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«INVOLYUTSIYA XOSSASIGA VA MAXSUS POTENSIALGA EGA BO’LGAN GIPERBOLIK TENGLAMA UCHUN ARALASH MASALA »

mavzusidagi

MAGISTRLIK DISSERTATSIYASI

“5A130101- Matematika (Differensial tenglamalar)”

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M U N D A R I J A

KIRISH

> Biz farzandlarimizni milliy tabiatimizga yot va zararli bo'lgan ta'sirlardan himoya qilishimiz , ularni hayotga , yon atrofda yuz berayotgan voqea-hodisalarga dahldorlik xissi bilan yashaydigan , mustaqil fikrlaydigan , imon-e'tiqodli , barkamol insonlar etib tarbiyalashimiz lozim.
>
> Islom Karimov

Vatanimiz mustaqillikka erishgandan so'ng shahdam odimlar bilanolg'a bormoqda, ilm-fan va texnikaning zamonaviy sohalari rivojlanmoqda va bu rivojlanish ilm aqli oldiga ko'plab zamonaviy muammolarni hal etishni ko'ndalang qilib qo'ymoqda. Ushbu fikrimizni prezidentimiz Islom Abduganiyevich Karimovning «O'zbekiston XXI asr bo'sag'asida: havfsizlikka tahdid, barqarorlik shartlari va taraqqiyot kafolatlari» nomli kitoblarida keltirilgan. Quyidagi so'zlardan ham bilib olsak bo'ladi:

«Respublikamizda quyidagi yo'nalishlar bo'yicha jahon darajasidagi ilmiy maktablar yaratilgan bo'lib, ularda tadqiqotlar muvoffoqiyotli olib borilmoqda. Jumladan, matematika, ehtimollar nazariyasi,tabiiy va ijtimoiy jarayonlarni matematik modellash, informatika va hisoblash texnikasi sohasidagi tadqiqotlar.

Matematika fanining ehtimollar nazariyasi va matemetik statistika, differensial tenglamalar va matematik fizika, funksional tahlil sohasidagi yutuqlari respublikadan ancha uzoqda ham mashxur».

Magistrlik dissertatsiyasi mavzusining asoslanishi va uning dolzarbligi: fizika, biologiya, texnika masalalarini matematik modellashtirish natijasida hosil qilingan chegaraviy masalalar o`rtasida xususiy hosilali differensial tenglamalar bilan yozilgan aralash masalalarning yechimi mavjudligi va yagonaligining ko'rsatilishi eng dolzarb masalalardan biridir.

Tadqiqot ob'ekti va predmeti: involyutsiyaga ega bo'lgan xususiy hosilali differensial tenglamalar.

Tadqiqot maqsadi va vazifalari: involyutsiya xossasiga ega bo'lgan giperbolik tipdagi aralash masala yechimi mavjudligi va yagonaligini ta'minlovchi yetarli shartlarni ishlab chiqish.

Tadqiqotning ilmiy jihatdan yangilik darajasi:giperbolik tipdagi tenglamalar uchun taxlil qilinayotgan aralash masalalarda tenglama tarkibida involyutsiyaning qatnashishidir.

Tadqiqotning asosiy masalalari va farazlari: Tekshirilayotgan masalaga Fur'ye usulini qo'llash bilan hosil qilingan Shturm-Liuvill masalasinining xos funksiyalari yordamida izlanayotgan masala yechimini qator ko'rinishida tasvirlash va bu qatorni tekis yaqinlashishga tekshirish bilan klassik yechim mavjudligini asoslash.

Tadqiqot mavzusi bo`yicha adabiyotlar sharhi. Tadqiqotda qo`llanilgan metodikaning tavsifi: involyutsiya qatnashgan aralash chegaraviy masalalar yechimi mavjudligi va yagonaligini ko`rsatishda chilarni ajratish usuli qo`llanilgan.

Tadqiqot natijalarining nazariy va amaliy ahamiyati.   bajarilgan ish nazariy ko`rininshda bo`lib, bu sohada ilmiy tadqiqotlar olib borish uchun foydalanilishi hamda mahsus kurslar va seminarlarda mavzu sifatida tanlanilishi mumkin.

Ishning tuzulmasining tavsifi. Ish mundarija, dissertatsiya mavzusining asoslanishi va dolzarbligi, tadqiqot ob'yekti va predmeti, tadqiqot maqsadi va vazifalari , tadqiqot natijalarining amaliy va nazariy ahamiyati, qo'llanilgan usullar, mavzu bo'yicha qisqacha adabiyotlar tafsifini o'z ichiga olgan kirish qismidan, uchta bobga ajratilgan asosiy qismdan,xulosa va foydalanilgan adabiyotlar ro'yxatidan iborat.

Ma'lumki fizika, kimyo,biologiya,mexanika kabi fan va texnika kabi ko'plab fan sohalarida taxlil qilinayotgan masalalarni matematik modellashtirish natijasida xususiy hosilali differensial tenglamalar bilan yozilgan turli ko'rinishdagi aralash chegaraviy masalalar yuzaga keladi. Chegaraviy masalalarga misollar sifatida matematik fizika kursidan o'rin olgan issiqlik tarqalish tenglamasi, tor tebranish tenglamasi kabi xususiy hosilali differensial tenglamalar bilan yozilgan chegaraviy masalalarni qarash mumkin. Chegaraviy masalani ifodalovchi tenglama , boshlang'ich yoki chegaraviy shartlarda involyutsiya qatnashishi mumkin.

Ma'lumki $\alpha(\alpha(x)=x)$, ya'ni $\alpha^2(x)=\alpha^{-1}(x)$ tenglikni qanoatlantiruvchi α akslantirishga involyutiv akslantirish yoki involyutsiya deyiladi. Involyutsiyaga misollar sifatida

$\alpha(x)=-x,\ \alpha(x)=c-x,\ \alpha(x)=\dfrac{1}{x},\ \alpha(x)=\sqrt{1-x^2},\ \alpha(x)=\sqrt[m]{1-x^m},\ \alpha(x)=\dfrac{ax+b}{cx-a}$ kabi

ko'plab funksiyalarni keltirishimiz mumkin.

Differensial tenglamalar kursida ham involyutsiya xossasiga ega bo'lgan differensial tenglamalar uchraydi. Misol uchun oddiy differensial tenglamalar kursida

$$y'(x)=y(\alpha(x))$$

ko'rinishdagi tenglamada $\alpha(x)=\dfrac{1}{x}$ akslantirish tadbiq etilsa, u holda $\alpha\circ\alpha(x)=x$

ko'rinishdagi ikkinchi tartibli involyutsiya mavjudligini ko'rish qiyin emas. Shuningdek, involyutsiya xossasiga ega bo'lgan

$$y'(x)=\frac{1}{y(a-x)}$$

Oddiy differensial tenglamani yozishimiz mumkin.

Xuddi shu kabi

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial u(\xi,t)}{\partial \xi}\bigg|_{\xi = c - x} + q(x)u(x,t)$$

involyutsiya qatnashgan xususiy hosilali differensial tenglama bo’ladi.

Involyutsiya xossasiga ega bo’lgan oddiy differensial tenglamalar haqida birinchi ish Zilbershteyin tomonidan Philosific Magazine 30 (1940) , ser.7. jurnalida e’lon qilingan “ Solution of the equation $f'(x) = f\left(\frac{1}{x}\right)$” mavzudagi maqolasidir.

Dissertatsiyaning asosiy qism uchta bobdan tashkil topgan bo’lib, birinchi bobda oddiy differensial tenglamalar nazariyasidan involyutsiya tushunchasini kiritish uchun zaruriy ma’lumotlar keltirilgan bo’lib, bu bob o’zgarmas koeffisiyentli chiziqli differensial tenglamalar , o’zgarmas koeffisiyentli tenglamalarga almashtirishlar vositasida keltiriluvchi Eyler va Lagranj tenglamalari to’g’risidagi ma’lumotlardan iborat. Shuningdek bu bobda involyutsiya tushunchasi, unig tiplari va xossalari keltirilgan.

Dissertatsiyaning ikkinchi bobida involyutsiya xossasiga ega bo’lgan oddiy differensial tenglamasi, chiziqli tenglama va tenglamalar sistemasi bo’yicha zaruriy ma’lumotlar keltirilgan hamda involyutsiya xossasiga ega bo’lgan oddiy differensial tenglamalarni yechishga oid misollar yechishdan namunalar keltirilgan.

Dissertatsiyaning uchinchi bobi taqdim etilayotgan ishning asosiy qismini tashkil qilgan bo’lib, bunda involyutsiya xossasiga ega bo’lgan xosilali eng sodda giperbolik tipdagi tenglama uchun aralash masala qaralgan hamda masalaning klassik yechimi keltirilgan.

A S O S I Y   Q I S M

I BOB. INVOLYUTSIYAGA EGA BO'LGAN ODDIY DIFFERENSIAL TENGLAMALAR

Involyutsiya xossasiga ega bo'lgan oddiy differensial tenglamalarni yechish jarayonida almashtirishlardan so'ng Eyler yoki Lagranj tenglamalari hosil bo'ladi. O'z navbatida bu tenglamalar almashtirishlar yordamida o'zgarmas koeffisiyentli chiziqli tenglamalarga olib o'tilishi va yechilishi mumkin. Shuning uchun bu bobda asosiy tushunchalar qatorida [11] adabiyotda yoritilgan o'zgarmas koeffisiyentli chiziqli differensial tenglamalar uchun Eyler va Lagranj tenglamalarini ko'rib chiqamiz.

1.1-§.Differensial tenglamalar nazariyasidan zaruriy tushunchalar

1.O'zgarmas koeffisiyentli chiziqli differensial tenglamalar

Biz $n-$tartibli o'zgarmas koeffisiyentli chiziqli

$$L[y] \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(t) \qquad (1.1.1)$$

differensial tenglamani qaraymiz, bu yerda $a_1, a_2, ..., a_n$ –o'zgarmas kompleks sonlar, $f(t)$ qandaydir oraliqda berilgan t o'zgaruvchining kompleks funksiyasi. (1.1.1) tenglamaning chap qismi $n-$*tartibli chiziqli differensial operator* deyiladi va $L[y]$ kabi belgilanadi. (1.1,1) tenglamaning o'zi esa

$$L[y] = f(t) \qquad (1.1.2)$$

ko'rinishda yoziladi.

Dastlab $n-$ tartibli bir jinsli o'zgarmas koeffisiyentli tenglamani qaraymiz.

2.Bir jinsli tenglamalar. Har bir

$$L[y] \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

operatorga yoki bir jinsli

$$L[y] = 0 \qquad (1.1.3)$$

tenglamaga (1.1.3) tenglamaning yoki $L[y]$ *operatorning xarakteristik ko'phadi* deb nomlangan

$$D(p) = p^n + a_1 p^{n-1} + \cdots + a_0 \qquad (1.1.4)$$

ko'phadni mos qo'yamiz.

1-lemma. Ixtiyoriy n marta uzluksiz differensiallanuvchi $f(t)$ funksiya uchun quyidagi:

$$L\left[e^{\lambda t} f(t)\right] = e^{\lambda t}\left(D(\lambda) f + \frac{D'(\lambda)}{1!} f' + \cdots + \frac{D^{(n)}(\lambda)}{n!} f^{(n)}\right) \qquad (1.1.5)$$

formula o'rinli.

Isboti. $L[y] = y^{(k)}$, $(0 \le k \le n)$ bo'lsin, u holda $D(p) = p^k$ bo'lib, agar $l > k$ bo'lganda $D^{(l)}(\lambda) = 0$ bo'lgani uchun

$$L\left[e^{\lambda t} f(t)\right] = \frac{d^k}{dt^k}\left(e^{\lambda t} f(t)\right) = \sum_{l=0}^{k} C_k^l \frac{d^{k-l}}{dt^{k-l}}(e^{\lambda t}) \cdot f^{(l)}(t) = \sum_{l=0}^{k} \frac{k(k-1)\ldots(k-l+1)}{l!} \lambda^{k-l} e^{\lambda t} f^{(l)}(t) =$$

$$= e^{\lambda t} \sum_{l=0}^{k} \frac{1}{l!} \frac{d^l}{d\lambda^l}(\lambda^k) \cdot f^{(l)}(t) = e^{\lambda t} \sum_{l=0}^{k} \frac{D^{(l)}(\lambda)}{l!} f^{(l)}(t) = e^{\lambda t} \sum_{l=0}^{n} \frac{D^{(l)}(\lambda)}{l!} f^{(l)}(t)$$

Biz bu yerda ikki funksiya ko'payitmasining $k-$ tartibli hosilasini hisoblashda Leybnits formulasidan foydalandik.

Demak (1.1.5) formula $L[y] = y^{(k)}$ xususiy hol uchun isbotlandi. (1.1.5) formulaning umumiy holda to'g'riligi $L[y]$ operatorning $L[y] = y^{(k)}$, $(0 \le k \le n)$ ko'rinishdagi operatorlarning chiziqli kombinatsiyasidan iborat ekanligidan kelib chiqadi.

Lemma isbotlandi.

2-lemma. $\lambda-$ soni $D(p)$ xarakteristik ko'phadning k karrali ildizi bo'lsa, u holda $y_1 = e^{\lambda t}$, $y_2 = te^{\lambda t}$,..., $y_k = t^{k-1} e^{\lambda t}$ funksiyalar bir jinsli (1.1.3) tenglamaning yechimlari bo'ladi.

Isboti. $\lambda-$ soni $D(p)$ xarakteristik ko'phadning k karrali ildizi bo'lgani uchun

$$D(\lambda) = D'(\lambda) = \cdots = D^{(k-1)}(\lambda) = 0, D^{(k)}(\lambda) \ne 0$$

(1.1.5) formulani $y_j = t^j e^{\lambda t}$ $(0 \le j \le k-1)$ funksiyalarga tadbiq qilib,

$$L\left[y_{j+1}\right]=L\left[e^{\lambda t}t^j\right]=e^{\lambda t}\left(\frac{D^{(k)}(\lambda)}{k!}\frac{d^k}{dt^k}\left(t^j\right)+\cdots+\frac{D^{(n)}(\lambda)}{n!}\frac{d^n}{dt^n}\left(t^j\right)\right)=0$$

chunki $l>j$ bo'lganda $\frac{d^l}{dt^l}\left(t^j\right)=0$. Lemma isbotlandi.

3-lemma. $r=0,1,2,...,k-1$ uchun $L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=0$ bo'lsin. U holda λ soni xarakteristik ko'phadning karraligi k dan kichik bo'lmagan ildizi bo'ladi.

Isboti. 1-lemmaga ko'ra

$$L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=e^{\lambda t}\left(D(\lambda)t^r+\frac{D'(\lambda)}{1!}\frac{d\left(t^r\right)}{dt}+\cdots+\frac{D^{(r)}(\lambda)}{r!}\frac{d^r\left(t^r\right)}{dt^r}\right)\Bigg|_{t=0}$$

chunki $k>r$ bo'lganda $\frac{d^k}{dt^k}\left(t^r\right)=0$. Bundan tashqari $k<r$ bo'lganda $\frac{d^k}{dt^k}\left(t^r\right)\Big|_{t=0}=0$. Shuning uchun

$$L\left[t^r e^{\lambda t}\right]\Big|_{t=0}=e^{\lambda t}\frac{D^{(r)}(\lambda)}{r!}r(r-1)...1\Big|_{t=0}=D^{(r)}=0 \qquad (r=0,1,...,k-1)$$

Demak, λ xarakteristik ko'phadning k dan kichik bo'lmagan ildizi bo'ladi. Lemma isbotlandi.

Endi bir jinsli (1.1.3) tenglamaga qaytamiz. Aytaylik $D(p)$ xarakteristik ko'phad m $(m\le n)$ ta turli $\lambda_1,\lambda_2,\ ...,\lambda_m$ ildizlarga ega bo'lsin. Bu ildizlarning karraliklarini $k_1,k_2,...,k_m$ bilan belgilaymiz. U holda 2-lemmaga ko'ra

$$\begin{aligned}
&y_1=e^{\lambda_1 t},\ y_2=te^{\lambda_1 t},\ ...,y_{k_1}=t^{k_1-1}e^{\lambda_1 t},\\
&y_{k_1+1}=e^{\lambda_2 t},\ y_{k_1+2}=te^{\lambda_2 t},\ ...,y_{k_1+k_2}=t^{k_2-1}e^{\lambda_2 t},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&y_{k_1+\cdots+k_{m-1}+1}=e^{\lambda_m t},\ y_2=te^{\lambda_1 t},\ ...,y_{k_1+\cdots+k_m}=t^{k_m-1}e^{\lambda_m t},
\end{aligned}\tag{1.1.6}$$

funksiyalar ham bir jinsli (1.1.3) tenglamaning yechimlari bo'ladi. $k_1+\cdots+k_m=n$ bo'lgani uchun (1.1.6) formula (1.1.3) tenglamaning n ta $y_j(t)$ yechimlarini aniqlaydi.

4-lemma. Agar $y_1,y_2,....y_n$ funksiyalar (1.6) tengliklar bilan aniqlansa, u holda

$$\begin{vmatrix} y_1(0) & y_2(0) & \dots & y_n(0) \\ y_1'(0) & y_2'(0) & \dots & y_n'(0) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(0) & y_2^{(n-1)}(0) & \dots & y_n^{(n-1)}(0) \end{vmatrix} \neq 0 \qquad (1.1.7)$$

Isboti. Faraz qilaylik (1.1.7) determinant nolga teng bo'lsin. U holda bu determinantning satrlari orasida

$$b_0 y_j^{(n-1)}(0) + b_1 y_j^{(n-2)}(0) + \dots + b_{n-1} y_j(0) = 0 \qquad (1.1.8)$$

$$(j = 1, 2, \dots, n)$$

chiziqli bog'liqlik mavjud, bu yerda $b_0, b_1, \dots, b_{n-1}$ koeffisiyentlarning barchasi bir vaqtda nolga teng emas.

$$L_1[y] = b_0 y^{(n-1)} + b_1 y^{(n-2)} + \dots + b_{n-1} y$$

differensial operatorni qaraymiz. Bu operatorga darajasi $n-1$ dan ortmagan

$$D_1(p) = b_0 p^{n-1} + b_1 p^{n-2} + \dots + b_{n-1}$$

xarakteristik ko'phad mos keladi.

$j = 1, 2, \dots, k_1$ bo'lsin, u holda (1.1.8) munosabatni

$$L\left[t^r e^{\lambda_1 t}\right]\Big|_{t=0} = 0 \qquad (j = 1, \dots, k_1)$$

yoki

$$L\left[t^r e^{\lambda_1 t}\right]\Big|_{t=0} = 0 \qquad (r = 0, 1, \dots, k_1 - 1;\ r = j-1)$$

ko'rinishda yozishimiz mumkin.

3-lemmaga ko'ra λ_1 soni $D_1(p)$ xarakteristik ko'phadning karraligi k_1 dan kichik bo'lmagan ildizi bo'ladi. Xuddi shu kabi $j = k_1 + 1, \dots, k_1 + k_2$ ko'rinishda tanlab

λ_2 soni $D_1(p)$ xarakteristik ko'phadning karraligi k_2 dan kichik bo'lmagan ildizi bo'lishini va hakozo λ_m soni $D_1(p)$ xarakteristik ko'phadning karraligi k_m dan kichik bo'lmagan ildizi bo'lishini isbotlashimiz mumkin. Shuning uchun har bir ildizni karraligi bilan hisoblab $D_1(p)$ xarakteristik ko'phadning ildizlari soni $k_1 + \dots + k_m = n$ ga teng bo'ladi degan xulosaga kelamiz. Ammo bu mumkin emas, chunki $D_1(p)$ xarakteristik ko'phadning darajasi $n-1$ dan ortmaydi. Hosil bo'lgan ziddiyat lemmani isbotlaydi.

3. Eyler va Lagranj tenglamalari. Differensial tenglamalar orasida oddiy almashtirishlar vositasida o'zgarmas koeffisiyentli tenglamalarga o'tuvchi o'zgaruvchi koeffisiyentli tenglamalar ham uchraydi.

$$a_0 t^n \frac{d^n y}{dt^n} + a_1 t^{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} t \frac{dy}{dt} + a_n y = 0 \qquad (1.1.9)$$

ko'rinishdagi tenglamaga Eyler tenglamasi deyiladi, bu yerda $a_0, a_1, ..., a_n$ o'zgarmas sonlar. Agar (1.1.9) tenglamada t ni Ct bilan almashtirsak tenglamaning ko'rinishi o'zgarmaydi. Demak, (1.1.9) tenglamada x erkli o'zgaruvchini

$$x = \ln t\,, \; t = e^x \qquad (1.1.10)$$

almashtirish bilan kiritsak, u holda x ni $x+C$ bilan almashtirishda tenglama o'zgarmaydi, ya'ni hosil bo'lgan yangi tenglama x ni oshkor ko'rinishda saqlamaydi. Erkli o'zgaruvchini almashtirishda tenglama chiziqli tenglamaga o'tmaganligi uchun, biz o'zgarmas koeffisiyentli chiziqli tenglamaga ega bo'lamiz.

Bu tasdiqni hisoblashlar vositasida bevosita tekshirishimiz mumkin. Biz y funksiyaning t bo'yicha hosilalarini (1.1.10) formula bo'yicha x bo'yicha hosilalari orqali ketma ket ifodalaymiz:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = e^{-x} \frac{dy}{dx} \;;$$

$$\frac{d^2 y}{dt^2} = e^{-x} \frac{d}{dx}\left(e^{-x} \frac{dy}{dx} \right) = e^{-2x}\left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right);$$

$$\frac{d^3 y}{dt^3} = e^{-x} \frac{d}{dx}\left[e^{-2x}\left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right) \right] = e^{-3x}\left(\frac{d^3 y}{dx^3} - 3x \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} \right);$$

Biz ko'ramizki, t bo'yicha olingan birinchi, ikkinchi va uchinchi tartibli hosilalarni qatnashgan ifodalar mos ravishda $e^{-ч}, e^{-2x}$ va e^{-3x} ko'paytuvchilarga ega. Faraz qilaylik t bo'yicha olingan $k-$ tartibli hosila

$$\frac{d^k y}{dt^k} = e^{-kx}\left(\frac{d^k y}{dx^k} + \alpha_1 \frac{d^{k-1} y}{dx^{k-1}} + \cdots + \alpha_{k-1}\frac{dy}{dx}\right)$$

ko'rinishga ega bo'lsin, bu yerda $\alpha_1, \alpha_2, ..., \alpha_{k-1} -$ o'zgarmas sonlar. U holda t bo'yicha olingan $(k+1)-$ tartibli hosila

$$\frac{d^{k+1} y}{dt^{k+1}} = e^{-x}\frac{d}{dx}\left(\frac{d^k y}{dt^k}\right) = e^{-(k+1)x}\left(\frac{d^{k+1} y}{dx^{k+1}} + (\alpha_1 - k)\frac{d^k y}{dx^k} + \cdots - k\alpha_{k-1}\frac{dy}{dx}\right)$$

ko'rinishga ega bo'ladi va yana qavs oldida $e^{-(k+1)x}$ ko'paytuvchi , qavslar ichida esa x bo'yicha birinchi tartibli hosiladan boshlab $(k+1)-$tartibli hosilagacha ifodalarning chiziqli kombinatsiyalari joylashgan. Demak ko'rsatilgan xossa ixtiyoriy k natural soni uchun isbotlandi. Biz hisoblangan hosilalarni (1.1.9) tenglamaga qo'ysak, har bir k uchun $\frac{d^k y}{dt^k}$ ifodani $a_k t^k = a_k e^{kx}$ ko'paytirishlozim bo'ladi va shu bilan birga x ni o'zida saqlovchi ko'rsatkichli ko'paytuvchilar qisqaradi hamda o'zgarmas koeffisiyentli chiziqli tenglama hosil bo'ladi.

1.2-§. Involyutsiya qatnashgan oddiy differensial tenglamalar

1. Differensial tenglamalar involyutsiyasi. Qandaydir f akslantirish berilgan bo'lib, bu akslantirishda P nuqtaning tasviri $P' = f(P)$ nuqta bo'lsin. O'z navbatida $P'' = f(P'), P'' = P$ ya'ni $f \circ f(P) = P$ bo'lsin. Demak, f akslantirish involyutiv akslantirish bo'lishi uchun quyidagi shartlarning biri o'rinli bo'lishi kerak:

1) ixtiyoriy P nuqta uchun

$$f(f(P)) = P \qquad (1.2.1)$$

tenglikning bajarilishi yoki

2) ixtiyoriy P nuqta uchun $P' = f'(x)$ munosabat bilan birgalikda

$$P' = f^{-1}(P) \qquad (1.2.2)$$

munosabat bajarilishi, ya'ni har qanday akslantirish o'ziga teskari akslantirish bilan ustma ust tushishi lozim.

Involyutsiya geometrік atama bo’lib o’ziga teskari akslantirishlar bilan bir xil bo’lgan akslantirishlarga *involyutiv akslantirishlar* deyiladi. Shuningdek geometriyada butun son o’qida aniqlangan haqiqiy argumentli (1.2.1) tenglikni qanoatlantiruvchi $f(x)$ funksiyaga *kuchli involyutsiya* deyiladi.

Dastlab [**Винер ….** kabi ko’plab ilmiy maqolalardan o’rin olgan oddiy differensial tenglamalar involyutsiyasining ta’rifini keltiramiz.

Ta’rif. Agar $f_1(x), f_2(x),..., f_m(x)$ akslantirishlar involyutsiyalar bo’lsa, u holda $F(x, y(f_1(x)), y(f_2(x)),..., y(f_m(x)),..., y^{(n)}(f_1(x)), y^{(n)}(f_2(x)),..., y^{(n)}(f_m(x)))=0$ ko’rinishdagi tenglamalarga involyutsiya xossasiga ega bo’lgan tenglamalar deyiladi.

Endi [Винер….] adabiyotlardan involyutsiya xossasiga ega bo’lgan differensial tenglamalar uchun isbotlangan ba’zi mulohazalarni keltiramiz.

1-teorema. Agar

$$y' = F(x, y(x), y(f(x))) \qquad (1.2.4)$$

tenglama uchun quyidagi shartlar bajarilsin:

1) $f(x)-$ yagona qo’zg’almas nuqtaga ega bo’lgan uzluksiz differensiallanuvchi funksiya;

2) $F(x, y(x), y(f(x)))-$ barcha argumentlari bo’yicha aniqlangan va bu argumentlar bo’yicha uzluksiz differensiallanuvchi funksiya;

3) (2.4) tenglama $y(f(x))$ argumentga nisbatan bir qiymatli yechimga ega, ya’ni

$$y(f(x)) = G(x, y(x), y'(x)) \qquad (1.2.5)$$

U holda

$$y''(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y(x)} \cdot y'(x) + \frac{\partial F}{\partial y(f(x))} \cdot f'(x) \cdot F(x, y(x), y'(f(x))) \qquad (1.2.6)$$

munosabat o’rinli bo’lib, bu yerda $y(f(x))$ (1.2.5) tenglik bilan beriladi hamda (1.2.4) tenglamaning

$$y(x_0) = y_0, y'(x_0) = F(x_0, y_0, y'_0) \qquad (1.2.7)$$

boshlang’ich shartlarni qanoatlantiruvchi yechimi bo’ladi.

Isboti. Dastlab (1.2.6) tenglama (1.2.4) tenglamani differensiallash yo'li bilan hosil qilinishini ko'rsatamiz. Buning uchun (1.2.4) tenglamani x bo'yicha differensiallaymiz:

$$y''(x)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+\frac{\partial F}{\partial y(f(x))}\cdot y'(f(x))\cdot f'(x)=$$

$$=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+\frac{\partial F}{\partial y(f(x))}\cdot F(x,y(f(x)),y(f(f(x))))\cdot f'(x)=$$

$$=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y(x)}\cdot y'(x)+\frac{\partial F}{\partial y(f(x))}\cdot F(x,y(f(x)),y(x))\cdot f'(x)$$

ya'ni (1.2.6) tenglama o'rinli ekanligini ko'ramiz. Teoremani isbotlashda (1.2.4) ning o'ng tomonidagi ifodadan hamda involyutsiyaning $f(f(x))=x$ xossasidan foydalandik. Boshlang'ich shartlardan (1.2.7) ni hosil qilish uchun esa

$$y'(f(x))=F(x,y(f(x)),y(x)) \qquad (1.2.8)$$

tenglamada x ning o'rniga $x=x_0$ qiymarni qo'yamiz va $f(x_0)=x_0$ ekanligini e'tiborga olamiz. Teorema isbotlandi.

Endi $y(x)$ funksiyani oshkormas holda saqlovchi

$$y'=F(y(f(x))) \qquad (1.2.9)$$

ko'rinishdagi differensial tenglamalarni qaraymiz.

2-teorema. Agar (1.2.9) tenglamada quyidagi shartlar bajarilsa:

1) $f(x)-$ yagona qo'zg'almas x_0 nuqtaga ega bo'lgan uzluksiz differensiallanuvchi kuchli involyutsiya;

2) $F(y(f(x)))-$butun son o'qida aniqlangan uzliksiz differensiallanuvchi qat'iy monoton funksiya bo'lsin. U holda

$$y''(f(x))=F'(y(f(x)))\cdot F(y(x))\cdot f'(x) \qquad (1.2.10)$$

oddiy differensial tenglamaning

$$y(x_0)=y_0,\ y'(x_0)=F(y_0)$$

boshlzng'ich shartlarni qanoatlantiruvchi yechimi (1.2.9) tenglamaning $y(x_0)=y_0$ boshlangich shartni qanoatlantiruvchi yechimidan iborat bo'ladi.

Isboti. Berilgan tenglamani x bo'yicha differensiallaymiz. Natijada

$$y''(x)=F'(y(f(x)))\cdot y'(f(x))\cdot f'(x)$$

tenglikni hosil qilamiz. Ammo

$$f(f(x))=x;\quad y'(x_0)=y'(f(x_0))=F(y(f(x_0)))=F(y(x_0))$$

bo'lgani uchun

$$y'(f(x))=F(y(f(f(x))))=F(y(x))$$

va shu bilan birga

$$y'=F(y(f(x)))$$

tenglikdan

$$y(f(x))=F^{-1}(y'(x))$$

kelib chiqadi. Teorema isbotlandi.

Yuqorida keltirilgan teoremalardan quyidagi natija kelib chiqadi.

Natija. Agar yuqoridagi 1-chi va 2-chi teoremalarda

$$f(x)=\frac{\alpha x+\beta}{\gamma x-\alpha},\ \ \alpha^2+\beta\gamma>0$$

akslantirishlar giperbolik involyutsiya bo'lib, (1.2.4) va (1.2.9) tenglamalar $(-\infty,\ \alpha/\gamma)$ yoki $(\alpha/\gamma,\ +\infty)$ oraliqlarda aniqlangan bo'lsa, u holda bu teoremalar o'z kuchini saqlaydi.

Eslatma.1) Agar x_0 nuqta $f(x)$ involyutsiyaning qo'zg'almas nuqtasi bo'lib, $x>x_0$ bo'lsa, (1.2.4) va (1.2.9) tenglamalar kechikkan argumentli differensial tenglamalar bo'ladi.

2) Agar x_0 nuqta $f(x)$ involyutsiyaning qo'zg'almas nuqtasi bo'lib, $x<x_0$ bo'lsa, (1.2.4) va (1.2.9) tenglamalar ortgan argumentli differensial tenglamalar bo'ladi.

2.Chiziqli differensial tenglamalar involyutsiyasi.

$$Ly(x)\equiv\sum_{k=0}^{n}a_k(x)y^{(k)}(x)=y(f(x))+\varphi(x) \qquad (1.2.11)$$

ko'rinishdagi chiziqli differensial tenglama

$$y(x_0)=y_0,\, y'(x_0)=y_1,\, y''(x_0)=y_2,\dots,\, y^{(n-1)}(x_0)=y_n \qquad (1.2.12)$$

bosglang'ich shartlar bilan berilgan bo'lsin. Bu yerda $f(x)$ involyutiv xossaga ega bo'lgan funksiya.

1-teorema. Aytaylik (1.2.11) tenglamaning $a_0(x), a_1(x),\dots, a_n(x)$ koeffisiyentlari, qo'zg'almas x_0 nuqtaga ega bo'lgan kuchli involyutsiya hosil qiluvchi $f(x)$ funksiya hamda $\varphi(x)$ funksiyalar $C^n(-\infty,+\infty)$ funksiyalar sinfiga tegishli. Agar

$$M=\frac{1}{f'(x)}\cdot\frac{d}{dx} \qquad (1.2.13)$$

ko'rinishdagi operator bo'lsa, u holda

$$\sum_{k=0}^{n} a_k(f(x))M^k Ly(x)-y(x)=\sum_{k=0}^{n} a_k(f(x))M^k\varphi(x)-\varphi(f(x)) \qquad (1.2.14)$$

chiziqli oddiy differensial tenglamaning

$$M^k Ly(x)\big|_{x=x_0}=y_k+M^k\varphi(x)\big|_{x=x_0},\ k=0,1,2,\dots,n-1 \qquad (1.2.15)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi (1.2.11) tenglamaning (1.2.12) boshlang'ich shartlarni qanoatlantiruvchi yechimi nilan bir xil bo'ladi.

Isboti. (1.2.11) differensial tenglamani n marta ketma ket differensiallash yo'li bilan uning ko'rinishini o'zgartiramiz.

$$Ly(x)\equiv a_n(x)y^{(n)}+a_{n-1}(x)y^{(ni1)}+\dots+a_1(x)y+a_0(x)$$

bo'lgani uchun (1.2.11) tenglamani

$$Ly(x)=y(f(x))+\varphi(x)$$

ko'rinishda yozamiz. Bu tenglikni

$$y(f(x))=Ly(x)-\varphi(x)\equiv M^0Ly(x)-M^0\varphi(x)$$

ko'rinishda yozamiz. Bu tenglamani har ikkala qismini x bo'yicha differensiallab,

$$y'(f(x))\cdot f'(x)=\frac{d}{dx}M^0Ly(x)-\frac{d}{dx}M^0\varphi(x),$$

yoki (1.2.13) operatorning aniqlanishiga ko'ra

$$y'(f(x))=\frac{1}{f'(x)}\cdot\frac{d}{dx}M^0Ly(x)-\frac{1}{f'(x)}\cdot\frac{d}{dx}M^0\varphi(x)=$$
$$=M^1Ly(x)-M^1\varphi(x)$$

tenglikka kelamiz. (1,2.13) munosabatni eltiborga olgan holda yana differensiallashni davom ettiramiz:

$$y''(f(x))\cdot f'(x)=\frac{d}{dx}M^1Ly(x)-\frac{d}{dx}M^1\varphi(x)$$

va bundan

$$y''(f(x))=\frac{1}{f'(x)}\cdot\frac{d}{dx}M^1Ly(x)-\frac{1}{f'(x)}\cdot\frac{d}{dx}M^1\varphi(x)=$$
$$=M^2Ly(x)-M^2\varphi(x)$$

tenglikni yozamiz. Bu jarayonni davom ettirib,

$$y^{(n)}(f(x))=\frac{1}{f'(x)}\cdot\frac{d}{dx}M^{n-1}Ly(x)-\frac{1}{f'(x)}\cdot\frac{d}{dx}M^{n-1}\varphi(x)=$$
$$=M^nLy(x)-M^n\varphi(x)$$

tenglikka kelamiz. Hosil qilingan tengliklar mos ravishda

$$a_0(f(x)),a_1(f(x)),...,a_n(f(x))$$

ifodalarga ko'paytirilib qo'shilsa

$$\sum_{k=0}^{n}a_k(f(x))y^k(f(x))=\sum_{k=0}^{n}a_k(f(x))M^kLy(x)-\sum_{k=0}^{n}a_k(f(x))M^k\varphi(x) \qquad (1.2.16)$$

tenglik hosil bo'ladi. Ammo (1.2.11) tenglamada $f(f(x))=x$ ekanligi e'tiborga olinsa, u holda (1.2.11) va (1.2.16) tenglamalardan (1.2.14) tenglama hosil bo'ladi.

Boshlang'ich shartlarni esa differensiallash yo'li bilan yuqorida keltirilgan ifodalardan hosil qilishimiz mumkin.

Haqiqatdan ham, $k=0,1,2,...,n-1$ uchun

$y^{(k)}(f(x))=M^kLy(x)-M^k\varphi(x)$ va $y^{(k-1)}(f(x_0))=y^{(k-1)}(x_0)=y_k$ ekanligidan

$$M^kLy(x)\big|_{x=x_0}=y_k+M^k\varphi(x)\big|_{x=x_0}$$

bo'ladi. Teorema isbotlandi.

2-teorema. Ushbu

$$y'(x)=\alpha x^{\beta}y\left(\frac{1}{x}\right), \quad 0<x<+\infty, \quad y(1)=y_0 \qquad (1.2.17)$$

ko'rinishdagi tenglamani integrallash mumkin.

Isboti. Berilgan tenglamani x bo'yicha differensiallash natijasida

$$y''(x)=\alpha\beta x^{\beta-1}y\left(\frac{1}{x}\right)-\alpha^2x^{-2}y(x) \qquad (1.2.18)$$

tenglama hosil bo'ladi. Berilgan (1.2.17) tenglamadan

$$y\left(\frac{1}{x}\right)=\frac{1}{\alpha}x^{-\beta}y'(x)$$

bo'lgani uchun bu tenglamani bir qator murakkab bo'lmagan hisoblaslar yordamida

$$x^2y''(x)-\beta xy'(x)+\alpha^2y(x)=0 \qquad (1.2.19)$$

Eyler tenglamasi ko'rinishida ifodalashimiz mumkin.

(1.2.19) tenglamada $x=e^t$, $y(x)=g(t)$ almashtirish kiritamiz. U holda

$$y(x)=y\left(e^t\right)=g(t)$$

bo'lgani uchun

$$y\left(\frac{1}{x}\right)=y\left(e^{-t}\right)=g(-t)$$

bo'ladi va bundan tashqari $t=\ln x$ ekanligi e'tiborga olinsa

$$y'(x)=g'(t)\cdot\frac{dt}{dx}=\frac{1}{x}g'(t),$$

$$y''(x)=-\frac{1}{x^2}g'(t)+\frac{1}{x^2}g''(t)$$

tengliklar hosil bo'ladi. Bu tengliklarga ko'ra (1.2.19) tenglama

$$g''(t)-(\beta-1)g'(t)+\alpha^2g(t)=0 \qquad (1.2.20)$$

ko'rinishni oladi.

Endi $y(1)=y_0$ boshlang'ich shart e'tiborga olinsa, u holda $x=e^t, x_0=1$ bo'lgani uchun $t=0$ bo'lganda $g(0)=y(1)=y_0$ va

$$y'(x)=\alpha x^{\beta}y\left(\frac{1}{x}\right)$$

berilgan tenglamadan $y'(1)=\alpha y_0$ bo'lgani uchun $g'(t)=xy'(x)$ tenglikdan $g'(0)=y'(1)=\alpha y_0$ ekanligini ko'ramiz.

Demak, (1.2.10) tenglamani

$$g(0)=y_0,\ g'(0)=\alpha y_0 \qquad (1.2.21)$$

boshlang'ich shartlar bo'yicha integrallash mumkin. Teorema isbotlandi. Isbotlangan teoremada hosil bo'lgan differensial tenglamani integrallaylik.

3.Chiziqli differensial tenglamalar sistemasining involyutsiyasi.

Biz endi involyutsiya xossasiga ega bo'lgan chiziqli tenglamalar sistemasini ko'rib chiqamiz. Quyidagi tasdiq o'rinli.

Teorema. Agar $y-n$ o'lchamli vector, A va $B-n\times n$ o'lchamli matritsalar bo'lsa, u holda

$$y'(x)=Ay(x)+By(c-x) \qquad (1.2.22)$$

sistemaning

$$y(c/2)=y_0$$

boshlang'ich shartni qanoatlantiruvchi yechimi

$$y''(x)=\left(A-BAB^{-1}\right)y'(x)+(BA^2B^{-1}-B^2)y(x)\text{y} \qquad (1.2.23)$$

sistemaning

$$y(c/2)=y_0,\ y'(c/2)=(A+B)y_0$$

boshlang'ich shartni qanoatlantiruvchi yechimidan iborat.

Isboti. Berilgan sistemani x bo'yicha differensiallab,

$$y''(x)=Ay'(x)-By'(c-x)$$

tenglamani hosil qilamiz. (1.2.22) tenglamada $x\to c-x$ almashtirish bajarsak,

$$y(c-x)=Ay(c-x)+By(x) \qquad (1.2.24)$$

tenglikni hosil qilamiz. (1.2.24) ni (1.23) ga qo'yib

$$y''(x)=Ay'(x)-BAy(c-x)-B^2y(x) \qquad (1.2.25)$$

tenglamani hosil qilamiz. Ammo (1.2.22) tenglamadan

$$y(c-x)=B^{-1}(y'(x)-Ay(x))=B^{-1}y'(x)-B^{-1}Ay(x)$$

bo’lgani uchun (1.2.25) bir qator sodda hisoblashlardan keyin (1.2.23) ko’rinishni oladi.

Ikkinchi tomondan $y(c/2)=y_0$ bo’lgani uchun $y'(c/2)=(A+B)y_0$ tenglik kelib chiqadi. Teorema isbotlandi.

Xususiy holda A va B kommutativ matritsalar bo’lsa, ya’ni $AB=BA$ bo’lsa, u holda $A=BAB^{-1}$, $B^{-1}A=AB^{-1}$ bo’lib,

$$A=BAB^{-1}=A-A=O,\ BA^2B^{-1}=BA\cdot AB^{-1}=BAB^{-1}A=A^2$$

bo’lgani uchun isbotlangan teoremadan quyidagi natija kelib chiqadi.

Natija. Agar $y-n$ o’lchamli vector, A va $B-n\times n$ o’lchamli kommutativ matritsalar bo’lsa, u holda

$$y''(x)=\left(A^2-B^2\right)y(x)$$

sistemaning

$$y(c/2)=y_0,\ y'(c/2)=(A+B)y_0$$

boshlang’ich shartni qanoatlantiruvchi yechimi

$$y'(x)=Ay(x)+By(c-x) \qquad (1.2.22)$$

sistemaning

$$y(c/2)=y_0$$

boshlang’ich shartni qanoatlantiruvchi Koshi masalasining yechimidan iborat.

1.**Misol.** Ushbu

$$\begin{cases} y_1'(x)=y_1(x)+2y_2(x)+5y_1(4-x)+2y_2(4-x), \\ y_2'(x)=3y_1(x)+y_2(x)+3y_1(4-x)+5y_2(4-x) \end{cases}$$

sistemaning

$$y(2)=y_0=\begin{pmatrix}2\\2\end{pmatrix}$$

boshlang’ich shartni qanoatlantiruvchi yechimini topish masalasi

$$\begin{cases} z_1''+24z_1+16z_2=0, \\ z_2''+24z_1+24z_2=0 \end{cases}$$

sistemaning

$$z(2) = y_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ z'(2) = \begin{pmatrix} 20 \\ 24 \end{pmatrix}$$

boshlang'ich shartni qanoatlantiruvchi yechimidan iborat ekanligi yuqorida keltirilgan natijadan kelib chiqadi, chunki keltirilgan misolda

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}$$

kommutativ matritsalar va

$$A^2 - B^2 = \begin{pmatrix} -24 & 16 \\ -24 & -24 \end{pmatrix}, \ A + B = \begin{pmatrix} 6 & 4 \\ 6 & 6 \end{pmatrix}$$

ekanligini e'tiborga olishimiz kifoya.

1.3-§. Oddiy differensial tenglamalar involyutsiyasiga oid misollar yechish

Biz bu bo'limda involyutsiya xossasiga ega bo'lgan birinchi va ikkinchi tartibli differensial tenglamalarni yechishga oid misollar yechishdan namunalar keltiramiz.

Dastavval agar $\alpha(\alpha(x)) = x$ bo'lsa, u holda $\alpha(x)$ involyutsoya eslagan holda involyutsiyalarga misollar keltiramiz.

Masalan $\alpha(x) = \sqrt[m]{1 - x^m}$ $m \in N$, $\alpha(x) = \dfrac{ax + b}{cx - a}$ funksiya va uning xususiy hollari bo'lgan $\alpha(x) = 1 - x, \alpha(x) = \dfrac{1}{x}, \alpha(x) = \dfrac{x}{x - 1}, \alpha(x) = \dfrac{x + 1}{x - 1}$ funksiyalar involyutsiyaga misol bo'la oladi. Shuning uchun birinchi darajali involyutsiyaga ega bo'lgan birinchi tartibli oddiy diferensial tenglamalarni umumiy holda

$$y'(x) = y(\alpha(x))$$

ko'rinishda berilishi mumkin. Agar x ni $\alpha(x)$ bilan almashtirsak

$$y'(\alpha(x)) = y(x)$$

ifodani va differensial tenglamani differensiallash bilan

$$y''(x) = \alpha'(x) y'(\alpha(x)) = \alpha'(x) y(x),$$

ya’ni ikkinchi tartibli Eyler tipidagi

$$y''(x)-\alpha'(x)y(x)=0$$

tenglamani hosil qilamiz. Endi involyutsiya qatnashgan oddiy differensial tenglamalarni yechishga misollar keltiramiz:

1-misol .Quyidagi Zilbershteyin tenglamasi [Zilbershteyin…] yeching:

$$y'(x)=y\left(\frac{1}{x}\right)$$

Yechilishi. Berilgan tenglamani yechish uchun $x=e^t,\ y(x)=g(t)$ almashtirish kiritamiz. U holda

$$y(x)=y(e^t)=g(t)$$

bo‘lgani uchun

$$y\left(\frac{1}{x}\right)=y(e^{-t})=g(-t)$$

bo‘ladi va bundan tashqari $t=\ln x$ bo‘lgani uchun

$$y'(x)=g'(t)\cdot\frac{dt}{dx}=\frac{1}{x}g'(t)=e^{-t}g'(t)$$

bo‘lib, Zilbershteyin tenglamasi

$$g'(t)=e^t g(-t)$$

ko‘rinishni oladi. Hosil bo‘lgan tenglamani yana bir bor differensiallash bilan

$$g''(t)=e^t g(-t)-e^t g'(-t)=g'(t)-g(t),$$

ya’ni

$$g''(t)-g'(t)+g(t)=0$$

o‘zgarmas koeffisiyentli differensial tenglamani hosil qilamiz. Bu tenglamaning xarakteristik tenglamasi

$$k^2-k+1=0$$

va uning ildizlari

$$k_{1,2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

bo‘lgani uchun hosil qilingan oddiy differensial tenglamaning umumiy yechimi

$$g(t)=e^{\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2}+C_2\sin\frac{t\sqrt{3}}{2}\right]$$

bo‘ladi. Bu tenglamadan

$$g'(t)=\frac{1}{2}g(t)+\frac{\sqrt{3}}{2}e^{\frac{t}{2}}\left[-C_1\sin\frac{t\sqrt{3}}{2}+C_2\cos\frac{t\sqrt{3}}{2}\right]=$$

$$=e^{\frac{t}{2}}\left[\frac{C_1+\sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2}+\frac{C_2-\sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

bo‘lgani uchun

$$e^{-t}g'(t)=g(-t)$$

tenglikka ko‘ra

$$e^{-\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2}-C_2\sin\frac{t\sqrt{3}}{2}\right]=e^{\frac{t}{2}}\left[\frac{C_1+\sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2}+\frac{C_2-\sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

Bu tenglikdan $C_1=\frac{C_1+\sqrt{3}C_2}{2}$, $\quad -C_2=\frac{C_2-\sqrt{3}C_1}{2}$, ya’ni $C_1=\sqrt{3}C_2$ ekanligini aniqlaymiz. Demak,

$$g(t)=C_2e^{\frac{t}{2}}\left[\sqrt{3}\cos\frac{t\sqrt{3}}{2}+\sin\frac{t\sqrt{3}}{2}\right]=2C_2e^{\frac{t}{2}}\left[\cos\frac{\pi}{6}\cos\frac{t\sqrt{3}}{2}+\sin\frac{\pi}{6}\sin\frac{t\sqrt{3}}{2}\right]=$$

$$=Ce^{\frac{t}{2}}\cos\left(\frac{\pi}{6}-\frac{t\sqrt{3}}{2}\right)$$

bu yerda $C=2C_2$. Ammo $g(t)=y(x),\ t=\ln x$ bo‘lgani uchun berilgan tenglamaning yechimi

$$y(x)=C\sqrt{x}\cos\left(\frac{\pi}{6}-\frac{\sqrt{3}}{2}\ln|x|\right)=C\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\ln|x|\right)$$

dan iborat.

Javob: $y(x)=C\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\ln x\right)$

2-misol. Ushbu

$$y'(x)=y(1-x)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $1-x$ bilan almashtirsak,

$$y'(1-x) = y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x) = -y'(1-x) = -y(x)$$

yoki

$$y''(x) + y(x) = 0$$

tenglamani hosil qilamiz. Bu tenglamaning umumiy yechimi

$$y(x) = C_1 cosx + C_2 \sin x$$

ko'rinishga ega va bu yechimda x ni $1-x$ bilan almashtirosh va differensiallash bizga

$$y(1-x) = C_1\cos(1-x) + C_2\sin(1-x) \quad \text{va} \quad y'(x) = -C_1\sin x + C_2\cos x$$

tengliklarni beradi. Berilgan tenglamaga ko'ra

$$-C_1\sin x + C_2\cos x = C_1\cos(1-x) + C_2\sin(1-x)$$

tenglikni hosil qilamiz. Bir qator hisoblashlardan so'ng bu tenglikdan

$$\begin{cases} C_1\cos 1 + C_2\sin 1 = C_2, \\ C_1\sin 1 - C_2\cos 1 = -C_1 \end{cases}$$

Bundan $C_2 = \dfrac{1+\sin 1}{\cos 1}C_1$ bo'lgani uchun berilgan tenglamaning umumiy yechimini

$$y(x) = C_1\left(\frac{1+\sin 1}{\cos 1}\sin x + \cos x\right) = C(\sin x + \cos(1-x))$$

ko'rinishda ifodalashimiz mumkin, bu yerda $C = \dfrac{1}{\cos 1}C_1$.

Javob: $y(x) = C(\sin x + \cos(1-x))$

3-misol. Ushbu

$$y'(x) = \frac{1}{y(a-x)}$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $a-x$ bilan almashtirsak,

$$y'(a-x)=\frac{1}{y(x)}$$

tenglikni hosil qilamiz. . $\alpha(x)=a-x$ akslantirishning qo'zg'almas nuqtasi

$a-x=x$ tenglikdan $x=\frac{a}{2}$ bo'lgani uchun berilgan tenglama uchun boshlang'ich

shartlarni $y\left(\frac{a}{2}\right)=y_0$, $y'\left(\frac{a}{2}\right)=\frac{1}{y\left(a-\frac{a}{2}\right)}=\frac{1}{y_0}$ ko'rinishida olishimiz mumkin.

Berilgan tenglamani differensiallash bilan

$$\frac{y''(x)y(x)-y'^2(x)}{y(x)}=0$$

tenglamaga kelamiz. Bu tenglamani

$$\left(\frac{y'(x)}{y(x)}\right)'=0$$

ko'rinishda yozib integrallash bilan

$$y'(x)=\frac{1}{y_0^2}y(x)$$

tenglamani hosil qilamiz. Bu tenglamani yana bir bor integrallash bilan

$$y(x)=Cexp\left(\frac{x}{y_0^2}\right)$$

umumiy yechimni va $y\left(\frac{a}{2}\right)=y_0$ boshlang'ich shartga ko'ra

$C=y_0exp\left(-\frac{a}{2y_0^2}\right)$ekanligini aniqlaymiz. Shuning uchun berilgan tenglamaning

yechimi

$$y(x)=y_0exp\left(\frac{x-\frac{a}{2}}{y_0^2}\right)$$

ko'rinishga ega.

$$\textbf{Javob:}\quad y(x) = y_0 exp\left(\frac{x-\frac{a}{2}}{y_0^2}\right)$$

4-misol. Ushbu

$$y'(x) = y\left(\frac{x}{x-1}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $\frac{x}{x-1}$ bilan almashtirsak,

$$y'\left(\frac{x}{x-1}\right) = y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x) = -\frac{1}{(x-1)^2}y'\left(\frac{x}{x-1}\right) = -\frac{1}{(x-1)^2}y(x)$$

yoki bundan

$$(x-1)^2 y''(x) + y(x) = 0$$

tenglamani hosil qilamiz.

Bu tenglamaning yechimini $y = (x-1)^k$ ko'rinishida izlasak, xarakteristik tenglama $k(k-1)+1=0$ ko'rinishda bo'lib, uning oldizlari $k = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ bo'lgani uchun bu tenglamaning umumiy yechimini

$$y(x) = \sqrt{x-1}\left[C_1 cos\sqrt{3}\ln\sqrt{x-1} + C_2 \sin\sqrt{3}\ln\sqrt{x-1}\right]$$

ko'rinishda ifodalashimiz mumkin.

Endi bu funksiyani berilgan tenglamaga qo'yamiz. Buning uchun x ni $\frac{x}{x-1}$ bilan almashtirsak bu funksiya

$$y\left(\frac{x}{x-1}\right) = \frac{1}{\sqrt{x-1}}\left[C_1 cos\sqrt{3}\ln\sqrt{x-1} - C_2 \sin\sqrt{3}\ln\sqrt{x-1}\right]$$

funksiyaga, va bu funksiyaning hosilasi

$$y'(x)=\frac{1}{2\sqrt{x-1}}\left[(C_1+\sqrt{3}C_2)cos\sqrt{3}\ln\sqrt{x-1}+(C_2-\sqrt{3}C_1)\sin\sqrt{3}\ln\sqrt{x-1}\right]$$

bo’lgani uchun berilgan tenglamaga ko’ra

$$2C_1=C_1+\sqrt{3}C_2,\ 2C_2=C_2-\sqrt{3}C_1$$

Bundan $C_1^2+C_2^2=0$ bo’lgani uchun berilgan tenglamaning yechimi $y(x)=0$ dan iborat.

Javob: $y(x)=0$

5-misol. Ushbu

$$y'(x)=y\left(\frac{x+1}{x-1}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $\frac{x+1}{x-1}$ bilan almashtirsak,

$$y'\left(\frac{x+1}{x-1}\right)=y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan

$$y''(x)=-\frac{2}{(x-1)^2}y'\left(\frac{x+1}{x-1}\right)=-\frac{2}{(x-1)^2}y(x)$$

yoki bundan

$$(x-1)^2y''(x)+2y(x)=0$$

tenglamani hosil qilamiz.

Bu tenglamaning yechimini $y=(x-1)^k$ ko’rinishida izlasak, xarakteristik tenglama $k(k-1)+2=0$ ko’rinishda bo’lib, uning oldizlari $k=\frac{1}{2}\pm\frac{i\sqrt{7}}{2}$ bo’lgani uchun bu tenglamaning umumiy yechimini

$$y(x)=\sqrt{x-1}\left[C_1cos\sqrt{7}\ln\sqrt{x-1}+C_2\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

ko’rinishda ifodalashimiz mumkin.

Endi bu funksiyani berilgan tenglamaga qo'yamiz. Buning uchun x ni $\frac{x+1}{x-1}$ bilan almashtirsak bu funksiya

$$y\left(\frac{x+1}{x-1}\right)=\frac{\sqrt{2}}{\sqrt{x-1}}\left[(C_1 cos\sqrt{7}\ln 2+C_2\sin\sqrt{7}\ln 2)\cos\sqrt{7}\ln\sqrt{x-1}\right]+$$

$$+\frac{\sqrt{2}}{\sqrt{x-1}}\left[(C_1\sin\sqrt{7}\ln 2-C_2\cos\sqrt{7}\ln 2)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

uning hosilasi esa

$$y'(x)=\frac{1}{2\sqrt{x-1}}\left[(C_1+\sqrt{7}C_2)cos\sqrt{7}\ln\sqrt{x-1}+(C_2-\sqrt{7}C_1)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

bo'lgani uchun berilgan tenglamadan

$$\begin{cases} C_1+\sqrt{7}C_2=2\sqrt{2}(C_1 cos\sqrt{7}\ln 2+C_2\sin\sqrt{7}\ln 2 \\ C_2-\sqrt{7}C_1=2\sqrt{2}(C_1\sin\sqrt{7}\ln 2-C_2\cos\sqrt{7}\ln 2 \end{cases}$$

tenglamalar sistemasini hosil qilamiz. Bu sistemadan $C_2=C_1\sqrt{\frac{\alpha+\sqrt{7}}{\alpha-\sqrt{7}}}$ munosabatni hosil qilamiz, bu yerda

$$\alpha=2\sqrt{2}\sin(\sqrt{7}\ln 2)-4\sin(2\sqrt{2}\ln 2)-2\sqrt{14}\cos(\sqrt{7}\ln 2)\,.$$

Shuning uchun berilgan tenglamaning umumiy yechimi

$$y(x)=C\sqrt{x-1}\left[\left(\alpha+\sqrt{7}\right)cos\sqrt{7}\ln\sqrt{x-1}+\left(\alpha-\sqrt{7}\right)\sin\sqrt{7}\ln\sqrt{x-1}\right]$$

ko'rinishga ega.

Javob: $y(x)=C\sqrt{x-1}\left[\left(\alpha+\sqrt{7}\right)cos\sqrt{7}\ln\sqrt{x-1}+\left(\alpha-\sqrt{7}\right)\sin\sqrt{7}\ln\sqrt{x-1}\right]$,

bu yerda $\alpha=2\sqrt{2}\sin(\sqrt{7}\ln 2)-4\sin(2\sqrt{2}\ln 2)-2\sqrt{14}\cos(\sqrt{7}\ln 2)$

6-misol. Ushbu

$$y''(x)=y\left(\frac{1}{x}\right)$$

tenglamani yeching.

Yechilishi. Berilgan tenglamani x bo'yicha ketma-ket differensiallab

$$y'''(x)=-\frac{1}{x^2}\,y'\left(\frac{1}{x}\right),$$

$$y^{(IV)}(x)=\frac{2}{x^3}y'\left(\frac{1}{x}\right)+\frac{1}{x^4}y''\left(\frac{1}{x}\right)$$

tengliklarni hosil qilamiz.

Endi berilgan tenglamada $f:x\to\frac{1}{x}$ almashtirish bajarsak, tenglama

$$y''\left(\frac{1}{x}\right)=y(x)$$

ko‘rinishni oladi. Agar bu tenglikni hisobga olsak, yuqorida hosil qilingan ikki tengliklardan

$$y^{(IV)}(x)=-\frac{2}{x}y'''(x)+\frac{1}{x^4}y(x)$$

ya’ni

$$x^4y^{(IV)}(x)+2x^3y'''(x)-y(x)=0$$

Eyler tenglamasini hosil qilamiz. Bu tenglama uchun xarakteristik tenglama

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-1=0,$$

yoki

$$k(k-1)(k-2)(k-1)-1=0$$

ko‘rinishda bo‘ladi. Bu tenglamani

$$\left(k^2-2k\right)^2+\left(k^2-2k\right)-1=0$$

ko‘rinishda yozsak,

$$k^2-2k=\frac{-1+\sqrt{5}}{2},\ k^2-2k=\frac{-1-\sqrt{5}}{2}$$

Bu tengliklarning har ikkala qismiga 1 ni qo‘shish natijasida

$$(k-1)^2=\frac{1+\sqrt{5}}{2},\ (k-1)^2=-\frac{\sqrt{5}-1}{2}$$

tengliklarni hosil qilamiz. Bundan xarakteristik tenglama ikkita haqiqiy va ikkita kompleks:

$$k_{1,2}=1\pm\sqrt{\frac{1+\sqrt{5}}{2}}=1\pm\frac{\sqrt{2+2\sqrt{5}}}{2},\ k_{3,4}=1\pm i\sqrt{\frac{-1+\sqrt{5}}{2}}=1\pm i\frac{\sqrt{-2+2\sqrt{5}}}{2}$$

ildizlarga ega bo‘lgani uchun berilgan differensial tenglamaning umumiy yechimi

$$y(x) = x\left[C_1 ch\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right) + C_2 sh\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)\right] +$$

$$+ x\left[C_3\cos\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right) + C_4\sin\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)\right]$$

ko‘rinishga ega bo‘ladi.

I BOB bo’yicha xulosalar

Bu bobda kelgusi tadqiqotlarni davom ettirish uchun zarur bo’lgan asosiy tushunchalar yoritilgan bo’lib, ular quyidagilardan iborat:

- differensial tenglamalar nazariyasidan zaruriy tushunchalar: o’zgarmas koeffisiyentli chiziqli differensial tenglamalar, Eyler Lagranj tenglamalari;
- oddiy differensial tenglamalar involyutsiyasi va uning asosiy xossalari;
- oddiy differensial tenglamalar involyutsiyasiga oid misollar yechilishi.

Adabiyotlar 1-bobga 1.Kartashev, Rojdestvenckiy
2/Viner ,

II BOB

INVOLYUTSIYAGA EGA BO’LGAN XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR

Ma’lumki o’tgan asrning 50-60 yillarida cheklangan argumentli oddiy differensial tenglamalar nazariyasida ‘’tadqiqotlar portlashi’’ bo’ldi. Bu davrda hozirda klassik adabiyotlar safidan joy olgan L.Els’golts, N.M.Krasovskiy, S.B.Norkin. A.D.Mishkis, E.Pinni, R.Bellman va K.Kuk, L.Halanay va N.Ogugoriyelli larning shu soha bo’yicha monografiyalari chop etildi. Ammo cheklangan argument bo’yicha xususiy hosillali differensial tenglamalarga bag’ishlangan ilmiy ishlar miqdori yetalicha oz bo’lgani uchun bu sohada tadqiqotlar davom etishi tabbiy holdir.

Differensial tenglamalar rivojlanishidagi tarixiy ma’lumotlardan kechikishga ega bo’lgan differensial tenglamalarga ko’p sondagi maqolalar bag’ishlangan.

2.1- §. Involyutiv cheklanishga ega bo’lgan chegaraviy masalalar uchun Fur’ye usulini asoslash

Ushbu

$$\frac{\partial^2}{\partial x^2}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(-x,t) \qquad (2.1.1)$$

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{\partial^2}{\partial t^2}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(-x,t) \qquad (2.1.2)$$

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{\partial}{\partial t}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(-x,t) \qquad (2.1.3)$$

korinishdagi tenglamalarga oid chegaraviy masalalarni o’rganish muhum o’rin tutadi. (2.1.2) tenglamalar uchun xarakteristikalar masalasi [2] maqolada o’rganilgan. [1] maqolada $\varepsilon=1$ bo’lgan holda Dirixle, Neyman va boshqa masalalar tadqiq qilingan. $\varepsilon\neq 0$ bo’lganda (2.1.1)-(2.1.3) tenglamalarni klassikatsiya qilinmaydi. $\varepsilon = 0$ bo'lsa, bu tenglamalat Laplas, to'lqin va issiqlik o'tkazuvchanlik tenglamalaridan iborat bo'ladi.

Quyida $(x,t): x \in [-l,l],\ t \in [0,T]$ to'g'ri to'rtburchak va $(x,t): x \in [-l,l],\ t \in [0,+\infty)$ yarim tasma sohalarida involyutiv cheklanishga ega bo'lgan (2.1.1)-(2.1.3) tenglamalar uchun Fur'ye usulini qo'llash bilan nol chegaraviy shartlar bo'yicha berilgan chegaraviy masalalar o'rganilgan. O'zgaruvchilarni ajratishning Fur'ye usuli bilan (2.1.1)-(2.1.3) tenglamalar bilan berilgan chegaraviy masalalar holat o'zgaruvchilari bo'yicha

$$\begin{aligned} &X''(x) - \varepsilon X''(-x) = \lambda X(x), \\ &X(-l) = X(l) = 0 \end{aligned} \tag{2.1.4}$$

Shturm-Liuvill masalasi hosil bo'ladi.

Har bir tenglama uchun ko'rsatilgan sohada klassik yechimlar hosil qilinadi va bu yechimlarning ε ga bog'liq tarzda korrektligi tekshiriladi. Mazkur bo'limning asosiy maqsadi (2.1.1)-(2.1.3) tenglamalar bilan berilgan masalalarda Fur'ye usulini qo'llash mumkinligini asoslashdir. Buning uchun birinchi masala sifatida (2.1.4) Shturm-Liuvill masalasi yechimi bo'lgan funksiyalar sistemasining to'laligini ko'rsatishdan iborat.

1. Trigonometrik sistemalarning to'laligi. Umumiylikni saqlagan holda $l = \pi$, ya'ni $x \in [-\pi,\pi]$ deb hisoblaymiz. Standart ko'rinishdagi hisoblashlarni bajarib (2.1.4) Shturm-Liuvill masalasi ixtiyoriy haqiqiy ε soni uchun

$$\sin nx,\ \cos\left(k + \frac{1}{2}\right)x\ , \quad n = \overline{1,\infty}, \quad k = \overline{0,\infty} \tag{2.1.5}$$

xos funksiyalarga ega ekanligini ko'rishimiz mumkin.

1-lemma. (2.1.5) funksiyalar sistemasi $L_2[-\pi,\pi]$ fazoda ortogonal va to'la sistema bo'ladi.

Isboti. bevosita hisoblashlar bilan (2.1.5) sistemaning ortogonalligini tekshirish mumkin.

Bu sistemaning to'laligini tekshirish uchun

$$\sin nx,\ \cos kx\ ,\ n=\overline{1,\infty},\quad k=\overline{0,\infty} \qquad (2.1.6)$$

sitemaning $L_2[-\pi,\pi]$ fazoda ortogonalligi va to'laligi haqidagi klassik natijadan foydalanamiz.

(2.1.5) va (2.1.6) sistemalarni taqqoslash bilan , agar $L_2[-\pi,\pi]$ fazoda

$$\cos nx=\sum_{k=0}^{\infty}a_k\cos\left(k+\frac{1}{2}\right)x\ ,\ n=\overline{0,\infty} \qquad (2.1.7)$$

yoyilma o'rinli bo'lsa , u holda (2.1.5) sistema to'la bo'ladi. (2.1.7) tenglik bajarilishi uchun faqat va faqat $L_2[-\pi,\pi]$ fazoda

$$\|\cos nx\|_{L_2}^2=\sum_{k=0}^{\infty}a_k^2\left\|\cos\left(k+\frac{1}{2}\right)x\right\|_{L_2}^2\ n=\overline{0,\infty} \qquad (2.1.8)$$

Parseval tengligi bajarilishi zarur va yetatrli, bu yerda

$$\|f\|_{L_2}^2=(f,g)_{L_2}\ ,\ (f,g)_{L_2}=\int_{-\pi}^{\pi}f(x)g(x)dx\ .$$

$$\|f\|_{L_2}^2=(f,g)_{L_2}\ ,\ (f,g)_{L_2}=\int_{-\pi}^{\pi}f(x)g(x)dx$$

(2.1.8) qatorga batafsil to'xtalamiz.

$$\|1\|_{L_2}^2=2\pi\ ,\ \|\cos nx\|_{L_2}^2=\left\|\cos\left(k+\frac{1}{2}\right)x\right\|_{L_2}^2=\pi\ ,n=\overline{1,\infty},\quad k=\overline{0,\infty} \qquad (2.1.9)$$

bo'lgani uchun yoyilmaning

$$a_k=\frac{\int_{-\pi}^{\pi}\cos nx\cos\left(k+\frac{1}{2}\right)xdx}{\cos\left(k+\frac{1}{2}\right)\ \frac{2}{L}}=\frac{4(-1)^{n+k}(2k+1)}{\pi(2k+1+2n)(2k+1-2n)},\ n=\overline{0,\infty} \qquad (2.1.10)$$

koeffisiyentlarini topamiz.

Demak, (2.1.8) qatorni

$$\|\cos nx\|_{L_2}^2=\sum_{k=0}^{\infty}\frac{4(-1)^{n+k}(2k+1)}{(2k+1+2n)(2k+1-2n)},n=\overline{0,\infty} \qquad (2.1.11)$$

ko'rinishda yozishimiz mumkin. Oxirgi munosabat $n=0$ bo'lganda

$$2\pi=\sum_{k=0}^{\infty}\frac{16}{\pi(2k+1)^2}$$

ko’rinishni oladi. Ammo $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi}{8}$ ayniyatga ko’ra bu tenglikning bajarilishi aniq. Endi n ixtiyoriy natural son bo’lsin. U holda (2.1.11) Parseval tengligi (2.1.9) ga ko’ra

$$1=\sum_{k=0}^{\infty}a_k^2 \qquad (2.1.12)$$

ko’rinishda ifodalashi mumkin.

Yarim gamma funksiyalarning quyidagi yoyilmasidan foydalanib,

$$\Psi(m,z)\equiv\frac{d^m}{dz^m}\Psi(z),\ \ \Phi(z)\equiv\frac{d^m+1}{dz^{m+1}}\equiv\ln\Gamma(z)$$

ifodalardan va (2.1.10) dagi a_k koeffisiyentlarning ko’rinishidan

$$\sum_{k=0}^{\infty}a_k^2=\frac{1}{\pi^2}\left\{\frac{\Phi\left(n+\frac{1}{2}\right)-\Phi\left(\frac{1}{2}-n\right)}{n}+\Psi\left(1,\ n+\frac{1}{2}\right)-\Psi\left(1,\ \frac{1}{2}-n\right)\right\}$$

tenglikni o’rinli ekanligiga ishonch hosil qilamiz. Gamma funksiyaning simmetriya formulalaridan

$$\Psi(1-z)-\Psi(z)=\pi ctgz,$$
$$\Psi(1,1-z)-\Psi(1,z)=\pi\frac{d}{dz}ctgz$$

(2.1.13) munosabatlar o’rinli ekanligi kelib chiqadi. (2.1.13) tarkibiga kirgan funksiyalarning analitikligidan yarim gamma funksiyalar uchun $z=n+\frac{1}{2}$ bo’lganda

$$\Psi\left(n+\frac{1}{2}\right)-\Psi\left(\frac{1}{2}-n\right)=0 \qquad \text{va} \qquad \Psi\left(1,\ \frac{1}{2}+n\right)-\Psi\left(1,\frac{1}{2}-n\right)=\pi^2$$

munosabatlarni hosil qilamiz. Bu bilan (2.1.12) tenglik o’rinli ekanligi va demak ixtiyoriy n natural soni uchun (2.1.8) Parseval tengligi ham o’rinli. Shuning uchun (2.1.5) funksiyalar sistemasi $L_2[-\pi,\ \pi]$ fazoda to’la ekanligi kelib chiqadi. Lemma isbotlandi.

Xuddi shu kabi quyidagi lemma asoslanadi:

2-lemma. Quyidagi

$$\sin\left(n+\frac{1}{2}\right)x,\ \cos nx\ ,\ n=\overline{0,\infty} \qquad (2.1.14)$$

$$\sin\left(n+\frac{1}{2}\right)x,\ \cos\left(n+\frac{1}{2}\right)x\ ,\ n=\overline{0,\infty} \qquad (2.1.15)$$

funksiyalar sistemasi $L_2[-\pi,\ \pi]$ fazoda orthogonal va to'liq bo'ladi.

2.Trigonometrik qatorlarning yaqinlashishi. Matematik fizika aralash masalalarini qo'yilishida $f(x)$ funksiya uchun yozilgan umumlashgan Fur'ye qatorining kompakt to'plamda shu $f(x)$ funksiyaga tekis yaqinlashuvchi bo'lishini ta'minlovchi shartlar muhum ahamiyatga ega. Birgina $f(x)$ funksiyaning uzluksiz bo'lishi Fur'ye qatorining bu funksiyaga tekis yaqilashishini emas balki olingan nuqtada oddiy yaqinlashishini ham ta'minlay olmasligi mumkin.

Oldingi bo'limda (2.1.5) (yoki, (2.1.14), yoki (2.1.15)) funksiyalar sistemasi bo'yicha Fur'ye qatorining $[-\pi,\pi]$ kesmada yaqilashishi ko'rsatildi. Quyidagi lemma bu funksiyalar sistemasi bo'yicha tuzilgan Fur'ye qatorining uzluksiz funksiyaga yaqinlashuvchi bo'lishining yetarli shartini beradi.

3-lemma. Agar $f(x)$ funksiya $[-\pi,\pi]$ kesmada uzluksiz, uning hosilasi bu kesmaning chekli sondagi nuqtalaridan tashqari barcha nuqtalarida uzluksiz va bu nuqtalarning har birida chekli chap yoki o'ng hosilalarga ega bo'lsa, u holda $f(x)$ funksiyaning (2.1.5),(2.1.14) yoki (2.1.15) funksiyalar bo'yicha tuzilgan trigonometrik qator $[-\pi,\pi]$ kesmada tekis metrika bo'yicha $f(x)$ funksiyaga absolyut yaqinlashadi

Isboti. Bu lemmaning isboti ham (2.1.6) funksiyalar sistemasi bo'yicha tuzilgan Fur'ye qatorining tekis yaqinlashishining klassik usuli kabi olib boriladi. Bunda

$$\gamma_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos\left(k+\frac{1}{2}\right)xdx\ ,\ \ \delta_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin\left(k+\frac{1}{2}\right)xdx,\ k=\overline{0,\infty}\ ,$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(k+\frac{1}{2}\right)xdx\ ,\ \ d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(k+\frac{1}{2}\right)xdx,\ k=\overline{0,\infty}$$

Fur'ye koeffisiyentlari uchun

$$\gamma_k = \left(k+\frac{1}{2}\right)d_k\ ,\ \ \delta_k = -\left(k+\frac{1}{2}\right)c_k\ ,\ \ k=\overline{0,\infty}\ ,$$

$$\frac{|\gamma_k|}{k+\frac{1}{2}} \le \frac{1}{2}\left(\gamma_k^2 + \frac{1}{\left(k+\frac{1}{2}\right)^2}\right),\ \frac{|\delta_k|}{k+\frac{1}{2}} \le \frac{1}{2}\left(\delta_k^2 + \frac{1}{\left(k+\frac{1}{2}\right)^2}\right), k=\overline{0,\infty}$$

munosabatlarni hosil qilamiz.

3-lemmaning natijasi sifatida quyidagi tasdiqni keltiramiz.

4-lemma. Agar $f(x)$ funksiya o'zining qandaydir $m-$tartibgacha bo'lgan hosilalari bilan birga $[-\pi,\pi\ ]$ kesmada uzluksiz,

$f(-\pi)= f(\pi)= f'(-\pi)= f'(\pi)=\cdots= f^{(m)}(-\pi)= f^{(m)}(\pi)=0$ bo'lsa, u holda uning $m+1$ chi tartibgacha hosilalari $[-\pi,\pi]$ kesmaning chekli sondagi nuqtalaridan tashqari barcha nuqtalarida mavjud va uzluksiz, bu nuqtalarda $m+1$ chi tartibgacha o'ng va chap hosilalari mavjud bo'lsa, u holada (2.1.5),(2.1.14),(2.1.15) funksiyalar sistemasi bo'yicha tuzilgan trigonometrik qator tekis metrika bo'yicha $[-\pi,\pi]$ kesmada absolyut yaqinlashadi. Shu bilan birga bu qatorlarni m marta hadlab differensiallash mumkin. Hosil bo'lgan qatorlar ham tekis metrika bo'yicha $[-\pi,\pi]$ kesmada absolyut yaqinlashadi.

Isboti. Bu lemmaning isboti 3-lemmani e'tiborga olgan holda xuddi matematik analiz kursidagidek klassik usul bilan amalga oshiriladi.

3.Aralash chegaraviy masalaning yechimi.

Aniqlik uchun quyidagi masalani qaraymiz .(2.1.3):

$$\frac{\partial^2}{\partial x^2}u(x,t)+\frac{\partial}{\partial t}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(-x,t)$$

tenglamani

$$u(-\pi,t)=u(\pi,t)=0,\ t\in[0,T] \qquad (2.1.16)$$

chegaraviy shartlarni

$$u(x,0)=\varphi(x)\,,\ \ x\in[-\pi,\pi] \qquad (2.1.17)$$

boshlang'ich shartni va

$$\varphi(-\pi)=\varphi(\pi)=0 \qquad (2.1.18)$$

qo'shmalik shartini qanoatlantiruvchi $u(x,t)\in C^{2,1}(Q)\cap C(\overline{Q})$ funksiyani toping, bu yerda $m=2$ bo'lganda φ funksiya 4-lemmaning shartlarini qanoatlantiradi.

Faraz qilaylik qo'yilgan masalaning yechimi mavjud bo'lsin. Demak, 4-lemmaga ko'ra ixtiyoriy t uchun yechimning (2.1.5) ifodadagi funksiyalarning X_n orthogonal sistemasi bo'yicha $[-\pi,\pi]$ kesmada tekis yaqinlashuvchi

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x) \qquad (2.1.19)$$

qator ko'rinishidagi ifodasi mavjud. U holda

$$T_n(t)=(u,X_n)_{L2} \qquad (2.1.20)$$

$u(x,t)$ funksiyani berilgan (2.1.3) tenglamaga qo'yilganda ayniyat hosil bo'ladi, u holda bu tenglamani X_n ga skalyar ko'paytirib

$$(L_{x,\varepsilon}u,X_n)_L+\left(\frac{\partial}{\partial t}u,X_n\right)_L=0 \qquad (2.1.21)$$

tenglamani hosil qilamiz, bu yerda $L_{x,\varepsilon}$ bilan

$$L_{x,\varepsilon}u(x,t)\equiv\frac{\partial^2u(x,t)}{\partial x2}-\varepsilon\frac{\partial^2u(1-x,t)}{\partial x^2}$$

tenglik bilan aniqlangan, aniqlanish sohasi

$$u(x)\in C[-\pi,\pi]:u(1-\pi)=u(\pi)=0$$

bo'lgan $L_{x,\varepsilon}:C_2\to C$ operator tushuniladi.

Bo'laklab integrallash bilan $L_{x,\varepsilon}$ operator uchun qo'shmalik sharti, ya'ni $(L_{x,\varepsilon}u,v)=(u,\,L_{x,\varepsilon}v)$ tenglikning bajarilishini tekshirish qiyin emas.

Demak,

$$(L_{x,\varepsilon}u,X_n)_L=(uL_{x,\varepsilon},X_n)_L=(u,\lambda_nX_n)=\lambda(u,X)$$

bu yerda λ_n bilan $L_{x,\varepsilon}$ operatorning $X_n,\ n=\overline{0,\infty}$ xos vektorlarga mos xos qiymatlari belgilangan.

Integral ostida t parameter bo'yicha differensiallashning Leybnits qoidasini qo'llashda masalaning qo'yilishida $\partial u/\partial t$ funksiyaning yetarlicha silliqligidan

$$\left(\frac{\partial u}{\partial t}, X_n\right)_L = \frac{\partial (u, X_n)_L}{\partial t}$$

tenglikni hosil qilamiz. Hosil bo'lgan munosabatni (2.1.21) ga qo'yib

$$\frac{d}{dt}T_n(t) + \lambda_n T_n(t) = 0 \qquad (2.1.22)$$

tenglamani hosil qilamiz. (2.1.17) ni e'tiborga olgan holda $T_n(t)$ funksiyani aniqlash uchun (2.1.20) dan $t = 0$ bo'lganda

$$T_n(0) = (\varphi, X_n)_L \qquad (2.1.23)$$

boshlang'ich shartni hosil qilamiz.

Demak, izlanayotgan $T_n(t)$ koeffisiyentlar (2.1.22)-(2.1.23) Koshi masalasini yechish orqali bir qiymatli topiladi.

Shuning uchun agar aralash masalaning yechimi mavjud bo'lsa, bu yechim (2.1.19) qator ko'rinishida bir qiymatli aniqlanadi va demak bu (2.1.3),(2.1.16)-(2.1.18) masalaning yechimini yagonaligini isbotlaydi.

Kelgusi mulohazalarda agar (2.1.3),(2.1.16)-(2.1.18) aralash masalaning yechimi mavjud bo'lsa, bu yechim x bo'yicha $[-\pi, \pi]$ kesmada tekis yaqinlashuvchi , $[0, T]$ kesmada aniqlangan t parametrning (2.1.19) qatori bilan tasvirlanadi. (2.1.19) qatorning yaqinlashishi ε ga bog'liq, chunki (2.1.3), (2.1.16)-(2.1.18) aralash masalaning yechimi

$$u(x,t) = \sum_{k=0}^{\infty} a_k e^{-(1-\varepsilon)\left(k+\frac{1}{2}\right)^2 t} \cos\left(k + \frac{1}{2}\right)x + \sum_{n=1}^{\infty} b_n e^{-(1+\varepsilon)n^2} \sin nx \qquad (2.1.24)$$

ko'rinishga ega, bu yerda

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x) \cos\left(n + \frac{1}{2}\right)x dx, \quad k = \overline{0, \infty} \qquad (2.1.25)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x) \sin nx dx, \quad n = \overline{n.\infty} \qquad (2.1.26)$$

$\varepsilon = 1$ bo'lganda (2.1.24) qatorning x bo'yicha $[-\pi, \pi]$ kesmada, t bo'yicha $[0, +\infty)$ oraliqda tekis yaqinlashuvchi ekanligi ravshan. Bu quyidagi

$$|a_k| e^{-(1-\varepsilon)\left(k+\frac{1}{2}\right)^2 t} \le |a_k|,\ k = \overline{0,\infty},\quad |b_n| e^{-(1+\varepsilon)n^2 t} \le |b_n|,\ n = \overline{1,\infty}$$

bahodan kelib chiqadi.

$\varepsilon < -1\ (\varepsilon > 1)$ bo'lganda (2.1.24) funksional qator x bo'yicha $\overline{Q}$ to'plamda yaqinlashmaydi, chunki ixtiyoriy $t > 0$ uchun doimo shunday $n_0\ (k_0)$ sonlar topiladiki barcha $n > n_0\ (k > k_0)$ sonlar uchun

$$|b_n| \exp\left(-(1+\varepsilon)n^2 t\right) > M\ ,\ \left(|a_k| \exp\left(-(1-\varepsilon)\left(k + \frac{1}{2}\right)^2 t\right) > M\right)$$

tengsizlik bajariladi, bu yerda $M -$ixtiyoriy yetarlicha katta musbat son. Demak, (2.1.24) funksional qator yaqinlashishining zaruriy sharti buziladi. Shuning uchun $\varepsilon > 1$ bo'lganda qo'yilgan masalaning yechimi mavjud emas.

Bu holatni namoyon etish uchun quyidagi misolni qaraymiz.

$\varepsilon = 2$ bo'lsin. (2.1.3),(2.1.16)-(2.1.18) aralash masalada boshlang'ich shartni ifodalovchi funksiya sifatida

$$\varphi(x) = \left(\pi^2 - x^2\right)^3$$

funksiyani olamiz. Bu funksiya $-\pi.\ \pi$ nuqtalarda ikkinchi tartibli hosilalarigacha nolga aylanishi ravshan.

$$\varphi(x) = -\frac{18432}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^л \left(-10 + \pi^2 (2л+1)^2\right)}{(2л+1)^7} cos\left(k + \frac{1}{2}\right)x$$

ekanligini tekshirish qiyin emas. Qaralayotgan masalaning (2.1.24) ko'rinishdagi

$$u(x,t) = -\frac{18432}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^л \left(-10 + \pi^2 (2k+1)^2\right)}{(2л+1)^7} e^{\left(k+\frac{1}{2}\right)^2 t} cos\left(k + \frac{1}{2}\right)x \qquad (2.1.27)$$

yechimini tuzamiz. Ixtiyoriy $t > 0$ uchun (2.1.27) qatorni uzoqlashishiga ishonch hosil qilish qiyin emas. Hosil qilingan natijalarni quyidagi teorema ko'rinishida ifodalashimiz mumkin:

Teorema. (2.1.3),(2.1.16)-(2.1.18) aralash masala $\varepsilon < 1$ bo'lganda , berilgan boshlang'ich shartlarga nisbatan uzluksiz (2.1.24) ko'rinishda ifodalanuvchi yagona yechimga ega. $\varepsilon > 1$ bo'lganda qo'yilgan masalaning yechimi mavjud emas.

2.2- §. Involyutsiyaga ega bo'lgan ikkinchi tartibli oddiy differensial tenglamalar xos funksiyalarining asosiy xossalari

1.Kirish Involyutsiyaga ega bo'lgan ikkinchi tartibli xususiy hosilali

$$w_t(t,x) = \alpha w_{xx}(t,x) + w_{xx}(t,-x),\ -1 < x < 1,\ t > 0 \tag{2.2.1}$$

tenglama

$$w(0,x) = f(x) \tag{2.2.2}$$

boshlang'ich shart va

$$\alpha_j w_x(t,-1) + \beta_j w_x(t,1) + \alpha_{j1} w(t,-1) + \beta_{j1} w(t,1) = 0,\ j = 1,2 \tag{2.2.3}$$

hegaraviy cshartlar bilan berilgan bo'lsin. u holda bu masalani yechish uchun Fur'ye usulini qo'llash bilan $f(x)$ funksiya qatnashmagan xos funksiyalar uchun ikkita spektral

$$\begin{gathered} -u''(-x) + \alpha u''(x) = \lambda u(x), \\ \alpha_j u'(-1) + \beta_j u'(1) + \alpha_{j1} u(-1) + \beta_{j1} u(1) = 0,\ j = 1,2 \end{gathered} \tag{2.2.4}$$

masalalar juftiga olib keladi.

Agar $f(x) \in L^2(-1,1)$ bo'lsa, u holda involyutsiyaga ega bo'lgan ikkinchi tartibli oddiy differensial operatorlar spektral masalasida xos funksiyalarning asosiy xossalari keltitiladi.

Involyutsiya qatnashgan funksional differensial tenglamalar turli ko'rinishdagi ko'plab masalalari [6,7] ishlarda o'rganilgan. Ikkinchi tartibli hosilalar bilan involyutsiya qatnashgan spektral masalalar [8-11] ishlarda o'rganilgan bo'lib, chegaraviy shartlar koeffisiyentlari hadlar sifatida qaralganda xos funksiyalarning Riss bazisi tashkil qilishi qaralgan.

Bu turdagi spektral masalalar involyutsiyaga ega bo'lgan xususiy hosilali differensial tenlamalarning yechilishi nazariyasini ham o'z ichiga olishi [7,265-bet] da keltirilgan.

Quyida keltiriladigan natijalar mualliflar tomonidan o'rganilgan [9-11] ishlarni davomidir.

2. Umumiy chegaraviy masala. Bu maqolada biz

$$Lu \equiv -u''(-x) + \alpha u''(x) + \beta u'(x) + \gamma u'(-x) + \eta u(-x) = \lambda u(x), \qquad (2.2.5)$$

$$\begin{aligned} &\alpha_1 u'(-1) + \beta_1 u'(1) + \alpha_{11} u(-1) + \beta_{11} u(1) = 0, \\ &\alpha_2 u'(-1) + \beta_2 u'(1) + \alpha_{21} u(-1) + \beta_{21} u(1) = 0 \end{aligned} \qquad (2.2.6)$$

ko'rinishdagi spektral masalalarni qaraymiz, bu yerda $\alpha, \beta, \gamma, \alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$ – qandaydir kompleks sonlar.

Bevosita hisoblash bilan operatorniong kvadrati

$$\begin{aligned} L^2 u = &\left(1+\alpha^2\right) u^{IV}(x) - 2\alpha u^{IV}(-x) + 2\gamma u'''(-x) + 2\alpha\beta u'''(x) + 2\alpha\eta u''(-x) + \\ &+ \left(-2\eta + \beta^2 - \gamma^2\right) u''(x) + \eta^2 u(x) \end{aligned} \qquad (2.2.7)$$

ko'rinishda bo'lishiga ishonch hosil qilishimiz mumkin.

Lu ifoda L operatorning aniqlanish sohasiga tegishli bo'lgani uchun, u holda Lu funksiya ham (2.2.6) chegaraviy shartlarni qanoatlantiradi:

$$\begin{aligned} &\alpha_1 (Lu)'(-1) + \beta_1 (Lu)'(1) + \alpha_{11} (Lu)(-1) + \beta_{11} (Lu)(1) = 0, \\ &\alpha_2 (Lu)'(-1) + \beta_2 (Lu)'(1) + \alpha_{21} (Lu)(-1) + \beta_{21} (Lu)(1) = 0 \end{aligned} \qquad (2.2.8)$$

Demak, L^2 operator oldingi differensial operatorning umumlashmasi bo'lib, (2.2.6) va (2.2.7) chegaraviy shartlarni qanoatlantiradi.

$\alpha = 0$ bo'lganda Lu ifoda oddiy differensial ifodadan iborat bo'ladi. Shuning uchun [8-10] da keltirilgan usulni qo'llab, biz quyidagi natijani hosil qilamiz.

2.1-teorema. Agar $\alpha = 0$ bo'lsa, u holda xos funksiyalarning (2.2.5) va (2.2.6) umumlashgan spektral masalasi quyidagi hollarda:

1) $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0;$

2) $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0;\ |\alpha_1| + |\beta_1| > 0,\ \alpha_1^2 \neq \beta_1^2,\ \alpha_{21}^2 \neq \beta_{21}^2;$

3) $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0;\ \alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0$

xos qiymatlar sistemasi $L_2(-1,1)$ fazoda Riss bazisini tashkil qiladi. A va A^2 operator xos qiymatlarining ildizlari qandaydir shartlarda ustma-ust tushadi,masalan [ ]. Shuning uchun oddiy differensial operator sifatida L operatorning kvadratini qarashimiz mumkin. [12-14] dan yaxshi ma'lumki, juft tartibli oddiy differensial operatorning xos vektorlari qat'iy regulyar chegaraviy shartlarda Riss bazisini tashkil qiladi. [10] kabi 2.2-teoremaning to'g'ri ekanligini keltirib chiqarish mumkin.

Bu usulni $\alpha = 0$ bo'lganida qo'llab bo'lmaydi, chunki L^2u oddiy differensial operator bo'lmaydi. Shuning uchun bu hollarni alohida qaraymiz.

3. Spektral ko'rinishdagi masalaning umumiy yechimi.

Involyutsiyaga ega bo'lgan Lu oddiy differensial operator (2.2.6) chegaraviy shartlar bilan

$$Lu \equiv -u''(-x) + \alpha u''(x) \qquad (2.2.9)$$

ko'rinishda berilgan bo'lsin.

Biz $Lu = \lambda u(x)$ spektral masalani Dirixle va Shturmning davriy, antidavriy chegaraviy shartlari bilan qaraymiz. Bu hollarda xos qiymatlarni va xos vektorlarni aniq ko'rinishda hisoblash mumkin. Bizning asosiy natijamiz quyidagicha ifodalanadi.

3.1-teorema. Agar bo'lsa, u holda

$$-u''(-x) + \alpha u''(x) = \lambda u(x) \qquad (2.2.10)$$

tenglamaning λ spektral parameter bo'yicha umumiy yechimi

$$u(x) = A\cos\sqrt{\frac{\lambda}{1-\alpha}}x + B\sin\sqrt{\frac{\lambda}{-1-\alpha}}x \qquad (2.2.11)$$

ko'rinishga ega, bu yerda A va B ixtiyoriy kompleks sonlar.

Agar $\alpha^2 = 1$ va $\lambda \neq 0$ bo'lsa, u holda (2.2.10) masala faqat ayniny nolga teng yechimga ega bo'ladi.

Isboti. (2.2.11) funksiya (2.2.10) tenglamaning yechimi ekanligini tekshirish qiyin emas. Boshqa yechimlar yo'q ekanligini isbotlaymiz. Har qanday

$u(x)$ funksiyani juft va toq funksiyalar yig'indisi ko'rinishda tasvirlash mumkin: $u(x)=u_1(x)+u_2(x)$. Bu ifodani (2.2.10) va

$$-u''(x)+\alpha u''(-x)=\lambda u(-x)$$

tenglamaga qo'yib $u_1(x)$ funksiya uchun

$$-(1-\alpha)u_1''(x)=\lambda u_1(x) \qquad (2.2.12)$$

va $u_2(x)$ funksiya uchun

$$-(-1-\alpha)u_2''(x)=\lambda u2(x) \qquad (2.2.13)$$

tenglamalarni hosil qilamiz. Teorema isbotlandi.

4. Dirixle masalasi. $\alpha^2 \neq 1$ uchun (2.2.10) spektral masalani

$$u(-1)=0, \ \ u(1)=0 \qquad (2.2.14)$$

chegaraviy shartlar bo'yicha qaraymiz. Haqiqiy α uchun (2.2.10) va (2.2.14) spektral masala o'z o'ziga qo'shma masala bo'lishini e'tiborga olamiz. (2.2.10) va (2.2.14) Dirixle spektral masalasining xos qiymati va xos funksiyalarini hisoblaymiz. 3.1-teoremadan foydalanib, (2.2.10) va (2.2.14) spektral masala ikkita oddiy xos qiymatlar ketma ketligiga ega ekanligini ko'rish qiyin emas.

Agar $\alpha \notin \{(8k^2+4k+1)/(4k+1) : k \in Z\}$ bo'lsa, u holda mos hos funksiyalar

$$u_{k,1}(x)=\cos\left(\frac{\pi}{2}+k\pi\right)x,\ k=0,1.2,....., \quad u_{k,2}(x)=\sin k\pi x,\ k=1,2,... \qquad (2.2.15)$$

formulalar bilan beriladi.

Agar qandaydir $k_0 \in Z$ uchun $\alpha \in \{(8k^2+4k+1)/(4k+1) : k \in Z\}$ bo'lsa, u holda (2.2.10) va (2.2.14) spektral masala mos hos funksiyalari

$$u_{k,1}(x)=\cos\left(\frac{\pi}{2}+k\pi\right)x,\ k=0,1.2,....., \quad u_{k,2}(x)=\sin k\pi x,\ k=1,2,...,k\neq k_0$$

$$u_{k_0,1}(x)=\cos\left(\frac{\pi}{2}+k_0\pi\right)x+\sin\sqrt{\frac{1-\alpha}{-1-\alpha}}\left(\frac{\pi}{2}+k_0\pi\right)x, \qquad (2.2.16)$$

$$u_{k_0 2}(x)=\sin k_0\pi x+\cos\sqrt{\frac{1-\alpha}{-1-\alpha}}\ k_0\pi x$$

formulalar bilan beriladi.

4.1-teorema. Agar $\alpha^2 \neq 1$ bo'lsa, u holda (2.2.10) va (2.2.14) spektral masalaning yuqorida berilgan xos funksiyalari $L_2(-1,1)$ fazoda to'la ortonormal sistemani tashkil qiladi.

Isboti. α ning haqiqiy qiymatlari uchun (2.2.10) va (2.2.14) spektral masala o'z o'ziga qo'shma masaladir. Shuning uchun (2.2.14) sistema o'z-o'ziga qo'shma operatorning xos qiymatlari bo'lgani uchun ular ortogonaldir. Xuddi shu kabi

$\alpha = (8k^2 + 4k + 1)/(4k + 1) : k \in Z$ hol ham qaraladi. Shu bilan birga har bir ortonormal bazis avtomatik holda Riss bazisi bo'ladi.

(2.4.2) sistema α ga bog'liq emas. Demak, 4-1-teorema isbotlandi.

5. Davriy va antidavriy masalalar. Endi (2.2.10) spektral masalani

$$u(-1) = u(1), \ \ u'(-1) = u'(1) \tag{2.2.17}$$

davriy chegaraviy shartlar bo'yicha qaraymiz. Bundan bevosita (2.2.10) va (2.2.17) spektral masalasining xos qiymatlari

$$(\lambda_{k1})^2 = -(1+\alpha)k^2\pi^2 \ , \ \ (\lambda_{k2})^2 = (1-\alpha)k^2\pi^2 \tag{2.2.18}$$

tengliklar bilan beriladi. Bu xos qiymatlar oddiy bo'lib, ularga

$$u_{k1}(x) = \sin k\pi x, \ k = 0,1,2,.... \ , \ u_{k2}(x) = \cos k\pi x, \ k = 0,1,2,.... \tag{2.2.19}$$

xos funksiyalar mos keladi.

Xuddi shu kabi (2.2.10) spektral masalani

$$u(-1) = -u(1), \ \ u'(-1) = -u'(1) \tag{2.2.20}$$

antidavriy chegaraviy shartlar bo'yicha qarab xos qiymatlar va ularga mos xos funksiyalar hisoblanadi.

Bu holda biz xos qiymatlarning ikkita

$$\begin{aligned} (\lambda_{k1})^2 &= (1-\alpha)\left(\frac{\pi}{2} + k\pi\right), \ k = 0,1,2,... \\ (\lambda_{k2})^2 &= (-1-\alpha)\left(\frac{\pi}{2} + k\pi\right), \ k = 0,1,2,... \end{aligned} \tag{2.2.21}$$

seriyasini topamiz.

Ularga mos xos funksiyalar

$$u_{k1}(x)=\cos\left(\frac{\pi}{2}+k\pi\right)x,\ k=1,2,....\ ,\ u_{k2}(x)=\sin\left(\frac{\pi}{2}+k\pi\right)x,\ k=0,1,2,.... \qquad (2.2.22)$$

tengliklar bilan aniqlanadi.

5.1-teorema. Agar $\alpha^2 \neq 1$ bo'lsa, u holda (2.2.10) spektral masalaning davriy va antidavriy chegaraviy shartlar bo'yicha xos funksiyalari $L_2(-1,1)$ fazoda to'la ortonormal sistemani tashkil qiladi.

Isboti. Bu teoremaning isboti ham 4.1-teorema isboti kabi amalga oshiriladi. Shu bilan birga davriy chegaraviy shartlar bo'yicha xos funksiyalari $L_2(-1,1)$ fazoda to'la ortonormal sistemani tashkil qiladi.

Xuddi shu kabi $\alpha^2 \neq 1$ bo'lganda (2.2.10) spektral masalaning Shturm ko'rinishidagi

$$u'(-1)=0,,\ \ u'(1)=0 \qquad (2.2.23)$$

va qo'shma bo'lmagan

$$u(-1)=0,\ \ u'(-1)=u'(1) \qquad (2.2.24)$$

chegaraviy shartlar bo'yicha xos funksiyalari $L_2(-1,1)$ fazoda to'la ortonormal bazisni tashkil etadi.

2.3- §. Involyutsiyaga ega bo'lgan nolokal to'lqin tenglamasi uchun teskari masala

1. Masalalarning qo'yilishi. $\Omega=\{(x,t):-\pi\leq x\leq\pi\ ,\ 0\leq t\leq T\}$ to'g'ri to'rtburchakli sohada

$$u_{tt}(x,t)-u_{xx}(x,t)+\varepsilon u_{xx}(-x,t)=f(x),\ \ (x,t)\in\Omega \qquad (2.3.1)$$

tenglamani qaraymiz, bu yerda ε noldan farqli $|\varepsilon|<1$ shartni qanoatlantiruvchi son.

Quyidagi ikkita masalani qaraymiz:

1-masala. Ω sohada (2.3.1) tenglamani va

$$u(x,0)=\phi(x),\ \ u_t(x,0)=\rho(x)\ ,\ u(x,T)=\psi(x),\ \ x\in[-\pi,\pi] \qquad (2.3.2)$$

$$u(-\pi,t)=0,\ u(\pi,t)=0,\ t\in[0,T] \tag{2.3.3}$$

shartlarni qanoatlantiruvchi $(u(x,t), f(x))$ funksiyalar juftini toping, bu yerda $\phi(x)$ va $\psi(x)$ yetarlicha silliqlikka ega bo'lgan berilgan funksiyalar.

2-masala. Ω sohada (2.3.1) tenglamani va

$$u(x,0)=\phi(x),\ u_t(x,0)=\rho(x),\ u(x,T)=\psi(x),\ x\in[-\pi,\pi] \tag{2.3.2}$$

$$u_x(-\pi,t)=0,\ u_x(\pi,t)=0,\ t\in[0,T] \tag{2.3.4}$$

shartlarni qanoatlantiruvchi $(u(x,t), f(x))$ funksiyalar juftini toping, bu yerda $\phi(x)$ va $\psi(x)$ yetarlicha silliqlikka ega bo'lgan berilgan funksiyalar.

1-va 2- masalalarda $(u(x,t), f(x))$ funksiyalar juftining regulyarlik sharti $f(x)\in C[-\pi,\pi]$ va $u(x,t)\in C^{2,2}_{x,t}(\Omega)$ shartlar bajarilishiga teng kuchli.

2.2. Spektral masalalar. 1- va 2- masalalarda qaralayotgan xususiy hosilali tenglamaga mos bir jinsli qismini yechishda o'zgaruvchilarni ajratish usuli bo'lgan Fur'ye usulini qo'llaymiz. Buning uchun 1- va 2- masalalarning $u(x,t)$ yechimini

$$u(x,t)=X(x)\cdot T(t)$$

ko'rinishda izlaymiz. U holda

$$u_{tt}(x,t)=X(x)\cdot T''(t),$$
$$u_{xx}(x,t)=X''(x)\cdot T(t),$$
$$u_{xx}(-x,t)=X''(-x)\cdot T(t)$$

bo'lgani uchun (3.2.1) tenglamaning bir jinsli qismi

$$X(x)\cdot T''(t)-X''(x)\cdot T(t)+\varepsilon X''(-x)\cdot T(t)=0,$$

yoki

$$\frac{T''(t)}{T(t)}=\frac{X''(x)-\varepsilon X''(-x)}{X(x)}$$

ko'rinishni oladi. Yuqoridagi masalaning berilishidan yechim chegaralangan bo'lishi uchun, bu nisbat manfiy o'zgarmas sondan iborat:

$$\frac{T''(t)}{T(t)}=\frac{X''(x)-\varepsilon X''(-x)}{X(x)}=-\lambda,\ \lambda>0.$$

Bu tenglikdan

$$T''(t)+\lambda T(t)=0 \text{ va } X''(x)-\varepsilon X''(-x)+\lambda X(x)=0$$

tenglamalarni hosil qilamiz. 1- va 2- masalalarda berilgan (2.3.3) va (2.3.4) chegaraviy shartlarga ko'ra va yechimning $u(x,t)=X(x)\cdot T(t)$ tanlanishiga ko'ra quyidagi ikkita:

$$\begin{cases} X''(x)-\varepsilon X''(-x)+\lambda X(x)=0, \ -\pi\le x\le\pi. \\ X(-\pi)=X(\pi)=0 \end{cases} \qquad (2.3.5)$$

va

$$\begin{cases} X''(x)-\varepsilon X''(-x)+\lambda X(x)=0, \ -\pi\le x\le\pi, \\ X'(-\pi)=X'(\pi)=0 \end{cases} \qquad (2.3.6)$$

spektral masalalarga ega bo'lamiz.

1.Birinchi spektral masalaning yechimi.

Birinchi spektral masalaning xos qiymati va xos vektorlarini topamiz. Buning uchun (2.3.5) :

$$X''(x)-\varepsilon X''(-x)+\lambda X(x)=0, \ -\pi\le x\le\pi$$

tenglamada x ni $-x$ ga almashtirib,

$$X''(-x)-\varepsilon X''(x)+\lambda X(-x)=0, \ -\pi\le x\le\pi$$

tenglamani hosil qilamiz.

Bu tenglamalarni hadlab qo'shib va hadlab ayirib

$$(1-\varepsilon)[X(x)+X(-x)]''+\lambda[X(x)+X(-x)]=0 \qquad (2.3.7)$$

va

$$(1+\varepsilon)[X(x)-X(-x)]''+\lambda[X(x)-X(-x)]=0 \qquad (2.3.8)$$

tenglamalarni hosil qilamiz.

Ma'lumki, har qanday funksiyani juft va toq funksiyalar yig'indisi ko'rinishida ifodalash mumkin bo'lgani uchun $X(x)=F(x)+G(x)$ belgilash kiritamiz, bu yerda $F(x)-$ juft funksiya, $G(x)-$toq funksiya, ya'ni $F(-x)=F(x)$ va $G(-x)=-G(x)$.

Shuning uchun yuqorida ko'rsatilgan birinchi spektral masalaning yechimi

$$\begin{cases} (1-\varepsilon)F''(x)+\lambda F(x)=0, \ -\pi \le x \le \pi \\ F(-\pi)=F(\pi)=0 \end{cases} \quad (2.3.9)$$

va

$$\begin{cases} (1+\varepsilon)G''(x)+\lambda G(x)=0, \ -\pi \le x \le \pi \\ G(-\pi)=G(\pi)=0 \end{cases} \quad (2.3.10)$$

spektral masalalar yechimlaridan tashkil topadi.

(2.3.9) va (2.3.10) spektral masalalarning umumiy yechimlari mos ravishda

$F(x)=A\cos\sqrt{\frac{\lambda}{1-\varepsilon}}x+B\sin\sqrt{\frac{\lambda}{1-\varepsilon}}x$ va $G(x)=C\cos\sqrt{\frac{\lambda}{1+\varepsilon}}x+D\sin\sqrt{\frac{\lambda}{1+\varepsilon}}x$

ko'rinishga ega, bu yerda A,B,C va $D-$ixtiyoriy o'zgarmas sonlar. U holda $F(-\pi)=F(\pi)=0$ va $G(-\pi)=G(\pi)=0$ chegaraviy shartlarga ko'ra $A=0, D=0$ bo'lgani uchun , birinchi spektral masala

$$\lambda_{k,1}=(1+\varepsilon)k^2 \ , k\in N, \quad \lambda_{k,2}=(1-\varepsilon)\left(k+\frac{1}{2}\right)^2 \ , k\in N\cup\{0\} \quad (2.3.11)$$

xos qiymatlarga ega va bu xos qiymatlarga mos xos funksiyalar mos ravishda

$$\chi_{k,1}=\sin kx, \ k\in N \ , \quad \chi_{k,2}=\cos\left(k+\frac{1}{2}\right)x, \ k\in N\cup\{0\}$$

xos qiymatlarga ega. Shu bilan birga

$$\left\|\chi_{k,1}\right\|^2=\int_{-\pi}^{\pi}\sin^2 kxdx=\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2kx)dx=\pi \quad \text{va}$$

$$\left\|\chi_{k,2}\right\|^2=\int_{-\pi}^{\pi}\cos^2\left(k+\frac{1}{2}\right)xdx=\frac{1}{2}\int_{-\pi}^{\pi}(1+\cos(2k+1)x)dx=\pi$$

bo'lgani uchun bu xos qiymatlarga mos normalangan xos funksiyalar mos ravishda

$$X_{k,1}=\frac{1}{\sqrt{\pi}}\sin kx, \ k\in N \ , \quad X_{k,2}=\frac{1}{\sqrt{\pi}}\cos\left(k+\frac{1}{2}\right)x, \ k\in N\cup\{0\} \quad (2.3.12)$$

ko'rinishga ega bo'ladi.

2.**Ikkinchi spektral masalaning yechimi.**

Ikkinchi spektral masalaning yechimi ham xuddi birinchi spektral masalaning xos qiymati va xos vektorlarini topishga o'xshash bo'lib, bu spektral masalaning yechimi

$$\begin{cases} (1-\varepsilon)F''(x)+\lambda F(x)=0, \ -\pi \le x \le \pi \\ F'(-\pi)=F'(\pi)=0 \end{cases} \tag{2.3.13}$$

va

$$\begin{cases} (1+\varepsilon)G''(x)+\lambda G(x)=0, \ -\pi \le x \le \pi \\ G(-\pi)=G(\pi)=0 \end{cases} \tag{2.3.14}$$

spektral masalalar yechimlaridan tashkil topadi.

(2.3.13) va (2.3.14) spektral masalalarning umumiy yechimlari mos ravishda

$$F(x)=A\cos\sqrt{\frac{\lambda}{1-\varepsilon}}x+B\sin\sqrt{\frac{\lambda}{1-\varepsilon}}x \text{ va } G(x)=C\cos\sqrt{\frac{\lambda}{1+\varepsilon}}x+D\sin\sqrt{\frac{\lambda}{1+\varepsilon}}x$$

ko'rinishga ega, bu yerda A,B,C va $D-$ixtiyoriy o'zgarmas sonlar. U holda

$$F'(x)=\sqrt{\frac{\lambda}{1-\varepsilon}}\left(-A\sin\sqrt{\frac{\lambda}{1-\varepsilon}}x+B\cos\sqrt{\frac{\lambda}{1-\varepsilon}}x\right)$$

va

$$G'(x)=\sqrt{\frac{\lambda}{1+\varepsilon}}\left(-A\sin\sqrt{\frac{\lambda}{1+\varepsilon}}x+B\cos\sqrt{\frac{\lambda}{1+\varepsilon}}x\right)$$

bo'lgani uchun $F'(-\pi)=F'(\pi)=0$ va $G'(-\pi)=G'(\pi)=0$chegaraviy shartlarga ko'ra $B=0,C=0$ bo'lgani uchun , ikkinchi spektral masalaning xos qiymatlari

$$\lambda_{k,1}=(1+\varepsilon)\left(k+\frac{1}{2}\right)^2, \ k\in N, \qquad \lambda_{k,2}=(1-\varepsilon)k^2, \ k\in N\bigcup\{0\} \tag{2.3.15}$$

ko'rinishga ega bo'lgani uchun bu xos qiymatlarning mos normalangan xos funksiyalari mos ravishda

$$X_{k,1}=\frac{1}{\sqrt{\pi}}\sin\left(k+\frac{1}{2}\right)x, \ k\in N \ , \quad X_{k,2}=\frac{1}{\sqrt{\pi}}\cos kx, \ k\in N\bigcup\{0\} \tag{2.3.16}$$

ko'rinishga ega ekanligini ko'rishimiz mumkin.

(3.2.8) va (3.2.12) funksiyalar sistemasi $L_2(-\pi,\pi)$ fazoda to'la ortonormal sistema tashkil qilishini ko'rsatishimiz mumkin.

3. Chegaraviy masalala yechimining mavjudligi haqidagi asosiy natijalar.

Biz bu bo'limda 1- va 2- masalalar uchun yechim mavjudligi va yagonaligi naqidagi natijalarni teoremalar ko'rinishida ifodalaymiz.

3.1-teorema. $\phi(x), \rho(x), \varphi(x) \in C^4[-\pi,\pi]$ va $\phi^{(i)}(\pm\pi) = \rho^{(i)}(\pm\pi) = \psi^{(i)}(\pm\pi) = 0$, $i = 0,1,2,3$ bo'lsin. u holda $|\varepsilon| < 1$ shartni qanoatlantiruvchi haqiqiy ε soni uchun 1- masala yagona yechimga ega bo'lib, u quyidagi ko'rinishga ega:

$$u(x,t) = \phi(x) + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{1k}\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t}{\left(k+\frac{1}{2}\right)^4}\cos\left(k+\frac{1}{2}\right)x +$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t - \left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4}\cos\left(k+\frac{1}{2}\right)x +$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{2k}\cos\sqrt{1+\varepsilon}kt - \left(C^{(4)}\right)_{4k}\sin\sqrt{1+\varepsilon}kt - \left(C^{(4)}\right)_{2k}}{k^4}\sin kx$$

va

$$f(x) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)\left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^2}\cos\left(k+\frac{1}{2}\right)x + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{k^2}\sin kx\,,$$

bu yerda

$$\left(C^{(4)}\right)_{1k} = \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}{1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}$$

$$\left(C^{(4)}\right)_{3k} = \frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)},$$

$$\left(C^{(4)}\right)_{2k} = \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k}\sin\sqrt{1+\varepsilon}kT}{1-\cos\sqrt{1+\varepsilon}kT},$$

$$\left(C^{(4)}\right)_{4k}=\frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon k}},$$

va $g=\phi,\psi,\rho$ uchun

$$\left(g^{(4)}\right)_{1k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\cos\left(k+\frac{1}{2}\right)xdx,\ \left(g^{(4)}\right)_{2k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\sin kxdx$$

3.2-teorema. $\phi(x),\rho(x),\varphi(x)\in C^4[-\pi,\pi]$ va $\phi^{(i)}(\pm\pi)=\rho^{(i)}(\pm\pi)=\psi^{(i)}(\pm\pi)=0$, $i=0,1,2,3$ bo'lsin. u holda $|\varepsilon|<1$shartni qanoatlantiruvchi haqiqiy ε soni uchun 2- masala yagona yechimga ega bo'lib, bu yechim quyidagi ko'rinishga ega:

$$u(x,t)=\phi(x)+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{1k}\cos\sqrt{1-\varepsilon}kt+\left(C^{(4)}\right)_{3k}\sin\sqrt{1-\varepsilon}kt-\left(C^{(4)}\right)_{1k}}{k^4}\cos\left(k+\frac{1}{2}\right)x$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\left(C^{(4)}\right)_{2k}\cos\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)t}{\left(k+\frac{1}{2}\right)^4}\sin\left(k+\frac{1}{2}\right)x$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\left(C^{(4)}\right)_{4k}\sin\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)t-\left(C^{(4)}\right)_{2k}}{\left(k+\frac{1}{2}\right)^4}\sin\left(k+\frac{1}{2}\right)x$$

va

$$f(x)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k}-\left(C^{(4)}\right)_{1k}\right)}{k^2}\cos kx+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k}-\left(C^{(4)}\right)_{2k}\right)}{\left(k+\frac{1}{2}\right)^2}\sin\left(k+\frac{1}{2}\right)x,$$

bu yerda

$$\left(C^{(4)}\right)_{1k}=\frac{\left(\phi^{(4)}\right)_{1k}-\left(\psi^{(4)}\right)_{1k}+\left(C^{(4)}\right)_{3k}\sin\sqrt{1-\varepsilon}kT}{1-\cos\sqrt{1-\varepsilon}kT}$$

$$\left(C^{(4)}\right)_{3k}=\frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon k}},$$

$$\left(C^{(4)}\right)_{2k}=\frac{\left(\phi^{(4)}\right)_{2k}-\left(\psi^{(4)}\right)_{2k}+\left(C^{(4)}\right)_{4k}\sin\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)T}{1-\cos\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)T},$$

$$\left(C^{(4)}\right)_{4k}=\frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)},$$

va $g=\phi,\psi,\rho$ lar uchun

$$\left(g^{(4)}\right)_{1k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\cos kxdx,\ \left(g^{(4)}\right)_{2k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\sin\left(k+\frac{1}{2}\right)xdx$$

4. Chegaraviy masala yechimining mavjudligi haqidagi teoremaning isboti.

Biz bu bo'limda 1- masala yechimining mavjudligi haqidagi 3.1-teoremaning isbotini to'la keltiramiz.

(2.3.1) masala o'z -o'ziga qo'shma bo'lgani uchun 1- masalaga mos keluvchi (2.3.12) xos funksiyalar sistemasi $L_2(-\pi,\pi)$ fazoda ortonormal bazis tashkil qiladi va $u(x,t)$ va $f(x)$ funksiyalar bu bazis bo'yicha

$$u(x,t)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}u_k(t)\cos\left(k+\frac{1}{2}\right)x+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}v_k(t)\sin kx \qquad (2.3.17)$$

va

$$f(x)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}f_{1k}\cos\left(k+\frac{1}{2}\right)x+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}f_{2k}\sin kx \qquad (2.3.18)$$

ko'rinishda qatorlar bilan ifodalanadi, bu yerda $u_k(t),v_k(t)$ funksiyalar va f_{1k}, f_{2k} o'zgarmas sonlar noma'lumlardir.

(2.3.17) va (2.3.18) ni (2.3.1) tenglamaga qo'yib, biz $u_k(t),v_k(t)$ funksiyalar va f_{1k}, f_{2k} o'zgarmas sonlarga nisbatan

$$u_k''(t)+(1-\varepsilon)\left(k+\frac{1}{2}\right)^2u_k(t)=f_{1k}$$

va

$$v_k''(t)+(1+\varepsilon)k^2v_k(t)=f_{2k}$$

tenglamalarni hosil qilamiz. Bu ikki tenglamani birgalikda yechib,

$$u_k(t)=C_{1k}\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t+C_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t+\frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2}$$

va

$$v_k(t)=C_{\underset{2k}{1}}\cos\sqrt{1+\varepsilon}kt+C_{4k}\sin\sqrt{1+\varepsilon}kt+\frac{f_{2k}}{(1+\varepsilon)k^2}$$

tengliklarni hosil qilamiz, bu yerda $C_{1k},C_{2k},C_{3k},C_{4k},f_{1k}$ va f_{2k} o'zgarmaslar (2.3.2) shartlar yordamida aniqlanadi; (2.3.12) xos funksiyalar sistemasi, $\phi(x),\rho(x)$ va $\psi(x)$funksiyalarning qiymatlaridan foydalanib,

$$C_{1k}+\frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2}=\phi_{1k}, \qquad C_{2k}+\frac{f_{2k}}{(1+\varepsilon)k^2}=\phi_{2k}$$

$$\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)C_{3k}=\rho_{1k}, \qquad \sqrt{1+\varepsilon}kC_{4k}=\rho_{2k}$$

$$C_{1k}\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T+C_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T+\frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2}=\psi_{1k}$$

va

$$C_{2k}\cos\sqrt{1+\varepsilon}kT+C_{4k}\sin\sqrt{1+\varepsilon}kT+\frac{f_{1k}}{(1+\varepsilon)k^2}=\psi_{2k}$$

tengliklarga ega bo'lamiz, bu yerda $\phi_{ik},\rho_{ik},\psi_{ik}$ koeffisiyentlar $\phi(x),\rho(x)$ va $\psi(x)$ funksiyalarning

$$g_{1k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g(x)\cos\left(k+\frac{1}{2}\right)xdx, \quad g_{2k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g(x)\sin kxdx$$

Funksiya uchun $g=\phi,\rho,\psi$ bo'lgandagi qiymatlaridir.

Yuqorida hosil qilingan tenglamalarni $C_{1k},C_{2k},C_{3k},C_{4k},f_{1k}$ va f_{2k} o'zgarmaslarga nisbatan yechib, biz

$$C_{1k}=\frac{\phi_{1k}-\psi_{1k}+C_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}{1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}$$

$$C_{3k}=\frac{\rho_{1k}}{\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)},$$

$$C_{2k}=\frac{\phi_{2k}-\psi_{2k}+C_{4k}\sin\sqrt{1+\varepsilon}kT}{1-\cos\sqrt{1+\varepsilon}kT},$$

$$C_{4k}=\frac{\rho_{2k}}{\sqrt{1+\varepsilon}k},$$

va

$$f_{1k}=(1-\varepsilon)\left(k+\frac{1}{2}\right)^2(\phi_{1k}-C_{1k}),\quad f_{2k}=(1+\varepsilon)k^2(\phi_{2k}-C_{2k}) \qquad (2.3.19)$$

Endi $u_k(t), v_k(t), f_{1k}, f_{2k}$ larni (2.3.17) va (2.3.18) ga qo'yib

$$u(x,t)=\phi(x)$$
$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\left(C_{1k}\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t+C_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t-C_{1k}\right)\cos\left(k+\frac{1}{2}\right)x$$
$$+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\left(C_{2k}\cos\sqrt{1+\varepsilon}kt+C_{4k}\sin\sqrt{1+\varepsilon}kt-C_{2k}\right)\sin kx$$

va

$$f(x)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}(1-\varepsilon)\left(k+\frac{1}{2}\right)^2(\phi_{1k}-C_{1k})\cos\left(k+\frac{1}{2}\right)x+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}(1+\varepsilon)k^2(\phi_{2k}-C_{2k})\sin kx$$

Bundan tashqari , agar $\phi^{(i)}(\pm\pi)=\rho^{(i)}(\pm\pi)=\psi^{(i)}(\pm\pi)$, $i=0,1,2,3$ bo'lsa, u holda bo'laklab integrallash bilan $\phi_{ik}, \rho_{ik}, \psi_{ik}, i=1,2$ koeffisiyentlar $g=\phi,\rho,\psi$ uchun

$$g_{1k}=\frac{\left(g^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^2} \text{ va } g_{2k}=\frac{\left(g^{(4)}\right)_{2k}}{k^2}$$

ko'rinishda ifodalanishi mumkin, bu yerda $\left(\phi^{(4)}\right)_{ik}, \left(\rho^{(4)}\right)_{ik}, \left(\psi^{(4)}\right)_{ik}, i=1,2$ sonlari $\phi^{(4)}(x), \rho^{(4)}(x)$ va $\psi^{(4)}(x)$ funksiyalarning

$$\left(g^{(4)}\right)_{1k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\cos\left(k+\frac{1}{2}\right)xdx,\ \left(g^{(4)}\right)_{2k}=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}g^{(4)}(x)\sin kxdx$$

funksiyalar vositasida $g=\phi,\rho,\psi$ bo'lgandagi qiymatlaridir.

U holda $C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}$ va f_{2k} o'zgarmas sonlar

$$C_{1k}=\frac{\left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4},\ C_{3k}=\frac{\left(C^{(4)}\right)_{3k}}{\left(k+\frac{1}{2}\right)^4}\quad va\ \ C_{2k}=\frac{\left(C^{(4)}\right)_{2k}}{k^4}, C_{4k}=\frac{\left(C^{(4)}\right)_{4k}}{k^4}$$

va

$$f_{1k}=\frac{(1-\varepsilon)}{\left(k+\frac{1}{2}\right)^2}\left(\left(\phi^{(4)}\right)_{1k}-\left(C^{(4)}\right)_{1k}\right),\qquad f_{2k}=\frac{(1+\varepsilon)}{k^2}\left(\left(\phi^{(4)}\right)_{2k}-\left(C^{(4)}\right)_{2k}\right)\quad (2.3.20)$$

ko'rinishda yozilishi mumkin, bu yerda

$$\left(C^{(4)}\right)_{1k}=\frac{\left(\phi^{(4)}\right)_{1k}-\left(\psi^{(4)}\right)_{1k}+\left(C^{\ (4)}\right)_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}{1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T},\qquad \left(C^{(4)}\right)_{3k}=\frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)}$$

$$\left(C^{(4)}\right)_{2k}=\frac{\left(\phi^{(4)}\right)_{2k}-\left(\psi^{(4)}\right)_{2k}+\left(C^{\ (4)}\right)_{4k}\sin\sqrt{1+\varepsilon}kT}{1-\cos\sqrt{1+\varepsilon}kT},\qquad \left(C^{(4)}\right)_{4k}=\frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon}k},$$

Demak, 1-masalaning yechimi

$$u(x,t)=\phi(x)+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{\ (4)}\right)_{1k}\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t}{\left(k+\frac{1}{2}\right)^4}\cos\left(k+\frac{1}{2}\right)x$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{3k}\sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t-\left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4}\cos\left(k+\frac{1}{2}\right)x\qquad (2.3.21)$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{2k}\cos\sqrt{1+\varepsilon}kt-\left(C^{(4)}\right)_{4k}\sin\sqrt{1+\varepsilon}kt-\left(C^{(4)}\right)_{2k}}{k^4}\sin kx$$

va

$$f(x)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k}-\left(C^{\ (4)}\right)_{1k}\right)\left(C^{\ (4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^2}\cos\left(k+\frac{1}{2}\right)x$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k}-\left(C^{\ (4)}\right)_{2k}\right)}{k^2}\sin kx,\qquad (2.3.22)$$

ko'rinishda yozilishi mumkin. Bu 3.1-teoremani isbotlaydi.

Xuddi shu tarzda 2- masalaning yechimini ifodalovchi 3.2-teorema isbotlanadi.

5.**Qatorlarning yaqinlashishi.** Formal ko'rinishda qurilgan yechim haqiqatdan han qaralayotgan masalaning aniq yechimi ekanligiga ishonch hosil qilish uchun hozirgacha bajarilgan amallar to'g'ri ekanligini isbotlashimiz kerak. (3.4.5) va (3.4.6) qatorlarning yaqinlashishi $u(x,t)$ va $f(x)$ funksiyalarning quyidagi baholariga asoslanadi.

$$|u(x,t)| \le |\phi(x)| + \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 4\left|\rho_{1k}^{(4)}\right|}{\left(\sqrt{1-\varepsilon}\right)\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^4}$$

$$+\frac{2}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right| + \sqrt{2}\left|\psi_{2k}^{(4)}\right| + 4\left|\rho_{2k}^{(4)}\right|}{\left(\sqrt{1+\varepsilon}\right)\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^4} \qquad (2.3.23)$$

va

$$|f(x)| \le \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{6\left|\phi_{1k}^{(4)}\right| + 2\left|\psi_{1k}^{(4)}\right| + 2\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2} + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{6\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + \sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2} \quad (2.3.24)$$

$\phi(x), \rho(x),\ \psi(x) \in C^4[-\pi,\pi]$ bo'lgani uchun , u holda trigonometrik qatorlar uchun Bessel tengsizligiga ko'ra

$$g=\phi,\rho,\psi \text{ uchun } \quad \sum_{k=0}^{\infty}\left|g_{1k}^{(4)}\right|^2 \le C\left\|g^{(4)}(x)\right\|^2_{L_2(-\pi,\pi)} \qquad (2.3.25)$$

va

$$g=\phi,\rho,\psi \text{ uchun } \quad \sum_{k=0}^{\infty}\left|g_{2k}^{(4)}\right|^2 \le C\left\|g^{(4)}(x)\right\|^2_{L_2(-\pi,\pi)} \qquad (2.3.26)$$

baholardan $\left\{\phi_{ik}^{(4)},\rho_{ik}^{(4)},\psi_{ik}^{(4)}\right\},\ k=1,2$ to'plamning chegaralanganligi kelib chiqadi. Shuning uchun Veyershtrass M-testiga ko'ra (2.3.23) va (2.3.24) qatorlar Ω sohada absolyut va tekis yaqinlashadi.

Endi (2.3.21) qatorni x va t o’zgaruvchilarga nisbatan ikki marta hadlab differensiallashdan foydalanib, $u_{xx}(x,t)$ va $u_{tt}(x,t)$ funksiyalar uchun quyidagi baholarni hosil qilamiz.

$$|u_{xx}(x,t)| \le |\phi''(x)| + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 4\left|\rho_{1k}^{(4)}\right|}{\left(\sqrt{1-\varepsilon}\right)\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2}$$

$$+\frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right| + \sqrt{2}\left|\psi_{2k}^{(4)}\right| + 2\left|\rho_{2k}^{(4)}\right|}{\left(\sqrt{1+\varepsilon}\right)\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2}$$

$$|u_{tt}(x,t)| \le \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 6\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + 3\sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1-\cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2}$$

bu qatorlar ni (2.3.25), (2.3.26) baholardan foydalanib, Veyershtrass M-testiga ko’ra Ω sohada absolyut va tekis yaqinlashishiga ishonch hosil qilamiz.

6. Yechimning yagonaligi. 1- va 2-masalalar yechimining yagonaligi yuqorida keltirilgan teoremalardan va (2.3.12) va (2.3.16) xos funksiyalar sistemasining to’laligidan kelib chiqadi.

II BOB bo’yicha xulosalar

Bu bobda involyutsiya qatnashgan giperbolik tipdagi xususiy hosilalali differensial tenglamalar uchun chegaraviy masalalarni Fur’ye usulini qo’llab yechishda yuzaga kelgan involyutsiya qatnashgan spektral masalalarni yechish va ular asosida teskari masalalarni yechish bo’yicha quyidagi jarayonlar bajarildi:

- involyutiv cheklanishga ega bo’lgan chegaraviy masalalar uchun o’zgaruvchilarni ajratishning Fur’ye usulining asoslash masalasi;
- involyutsiyaga ega bo’lgan ikkinchi tartibli oddiy differensial tenglamalar uchun xos qiymatlar va xos funksiyalarining asosiy xossalari haqidagi spektral masalalarning yechilishi;
- involyutsiyaga ega bo’lgan nolokal to’lqin tenglamasi uchun teskari masalani Fur’ye usuli bilan yechishni va asoslash.

III BOB INVOLYUTSIYA XOSSASIGA EGA BO'LGAN ARALASH MASALALAR

3.1- §. Masalaning qo'yilishi.

Biz quyidagi:

$$\frac{1}{\beta i}\frac{\partial u(x,t)}{\partial t}=\left.\frac{\partial u(\xi,t)}{\partial \xi}\right|_{\xi=1-x}+q(x)u(x,t),\ x\in[0,1],\ t\in(-\infty;+\infty) \quad , \ (3.1.1)$$

$$u(x,0)=\varphi(x), \qquad u(0,t)=0 \qquad (3.1.2)$$

aralash masalani qaraymiz. Bu yerda quyidagi shartlar :

1) $\beta -$ haqiqiy son va $\beta\neq 0$;

2) $q(x)\in C^1[0,1]$, $q(x)=q(1-x)$, $q(x)-$haqiqiy funksiya;

3) $\varphi(x)\in C^1[0,1]$ va $\varphi(0)=0,\ \varphi'(1)=0$

bajariladi deb hisoblaymiz.

(3.1.1)tenglama $\nu(x)=1-x$ involyutsiyani o'zida saqlovchi eng soda xususoy hosilali differensial tenglamadir. Involyutsoyaga ega bo'lgan tenglamalar ustida ko'plab tadqiqotlar olib borilmoqda (masalan [1] va unda keltirilgan adabiyotlarga qarang) .

(3.1.1)-(3.1.2) masalaning yechimini topish uchun Fur'ye usulidan foydalanamiz. Biz keltirgan shartlar masalaning klassik yechimini, ya'ni har ikkala argument bo'yicha uzluksiz differensialanuvchi yechimni toppish imkoniyatini beradi. $\varphi(x)$ funksiyaga nisbatan qo'yilgan shartlar tabbiy bo'lib , bu funksiya (3.1.1)-(3.1.2) chegaraviy masaladan kelib chiquvchi xos funksiani qanoatlantiradi. $q(x)$ funksiyaga qo'yolgan shart esa masalani tadqiq qilishdagi ko'plab muammolarni osonlashtiradi va yechim ko'rinishining yaxshi shaklini beradi.

Ishda [2] ning usullaridan foydalaniladi va bu usul funksional qatorni hadlab differensiallashdan chetlanishga imkoniyat yaratadi.

3.2- §. Masalaning yechilishi.

Fur'ye usuliga ko'ra $u(x,t)=y(x)\cdot T(t)$ belgilash kiritamiz. Natijada (3.1.1) tenglama

$$\frac{1}{\beta i}y(x)\cdot T'(t)=y'(1-x)\cdot T(t)+q(x)y(x)\cdot T(t) \qquad (3.2.1)$$

yoki

$$\frac{T'(t)}{\beta iT(t)}=\frac{y'(1-x)+q(x)y(x)}{y(x)} \qquad (3.2.2)$$

ko'rinishda yozilishi mumkin. Oxirgi tenglikning chap qismi faqat t ga , o'ng qismi esa faqat x ga bog'liq funksiyalar bo'lgani uchun , bu tenglik o'zgarmas sondan iborat, ya'ni

$$\frac{T'(t)}{\beta iT(t)}=\frac{y'(1-x)+q(x)y(x)}{y(x)}=\lambda$$

Bundan $y(x)$ funksiya uchun

$$y'(1-x)+q(x)y(x)=\lambda y(x), \qquad (3.2.3)$$

$$y(0)=0 \qquad (3.2.4)$$

xos qiymatlar masalasini, $T(t)$ funksiya uchun esa $T(t)=ce^{\lambda i\beta t}$ ifodani hosil qilamiz.

2. (1.3)- (1.4) masalaning yechimini topamiz. (1.3) tenglamada x ni $1-x$ ga almashtirib, $y(x)=z_1(x)$, $y(1-x)=z_2(x)$ belgilashlar kiritsak,

$$z_1'(x)+q(1-x)z_2(x)=\lambda z_2(x),$$

$$-z_2'(x)+q(x)z_1(x)=\lambda z_1(x)$$

tenglamalarni hosil qilamiz. Agar $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x)\end{pmatrix}$ belgilash kiritsak, bu

tenglamalarni vector matritsa usuli bilan

$$\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} z_1'(x) \\ z_2'(x)\end{pmatrix}+\begin{pmatrix} q(x) & 0 \\ 0 & q(1-x)\end{pmatrix}\cdot\begin{pmatrix} z_1(x) \\ z_2(x)\end{pmatrix}=\lambda\begin{pmatrix} z_1(x) \\ z_2(x)\end{pmatrix},$$

yoki $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x)\end{pmatrix}$ ga nisbatan

$$Bz'(x)+P(x)z(x)=\lambda z(x) \qquad (3.2.5)$$

Dirak sistemasi ko'rinishida yozishimiz mumkin, bu yerda

$$B=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ P(x)=\begin{pmatrix} q(x) & 0 \\ 0 & q(1-x) \end{pmatrix} \quad \text{va} \quad z_2(x)=z_1(1-x).$$

Teskarisi ham o'rinli: agar $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ funksiya (3.2.5) sistemaning yechimi bo'lib, $z_1(x)=z_2(1-x)$ tenglik bajarilsa, u holda $y(x)=z_1(x)$ funksiya (3.2.3) tenglamaning yechimi bo'ladi.

1-lemma. (3.2.5) tenglamaning umumiy yechimi

$$z(x)=z(x,\lambda)=TV(x,\lambda)c \qquad (3.2.6)$$

ko'rinishga ega, bu yerda

$$T=\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix},\ V(x,\lambda)=diag\left(u_1(x)e^{-\lambda ix},\ u_2(x)e^{\lambda ix}\right),$$

$$u_1(x)=\exp\left(i\int_0^x q(t)dt\right), \qquad u_2(x)=\exp\left(-i\int_0^x q(t)dt\right),$$

$c=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, c_k, c_2- ixtiyoriy o'zgarmas sonlar.

Eslatma 1)**.** B matritsani diognal ko'rinishga keltiruvchi, ya'ni

$T^{-1}BT=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ tenglikni qanoatlantiruvchi

$$T_1=\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, T_2=\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, T_3=\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, T_4=\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

matritsalar mavjud bo'lib, bu matritsalarga teskari matritsalar mos ravishda

$$T_1^{-1}=\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, T_2^{-1}=\frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, T_3^{-1}=\frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, T_4^{-1}=\frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$$

ko'rinishga ega.

Eslatma 2)**.** (3.2.5) tenglamaning umumiy yechimi komponentalari bo'yicha quyidagi ko'ronoshga ega:

$$z_1(x)=c_1u_1(x)e^{-\lambda ix}-c_2iu_2(x)e^{\lambda ix},\ \ z_2(x)=-c_1iu_1(x)e^{-\lambda ix}+c_2iu_2(x)e^{\lambda ix}$$

bu yerda

$$u_1(x)=\exp\left(i\int_0^x q(t)dt\right),\ u_1(x)=\exp\left(-i\int_0^x q(t)dt\right)$$

Isboti. Dastlab (3.2.5) sistemadagi $B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ matritsani kanonik ko'ronoshga keltiramiz. B matritsaning xarakteristik tenglamasi $|B-kE|=0 \Leftrightarrow k^2+1=0$ tenglama $\pm i$ ildizlarga ega bo'lgani uchun uning kanonik shakli $D=diag(i,-i)$ ko'rinishda bo'lib bunga $T^{-1}BT=D$ tenglik orqali erishiladi.

Agar $T=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ almashtirish matritsasi bo'lsa, u holda bu tenglikni

$$BT=TD \Leftrightarrow \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\cdot\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\Leftrightarrow\begin{pmatrix}-a_{21} & -a_{22}\\ a_{11} & a_{12}\end{pmatrix}=\begin{pmatrix}a_{11}i & -a_{12}i\\ a_{21}i & -a_{22}i\end{pmatrix}$$

ko'rinishda ifodalashimiz mumkin. Oxirgi tenglikdan $a_{11}=1, a_{21}=-i, a_{12}=-i, a_{22}=1$ bo'lgani uchun almashtirish atritsasi $T=\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$ dan iborat/ Bu matritsaga teskari matritsani topamiz.

$$\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ -i & 1 & 0 & 1\end{array}\right)\to\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ 0 & 2 & i & 1\end{array}\right)\to\left(\begin{array}{cc|cc}1 & -i & 1 & 0\\ 0 & 1 & i/2 & 1/2\end{array}\right)\to\left(\begin{array}{cc|cc}1 & 0 & 1/2 & i/2\\ 0 & 1 & i/2 & 1/2\end{array}\right)$$

bo'lgani uchun bu tenglikdan $T^{-1}=\begin{pmatrix}1/2 & i/2\\ i/2 & 1/2\end{pmatrix}$ ekanligini aniqlaymiz. Endi (1.5) tenglamada $z=Tv$ almashtirish kiritamiz. U holda

$$BTv'(x)+P(x)Tv(x)=\lambda Tv(x)$$

tenglik hosil bo'ladi. Hosil bo'lgan tenglikni chapdan T^{-1} matritsaga ko'paytirilsa

$$T^{-1}BTv'(x)+T^{-1}P(x)Tv(x)=\lambda T^{-1}Tv(x),$$

yoki

$$Dv'(x)+T^{-1}P(x)Tv(x)=\lambda v(x),$$

yoki nihoyat

$$v'(x)+P_1(x)v(x)=\lambda D^{-1}v(x) \qquad (3.2.7)$$

tenglamani hosil qilamiz, bu yerda

$$P_1(x) = D^{-1}T^{-1}P(x)T = D^{-1}q(x),$$

chunki $q(x) = q(1-x)$ simmetriklikdan $P_1(x)$ ham diognal matritsa bo'ladi.

Endi (1.7) sistemani komponentalari bo'yicha yozsak bu sistema ikkita

$$v_1'(x) - iq(x)v_1(x) = -\lambda v_1(x), \qquad v_2'(x) + iq(x)v_2(x) = \lambda v_2(x)$$

tenglamalarga ajraladi. Bu tenglamalarning umumiy yechimlari mos ravishda

$$v_1(x) = v_1(x,\lambda) = c_1 u_1(x) e^{-\lambda ix} \quad , \ v_2(x) = v_2(x,\lambda) = c_2 u_2(x) e^{\lambda ix}$$

Bu yerda $u_1(x), u_2(x)$ lemma shartida aniqlangan funksiyalar, $c_1, c_2 -$ ixtiyoriy o'zgarmas sonlar.

Nihoyat (3.2.7) tenglamaning yechimini matritsa ko'rinishida yozib (3.2.6) formulani hosil qilamiz. Lemma isbotlandi.

2-lemma. (3.2.3) sistemaning umumiy yechimi

$$y(x) = y(x,\lambda) = c\varphi(x,\ \lambda) \tag{3.2.8}$$

ko'rinishga ega, bu yerda

$$\varphi(x,\ \lambda) = u_1(x) e^{-i\int_0^1 q(t)dt} e^{\lambda i(1-x)} - iu_2(x) e^{\lambda i x},\ c - ixtiyoriy \ \ o'zgarmas\ son$$

Isboti. Agar $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$ funksiya (3.2.5) sistemaning yechimi bo'lsa va

$z_2(x) = z_1(1-x)$ bo'lsa, u holda $y(x) = z_1(x)$ funksiya (3.2.3) sistemaning yechimi bo'lishi yuqorida ko'rsatolgan edi. Xususiy holda bundan

$$z_2(1) = z_1(0) \tag{3.2.9}$$

tenglikni hosil qilamiz. (3.2.6) va (3.2.9) dan

$$c_1 u_1(0) - ic_2(0) = -c_1 iu_1(1) e^{-\lambda i} + c_2 u_2(1) e^{\lambda i}$$

yoki

$$c_1\left[u_1(0) + iu_1(1)e^{-\lambda i}\right] = c_2\left[u_2(1)e^{\lambda i} + iu_2(0)\right] \tag{3.2.10}$$

$u_1(0) = 1,\ u_2(0) = 1,\ u_2(1) e^{-i\int_0^1 q(t)dt} = u_1^{-1}(1)$ bo'lgani uchun (1.10) dan

$$c_1\left[1 + iu_1(1)e^{-\lambda i}\right] = c_2 u_2(1) e^{\lambda i}\left[1 + iu_1(1)e^{-\lambda i}\right]$$

Bundan $c_1 = c_2 u_2(1) e^{\lambda i}$ bo'lgani uchun

$$y(x)=z_1(x)=c_1u_1(x)e^{-\lambda i x}-c_2iu_2(x)e^{\lambda i x}=c_2\varphi(x,\lambda)$$

bo'lib, bu (1.8) ni isbotlaydi.

Endi masalaning xos qiymati va xos funksiyalarini topamiz.

3-lemma. (3.2.3)-(3.2.4) chegaraviy masalaning xos qiymati

$$\lambda_n=2\pi n+a, n\in N, \text{ bu yerda } \quad a=\frac{\pi}{2}+\int_0^1 q(t)dt, \qquad (3.2.11)$$

va unga mos xos funksiya

$$y_n(x)=p(1-x)e^{2\pi n i(1-x)}-ip(x)e^{2\pi n i x}, \qquad (3.2.12)$$

bu yerda $p(x)=u_2(x)e^{aox}$.

Isboti. (3.2.4) va (3.2.8) ga ko'ra xos qiymatlarni topish uchun $\varphi(0,\lambda)=0$, yoki

$$u_1(0)e^{-i\int_0^1 q(t)dt}e^{\lambda i(1-0)}-iu_2(0)e^{\lambda i 0}=0, \text{ yoki } e^{i(\frac{\pi}{2}+\int_0^1 q(t)dt)}=e^{i\lambda}$$

Bu tenglikdan

$$\lambda_n=2\pi n+a,\ n\in Z$$

xos qiymatlarni topamiz, bu yerda $a=\frac{\pi}{2}+\int_0^1 q(t)dt$.

Endi masalaning $y_n(x)=\varphi(x,\lambda_n)$ xos funksiyalarini topamiz.

$$u_1(x)=\exp\left(i\int_0^x q(t)dt\right)=\exp\left(i\int_0^1 q(t)dt-i\int_x^1 q(t)dt\right)=\exp\left(a-i\int_0^{1-x} q(t)dt\right)=e^{ix}u_2(1-x)$$

Biz bu yerda yana $q(x)$ potensialni simmetrikligidan foydalandik.

Demak, yuqoridagi belgilashlarga asosan

$$y_n(x,\lambda_n)=u_1(x)e^{-i\int_0^1 q(t)dt}e^{\lambda_n i(1-x)}-iu_2(x)e^{\lambda_n i x}=u_1(x)e^{-ia}\cdot e^{2n\pi i(1-x)}\cdot e^{ia(1-x)}-iu_2(x)\cdot e^{2\pi n i x}\cdot e^{aix}=$$
$$=u_2(1-x)e^{ia(1-x)}e^{2\pi n i(1-x)}-iu_2(x)e^{aix}e^{2\pi n i x}=p(1-x)e^{2\pi n i(1-x)}-ip(x)e^{2\pi n i x}$$

Lemma isbotlandi.

3.3- §. Masala xos qiymati va xos funksiyalarining xossalari.

Dastlab $\{y_n(x)\}$ funksiyalar sistemasining xossalarini tekshiramiz.

4-lemma. $\{y_n(x)\}$ funksiyalar sistemasi $L_2[0,1]$ fazoda to'liq ortonormal sistemani tashkil qiladi.

Isboti. Xos funksiyalari $\{y_n(x)\}$ bo'lgan L operatorni qaraymiz:

$$Ly \equiv y'(1-x)+q(x)y(x), \quad y(0)=0$$

Bu operatorga qo'shma bo'lgan L^* operatorni topamiz. Aytaylik $z(x)\in W_2^1[0,1]$ bo'lsin. U holda

$$(Ly,z)=\int_0^1[y'(1-x)+q(x)y(x)]\overline{z(x)}dx=\int_0^1 y'(1-x)\overline{z(x)}dx+\int_0^1 q(x)y(x)\overline{z(x)}dx=$$

$$=\int_0^1 y'(x)\overline{z(1-x)}dx+\int_0^1 q(x)y(x)\overline{z(x)}dx=y(1)\overline{z(0)}+\int_0^1 y(x)\overline{z'(1-x)+z(x)\overline{q(x)}}dx$$

Bundan $L^*z(x)=z'(1-x)+\overline{q(x)}z(x), \quad z(0)=0$, ammo $q(x)$ haqiqiy funksiya bo'lgani uchun, u holda $L=L^*$ bo'lib bundan xos funksiyalarning ortogonalligi kelib chiqadi.

Endi $\{y_n(x)\}$ funksiyalar sistemasining to'laligini ko'rsatamiz.

Aytaylik $f\in L[0,1]$ bo'lib, f funksiya $y_n, n\in Z$ funklsiyaga orthogonal bo'lsin. U holda

$$(y_n,f)=\int_0^1 y_n(x)\overline{f(x)}dx=\int_0^1\overline{f(x)}\left[p(1-x)e^{2\pi ni(1-x)}-ip(x)e^{2\pi nix}\right]dx=\int_0^1\overline{f(1-x)}p(x)e^{2\pi nix}-$$

$$-i\int_0^1\overline{f(x)}p(x)e^{2\pi nix}dx=\int_0^1\left[\overline{f(1-x)}-i\overline{f(x)}\right]p(x)e^{2\pi nix}=0$$

$\{e^{2\pi nix}\}$ trigonometrik sistema tola bo'lgani uchun oxirgi tenglikdan

$$f(1-x)+if(x)\equiv 0\ ,x\in[0,1] \qquad (3.3.1)$$

ayniyatga ega bo'lamiz. (3.3.1) tenglikda x ni $1-x$ ga almashtirib hosil bo'lgan tenglikni i ga ko'paytirib

$$if(x)+i^2f(1-x)\equiv 0 \qquad (3.3.2)$$

ayniyatga ega bo'lamiz. (3.3.1) va (3.3.2) tengliklarni hadlab qo'shib $f(x)\equiv 0$ ekanligini ko'ramiz. Lemma isbotlandi.

Eslatma. 4-lemmadan (1.11) xos qiymatlar bir karralliligi kelib chiqadi.

5-lemma. Aytaylik $y_n^0(x) = \dfrac{y_n(x)}{\|y_n\|}$ bo'lsin, bu yerda $\|\cdot\| - L_2[0;1]$ fazodagi norma.

U holda $y_n^0(x) = \gamma y_n(x)$, bu yerda

$$\gamma_n = \frac{1}{\sqrt{2}}[1],\ [1] = 1 + O\left(\frac{1}{n}\right)$$

Isboti.

$$\|y_n\|^2 = \int_0^1 y_n(x)\overline{y_n(x)}dx = \int_0^1 p(1-x)\overline{p(1-x)}dx + \int_0^1 p(x)\overline{p(x)}dx +$$
$$+ i\int_0^1 p(1-x)\overline{p(x)}e^{-4\pi inx}dx - \int_0^1 p(x)\overline{p(1-x)}e^{4\pi nix}dx$$

Uchinchi va to'rtinchi integrallarni bo'laklab integrallash, ularga kiruvchi eksponentalarni chegaralanganligi hamda $\|p(x)\| = 1$ ekanligini e'tiborga olsak $\|y_n\|^2 = 2 + O(1/n)$ ekanligini ko'ramiz, bundan esa lemmaning tasdig'I kelib chiqadi.

3.4- §. Xos funksiyalar bo'yicha Fur'ye qatori.

L operatorning $L_2[0;1]$ fazodagi aniqlanish sohasini D_L bilan belgilaymiz.

6-lemma. Agar $f(x) \in D_L$ bo'lsa, u holda bu funksiyaning $\{y_n(x)\}$ xos funksiyalar bo'yicha Fur'ye qatori [0;1] kesmada absolyut va tekis yaqinlashadi.

Isboti. $f(x)$ funksiyaning $\{y_n(x)\}$ xos funksiyalari bo'yicha Fur'ye qatori

$$\sum_{-\infty}^{\infty}(f, y_n)\gamma_n^2 y_n(x) = \sum_{-\infty}^{\infty}(f, y_n^0)\gamma_n^2 y_n^0(x)$$

ko'rinishga ega. Aytaylik μ_0 haqiqiy soni L operatorning xos qiymati bo'lmasin. E birlik operator uchun $(L - \mu_0 E)f = g$ bo'lsin. U holda $f = R_{\mu_0} g$ bu yerda R_λ $- L$ operatorning rezolventasi. Shu bilan birga

$$(L - \mu_0 E)y_n^0 = (\lambda_n - \mu_0)y_n^0,$$

bndan $y_n^0 = (\lambda_0 - \mu_0)R_{\mu_0} y_n^0$ va

$$\left(f, y_n^0\right) = \left(R_{\mu_0} g, y_n^0\right) = \left(g, R_{\mu_0} y_n^0\right) = \frac{1}{\lambda_0 - \mu_0}\left(g, y_n^0\right)$$

Shuning uchun

$$\sum_{-\infty}^{\infty}\left(f, y_n^0\right) y_n^0(x) = \sum_{-\infty}^{\infty}\frac{1}{\lambda_0 - \mu_0}\left(g, y_n^0\right) y_n^0(x)$$

$\dfrac{1}{\lambda_0 - \mu_0} = O\left(\dfrac{1}{n}\right)$ va $\sum_{-\infty}^{\infty}\left|\left(g, y_n^0\right)\right|^2 < \infty$ bo'lgani uchun lemmaning tasdig'i Koshi Bunyakovskiy tengsizligi va $y_n^0(x)$ xos funksiyaning chegaralanganligidan kelib chiqadi.

6-lemmadan $\sum|c_n|$ qatorning yaqinlashishi kelib chiqadi. Shuning uchun $f_0(x) = \sum c_n e^{2\pi n i x}$ funksiya $(-\infty, +\infty)$ oraliqda uzluksiz va davri 1 ga teng bo'lgan davriy funsiyadir, bu yerda $c_n = (f(x), y_n(x))\gamma_n^2$.

7-lemma. $x \in [0;1]$ bo'lganda

$$f_0(x) = \frac{1}{2p(x)}[i\varphi(x) + \varphi(1-x)] \qquad (3.4.1)$$

formula o'rinli.

Isboti. 6-lemmaga ko'ra $x \in [0;1]$ bo'lganda

$$\varphi(x) = \sum_{-\infty}^{\infty}(\varphi, y_n)\gamma_n^2 y_n(x) = \sum_{-\infty}^{\infty} c_n y_n(x) = \sum_{-\infty}^{\infty} c_n\left[p(1-x)e^{2\pi n i(1-x)} - ip(x)e^{2\pi n i x}\right],$$

bundan

$$\varphi(x) = p(1-x) f_0(1-x) - ip(x) f_0(x) \qquad (3.4.2)$$

va

$$\varphi(1-x) = p(x) f_0(x) - ip(1-x) f_0(1-x) \qquad (3.4.3)$$

(3.4.2) va (3.4.3) dan

$$i\varphi(x) + \varphi(1-x) = 2p(x) f_0(x) \qquad (3.4.4)$$

(3.4.4) dan esa (3.4.1) kelib chiqadi.

Eslatma. $f_0(x)$ funksiya davriy bo'lgani uchun [0;1] kesmadagina berilgan qiymati bilan butun son o'qida bir qiymatli aniqlanadi. Shuning uchun $f_0(x)$ funksiya qator bilan emas balki (1.15) formula bilan beriladi.

8-lemma. Agar $\varphi(x)\in C^1[0;1],\ \varphi(0)=\varphi'(1)=0$ bo'lsa, u holda $f_0(x)$ funksiya butun son o'qida uzluksiz differensiallanuvchi bo'ladi.

Isboti. (3.4.1) formuladan $f_0(x)$ funksiyaning [0;1] kesmada uzluksiz differensiallanuvchanligi (kesma chetlarida bir tonlamali hosilalar tushuniladi) kelib chiqadi. $f_0(x)$ davriy funksiya bo'lgani uchun, bu funksiya $x=n, n\in N$ nuqtalardan boshqa butun $(-\infty,+\infty)$ son o'qida uzluksiz differensiallanuvchi bo'ladi. $f_0'(n-0)=f_0'(n+0)$ ekanligini ko'rsatamiz. $f_0(x)$ funksiyaning davriyligiga ko'ra

$$f_0'(0+0)=f_0'(0-0) \qquad (3.4.5)$$

ekanligini ko'rsatish yetarli. (3.4.4) ifodani differensiallab

$$i\varphi'(x)-\varphi'(1-x)=2p'(x)f_0(x)+2p(x)f_0'(x) \qquad (3.4.6)$$

tenglikni hosil qilamiz. (3.4.6), lemma shartlari va

$$f_0(0)=f_0(1),\ f_0'(1-0)=f_0'(0-0)$$

shartlardan

$$2p'(0)f_0(0)+2p(0)f_0'(0+0)=i\varphi'(0),\quad 2p'(1)f_0(0)+2p(1)f_0'(0-0)=-i\varphi'(0)$$

tengliklarga va bulardan

$$2[p'(0)+ip'(1)]f_0(0)+2[p(0)f_0'(0+0)+ip'(1)f_0'(0-0)]=0 \qquad (3.4.7)$$

tenglikni hosil qilamiz. Shu bilan birga

$$p(0)=1,\ p(1)=\exp\left(-i\int q(t)dt\right)e^{ia}=e^{\pi i/2}=i,\ u_2'(x)=-iq(x)u_2(x),$$

$$p'(0)=-iq(0)+ia,\ \ p'(1)=q(1)=a$$

hamda $q(0)=q(1)$. U holda $p'(0)+ip'(1)=0$ bo'lib (3.4.7) dan (3.4.5) kelib chiqadi. Lemma isbotlandi.

Eslatma. $\varphi'(1)=0$ shart tabbiy, chunki barcha xos funksiyalar bu shartni qanoatlantiradi.

3.5- §. Masalaning klassik yechimi.

Fur'ye usuliga ko'ra (3.1.1),(3.1.2) masalaning $u(x,t)$ yechimi formal ko'rinishda

$$\sum_{-\infty}^{\infty}(\varphi, y_n^0)y_n^0(x)e^{\lambda_n \beta it} = \sum_{-\infty}^{\infty} c_n y_n(x)e^{\lambda_n \beta it}, \tag{3.5.1}$$

qator ko'rinishida ifodalanadi, bu yerda $c_n = (f(x), y_n(x))\gamma_n^2$.

9-lemma. Barcha $x \in [0;1]$ va $t \in (-\infty,+\infty)$ lar uchun (3.5.1) qator absolyut va tekis yaqinlashadi vauning uchun quyidagi

$$\sum_{-\infty}^{\infty} c_n y_n(x)e^{\lambda_n \beta it} = e^{a\beta it}\left[p(1-x)f_0(1-x+\beta t) - ip(x)f_0(x+\beta t)\right] \tag{3.5.2}$$

formula o'rinli, bu yerda $p(x) = u_2(x)e^{iax}$.

Isboti. (3.5.1) qatorning yaqinlashishi 6-lemmadan kelib chiqadi. Keyin

$$\sum_{-\infty}^{\infty} c_n y_n(x)e^{\lambda_n \beta ie} = \sum_{-\infty}^{\infty} c_n\left[p(1-x)e^{2\pi ni(1-x)} - ip(x)e^{2\pi nix}\right]e^{\lambda_n \beta it} =$$

$$= e^{a\beta it}\left[p(1-x)\sum_{-\infty}^{\infty} c_n e^{2\pi ni(1-x+\beta t)} - ip(x)\sum_{-\infty}^{\infty} c_n e^{2\pi ni(x+\beta t)}\right]$$

Bundan (3.5.2) kelib chiqadi.

Teorema. Agar $\varphi(x) \in C^1[0;1]$, $\varphi(0) = \varphi'(1) = 0$. $q(x) \in C[0,1], q(x) = q(1-x)$ bo'lsa, u holda (3.1.1),(3.1.2) masalaning klassik yechimi mavjud va u

$$u(x,t) = e^{a\beta it}\left[p(1-x)f_0(1-x+\beta t) - ip(x)f_0(x+\beta t)\right], \tag{3.5.3}$$

ko'rinishga ega, bu yerda $p(x) = \exp\left(aix - i\int_0^x q(t)dt\right)$, $f_0(x)$ davri 1 ga teng bo'lgan davriy funksiya bo'lib [0,1] kesmada

$$f_0(x) = \frac{1}{2p(x)}\left[i\varphi(x) + \varphi(1-x)\right] \tag{3.5.4}$$

Isboti. Yuqorida agar (3.5.4) tenglik bilan berilgan $f_0(x)$ funksiyani butun son o'qida 1 ga teng davr bo'yicha davom ettirsak, uholda son o'qining barcha nuqtasida uzluksiz differensiallanuvchi funksiya bo'lishi ko'rsatilgan edi. Endi (3.5.3) formula bilan berilgan $u(x,t)$ funksiya (3.1.1),(3.1.2) aralash masalaning yechimi bo'lishini ko'rsatamiz.

Dastlab $u(x,t)$ funksiya (1.1) tenglamani qanoatlantirishini ko'rsatamiz.

$$\frac{1}{\beta i}u(x,t)=ae^{a\beta it}\left[p(1-x)f_0(1-x+\beta t)-ip(x)f_0(x+\beta t)\right]+$$
$$+\frac{1}{i}e^{a\beta it}\left[p(1-x)f_0'(1-x+\beta t)-ip(x)f_0'(x+\beta t)\right],$$

$$u_\xi(\xi,t)\Big|_{\xi=1-x}=$$
$$=ae^{a\beta it}\left[-p'(1-\xi)f_0(1-\xi+\beta t)-p(1-\xi)f_0'(1-\xi+\beta t)-p'(1-x)f_0(\xi+\beta t)-ip(\xi)f_0'(\xi+\beta t)\right]\Big|_{\xi=1-x}=$$
$$=e^{a\beta it}\left[-p'(x)f_0(x+\beta t)-p(x)f_0'(x+\beta t)-p'(1-x)f_0(1-x+\beta t)-ip(1-x)f_0'(1-x+\beta t)\right]$$

Hisoblanganlarni (3.1.1) formulaga qo'yib

$$e^{a\beta it}\{f_0(1-x+\beta t)[ap(1-x)+ip'(1-x)-p(1-x)q(x)]+f_0(x+\beta t)[-aip(x)+p'(x)+ip(x)q(x)]+\}$$
$$+f_0'(1-x+\beta t)\left[\frac{1}{i}p(1-x)f_0+ip(1-x)\right]+f_0'(x+\beta t)[-p(x)+p(x)]\ \}$$

tenglikni hosil qilamiz.Oxirgi ikkita kvadrat qavslardagi ifodalar nolga teng. $p(x)$ va $p'(x)$ ifodalarinig oshkor ko'rinishlarini qo'ysak birinchi va ikkinchi kvadratdagi ifodalarining ham nolga tengligini ko'ramiz. Demak $u(x,t)$ funksiya (1.1) tenglamani qanoatlantiradi.

Shu bilan birga $x\in[0,1]$ bo'lganda

$$u(x,0)=p(1-x)f_0(1-x)-ip(x)f_0(x)=\varphi(x)$$

ya'ni boshlang'ich shart bajariladi.

Nihoyat,

$$u(0,t)=e^{a\beta it}\left[p(1)f_0(\beta t)-ip(0)f_0(\beta t)\right]=0,$$

ya'ni chegaraviy shart ham bajariladi. Teorema isbotlandi.

III BOB bo’yicha xulosalar

Bu bobda maxsus potensialga ega bo’lgan involyutsiya qatnashgan xususiy hosilali birinchi tartibli differensial tenglama uchun aralash masalaning yechimi Fur’ye usuli bilan yechish jarayonida quyidagi natijalar bayon qilindi:

- Fur’ye usulini qo’llash bilan involyutsiyaga ega bo’lgan speкtral masala hosil qilindi;
- spektral masalaning xos qiymat va xoc funksiyalarining xossalari tekshirildi;
- xos funksiyalar bo’yicha Fur’ye qatori tuzildi va yaqinlashishga tekshirildi;
- aralash masalaning klassik yechimining ifodalanishi aniqlandi.

X u l o s a

Magistrlik dissertatsiyasida involyutsiya xossasisiga ega bo’lgan differensial tenglamalar bayon etilgan bo’lib, dastlab involyutsiya, kasr chiziqli akslantirishlar uchun involyutsiya hosil bo’lishining yetarli shartlari, involyutsiya hosil qiluvchi dtenglamalarni yuqori tartibli Eyler tenglamalariga keltirish usullari turli xil misollar vositasida bayon etildi.

Dissertatsiyada involyutsiya xossasiga ega bo’lgan differensial tenglamalarning eng sodda hollarini muhokama qilamiz va ularning xususiyatlarini o’rganamiz. Involyutsiya xossasiga ega bo’lgan differensial tenglamalar haqida birinchi ish [3] adabiyotda ko’rsatilgan 1940 yilda Philosific Magazine jurnalining 30- tomining 7-seriyasida e’lon qilingan Silberstein R.ning “ Solution of the equation $f'(x)=f\left(\frac{1}{x}\right)$” mavzudagi hamda И.Я. Винер ning “Дифференциальные уравнения” jurnalining 1969 yil 5-tomida e’lon qilingan “Дифференциальные уравнения с инволюциями” mavzusidagi maqolalaridir. Ammo internet sayitlarida bu sohalar bo’yicha e’lon qilingan maqolalarni deyarli uchratmadik va mustaqil tarzda

$$f'(x)=f\left(\frac{1}{x}\right)$$

tenglamaning umumiy ko’rinishi

$$y(x)=C\sqrt{x}\cos\left(\frac{\pi}{6}-\frac{\sqrt{3}}{2}\ln|x|\right)$$

funksiyadan iborat bo’lishini aniqladik, shundan so’ng И.Я.Винерning maqolasi bo’yicha

$$f'(x)=f\left(\frac{1}{x-a}\right)$$

ko'rinishdagi tenglamalar taxlil qilindi. Shu bilan birga ikkinchi tartibli involyutsya xossasiga ega bo'lgan $f''(x) = f\left(\frac{1}{x}\right)$ tenglamaning umumiy yechimi aniqlash masalari ko'rildi. Involyutsiya xossasiga ega bo'lgan

$$f'(x) = f\left(\frac{1}{x}\right)$$

tenglama birinchi bor Zilbershteyin tomonidan taxlil qilingan bo'lib, taqdim etilayotgan ishda

$$f'(x) = p(x) f\left(\frac{1}{x}\right)$$

ko'rinishdagi tenglamalarga o'tkazish masalasi ko'plab misollarda qarab o'tildi. Tadqiqotlarni involyutsiya xossasiga ega bo'lgan bir jinsli bo'lmagan

$$f'(x) = p(x) f\left(\frac{1}{x}\right) + g(x) + \frac{1}{g(x)}$$

tenglamalar ustida ham olib borish mumkin deb o'ylaymiz.

Umuman dissertatsiyada olingan asosiy natijalar:

a) involyutsiya va uning xossalarini ifodalovchi birinchi tartibli chiziqli differensial tenglamalar tadqiq qilinishi;

b) involyutsiya xossasiga ega bo'lgan Zilbershteyin tipidagi ikkinchi tartibli differensial tenglamasining yechimi keltirilishi;

c) Involyutsiya xossasiga ega bo'lgan

$$f'(x) = p(x) f\left(\frac{1}{x}\right)$$

ko'rinishdagi tenglamalarga oid ko'plab misollarni ko'rilganligi.

Kelgusida involyutsiya xossasiga ega bo'lgan xususiy hosilalari differensial tenglamalar ustida tadqiqotlarni olib borish mumkin, chunki biz internet sayitlaridan olgan [15-20] maqolalarda shu xususda tadqiqotlar o'tkazilganligini aniqladik.

Foydalanilgan adabiyotlar

RO’YXATI

1. Andreev A.A., Analogs of Classical Boundary Value Problems for a Second Order Differential Equation with Deviating Argument. // Differential Equations, Vol. 40, No.8,2004,pp.1192-1194.
2. Андреев А.А.,Шиндяпин И.П. О корректности граничных задач для одного дифференциального уравнения в частных производных с отклоняюшимся аргументом.// аналитические методы в теории дифференциальных и интегральных уравнений. Куйбишев. Куйбишев гос. ун-та,1987, С.51-53.
3. Aleroev T.S., Kurane M., and Malik S.A. Determination of a source term for a time fractional diffusion equation with an integral type over – determining condition. // Electronic Journal of Differential Equations. 2013.V.2013. No.270. P.1-16.
4 Бурлуцкая М.Ш. , Хромов А.П. О классическом решении смешанной задачи для уравнения первого порядка с инволюцией. // Вестник Воронежского университета ,Серия Физика.Математика 2010 № 2 , с.26-33.
5 Бурлуцкая М.Ш. , Хромов А.П. Классическом решение методом Фурье смешанных задач при минимальные требованиях на исходные данные. // Изв.Сарат.ун-та.Нов.сер.Математика.Механика.Информатика.2014.Т.14, вып.2, с.171-198.
6. Винер И.А., Дифференциальные уравнения с инволюцией. // Дифференциальные уравнения 1969, № 5, с.1131-1137.
7. Sadybekov M.A.and Sarsenbi A.M. On the notion of regularity of boundary

value problems for differential equation of second order with dump argument, Mathematical Journal, vol.7,No.1, article 23, 2007 (Russian).

8. Sarsenbi A.M. Unconditional bases related to a nonclassical second –order differential operator, Differential Equations, vol.46, no.4.pp.506-511,2010.

9. Sarsenbi A.M. Unconditional bases related to a nonclassical second –order differential operator, Differential Equations, vol.46, no.4.pp.506-511,2010.

10. Silberstein R. Solution of the equation $f'(x)=f\left(\frac{1}{x}\right)$. // Phil. Mag.7th Series, 30(1940), p.185-186.

11.Карташев А.П.,Рождественский В.Л. Обыкновенные дифференциальные уравнения и вариационного исчисления. М.: Наука,1980.

12. Keselman G.M. On the unconditional convergence of eigenfunction expansions of certain differential operators.// Proceedings of Institutes of Higher Education, vol.2. no.39, pp. 82-93, 1964.

13. Kirane M., Malik S.A. Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time, Applied Mathematics and Computation. 2011, V.218. No.1, 163-170.

14. Kirane M., Malik S.A., Al-Gwaiz M.A. An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions, Mathematical Methods in the Applied Sciences . 2013, V.36, No.9, P.1056-1069.

15. Kopzhassarova A.and Sarsenbi A. Basis Properties of Eigenfunctions of Second-Order Differential Operators wiyh Involyution, Abstract and Applied Analysis Volume 2012 (2012), Article ID 576-843,6 pages. 1,3.

16. Линьков А.В. Обоснование метода Фурье для краевых задач математической физики с инволютивным отклонениям. // Вестн. Самар.ГУ,1999,№ 2, С.60-66.

17. Mihailov V.P. On Riss bases in $L_2[0,1]$. Reports of the Academy of Sciences

of the USSR, vol.144,pp.981-984,1962.

18. Naimark M.A. Linear Differential Operators, Part. II.Ungar.New-York, 1968.

19. Хромов А.П. Смешанная задача для дифференциального уравнения с инволюцией и потенциалом специального вида. // Известия Саратовского университета, Нов.сер. Математика, Механика, Информатика. 2010, ,т.10. вып № 4 ,с.17-22.

20. Watkins W. Modified Wiener equations , IJMMS 27:6 (2001) ,347-356.

Muallif tomonidan mavzu bo`yicha e`lon qilingan ilmiy ishlar.

1. ,,Invalyutsiya qatnashgan tenglamalar uchun xos qiymatlar masalasi.’’ G’.M Mo’minov, T.T Ubaydullayev, L.A Mo’ydinov, N.M Xakimova “Pedogog o’qishlar” yosh olimlar va magistrlar va iqtidorli talabalar maqola va tezislar to’plami. 52-57 bet.

2. ,,Qizdirilgan jismning sovutish haqidagi chegaraviy masalani yechishda Fur’ye usulining qo’llanilishi’’. Mo’minov G’.M, Ubaydullayev T.T, Mo’ydinov L.A Xakimova N.M. “Mikroelektronika, Nanozarralar fizikasi va texnologiyalari” Respublika ilmiy-amaliy anjumani materiallari. 356-358 bet.