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BITIRUV MALAKAVIY ISHI

Mavzu: “ Garmonik funksiyaning asosiy xossalari va integral ko’rinishi”

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Kirish

1.Masalaning qo’yilishi. Kompleks o’zgaruvchili funksiyalar nazariyasi fanining umumiy kursidan ma’lum bo’lgan bir qiymatli analitik funksiyaning muhim xossalaridan biri shundan iboratki, agar bu funksiyaning noli bo’lmasa, uning modulini logarmi garmonik funksiyadan iborat bo’ladi. Komplers o’zgaruvchili funksiyalar nazariyasi, asosan uning geometrik qismi-konform akslantirish nazariyasi-fizik tasvirda paydo bo’lgan va rivojlangan.

Aksincha, kompleksli o’zgaruvchili funksiyalar nazariyasining rivojlanishi matematikaning turli yo’nalishlarida muhim amaliy ahamiyatga ega bo’lgan masalalarni yechishda yangi usullarni yaratilishiga olib kelindi. Ikki o’zgaruvchili garmonik funksiyalar bir qiymatli analitik funksiyaning haqiqiy yoki mavhum qismidan iborat bo’lib, Laplas tenglamasining yechimi bo’ladi. Laplas tenglamasi uchun qo’yilgan chegaraviy masalani konform akslantirish orqali aniq sohalarda yechimning yaqqol ko’rinishga tasvirlash muhimdir. Garmonik funksiya tekislikdagi vektor maydon potensiali va chegaraviy masalalar bilan bog’liqdir.

2.Mavzuning dolzarbligi. Matematik fizika tenglamalarini yechishda Laplas tenglamasi o’zining amaliy ahamiyati jihatidan muhim o’rin egallaydi. Tenglama yechimini topishda yechimning mavjudlik va yagonaligi, qo’yilgan chegaraviy shartlarni qanoatlantirishini tekshirish muhimdir. Tekislikda qaralayotgan Laplas tenglamasi uchun Dirixli masalasining yechimini topishda yechim garmonik funksiya bo’lishi kerakligidan kompleks o’zgaruvchili bir qiymatli analitik funksiyadan bog’liqdir. Qaralayotgan soha murakkab ko’rinishga ega bo’lganda buni konform akslantirish orqali yechimni topish mumkin bo’lgan sohaga o’tkaziladi. Konform akslantirilgan sohada Laplas tenglamasi uchun qo’yilgan Dirixle masalasi yechimi Grin funksiyasi orqali qulay ko’rinishda ifodalanadi. Ushbu malakaviy bitiruv ishida matematik fizika tenglamasini yechishda kompleks o’zgaruvchili funksiyalar nazariyasi usulidan foydalanish qulay imkoniyatlarga ega bo'lishligi ushbu mavzuning dolzarbligini ifodalaydi.

3.Ishning maqsadi va vazifalari. Bitiruv malakaviy ishning maqsadi kompleksli o'zgaruvchili funksiyalar nazariyasi kursining muhim tushchalari asosida garmonik funksiyaning asosiy xossalani o'rganishdan iborat.

4.Ilmiy tadqiqot usullari. Kompleksli o'zgaruvchili funksiyalar nazariyasi kursidan bir qiymatli analitik funksiyaning haqiqiy va mavhum qismi, Koshining integral formulasi, o'rta qiymat haqidagi teorema, konform akslantirish tushunchasi, garmonik funksiyalar uchun Grin formulasi, matematik fizikaning Laplas tenglamasi uchun qo'yilgan chegaraviy masalalaridan Dirixle masalasini yechish.

5.Ishning ilmiy ahamiyati. Bitiruv malakaviy ishda olingan natijalar referativ xarakterda bo'lishiga qaramasdan kelgusida elliptik tipdagi tenglamalar uchun qo'yilgan turli xil chegaraviy masalarni yechishda yordam beradi.

6.Ishning amaliy ahamiyati. Bitiruv malakaviy ishda olingan natijalar matematik fizika tenglamari uchun qo'yilgan chegaraviy masalalarni kompleksli o'zgaruvchili funksiyalar nazariyasi elementlari orqali yechish ahamiyatiga ega.

7.Ishning tuzilishi. Bitiruv malakaviy ishi kirish, 2 ta bob, xulosa qismi va foydalanilgan adabiyotlar ro'yxatidan iborat. Ushbu ish … matnli sahifan tashkil topgan va har bir bob paragraflarga ajratilgan, ular o'zining nomerlanish hamda belgilanishiga ega.

Ishning 1-bobida garmonik funksiyaning asosiy xossalai, Koshi-Riman shartlari, qo'shma garmonik funksiyalar o'rganilgan. 2-bobda garmonik funksiyaning integral tasviri yordamida, ya'ni o'rta qiymat haqidagi teorema yordamida konform akslantirish orqali doira va yuqori yarim tekislikda Dirixle masalasini yechish.

8. Olingan natijaning qisqacha mazmuni. **§1.1. da**

1.1–Teorema([1]). $f(z) = u(x, y) + iv(x, y)$ funksiya $z = x + iy$ nuqtada differensiallanuvchi bo'lishi uchun

1) $u(x,y)$ va $v(x,y)$ funksiyalar (x,y) nuqtada differensiallanuvchi;

2) (x,y) nuqtada

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad (1.2)$$

Koshi – Riman shartining bajarilishi zarur va yetarlidir. $f'(z)$ hosila uchun

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \qquad (1.3)$$

formula o’rinlidir.

§1.2.da garmonik funksiyaning asosiy xossalari o’rganilgan

1.1-xossa. Ixtiyoriy $u(x,y)$ garmonik funksiya o’zining x,y argumentlarini analitik funksiyasidan iborat bo’ladi, yani D sohaning har bir $z_0=x_0+iy_0$ nuqtaning yig’indisi ko’rinishda tasvirlanadi.

$$u(x,y)=\sum_{m,n=0}^{\infty}C_{mn}(x-x_0)^m(y-y_0)^n \qquad (1.9)$$

1.2-xossa (o’rta qiymat haqida). Agar $u(z)$ funksiya markazi z nuqtada, radiusi r ga teng bo’lgan yopiq doirada uzluksiz va bu doirada garmonik bo’lsa, u holda

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\varphi})d\varphi. \qquad (1.12)$$

1.3-xossa. O’zgarmasdan farqli bo’lgan garmonik funksiya o’zining eng katta va eng kichik qiymatiga aniqlanish sohasining ichki nuqtasida erishishi mumkin emas.

1.4-xossa. Agar D sohada garmonik $u(x,y)$ funksiya hech bo’lmaganda yuqori yoki quyidan chegaralangan bo’lsa, u holda u o’zgarmas.

1.5-xossa. Agar $u(z)$ funksiya D sohada uzliksiz va yetarlicha kichik r -lar uchun ixtiyoriy z nuqtada

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\varphi})d\varphi$$

bo’lsa, u holda $u(z)$ funksiya D sohada garmonikdir

1.6-xossa. D sohada garmonik va $\overline{D}$ da uzluksiz bo'lgan $u_0(z), u_1(z), ..., u_n(z), ...$ garmonik funksiyalar ketma-ketligi berilgan bo'lsin. Agar $\sum_{k=0}^{\infty} u_k(z)$ qator D ning chegarasida tekis yaqinlashsa, u holda bu qator D-ning ichida ham tekis yaqinlashadi va uning yig'indisi ham D sohada garmonik funksiya bo'ladi.

1.7-xossa. Agar $u(z)$ funksiya bir bog'lamli D sohada garmonik va o'zining xususiy hosilasi bilan $\overline{D}$ da uzluksiz bo'lsa, u holda

$$\int_{\partial D} \frac{\partial u}{\partial n} dS_y = 0,$$

bu yerda $\frac{\partial u}{\partial n}$-normal bo'yicha hosilasi, dS-yoyning differensiali.

§1.3 da Laplas tenglamasining qutb, silindrik va sferik koordinatalardagi ifodasi, ya'ni

$$\Delta_{\rho,\varphi} u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} = 0 \tag{1.18}$$

$$\Delta_{\rho,\varphi,z} u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1.19}$$

$$\Delta_{r,\varphi,\psi} u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\varphi} \frac{\partial}{\partial \varphi} \left(\sin\varphi \frac{\partial u}{\partial \varphi} \right) + \frac{1}{r^2 \sin\varphi} \frac{\partial^2 u}{\partial \psi^2} = 0. \tag{1.20}$$

lar $\nabla u = grad u, \nabla = \frac{\partial}{\partial x} \bar{i} + \frac{\partial}{\partial y} \bar{j} + \frac{\partial}{\partial z} \bar{k}$

$$(\nabla, \nabla u) = (\nabla, \nabla) u = div grad u = \left(\frac{\partial}{\partial x} \bar{i} + \frac{\partial}{\partial y} \bar{j} + \frac{\partial}{\partial z} \bar{k}, \frac{\partial u}{\partial x} \bar{i} + \frac{\partial u}{\partial y} \bar{j} + \frac{\partial u}{\partial z} \bar{k} \right) =$$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, (\nabla, \nabla) = \Delta$$

orqali ifodalashdan hosil qilingan.

Birinchi bobning 4 paragrafida Laplas tenglamasining **elementar** yoki **fundamental** yechimi yozilishi ko'rsatilgan:

1**.2-ta’rif.** ***([3],[6])*** *Laplas tenglamasining berilgan sohaning ajralgan maxsus nuqtalari yoki o’zi-o’zini kesmaydigan silliq sirtlarda maxsuslikka ega bo’lgan yechimiga* ***fundamental*** *yechimi deyiladi. Laplas yenglamasining sohada maxsuslikka ega bo’lmagan va ozining iikinchi tartibli xususiy hosilalari bilan uzluksiz yechimiga esa* ***regulyar*** *yechimi deyiladi.*

$$u(r)=\begin{cases}\ln r, \text{ agar } n=2 \text{ bo'lsa,}\\ \dfrac{1}{r^{n-2}}, \text{ agar } n>2 \text{ bo'lsa}\end{cases}$$

Grinning asosiy integral formulasi kelitirilgan:

$$u(M_0)=\frac{1}{\Omega}\iint_{\Sigma}\left[\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M}-u(M)\frac{\partial}{\partial n_M}\left(\frac{1}{R_{MM_0}}\right)\right]d\sigma_M-\iiint_D\frac{\Delta u(M)}{R_{MM_0}}d\tau_M . \qquad 1.23)$$

Bunda

$$\Omega=\begin{cases}4\pi, \text{ agar } M_0\in D \text{ bo'lsa,}\\ 2\pi, \text{ agar } M_0\in\Sigma \text{ bo'lsa,}\\ 0, \quad \text{agar } M_0\notin D\cup\Sigma \text{ bo'lsa.}\end{cases}$$

Agar u garmonik funksiya bo’lsa, (1.23) formula quyidagi ko’rinishni oladi:

$$u(M_0)=\frac{1}{\Omega}\iint_{\Sigma}\left[\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M}-u(M)\frac{\partial}{\partial n_M}\left(\frac{1}{R_{MM_0}}\right)\right]d\sigma_M \qquad (1.24)$$

Qaralayotgan soha tekislikda biror silliq C yopiq chiziq bilan chegaralangan D sohadan iborat bo’lsa, u holda yuqoridagi mulohazalarda v o’rnida Laplas tenglamasining tekislikdagi fundamental yechimi

$$v=\ln\frac{1}{R_{MM_0}}$$

funksiyani ishlatsak (1.23) va (1.24) ga ox’shash formulalarni olamiz:

$$u(M_0)=\frac{2}{\Omega}\int_C\left[\ln\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M}-u(M)\frac{\partial}{\partial n_M}\left(\ln\frac{1}{R_{MM_0}}\right)\right]ds_M-\iint_S\frac{\Delta u(M)}{R_{MM_0}}d\sigma_M$$

(1.23’)

$$u(M_0)=\frac{2}{\Omega}\int_C\left[\ln\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M}-u(M)\frac{\partial}{\partial n_M}\left(\ln\frac{1}{R_{MM_0}}\right)\right]ds_M\ . \qquad (1.24')$$

II-bob . Garmonik funksiya uchun Dirixli masalasini yechishga bag'ishlangan bo'lib §2.1. da Laplas tenglamasi uchun Dirixle masalasining qo'yilishi, yechimning mavjudligi va yagonaligi ko'rsatilgan.

§2.2. Laplas tenglamasini konform akslantirishga nisbatan invariantligi ko'rsatilgan.

§ 2.3 doirada Laplas tenglamasi uchun Dirixle masalasi o'rganilgan

2.2-teorema. $u(z)$ funksiya $|z|<1$ doirada garmonik, $|z|\le 1$ yopiq doirada uzluksiz bo'lsin. U holda Puasson formulasi

$$u(re^{i\varphi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}u(re^{i\varphi})d\theta \qquad (2.8)$$

o'rinli bo'ladi.

§ 2.4. Yuqori yarim tekislikda Laplas tenglamasi uchun Dirixle masalasiga bag'ishlangan.

2.3-teorema. $u(z)$ funksiya $\operatorname{Im} z>0$ yuqori yarim tekislikda garmonik va chegaralangan, chekli sondagi nuqtalardan tashqari $\operatorname{Im} z=0$ to'g'ri chiziqqacha uzluksiz bo'lsin. U holda Puasson formulasi

$$u(z)=\frac{1}{\pi}\int\frac{yu(t)}{(t-x)^2+y^2}dt \qquad (2.17)$$

o'rinlidir, bu yerda $z=x+iy$, $y>0$.

1-Bob. Garmonik funksiya va uning xossalari

§1.1. Koshi-Riman sharti. Qo'shma garmonik funksiyalar

$w = f(z)$ funksiya z_0 nuqtaning biror atrofida aniqlangan bo'lsin. Agar $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ nisbat $\Delta z \to 0$ da aniq chekl limitga ega bo'lsa, bu limitga $w = f(z)$) funksiyaning z_0 nuqtadagi hosilasi deyiladi va $f'(z_0)$ belgilanadi, $w = f(z)$ funksiyaga z_0 nuqtada differensiallanuvchi deyiladi. Shunday qilib,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{1.1}$$

$w = f(z)$ funksiya sohada differensiallanuvchi deyiladi, agar shu sohaning har bir nuqtasida differensiallanuvchi bo'lsa.

1.1–Teorema([1]). $f(z) = u(x,y) + iv(x,y)$ funksiya $z = x + iy$ nuqtada differensiallanuvchi bo'lishi uchun

1) $u(x,y)$ va $v(x,y)$ funksiyalar (x,y) nuqtada differensiallanuvchi;

2) (x,y) nuqtada

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.2}$$

Koshi – Riman shartining bajarilishi zarur va yetarlidir. $f'(z)$ hosila uchun

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{1.3}$$

formula o'rinlidir.

1.1-misol. a) $f(z) = e^z$ funksiyani differensiallanuvchanlikka tekshiring.

Yechish. $f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x \cos y + ie^x \sin y$ funksiya butun kompleks tekislikda differensiallanuvchi, chunki $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$ funksiyalar (1.2) Koshi – Riman shartini qanoatlantiradi, ya'ni

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

(1.3) formulaga ko'ra

$$f'(z)=(e^z)'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=e^x\cos y+ie^x\sin y=e^x(\cos y+i\sin y)=e^xe^{iy}=e^{x+iy}=e^z .$$

Demak, $$f'(z)=(e^z)'=e^z .$$

b) $f(z)=\bar{z}^{2}$ funksiyani differensiallanuvchanlikka tekshiring.

Yechish. $$f(z)=\bar{z}^2=(x-iy)^2=x^2-y^2-i2xy .$$

$$u(x,y)=x^2-y^2 \text{ va } v(x,y)=-2xy$$

$$\frac{\partial u}{\partial x}=2x,\frac{\partial u}{\partial y}=-2y,\frac{\partial v}{\partial x}=2y,\frac{\partial v}{\partial x}=-2x$$

(1.2) shartdan

$\begin{cases}2x=-2x\\-2y=2y\end{cases}\Rightarrow\begin{cases}4x=0\\4y=0\end{cases}\Rightarrow\begin{cases}x=0\\y=0\end{cases}$ bundan $f(x)=\bar{z}^2$ funksiya z=0 nuqtada differensiallanuvchi ekan.

$f(z)=u(x,y)+iv(x,y)$ funksiya D sohada differensiallanuvchi va $u(x,y)$, $v(x,y)$ funksiyalar ikkinchi shartigacha uzluksiz xususiy hosilalarga ega bo’lsin, uzbekiston holda , (1.2) tenglikning birinchisini x bo’yicha, ikkinchisini uzbekiston bo’yicha differensiallab

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y},\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 u}{\partial y\partial x}$$

hosil qilamiz.

Bu tengliklarni qo’shib, $\frac{\partial^2 v}{\partial x\partial y}$ va $\frac{\partial^2 u}{\partial y\partial x}$ hosilalarini uzluksiz ekanligidan tengligini inobatga olib ,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{1.4}$$

Huddi shunga o’xshash (1.2) tenglikning birinchisini uzbekiston bo’yicha ikkinchisini x bo’yicha differensiallab

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 v}{\partial y^2},\frac{\partial^2 u}{\partial y\partial x}=-\frac{\partial^2 v}{\partial x^2}$$

Hosil qilamiz. Bu tengliklarni birinchidan ikkinchisini ayirib

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

ega bo’lamiz.

1.1-tarif. D sohada uzluksiz ikkinchi tartibli xususiy hosilaga ega bo’lgan va (1.4) tenglamani qanoatlantiruvchi $u(x,y)$ haqiqiy funksiyaga D sohada **garmonik funksiya**, (1.4) tenglamaga esa **Laplas tenglamasi** deyiladi.

Odatda $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ - Laplas operatorini ifodalaydi.

Bir qiymatli analitik funksiyaning xossalaridan ko;rinadiki, D sohada differensiallanuvchi funksiya shu sohada ixtiyoriy tartibli hosilaga ega va haqiqatandan, ixtoyoriy tartibli uzluksiz xususiy hosilaga ega bo’ladi. Shuning uchun D sohada differensiallanuvchi $f(z)=u(x,y)+iv(x,y)$ funksiyaning haqiqiy va mavhum qismlari shu sohada garmonik funksiyalardan iboratdir.

1.2-tarif. O’zaro Koshi-Riman sharti bilan bog’langan $u(x,y)$ va $v(x,y)$ garmonik funksiyalarga qo’shma garmonik funksiya deyiladi.

Shunday qilib, sohada differensiallanuvchi funksiyaning haqiqiy va mavhum qismlari shu sohada qo’shma garmonik funksiyalardan iboratdir.

Teskarisi, agar D bo’lsa, 1.1-teoremaga ko’ra $f(z)=u(x,y)+iv(x,y)$ funksiya D sohada differensiallanuvchidir. Bundan, quyidagi teorema o’rinlidir.

1.2-teorema. $f(z)=u(x,y)+iv(x,y)$ funksiyani D sohada differensial-lanuvchi bo’lishi uchun $u(x,y)$ va $v(x,y)$ funksiyalarni shu sohada qo’shma garmonik bo’lish zarur va yetarlidir.

Bir bog’lamli sohada $u(x,y)$, $v(x,y)$ funksiyalardan birini bilgan holda ikkinchisini topish mumkin.

1.3-teorema. Bir bog’lamli D sohada garmonik $u(x,y)$ funksiya uchun unga qo’shma bo’lgan garmonik funksiyani ixtiyoriy o’zgarmas qo’shiluvchi aniqligida topish mumkin.

Isbot. $u(x,y)$ funksiya bir bog’lamli D sohada garmonik ekanligidan

$$\frac{\partial}{\partial y}(-\frac{\partial u}{\partial y})=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})$$

$-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$ ifoda ixtiyoriy o'zgarmas C qo'shiluvchi aniqligida aniqlanadigan bir qiymatli $v(x,y)$ funksiyaning to'la differensialidan iborat, yani

$$dv(x,y)=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ekanligini etiborga olib,

$$dv(x,y)=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy,$$

$$v(x,y)=\int_{(x_0,y_0)}^{(x,y)}-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy+C,\qquad (x_0,y_0)\in D,\ (x,y)\in D \tag{1.5}$$

bu yerda integral (x_0,y_0) va (x,y) nuqtalarni tutashtiruvchi chiziqdan bog'liq emas, agar (x_0,y_0) nuqta fiksirlangan bo'lsa, u holda faqat (x,y) nuqtadan bog'liq.

(1.5) dan

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$$

bundan kelib chiqadiki, $v(x,y)$ D sohada $u(x,y)$ funksiyaga qo'shma bo'lgan garmonik funksiya .

1.2 va 1.3 teoremalardan kelib chiqadiki, agar bir bog'lamli D sohada u(x,y)garmonik funksiya berilgan bo'lsa, u holda o'zgarmas qo'shiluvchi aniqligida D sohada differensiallanuvchi $f(z)=u(x,y)+iv(x,y)$ funksiyani topish mumkin, yani berilgan haqiqiy (yoki mavhum) funksiyani tiklash mumkin. Agar D soha ko'p bog'lamli bo'lsa, (1.5) integral bilan aniqlanuvchi $v(x,y)$, huddi shunday $f(z)=u(x,y)+iv(x,y)$ funksiya bir qiymatli bo'lmasligi mumkin.

Berilgan $u(x,y)$ funksiya (aksincha), $v(x,y)$ funksiyani topishga (1.5) formulaga ko'ra Koshi-Riman shartidan foydalanish ham qulaydir.

1.2-misol. Agar $f(z)=v(x,y)=x^3-3xy^2$ funksiya berilgan bo'lsa, $f(z)$ differensiallanuvchi funksiyani toping.

Yechish. $v(x,y)=x^3-3xy^2$ funksiya butun kompleks tekislikda garmonik ekanligini yani

$$\frac{\partial v}{\partial x}=3x^2-3y^2, \frac{\partial^2 v}{\partial x^2}=6x, \frac{\partial v}{\partial y}=-6xy, \frac{\partial^2 v}{\partial y^2}=-6x$$

$$\Delta v=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=6x-6x=0$$

ko’rish mumkin.

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-3x^2+3y^2$$

bundan

$$\begin{aligned} u(x,y)&=\int(-3x^2+3y^2)dy+g(x) \\ u(x,y)&=-3x^2y+y^3+g(x) \end{aligned} \tag{1.6}$$

(1.6) dan

$$\frac{\partial u}{\partial x}=-6xy+g'(x) \tag{1.7}$$

topamiz. Ikkinchi tomondan, (1.2) ga ko’ra

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=-6xy \tag{1.8}$$

(1.7) va (1.8) larni tenglashtirib,

$$g'(x)=0 \Rightarrow \begin{cases} -6xy+g'(x)=-6xy \quad dan \\ g(x)=C, c-haqiqiy \ \ a'zgarmas \end{cases}$$

(1.6) dan $u(x,y)=-3x^2y+y^3+C$ ni hosil qilamiz. Izlanayotgan funksiya

$$f(z)=u(x,y)+iv(x,y)=-3x^2y+y^3+i(-3x^2y+y^3)+C=iz^3+C$$

butun kompleks tekislikda differensiallanuvchi.

§1.2. Garmonik funksiyaning asosiy xossalari

1.1-xossa. Ixtiyoriy $u(x,y)$ garmonik funksiya o'zining x,y argumentlarini analitik funksiyasidan iborat bo'ladi, yani D sohaning har bir $z_0 = x_0 + iy_0$ nuqtaning yig'indisi ko'rinishda tasvirlanadi.

$$u(x,y) = \sum_{m,n=0}^{\infty} C_{mn}(x-x_0)^m (y-y_0)^n \tag{1.9}$$

Isbot. $u(x,y)$ ni 1.2-teoremaga ko'ra z_0 nuqtaning $|z-z_0| < R$ atrofida bir qiymatli analitik $f(z)$ funksiyaning haqiqiy qismi sifatida qarash mumkin. Bu nuqtaning atrofida

$$f(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n \tag{1.10}$$

Bu yerda $C_n = \alpha_n + i\beta_n$ (1.10) qatorning umumiy hadini haqiqiy qismi

$$\begin{aligned} \alpha_n &= \left\{ (x-x_0)^n - \frac{n(n-1)}{2!}(x-x_0)^{n-2}(y-y_0)^2 + \ldots \right\} + \\ \beta_n &= \left\{ -n(x-x_0)^{n-1}(y-y_0) + \frac{n(n-1)(n-2)}{3!}(x-x_0)^{n-3}(y-y_0)^3 + \ldots \right\} \end{aligned} \tag{1.11}$$

Absolyut qiymati bo'yicha

$$|C_n| \{ |x-x_0| + |y-y_0| \}^n$$

Oshmaydi, Abel teoremasiga ([1],19-n.) ko'ra (1.10) qator ixtiyoriy $|z-z_0| \le r < R$ doirada absolyut yaqinlashadi, yani $\sum_{n=0}^{\infty} |C_n| r^n$ qator $r < R$ da yaqinlashadi, umumiy hadi (1.11) bo'lgan qator $|x-x_0| + |y-y_0| < R$ bo'lganda absolyut yaqinlashadi. Bu qator $u(x,y)$ funksiya uchun qatorni ifodalaydi. Buning hadlarini guruhlarga talab qilingan (1.9) qatorlarni hosil qilamiz. Teorema isbot bo'ldi.

1.2-xossa (o'rta qiymat haqida). Agar $u(z)$ funksiya markazi z nuqtada, radiusi r ga teng bo'lgan yopiq doirada uzluksiz va bu doirada garmonik bo'lsa, u holda

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\varphi}) d\varphi . \tag{1.12}$$

Bu xossaning isboti bir qiymatli analitik funksiya uchun o'rta qiymatli haqidagi [2] teoremadan haqiqiy qismini ajratishdan osongina kelib chiqadi.

1.3-xossa. O'zgarmasdan farqli bo'lgan garmonik funksiya o'zining eng katta va eng kichik qiymatiga aniqlanish sohasining ichki nuqtasida erishishi mumkin emas.

Isbot. Xossani maksimum nuqta bo'lgan hol uchun isbotlash yetarli, chunki $u(z)$ garmonik funksiyaning minimum nuqtasi $-u(z)$ funksiyani maksimum nuqtasi bo'ladi. Teskaridan faraz qilib, $u(z)$ garmonik funksiya z_0 ichki nuqtada o'zining maksimumiga erishsin. z_0 nuqtaning atrofida shunday bir qiymatli $f(z)$ funksiyani quramizki, $u(x,y) = \operatorname{Re} f(z)$ funksiya analitik va o'zgarmas , uning moduli $e^{u(z)}$ funksiya bizning farazimizga ko'ra sohaning z_0 ichki nuqtasida maksimumiga erishadi. Bu esa modulning maksimum prinsipiga ([1],15-n) ziddir. Xossa isbot bo'ldi.

1.3-misol. $u(x,y) = x^2 - y^2 + 2y$ funksiyaning $x^2 + y^2 \le 4$ doiradagi extremal nuqtalari va extremal qiymatlarini toping.

Yechish. Berilishiga ko'ra $u(x,y)$ funksiya sonlar tekisligida, xususan berilgan doirada uzluksiz va istalgan tartibli uzluksiz xususiy hosilalarga ega. Dastlab berilgan funksiyani $x^2 + y^2 \le 4$ doirada garmoniklikka tekshiramiz. Buning uchun uning xususiy hosilalarini hisoblaymiz:

$$u_x = \frac{\partial u(x,y)}{\partial x} = 2x, \quad u_{xx} = \frac{\partial^2 u(x,y)}{\partial x^2} = 2,$$

$$u_y = \frac{\partial u(x,y)}{\partial y} = -2y + 2, \quad u_{yy} = \frac{\partial^2 u(x,y)}{\partial y^2} = -2.$$

U holda bu funksiya uchun

$$\Delta u(x,y) = u_{xx} + u_{yy} = 2 - 2 = 0$$

bo'lib, u Laplas tenglamasining regulyar yechimi, ya'ni tekislikdagi barcha nuqtalarda, xususan $x^2 + y^2 \le 4$ doirada ham garmonik funksiya bo'ladi.

Demak bu funksiya uchun garmonik funksiya uchun maksimum qiymat prinsipini qo'llash mumkin. Bu funksiya garmonik bo'lgan $x^2 + y^2 \le 4$ doira chegarasi $x^2+y^2=4$ aylanadan iborat bo'lib, chegaraviy nuqtalarda $x^2=4-y^2$ tenglik o'rinli bo'lib, bu nuqtalar to'plamida qaralayotgan funksiya

$$w(y)=u(\pm\sqrt{4-y^2},y)=-2y^2+2y+4,\ -2\le y\le 2$$

ko'rinish oladi. Bu kvadrat funksiya bo'lib, u $y=\frac{1}{2}$ nuqtada $w=4{,}5$ maksimum qiymatga va $y=-2$ nuqtada esa $w=-8$ minimum qiymatga erishadi. Erkli o'zgaruvchi y ning $y=\frac{1}{2}$ qiymatiga x ning $x=\pm\frac{\sqrt{15}}{2}$ va $y=-2$ qiymatga esa $x=0$ qiymati mos keladi.

Shunday qilib berilgan $u(x,y)=x^2-y^2+2y$ funksiya $\left(\pm\frac{\sqrt{15}}{2};\frac{1}{2}\right)$ chegaraviy nuqtalarda

$$u_{\max}=u\left(\pm\frac{\sqrt{15}}{2};\frac{1}{2}\right)=\frac{9}{2}$$

eng katta (maksimum) qiymatiga va $(0;-2)$ chegaraviy nuqtada esa

$$u_{\min}=u(0;-2)=-8$$

eng kichik (minimum) qiymatiga erishadi.

1.4-xossa. Agar D sohada garmonik $u(x,y)$ funksiya hech bo'lmaganda yuqori yoki quyidan chegaralangan bo'lsa, u holda u o'zgarmas.

Isbot. $u(x,y)$ funksiya D sohada yuqoridan chegaralangan bo'lsin: $u(x,y)<M$ dir. Butun D sohada shunday bir qiymatli analitik $f(z)$ funksiyani quramizki, $u(x,y)=\operatorname{Re} f(z)$. Shartiga ko'ra $w=f(z)$ funksiyaning barcha qiymatlari $u(x,y)<M$ yarim tekislikda yotadi. $f(z)$ funksiya o'zgarmas, demak, $u(x,y)$ ham o'zgarmasdir. Xossa isbot qilindi.

Quyidagi o'rta qiymat haqidagi teoremaga teskari teoremalardan iboratdir.

1.5-xossa. Agar $u(z)$ funksiya D sohada uzliksiz va yetarlicha kichik r -lar uchun ixtiyoriy z nuqtada

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\varphi})d\varphi$$

bo'lsa, u holda $u(z)$ funksiya D sohada garmonikdir.

Quyidagi xossa kompleks o'zgaruvchi funksional qatorlar uchun o'rganilgan Veyeritrass teoremasiga o'xshashdir.

1.6-xossa. D sohada garmonik va $\overline{D}$ da uzluksiz bo'lgan $u_0(z), u_1(z), ..., u_n(z), ...$ garmonik funksiyalar ketma-ketligi berilgan bo'lsin. Agar $\sum_{k=0}^{\infty} u_k(z)$ qator D ning chegarasida tekis yaqinlashsa, u holda bu qator D-ning ichida ham tekis yaqinlashadi va uning yig'indisi ham D sohada garmonik funksiya bo'ladi.

1.7-xossa. Agar $u(z)$ funksiya bir bog'lamli D sohada garmonik va o'zining xususiy hosilasi bilan $\overline{D}$ da uzluksiz bo'lsa, u holda

$$\int_{\partial D} \frac{\partial u}{\partial n} dS_y = 0,$$

bu yerda $\frac{\partial u}{\partial n}$ -normal bo'yicha hosilasi, dS -yoyning differensiali.

§1.3. Laplas tenglamasining qutb, silindrik va sferik

koordinatalardagi ifodasi

Bizga ma'lumki, $\nabla u = gradu, \nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$

$$(\nabla, \nabla u) = (\nabla, \nabla)u = divgradu = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}, \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}\right) =$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, (\nabla, \nabla) = \Delta$$

deb belgilaymiz va Laplas operatori deyiladi.

$$\Delta = (\nabla, \nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta u = 0, \Delta u = (\nabla, \nabla)u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.13}$$

Laplas tenglamasi deyiladi.

Laplas tenglamasi silindrik va sferik koordinatalarda mos ravishda quyidagi ko'rinishda bo'ladi

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 u}{\partial \varphi^2} + \rho \frac{\partial^2 u}{\partial z^2}\right) = 0, \tag{1.14}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0 \tag{1.15}$$

silindrik koordinatalar sistemasida (1.14) ning yechimi Bessel funksiyalarini, (1.15) tenglamani sferik koordinatlar sistemasidagi yechimi sferik funksiyadan iborat bo'ladi.

Ma'lumki, Dekart koordinatalar sistemasida

$$gradu = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}$$

ko'rinishda bo'ladi.

Silindrik koordinatalarda

$$gradu = \frac{\partial u}{\partial r}\bar{i} + \frac{1}{r}\frac{\partial u}{\partial \varphi}\bar{j} + \frac{\partial u}{\partial z}\bar{k},$$

sferik koordinatalarda esa

$$gradu = \frac{\partial u}{\partial \rho}\bar{i} + \frac{1}{\rho}\frac{\partial u}{\partial \theta}\bar{j} + \frac{1}{\rho \sin\theta}\frac{\partial u}{\partial \varphi}\bar{k}$$

ko'rinishda bo'ladi.

Dekart koordinatalar sistemasi $a=a_x i + a_y j + a_z k$ da vector maydon divergensiyasi

$$div\bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$ ko'rinishda bo'ladi.

Silindrik va sferik koordinatalarda a vector maydon divergensiyasi mos ravishda quyidagi

$$div\bar{a} = \frac{1}{r}\left(\frac{\partial (ra_x)}{\partial r} + \frac{\partial a_\varphi}{\partial \varphi} + r\frac{\partial a_z}{\partial z}\right)$$

$$div\bar{a} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 a_\rho\right) + \frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta a_\theta) + \frac{1}{\rho \sin\theta}\frac{\partial a_\varphi}{\partial \varphi}$$

ko'rinishda bo'ladi.

Laplas tenglamasining qutb koordinatalar sistemasida ko'rinishi ifodalash uchun matematik analiz kursidan ma'lum bo'lgan (x, y) Dekart koordinatalar sistemasidan egri chiziqli (ρ, φ) qutb koordinatalar sistemasiga o'tish formulalari

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \end{aligned}\right\} \qquad (1.16)$$

ko'rinishga o'tish zarur. Bunda ρ - koordinata boshidan berilgan $A(x, y)$ nuqtagacha masofa bo'lib, uni odatda nuqtaning radius vektori deyiladi, φ - esa OX o'qining musbat yo'nalishi bilan nuqtaning radius vektori orasidagi (soat strelkasi harakatiga teskari yo'nalishda aniqlangan) burchak bo'lib, uni odatda berilgan nuqtaning bosh argumenti deyiladi.

$$u(\rho\cos\varphi, \rho\sin\varphi) = v(\rho, \varphi)$$

deb belgilaymiz.

Aytilganlarga asosan $0 \le \varphi < 2\pi$ bo'lganda (1.16) o'zaro bir qiymatli akslantirish bo'lib, unga mos teskari almashtirishlar quyidagicha aniqlanadi [4]:

$$\left.\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \varphi &= arctg\,\frac{y}{x} \end{aligned}\right\}. \qquad (1.17)$$

(ρ,φ) koordinatalar sistemasida Laplas tenglamasining ko'rinishini topish uchun dastlab birinchi tartibli u_x va u_y xususiy hosilalarni hisoblaymiz:

$$u_x = v_\rho \rho_x + v_\varphi \varphi_x = \frac{x}{\sqrt{x^2+y^2}} v_\rho - \frac{y}{x^2+y^2} v_\varphi = \cos\varphi \cdot v_\rho - \frac{\sin\varphi}{\rho} v_\varphi,$$

$$u_y = v_\rho \rho_y + v_\varphi \varphi_y = \frac{y}{\sqrt{x^2+y^2}} v_\rho + \frac{x}{x^2+y^2} v_\varphi = \sin\varphi \cdot v_\rho + \frac{\cos\varphi}{\rho} v_\varphi.$$

Bu xususiy hosilalar yordamida Laplas tenglamasi uchun kerakli bo'lgan ikkinchi tartibli u_{xx} va u_{yy} xususiy hosilalarni hisoblaymiz:

$$u_{xx} = (u_x)_x = (u_x)_\rho \rho_x + (u_x)_\varphi \varphi_x =$$

$$= \cos\varphi \frac{\partial}{\partial\rho}\left(\cos\varphi \cdot v_\rho - \frac{\sin\varphi}{\rho} v_\varphi\right) - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\cos\varphi \cdot v_\rho - \frac{\sin\varphi}{\rho} v_\varphi\right) =$$

$$= \cos^2\varphi \cdot v_{\rho\rho} - \frac{\sin 2\varphi}{\rho} v_{\rho\varphi} + \frac{\sin^2\varphi}{\rho^2} v_{\varphi\varphi} + \frac{\sin^2\varphi}{\rho} v_\rho + \frac{\sin 2\varphi}{\rho^2} v_\varphi.$$

$$u_{yy} = (u_y)_y = (u_y)_\rho \rho_y + (u_y)_\varphi \varphi_y =$$

$$= \sin\varphi \frac{\partial}{\partial\rho}\left(\sin\varphi \cdot v_\rho + \frac{\cos\varphi}{\rho} v_\varphi\right) + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\sin\varphi \cdot v_\rho + \frac{\cos\varphi}{\rho} v_\varphi\right) =$$

$$= \sin^2\varphi \cdot v_{\rho\rho} + \frac{\sin 2\varphi}{\rho} v_{\rho\varphi} + \frac{\cos^2\varphi}{\rho^2} v_{\varphi\varphi} + \frac{\cos^2\varphi}{\rho} v_\rho - \frac{\sin 2\varphi}{\rho^2} v_\varphi.$$

Topilgan bu ifodalarni $\Delta u = 0$, ya'ni $u_{xx} + u_{yy} = 0$ Laplas tenglamsiga qo'yib, uning qutb koordinatalardagi ko'rinishini olamiz:

$$(\Delta_{\rho,\varphi} v)(\rho,\varphi) == \cos^2\varphi \cdot v_{\rho\rho} - \frac{\sin 2\varphi}{\rho} v_{\rho\varphi} + \frac{\sin^2\varphi}{\rho^2} v_{\varphi\varphi} + \frac{\sin^2\varphi}{\rho} v_\rho + \frac{\sin 2\varphi}{\rho^2} v_\varphi +$$

$$= \sin^2\varphi \cdot v_{\rho\rho} + \frac{\sin 2\varphi}{\rho} v_{\rho\varphi} + \frac{\cos^2\varphi}{\rho^2} v_{\varphi\varphi} + \frac{\cos^2\varphi}{\rho} v_\rho - \frac{\sin 2\varphi}{\rho^2} v_\varphi = 0.$$

Ushbu tenglamani soddalashtirsak u quyidagi tenglamaga teng kuchli bo'ladi:

$$(\Delta_{\rho,\varphi} v)(\rho,\varphi) = v_{\rho\rho} + \frac{1}{\rho^2} v_{\varphi\varphi} + \frac{1}{\rho} v_\rho = 0.$$

Agar differensiallah uchun

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v}{\partial\rho}\right) = \rho v_{\rho\rho} + v_\rho$$

tenglikning o'rinli ekanligini hisobga olsak yuqoridagi tenglamani

$$\Delta_{\rho,\varphi} u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0 \qquad (1.18)$$

ko'rinshda yozish mumkin bo'ladi. Odatda (1.18) tenglama Laplas tenglamasining qutb koordinatalar sistemasidagi tasviri hisoblanadi.

Koshi-Riman shartining qutb koordinatalar orqali ifodasini ko'rsatamiz. Buning uchun $x = r\cos\varphi$, $y = r\sin\varphi$, $z = x + iy$, $|z| = r$, $\varphi = \arg z$, $z \neq 0$ deb olib, $f(z)$ funksiyaning moduli va argumenti orqali ifodalaymiz. U holda,

a) $f(z) = u(x,y) + iv(x,y) = u(r,\varphi) + iv(r,\varphi) = f(r,\varphi)$ funksiya r, φ ga nisbatan differensiallanuvchi bo'ladi.

b) $u(r,\varphi)$, $v(r,\varphi)$ funksiyalar Koshi-Riman

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}$$

sistemani qanoatlantiradi. Buni isbotlash uchun hosilalarni hisoblaymiz, bunda (1.5) sistemadan foydalanib:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial y}\cos\varphi - \frac{\partial v}{\partial x}\sin\varphi = \frac{1}{r}\left[-\frac{\partial v}{\partial x} r\sin\varphi + \frac{\partial v}{\partial y} r\cos\varphi\right] =$$

$$= \frac{1}{r}\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \varphi}\right] = \frac{1}{r}\frac{\partial v}{\partial \varphi},$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\varphi + \frac{\partial v}{\partial y}\sin\varphi = -\frac{\partial u}{\partial y}\cos\varphi + \frac{\partial u}{\partial x}\sin\varphi =$$

$$= -\frac{1}{r}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}\right] = -\frac{1}{r}\frac{\partial u}{\partial \varphi}.$$

yuqoridagi sistemani bajarilishi isbot bo'ldi.

1.4-misol [11]. Berilgan $|f(z)| = e^{r^2 \cos 2\varphi}$ $(z = re^{i\varphi})$ funksiyaga ko'ra bir qiymatli analitik $f(z)$ funksiyani tiklang.

Yechish. Berilgan funksiyadan $\ln|f(z)| = r^2\cos 2\varphi$ topamiz. Endi

$$\ln f(z) = \ln|f(z)| + iArgf(z) = r^2\cos 2\varphi + iv(r,\varphi)$$

bir qiymatli analitik funksiya uchun uning haqiqiy qismi

$u(r,\varphi) = r^2\cos 2\varphi$ berilgan va $\ln f(z) = u(r,\varphi) + iv(r,\varphi)$ funksiyani tiklash kerak.

Bundan $f(z) = e^{u(r,\varphi)+iv(r,\varphi)}$. formuladan $v(r,\varphi)$ funksiyani tiklaymiz:

Demak, $v(r,\varphi) = r^2\sin 2\varphi + c \Rightarrow \ln f(z) = r^2\cos 2\varphi + ir^2\sin 2\varphi + ic = z^2 + ic$, $z = re^{i\varphi}$ va

$$f(z)=e^{z^2+ic},\ v=\int 2r^2\cos 2\varphi d\varphi+\psi(r)=r^2\sin 2\varphi+\psi(r),\quad \frac{\partial v}{\partial r}=2r\sin 2\varphi+\psi'(r)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \varphi}=2r\sin 2\varphi=2r\sin 2\varphi+\psi'(r)\Rightarrow\quad \psi'(r)=c=const\quad\Rightarrow f(z)=e^{z^2}.$$

Endi uch o'lchovli fazodagi biror sohada qaralgan Laplas tenglamasining silindrik va sferik koordinatalardagi tasvirlarini keltirib chiqaramiz. Ushbu tenglamalar qaralayotgan soha silindrsimon va sharsimon ko'rinishda bo'lganda Laplas tenglamasining yechimlarini topishda qulay hisoblanadi.

Ma'lumki, (x,y,z) Dekart koordinatalar sistemasidan egri chiziqli (ρ,φ,z) silindrik koordinatalar sistemasiga o'tish formulalari

$$\left.\begin{aligned} x&=\rho\cos\varphi\\ y&=\rho\sin\varphi\\ z&=z\end{aligned}\right\}$$

ko'rinishga ega bo'lib, yuqoridagi hisoblashlarga asosan $u_{xx}+u_{yy}+u_{zz}=0$ Laplas tenglamasining silindrik koordinatalardagi ko'rinishi

$$\Delta_{\rho,\varphi,z}u=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=0. \qquad (1.19)$$

kabi ekanligiga ishonch hosil qilamiz.

Xudddi shu kabi hisoblashlarni (x,y,z) Dekart koordinatalar sistemasidan egri chiziqli (r,φ,ψ) sferik koordinatalar sistemasiga o'tish formulalari

$$\left.\begin{aligned} x&=r\sin\varphi\cos\psi\\ y&=r\sin\varphi\sin\psi\\ z&=r\cos\varphi\end{aligned}\right\}$$

kabi bo'lib, unga teskari almashtirish

$$\left.\begin{aligned} &r=\sqrt{x^2+y^2+z^2},\ \varphi=\arccos\frac{z}{\sqrt{x^2+y^2+z^2}},\\ &\psi=arctg\frac{y}{x}\end{aligned}\right\}$$

formula bilan aniqlanadi. Bu holda ham xuddi qutb koordonatalardagi kabi

$$u(r\sin\varphi\cos\psi,\ r\sin\varphi\sin\psi,\ r\cos\varphi)=w(r,\varphi,\psi)$$

belgilash kiritib, kerakli xusuiy hosilalarni hisoblash bilan Laplas tenglamasining sferik koordinatalardagi ko'rinishini olamiz:

$$\Delta_{r,\varphi,\psi}u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial^2 u}{\partial\psi^2} = 0. \qquad (1.20)$$

1.5-misol [12]. $u(x,y) = chx\cdot\sin y$ funksiyaga qo'shma garmonik bo'lgan $v(x,y)$ funksiya'ni Koshi-Riman sistemasidan foydalanib toping.

Yechish. (1.5) Koshi-Riman sistemasidan foydalanib $u(x,y) = shx\cdot\cos y$ funksiyadan $u_x(x,y) = \cos y\cdot chx = v_y(x,y)$ ni olamiz.Bundan esa

$$v(x,y) = \sin y\cdot chx + \varphi(x)$$

ga ega bo'lamiz.

$v(x,y) = \sin y\cdot chx + \varphi(x)$ funksiyani x bo'yicha differensiallab, berilgan funksiya va (1.5) formulani e'tiborga olib, quydagiga

$$v_x(x,y) = \sin y\cdot shx + \varphi(x) = -u_y(x,y) = shx\cdot\sin y$$

ega bo'lamiz. Bundan

$$\varphi'(x) = 0 \Rightarrow \varphi(x) = const.$$

Shunday qilib $v(x,y)$ funksiya $v(x,y) = \sin y\cdot shx + c$ ko'rinishga ega bo'ladi.

1.6-misol: $u_x(x,y) = xy$ bo'lsa $v(x,y)$ qo'shma garmonik funksiya'ni toping.

Yechish: (1.5) Koshi-Riman sistemasiga ko'ra $u_x(x,y) = xy = v_y(x,y)$ ga teng. Bundan $v(x,y) = \frac{xy^2}{2} + \varphi(x)$ ga ega bo'lamiz . Izlanayotgan $v(x,y)$ funksiya garmonik funksiya bo'lgani uchun Laplas tenglamasini qanoatlantiradi, ya'ni $v_{yy}(x,y) = x,\quad v_{xx}(x,y) = \varphi''(x)$ larni etiborga olib quyidagilarni $v_{yy}(x,y) + v_{xx}(x,y) = \varphi''(x) + x = 0 \Rightarrow \varphi''(x) = -x$ olamiz. Bundan $\varphi(x) = -x^3/6 + c_1x + c_2$ ni topamiz .Shunday qilib $v(x,y)$ qo'shma garmonik funksiya $v(x,y) = \frac{xy^2}{2} - \frac{x^3}{6} + c_1x + c_2$ ko'rinishsda bo'ladi .

1.7-misol. Agar $f(z)$ analitik funksiya'ning $u(x,y) = \operatorname{Re} f(z) = \sin x\cdot chy$ haqiqiy qismi berilgan bo'lsa, u holda bir bog'lamli D sohada egri chiziqli integrallash yordamida $f(z)$ analitik funksiya'ni toping.

Yechish. Ma'lumki $f(z) = u(x,y) + iv(x,y)$ bo'lib $u(x,y)$ va $v(x,y)$ funksiyalar

$$\begin{cases} u_x - v_y = 0 \\ u_y - v_x = 0 \end{cases}$$

Koshi-Riman sistemasini qanoatlantiradi. Demak, (1.7) formulaga ko’ra quydagiga ega bo’lamiz:

$$u(x,y) = -\int_{x_0}^{x} \frac{\partial v(x,y)}{\partial y} dx + \int_{y_0}^{y} \frac{\partial v(x,y)}{\partial x} dy + c = -\int_{x_0}^{x} \sin x \cdot shy_0 dx + \int \cos xchydy = \cos xshy_0 -$$
$$- \cos x_0 shy_0 + \cos xshy - \cos xshy_0 + c = -\cos x_0 shy_0 + \cos xshy + c.$$

Shunday qilib, $f(z)$ analitik funksiya quydagi

$$f(z) = \sin xchy + i(\cos xshy - \cos x_0 shy_0 + c)$$

ko’rinishida tuziladi.

1.8-misol. $v(x,y) = \frac{e^{2x} - 1}{e^x} \sin y$ funksiya va $f(0) = 2$ shart orqali

$$f(z) = u(x,y) + iv(x,y)$$

analitik funksiya’ni toping.

Yechish. $v(x,y)$ funksiya garmonik funksiya ekanligini tekshirish maqsadida $\frac{\partial^2 v}{\partial x^2} = 2shx\sin y$ va $\frac{\partial^2 v}{\partial y^2} = -2shx\sin y$ hosilalarini topamiz va barcha (x,y) lar uchun Laplas tenglamasi qanoatlantirilishiga ishonch hosil qilamiz, ya’ni

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

tenglik bajariladi. Demak $v(x,y)$ funk siya garmonik funksiyadir. Bundan va (1.7) formuladan hamda $x_0 = 0, \quad y_0 = 0$ nuqtani tanlab, quyidagiga

$$u(x,y) = 2\int_0^x sh\, t \cos y\, dt - 2\int_0^y ch0 \cdot \sin s\, ds = 2chx \cos y - 2 + c$$

ega bo’lamiz.

Shunday qilib, $f(z)$ analitik funksiya quyidagi

$$f(z) = 2chx \cdot \cos y + 2i \cdot shx \cdot \sin y - 2 + c$$

ko’rinishda tuziladi. Demak, izlanayotgan $f(z)$ funksiya

$$f(z) = 2(chx \cdot \cos y + i \cdot shx \cdot \sin y)$$

ko'rinishda bo’ladi.

§ 1.4. Laplas tenglamasining fundamental yechimi va Grin formulalari

Fazodagi silindrik hamda sharsimon sohalarda berilgan Laplas tenglamasining faqat radius vektorlardan, ya'ni (1.18) yoki (1.19) tenglamaning faqat ρ yoki r dan bog'liq bo'lgan (qolgan z,φ,ψ o'zgaruvchilarga bog'liq bo'lmagan) yechimiga silindrik va sferik simmetrik yechimi deb yuritiladi. Ushbu yechimlar garmonik funksiyalar va umuman elliptik tipli differensial tenglamalar nazariyasida muhim ahamiyatga ega. Shuning uchun ham biz Laplas tenglamasining sferik va silindrik simmetrik yechimlarinining ko'rinishini topish masalasi bilan shug'ullanamiz.

1**.2-ta'rif.** ***([3],[6])*** *Laplas tenglamasining berilgan sohaning ajralgan maxsus nuqtalari yoki o'zi-o'zini kesmaydigan silliq sirtlarda maxsuslikka ega bo'lgan yechimiga fundamental yechimi deyiladi. Laplas yenglamasining sohada maxsuslikka ega bo'lmagan va ozining iikinchi tartibli xususiy hosilalari bilan uzluksiz yechimiga esa regulyar yechimi deyiladi.*

Faraz qilaylik Laplas tenglamasining silindrik simmetrik (fazoda) yoki doiraviy simmetrik (tekislikda) yechimini topish lozim bo'lsin. Aytilganlarga asosan bu holda (1.18) tenglamaning faqat ρ dan bog'liq $u=u(\rho)$ yechimini topish lozim. Ushbu hollarda $u_\varphi=u_z=0$ bo'lganligi uchun (1.18) tenglama

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)=0$$

ko'rinishga keladi. Uni integrallab

$$\rho\frac{\partial u}{\partial\rho}=C_1$$

yoki

$$\frac{\partial u}{\partial\rho}=C_1\frac{1}{\rho}$$

tenglamaga kelamiz. Uni integrallash natijasida Laplas tenglamasining silindrik simmetrik yechimining umumiy ko'rinishi

$$u(\rho)=C_1\ln\rho+C_2$$

hosil qilamiz. Agar ushbu umumiy yechimda $C_1 = -1, C_2 = 0$ deb tanlab Laplas tenglamasining silindrik yoki doiraviy simmetrik yechimlardan bittasini hosil qilamiz:

$$u(\rho) = \ln \frac{1}{\rho}.$$

Ushbu yechimga odatda tekislikda Laplas tenglamasining ***fundamental yechimi*** deyiladi.

Xuddi shu kabi Laplas tenglamasining sferik simmetrik $u = u(r)$ yechimini topamiz. Bu holda Lapals tenglamasining sferik koordinatalardagi (1.19) ko'rinishidan foydalanamiz. Qaralayotgan holda $u_\varphi = u_\psi = 0$ bo'lib, Laplas tenglamasi quyidagi ko'rinishga keladi:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0.$$

Bu tenglamani integrallab

$$u(r) = C_1 \frac{1}{r} + C_2$$

umumiy yechimni hosil qilamiz. Agar bunda $C_1 = 1, C_2 = 0$ deb faraz qilsak, Laplas tenglamasining fazodagi fundamental yechimi deb ataluvchi

$$u(r) = \frac{1}{r}$$

yechimni hosil qilamiz.

Shunday qilib Lapals tenglamasining **elementar** yoki **fundamental** yechimi umumiy holda quyidagi ko'rinishda yozilishu mumkin degan xulosaga kelamiz:

$$u(r) = \begin{cases} \ln r, \text{ agar } n = 2 \text{ bo'lsa,} \\ \dfrac{1}{r^{n-2}}, \text{ agar } n > 2 \text{ bo'lsa} \end{cases}$$

bunda

$$r = |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Bu funksiya uchun cheksizlikda $(|x| \to \infty)$

$$u(r)=O\left(\frac{1}{|r|^{n-2}}\right)$$

baho o'rinlidir. Agar $n=2$ bo'lsa, u holda $u(r)$ funksiya cheksizlikda **chegaralangan** bo'ladi.

1.14-misol. $E(x,y)=\frac{1}{r}$ funksiyani $u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3}=0$ tenglama uchun fundamental yechimi bo'lishini isbotlang.

Isbot. Berilgan $E(x,y)=\frac{1}{r}$ funksiya $x\neq y$ bo'lganda x bo'yicha ham, ξ bo'yicha ham berilgan tenglamani qanoatlantiradi.

Haqiqatan ham,

$$\frac{\partial E}{\partial x_1}=-\frac{1}{r^3}(x_1-y_1),\quad \frac{\partial^2 E}{\partial x_1^{\,2}}=\frac{3}{r^5}(x_1-y_1)^2-\frac{1}{r^3},$$

$$\frac{\partial E}{\partial x_2}=-\frac{1}{r^3}(x_2-y_2),\quad \frac{\partial^2 E}{\partial x_2^{\,2}}=\frac{3}{r^5}(x_2-y_2)^2-\frac{1}{r^3},$$

$$\frac{\partial E}{\partial x_3}=-\frac{1}{r^3}(x_3-y_3),\quad \frac{\partial^2 E}{\partial x_3^{\,2}}=\frac{3}{r^5}(x_3-y_3)^2-\frac{1}{r^3}.$$

Bu ifodalarni berilgan tenglamaning chap tomoniga qo'yib, quyidagiga

$$\frac{\partial^2 E}{\partial x_1^{\,2}}+\frac{\partial^2 E}{\partial x_2^{\,2}}+\frac{\partial^2 E}{\partial x_3^{\,2}}=\frac{3}{r^5}\left[(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right]-\frac{3}{r^3}=$$

$$=\frac{3}{r^5}r^2-\frac{3}{r^3}=3(\frac{3}{r^3}-\frac{3}{r^3})=0$$

ega bo'lamiz. Shunday qilib, $E(x,y)$ funksiya $x\neq y$ bo'lganda berilgan Laplas tenglamasini qanoatlantiradi.

Demak, 2.1-ta'rifga ko'ra $E(x,y)=\frac{1}{r}$ fundamental yechimdir.

1.15-misol [2]. $E(x,y)=\frac{1}{\pi z}=\frac{1}{\pi(x+iy)}$ funksiya $\frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$ Koshi – Riman operatorining fundamental yechimi ekanligini isbotlang.

Isbot. $E(x,y)=\frac{1}{\pi(x+iy)}$ funksiya $z=x+iy\neq 0$ da $\frac{\partial}{\partial \bar z}=0$ Koshi – Riman tenglamasini qanoatlantiradi.

Haqiqatan,

$$\frac{\partial E}{\partial x} = -\frac{1}{\pi(x+iy)^2}, \quad \frac{\partial E}{\partial y} = -\frac{i}{\pi(x+iy)^2}$$

ifodaga ko'ra

$$\frac{\partial E}{\partial \bar{z}} = -\frac{1}{2\pi}\left[\frac{1}{(x+iy)^2} + i\frac{i}{(x+iy)^2}\right] = -\frac{1}{2\pi}\left[\frac{1}{(x+iy)^2} - \frac{1}{(x+iy)^2}\right] = 0$$

tenglikga ega bo'lamiz. Shunday qilib, $E(x,\xi) = \dfrac{1}{\pi(x+iy)}$ funksiya Koshi – Riman tenglamasini qanoatlantiradi. Demak, bu funksiya fundamental yechimdir.

Faraz qilaylik, D soha va uning chegarasida o'zining birinchi tartibli xususiy hosilalari bilan uzluksiz, D sohada esa ikkinchi tartibli uzluksiz xusuiy hosilalarga ega bo'lgan $u(x,y,z)$ va $v(x,y,z)$ funksiyalar berilgan bo'lsin. Bizga matematik analiz kursidan ma'lum bo'lgan va biror D hajm bo'yicha integralni uning Σ sirti bo'yicha olingan integralga keltiruvchi Ostragradskiy formulasi quyidagicha edi:

$$\iiint\limits_D \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z}\right) d\tau =$$

$$= \iint\limits_\Sigma \left(P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma\right) d\sigma.$$

Bunda $d\tau$ va $d\sigma$ mos ravishda birlik hajm va yoy elementlari. Ushbu formulada $P = u\dfrac{\partial v}{\partial x}$, $Q = u\dfrac{\partial v}{\partial y}$, $R = u\dfrac{\partial v}{\partial z}$ deb olsak Grinning 1-formulasi deb ataluvchi

$$\iiint\limits_D u\Delta v d\tau = \iint\limits_\Sigma u\frac{\partial v}{\partial n} d\sigma - \iiint\limits_T \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) d\tau. \quad (1.21)$$

Agar bu formulada u va v larning o'rnini almashtirib hosil bo'lgan tenglikni (1.21) dan ayirsak Grinning 2-formulasini hosil qilamiz

$$\iiint\limits_D (u\Delta v - v\Delta u) d\tau = \iint\limits_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma. \quad (1.22)$$

Grinning ushbu 2-formulasi garmonik funksiyalar uchun o'rta qiymat haqidagi teoremani isbotlashda muhim ahamiyatga ega. Faqaz qilaylik, $M_0(x_0, y_0, z_0)$ nuqta silliq Σ sirt ichida yotsin. $U_\varepsilon = U_\varepsilon(M_0)$ esa markazi M_0

nuqtada, radiusi ε ga teng bo'lgan va butunlay D sohada yotuvchi shar bo'lsin. Grinning 2-formulasi (1.22) ni $D \setminus U_\varepsilon$ sohada v va $v = \frac{1}{R}\left(R = R_{MM_0}\right)$ funksiyalar uchun tatbiq etishimiz mumkin, bunda $\forall M(x,y,z) \in D \cup \Sigma$. Sharda normal bo'yicha hosila radius vektor bo'yicha hosilaga teng bo'lganligini, sferada funksiyaning o'rta qiymatini hisobga olsak va $\varepsilon \to 0$ da limitga o'tib Grinning asosiy formulasi deb ataluvchi quyidagi formulaga kelishamiz:

$$u(M_0) = \frac{1}{\Omega}\iint_\Sigma \left[\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M} - u(M)\frac{\partial}{\partial n_M}\left(\frac{1}{R_{MM_0}}\right)\right] d\sigma_M - \iiint_D \frac{\Delta u(M)}{R_{MM_0}} d\tau_M \,. \qquad 1.23)$$

Bunda

$$\Omega = \begin{cases} 4\pi, & \text{agar } M_0 \in D \text{ bo'lsa,} \\ 2\pi, & \text{agar } M_0 \in \Sigma \text{ bo'lsa,} \\ 0, & \text{agar } M_0 \notin D \cup \Sigma \text{ bo'lsa.} \end{cases}$$

Agar u garmonik funksiya bo'lsa, (1.23) formula quyidagi ko'rinishni oladi:

$$u(M_0) = \frac{1}{\Omega}\iint_\Sigma \left[\frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M} - u(M)\frac{\partial}{\partial n_M}\left(\frac{1}{R_{MM_0}}\right)\right] d\sigma_M \qquad (1.24)$$

Agarda qaralayotgan soha tekislikda biror silliq C yopiq chiziq bilan chegaralangan D sohadan iborat bo'lsa, u holda yuqoridagi mulohazalarda v o'rnida Laplas tenglamasining tekislikdagi fundamental yechimi

$$v = \ln \frac{1}{R_{MM_0}}$$

funksiyani ishlatsak (1.23) va (1.24) ga ox'shash formulalarni olamiz:

$$u(M_0) = \frac{2}{\Omega}\int_C \left[\ln \frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M} - u(M)\frac{\partial}{\partial n_M}\left(\ln\frac{1}{R_{MM_0}}\right)\right] ds_M - \iint_S \frac{\Delta u(M)}{R_{MM_0}} d\sigma_M \quad 1.23')$$

$$u(M_0) = \frac{2}{\Omega}\int_C \left[\ln \frac{1}{R_{MM_0}}\frac{\partial u(M)}{\partial n_M} - u(M)\frac{\partial}{\partial n_M}\left(\ln\frac{1}{R_{MM_0}}\right)\right] ds_M \,. \qquad (1.24')$$

II-bob. Garmonik funksiya uchun Dirixli masalasi

§2.1. Laplas tenglamasi uchun Dirixle masalasining qo'yilishi. Yechimning mavjudligi va yagonaligi.

Ko'pgina statsionar (vaqtdan bog'liq bo'lmagan) fizikaviy masalalar ma'lum chegaraviy shartlarni qanoatlantiruvchi garmonik funksiyalarni topishga keltirladi.

Chegaralangan D sohaning chegarasida $u_0(z)$ uzluksiz funksiya berilgan bo'lsin.

Garmonik funksiyalar to'plami – bu eng sodda ikkinchi tartibli xusuxiy hosilali differensial tenglamalardan biri bo'lgan (1.4) Laplas tenglamasining barcha yechimlari to'plamidir. Oddiy differensial tenglamalar kabi aniq bitta yechimni ajratib olish uchun qo'shimcha shart qo'yilganidek, Laplas tenglamasi yechimini ham to'la aniqlash uchun qoshimcha shart talab qilinadi. Laplas tenglamasi uchun chegarviy shart ko'rinishida, yani berilgan munosabat izlanayotgan yechim sohaning chegarasida qanoatlantirish kerak.

Mana shunday shartlardan eng soddasi izlanayotgan garmonik funksiyani cheganing har bir nuqtasidagi qiymatini berilishidir. Shunday qilib, birinchi chegarviy masala, yoki "klassik" Dirixle masalasi (Lejin Dirixle (1805-1859)- nemis matematigi):

D sohada garmonik , uning chegarsi $\overline{D} = D \cup \partial D$ gacha uzluksiz va uning chegarsi ∂D da $u_0(z)$- qiymatni qabul qiluvchi $u(z)$ funksiyani toping:

$$\Delta u = 0, \qquad z \in D; \; u|_{z\in\partial D} = u_0(z). \tag{2.1}$$

Bu yerda va keyin $u(z) = u(x, y),\ u_0(z) = u_0(x, y) -$ haqiqiy funksiyalar,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ - Laplas operatori.

Klassik Dirixle masalasi (2.1) ning yechimi mavjud va yagonadir. Yechimning mavjudligi matematik fizik tenglamalari kursida isbotlangan [1]. Yechishning prinsipi (1.3-xossa) dan kelib chiqadi.

Umuman olganda, $u_1(z), u_2(z) - D$ sohada garmonik chegarsi ∂D gacha uzluksiz funksiyalar bo'lib, $u_1|_{z\in\partial D} = u_2|_{z\in\partial D}$ u holda $u_1(z) - u_2(z) - D$ sohada garmonik , chegarasi ∂D gacha uzluksiz va $z \in \partial D$ nolga teng. 1.3- xossaga ko'ra $u_1(z) - u_2(z) = 0, \qquad z \in D$, yani $u_1(z) = u_2(z), \quad z \in D,$

Quyidagi misoldan ko'rish mumkinki, Dirixle masalasini qarashda izlanuvchi $u(z)$ funksiyani chegaralanganlik shartini bekor qilishda yagonalik teoremasi o'rinli bo'lmaydi.

2.1-misol. $u(x, y) = y$ funksiya $y > 0$ yarim tekislikda garmonik, chegaragacha uzluksiz va $y = 0$ $(x \neq \infty)$ da nolga teng. $u(x, y) = 0$ funksiya ham shu shartni qanoatlantiradi.

§2.2. Laplas tenglamasini konform akslantirishga nisbatan invariantligi

2.1-teorema. $z=g(\varsigma)$ regulyar funksiya G sohani D sohaga konform akslantirsin va $u(z)$ D sohada garmonik bo’lsin. U holda $\tilde{u}(\zeta)=u(g(\zeta))$ funksiya G sohada garmonikdir.

Isbot. $G_1 \subset G$ bi rbog’lamli sohani qaraymiz. $z=g(\varsigma)$ konform akslantirishida G_1 sohaning aksi $D_1 \subset D$ bir bog’lamli soha bo’ladi. $f(z)$ funksiya D_1 sohada regulyar bo’lsin, u holda $\operatorname{Re} f(z)-u(z)$ (bunday $f(z)$) funksiyaning mavjudligi 1.2-teoremaga asosan. U holda $\tilde{f}(\zeta)=f(g(\zeta))$ funksiya G_1 sohada regulyar va shuning uchun $\tilde{u}(\zeta)=\operatorname{Re}\tilde{f}(\zeta)$ G_1 da garmonik funksiya . G_1-G sohaning ixtiyoriy bir bog’lamli qism sohasi, bundan $\tilde{u}(\zeta)$ G sohada garmonik funksiya.

2.1-teoremani quyidagicha ham isbotlash mumkin.

x(ξ,η)=Reg(ζ), y(ξ,η)=Img(ξ), ζ=ξ +iη deb belgilaymiz. U holda z=g(ζ) ($z=x+iy$) akslantirishni

$$x=x(\xi,\eta),\ y=y(\xi,\eta) \tag{2.2}$$

ko’rinishda yozish mumkin.

g(ζ)- regulyar funksiya, demak x(ξ,η),y(ξ,η) funksiyalar Koshi-Riman shartini qanoatlantiradi. Shuning uchun (2.2) o’zgaruvchilarni almashtirishdan

$$\frac{\partial^2\tilde{u}}{\partial\xi^2}+\frac{\partial^2\tilde{u}}{\partial\eta^2}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)|g'(\zeta)|^2. \tag{2.3}$$

(2.3) formuladan kelib chiqadiki, agar u(z) x,y o’zgaruvchi bo’yicha garmonik bo’lsa, u holda $\tilde{u}(\zeta)=u(g(\zeta))$ funksiya ξ,η o’zgaruvchilar bo’yicha garmonik, yani Laplas tenglamasi conform akslantirishga nisbatan invariantdir. Bu o’z navbatida Dirixle masalasini konform akslantirish yordamida yechish metodi uchun asos bo’ladi.

2.2-misol. D -Jmz<0, |z+i|>l soha bo’lsin, bu yerda l>R>0

Bu sohada konsentrik bo’lamagan halqa (to’g’ri chiziq- radiusi cheksizga teng bo'lgan aylana). Imz=0 to'g'ri chiziqqa nisbatan simmetriklikka ko'ra bu nuqtalar ±ia iborat a>0/ |z+il|=R aylanaga nisbatan simmetriyaga ko'ra

$$(l+a)(l-a)=R^2, \qquad a=\sqrt{l^2-R^2}.$$

$$\Delta u = 0, \qquad z \in D; \tag{2.4}$$

$$u|_{Jmz=0}=0, \qquad u|_{|z+il|=R}=T=const \tag{2.5}$$

Dirixle masalasini yechamiz.

Buning uchun D sohani k:R_1<|ζ|<1 konsentrik halqaga $\zeta = h(z) = \dfrac{z+ia}{z-ia}$ qaraymiz, bu yerda $a=\sqrt{l^2-R^2}$, $R_1=(R+l-a)/(R+l-a)$. Bu akslantirishda Jmz=0 to'g'ri chiziq |ζ|=1 aylanaga o'tadi, |z+il|=R aylana-|ζ|=R_1 aylanaga o'tadi. Z=g(ζ) funksiya ζ=h(z) funksiyaga teskari funksiya bo'lsin.2.1-teoremaga ko'ra $\tilde{u}(\zeta)=u(g(\zeta))$ k halqada garmonik funksiya :

$$\Delta\tilde{u}=0, \zeta \in k \tag{2.6}$$

(2.5) shartdan

$$\tilde{u}|_{|\zeta|=1}=0, \qquad \tilde{u}_{|\zeta|=R_1}=T. \tag{2.7}$$

Shunday qilib, (2.4)-(2.5) masala (2.6)-(2.7) Dirixle masalasiga keltirildi. Bu masalani yechamiz.

ζ =ξ+iη =ρe$^{i\theta}$ bo'lsin. ξ = ρcosθ, η=ρsinθ almashtirishdan keyin (2.6) Laplas tenglamasi

$$\frac{\partial^2\tilde{u}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial^2\tilde{u}}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\tilde{u}}{\partial\theta^2}=0$$

ko'rinishda yoziladi.

(2.7) shartda chegaraviy funksiya θ dan bog'liq emas, u holda tabiyki, (2.6)-(2.7) masalaning yechimi ham θ dan bog'liq bo'lmaydi deb faraz qilish mumkin, yani $\tilde{u}(\zeta)$ funksiya faqat bitta ρ o'zgaruvchining funksiyasi bo'ladi. Dirixle masalasi yechimining yagonaligidan isbotlanadiki, (2.6)-(2.7) masala yechimi θdan bog'liq bo'lmaydi. $\tilde{u}(\zeta)$ funksiya θ dan bog'liq bo'lmaganda (2.6) oddiy differensial tenglamadan iborat

$$\frac{d^2\tilde{u}}{d\rho^2}+\frac{1}{\rho}\frac{d\tilde{u}}{d\rho}=0$$

Bu tenglamani umumiy yechimi $\tilde{u}(\zeta) = C_1 + C_2 \ln\rho = C_1 + C_2 \ln|\zeta|$. (2.7) shartdan $C_1 = 0, C_2 = \dfrac{T}{\ln R_1}$ topamiz, yani $\tilde{u}(\zeta) = \dfrac{T}{\ln R_1}\ln|\zeta|$ funksiya (2.6)-(2.7) masala yechimini toppish uchun z=x+iy koordinataga o’tish kerak bo’ladi.

$$u(z) = \tilde{u}(h(z)) \qquad va$$

$$|\zeta| = |h(z)| = \left|\frac{z+ia}{z-ia}\right| = \frac{\left|x^2+y^2-a^2+i2ax\right|}{x^2+(y-a)^2}$$

Ekanligidan (2.4)-(2.5) masalaning yechimi

$$u(x,y) = \frac{T}{\ln R_1}\ln\frac{\sqrt{(x^2+y^2-a^2)^2+4a^2x^2}}{x^2+(y-a)^2}$$

Funksiyadan iboratdir, bu yerda $a = \sqrt{l^2-R^2}$, $R_1 = \dfrac{R+l-a}{R+l+a}$.

2.3-§ Doirada Laplas tenglamasi uchun Dirixle masalasi

2.2-teorema. $u(z)$ funksiya $|z|<1$ doirada garmonik, $|z|\le 1$ yopiq doirada uzluksiz bo'lsin. U holda Puasson formulasi

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}u(re^{i\varphi})d\theta \qquad (2.8)$$

o'rinli bo'ladi.

Isbot. $z=0$ bo'lganda (2.8) formula (1.12) bilan mos tushadi (garmonik funksiya uchun o'rta qiymat haqidagi teorema). $z=0$ ham $u(z)$ funksiyaning qiymatini konform akslantirish yordamida o'rta qiymat haqidagi teorema orqali topishni ko'rsatamiz.

$z_0 = r_0 e^{i\varphi_0}$, $0\le r_0<1$ nuqtani fiksirlab va $|z|<1$ doirani $|\zeta|<1$ doiraga, $h(z_0)=0$ konform akslantirishni qaraymiz.

$$\zeta = h(z) = \frac{z-z_0}{1-z\bar{z}_0} \qquad (2.9)$$

(2.9) dan

$$z = h(\zeta) = \frac{\zeta+z_0}{1+\zeta\bar{z}_0} \qquad (2.10)$$

Z=g(ζ) funksiya $|\zeta|<1$ doirani $|z|<1$ doiraga shunday conform akslantiradiki, g(0)=z_0.

$u(z)$ funksiya $|z|<1$ doirada garmonik, $|z|\le 1$ yopiq doirada uzluksiz haqiqatdan, $\tilde{u}(\zeta)=u(g(\zeta))$ funksiya $|\zeta|<1$doirada garmonik (2.1-teorema) va $|\zeta|\le 1$ yopiq doirada uzluksiz. Garmonik funksiya uchun o'rta qiymat haqidagi teoremaga asosan

$$u(z_0) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi}\tilde{u}(e^{i\psi})d\psi \qquad (2.11)$$

Avvalgi o'zgaruvchiga qaytamiz. (2.11) integralda

$$e^{i\psi} = h(e^{i\theta}) = \frac{e^{i\theta}-z_0}{1-e^{i\theta}\bar{z}_0} \qquad (2.12)$$

(2.12) dan topamiz

$$d\psi = \frac{1-|z_0|^2}{(e^{i\theta}-z_0)(e^{-i\theta}-\bar{z}_0)}d\theta = \frac{1-r_0^2}{1-2r_0\cos(\varphi_0-\theta)+r_0^2}d\theta \qquad (2.13)$$

$z_0 = r_0 e^{i\psi_0}$ ni $z = re^{i\varphi}$ ga almashtirib (2.11)-(2.13) dan (2.8) formulani hosil qilamiz.

Puasson formulasini boshqacha ko'rinishga ham almashtirish mumkin. Ko'rish mumkinki,

$$\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2} = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = \mathrm{Re}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}$$

Shuning uchun (2.8) formulani

$$u(z) = \mathrm{Re}\,\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta, \qquad (2.14)$$

Ko'rinishda yozish mumkin, chunki $u(e^{i\theta})$ - haqiqiy funksiya. (2.14) integralda

$$e^{i\theta}=\zeta \text{ deb, } d\theta = \frac{d\zeta}{i\zeta}, \qquad u(z) = \mathrm{Re}\,\frac{1}{2\pi i}\int_{|\zeta|=1} u(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}, \qquad |z|<1 \qquad (2.15)$$

Hosil qilamiz. Shunday qilib (2.8) Puasson formulasini (2.15) ko'rinishda yozish mumkin.

Puasson formulasi yordamida |z|<1 doirada Laplas tenglamasi uchun Dirixle masalamasini yechish mumkin. Xususiy holda, agar chegaraviy funksiya sinφ va cosφ bog'liq ratsional funksiya bo'lsa, u holda (2.15) formuladagi integral qoldiq yordamida hisoblanadi.

2.3-misol.

$$\left.\begin{aligned} &\Delta u = 0, |z|<1; \\ &u|_{|z|=1} = \frac{\sin\varphi}{5+4\cos\varphi} \end{aligned}\right\} \qquad (2.16)$$

Masalaning yechimini toping, bu yerda $z=re^{i\varphi}$

Yechish. (2.15) formuladan foydalanamiz. $\zeta = e^{i\theta}$ bo'lsin, u holda

$$\sin\theta = \frac{1}{2i}(\xi - \frac{1}{\zeta}), \qquad \cos\theta = \frac{1}{2}(\xi + \frac{1}{\zeta}) \qquad va$$

$$u(\xi) = \frac{\sin\theta}{5+4\cos\theta} = \frac{\xi^2-1}{2i(2\xi^2+5\zeta+2)}.$$

$$J = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{(\xi^2-1)(\xi+z)}{2i(2\xi^2+5\zeta+2)(\zeta-z)\xi}d\zeta$$

Bu yerda $|\zeta|=1$ aylana soat strelkasiga teskari yo'nalishda yo'nalgan. Integral ostidagi $f(\zeta)$ funksiya $|\zeta|>1$ sohada bitta chekli maxsus nuqta birinchi tartibli ξ =-2 - qutb va ξ =∞ tuzatib bo'ladigan maxsus nuqtaga ega. Qoldiqlar nazariyasining asosi teoremasiga ko'ra ([2],§28)

$$J = -\underset{z=-2}{res}\, F(\zeta) - \underset{z=\infty}{res}\, F(\zeta).$$

$$\underset{\zeta=2}{res}\, F(\zeta) = \lim_{\xi\to 2}(\xi - 2)F(\zeta) = \frac{2-z}{4i(z+2)},$$

$$\underset{\zeta=\infty}{res} F(\zeta) = -\frac{1}{4i}, \qquad J = \frac{z}{2i(z+2)}$$

(2.15) formula orqali (2.16) masalaning yechimi hosil qilamiz.

$$u(x,y) = \mathrm{Re}\frac{z}{2i(z+2)} = \frac{y}{(z+2)^2 + y^2} = \frac{r\sin\varphi}{r^2 + 4r\cos\varphi + \theta}.$$

2.4-§. Yuqori yarim tekislikda Laplas tenglamasi uchun Dirixle masalasi.

2.3-teorema. $u(z)$ funksiya $\operatorname{Im} z > 0$ yuqori yarim tekislikda garmonik va chegaralangan, chekli sondagi nuqtalardan tashqari $\operatorname{Im} z = 0$ to'g'ri chiziqqacha uzluksiz bo'lsin. U holda Puasson formulasi

$$u(z) = \frac{1}{\pi}\int \frac{yu(t)}{(t-x)^2 + y^2}dt \tag{2.17}$$

o'rinlidir, bu yerda $z = x + iy$, $y > 0$.

Isbot. $z_0 = x_0 + iy_0$, $y > 0$ nuqtani feksirlab, Jmz>0 yuqori tekislikni |ζ|<1 , h(z_0)=0 doiraga

$$\xi = h(z) = \frac{z - z_0}{z - \bar{z}_0} \tag{2.18}$$

konform akslantirishni qaraymiz.

(2.18) dan

$$z = g(\xi) = \frac{\zeta \bar{z}_0 - z}{\xi - 1} \tag{2.19}$$

topamiz. $z = g(\xi)$ funksiya |ξ |<1 doirani $\operatorname{Im} z > 0$ yuqori yarim tekislikka shunday konform akslantiradiki, g(0)=z_0. 2.1-teoremaga ko'ra $\tilde{u}(\zeta) = u(g(\zeta))$ |ζ|<1 doirada chegaralangan va chekli sondagi nuqtalardan tashqari |ζ|=1 aylanagacha uzluksiz. O'rta qiymat haqidagi teoremaga ko'ra

$$u(z_0) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{i\psi})d\psi \tag{2.20}$$

Avvagi o'zgaruvchiga qaytamiz. (2.20) integralda

$$e^{i\psi} = h(t) = \frac{t - z_0}{t - \tilde{z}_0} \tag{2.21}$$

almashtirish olamiz.

$$\tilde{u}(e^{i\psi}) = \tilde{u}(h(t)) = u(t).$$

(2.21) dan

$$d\psi = \frac{z_0 - \bar{z}_0}{i|t - z_0|^2} = \frac{2y_0}{(t - x_0)^2 + y_0^2}dt \tag{2.22}$$

$z_0 = x_0 + iy_0$ ni $z = x + iy$ ga almashtirib (2.20)-(2.22) dan (2.17) ni hosil qilamiz.

$\dfrac{y}{(t+x) + y^2} = \operatorname{Re}\dfrac{1}{t(t-z)}$ ekanligidan (2.17) Puasson formulasini

$$u(z)=\operatorname{Re}\frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{u(t)}{t-z}dt \tag{2.23}$$

ko'rinishda yozish mumkin.

(2.17) yoki (2.23) formulalar yordamida Laplas tenglamasi uchun Dirixle masalasini $\operatorname{Im} z>0$ yuqori yarim tekislikda yechish mumkin.

$$\Delta u=0,\qquad y>0;\ \ u|_{y=0}=R(x) \tag{2.24}$$

Masalani qaraymiz, bu yerda R(x) haqiqiy ratsional funksiya, haqiqiy o'qda qutbi yo'q va $z\to\infty$ z⟶∞ da $\operatorname{Re} z\to 0$.

Bu masalaning yechimi (2.23) ga ko'ra

$$u(z)=\operatorname{Re}\frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{R(t)}{t-z}dt,$$

bu yerda $\operatorname{Im} z>0$. Bu integralni qoldiq yordamida hisoblash mumkin.

$$u(z)=-2\operatorname{Re}\sum_{Jm\xi_k<0}\underset{\zeta=\xi_k}{res}\frac{R(\xi)}{\xi-z}. \tag{2.25}$$

Qoldiq R(ξ) funksiyaning Jmξ <0 yarim tekislikdagi barcha qutblari bo'yicha olinadi.

2.4-misol.

$$\Delta u=0,\qquad y=0;\ \ u|_{y=0}=\frac{1}{1+x^2}$$

masalani qaraymiz.

(2.25) formulaga ko'ra

$$u(z)=-2\operatorname{Re}\underset{\xi=-i}{res}\frac{1}{(1+\xi^2)(\xi-z)}=-2\operatorname{Re}\frac{1}{2i(z+i)}=\frac{y+1}{x^2+(y+1)^2}$$

Ixtiyoriy bir bog'lamli sohada Laplas tenglamasi uchun Dirixle masalasini yechish shu sohani doira yoki yuqori yarim tekislikka konform akslantirish va Puasson formulasi orqali yechiladi.

2.5-misol.

$$\Delta u=0,\qquad 0<y<\pi; \tag{2.26}$$

$$u|_{y=0} = \begin{cases} 1, & x > 0 \quad bo'lg\, anda, \\ 0, & x < 0 \quad bo'lg\, anda \end{cases}$$

$$u|_{y=\pi} = 0, \qquad z = x + iy \qquad (2.27)$$

masalani yeching.

Yechish. $\zeta = e^z (\zeta = \xi + i\eta)$ funksiya 0<y<π yo'lakni η>0 yuqori tekislikka konform akslantiradi. Bunda (2.27) shart

$$\tilde{u}|_{\eta=0} = \begin{cases} 1, & \xi > 1 \quad bo'lg\, anda, \\ 0 & \xi < 1 \quad bo'lg\, anda \end{cases}$$

(2.17) formulaga ko'ra

$$\tilde{u}(\zeta) = \frac{1}{\pi}\int_1^{\infty} \frac{\eta dt}{(\xi - t)^2 + \eta^2} = \frac{1}{2} - \frac{1}{\pi} arctg \frac{1-\xi}{\eta}.$$

$\xi = e^x \cos y, \quad \eta = e^x \sin y$ almashtirish olib (2.26)-(2.27) masalaning yechimini

$$u(z) = \frac{1}{2} - \frac{1}{\pi} arctg \frac{e^{-x} - \cos y}{\sin y}.$$

hosil qilamiz.

X U L O S A

Kompleks o'zgaruvchili funksiyalar nazariyasi kursi o'zining qo'llanishi jihatidan nafaqat matematikaning yo'nalishlari, balkim fizika, mexanika va boshqa fan yo'nalishlaridagi ko'plab masalalarni yechishda asosiy xizmatchi vositasini o'taydi. Kompleks o'zgaruvchili funksiyalar nazariyasining asosiy ish vositasi hisoblangan bir qiymatli analitik funksiya, uning haqiqiy va mavhum qismi qo'shma garmonik funksiyalar tushunchasi o'zining qo'llanishi jihatidan muhim o'rin egallaydi. Garmonik funksiya matematik fizikaning asosiy tenglamalaridan biri bo'lgan Laplas tenglamasining yechimidir. Laplas tenglamasi uchun qo'yilgan chegaraviy masalalarni (Dirixle, Neyman) yechishda Grin funksiyasi muhim rol o'ynaydi.

Ushbu malakaviy bitiruv ishi garmonik va subgarmonik funksiyalarning asosiy xossalarini o'rganishga bag'ishlangan bo'lib, subgarmonik funksiyani bir qiymatli analitik funksiyaning moduli yoki modulining logarfmi sifatida qaraladi. Shu maqsadda ishning I-bobida garmonik funksiyani bir qiymatli analitik funksiyaning haqiqiy yoki mavhum qismi sifatida qaralib, asosiy xossalaridan o'rta qiymat haqidagi teorema, maksimum prinsiplariga doir misollar yechib o'rganilgan. Bu bir tomondan kompleks o'zgaruvchili funksiyalar nazariyasining asosiy tushunchalarini chuqurroq o'rganib mustaqil tarzda boshqa masalalarni yechishga qollash imkoniyatini bersa, ikkinchi tomondan bu orttirilgan tajribalar asosida umumiy o'rta ta'lim, akademik litsey va kasb-unar kollejlarida bu bilimlarni keng qamrovda qo'llay olish imkoniyatiga ega bo'linadi.

Ishning II-bobida garmonik funksiya, ya'ni Laplas trnglamasi uchun qo'yilgan chegaraviy (Dirixle) masalasining yechimi sifatida, konform akslantirish orqali yechimni integral ko'rinishda ifodalash masalalari o'rganilgan.

Xulosa qilib aytganda, kompleks va haqiqiy o'zgaruvchili funksiyalar orasidagi uzviy bo'lanishni bu funksiyalar orqali izchil o'rganish imkoniyatini beradi.

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MUNDARIJA

A.Navoiy nomidagi SamDU mexanika-matematika fakul'teti matematika bo'limining

IV-kurs talabasi ***Mardiyev Olmosning*** **"Garmonik funksiyaning asosiy xossalari va integral ko'rinishi"** mavzusidagi bitiruv malakaviy ishiga

B A H O

Ushbu bitiruv malakaviy ishi garmonik funksiyalarning asosiy xossalari o'rganish va integral ko'rinishda tasvirlashga bag'ishlangan bo'lib, kirish va ikkita bobdan iborat. Kompleks o'zgaruvchili funksiyalar nazariyasining asosiy elementlaridan biri bir qiymatli analitik funksiyaning haqiqiy va mavhum qismlari bo'lgan qo'shma garmonik funksiyalarning xossalarini o'rganish muhim ahamiyat kasb etadi.

Kompleks o'zgaruvchili funksiyalar nazariyasi o'zining qo'llanish jihatidan haqiqiy o'zgaruvchili funksiyalar nazariyasiga qaraganda ko'proq imkoniyatga egadir. Bir qiymatli analitik funksiyaning haqiqiy va mavhum qismlari Koshi-Riman shartini qanonatlantirishi, qo'shma garmonik funksiyalardan iborat ekanligidan matematik fizika tenglamalari uchun qo'yilgan chegaraviy masalalarni tekislikdagi sohalarda yechishda bir qator qulayliklar yaratadi. Elliptik tipdagi tenglamalarning eng sodda ko'rinishlaridan biri bo'lgan Laplas tenglamasi o'zining qo'llanish jihatidan muhim o'rin egallasa, ikkinchi tomondan bu tenglamaning yechimi garmonik funksidan iboratdir. Laplas tenlamasi uchun qo'yilgan Dirixle masalasining yechimini topishda bir qiymatli analitik funksiyalar orqali murakkabroq ko'rinishdagi sohalarda qaralayotgan masala yechimini topishda konform akslantirish yordamida soddaroq ko'rinishdagi sohaga o'tkazilib yechish qulaylik tug'diradi. .

Ishning II-bobida Laplas tenglamasi uchun qo'yilgan Dirixle masalasi yechimini mavjudligi va yagonaligi, garmonik funksiyani konform akslantirishga nisbatan invariantligi, doira va yuqori yarim tekislikda Dirixle masalasi yechumining integral ko'rinishi yozilgan.

Malakaviy bitiruv ishini bajarishda talaba O.Mardiyev yil davomida berilgan topshiriqlarni o'z vaqtida bajarishga harakat qildi.

Ushbu ish malakaviy bitiruv ishlari uchun qo'yilgan barcha talablarga javob beradi va uning bajaruvchisi Mardiyev Olmosning bilimini yaxshi deb baholash mumkin.

Fizika-matematika fanlari nomzodi, dotsent E.N.Sattorov

A.Navoiy nomidagi SamDU mexanika-matematika fakul'teti matematika bo'limining
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T A Q R I Z

Bitiruv malakaviy ishi кompleks o'zgaruvchili funksiyalar nazariyasining asosiy tushunchalarigan biri bir qiymatli analitik funksiyaning haqiqiy va mavhum qismidan iborat bo'lgan qo'shma garmonik funksiyani ifodalovchi garmonik funksiyaning asosiy xossalari va integral ko'rinishda tasvirlashga bag'ishlangan.

Malakaviy bitiruv ishi kirish va ikkita bobdan iborat bo'lib, 1-bobda garmonik funksiya tushunchasi, bir qiymatli analitik funksiya bilan bog'lanishi, uning asosiy xossalari (o'rta qiymat haqidagi teorema, maksimum prinsiplari) o'rganilgan. Laplas tenglamasining fundamental yechimi, shu asosda garmonik funksiyaning integral tasvir (Grin) formulasi o'rganilgan.

II-bobida subgarmonik funksiyaning asosiy xossalaridan uning chegaraviy qiymati orqali integral ko'rinishi, ya'ni Dirixle masalasining doira, yuqori yarim tekislikdagi yechimini konform akslantirish orqali qurilgan.

Bu bir tomondan kompleks o'zgaruvchili funksiyalar nazariyasining asosiy tushunchalarini chuqurroq o'rganib mustaqil tarzda boshqa masalalarni yechishga qo'llash imkoniyatini bersa, ikkinchi tomondan bu orttirilgan tajribalar asosida umumiy o'rta ta'lim, akademik litsey va kasb-unar kollejlarida bu bilimlarni keng qamrovda qo'llay olish imkoniyatiga ega bo'linadi.

Malakaviy bitiruv ishini bajarishda talaba Mardiyev Olmos mavzuga doir bir qator materiallarni o'rganganligi ma'lum bo'ldi.

Ushbu ish malakaviy bitiruv ishlari uchun qo'yilgan barcha talablarga javob beradi va uning bajaruvchisi Mardiyev Olmosning bilimini muffaqiyatli himoyadan so'ng yaxshi deb baholash mumkin.

SamDU differensial tenglamalar kafedrasi assisenti   F.Tursunov