorini bilan bog'liq ekan.

2.2.2. Musbat gessianlar.

Berilgan $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ uchun

$$H_m(\lambda) = \sum_{1 \leq j_1 < \dots < j_m \leq n} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_m} \quad (2.2.7)$$

Gessianni tushunish uchun quyidagi yordamchi ifodadan foydalanamiz:

$$(t + \lambda_1)(t + \lambda_2) \dots (t + \lambda_n) = t^n + H_1(\lambda)t^{n-1} + \dots + H_n(\lambda), \quad t \in \mathbb{R} \quad (2.2.8)$$

(2.2.8) ni $n=2$ uchun yozib ko'ramiz:

$$(t + \lambda_1)(t + \lambda_2) = t^2 + (\lambda_1 + \lambda_2)t + \lambda_1 \lambda_2 \Rightarrow H_1(\lambda) = (\lambda_1 + \lambda_2), \quad H_2(\lambda) = \lambda_1 \lambda_2$$

(2.2.7) formula bo'yicha yozadigan bo'lsak,

$$H_1(\lambda) = \sum_{1 \leq j_1 \leq 2} \lambda_{j_1} = \lambda_1 + \lambda_2, \quad H_2(\lambda) = \sum_{1 \leq j_1 < j_2 \leq 2} \lambda_{j_1} \lambda_{j_2} = \lambda_1 \lambda_2$$

$n=3$ uchun :

$$(t + \lambda_1)(t + \lambda_2)(t + \lambda_3) = t^3 + (\lambda_1 + \lambda_2 + \lambda_3)t^2 + (\lambda_1 \lambda_2 + \lambda_3 \lambda_2 + \lambda_1 \lambda_3)t + \lambda_1 \lambda_2 \lambda_3$$

(2.2.7) formula bo'yicha yozadigan bo'lsak,

$$\begin{aligned} H_1(\lambda) &= \sum_{1 \leq j_1 \leq 3} \lambda_{j_1} = \lambda_1 + \lambda_2 + \lambda_3, \\ H_2(\lambda) &= \sum_{1 \leq j_1 < j_2 \leq 3} \lambda_{j_1} \lambda_{j_2} = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\ H_3(\lambda) &= \sum_{1 \leq j_1 < j_2 < j_3 \leq 3} \lambda_{j_1} \lambda_{j_2} \lambda_{j_3} \end{aligned}$$

Endi biz $H_m(\lambda) \geq 0$ bo'ladigan $\lambda \in \mathbb{R}^n$ vektorlar bilan qiziqamiz va shunday vektorlarni birlashtiramiz, ya'ni $S_m = \{\lambda \in \mathbb{R}^n, H_m(\lambda) \geq 0\}$

$m > 1$ da bu to'plam qavariq konusni hosil qilmaydi, ya'ni $\lambda', \lambda'' \in S_m$

ekanligidan $\lambda' + \lambda'' \in S_m$ ekanligi kelib chiqmaydi. Shuning uchun bu to'plamni toraytiramiz, ya'ni $\lambda \in S_m$ vektorlar uchun shunday $\Gamma_m \subset S_m$ to'plamni olamizki,

$\{\lambda + tI = (\lambda_1 + t_1, \dots, \lambda_n + t_n), t \in \mathbb{R}_+^n\} \subset S_m$ bo'ladi. Unda Γ_m ning ko'rinishi $\Gamma_m = \{H_1(\lambda) \geq 0, \dots, H_m(\lambda) \geq 0\}$ bo'ladi. Undan tashqari Γ_m $(1, 1, \dots, 1)$ vektorni o'z ichiga oluvchi S_m to'plamni $S_m^0 = \{\lambda \in \mathbb{R}^n : H_m(\lambda) > 0\}$ ochiq yadrosining bog'lamli komponentasini yopig'i bilan ustma-ust tushadi.

Γ_m quyidagi xossalarga ega:

$$1) \Gamma_n \subset \Gamma_{n-1} \subset \dots \subset \Gamma_1 \quad \Gamma_n = \{\lambda_1 \geq 0, \dots, \lambda_n \geq 0\} = \mathbb{R}_+^n$$

Biz bu xossani $n=3$ bo'lgan holda qaraymiz va

$$\Gamma_3 = \{H_1(\lambda) \geq 0, H_2(\lambda) \geq 0, H_3(\lambda) \geq 0\} \text{ ekanligidan}$$

$$H_1(\lambda) = \sum_{1 \leq j_1 \leq 3} \lambda_{j_1} = \lambda_1 + \lambda_2 + \lambda_3 \geq 0$$

$$H_2(\lambda) = \sum_{1 \leq j_1 < j_2 \leq 3} \lambda_{j_1} \lambda_{j_2} = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \geq 0 \Rightarrow \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$$

$$H_3(\lambda) = \sum_{1 \leq j_1 < j_2 < j_3 \leq 3} \lambda_{j_1} \lambda_{j_2} \lambda_{j_3} = \lambda_1 \lambda_2 \lambda_3 \geq 0$$

bo'lishi kelib chiqadi.

$$\Gamma_2 = \{H_1(\lambda) \geq 0, H_2(\lambda) \geq 0\} \text{ esa}$$

$$H_1(\lambda) = \sum_{1 \leq j_1 \leq 3} \lambda_{j_1} = \lambda_1 + \lambda_2 + \lambda_3 \geq 0$$

$$H_2(\lambda) = \sum_{1 \leq j_1 < j_2 \leq 3} \lambda_{j_1} \lambda_{j_2} = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \geq 0$$

shartlarni qanoatlantiradi.

$$\Gamma_1 = \{H_1(\lambda) \geq 0\} \text{ ekanligidan esa } H_1(\lambda) = \sum_{1 \leq j_1 \leq 3} \lambda_{j_1} = \lambda_1 + \lambda_2 + \lambda_3 \geq 0 \text{ tengsizlikning}$$

bajarilishi yetarli ekan.

2) Γ_m qavariq konusdan iborat, ya'ni agar $\lambda, \mu \in \Gamma_m$ bo'lsa, u holda $\forall a, b \in \mathbb{R}^+$ uchun $a\lambda + b\mu \in \Gamma_m$ o'rinli.

3) $\{H_m^{1/m}(\lambda), \lambda \in \Gamma_m\}$ o'suvchi va botiq konusdan iborat, ya'ni

$$H_m^{1/m}(\mu) \leq H_m^{1/m}(\lambda) + \sum_j (\mu_j - \lambda_j) \frac{\partial}{\partial \lambda_j} H_m^{1/m}(\lambda) \quad \forall \lambda, \mu \in \Gamma_m$$

Soddalik uchun $m=1, n=1$ bo'lgan holni qarab chiqamiz:

$$H_1(\mu) \leq H_1(\lambda) + (\mu - \lambda) \frac{\partial}{\partial \lambda} H_1(\lambda) \quad \forall \lambda, \mu \in \Gamma_m \Rightarrow$$

$$\mu \leq \lambda + (\mu - \lambda) \Rightarrow \mu = \mu$$

Endi $m=1, n=2$ bo'lgan holni qaraymiz:

$\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2 + (\mu_1 - \lambda_1) + (\mu_2 - \lambda_2)$ demak, $n=k, m=1$ uchun hamma vaqt tenglik bajarilar ekan. $m=2, n=2$ bo'lgan holda esa quyidagicha ko'rinishga o'tadi:

$$\sqrt{\mu_1 \mu_2} \leq \sqrt{\lambda_1 \lambda_2} + (\mu_1 - \lambda_1) \frac{\lambda_2}{2\sqrt{\lambda_1 \lambda_2}} + (\mu_2 - \lambda_2) \frac{\lambda_1}{2\sqrt{\lambda_1 \lambda_2}}$$

$$\sqrt{\mu_1 \mu_2} \leq \frac{2\lambda_1 \lambda_2 + \mu_1 \lambda_2 - 2\lambda_1 \lambda_2 + \mu_2 \lambda_1}{2\sqrt{\lambda_1 \lambda_2}} \Rightarrow$$

$$2\sqrt{\lambda_1 \lambda_2 \mu_1 \mu_2} \leq \mu_1 \lambda_2 + \mu_2 \lambda_1, \quad a = \mu_1 \lambda_2, b = \mu_2 \lambda_1$$

$$\Rightarrow a + b \geq 2\sqrt{ab}$$

$$4) \sum_j \mu_j \frac{\partial}{\partial \lambda_j} H_m(\lambda) \geq m H_m^{1/m}(\mu) H_m^{1-1/m}(\lambda) \quad \forall \lambda, \mu \in \Gamma_m$$

$$m=1, n=1 \text{ bo'lsa, } \mu \cdot 1 \geq \mu, \Rightarrow \mu = \mu$$

$$m=1, n=2 \Rightarrow \mu_1 + \mu_2 \geq \mu_1 + \mu_2$$

$$m=2, n=2 \Rightarrow \mu_1 \lambda_2 + \mu_2 \lambda_1 \geq 2\sqrt{\mu_1 \mu_2} \sqrt{\lambda_1 \lambda_2}$$

$$\begin{aligned}
m=2, n=3 &\Rightarrow \mu_1 \frac{\partial}{\partial \lambda_1} (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2) + \\
&+ \mu_2 \frac{\partial}{\partial \lambda_2} (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2) + \mu_3 \frac{\partial}{\partial \lambda_3} (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2) \geq \\
&\geq 2\sqrt{\mu_1 \mu_2 + \mu_1 \mu_3 + \mu_3 \mu_2} \cdot \sqrt{\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2} \\
&\mu_1 (\lambda_2 + \lambda_3) + \mu_2 (\lambda_1 + \lambda_3) + \mu_3 (\lambda_1 + \lambda_2) \geq 2\sqrt{\mu_1 \mu_2 + \mu_1 \mu_3 + \mu_3 \mu_2} \cdot \sqrt{\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2}
\end{aligned}$$

5) $\left[\frac{H_m}{\binom{n}{m}} \right]^{1/m} \leq \left[\frac{H_m}{\binom{n}{k}} \right]^{1/k} \quad \forall 1 \leq k \leq m \leq n$ Bu tengsizlik Makloren tengsizligi deb yuritiladi.

2.2.3. Musbat gessianlar bilan differensial formalar

Bizga (1,1) - bidarajali $\alpha = \frac{i}{2} \sum_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k$ haqiqiy differensial forma berilgan bo'lsin. Bunda $(a_{j\bar{k}})$ - ermit matrisa. Agar $\lambda(\alpha) = (\lambda_1(\alpha), \dots, \lambda_n(\alpha))$ - bu matrisaning xos qiymati bo'lsa, u holda $H_m(\alpha) = H_m(\lambda(\alpha))$ gessian aniqlash mumkin. Bu esa (1,1) - ermit formasi va $\lambda \in \mathbb{R}^n$ vektor orasidagi bog'liklikni aniqlashga yordam beradi:

$$\begin{aligned}
\hat{\Gamma}_m &= \{ \alpha : \lambda(\alpha) \in \Gamma_m \} = \left\{ H_m \left(\alpha + t_1 dd^c |z_1|^2 + \dots + t_n dd^c |z_n|^2 \right) \geq 0, \right. \\
&\quad \left. \forall t = (t_1, \dots, t_n) \in \mathbb{R}_+^n \right\} = \\
&= \{ \alpha \wedge \beta^{n-1} \geq 0, \dots, \alpha^m \wedge \beta^{n-m} \geq 0 \}.
\end{aligned} \tag{2.2.9}$$

Gessianning 2-xossasiga ko'ra agar $\lambda', \lambda'' \in \Gamma_m$ bo'lsa, u holda $a, b \in \mathbb{R}_+$ uchun $a\lambda' + b\lambda'' \in \Gamma_m$ bo'ladi. Biz λ' sifatida α differensial formaning xos qiymatlarini, λ'' sifatida $\lambda'' = (t_1, t_2, \dots, t_n) \in \mathbb{R}_+^n$ ni belgilasak va $a, b = 1$ xususiy holni qarasak

$$H_m \left(\alpha + t_1 dd^c |z_1|^2 + \dots + t_n dd^c |z_n|^2 \right) \geq 0 \quad \forall t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$$

ekanligiga ishonch hosil qilishimiz mumkin. Chunki Γ_m ni

$\Gamma_m = \{H_1(\lambda) \geq 0, \dots, H_m(\lambda) \geq 0\}$ ko`rinishda aniqlagan edik.

Endi boshqa holatlarni ko`rib chiqamiz, ya'ni  $\lambda' \notin \Gamma_m$  bo`lsin. Buni misollar yordamida tushunib olishimiz osonroq bo`ladi:

2.2.1-misol $\alpha = \frac{i}{2}(-7dz_1 \wedge d\bar{z}_1 - 11dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3)$ bo`lsin.

$\lambda(\alpha) = (\lambda_1, \lambda_2, \lambda_3) = (-7, -11, 3) \notin \Gamma_2 = \{H_1(\lambda) \geq 0, H_2(\lambda) \geq 0\}$ Chunki

$$H_1(\lambda(\alpha)) = \lambda_1 + \lambda_2 + \lambda_3 = -7 - 11 + 3 = -15 < 0,$$

$$H_2(\lambda(\alpha)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 77 - 21 - 33 = 23 > 0$$

$$H_3(\lambda(\alpha)) = \lambda_1\lambda_2\lambda_3 = 231 > 0$$

Biz $t = (t_1, t_2, t_3) \in \mathbb{R}_+^3$ ning ixtiyoriyligidan foydalanib $t = (6, 1, 1) \in \mathbb{R}_+^3$ deb tanlab olamiz va

$$\begin{aligned} H_2\left(\frac{i}{2}(-7dz_1 \wedge d\bar{z}_1 - 11dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) + 6dd^c|z_1|^2 + dd^c|z_2|^2 + dd^c|z_3|^2\right) = \\ = H_2(-1, -10, 4) = -44 + 10 = -34 < 0, \end{aligned}$$

bo`lishini ko`ramiz. Xulosa qilishimiz uchun yana bir misolni ko`rib o`taylik:

2.2.2-misol. $\alpha = \frac{i}{2}(-2dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3)$ bo`lsin.

$\lambda(\alpha) = (\lambda_1, \lambda_2, \lambda_3) = (-2, 1, 3) \notin \Gamma_2, \Gamma_3$ Chunki

$$H_1(\lambda(\alpha)) = \lambda_1 + \lambda_2 + \lambda_3 = -2 + 1 + 3 = 2 > 0$$

$$H_2(\lambda(\alpha)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -2 - 6 + 3 = -5 < 0$$

$$H_3(\lambda(\alpha)) = \lambda_1\lambda_2\lambda_3 = -6 < 0$$

Biz $t = (t_1, t_2, t_3) \in \mathbb{R}_+^3$ ning ixtiyoriyligidan foydalanib $t = (0, 1, 1) \in \mathbb{R}_+^3$ deb tanlab olamiz va

$$\begin{aligned}
& H_2 \left(\frac{i}{2} (-2dz_1 \wedge d\bar{z}_1 + 1dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) + dd^c |z_2|^2 + dd^c |z_3|^2 \right) = \\
& = H_2 (-2, 2, 4) = -4 - 8 + 8 = -4 < 0
\end{aligned}$$

tengsizlik o`rinli bo`ladi. Agar $t = (1, 3, 5) \in \mathbb{R}_+^3$ deb olsak, unda ushbu

$$\begin{aligned}
& H_3 \left(\frac{i}{2} (-2dz_1 \wedge d\bar{z}_1 + 1dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) + dd^c |z_1|^2 + 3dd^c |z_2|^2 + 5dd^c |z_3|^2 \right) = \\
& = H_3 (-1, 4, 8) = -32 < 0
\end{aligned}$$

ifodaga ega bo`lamiz. Shunday qilib, biz quyidagi xulosaga keldik:

$\lambda(\alpha) \in \Gamma_m$ bo`lgandagina

$$H_m \left(\alpha + t_1 dd^c |z_1|^2 + \dots + t_n dd^c |z_n|^2 \right) \geq 0 \quad \forall t = (t_1, \dots, t_n) \in \mathbb{R}_+^n \text{ tengsizlik o`rinli bo`lar ekan.}$$

Boshqa hollarda bu tengsizlik bajarilmas ekan.

2.2.1-teorema.[18] Agar $\alpha_1, \dots, \alpha_k \in \hat{\Gamma}_m$, $1 \leq k \leq m$ bo`lsa, u

holda $\alpha_1 \wedge \dots \wedge \alpha_k \wedge \beta^{n-m} \geq 0$, bo`ladi.

II BOB bo'yicha xulosa

Ushbu bobda differensial forma va oqimlar, musbat aniqlangan differensial formalar o'rganildi. Xususan, musbat gessianlar bilan differensial formalar orasidagi bog'lanish o'rganildi. Musbat gessianlardan biz ikki marta silliq $m-sh$ funksiyalarga ta'rif berishda va ularning xossalarini isbotlashda hamda qat'iy $m-sh$ (ikki marta silliq funksiyalar uchun) funksiyalarni ta'riflashda foydalanamiz.

III BOB. QAT'ITY m -sh FUNKSIYALAR

3.1. m -sh funksiyalar va ularning xossalari

3.1.1-ta'rif. Ikki marta silliq $u \in C^2(D)$ funksiya ($D \subset \mathbb{C}^n$) $z^0 \in D$ nuqtada m -sh ($1 \leq m \leq n$) deyiladi, agar $(u_{j\bar{k}})|_{z=z^0}$ matrisani $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ xos qiymatlari Γ_m ga tegishli bo'lsa yoki $dd^c u \in \hat{\Gamma}_m$ bo'lsa. $u \in C^2(D)$ funksiya $D \subset \mathbb{C}^n$ sohada m -sh, ($1 \leq m \leq n$) deyiladi, agar u har bir $z^0 \in D$ nuqtada m -sh bo'lsa.

Boshqacha aytganda, $u \in C^2(D)$ funksiya m -sh ($1 \leq m \leq n$) deyiladi, agar $dd^c u \in \hat{\Gamma}_m$ bo'lsa yoki unga quyidagi tengsizlik teng kuchli:

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad \forall k = 1, 2, \dots, m \quad (3.1.1)$$

(3.1.1) ga quyidagi tengsizlik ham ekvivalent:

$$\left[\left(dd^c u + t_1 dd^c |z_1|^2 + \dots + t_n dd^c |z_n|^2 \right) \right]^m \wedge \beta^{n-m} \geq 0 \quad \forall t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$$

3.1.1-misol. $u(z) = -|z_1|^2 + 2|z_2|^2 + 3|z_3|^2$ funksiya ($m = 1, 2$ bo'lgan hol uchun) m -sh bo'ladi.

$$dd^c u = \frac{i}{2} (-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3)$$

$k = 1$ bo'lganda

$$\begin{aligned} (dd^c u) \wedge (dd^c |z|^2)^2 &= \frac{i}{2} (-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) \wedge \\ &\wedge \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) \wedge \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) = \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{i}{2}\right)^3 \left(-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3\right) \wedge (dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 + \\
&\quad + dz_1 \wedge d\bar{z}_1 \wedge dz_3 \wedge d\bar{z}_3 + dz_2 \wedge d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 + \\
&\quad + dz_3 \wedge d\bar{z}_3 \wedge dz_1 \wedge d\bar{z}_1 + dz_3 \wedge d\bar{z}_3 \wedge dz_2 \wedge d\bar{z}_2)
\end{aligned}$$

Tashqi ko‘paytmada

$dz_k \wedge dz_j = -dz_j \wedge dz_k$ bo‘lishidan foydalansak,

$$\begin{aligned}
&\left(dd^c u\right) \wedge \left(dd^c |z|^2\right)^2 = \left(\frac{i}{2}\right)^3 \left(-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3\right) \\
&\wedge 2(dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 + \\
&\quad + dz_1 \wedge d\bar{z}_1 \wedge dz_3 \wedge d\bar{z}_3 + dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3) = \\
&= \left(\frac{i}{2}\right)^3 4dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3
\end{aligned}$$

$$dz_1 = dx_1 + idy_1, \quad d\bar{z}_1 = dx_1 - idy_1$$

$$\begin{aligned}
dz_1 \wedge d\bar{z}_1 &= (dx_1 + idy_1)(dx_1 - idy_1) = -idx_1 \wedge dy_1 + idy_1 \wedge dx_1 = \\
&= -2idx_1 \wedge dy_1
\end{aligned}$$

ekanidan foydalanib

$$dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 = (-2i)^3 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3$$

SHunday qilib,

$$\begin{aligned}
&2\left(\frac{i}{2}\right)^3 (8i) dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 = \\
&= 2dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 \geq 0
\end{aligned}$$

Demak, $\left(dd^c u\right) \wedge \left(dd^c |z|^2\right)^2 \geq 0$ bo‘lishi uchun $(\lambda_1 + \lambda_2 + \lambda_3) \geq 0$ bo‘lishi kerak

ekan.

Endi $k = 2$ uchun (3.1.1) tengsizlikni tekshiramiz:

$$\begin{aligned}
& (dd^c u)^2 \wedge (dd^c |z|^2) = dd^c u \wedge dd^c u \wedge dd^c |z|^2 = \\
& = \left(\frac{i}{2}\right)^2 (-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) \wedge \\
& \quad \wedge (-dz_1 \wedge d\bar{z}_1 + 2dz_2 \wedge d\bar{z}_2 + 3dz_3 \wedge d\bar{z}_3) \wedge \\
& \quad \wedge \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) = \\
& = 2 \left(\frac{i}{2}\right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 \geq 0 \\
& \Rightarrow \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \geq 0
\end{aligned}$$

$k=3$ da esa:

$$(dd^c u)^3 = -6 \left(\frac{i}{2}\right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3$$

Demak , $\lambda_1 \lambda_2 \lambda_3 < 0$ ekan.

3.1.2-misol. $u(z) = -7|z_1|^2 - 11|z_2|^2 + 3|z_3|^2$. Bu funksiya uchun ham yuqoridagi mulohazalarni yuritish mumkin.

$\lambda_1 = -7, \lambda_2 = -11, \lambda_3 = 3$ bo'lganda $k=2$ uchun bajarilgan tengsizlik

$k=1$ uchun bajarilmaydi, ya'ni $(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2) = 77 - 21 - 33 = 23 > 0$

$(\lambda_1 + \lambda_2 + \lambda_3) = -15 < 0, \quad k=3$ uchun $\lambda_1 \lambda_2 \lambda_3 = 231 > 0$ o'rinli. Ammo bu

funksiya plyurisubgarmonik ham , $m-sh$ ham emas. Demak, (3.1.1) tengsizlik

$k=m$ uchun bajarilishidan qolgan barcha $k = \overline{1, m-1}$ lar uchun ham bajarilishi

kelib chiqmas ekan.

Undan tashqari quyidagi tasdiq ham o`rinli:

$$dd^c u \wedge \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \beta^{n-m} \geq 0 \quad \forall \alpha_1, \dots, \alpha_{k-1} \in \hat{\Gamma}_m, 1 \leq k \leq m \quad (3.1.2)$$

Bu tasdiq ikki xil xarakterga ega: agar u ikki marta silliq bo`lib,  $\alpha_1, \dots, \alpha_{m-1} \in \hat{\Gamma}_m$  uchun (3.1.2) shartni qanoatlantirsa,  $u$  holda  $u$  funksiya  $m$ -sh bo`ladi.

u funksiyaning m -sh ligini tekshirish uchun (3.1.2) da faqat $\alpha_j = dd^c u_j, u_j \in C^2(D), j = 1, 2, \dots, m-1$ differensial formani chegaralangan bo`lishi yoki  $m$ -sh ko`phad bo`lishi yetarli.

Bu esa bizga L^1_{loc} funksiyalar sinfiga m -sh funksiyalarni aniqlashga yordam beradi.

3.1.2-ta`rif. $u \in L^1_{loc}(D)$ funksiya $D \subset \mathbb{C}^n$ sohada m -sh deyiladi, agar u yuqoridan yarim uzluksiz va ixtiyoriy ikki marta silliq $v_1, \dots, v_{m-1}$ m -sh funksiyalar uchun

$$dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \text{ oqim}$$

$$\begin{aligned} & \left[dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \right](\omega) = \\ & \int u dd^c v_1 \wedge \dots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \wedge dd^c \omega, \omega \in F^{0,0} \end{aligned} \quad (3.1.3)$$

musbat aniqlangan bo`lsa.

Biz bilamizki, (p, p) -finit differensial formalar fazosidagi chiziqli, uzluksiz T funksional (kompleks qiymatli)

$$F^{p,p} = F^{p,p}(D) = \{ \omega \in F^{p,p}(D) \cap C^\infty(D) : \text{supp } \omega \subset\subset D \}, (n-p, n-p) = (q, q)$$

bidarajali oqim deyilar edi.

$m-sh$ funksiyalarning xossalari:

1) $u \in C^2(D)$ funksiya $m-sh$ ($1 \leq m \leq n$) bo`ladi, faqat va faqat shu holdaki,

qachonki $dd^c u \in \hat{\Gamma} \forall z^0 \in D$ bo`lsa.

2) agar $u \in m-sh(D)$ bo`lsa, u holda $u_j(z) = u * K_j(z-w)$ standart

approksimatsiya (o`rama) ham  $m-sh$  bo`ladi, ($j \rightarrow \infty$ da $u_j \downarrow u$ bo`ladi)

3) agar $u, v \in m-sh$ bo`lsa, u holda ixtiyoriy $a, b \geq 0$ uchun $au + bv \in m-sh(D)$.

$au + bv \in m-sh(D) \Rightarrow \lambda(u), \lambda(v) \in \Gamma_m \quad a, b \geq 0$ ekanligidan

$a\lambda(u) + b\lambda(v) \in \Gamma_m \Rightarrow au + bv \in m-sh(D)$ bo`lishi kelib chiqadi.

4) $psh = n-sh \subset \dots \subset 1-sh = sh$

Bu xossani $n=3$ hol uchun isbotlaymiz:

$$(dd^c u)^k \wedge (dd^c |z|^2)^{3-k} \geq 0, \quad k=1,2,3$$

$k=1$ bo`lganda  $(dd^c u) \wedge (dd^c |z|^2)^2 \geq 0$  bo`lishi uchun $(\lambda_1 + \lambda_2 + \lambda_3) \geq 0$

bo`lishi kerak ekan.

Endi $k=2$ uchun tengsizlikni tekshiramiz:

$$(dd^c u)^2 \wedge (dd^c |z|^2) \geq 0 \Rightarrow \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \geq 0$$

$k=3$ uchun:

$$\begin{aligned} (dd^c u)^3 &= dd^c u \wedge dd^c u \wedge dd^c u = 6 \left(\frac{i}{2} \right)^3 \lambda_1 \lambda_2 \lambda_3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 \\ &\Rightarrow \lambda_1 \lambda_2 \lambda_3 \geq 0 \end{aligned}$$

Yuqoridagi uchala tengsizlikni birgalikda yechamiz:

$$\begin{cases} (\lambda_1 + \lambda_2 + \lambda_3) \geq 0 \\ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \geq 0 \\ \lambda_1 \lambda_2 \lambda_3 \geq 0 \end{cases} \quad (3.1.4)$$

(3.1.4) tengsizlikning bajarilishi uchun $\lambda_1, \lambda_2, \lambda_3$ larning 2 tasi manfiy, bittasi musbat yoki uchalasi ham musbat bo'lishi kerak. Ikkala holni ham tekshirib ko'ramiz:

1) $\lambda_1 \geq 0, \lambda_2 \leq 0, \lambda_3 \leq 0$ bo'lsin.

$$\begin{cases} (\lambda_1 + \lambda_2 + \lambda_3) \geq 0 \\ \lambda_1 \lambda_2 + \lambda_3 \lambda_2 + \lambda_1 \lambda_3 \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 \geq -(\lambda_2 + \lambda_3) \\ \lambda_1(\lambda_2 + \lambda_3) + \lambda_2 \lambda_3 \geq 0 \end{cases}$$

$$\begin{cases} \lambda_1 \geq -(\lambda_2 + \lambda_3) \\ \lambda_2 \lambda_3 \geq -\lambda_1(\lambda_2 + \lambda_3) \geq (\lambda_2 + \lambda_3)^2 \end{cases} \Rightarrow \begin{cases} \lambda_1 \geq -(\lambda_2 + \lambda_3) \\ (\lambda_2 + \lambda_3)^2 \leq \lambda_2 \lambda_3 \end{cases}$$

$\lambda_2^2 + 2\lambda_2 \lambda_3 + \lambda_3^2 \leq \lambda_2 \lambda_3 \Rightarrow \lambda_2^2 + \lambda_3^2 \leq -\lambda_2 \lambda_3$ Bu esa ziddiyat. Bu holda

tengsizliklar sistemasi yechimga ega emas ekan. 2-holda tengsizliklar sistemasi doimo yechimga ega. Ravshanki, $m=1$ bo'lganda m -subgarmonik funksiya subgarmonik funksiyaga aylanadi va $(\lambda_1 + \lambda_2 + \lambda_3) \geq 0$ tengsizlikning bajarilishi yetarli. Yuqoridagilardan ko'rinadiki, pyurisubgarmonik bo'lgan funksiya albatta m -sh va sh funksiya bo'la oladi. O'z navbatida m -sh bo'lgan funksiya subgarmonik funksiya bo'la oladi.

Endi quyidagi teoremani isbotlaymiz:

3.1.1-teorema. ([24] ga qarang) $D \subset \mathbb{C}^n$ sohada n -sh bo'lgan ixtiyoriy funksiya psh dir.

Isboti. Faraz qilaylik, (3.1.1) tengsizliklar barcha k lar uchun

bajarilsin, $k = 1, 2, \dots, n$. Natijada quyidagi tengsizliklar sistemasi hosil bo`ladi:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n \geq 0 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n \geq 0 \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_{n-2}\lambda_{n-1}\lambda_n \geq 0 \\ \dots \\ \lambda_1\lambda_2\lambda_3\dots\lambda_{n-1}\lambda_n \geq 0 \end{cases} \quad (3.1.5)$$

$\lambda_1, \lambda_2, \dots, \lambda_n$ larning haqiqiy sonlar ekanligidan (3.1.5) tengsizliklar sistemasini o`rganish uchun biz quyidagicha belgilashlarni kiritaylik:

$$\begin{cases} a_1 = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n \geq 0 \\ a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n \geq 0 \\ a_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_{n-2}\lambda_{n-1}\lambda_n \geq 0 \\ \dots \\ a_n = \lambda_1\lambda_2\lambda_3\dots\lambda_{n-1}\lambda_n \geq 0 \end{cases} \quad (3.1.6)$$

Endi koeffitsiyentlari $1, -a_1, a_2, \dots, (-1)^n a_n$ bo`lgan

$$f(\lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - a_3\lambda^{n-3} + \dots + (-1)^n a_n \quad (3.1.7)$$

ko`rinishdagi ko`phadni qaraylik. Bu ko`phad uchun Viyet teoremasini qo`llasak (3.1.6) sistema hosil bo`ladi.

Bu ko`phadning ildizlari xarakterini aniqlash uchun quyidagi teoremadan foydalanamiz:

3.1.2-teorema (Dekart) ([5] ga qarang) $f(\lambda)$ ko`phadning karrasini hisobga olgan holda hisoblangan musbat ildizlarining soni bu ko`phadning koeffitsiyentlaridan tuzilgan sistemadagi ishora o`zgarishlar soniga teng yoki bu sondan juft songa kam bo`ladi. (Bunda nolga teng bo`lgan koeffitsiyentlar

hisobga olinmaydi)

SHubhasiz, $f(\lambda)$ ko`phadning manfiy ildizlarining sonini topish uchun Dekart teoremasini  $f(-\lambda)$  ko`phadga qo`llash kifoyadir.

Ko`phadning barcha ildizlari haqiqiy ekanligi avvaldan ma'lum bo`lgan xususiy holda Dekart teoremasiga birmuncha aniqlik kiritish imkoni tug`iladi: agar  $f(\lambda)$  ko`phadning barcha ildizlari haqiqiy bo`lib, ozod had noldan farqli bo`lsa, u holda bu ko`phad musbat ildizlarining soni  $k_1$  uning koeffitsiyentlari sistemasidagi ishora o`zgarishlar soni s_1 ga teng, manfiy ildizlarining soni k_2 esa $f(-\lambda)$ ko`phadning koeffitsiyentlari sistemasidagi ishora o`zgarishlar soni s_2 ga teng bo`ladi.

Endi yuqoridagi tasdiqni biz qurgan (3.1.7) ko`phad uchun qo`llaylik. Oldin biz $a_1, a_2, \dots, a_n \neq 0$ bo`lgan holni qaraymiz. Agar  $a_1, a_2, \dots, a_n \neq 0$  bo`lsa, u holda (3.1.7) ko`phadning koeffitsiyentlari sistemasi

$$1, -a_1, a_2, \dots, (-1)^n a_n$$

uchun $a_1, a_2, \dots, a_n > 0$ ligidan bu sistemadagi ishora o`zgarishlar soni  $n$  ga teng bo`ladi. Bundan (3.1.7) ko`phad  $n$  ta musbat ildizga ega bo`lishi kelib chiqadi. Bunda $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ ekanini ko`ramiz. Endi ixtiyoriy

$u \in C^2(D) \cap (m - sh(D))$ funksiyani quyidagi yordamchi funksiya bilan almashtiramiz:

$$v_\varepsilon(z) = u(z) + \varepsilon |z|^2 \in C^2(D) \cap (m - sh(D))$$

Bu funksiya uchun ham yuqoridagi mulohazalarni yuritish orqali

koeffitsiyentlari $1, -a_1', a_2', \dots, (-1)^n a_n'$ bo'lgan

$$f(\lambda) = \lambda^n - a_1' \lambda^{n-1} + a_2' \lambda^{n-2} - a_3' \lambda^{n-3} + \dots + (-1)^n a_n' \quad (3.1.8)$$

ko'rinishdagi ko'phadni hosil qilamiz. Bu ko'phad uchun ham Viyet teoremasini qo'llasak ushbu

$$\begin{cases} a_1' = \lambda_1 + \varepsilon + \lambda_2 + \varepsilon + \dots + \lambda_n + \varepsilon > 0 \\ a_2' = (\lambda_1 + \varepsilon)(\lambda_2 + \varepsilon) + (\lambda_1 + \varepsilon)(\lambda_3 + \varepsilon) + \dots + (\lambda_{n-1} + \varepsilon)(\lambda_n + \varepsilon) > 0 \\ \dots \\ a_n' = (\lambda_1 + \varepsilon)(\lambda_2 + \varepsilon)(\lambda_n + \varepsilon) > 0 \end{cases} \quad (3.1.9)$$

sistema hosil bo'ladi. $a_1', a_2', \dots, a_n' > 0$ bo'lgani uchun (3.1.8) ko'phad koeffitsiyentlari sistemasida ishora o'zgarishlar soni n ga teng bo'ldi.

Demak, bu ko'phad n ta musbat $\lambda_1(v_\varepsilon), \lambda_2(v_\varepsilon), \dots, \lambda_n(v_\varepsilon)$ ildizga ega ekan.

Bunda $\lambda_j = \lim_{\varepsilon \rightarrow 0} \lambda_j(v_\varepsilon) \geq 0, j = \overline{1, n}$. SHunday qilib, biz

$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \dots, \lambda_n \geq 0$ ekanligini aniqladik. Demak, agar

$u \in C^2(D)$ funksiyasi uchun (1) tengsizliklar barcha k lar uchun bajarilsa,

u holda $u(z)$ funksiya *psh* bo'lar ekan, ya'ni

$$n - sh(D) \cap C^2(D) = psh(D) \cap C^2(D). \quad (3.1.10)$$

Endi biz (3.1.10) ni ixtiyoriy $n - sh$ funksiyalar uchun ko'rsatamiz: faraz

qilaylik $u -$ ixtiyoriy $n - sh$ funksiya bo'lsin, $u \in n - sh(D)$. U holda

$u_j = u * K_j(z - w)$ standart approksimatsiya olamiz,

$$u_j \in n-sh(D_j) \cap C^\infty(D_j), \quad u_j \downarrow u, \quad D_j \subset\subset D_{j+1} \subset\subset D, \quad D = \bigcup_{j=1}^{\infty} D_j.$$

Unda (3.1.10) ga ko`ra  $u_j \in psh(D_j)$  bo`lib, plyurisubgarmonik funksiyalarning xossasiga ko`ra  $j \rightarrow \infty$  da  $u_j \searrow u$  bo`lganligidan $u \in psh(D)$ bo`ladi. **Teorema isbotlandi.**

5) agar $\gamma(t)$ — $t \in \mathbb{R}_+$ parametrning o`suvi qavariq funksiyasi va  $u \in m-sh$  bo`lsa, u holda $\gamma \circ u \in m-sh$ bo`ladi.

6) tekis yaqinlashuvchi yoki kamayuvchi $m-sh$ funksiyalar ketma-ketligining limiti yana $m-sh$ bo`ladi.

7) chekli sondagi $m-sh$ funksiyalarning maksimumi $m-sh$ funksiyadan iborat bo`ladi; ixtiyoriy lokal tekis chegaralangan  $\{u_\theta\} \subset m-sh$  m-subgarmonik funksiyalar oilasining supremumi  $u(z) = \left\{ \sup_{\theta} u_\theta(z) \right\}$  regulyarizatsiyasi  $u^*(z)$  ham  $m-sh$  bo`ladi. $m-sh \subset sh$ ekanligidan $\{u(z) < u^*(z)\}$ — to'plam $\mathbb{C}^n \approx \mathbb{R}^{2n}$ da polyar to'plam bo`ladi. Qisqacha aytganda uning Lebeg o`lchovi nolga teng. Ixtiyoriy lokal tekis chegaralangan $\{u_j\} \subset m-sh$ funksiyalar ketma-ketligining limiti $u(z) = \overline{\lim_{j \rightarrow \infty}} u_j(z)$ regulyarizatsiyasi $u^*(z)$ ham $m-sh$ va $\{u(z) < u^*(z)\}$ — polyar to'plam bo`ladi.

8) agar $u \in m-sh$ bo`lsa, u holda ixtiyoriy  $P \subset \mathbb{C}^n$  kompleks gipertekislik uchun  $u|_P \in sh_{m-1}$  bo`ladi.

Haqiqatdan, gessianning 4-xossasidan foydalanib, $\lambda, \mu \in \Gamma_m$ vektor uchun

$$\sum_j \mu_j \frac{\partial}{\partial \lambda_j} H_m(\lambda) \geq m H_m^{1/m}(\mu) H_m^{1-1/m}(\lambda)$$

$\mu = (0, \dots, 0, 1)$ deb oladigan bo'lsak, $\frac{\partial}{\partial \lambda_n} H_m(\lambda) = H_{m-1}(\lambda) \geq 0$. Bu shuni

anglatadiki, agar $\lambda = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n) \in \Gamma_m$ bo'lsa, u holda $\lambda' = (\lambda_1, \dots, \lambda_{n-1}) \in \Gamma_{m-1}$

bo'ladi. Endi $u \in m-sh(D)$ va $P \subset \mathbb{C}^n$ gipertekislik bo'lsin. Umumiylikka zid

ish qilmagan holda biz $u \in C^2(D)$, $P = \{z_n = 0\}$ deb faraz qilamiz va

$u|_P = u(\lambda', 0) \in sh_m$ ekanini isbotlaymiz.

$\lambda = (\lambda_1, \dots, \lambda_n) - U = (u_{j\bar{k}})$, $j, k = 1, 2, \dots, n$ matrisani xos qiymatlari bo'lsin.

$\mu = (\mu_1, \dots, \mu_{n-1}) -$ esa $\lambda' U = (u_{j\bar{k}})$, $j, k = 1, 2, \dots, n-1$ matrisani xos qiymatlari

bo'lsin. Unitar almashtirishlardan keyin (tayinlangan $(\lambda', 0) \in D$ nuqtada) U

quyidagi ko'rinishga keladi:

$$U = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & u_{1\bar{n}} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_{n-1} & u_{n-1\bar{n}} \\ u_{n\bar{1}} & u_{n\bar{2}} & \dots & u_{n\bar{n-1}} & u_{n\bar{n}} \end{pmatrix}$$

$u(z) \in m-sh$ ekan, u holda (ixtiyoriy tayinlangan $t \in \mathbb{R}_+$ parametr uchun)

$u_t(z) = u(z) + t|z_n|^2 \in m-sh$ bo'ladi.

Natijada $\lambda(t) = \lambda(u_t) \in \Gamma_m$ vektor yuqorida isbotlanganiga ko'ra

$\lambda'(t) = (\lambda_1(t), \dots, \lambda_{n-1}(t)) \in \Gamma_{m-1}$. Boshqa tomondan $\lambda(t)$ quyidagi

$$\begin{vmatrix} \lambda - \mu_1 & 0 & \dots & 0 & u_{1\bar{n}}^- \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda - \mu_{n-1} & u_{n-1\bar{n}}^- \\ u_{n\bar{1}} & u_{n\bar{2}} & \dots & u_{n\bar{n-1}} & \lambda - t - u_{n\bar{n}}^- \end{vmatrix} = 0$$

tenglamaning ildizi. Yana almashtirishlar bajarsak,

$$\begin{vmatrix} \lambda - \mu_1 & 0 & \dots & 0 & \frac{u_{1\bar{n}}^-}{t} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda - \mu_{n-1} & \frac{u_{n-1\bar{n}}^-}{t} \\ u_{n\bar{1}} & u_{n\bar{2}} & \dots & u_{n\bar{n-1}} & \frac{\lambda - u_{n\bar{n}}^-}{t} - 1 \end{vmatrix} = 0$$

$t \rightarrow \infty$ da $\lambda_j(t) \rightarrow \mu_j$, $\lambda(t) \in \Gamma_{m-1}$ ekan, u holda Γ_{m-1} ning yopiqligidan $\mu \in \Gamma_{m-1}$ ya'ni $u|_p = u(z, 0) \in m-sh$ ekanligi kelib chiqadi. **8-xossa isbotlandi.**

3.2. Qat'iy $m-sh$ funksiyalar

3.2.1-ta'rif. Agar $u \in m-sh(D) \cap C^2(D)$, ($D \subset \mathbb{C}^n$) funksiya uchun $\varepsilon > 0$ son topilib, $u - \varepsilon \|z\|^2$ funksiya ham D sohada $m-sh$ funksiya bo'lsa, u holda $u(z)$ funksiya D sohada qat'iy $m-sh$ funksiya deyiladi.

Biz ikki marta silliq $u \in C^2(D)$ funksiya uchun qat'iy $m-sh$ funksiya ta'rifini boshqacha ko'rinishda ham ifodalashimiz mumkin:

3.2.2-ta'rif. Agar $u \in C^2(D)$, ($D \subset \mathbb{C}^n$) funksiya uchun ushbu

$$(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} > 0, \quad \forall k = 1, 2, \dots, m \quad (3.2.1)$$

qat'iy tengsizliklar o'rinli bo'lsa, u holda bu funksiya qat'iy $m-sh$ funksiya deyiladi. Ravshanki, bu ikki ta'rif bir biriga ekvivalent. Quyidagi lemmani

qaraymiz:

3.2.1-lemma. Agar

$$\begin{cases} H_1(\lambda) = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n \geq 0 \\ H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n \geq 0 \\ H_3(\lambda) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_{n-2}\lambda_{n-1}\lambda_n \geq 0 \\ \dots \\ H_{m-1}(\lambda) = \lambda_1\dots\lambda_{m-1} + \dots + \lambda_{n-m+2}\dots\lambda_n \geq 0 \\ H_m(\lambda) = \lambda_1\dots\lambda_m + \dots + \lambda_{n-m+1}\dots\lambda_n > 0. \end{cases} \quad (3.2.2)$$

bo'lsa, u holda $H_{m-1} > 0$ bo'ladi.

Isboti. a) Bu lemma $m=1$ uchun trivial bo'lib, uni $m=2$ uchun isbotlaylik. Haqiqatdan, $u \in 2-sh(D) \cap C^2(D)$ ($D \subset \mathbb{C}^n$) funksiya uchun

$$\begin{cases} (dd^c u)^2 \wedge (dd^c |z|^2) > 0 \Rightarrow H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n > 0 \\ (dd^c u) \wedge (dd^c |z|^2)^2 \geq 0 \Rightarrow H_1(\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n \geq 0 \end{cases}$$

bo'lsin. Ushbu

$$\begin{cases} (dd^c u)^2 \wedge (dd^c |z|^2) > 0 \Rightarrow H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n > 0 \\ (dd^c u) \wedge (dd^c |z|^2)^2 > 0 \Rightarrow H_1(\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n > 0 \end{cases}$$

bo'lishini isbot qilaylik. Agar biz

$$\begin{cases} H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n > 0 \\ H_1(\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n = 0 \end{cases}$$

deb faraz qilsak, u holda

$$H_1^2(\lambda) = (\lambda_1 + \lambda_2 + \dots + \lambda_n)^2 = \lambda_1^2 + \dots + \lambda_n^2 + 2H_2(\lambda) = 0 \Rightarrow$$

$$H_2(\lambda) = -\frac{\lambda_1^2 + \dots + \lambda_n^2}{2} \leq 0$$

bo`ladi. Bu esa ziddiyatga olib keladi. Demak, $H_1(\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n > 0$.

Endi bu lemmani umumiy holda, ya`ni ixtiyoriy $2 < m \leq n$ uchun isbotlaymiz:

$$\begin{aligned} H_2^2(\lambda) &= [(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_1\lambda_n) + (\lambda_2\lambda_3 + \dots + \lambda_2\lambda_n) + (\lambda_3\lambda_4 + \dots + \lambda_3\lambda_n) + \dots + \lambda_{n-1}\lambda_n]^2 = \\ &= (\lambda_1\lambda_2)^2 + \dots + (\lambda_{n-1}\lambda_n)^2 + 2\lambda_1^2\lambda_2\lambda_3 + 2\lambda_1^2\lambda_2\lambda_4 + \dots + 2\lambda_1^2\lambda_{n-1}\lambda_n + \\ &+ 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3 + \dots + \lambda_n) + 2\lambda_1\lambda_2\lambda_4(\lambda_2 + \lambda_3 + \dots + \lambda_n) + \dots + \\ &+ 2\lambda_1\lambda_{n-1}\lambda_n(\lambda_2 + \lambda_3 + \dots + \lambda_n) + 2\lambda_2^2\lambda_3\lambda_4 + \dots + 2\lambda_2^2\lambda_{n-1}\lambda_n + \\ &+ 2\lambda_2\lambda_3\lambda_4(\lambda_3 + \dots + \lambda_n) + \dots + 2\lambda_2\lambda_{n-1}\lambda_n(\lambda_3 + \dots + \lambda_n) + \dots + \\ &+ 2\lambda_{n-2}^2\lambda_{n-1}\lambda_n + 2\lambda_{n-2}\lambda_{n-1}\lambda_n(\lambda_{n-1} + \lambda_n) = (\lambda_1\lambda_2)^2 + \dots + (\lambda_{n-1}\lambda_n)^2 + \\ &+ 2(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_1\lambda_{n-1}\lambda_n)(\lambda_1 + \dots + \lambda_n) + \\ &+ 2(\lambda_2\lambda_3\lambda_4 + \dots + \lambda_2\lambda_{n-1}\lambda_n)(\lambda_2 + \dots + \lambda_n) + \dots + 2\lambda_{n-2}\lambda_{n-1}\lambda_n(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = \\ &= (\lambda_1\lambda_2)^2 + \dots + (\lambda_{n-1}\lambda_n)^2 + 2(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_1\lambda_{n-1}\lambda_n)H_1(\lambda) + \\ &+ 2(\lambda_2\lambda_3\lambda_4 + \dots + \lambda_2\lambda_{n-1}\lambda_n)(H_1(\lambda) - \lambda_1) + \dots + 2\lambda_{n-2}\lambda_{n-1}\lambda_n(H_1(\lambda) - \lambda_{n-3} - \lambda_{n-4} - \dots - \lambda_1) = \\ &= (\lambda_1\lambda_2)^2 + \dots + (\lambda_{n-1}\lambda_n)^2 + (\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \dots + \lambda_{n-2}\lambda_{n-1}\lambda_n)H_1(\lambda) - \\ &- 2(\lambda_1\lambda_2\lambda_3\lambda_4 + \dots + \lambda_{n-3}\lambda_{n-2}\lambda_{n-1}\lambda_n) = 2H_3(\lambda)H_1(\lambda) - 2H_4(\lambda) + (\lambda_1\lambda_2)^2 + \dots + (\lambda_{n-1}\lambda_n)^2 \\ &\Rightarrow H_3(\lambda) = \frac{H_2^2(\lambda) - (\lambda_1\lambda_2)^2 - \dots - (\lambda_{n-1}\lambda_n)^2 + 2H_4(\lambda)H_0(\lambda)}{2H_1(\lambda)} \end{aligned}$$

Yuqoridagi kabi hisoblashlar bajarib, $H_4(\lambda)$ ni quyidagi ko`rinishda ifodalash mumkin:

$$H_4(\lambda) = \frac{H_3^2(\lambda) - \sum_{1 \leq j_1 < j_2 < j_3 \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \lambda_{j_3}^2 + 2H_5(\lambda)H_1(\lambda)}{2H_2(\lambda)}$$

Xuddi shunday usul bilan $H_m(\lambda)$ ni

$$H_m(\lambda) = \frac{H_{m-1}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{m-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{m-1}}^2 + 2H_{m+1}(\lambda)H_{m-3}(\lambda)}{2H_{m-2}(\lambda)}, \quad 3 \leq m \leq n$$

(3.2.3)

ko`rinishda aniqlashimiz mumkin. (3.2.3) tenglikning to`g`ri ekanligini matematik induktsiyadan foydalanib isbotlaymiz:

$m=3$ uchun uchun to`g`ri ekanligini yuqorida tekshirib ko`rdik.

$m=n-1$ uchun to`g`ri deb faraz qilib, (3.2.3) tenglikni $m=n$ uchun to`g`ri ekanligini tekshirib ko`ramiz:

$$H_n(\lambda) = \frac{H_{n-1}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{n-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{n-1}}^2 + 2H_{n+1}(\lambda)H_{n-3}(\lambda)}{2H_{n-2}(\lambda)} \quad (3.2.4)$$

$H_{n+1}(\lambda) = 0$ bo`lgani uchun (14) ni quyidagi ko`rinishda ifodalashimiz mumkin:

$$H_n(\lambda) = \frac{H_{n-1}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{n-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{n-1}}^2}{2H_{n-2}(\lambda)} \quad (3.2.5)$$

$$\begin{aligned} H_{n-1}^2(\lambda) &= (\lambda_1 \lambda_2 \dots \lambda_{n-1} + \lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_n + \lambda_1 \lambda_2 \dots \lambda_{n-3} \lambda_{n-1} \lambda_n + \dots + \lambda_2 \lambda_3 \dots \lambda_{n-1} \lambda_n)^2 = \\ &= (\lambda_1 \lambda_2 \dots \lambda_{n-1})^2 + (\lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_n)^2 + \dots + (\lambda_2 \lambda_3 \dots \lambda_{n-1} \lambda_n)^2 + 2(\lambda_1 \lambda_2 \dots \lambda_{n-2})^2 \lambda_{n-1} \lambda_n + \\ &+ 2(\lambda_1 \lambda_2 \dots \lambda_{n-3})^2 \lambda_{n-2} \lambda_n + \dots + 2\lambda_1 \lambda_2 (\lambda_3 \lambda_4 \dots \lambda_{n-1} \lambda_n)^2 = (\lambda_1 \lambda_2 \dots \lambda_{n-1})^2 + (\lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_n)^2 + \\ &+ \dots + (\lambda_2 \lambda_3 \dots \lambda_{n-1} \lambda_n)^2 + 2\lambda_1 \lambda_2 \lambda_3 \lambda_4 \dots \lambda_{n-1} \lambda_n (\lambda_1 \lambda_2 \dots \lambda_{n-2} + \lambda_1 \lambda_2 \dots \lambda_{n-3} \lambda_{n-1} + \dots + \lambda_3 \lambda_4 \dots \lambda_n) = \\ &= 2H_n(\lambda)H_{n-2}(\lambda) + (\lambda_1 \lambda_2 \dots \lambda_{n-1})^2 + (\lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_n)^2 + \dots + (\lambda_2 \lambda_3 \dots \lambda_{n-1} \lambda_n)^2 \end{aligned}$$

Yuqoridagi ifodadan $H_n(\lambda)$ ni topsak, (3.2.5) tenglik kelib chiqadi.

Endi yuqoridagilardan quyidagicha xulosa qilish mumkin:

Agar $H_1(\lambda) \geq 0, H_2(\lambda) \geq 0$ bo'lsa, u holda

$$H_2(\lambda) = \frac{H_1^2(\lambda) - \sum_{1 \leq i \leq n} \lambda_i^2}{2} \geq 0$$

ekanligidan $H_1^2(\lambda) - \sum_{1 \leq i \leq n} \lambda_i^2 \geq 0$ bo'lishi kelib chiqadi. Lemma shartiga ko'ra

$H_1(\lambda) \geq 0, \dots, H_{m-1}(\lambda) \geq 0$ bo'lgani uchun

$$H_k^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{m-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_k}^2 \geq 0, \quad 3 \leq k \leq m-2$$

bo'lishini bilib olish mumkin. Teskarisini faraz qilsak, ya'ni

$$H_k^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_k}^2 < 0, \quad 3 \leq k \leq m-2$$

bo'lsa, u holda $H_{k+1}(\lambda) = 0$ yoki $H_{k-3}(\lambda) = 0, \quad 3 \leq k \leq m-1$ bo'lganda

$$\begin{aligned} H_k(\lambda) &= \frac{H_{k-1}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{k-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{k-1}}^2 + 2H_{k+1}(\lambda)H_{k-3}(\lambda)}{2H_{k-1}(\lambda)} = \\ &= \frac{H_{k-1}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{k-1} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{k-1}}^2}{2H_{k-2}(\lambda)} < 0, \quad 3 \leq k \leq m-2 \end{aligned}$$

ekanligi kelib chiqadi. Bu esa lemma shartiga zid bo'ladi. Demak, $H_i(\lambda) \geq 0$,

$i = \overline{1, m-1}$ bo'lganda $H_k^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_k}^2 \geq 0, \quad 3 \leq k \leq m-2$ ekan.

Agar $H_{m-1}(\lambda) = 0$ bo'lsa, u holda

$$\frac{H_{m-2}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{m-2} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{m-2}}^2 + 2H_m(\lambda)H_{m-4}(\lambda)}{2H_{m-3}(\lambda)} = 0 \Rightarrow$$

$$H_m(\lambda) = - \frac{H_{m-2}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{m-2} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{m-2}}^2}{2H_{m-4}(\lambda)} \leq 0$$

bo`lishi kelib chiqadi. Bunda  $H_{m-4} \neq 0$ , agar  $H_{m-4} = 0$  bo`lib qolsa, u holda

$$H_{m-2}^2(\lambda) - \sum_{1 \leq j_1 < j_2 < \dots < j_{m-2} \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 \dots \lambda_{j_{m-2}}^2 = 0 \Rightarrow$$

$$\lambda_1 = \lambda_2 = \dots \lambda_n = 0 \Rightarrow H_m = 0$$

Bu esa ziddiyatga olib keladi. Demak, $H_{m-1}(\lambda) > 0$ ekan.

Lemma isbotlandi.

Endi yuqoridagi lemmaga asoslanib quyidagi teoremani isbotlash mumkin:

3.2.1-teorema. Agar $u \in m-sh(D) \cap C^2(D)$ bo`lib, (3.1.1)

tengsizliklarda $k = m$ uchun $(dd^c u)^m \wedge (dd^c |z|^2)^{n-m} > 0$ bo`lsa, u holda

qolgan barcha $k = \overline{1, m-1}$ lar uchun ham $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} > 0$ bo`ladi,

ya'ni $u \in m-sh(D) \cap C^2(D)$ funksiyaning qat'iy $m-sh$ bo`lishligi uchun

$(dd^c u)^m \wedge (dd^c |z|^2)^{n-m} > 0$ ning bajarilishi zarur va yetarlidir.

B.I. Abdullaev tomonidan [15] maqolada ushbu

$$u = \ln(x_1^2 + x_2^2 + x_3^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2] - 3\ln 2, \quad (3.2.6)$$

bunda $z_j = x_j + iy_j, j = 1, 2, 3$, funksiyaning $1-wsh$ va $2-wsh$ sinflariga

tegishli ekanligi isbot qilingan. Jumladan Abdullaev tomonidan, bu funksiyaning 2 –subgarmoniklikka, ya'ni 2–sh sinfiga tegishliligi masalasi ko'tarilgan edi. Quyida biz umumiy holda $\mathbb{C}^n$ fazosida

$$u = \ln(x_1^2 + x_2^2 + \dots + x_n^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + \dots + (z_n + \bar{z}_n)^2] - n \ln 2 \quad (3.2.7)$$

funksiyaning m –sh ($1 \leq m \leq n$) masalasini ko'ramiz.

Oldin (3.2.6) funksiyaning 2–sh likka tekshirib ko'ramiz: 1-ta'rifga ko'ra (3.2.6) funksiya 2–sh bo'lishi uchun

$$\begin{cases} (dd^c u) \wedge (dd^c |z|^2)^2 \geq 0 \\ (dd^c u)^2 \wedge (dd^c |z|^2) \geq 0 \end{cases} \quad (3.2.8)$$

tengsizliklar bir vaqtda o'rinli bo'lishi kerak edi. Dastlab quyidagi

hisoblashlarni bajaramiz:

$$\frac{\partial u}{\partial z_1} = \frac{1}{|z + \bar{z}|^2} 2(z_1 + \bar{z}_1) = 2 \frac{z_1 + \bar{z}_1}{|z + \bar{z}|^2};$$

$$\frac{\partial u}{\partial z_2} = \frac{1}{|z + \bar{z}|^2} \cdot 2(z_2 + \bar{z}_2) = 2 \frac{z_2 + \bar{z}_2}{|z + \bar{z}|^2};$$

$$\frac{\partial u}{\partial z_3} = \frac{1}{|z + \bar{z}|^2} \cdot 2(z_3 + \bar{z}_3) = 2 \frac{z_3 + \bar{z}_3}{|z + \bar{z}|^2};$$

.....

$$\frac{\partial u}{\partial z_n} = \frac{1}{|z + \bar{z}|^2} \cdot 2(z_n + \bar{z}_n) = 2 \frac{z_n + \bar{z}_n}{|z + \bar{z}|^2};$$

$$1) \quad u_{11} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} = \frac{2|z + \bar{z}|^2}{|z + \bar{z}|^4} - \frac{2(z_1 + \bar{z}_1) \cdot 2(z_1 + \bar{z}_1)}{|z + \bar{z}|^4} = \frac{2}{|z + \bar{z}|^2} - \frac{4(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^4};$$

$$2) \ u_{1\bar{2}} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} = -\frac{2(z_1 + \bar{z}_1) \cdot 2(z_2 + \bar{z}_2)}{|z + \bar{z}|^4} = -4 \frac{(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4};$$

$$3) \ u_{1\bar{3}} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_3} = -4 \frac{(z_1 + \bar{z}_1)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4};$$

$$4) \ u_{2\bar{1}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} = -4 \frac{(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4};$$

$$5) \ u_{2\bar{2}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} = \frac{2}{|z + \bar{z}|^2} - \frac{4(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^4};$$

$$6) \ u_{2\bar{3}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} = -4 \frac{(z_2 + \bar{z}_2)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4};$$

$$7) \ u_{3\bar{1}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} = -4 \frac{(z_1 + \bar{z}_1)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4};$$

$$8) \ u_{3\bar{2}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} = -4 \frac{(z_2 + \bar{z}_2)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4};$$

$$9) \ u_{3\bar{3}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} = \frac{2}{|z + \bar{z}|^2} - \frac{4(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^4};$$

$$\Rightarrow u_{s\bar{t}} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_t} = -4 \frac{(z_s + \bar{z}_s)(z_t + \bar{z}_t)}{|z + \bar{z}|^4}, \quad s, t = \overline{1, n}; \quad s \neq t$$

$$u_{ss} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_s} = \frac{2}{|z + \bar{z}|^2} - \frac{4(z_s + \bar{z}_s)^2}{|z + \bar{z}|^4}, \quad s = \overline{1, n}$$

Endi (3.2.8) tengsizliklar sistemasidagi 1-tengsizlikni tekshirib ko`ramiz:

$$(dd^c u) \wedge (dd^c |z|^2)^2 = \left(\frac{i}{2}\right)^3 (u_{1\bar{1}} + u_{2\bar{2}} + u_{3\bar{3}}) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 =$$

$$\left(\frac{i}{2}\right)^3 \frac{2}{|z + \bar{z}|^2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 > 0$$

2-tengsizlikni ham tekshirib ko`ramiz:

$$\begin{aligned}
& (dd^c u) \wedge (dd^c |z|^2)^2 = 2 \left(\frac{i}{2} \right)^3 (u_{1\bar{1}} \cdot u_{2\bar{2}} + u_{1\bar{1}} \cdot u_{3\bar{3}} + u_{2\bar{2}} \cdot u_{3\bar{3}} - \\
& - u_{1\bar{2}} \cdot u_{2\bar{1}} - u_{1\bar{3}} \cdot u_{3\bar{1}} - u_{2\bar{3}} \cdot u_{3\bar{2}}) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 = \\
& = \left[\begin{aligned} & \frac{4}{|z + \bar{z}|^4} - \frac{8(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^6} - \frac{8(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^6} + \frac{16(z_1 + \bar{z}_1)^2 (z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} + \\ & + \frac{4}{|z + \bar{z}|^4} - \frac{8(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^6} - \frac{8(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^6} + \frac{16(z_3 + \bar{z}_3)^2 (z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} + \\ & + \frac{4}{|z + \bar{z}|^4} - \frac{8(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^6} - \frac{8(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^6} + \frac{16(z_1 + \bar{z}_1)^2 (z_3 + \bar{z}_3)^2}{|z + \bar{z}|^8} - \\ & - \frac{16(z_1 + \bar{z}_1)^2 (z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} - \frac{16(z_1 + \bar{z}_1)^2 (z_3 + \bar{z}_3)^2}{|z + \bar{z}|^8} - \frac{16(z_2 + \bar{z}_2)^2 (z_3 + \bar{z}_3)^2}{|z + \bar{z}|^8} \end{aligned} \right] dV = \\
& = \frac{12}{|z + \bar{z}|^4} - \frac{16 \left[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2 \right]}{|z + \bar{z}|^8} dV = \frac{12 - 16}{|z + \bar{z}|^4} dV = -\frac{4}{|z + \bar{z}|^4} dV < 0
\end{aligned}$$

bunda $dV = 2 \left(\frac{i}{2} \right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3$. Demak,

$$\begin{cases} (dd^c u) \wedge (dd^c |z|^2)^2 > 0 \\ (dd^c u) \wedge (dd^c |z|^2)^2 < 0 \end{cases}$$

SHunday qilib, (3.2.6) funksiya 2-sh funksiya emas ekan.

Endi (3.2.7) funksiyaning m - subgarmoniklikka tekshirishda

$dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge d\bar{z}_1 + \dots + \lambda_n dz_n \wedge d\bar{z}_n]$ diagonal formadan foydalanamiz.

$(dd^c u)^m \wedge \beta^{n-m} = m!(n-m)!H_m(u)\beta^n$ bo`lgani uchun

$H_m(u) = \sum_{1 \leq j_1 < \dots < j_m \leq n} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_m}$ – gessian vektori musbat aniqlangan bo`lsa, u

holda $(dd^c u)^m \wedge (dd^c |z|^2)^{n-m} \geq 0$ bo`ladi.

Dastlab $\lambda_1, \lambda_2, \dots, \lambda_n$ xos qiymatlarni topamiz:

a) $n=1$ da (3.2.7) funksiya $u = \ln x^2 = \ln(z + \bar{z})^2 - 2\ln 2$ ko`rinishda bo`ladi va

$$dd^c u = \frac{\partial^2 u}{\partial z \partial \bar{z}} = -\frac{2}{|z + \bar{z}|^2} dz \wedge d\bar{z} \Rightarrow \lambda = -\frac{2}{|z + \bar{z}|^2} < 0 \Rightarrow H_1(\lambda) = -\frac{2}{|z + \bar{z}|^2} < 0.$$

Demak, bu funksiya subgarmonik emas ekan.

b) $n=2$ da:

$$u = \ln(x_1^2 + x_2^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2] - 2\ln 2 \text{ va}$$

$$\begin{vmatrix} u_{1\bar{1}} - \lambda & u_{1\bar{2}} \\ u_{2\bar{1}} & u_{2\bar{2}} - \lambda \end{vmatrix} = u_{1\bar{1}}u_{2\bar{2}} - (u_{1\bar{1}} + u_{2\bar{2}})\lambda + \lambda^2 - u_{1\bar{2}}u_{2\bar{1}} = 0 \Rightarrow$$

$$\Rightarrow \frac{4}{|z + \bar{z}|^4} - \frac{8(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^6} - \frac{8(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^6} + \frac{16(z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} -$$

$$- \frac{16(z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} + \lambda^2 = 0 \Rightarrow \lambda^2 - \frac{4}{|z + \bar{z}|^4} = 0 \Rightarrow \lambda_{1,2} = \pm \frac{2}{|z + \bar{z}|^2},$$

$$\begin{cases} H_1(\lambda(u)) = \lambda_1 + \lambda_2 = \frac{2}{|z + \bar{z}|^2} - \frac{2}{|z + \bar{z}|^2} = 0 \\ H_2(\lambda(u)) = \lambda_1 \lambda_2 = -\frac{4}{|z + \bar{z}|^4} < 0 \end{cases}$$

Demak, $u = \ln(x_1^2 + x_2^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2] - 2\ln 2$ funksiya 1-sh bo`la oladi, lekin 2-sh bo`la olmas ekan.

$$v) n=3 \text{ da: } u = \ln(x_1^2 + x_2^2 + x_3^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2] - 2\ln 2$$

$$\begin{vmatrix} u_{1\bar{1}} - \lambda & u_{1\bar{2}} & u_{1\bar{3}} \\ u_{2\bar{1}} & u_{2\bar{2}} - \lambda & u_{2\bar{3}} \\ u_{3\bar{1}} & u_{3\bar{2}} & u_{3\bar{3}} - \lambda \end{vmatrix} = -\lambda^3 + (u_{1\bar{1}} + u_{2\bar{2}} + u_{3\bar{3}})\lambda^2 +$$

$$+ (-u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{1}}u_{3\bar{3}} - u_{2\bar{2}}u_{3\bar{3}} + u_{1\bar{2}}u_{2\bar{1}} + u_{1\bar{3}}u_{3\bar{1}} + u_{2\bar{3}}u_{3\bar{2}})\lambda +$$

$$+ u_{1\bar{1}}u_{2\bar{2}}u_{3\bar{3}} + u_{1\bar{2}}u_{2\bar{3}}u_{3\bar{1}} + u_{2\bar{1}}u_{3\bar{2}}u_{1\bar{3}} - u_{1\bar{3}}u_{2\bar{2}}u_{3\bar{1}} - u_{1\bar{1}}u_{2\bar{3}}u_{3\bar{2}} - \\ - u_{2\bar{1}}u_{1\bar{2}}u_{3\bar{3}} = 0 \Rightarrow$$

$$u_{1\bar{1}} + u_{2\bar{2}} + u_{3\bar{3}} = \frac{2}{|z + \bar{z}|^2} - \frac{4(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^4} + \frac{2}{|z + \bar{z}|^2} - \frac{4(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^4} + \frac{2}{|z + \bar{z}|^2} - \frac{4(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^4} = \\ = \frac{6}{|z + \bar{z}|^2} - \frac{4((z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2)}{|z + \bar{z}|^4} = \frac{6}{|z + \bar{z}|^2} - \frac{4|z + \bar{z}|^2}{|z + \bar{z}|^4} = \frac{2}{|z + \bar{z}|^2}$$

$$-u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{1}}u_{3\bar{3}} - u_{2\bar{2}}u_{3\bar{3}} + u_{1\bar{2}}u_{2\bar{1}} + u_{1\bar{3}}u_{3\bar{1}} + u_{2\bar{3}}u_{3\bar{2}} = \\ - \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^4} \right) \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^4} \right) - \\ - \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^4} \right) \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^4} \right) - \\ - \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^4} \right) \left(\frac{2}{|z + \bar{z}|^2} - \frac{4(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^4} \right) +$$

$$\begin{aligned}
& + \left(-4 \frac{(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4} \right) \left(-4 \frac{(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4} \right) + \\
& + \left(-4 \frac{(z_1 + \bar{z}_1)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4} \right) \left(-4 \frac{(z_1 + \bar{z}_1)(z_3 + \bar{z}_3)}{|z + \bar{z}|^4} \right) + \\
& + \left(-4 \frac{(z_3 + \bar{z}_3)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4} \right) \left(-4 \frac{(z_3 + \bar{z}_3)(z_2 + \bar{z}_2)}{|z + \bar{z}|^4} \right) = \\
& = - \left(\frac{4}{|z + \bar{z}|^4} - \frac{8(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^6} - \frac{8(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^6} + \frac{16(z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} \right) - \\
& - \left(\frac{4}{|z + \bar{z}|^4} - \frac{8(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^6} - \frac{8(z_1 + \bar{z}_1)^2}{|z + \bar{z}|^6} + \frac{16(z_1 + \bar{z}_1)^2(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^8} \right) - \\
& - \left(\frac{4}{|z + \bar{z}|^4} - \frac{8(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^6} - \frac{8(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^6} + \frac{16(z_3 + \bar{z}_3)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} \right) + \\
& + \frac{16(z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} + \frac{16(z_1 + \bar{z}_1)^2(z_3 + \bar{z}_3)^2}{|z + \bar{z}|^8} + \frac{16(z_3 + \bar{z}_3)^2(z_2 + \bar{z}_2)^2}{|z + \bar{z}|^8} = \\
& = -\frac{12}{|z + \bar{z}|^4} + \frac{16((z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2)}{|z + \bar{z}|^6} = \frac{4}{|z + \bar{z}|^4}
\end{aligned}$$

$$\begin{aligned}
& u_{1\bar{1}}u_{2\bar{2}}u_{3\bar{3}} + u_{1\bar{2}}u_{2\bar{3}}u_{3\bar{1}} + u_{2\bar{1}}u_{3\bar{2}}u_{1\bar{3}} - u_{1\bar{3}}u_{2\bar{2}}u_{3\bar{1}} - u_{1\bar{1}}u_{2\bar{3}}u_{3\bar{2}} - \\
& - u_{2\bar{1}}u_{1\bar{2}}u_{3\bar{3}} = +u_{1\bar{1}}u_{2\bar{2}}u_{3\bar{3}} + u_{1\bar{2}}u_{2\bar{3}}u_{3\bar{1}} + u_{2\bar{1}}u_{3\bar{2}}u_{1\bar{3}} - u_{1\bar{3}}u_{2\bar{2}}u_{3\bar{1}} - u_{1\bar{1}}u_{2\bar{3}}u_{3\bar{2}} - \\
& - u_{2\bar{1}}u_{1\bar{2}}u_{3\bar{3}} = -\frac{8}{|z + \bar{z}|^6} \Rightarrow
\end{aligned}$$

$$\begin{aligned}
& -\lambda^3 + \frac{2}{|z + \bar{z}|^2}\lambda^2 + \frac{4}{|z + \bar{z}|^4}\lambda - \frac{8}{|z + \bar{z}|^6} = - \left(\lambda - \frac{2}{|z + \bar{z}|^2} \right) \left(\lambda^2 - \frac{4}{|z + \bar{z}|^4} \right) = 0 \Rightarrow \\
& \lambda_{1,2} = \frac{2}{|z + \bar{z}|^2}, \quad \lambda_3 = -\frac{2}{|z + \bar{z}|^2}
\end{aligned}$$

$$\begin{cases} H_1(\lambda(u)) = \lambda_1 + \lambda_2 + \lambda_3 = \frac{2}{|z + \bar{z}|^2} > 0 \\ H_2(\lambda(u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -\frac{4}{|z + \bar{z}|^4} < 0 \end{cases}$$

Demak, $u = \ln(x_1^2 + x_2^2 + x_3^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + (z_3 + \bar{z}_3)^2] - 2\ln 2$ funksiya

ham $1-sh$ bo`la oladi, lekin  $2-sh$  bo`la olmas ekan.

g) $n=4$ da esa (3.2.7) funksiya $2-sh$ bo`la oladi :

$$\begin{vmatrix} u_{1\bar{1}} - \lambda & \cdots & u_{1\bar{4}} \\ \vdots & \ddots & \vdots \\ u_{4\bar{1}} & \cdots & u_{4\bar{4}} - \lambda \end{vmatrix} = \lambda^4 - \frac{4}{|z + \bar{z}|^2} \lambda^3 + \frac{16}{|z + \bar{z}|^6} \lambda - \frac{16}{|z + \bar{z}|^8} =$$

$$= \left(\lambda - \frac{2}{|z + \bar{z}|^2} \right)^3 \left(\lambda + \frac{2}{|z + \bar{z}|^2} \right) = 0 \Rightarrow \lambda_{1,2,3} = \frac{2}{|z + \bar{z}|^2}, \lambda_4 = -\frac{2}{|z + \bar{z}|^2}$$

$$\begin{cases} H_1(\lambda) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{4}{|z + \bar{z}|^2} > 0 \\ H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 0 \\ H_3(\lambda) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = -\frac{16}{|z + \bar{z}|^6} < 0 \\ H_4(\lambda) = \lambda_1\lambda_2\lambda_3\lambda_4 = -\frac{16}{|z + \bar{z}|^8} < 0. \end{cases}$$

d) $n=5$ da ham (3.2.7) funksiya $2-sh$ bo`ladi :

$$\lambda_{1,2,3,4} = \frac{2}{|z + \bar{z}|^2}, \lambda_5 = -\frac{2}{|z + \bar{z}|^2}$$

$$H_1(\lambda(u)) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = \frac{6}{|z + \bar{z}|^2} > 0$$

$$H_2(\lambda(u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_4 + \\ + \lambda_3\lambda_5 + \lambda_4\lambda_5 = \frac{12}{|z + \bar{z}|^4} - \frac{8}{|z + \bar{z}|^4} > 0$$

$$H_3(\lambda(u)) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \\ + \lambda_2\lambda_4\lambda_5 + \lambda_3\lambda_4\lambda_5 = \frac{32}{|z + \bar{z}|^6} - \frac{48}{|z + \bar{z}|^6} < 0$$

e) $n=6$ da esa (3.2.7) funksiya 3-sh bo`ladi: $\lambda_{1,2,3,4,5} = \frac{2}{|z + \bar{z}|^2}, \lambda_6 = -\frac{2}{|z + \bar{z}|^2}$

$$H_1(\lambda(u)) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \frac{8}{|z + \bar{z}|^2} > 0$$

$$H_2(\lambda(u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_1\lambda_6 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_2\lambda_6 + \\ \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6 = \frac{20}{|z + \bar{z}|^4} - \frac{10}{|z + \bar{z}|^4} > 0$$

$$H_3(\lambda(u)) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_2\lambda_6 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_3\lambda_6 + \lambda_1\lambda_4\lambda_5 + \\ + \lambda_1\lambda_4\lambda_6 + \lambda_1\lambda_5\lambda_6 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_2\lambda_3\lambda_6 + \lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_4\lambda_6 + \lambda_2\lambda_5\lambda_6 + \lambda_3\lambda_4\lambda_5 + \\ \lambda_3\lambda_4\lambda_6 + \lambda_3\lambda_5\lambda_6 + \lambda_4\lambda_5\lambda_6 = \frac{20}{|z + \bar{z}|^6} - \frac{20}{|z + \bar{z}|^6} = 0 \Rightarrow (dd^c u)^3 \wedge (dd^c |z|^2)^3 = 0$$

$H_4(\lambda(u)) < 0$ ekanligi bizga ma'lum. Demak, $n=2$ va $n=3$ bo`lgan xususiy holda (3.2.7) funksiya subgarmonik funksiya,  $n=4$  va  $n=5$  da 2-sh funksiya  $n=6$  va  $n=7$  da 3-sh funksiya bo`ladi.

SHunday qilib, biz quyidagi xulosaga keldik:

$u = \ln(x_1^2 + x_2^2 + \dots + x_n^2) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + \dots + (z_n + \bar{z}_n)^2] - 2 \ln 2$ funksiya

$1 \leq m \leq \left\lceil \frac{n}{2} \right\rceil, n \geq 2$ bo'lganda $m - sh$ ekan.

Buni matematik induksiyadan foydalanib isbotlaymiz:

1) $n = 2$ da to'g'riligini yuqorida ko'rsatib o'tdik.

2) $n = 2k$ da to'g'ri deb faraz qilamiz, ya'ni $H_k(\lambda) \geq 0, H_{k+1}(\lambda) < 0$

bunda $\lambda_{1,2,\dots,2k-1} = \frac{2}{|z + \bar{z}|^2}, \lambda_{2k} = -\frac{2}{|z + \bar{z}|^2}$

3) $n = 2k + 1$ da to'g'riligini, ya'ni $H_k(\lambda) \geq 0, H_{k+1}(\lambda) < 0$ ekanligini

isbotlaymiz, bunda $\lambda_{1,2,\dots,2k-1,2k} = \frac{2}{|z + \bar{z}|^2}, \lambda_{2k+1} = -\frac{2}{|z + \bar{z}|^2}$

$H_k(\lambda)$ yig'indida C_{2k+1}^k ta had ishtirok etadi va shulardan C_{2k}^k ta had musbat

bo'ladi, chunki bu hadlarda λ_{2k+1} ishtirok etmaydi. λ_{2k+1} ishtirok etganlari

$$C_{2k+1}^k - C_{2k}^k = \frac{(2k+1)!}{k!(k+1)!} - \frac{(2k)!}{k!k!} = \frac{(k+2) \cdot \dots \cdot (2k+1)}{k!} - \frac{(k+1) \cdot \dots \cdot (2k)}{k!} =$$

ga teng.

$$\frac{(k+2) \cdot \dots \cdot (2k)}{k!} (2k+1-k-1) = \frac{k(k+2) \cdot \dots \cdot (2k)}{k!}$$

Demak, $\frac{k(k+2) \cdot \dots \cdot (2k)}{k!}$ ta had manfiy, $\frac{(k+1) \cdot \dots \cdot (2k)}{k!}$ ta had musbat va

barcha hadlar bir xil qiymatga ega: $\frac{k(k+2) \cdot \dots \cdot (2k)}{k!} < \frac{(k+1) \cdot \dots \cdot (2k)}{k!}$.

SHuning uchun $H_k(\lambda) > 0$. Endi $H_{k+1}(\lambda) < 0$ ekanligini ko'rsatamiz:

$H_{k+1}(\lambda)$ yig'indida esa C_{2k+1}^{k+1} ta had ishtirok etadi va shulardan C_{2k}^{k+1} ta had

musbat bo`ladi, chunki bu hadlarda λ_{2k+1} ishtirok etmaydi. λ_{2k+1} ishtirok etganlari esa

$$C_{2k+1}^{k+1} - C_{2k}^{k+1} = \frac{(2k+1)!}{k!(k+1)!} - \frac{(2k)!}{(k+1)!(k-1)!} = \frac{(k+1) \cdot \dots \cdot (2k+1)}{(k+1)!} - \frac{k(k+1) \cdot \dots \cdot (2k)}{(k+1)!} =$$

$$\frac{(k+1)^2(k+2) \cdot \dots \cdot (2k)}{k!} (2k+1-k-1) = \frac{(k+1)^2(k+2) \cdot \dots \cdot (2k)}{k!}$$

Demak, $\frac{(k+1)^2(k+2) \cdot \dots \cdot (2k)}{k!}$ ta had manfiy, $\frac{k(k+1) \cdot \dots \cdot (2k)}{k!}$ ta had musbat

va barcha hadlar bir xil qiymatga ega:

$$\frac{k(k+1) \cdot \dots \cdot (2k)}{k!} < \frac{(k+1)^2(k+2) \cdot \dots \cdot (2k)}{k!}. \text{ SHuning uchun } H_{k+1}(\lambda) < 0$$

bo`ladi.

SHunday qilib, bu funksiya $n = 2k + 1$ bo`lganda qat'iy $m - sh(m = k)$

funksiya bo`lar ekan.

III BOB bo'yicha xulosa

Ushbu bobda m -sh funksiyalar va ularning xossalari hamda qat'iy m -sh funksiyalar o'rganildi. Ixtiyoriy n -sh funksiyalar plyurisubgarmonik bo'lishi polinomlar nazariyasining asosiy teoremlaridan biri bo'lgan Dekart teoremasi yordamida isbotlandi. m -sh funksiyalarning qat'iy m -sh bo'lish shartlari o'rganildi. Olingan natijalar va usullarni funksiyalar nazariyasining keyingi rivojlanishida va kompleks analizning tadbirlarida qo'llanilishi mumkin.

XULOSA

Dissertatsiya Qat'iy m -sh funksiyalarni o'rganishga bag'ishlangan.

Dissertatsiyada olingan asosiy natijalar yuzasidan quyidagi xulosaga kelindi:

1. $\forall u \in C^2(D) \ (D \subset \mathbb{C}^n)$ funksiya uchun $(dd^c u)^k \wedge (dd^c |z|^2)^{n-k} \geq 0 \ \forall k = \overline{1, m}$ tengsizlikning $k = m$ uchun bajarilishidan qolgan barcha $\forall k = \overline{1, m-1}$ lar uchun bajarilmasligini aniqlandi.

2. m -sh funksiyalar sinfi $m = n$ bo'lgan holda plyurisubgarmonik funksiyalar sinfi bilan ustma-ust tushishi isbotlandi.

3. m -sh funksiyalarning qat'iy m -sh bo'lish shartlari o'rganildi.

Umuman, dissertatsiya ishidagi olingan asosiy natijalar yangi bo'lib, kompleks o'zgaruvchining funksiyalar nazariyasining keyingi rivojida qo'llanilishi mumkin.

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