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**«Aralash tipdagi tenglamalar nazariyasining ba’zi masalalari»**

**MAVZUSIDA**

**MAGISTRLIK DISSERTTIYASI**

**“5A130101- Matematika (Differensial tenglamalar)” yo’nalishi magistri**

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# Kirish

***Mavzuning dolzarbligi***. Matematik fizika tenglamalari kursidan bizga ma'lumki, aralash tipdagi tenglamalar nazariyasi amaliy ahamiyatga egadir. Bu nazariya, ayniqsa, XXI asrda tez rivojlanib borayotgan zamonaviy differensial tenglamalar nazariyasining bir bo'limi hisoblanadi.

Aralash tipdagi tenglamalar nazariyasidagi masallalarni hal etishda o'zbek olimlaridan akademiklar Salohiddinov M.S.[1-4], Jo'rayev T.J.[5-9] va ularning shogirdlarining ilmiy ishlari muhim o'rin egallaydi.

Aralash parabolik tipdagi 2- tartibli xususiy hosilali chiziqli tenglamalar uchun chegaraviy masalalar bo'yicha birinchi fundamental tadqiqot ishlari fransuz matematigi M.Gevre[10-11] tomonidan hal etilgan.

Ushbu magistrlik dissertatsiyasida aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar tadqiq etilgan bo'lib, u kirish qismi, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yxatidan tuzilgan.

***Ishning maqsadi*** . Aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar yechimining mavjudligi va yagonaligini isbotlash.

***Tadqiqot usuli***. Aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar yechimining yagonaligi A.V.Bitsadze ekstremum prinsipi yordamida ko'rsatiladi, yechimning mavjudligi esa integral tenglamalar nazariyasi asosida isbotlanadi.

***Ilmiy yangiliklar***. Dissertatsiyada olingan natijalar quyidagicha:

1) Quyidagi

$$u_{xx} = k(x)u_t, \qquad (1)$$

- aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar o'rganildi. Bu yerda $x>0$ da $k(x)=x^{\lambda_1}$ , $x<0$ da $k(x)=(-x)^{\lambda_2}$ , $\lambda_1,\lambda_2>0.$

2) A.V. Bitsadzening ekstremum prinsipi yordamida, qo'yilgan masalalar yechimining yagonaligi ko'rsatildi.

3) Grin funksiyasi yoradamida aralash tipdagi buziluvchan parabolik tenglama uchun chegaraviy masalalar yechimi topildi.

4) Lokal va nolokal chegaraviy masalalar yechimining mavjudligi integral tenglamalar nazariyasiga ko'ra isbotlandi.

***Amaliy va nazariy ahamiyati***. Ushbu magistrlik ishi mavzusi fizika, mexanika, biologiya fanlarida uchraydigan amaliy masalalarni o'rganishda katta amaliy ahamiyatga ega bo'lib, dissertatsiyada amaliy va nazariy jihatdan ahamiyatli bo'lgan yangi natijalar olindi. Bu natijalar xususiy hosilali tenglamalar nazariyasidagi aralash tipdagi tenglamalar hamda shu tenglamalarga keltiriladigan amaliy masalalarni hal etishda qo'llanilishi mumkin.

***Ishning aprobatsiyasi***. Mazkur dissertatsiyada olingan natijalar fizika-matematika fakulteti, Matematika kafedrasining fizika-matematika fanlari doktori, dotsent G'.Mo'minov rahbarligidagi ilmiy seminarlarda, ilmiy-amaliy anjumanlarida ma'ruza qilingan.

***E'lon qilingan ishlari.*** Dissertatsiyaning asosiy natijalari [18-21] ishlarda e'lon qilingan.

***Dissertatsiyaning tuzilishi va hajmi.*** Dissertatsiya kirish qismi, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yhatidan tuzilgan.

Ishning umumiy xajmi 65 betdan iborat. Adabiyotlar ro'yhati bo'yicha 22 ta.

***Dissertatsiyaning mazmuni***. Magistrlik dissertatsiyasining kirish qismida dissertatsiya mavzusi bo'yica foydalanilgan adabiyotlarning qisqacha mazmuni, uning dolzarbligi, tadqiqot ishida erishilgan asosiy natijalarning qisqacha mazmuni keltirilgan.

I-bobda dissertatsiya mavzusi uchun zarur boshlang'ich ma'lumotlar berilgan bo'lib, bunda xususiy hosilali differensial tenglamalar haqida bshlang'ich ma'lumotlar, maxsus funksiyalar nazariyasidagi Bessel funksiyalari va ularning ba'zi muhim xossalari keltirilgan.

II-bobda buziluvchan parabolik tipdagi tenglamalar uchun 1- va 2 - chegaraviy masalalar qo'yilgan va Grin funksiyasi yordamida bu masalalar yechimlari ko'rinishi aniqlangan.

III-bobda quyidagi

$$u_{xx} = k(x)u_t, \qquad (1)$$

aralash tipdagi buziluvchan parabolic tipdagi tenglama uchun lokal va nolokal chegaraviy masalalar o'rganilgan. Bu yerda $x > 0$ da $k(x) = x^{\lambda_1}$ , $x < 0$ da $k(x) = (-x)^{\lambda_2}$, $\lambda_1, \lambda_2 > 0$.

3.1-§.da (1) tenglama uchun lokal va nolokal chegaraviy masalalarning qo'yilishi berilgan.

$D$ - soha $x > 0$ da $t = 0,\ x = a_1, t = T$ va $x < 0$ da esa $t = T, x = -a_2$, $t = 0,$ to'g'ri chiziqlarning mos ravishda $OA_1, A_1B_1, B_1B$ va $BB_2, B_2A_2, A_2O$ kesmalari bilan chegaralangan bo'lsin. Bunda $a_i = (2q_i)^{1/2q_i}$, $2q_i = \lambda_i + 2 (i = 1,2).$

$D^+$ va $D^-$ orqali $D$- sohaning mos ravishda $x > 0$ va $x < 0$ bo'lgan holdagi qismlarini belgilaymiz:

$$D^+ = \{(x,t) : 0 < x < a_1, 0 < t \le T\},$$

$$D^{-}=\{(x,t):-a_2<x<0, 0<t\le T\},$$

$$I=\{(x,t):x=0, 0<t\le T\},$$

$$J^{+}=\{(x,t):0<x<a_1, t=0\},\quad J^{-}=\{(x,t):-a_2<x<0, t=0\},$$

$$L^{+}=\{(x,t):x=a_1, 0<t\le T\},\quad L^{-}=\{(x,t):x=-a_2, 0<t\le T\}.$$

Demak, $D=D^{+}\cup I\cup D^{-}$.

**T a' r i f**. (1) tenglamaning $D$- sohadagi regulyar yechimi deb, quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani ataymiz:

1) $u(x,t)\in C^{2,1}(D)$;

2) $u(x,t)$ - $D^{+}\cup D^{-}$ sohada (1) tenglamani qanoatlantiradi.

$D$- sohada (1) tenglama uchun quyidagi masalani o'rganamiz.

**M a s a l a I**. Quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani aniqlang:

1) $u(x,t)\in C(\overline{D})$;

2) $u(x,t)$- (1) tenglamaning $D$- sohadagi regulyar yechimi;

3) $u(x,t)$ - chegaraviy shartlarni qanoatlantiradi:

$$u\big|_{J^{+}}=0,\ u\big|_{J^{-}}=0; \tag{2}$$

$$u(a_1,t)=\varphi_1(t),\ \ 0\le t\le 1, \tag{3}$$

$$u(-a_2,t)=\varphi_2(t),\qquad 0\le t\le 1; \tag{4}$$

4) $\lim\limits_{x\to+0}u_x(x,t_0)=\lim\limits_{x\to-0}u_x(x,t_0)$ , (5)

bu yerda $\varphi_i(t)(i=1,2)$ - berilgan funksiyalar bo'lib,

$$\varphi_i(t) \in C[0,1] \cap C^2(0,1). \qquad (6)$$

**M a s a l a II**. Quyidagi xossalarga ega $u(x,t)$ funksiyani aniqlang:

1) $u(x,t) \in C(\overline{D})$;

2) $u(x,t)$ - (1) tenglamaning $D$ - sohadagi regulyar yechimi;

3) ushbu chegaraviy shartlarni qanoatlantiradi:

$$u\big|_{J^+} = 0, \; u\big|_{J^-} = 0; \qquad (2)$$

$$u(a,t) + \mu_1(t)u(\alpha_1(t),t) = \chi_1(t), \qquad 0 \le t \le T, \qquad (7)$$

$$u(-a,t) + \mu_2(t)u(\alpha_2(t),t) = \chi_2(t), \qquad 0 \le t \le T; \qquad (8)$$

4) $\lim\limits_{x\to+0} u_x(x,t_0) = \lim\limits_{x\to-0} u_x(x,t_0)$, (9)

bu yerda $\mu_i(t), \chi_i(t), \alpha_i(t)(i=1,2)$ - berilgan funksiyalar bo'lib,

$$\mu_i(t), \chi_i(t), \alpha_i(t) \in C[0,T] \cap C^2(0,T]. \qquad (10_i)$$

**M a s a l a III**. Quyidagi xossalarga ega $u(x,t)$ funksiyani aniqlang:

1) $u(x,t) \in C(\overline{D})$;

2) $u(x,t)$ - (1) tenglamaning $D$ - sohadagi regulyar yechimi;

3) $u(x,t)$ - (2), (5) shartlarni va ushbu nolokal shartlarni qanoatlantiradi:

$$u(a,t) + \mu_{01}(t)u(0,t) + \mu_1(t)u(\alpha_1(t),t) = \chi_1(t), \; 0 \le t \le T, \quad (11)$$

$$u(-a,t) + \mu_{02}(t)u(0,t) + \mu_2(t)u(\alpha_2(t),t) = \chi_2(t), \; 0 \le t \le T; \quad (12)$$

bu yerda $\mu_i(t), \mu_{0i}(t), \chi_i(t), \alpha_i(t) (i=1,2)$ - berilgan funksiyalar bo'lib,

$$\mu_i(t), \mu_{0i}(t), \chi_i(t), \alpha_i(t) \in C[0,T] \cap C^2(0,T]. \qquad (13_i)$$

Ko'rinib turibdiki, $\mu_{0i}(t) = \mu_i(t) = 0 (i=1,2)$ bo'lgan holda, masala III dan masala I hosil bo'ladi.

3.2-§.da (1) tenglama uchun qo'yilgan lokal va nolokal chegaraviy masalalar yechimining yagonaligi isbotlangan.

Quyidagi teorema o'rinli.

**Teorema 1(Ekstremum prinsipi).** (1) tenglamaning $\overline{D}$ yopiq sohada uzluksiz $u(x,t)$ yechimi, bu sohada o'zining maksimum va minimum qiymatlariga $L^{+} \cup L^{-} \cup J^{+} \cup J^{-}$ da erishishi mumkin.

Bu teoremani isbotlashda quyidagi lemmalardan foydalanamiz[2]:

**L e m m a 1**. Agar (1) tenglamaning $u(x,t)$ yechim $\overline{D^{+}}$ yopiq sohada musbat maksimumga (manfiy minimumga) $(+0, t_0) \in I$ nuqtada erishsa, u holda

$$\lim_{x \to +0} u_x(x, t_0) < 0 \ \left( \lim_{x \to +0} u_x(x, t_0) > 0 \right)$$

**L e m m a 2**. Agar (1) tenglamaning $u(x,t)$ yechim $\overline{D^{-}}$ yopiq sohada musbat maksimumga (manfiy minimumga) $(-0, t_0) \in I$ nuqtada erishsa, u holda

$$\lim_{x \to -0} u_x(x, t_0) > 0 \ \left( \lim_{x \to -0} u_x(x, t_0) < 0 \right)$$

Teorema 1dan masala I yechimining yagonaligi kelib chiqadi.

**T e o r e m a 2**. Agar

$$|\mu_i(t)| \le 1 \quad (i=1,2) \qquad (14)$$

tengsizliklar bajarilsa, u holda (1) tenglama uchun qo'yilgan masala II yagona yechimga ega.

**T e o r e m a 3**. Agar

$$|\mu_{0i}(t)| + |\mu_i(t)| \leq 1 \quad (i = 1,2) \tag{15}$$

tengsizliklar o'rinli bo'lsa, u holda (1) tenglama uchun qo'yilgan masala III yagona yechimga ega.

Oxirgi 3.3-§ quyidagi yechimning mavjudligi haqidagi teoremalarni isbotlashga bag'ishlangan.

**T e o r e m a 4.** Agar $(3.6_1)$, $(3.6_2)$ shartlar bajarilsa, u holda $D$ sohada masala I ning yagona yechimi mavjud.

**T e o r e m a 5.** Agar $(10_1)$, $(10_2)$ shartlar bajarilsa, u holda $D$ sohada masala II ning yagona yechimi mavjud.

**T e o r e m a 6.** $(13_1)$, $(13_2)$ shartlar o'rinli bo'lsin. U holda $D$ sohada (3.1) tenglama uchun qo'yilgan masala III ning yagona yechimi mavjud.

(1) tenglama uchun qo'yilgan masalalarning $D^+$ va $D^-$ sohadagi yechimlari ushbu

$$u_{xx} = k(x)u_t$$

-buziluvchan parabolik tenglama uchun 1- yoki 2- chegaraviy masalaning yechimi sifatida aniqlanadi.

# I-bob. Boshlang'ich ma'lumotlar

## 1.1-§. Xususiy hosilali differensial tenglamalar nazaruyasidan ba'zi ma'lumotlar

1. $E^n$-Evklid fazosida $D$-ochiq bog'limli sohani qaraymiz va bu sohada biror $x=(x_1,x_2,...,x_n)\in D$ nuqtani olamiz, bunda $x_1,x_2,...,x_n$ - ortogonal dekart koordinatalar sistemasidagi $x$ nuqtaning koordinatalari. $D$ sohada aniqlangan $u=u(x)$ funksiyani qaraymiz.

**Ta'rif 1.** Tartiblangan manfiy bo'lmagan $n$ ta butun sonning $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ ketma-ketligi $n$-tartibli mu'lteindeks deyiladi, $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$ son esa mu'lteindeksning uzunligi deyiladi.

Quyidagi belgilashlarni kiritamiz: $u(x)$ funksiyaning $x\in D$ nuqtadagi $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$ - tartibli hosilasi:

$$D^\alpha u=D^{\alpha_1}{}_1,D^{\alpha_2}{}_2,\dots,D^{\alpha_n}{}_n u=\frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1,\partial^{\alpha_2}x_2,\dots,\partial^{\alpha_n}x_n},$$

u holda $D^\circ u=u(x)$. Xususan, $\alpha=\alpha_i$ da:

$$D^\alpha u=\frac{\partial^{\alpha_i}u}{\partial x_i^{\alpha_i}},\quad D_i u=\frac{\partial u}{\partial x_i},\qquad D_i^2 u=\frac{\partial^2 u}{\partial x_i^2}=u_{x_i x_i}$$

Faraz qilamiz, $F=F(x,...p_\alpha,...)$ funksiya $D$ sohada $x$ nuqtaning va $p_\alpha=p_{\alpha_1},p_{\alpha_2},\dots,p_{\alpha_n}=D^\alpha u$, $\alpha_i=0,1,\dots$ haqiqiy o'zgaruvchining berilgan funksiyasi bo'lib, kamida bitta $\frac{\partial F}{\partial p_m}\neq 0$ ($m=\max|\alpha|$) bo'lsin.

Bundan tashqari, $C^K(D)$ orqali $D$ sohada aniqlangan va k - tartibgacha xususiy hosilalri bilan uzluksiz bo'lgan haqiqiy $u(x)$ funksiyalarning to'plamini belgilaymiz.

**Ta'rif 2.** Quyidagi munosabat

$$F(x,\ldots,D^{\alpha}u,\ldots)=0 \tag{1.1}$$

noma'lum $u(x)=u(x_1,x_2,...,x_n)$ funksiyaga nisbatan $m-$tartibli $n$ o'zgaruvchili xususiy hosilali differensial tenglama deyiladi. Bu yerda $F$ -berilgan funksya.

Agar $F$ funksiya barcha $p_{\alpha}$ $(|\alpha|=0,1,\ldots,m)$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (1) tenglama chiziqli differensial tenglama deyiladi.

Agarda $F$ , $|\alpha|=m$ bo'lganda barcha $p_{\alpha}$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (1) tenglama kvazichiziqli differentsial tenglama deyiladi.

**Ta'rif 3.** Agar $D$ sohada aniqlangan $u(x)$ funksiya (1.1) tenglamada qatnashgan barcha tartibdagi hosilalarri bilan uzluksiz bo'lib, uni ayniyatga aylantirsa, u holda $u(x)$ funksiya (1.1) tenglamaning regulyar yoki klassik yechimi deyiladi.

Xususiy hosilali $m-$tartibli chiziqli differensial tenglamani quyidagicha yozlish mumkin:

$$Lu=\sum_{|\alpha|\le m}a_{\alpha}(x)D^{\alpha}u=f(x) \tag{1.2}$$

ko'rinishda yozib olish mumkun.

Agar $\forall x \in D$ lar uchun (1.2) tenglamaning o'ng tomoni $f(x)=0$ bolsa, u holda (1.2) tenglama bir jinsli, $f(x) \neq 0$ shart bajarilsa, - bir jinsli bo'lmagan tenglama deyiladi.

Agar $u(x)$ va $v(x)$ funktsiyalar bir jinsli bo'lmagan (2) tenglamaning yechimlari bo'lsa, $w(x)=u(x)-v(x)$ ayirma funksiya bir jinsli $(f=0)$ tenglamaning yechimi bo'ladi.

Agarda $u_i(x), i=1,2,\dots,k$ funktsiyalar bir jinsli $(f=0)$ tenglamaning yechimlari bo'lsa, $u(x)=\sum_{i=1}^{k} c_i u_i(x)$ funksiya ham, bu yerda $c_i -$ haqiqiy o'zgarmaslar, shu tenglamaning yechimi bo'ladi.

Xususiy hosilali ikkinchi tartibli chiziqli differensial tenglama

$$\sum_{i,j=1}^{m} A_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} B_i(x)\frac{\partial u}{\partial x_i}+C(x)u=F(x) \qquad (1.3)$$

ko'rinishda yozish mumkin, bu yerda $A_{i,j}$, $B_i$, $C$, $F-D$ sohada berilgan haqiqiy funksiyalar. Bu yerda barcha $x \in D$ da $A_{i,j}=A_{j,i}$ va

$$\sum_{i,j=1}^{n} A_{i,j}^2(x) \neq 0.$$

Shuni aytish lozimki, (1.3) tenglamaning barcha $A_{i,j}$, $i,j=1,\dots,n$ koeffiisientlari nolga teng bo'lgan $x \in D$ nuqtalarda tenglama 2- tartibli bo‘lmay qoladi, ya'ni bu nuqtalarda tenglamaning tartibi buziladi.

2. Xarakteristik forma tushunchasi. n- o'lchovli fazoda 2-tartibli differensial tenglamalarning klassifikasiyasi. Faraz qilamiz, (1.1) tenglamada ishtirok etayotgan $F=F(x,...p_\alpha,...)$ funksiya, $p_\alpha = p_{\alpha_1,\alpha_2,...\alpha_n}$, $|\alpha|=m$ o'zgaruvchilar

bo'yicha uzluksiz 1- tartibli hosilalarga ega bo'lsin. (1.1) tenglama nazariyasida $\lambda_1, \lambda_2, \ldots, \lambda_n$ haqiqiy o'zgaruvchilarga nisbatan ushbu

$$K(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{|\alpha|=m} \frac{\partial F}{\partial p_\alpha} \lambda^\alpha, \quad \lambda^\alpha = \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \ldots, \lambda_n^{\alpha_n} \tag{1.4}$$

$m-$tartibli forma - $m$ darajali bir jinsli ko'phad muhim ro'l o'ynaydi. Bu forma (1) tenglamaga mos bo'lgan *xarakteristik forma* deyiladi.

Ikkinchi tartibli kvazichiziqli

$$\sum_{i,j=1}^{n} A_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \Phi(x, u, u_{x_1}, u_{x_2}, ..., u_{x_n}) = 0 \tag{1.5}$$

differensial tenglama uchun (1.4) forma

$$Q(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i,j=1}^{n} A_{i,j}(x) \lambda_i \cdot \lambda_j \tag{1.6}$$

kvadratik formadan iborat, bu yerda $A_{ij}(x) \in C(D)$.

(1.5) tenglamada erkli o'zgaruvchilarni almashtirib uni soddaroq ko'rinishga keltirishga xarakat qilamiz. Shu maqsadda, teglamada $x = (x_1, x_2, ..., x_n)$ o'zgaruvchilar o'rniga yangi $y = y(x)$, ya'ni

$$y_k = y_k(x_1, x_2, ..., x_n) \qquad k = \overline{1,n}$$

o'zgaruvchilarni

$$y_k = \beta_{k_1} x_1 + \beta_{k_2} x_2 + ... + \beta_{k_n} x_n \quad k = \overline{1,n}$$

tenglik orqali kiritamiz. x – o'zgaruvchilarning biror atrofida $y_k(x) \in C^2(D)$ bo'lsin va ushbu yakobian

$$\frac{D(y_1, y_2, ..., y_n)}{D(x_1, x_2, ..., x_n)} \neq 0$$

deb hisoblaymiz. Demak, bu shartga asosan, x o'zgaruvchilarni y orqali ifodalashimiz mumkin, ya'ni x=x(y). U holda y - yangi o'zgaruvchilar bo'yicha u(x) funksiyaning hosilalari:

$$\frac{\partial u}{\partial x_1} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i},$$

$$\frac{\partial^2 u}{\partial x_j \partial x_i} = \sum_{k,e=1}^{n} \frac{\partial^2 u}{\partial y_e \partial y_k} \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i}.$$

Bularni (1.5) tenglamaga qo'ysak, quyidagi tenglamaga kelamiz

$$\sum_{i,j=1}^{n} A_{i,j}(x) \left\{ \sum_{k,e=1}^{n} \frac{\partial^2 u}{\partial y_e \partial y_k} \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i} \right\} +$$

$$+ \overline{\Phi_0}(y, u, u_{y_1}, ..., u_{y_n}) = 0$$

yoki

$$\sum_{k,e=1}^{n} \overline{A}_{k,e}(y) \frac{\partial^2 u}{\partial y_e \partial y_k} + \Phi_1(y, u, u_{y_1}, ..., u_{y_n}) = 0 \qquad (1.7)$$

Bu yerda

$$\overline{A}_{k,e}(y) = \sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i}, \qquad (1.8)$$

$$\Phi_1(y, u, u_{y_1}, ..., u_{y_n}) =$$

$$= \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i} + \overline{\Phi_0}(y, u, u_{y_1}, ..., u_{y_n}).$$

(1.5) tenglama aniqlangan $D$ sohada $x_0$ nuqtani olamiz, va ushbu belgilashlarni kiritamiz.

$$y_0 = y(x_0), \qquad \beta_{ki} = \frac{\partial y_k(x_0)}{\partial x_i}.$$

U holda (8) formani $x_0$ nuqtada quyidagicha yozish mumkin:

$$\overline{A}_{k,e}(y_0) = \sum_{i,j=1}^{n} A_{i,j}(x) \beta_{e,j} \beta_{k,i} \qquad (1.9)$$

(1.6) kvadratik forma $x_0$ nuqtada

$$Q = \sum_{i,j=1}^{n} A_{i,j}(x_0)\lambda_i \cdot \lambda_j \tag{1.10}$$

ko'rinishda yoziladi.

Ushbu

$$\lambda_i = \sum_{k=1}^{n} \beta_{k,i}\zeta_k \text{ , } \det(\beta_{k,i}) \neq 0 \tag{1.11}$$

maxsus bo'lmagan affin almashtirish yordamida (1.10) kvadratik forma quyidagi ko'rinishga keltiriladi:

$$Q = \sum_{i,j=1}^{n} A_{i,j}(x_0)\left(\sum_{k=1}^{n} \beta_{k,i}\zeta_k\right)\cdot\left(\sum_{e=1}^{n} \beta_{e,i}\zeta_e\right) = \sum_{k,e=1}^{n} \zeta_k\zeta_e\left(\sum_{i,j=1}^{n} A_{i,j}(x_0)\cdot\beta_{ki}\cdot\beta_{ej}\right) =$$

$$= \sum_{k,e=1}^{n} \overline{A}_{ke}(y_0)\zeta_k\zeta_e \tag{1.12}$$

Bu kvadratik formaning koeffitsiyentlari ham (1.9) formula bilan aniqlanadi.

Shunday qilib, (1.5) tenglamani $x_0$ nuqtada $x$ o'zgaruvchilar o'rniga yangi $y = y(x)$ o'zgaruvchilar kiritib soddalashtirish uchun, shu nuqtada (1.10) kvadratik formani maxsus bo'lmagan (1.11) chiziqli almashtirish yordami bilan soddalashtirish yetarlidir.

Bizga algebra kursidan ma'lumki, hamma vaqt shunday maxsus bo'lmagan (1.11) almashtirish mavjud bo'lib, uning yordami bilan (10) kvadratik forma quyidagi ko'rinishga olib kelinadi.

$$Q = \sum_{k=1}^{n} \mu_k \zeta_k^{\ 2}, \tag{1.13}$$

bu yerda $\mu_k$ $k = 1,\ldots,n$ koeffitsientlar 1, -1, 0 qiymatlarni qabul qiladi. Shu bilan birga masbat (manfiy) koeffitsiyentlar soni (inertsiya indeksi) va nolga teng bo'lgan koeffitsiyentlar soni (forma defekti) affin almashtirishga nisbatan invariant, ya'ni bu sonlar faqat (1.10) forma bilan aniqlanib, (1.11) almashtirishning tanlab olinishiga bog'liq bo'lmaydi.

Bu esa (1.5) differensial tenglamani, $A_{i,j}(x)$ koeffitsiyentlarning $x_0$ nuqtada qabul qiladigan qiymatlariga qarab, klassifikatsiya qilish imkonini beradi.

Yuqorida fikrlar asosida (1.7) tenglama

$$\sum_{k=1}^{n} \mu_k u_{y_k y_k} + \overline{\Phi}(y,u,u_{y_1},...,u_{y_n}) = 0 \tag{1.14}$$

ko'rinishda yoziladi.

Ikkinchi tartibli differensial tenglamaning aralash hosilalar qatnashmagan bunday ko'rinishi, odatda uning *kanonik ko'rinishi* deyiladi.

(1.5) tenglamani faqat $n = 2$ bo'lgan holdagina, bitta nuqtada emas, hech bo'lmaganda $x_0 \in D$ nuqtaning biror kichik atrofida kanonik ko'rinishga olib kelish mumkin. Bu holni biz alohida ko'rib chiqamiz chiqamiz.

Agar barcha $\mu_k = 1$ yoki barcha $\mu_k = -1$ $k = 1,\dots,n$ bolsa, yani $Q$ forma mos ravishda musbat yoki manfiy aniqlangan (gefinit) bo'lsa, (1.5) tenglama $x \in D$ nuqtada *elliptik tipdagi* yoki *elliptik tenglama* deyiladi.

Agar $\mu_k$ koeffisiyentlardan bittasi manfiy, qolganlari musbat (yoki aksincha) bo'lsa, (1.5) tenglama $x \in D$ nuqtada *giperbolik tenglama* deb ataladi.

$\mu_k$ koeffisiyentlardan $l$ tasi, $1 < l < n-1$, musbat, qolgan $n-l$ tasi manfiy bo'lsa, (1.5) tenglamaga *ultragiperbolik tipdagi tenglama* deyiladi.

Agar $\mu_k$ koeffisiyentlardan bittasi nolga teng, qolganlari noldan farqli va bir hil ishorali bo'lsa, (1.5) tenglama keng manoda $x \in D$ nuqtada *parabolik tenglama* deb ataladi.

Agar $\mu_k$ koeffisiyentlardan kamida bittasi nolga teng bo'lsa, (1.5) tenglama *keng manoda* $x \in D$ nuqtada *parabolik tenglama* deb ataladi.

Agar (5) tenglama $D$ sohaning xar bir nuqtasida elliptik, giperbolik yoki parabolik bo'lsa, u holda $D$ sohada mos ravishda elliptik, giperbolik yoki parabolik tipdagi tenglama deb ataladi.

*Eslatma*. $A$ matritsaning *xarakteristik sonlari* ushbu $\det(A - \lambda E) = 0$ algebraik tenglamaning ildizlaridan iborat, bu yerda $E$ - birllik matritsa.

(1.5) tenglama berilgan $D$ sohaning ixtiyoriy $x$ nuqtasida $A$ matritsa xarakteristik sonlarning ishorasini aniqlab, (1.5) tenglamani qaysi tipga tegishli ekanligini aniqlab olish mumkin.

Ushbu

$$\sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial \varpi}{\partial x_j} \cdot \frac{\partial \varpi}{\partial x_i} = 0$$

tenglama (1.5) differensial tenglamaga mos xarakteristik tenglama deyiladi.

Agar $\varpi(x_1, x_2, ..., x_n)$ funksiya xarakteristikalar tenglamasini qanoatlantirsa,

$$\varpi(x_1, x_2, ..., x_n) = c, \ c = const$$

tenglama bilan aniqlangan sirt berilgan (1.5) differensial tenglamani *xarakteristik sirti* yoki *xarakteristikasi* deyiladi.

O'zgaruvchlar soni ikkita bo'lganda "xarakteristik sirt" iborasi o'rniga "xarakteristik egri chiziq" iborasi qo'llaniladi.

3. Ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali differensial tenglamalarning klassifikatsiyasi va kanonik ko'rinishlari. $x, y$ - erkli o'zgaruvchilar, $u = u(x, y)$ esa noma'lum funksiya bo'lsin.

Ushbu tenglik

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + F_1(x,y,u,u_x,u_y) = 0 \qquad (1.15)$$

yuqori tartibli hosilalarga nisbatan tenglama chiziqli tenglama deyiladi. $A, B, C, F_1 -$ berilgan funksiyalar.

Agar (1.15) tenglamada $A, B, C -$ koeffisiyentlar $x, y$ lardan tashqari $u, u_x, u_y$ larga ham bog'liq bo'lsa, u holda bunday tenglama kvazi chiziqli tenglama deyiladi.

Agar tenglama $u_{xx}, u_{xy}, u_{yy}$ yuqorori tartibdagi hosilalarga nisbatan chiziqli bo'lgani kabi, $u$ funksiya va uning $u_x, u_y$ 1- tartibli hosilalariga nisbatan chiziqli bo'lsa, tenglama chiziqli tenglama deyiladi:

$$\begin{gathered} A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + a(x,y)u_x + \\ + b(x,y)u_y + c(x,y)u + f(x,y) = 0, \end{gathered} \qquad (1.16)$$

bu yerda $A, B, C, a, b, c, f$ - berilgan funksiyalar.

Agar $f = 0$ bo'lsa, chiziqli tenglama bir jinsli, $f \neq 0$ bo'lsa, - bir jinsli bo'lmagan tenglama deyiladi.

Quyidagi

$$Ady^2 - 2Bdydx + Cdx^2 = 0 \qquad (1.17)$$

yoki

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A}, \qquad (1.18)$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A}. \qquad (1.19)$$

tenglamalar (1.15) tenglamaning xarekteristik tenglamalari deyiladi, ularning yechimlari esa

$$\varphi(x, y) = c, \qquad \psi(x, y) = c \tag{1.20}$$

(1.15) tenglamaning xarakteristikalari deyiladi.

**Ta'rif.** (1.15) tenglama biror $M$ nuqtada

agar $B^2 - AC > 0$ bo'lsa, giperbolik tipdagi,

agar $B^2 - AC = 0$ bo'lsa, parabolik tipdagi,

agar $B^2 - AC < 0$ bo'lsa, elliptik tipdagi tenglama deyiladi.

4. Buzuluvchan differensial tenglamalar. (1.15) tenglamani qaraymiz.

(1.15) tenglama $D$ sohada elliptik tipga tegishli bo'lsin. $D$ sohaning butun chegarasida yoki bu chegaraning biror qismida $B^2 - AC = 0$ bo'lsa, u holda (1.15) tenglama $\overline{D}$ da *buziluvchan elliptik tenglama* deyiladi. $D$ soha chegarasining bu qismi esa (1.15) tenglamaning *buzilish chizig'i* yoki *parabolik chizig'i* deyiladi.

Bunda parabolic chiziq nuqtalarida quyidagi ikkita hol bilan chegaralanamiz, ya'ni $\gamma : B^2 - AC = 0$ parabolik chiziq nuqtalarida :yoki

$$Ady^2 - 2Bdxdy + Cdx^2 \neq 0. \tag{1.21}$$

yoki

$$Ady^2 - 2Bdxdy + Cdx^2 = 0 \tag{1.22}$$

(1.21) tengsizlik shuni bildiradiki, (1.15) tenglama uchun xarakteristik yo'nalish $\gamma : B^2 - AC = 0$ chiziq nuqtalarida bu chiziqning urinma yo'nalishi bilan ustma-ust tushmaydi, (1.22) tenglik esa bu yo'nalishlarni ustma-ust tushishini ko'rsatadi.

Birinchi holda (1.15) tenglama birinchi tur buzuluvchan elliptik tenglama deyiladi, ikkinchi holda esa - ikkinchi tur buzuluvchan elliptik tenglama deyiladi.

Bunday tenglamalarning eng soddalariga

$$yu_{xx} + u_{yy} = 0, \quad (y \geq 0)$$

va

$$yu_{yy} + u_{xx} = 0(y \geq 0)$$

tenglamalarni misol qilib keltirishimiz mumkin. Bu tenglamalar uchun $y = 0$ parabolic buzilish chizig'I bo'ladi. $y > 0$ da tenglamalar elliptic tipga tegishli va mos ravishda ular birinchi tur va ikkinchi tur buzuluvchan elliptik tenglamalardir.

(1.15) tenglama $D$ sohada giperbolik tipga tegishli bo'lsin. $D$ sohaning butun chegarasida yoki bu chegaraning biror qismida $B^2 - AC = 0$ bo'lsa, u holda (1.15) tenglamani $\overline{D}$ yopiq sohada buziluvchan giperbolik tenglama deyiladi; $D$ soha chegarasining bu qismi esa (1.15) tenglama uchun buzilish chizig'i yoki parabolik chizig'i deyiladi. Bunda (1.15) tenglama xarakteristik yo'nalishlari maydoni chegarasida $\gamma : B^2 - AC = 0$ parabolik chiziq holatining ikki holi bilan chegaralanamiz:

1) $\gamma$ parabolik chiziqning xech bir nuqtasida urinma (1.15) tenglamaning xarakteristik yo'nalishi bilan ustma-ust tushmasin, ya'ni $\gamma$ ning hamma yerida

$$Ady^2 - 2Bdxdy + Cdx^2 \neq 0. \tag{1.23}$$

U holda $\gamma$ - (1.15) tenglama xarakteristikalarining o'sish nuqtalarining geometric o'rni bo'ladi;

2) $\gamma$ parabolik chiziqning xar bir nuqtasida urinma (1.15) tenglamaning xarakteristik yo'nalishi bilan ustma-ust tushsin. Bu shuni bildiradiki, $\gamma$ da

$$Ady^2 - 2Bdxdy + Cdx^2 = 0 \qquad (1.24)$$

tenglik o'rinli, ya'ni $\gamma$ chiziq bir vaqtda (1.15) tenglamaning xarakteristikasi bo'ladi [(1.15) tenglama xarakteristikalar oilasining eguvchilari].

1-holda (1.15) tenglama birinchi tur buziluvchan giperbolik tenglama, 2-holda ikkinchi tur buziluvchan giperbolik tenglama deb ataladi.

Misol uchun,

$$yu_{xx} - u_{yy} = 0,\ yu_{yy} - u_{xx} = 0(y \geq 0)$$

tenglamalar mos ravishda 1-tur va 2-tur buziluvchan giperbolik tipdagi tenglamalar bo'lib, $y = 0$ parabolic buzilish chizig'i bo'ladi.

## 1.2-§. Bessel funksiyalari va ularning xossalari

1. Eyler integrallari(Gamma va Beta funksiyalar). Quyidagi

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \quad (\ a > 0,\ b > 0\ ) \qquad (1.25)$$

integral Beta funksiya yoki birinchi tur Eyler integrali deyiladi. U ikki ozgaruvchili *a* va *b* parametrlarning funksiyasini ifodalaydi.

Beta funksiya quyidagi xossalarga ega:

$1^0$. Beta funksiya *a* va *b* ga nisbatan simmetrik, ya'ni

$$B(a,b) = B(b,a)$$

$2^0$. Quyidagi formulalar o'rinli

$$a>1 \ \text{ da } \ B(a,b) = \frac{a-1}{a+b-1} B(a-1,b),$$

$$b>1 \text{ da } B(a,b)=\frac{b-1}{a+b-1}B(a,b-1)$$

Agar $a$ va $b$ lar mos ravishda $m$ va $n$ natural sonlardan iborat bo'lsa, u holda

$$B(m,n)=\frac{(n-1)!(m-1)!}{(m+n-1)!}.$$

$3^0$. Beta funksiya quyidagicha ham ifodalanadi:

$$B(a,b)=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

Xususan,

$$B(a,1-a)=\int_0^{+\infty}\frac{t^{a-1}}{1+t}dt=sin\frac{\pi}{a\pi},\ \ B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$$

bo'ladi.

Ushbu

$$\Gamma(a)=\int_0^{\infty}e^{-t}t^{a-1}dt \qquad (a>0) \qquad (1.26)$$

integral, Lejandr tomonidan Eylerning ikkinchi tur integrali deb nomlangan

va u Г ("Gamma") funksiyani aniqlaydi.

Г funksiyaning sodda xossalarini keltiramiz:

$1^0$. $\Gamma(a)$ funksiya, barcha $a>0$ qiymatlarda, uzluksiz va barcha tartibdagi hosilalarga ega.

$2^0$. $\Gamma(a)$ funksiya uchun $\Gamma(a+1)=a\Gamma(a)$ formula o'rinli.

Bu formula yordamida

$$\Gamma(a+n)=(a+n-1)(a+n-2)...(a+1)a\Gamma(a)$$

tenglikni olish mumkin. Bundan esa $\Gamma(n+1)=n!$

$3^0$. $a\to 0$ da va $a\to+\infty$ da ham $\Gamma(a)=\dfrac{\Gamma(a+1)}{a}\to+\infty.$

$4^0$. Beta va Gamma funksiyalar orasida quyidagicha bog'lanish mavjud

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$5^0$. Quyidagi formula o'rinli

$$\Gamma(a)\Gamma(a-1)=\frac{\pi}{sina\pi}.$$

$a=\frac{1}{2}$ da, bu yerdan

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

$6^0$. Ushbu Lejandr formulasi o'rinli

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a).$$

2.Bessel funksiyalari va ularning xossalari. *Birinchi tur Bessel funksiyalari*. Ushbu tenglama

$$x^2y''+xy'+(x^2-\mu^2)y=0 \qquad (1.27)$$

yoki

$$y''+\frac{1}{x}y'+\left(1-\frac{\mu^2}{x^2}\right)y=0$$

Bessel tenglamasi deyiladi. Bu yerda $\mu$ - o'zgarmas son (1.27) tenglamaning indeksi deyiladi;

$$p(x) = \frac{1}{x}, \qquad q(x) = 1 - \frac{\mu^2}{x^2},$$

Quyidagi

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

gipergeometrik tenglamadan (1.27) tenglamani keltirib chiqaramiz. Shu maqsadda gipergeometrik teglamada quyidagi

$$x = \frac{\xi^2}{4ab}, \qquad dx = \frac{\xi d\xi}{2ab}, \qquad dx^2 = \frac{\xi^2 d\xi^2}{4a^2b^2}$$

almashtirishni bajaramiz. U holda

$$\left(1 - \frac{\xi^2}{4ab}\right)\frac{d^2y}{d\xi^2} + \left[c - \left(\frac{1}{b} + \frac{1}{a} + \frac{1}{ab}\right)\frac{\xi^2}{4}\right]\frac{dy}{d\xi}\frac{1}{\xi} + y = 0$$

tenglama hosil bo'ladi. Bu yerdan $a \to \infty$ va $b \to \infty$ da

$$\xi y'' + 2cy' + \xi y = 0$$

tenglamani olamiz. Endi bu tenglamada

$$y = \xi^{-\mu} z$$

almashtirishni bajaramiz va bu yerda $c = \mu + \frac{1}{2}$ deb olsak,

$$\xi^2 z'' + \xi z' + (\xi^2 - \mu^2)z = 0$$

(1.27) Bessel tenglamasining o'zi hosil bo'ladi.

(1.27) tenglamada ushbu

$$y = x^{\mu} z$$

almashtirish bajarsak, z ga nisbatan quyidagi tenglamani hosil qilamiz:

$$z'' + \frac{2\mu + 1}{x} z' + z = 0 \tag{1.28}$$

(1.27)tenglamaning yechimini ushbu darajali qator o'rinishida izlaymiz

$$z = \sum_{n=0}^{\infty} C_n x^n. \tag{1.29}$$

Bundan $z'$ va $z''$ hosilalarni hisoblaymiz:

$$z' = C_1 + 2C_2 x + 3C_3 x^2 + \cdots + (n+2)C_{n+2} x^{n+1} + \cdots$$

$$\frac{z'}{x} = \frac{C_1}{x} + 2C_2 + 3C_3 x + \cdots + (n+2)C_{n+2} x^n + \cdots$$

$$z'' = 2C_2 + 2 \cdot 3C_3 x + 3 \cdot 4C_4 x^2 + \cdots + (n+1)(n+2)C_{n+2} x^n + \cdots$$

Bularni (1.28) tenglamaga o'rniga qo'yamiz, u holda

$$\frac{2\mu + 1}{x} C_1 + [2C_2 + (2\mu + 1)2C_2 + C_0] +$$

$$+[2 \cdot 3C_3 + (2\mu + 1)3C_3 + C_1]x + [3 \cdot 4C_4 + (2\mu + 1)4C_4 + C_2]x^2 + \cdots$$

$$+[(n+1)(n+2)C_{n+2} + (2\mu + 1)(n+2)C_{n+2} + C_n]x^n + \cdots = 0.$$

Endi aniqmas koeffitsiyentlar usulini qo'llagan holda, $x$ ning barcha darajalari oldidagi koeffitsiyentlarni nolga tenglaymiz:

$$C_1 = 0 \tag{1.30}$$

$$(n+1)(n+2)C_{n+2} + (2\mu + 1)(n+2)C_{n+2} + C_n = 0, n = 1,2, \ldots$$

Oxirgi tenglikdan

$$C_{n+2} = -\frac{C_n}{(n+2)(n+2\mu+1)}, n = 0,1,2, \dots \tag{1.31}$$

(1.30) va (1.31) tengliklardan

$$C_1 = C_3 = C_5 = \dots = C_{2n-1} = \dots = 0$$

$$C_2 = -\frac{C_0}{2(2\mu+2)},$$

$$C_4 = -\frac{C_2}{4(2\mu+4)} = \frac{C_0}{2 \cdot 4(2\mu+2)(2\mu+4)},$$

$$C_{2n} = (-1)^n \frac{C_0}{2 \cdot 4 \cdot \dots \cdot 2n(2\mu+2)(2\mu+4) \dots (2\mu+2n)} =$$

$$= (-1)^n \frac{C_0}{2^{2n} \cdot 1 \cdot 2 \cdot \dots \cdot n(\mu+1)(\mu+2) \dots (\mu+n)}$$

ifodalarni hosil qilish mumkin. Koeffitsiyentlarning bu qiymatlarini (1.29) ga olib borib qo'ysak, (1.28) tenglamaning yechimini quyidagi qator ko'rinishida topamiz:

$$z = C_0 \left[ 1 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} \cdot 1 \cdot 2 \cdot \dots \cdot n(\mu+1)(\mu+2) \dots (\mu+n)} \right]. \tag{1.32}$$

Bu yerda $C_0$ - o'zgarmas sonni ixtiyoriy tanlab olishimiz mumkin. Matematik analiz kursidan ma'lum bo'lgan Dalamber alomatiga ko'ra, (1.32) qator $x$ ning barcha qiymatlarida yaqinlashuvchi bo'ladi.

Darajali qatorni, bu qator yaqinlashish o'ralig'ida doimo hadma-had differesiallash mumkin bo'lgani uchun, (1.32) qator bilan ifodalangan $z$ funksiya (1.28) tenglamaning yechimi bo'ladi.

$C_0$ - o'zgarmasni quyidagicha tanlab olamiz:

$$C_0 = \frac{1}{2^\mu \Gamma(\mu+1)}.$$

Ma'lumki,

$$1\cdot 2\cdot \ldots \cdot n = n! = \Gamma(n+1),$$

$$(\mu+1)(\mu+2)\ldots(\mu+n)\Gamma(\mu+1) = (\mu+2)(\mu+3)\ldots$$

$$(\mu+n)\Gamma(\mu+2) = \cdots = (\mu+n)\Gamma(\mu+n) = \Gamma(\mu+n+1)$$

tengliklar o'rinli. Bu tengliklarga asosan (1.32) ni quyidagicha yozamiz:

$$z = \frac{1}{2^{\mu}\Gamma(\mu+1)} +$$

$$+\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2^{2n+\mu}\cdot 1\cdot 2\cdot \ldots \cdot n(\mu+1)(\mu+2)\ldots(\mu+n)\Gamma(\mu+1)} =$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2^{2n+\mu}\Gamma(n+1)\Gamma(\mu+n+1)}.$$

Shunday qilib, (1.28) tenglamaning yechimini aniqladik. Endi bu yechimni $y = x^{\mu}z$ tenglikka qo'ysak, (1.27) tenglamaning yechimini hosil qilamiz va uni $J_{\mu}(x)$ orqali belgilab olamiz:

$$J_{\mu}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\mu}}{\Gamma(n+1)\Gamma(\mu+n+1)}. \qquad (1.33)$$

$J_{\mu}(x)$ funksiya *1- turdagi $\mu$ indeksli* yoki *$\mu$ tartibli Bessel funksiyasi* deyiladi. Bu funksiya (1.27) Bessel tenglamasining yechimlaridan biridir. Xususan, 1)$\mu = 0$ bo'lganda

$$J_0(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{\Gamma^2(n+1)} =$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}. \qquad (1.34)$$

2)$\mu = 1$ bo'lsa,

$$J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{n!\,(n+1)!}.$$

Umuman $\mu$ ning butun musbat qiymatlarida

$$J_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\mu}}{n!\,(n+\mu)!}. \qquad (1.35)$$

(1.34) va (1.35) formulalardan ko'rinadiki, $\mu$ ning $\mu = 0$ yoki ixtiyioriy butun va juft qiymatlarida $J_\mu(x)$ Bessel funksiyasi juft funksiya bo'ladi.

$\mu$ kasr son bo'lganda $J_\mu(x)$ funksiyani $x \geq 0$ lar uchun tekshiramiz. (1.27) tenglamada $\mu^2$ qatnashayotgani sababli, yuqoridagi mulohazalar $\mu$ ni $-\mu$ ga almashtirganda ham (1) tenglamaning yechimiga olib keladi. Shunday qilib, (1.33) funksiyada $\mu$ ni $-\mu$ ga almashtirtirsak, quyidagi funksiyani olamiz:

$$J_{-\mu}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n-\mu}}{\Gamma(n+1)\Gamma(-\mu+n+1)}. \qquad (1.36)$$

$J_{-\mu}(x)$ funksiya ham *1-turdagi* $-\mu$ *indeksli Bessel funksiyasi* deyiladi.

$\mu$ indeks butun son bo'lmagan holda, $J_\mu(x)$ va $J_{-\mu}(x)$ funksiyalar chiziqli bog'liq bo'lmaydi, chunki bu holda ularni ifodalovchi (1.33) va (1.36) qatorlarning boshlang'ich hadlarining koeffitsiyentlari noldan farqli bo'ladi va shuning uchun bu qatorlar $x$ ning turli darajalarini o'z ichiga oladi. Shunday qilib, (1.27) tenglamaning butun bo'lmagan $\mu$ indeks uchun, umumiy yechimi:

$$y = C_1 J_\mu(x) + C_2 J_{-\mu}(x), \qquad (1.37)$$

bu yerda $C_1, C_2$ - ixtiyoriy o'zgarmas sonlar.

Endi turli indeksli Bessel funksiyalari orasidagi munosabatlarni o'rganamiz. Ixtiyoriy $\mu$ da 1-tur Bessel funksiyalari uchun quyidagi formulalar o'rinli:

$$\frac{d}{dx}\left[x^{\mu}J_{\mu}(x)\right] = x^{\mu}J_{\mu-1}(x), \tag{1.38}$$

$$\frac{d}{dx}\left[x^{-\mu}J_{\mu}(x)\right] = -x^{-\mu}J_{\mu+1}(x). \tag{1.39}$$

Haqiqatan ham, 1-tur Bessel funksiyasining ushbu

$$J_{\mu}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\mu}}{\Gamma(n+1)\Gamma(\mu+n+1)}$$

ifodasini (1.38) ni chap tomoniga qo'ysak,

$$\frac{d}{dx}\left[x^{\mu}J_{\mu}(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+2\mu}}{2^{2n+\mu}\Gamma(n+1)\Gamma(\mu+n+1)}\right] =$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{2(n+\mu)x^{2n+2\mu-1}}{2^{2n+\mu}\Gamma(n+1)(\mu+n)\Gamma(\mu+n)} =$$

$$= x^{\mu}\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{x}{2}\right)^{2n+2\mu-1}}{\Gamma(n+1)\Gamma(\mu+n+1-1)} = x^{\mu}J_{\mu-1}(x).$$

(1.39) formula ham xuddi (1.38) formula kabi ko'rsatiladi.

(1.38) formuladan bevosita

$$x^{\mu}J_{\mu}'(x) + \mu x^{\mu-1}J_{\mu}(x) = x^{\mu}J_{\mu-1}(x)$$

yoki

$$xJ_{\mu}'(x) + \mu J_{\mu}(x) = xJ_{\mu-1}(x), \tag{1.40}$$

(1.39) formuladan esa

$$x^{-\mu}J'_{\mu}(x) - \mu x^{-\mu-1}J_{\mu}(x) = -x^{-\mu}J_{\mu+1}(x)$$

yoki

$$xJ'_{\mu}(x) - \mu J_{\mu}(x) = -xJ_{\mu+1}(x) \tag{1.41}$$

formulani hosil qilamiz.

Endi (1.40) va (1.41) formulalarni qo'shamiz, u holda

$$J_{\mu-1}(x) - J_{\mu+1}(x) = 2J'_{\mu}(x), \tag{1.42}$$

ularni ayirish natijasida

$$J_{\mu-1}(x) + J_{\mu+1}(x) = \frac{2\mu}{x}J_{\mu}(x) \tag{1.43}$$

formulalarni olamiz.

## 1.3-§. Ikkinchi tartibli chiziqli parabolik tipdagi tenglamalar. Ekstremum prinsipi

Bizga ma'lumki, issiqlik tarqalish yoki muhitda zarrachalarning diffuziya jarayonlari ushbu umumiy differensial tenglama bilan ifodalanadi

$$\rho\frac{\partial u}{\partial t} = div(pgradu) - qu + F(x,t),$$

bu yerda $gradu = \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\vec{e_k}$ va agar $\vec{p} = p(p_1, p_2, \ldots, p_n)$ - bo'lsa, $div\vec{p} = \sum_{i=1}^{n}\frac{\partial p_i}{\partial x_i}$.

Demak,

$$div(pgradu) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i}\right).$$

Quyidagi tenglama

$$u_t = a^2u_{xx} + f(x,t), \qquad 0 < x < l, t > 0 \tag{1.44}$$

issiqlik o`tkazish tenglamasi deyiladi. Bu yerda $a^2 -$ issiqlik tarqalish koeffitsiyenti.

1-chegaraviy masala. Quydagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiya topilsin:

1)$u(x,t) \in C(0 \leq x \leq 1,\ 0 \leq t \leq T) \cap C^{2,1}(0 < x < l, 0 < t \leq T)$ , $\forall\, T$ - o`zgarmas son:

2) $u(x,t)$ funksiya $(0 < x < l, 0 < t \leq T)$ ochiq sohada (1.44) tenglamani qanoatlantiradi.

3) ushbu

$$u(x,0) = \varphi(x), 0 \leq x \leq l, \tag{1.45}$$

boshlang'ich va

$$u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t), 0 \leq t \leq T \tag{1.46}$$

chegaraviy shartlarni qanoatlantiradi.

2-chegaraviy masala. (1.44) tenglamaning 1), 2), (1.45)_boshlang'ich shartni va

$$u_x(0,t) = \nu_1(t), \qquad u_x(l,t) = \nu_2(t), 0 < t \leq T \tag{1.46'}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimi topilsin.

Maksimum qiymat prinsipini. Quyidagi

$$\Pi = \{x = 0, 0 \leq t \leq T\} \cup \{x = 1, 0 \leq t \leq T\} \cup \{t = 0, 0 \leq x \leq 1\}$$

to'plam berligan bo'lsin. $u(x,t)$ funksiya $0 \leq x \leq 1, 0 \leq t \leq T$ sohada aniqlangan , uzluksiz bo'lib $0 < x < 1, 0 < t \leq T$ ochiq sohada (1.44) tenglamani qanoatlantirsin, u holda $u(x,t)$ funksiya o'zining maksimum va minimum qiymatlariga $x = 0$ yoki $x = 1$ chegaraviy nuqtalarda yoki boshlang'ich momentda (ya'ni Π-sohada) erishadi.

Isbot. Belgilashni kiritamiz:

$$max[u(x,t)] = M.$$

Teskarisini faraz qilaylik. $u(x,t)$ funksiya biror $(x_0, t_0) \notin \Pi$ nuqta o'zining maksimum qiymatiga erishsin.Ya'ni , $u(x_0,t_0) = M + \varepsilon, \varepsilon > 0,$ u holda matematik kursidan malumki,

$u_x(x_0,t_0)=0$ , $u_{xx}(x_0,t_0)\leq 0$

$u_t(x_0,t_0)\geq 0$ ($t_0 < T$ da $u_t(x_0,t_0)=0$ $t_0$Tda),

$u_t(x_0,t_0)\geq 0$)

Quyidagi yordamchi funksiyani kiritamiz.

$v(x,t) = u(x,t) + k(t_0 - t)$, bu yerda $k$ - o'zgarmas son.

Bir tomondan $v(x_0,t_0) = u(x_0,t_0) = M + \varepsilon$, $k$ - o'zgarmas sonni quyidagicha tanlab olamiz.

$$k(t_0 - t) < kT < \varepsilon,\ 0 < k < \frac{\varepsilon}{T}.$$

Shunga asosan max $\lfloor v(x,t)\rfloor \leq M + \varepsilon$. .Shunday qilib , ikkinchi tomondan quyidagini hosil qildik.

$$\lfloor v(x,t)\rfloor_{\Pi} < \lfloor u\rfloor_{\Pi} + \varepsilon.|$$

$v(x,t)$ funksiya haqidagi quyidagi malumotlarga ega bo'ldik.

1)П sohada $v(x,t)$ funksiya M+ε dan aniq kichkina.

2)П sohaga tegishli bo'lmagan $(x_0,t_0)$nuqtalarda M+ε gat eng.

Demak , $v(x,t)$fuksiya П ga tegishli bo'lmagan nuqtada maksimumga erishadi . Ya'ni , shunday $(x_1,t_1) \notin \Pi$ nuqta topiladiki, $v_{xx}(X_1,t_1)\leq 0$ va $v_t(X_1,t_1)\geq 0$

Bundan $u_{xx}(x_1,t_1)\leq 0$ va $u_t(x_1,t_1) = v_t(x_1,t_1)+k>0$.

Zidlikka duch keldik. Chunki, bu yerdan $u_t>a^2u_{xx}$ kelib chiqadi.

Demak , farazimiz noto'g'ri ekan. $u(x,t)$funksiya o'zining maksimum nuqtaga faqat П da erishadi. Minimum qiymati uchun ham huddi shunday isbotlanadi. Bunda –u=v almashtirish bajariladi.

Yagonalik teoremasi. Agar $u_1$(x,t), $u_2(x,t)$ funksiyalar

$0 \leq x \leq 1, 0 \leq t \leq T$ sohada aniqlangan, uzluksiz bo'lib ,

$$u_t = a^2u_{xx} + f(x,t), \qquad 0 < x < l, t > 0$$

issqlik o'tkazish tenglamasini hamda bir hil boshlang'ich va chegaraviy shartlarni qanoatlantirsa yani

$u_1(x,0)=u_2(x,0)=\varphi(x)$, $u_1(0,t)= u_2(0,t)=\mu_1(t)$, $u_1(1,t)=u_2(1,t)=\mu_2(t)$

bo'lsa , u holda $u_1(x,t)=u_2(x,t)$.

Isbot. Quyidagi funksiyani qaraymiz

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

$u_1(x,t)$ va $u_2(x,t)$lar $0 \le t \le T$ da uzluksiz bo'lgan uchun, v(x,t)funksiya ham shu sohada uzluksiz bo'ladi.

$v(x,t)$ funksiya $0 < x < 1, t > 0$ sohada $v_t = a^2 v_{xx}$-issiqlik o'tkazish tenglamasini qanoatlantiradi. Shunday qilib, $v(x,t)$ uchun maksimum prinsipini qo'llasak , bu funksiya o'zining maksimum qiymatlariga t=0 yoki x=0 yoki x=1 da erishadi. Shu bilan birga shartga ko'ra

$$v(x,0) = 0, v(0,t) = 0, v(l,t) = 0,$$

shuning uchun $v(x,t) = 0$, bundan $u_1(x,t) \equiv u_2(x,t)$. Bu yerdan kelib chiqadiki , 1-chsgaraviy masalaning yechimi yagona . Endi maksimum prinsipidan kelib chiqadigan natijalarni isbot qilamiz.

1.Agar issiqlik o'tkazish tenglamasining $u_1(x,t)$, $u_2(x,t)$ ikkita yechimlarni ushbu

$$u_1(x,0) \le u\ (x,0),$$

$$u_1(0,t) \le u_2(0,t)\ ,\quad u_1(l,t) \le u_2(l,t)$$

shartlarni qanoatlantirsa , u holda

$$u_1(x,t) \le u_2(x,t), \qquad \forall(x,t) \in (\ 0 \le x \le 1, 0 \le t \le T)$$

Isbot. Haqiqatdan ham, $v(x,t)=u_2(x,t)-u_1(x,t)$ funksiya uchun maksimum prinspni o`rinli bo`lib,

$$v(x,0) \ge 0, \qquad v(o,t) \ge 0, \quad v(l,t) \ge 0.$$

Shuning uchun maksimum prinspiga asosan, barcha 0<x<1, 0<t<T lar uchun $v(x,t) \ge 0$ bo`ladi. Aks holda v(x,t) funksiya manfiy mimimum qiymatga 0<x<1, 0<t<T sohada ega bo`ladi.

2. Agar issiqlik o`tkazish tenglamasining uchta u(x,t), $\underline{u}(x,t)$, $\bar{u}(x,t)$ yechimlari ushbu

$\underline{u}(x,t) \leq u(x,t) \leq \bar{u}(x,t), t = 0, x = 0$, va $x = 1$ da shartlarni qanoatlantirsa, u holda bu tengsizliklar barcha $(x,t) \in (0 \leq x \leq 1, 0 \leq t \leq T$ lar uchun ham o`rinli bo`ladi.

Isbot: bu tasdiqni isbotlash uchun 1-tasdiqni $u(x,t) \leq \bar{u}(x,t)$ $u(x,t) \leq \underline{u}(x,t)$, funksiyalarga qo`llash kerak.

1.Agar issiqlik o`tkazish tenglamasining ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimlarri uchun ushbu

$| u_1(x,t) - u_2(x,t)| \leq \varepsilon$, t=0, x=0, x=1 da tengsizliklar o`rinli bo`lsa, u holda bu tengsizlik barcha

$(x,t) \in (0 \leq x \leq 1, 0 \leq t \leq T$ lar uchun o`rinli bo`ladi.

Isbot: Bu tasdiq 2-natijadan kelib chiqadi. Buning uchun uni issiqlik o`tkazish tenglamasining quydagi

$$\underline{u}(x,t) = -\varepsilon$$

$$\underline{u}(x,t) = u_1(x,t) - u_2(x,t)$$

$$\bar{u}(x,t) = \varepsilon$$

yechimlari uchun qo`llash kerak.

Cheksiz to`g`ri chiziq uchun yagonalik teoremasi.

$$u_t = a^2 u_{xx} + f(x,t), \qquad -\infty < x < +\infty, t > 0 \tag{1.44'}$$

cheksiz to`g`ri chiziq uchun masalalarni yechishda nomalum funksiyani $-\infty < x < +\infty, t > 0$ sohada chegaralanmagani talab qilish muhimdir, ya`ni

$$|u(x,t)| < M, -\infty < x < +\infty, t > 0$$

Teorema. Agar (1) issiqlik o`tkazish tenglamasining $u_1(x,t)$ va $u_2(x,t)$ uzluksiz va barcha

$(x,t) \in (-\infty < x < +\infty,\ t > 0)$lar uchun chegaralangan va

$$u_1(x,0) = u_2(x,0) \quad -\infty < x < +\infty,$$

bo`lsa, u holda

$$u_1(x,t) = u_2(x,t) \quad -\infty < x < +\infty,\ t \geq 0$$

Isbot. Ushbu $v(x,t)=u_1(x,t)=u_2(x,t)$ funksiyani qaraymiz $v(x,t)$ funksiya uzluksiz,

$$u_t = a^2 u_{xx}$$

issiqlik o`tkazish tenglamasini qanoatlantiradi.

$-\infty < x < +\infty, t > 0$ butun sohada chegaralanmagan

$$|v(x,t)| \leq |u_1(x,t)| + |u_2(x,t) < 2M$$

Va v(x,t)=0 shartni qanoatlantiradi.

Oraliq uchun masala yechimining yagonaligini isbot qilishda maksimum prinsipini qo`llaylik. Bu yerda esa, yani chegaralanmagan soha uchun uni qo`llash mumkin emas. Maksimum prinsipini qo`llash uchun

$$|x| \leq L$$

sohani qaraylik, bu yerda L-yordamchi o`zgarmas son. Ushbu

$$V(x,t) = \left(\frac{4M}{L^2}\right)\left(\frac{x^2}{2v} + a^2 t\right)$$

funksiyani qaraylik. V(x,t)-uzluksiz, fifferensiallanuvchi (1) ssiqlik o`tkazish tenglamasini qanoatlantiradi: $V_t = a^2 V_{xx}$ va quydagi hossalarga ega:

$$V(x,0) \geq |V(x,0)=0, \quad V(\pm L, t) \geq 2M \geq V(\pm L, t).$$

$|x| \leq L$ soha uchun maksimum prinsipini qo`llasak,

$$-\frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right) \leq V(x,t) \leq -\frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

(x,t)ni biror qiymatini fikrlab va L-ni ihtiyoriy ekanini hisobga olgan holda, uni cheksizlikka intiltirsak, $L \to \infty$ da v(x,t)=0 kelib chiqadi. Teorema isbot bo`ldi.

## **I-bob** bo'yicha xulosa

Aralash parabolik tipdagi 2- tartibli xususiy hosilali chiziqli tenglamalar nazariyasi amaliy jihatdan katta iahamiyatga ega bo'lgan, ayniqsa, XXI asrda tez rivojlanib borayotgan zamonaviy differensial tenglamalar nazariyasining bir bo'limi hisoblanadi.

I-bob uchta paragrafdan iborat bo'lib, mazkur bobning 1.1-§-da xususiy hosialali differensial tenglamalar nazaruyasidan ba'zi ma'lumotlar keltirilgan. Bunda xarakeristik forma, xarakteristik sirt, tenglamani tiplarga ajratish bo'yicha tushunchalar, va shu bilan birga xususan, ikkinchi tartibli ikki o'zgaruvchili, yuqori taribli hosilalarga nisbatan chiziqli xususiy hosilali tenglamalar bo'yicha boshlang'ich ma'lumotlar yoritilgan.

1.2-§-da muhim maxsus funksiyalardan Bessel funksiyalari va ularning xossalari o'rganilgan. Bessel funksiyalarini o'rganish katta ahamiyatga ega bo'lib, dissertasiyada qaraladigan buziluvchan parabolik tipdagi tenglamalar uchun chegaraviy masalalarni yechimini aniqlashda ahamiyatlidir.

1.3-§-da esa ikkinchi tartibli chiziqli parabolik tipdagi tenglamalar haqida dissertasiya mavzusini o'rganishda ahamiyatli bo'lgan zarur ma'lumotlar berilgan bo'lib, parabolik tipdagi tenglamalar uchun ekstremum prinsipi va undan kelib chiqadigan natijalar o]rganilgan.

## II-bob.Buziluvchan parabolik tipdagi tenglamalar

Ushbu bobda buziluvchan parabolik tipdagi tenglama uchun 1-va 2-chegaraviy masalalar qo'yilgan va bu masalalar yechimlari Grin funksiyalari yordamida aniqlangan.

### 2.1-§. Buziluvchan parabolik tipdagi tenglama uchun 1-chegaraviy masala. Grin funksiyasi

Ushbu

$$u_{xx} = k(x)u_t , \qquad (2.1)$$

tenglamani qaraymiz, bu yerda $x > 0$ da $k(x) = x^{\lambda_1}$ , $x < 0$ da $k(x) = (-x)^{\lambda_2}$ , $\lambda_1, \lambda_2 > 0.$

(1)tenglama parabolic tipga tegishli bo'lib, bu tenglama uchun $D^+$ sohada 1-chegaraviy masalani o'rganamiz. $D^+$ sohada tenglama ushbu

$$u_{xx} = x^{\lambda_1} u_t \qquad (2.1_1)$$

ko'rinishda bo'ladi.

**1-chegraviy masala.** Quyidagi xossalarga ega $u(x,t)$ funksiya aniqlansin:

1) $u(x,t) \in C\left(\overline{D^+}\right) \cap C^{2,1}\left(D^+\right)$;

2) $u(x,t)$ funksiya $(2.1_1)$ tenglamani $D^+$ sohada qanoatlantiradi;

3) $u(x,t)$ quyidagi chegaraviy shartlarni qanoatlantiradi:

$$u\,|_{x=0} = \tau(t), \quad 0 \le t \le 1, \qquad (2.2)$$

$$u|_{t=0} = 0, \quad 0 \le x \le a_1 \qquad (2.3)$$

$$u|_{x=a_1} = \varphi_1(t), \quad 0 \le t \le 1, \qquad (2.4)$$

$$\tau(0) = \varphi_1(0) = 0,$$

bu yerda $\tau(t), \varphi_1(t)$ - berilgan funksiyalar bo'lib,

$$\tau(t) \in C[0,1] \cap C^2(0,1) \qquad (2.5)$$

$$\varphi_1(t) \in C[0,1] \cap C^2(0,1).$$

$(2.1_1)$ tenglamada va (2.2) - (2.4) chegaraviy shartlarda quyidagi almashtirishni bajaramiz:

$$s = \frac{1}{(\lambda_1 + 2)^2} x^{\lambda_1 + 2}. \qquad t = t \qquad (2.6)$$

U holda

$$u_t = su_{ss} + \alpha_1 u_s \qquad (2.7)$$

ko'rinishdagi tenglama va

$$u(s,t)|_{s=0} = \tau(t), \quad 0 \le t \le T, \quad , \qquad (2.8)$$

$$u(s,t)|_{t=0} = 0, \quad 0 \le s \le 1, \qquad (2.9)$$

$$u(s,t)|_{s=1} = \varphi_1(t), \quad 0 \le t \le T, \qquad (2.10)$$

$$\tau(0) = \varphi_1(0) = 0$$

chegaraviy shartlar hosil bo'ladi, bu yerda

$$\alpha_1 = \frac{\lambda_1 + 1}{\lambda_1 + 2}.,$$

$D^+$ - soha (2.6) almashtirish natijasida $\Omega^+ = \{(s,t): 0 < s < 1, 0 < t \le 1\}$ sohaga o'tadi.

Quyidagi ko'rinishdagi

$$G_1(s,\eta,t-\theta;\alpha_1) = \sum_{k=0}^{\infty} e^{-\frac{\beta_k^2(t-\theta)}{4}} \left(\frac{s}{\eta}\right)^{\frac{1-\alpha_1}{2}} \frac{J_{1-\alpha_1}(\beta_k\sqrt{s}) J_{1-\alpha_1}(\beta_k\sqrt{\eta})}{J_{2-\alpha_1}^2(\beta_k)} \qquad (2.11)$$

funksiya $\Delta$ sohada (2.7) tenglama uchun 1- chegaraviy masalaning Grin funksiyasi[15] bo'lib, bu yerda

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{\nu + 2k}$$

-birinchi tur Bessel funksiyasi, $\beta_k$ – sonlar esa $J_{1-\alpha_1}(\beta_k)=0$ tenglamaning musbat ildizlari[1-2].

Grin funksiyasi quyidagi xossalarga ega:

1º. $G_1(s,\eta,t-\theta\ ;\alpha_1)$ funksiya $(s,t)\in\Delta$ ning funksiyasi sifatida (2.7) tenglamani qanoatlantiradi:

$$G_{1t}=sG_{1ss}+\alpha_1 G_{1s}\,;$$

2º. $G_1(s,\eta,t-\theta;\alpha_1)$ $(\eta,\theta)\in\Delta$ o'zgaruvchilar bo'yicha (2.7) ga mos qo'shma tenglamani qanoatlantiradi:

$$\eta G_{1\eta\eta}+(2-\alpha_1)G_{1\eta}+G_{1\bar{t}}=0\,;$$

3º. Har qanday $f(s)\in C(0,1)$ va $s\in[0,1]$ uchun ushbu tenglik o'rinli:

$$\lim_{t\to 0}\int_0^1 G_1(s,\eta,t;\alpha_1)\,f(\eta)d\eta=f(s) \qquad (2.12)$$

Belgilashlar kiritamiz:

$$Lu\equiv su_{ss}+\alpha_1 u_s-u_t\,,\,Mv\equiv sv_{ss}+(2-\alpha_1)v_s+v_t\,.$$

(2.7) tenglamaning (2.2)-(2.4) shartlarni qanoatlantiruvchi yechimini topamiz. $\Delta_\varepsilon\subset\Delta$ sohani qaraymiz, bu yerda $\Delta_\varepsilon$ - $AA_1:\ t=0,\ \ A_1B_{1\varepsilon}:\ \ s=1,$ $B_{1\varepsilon}B_{2\varepsilon}:\ \ t=1-\varepsilon$ va $B_{2\varepsilon}A_1: s=0$ kesmalar bilan chegaralangan soha. Bu soha bo'yicha

$$\mathrm{G_1Lu - uMG_1}=0$$

tenglikni $\Delta_\varepsilon$ soha bo'yicha integrallaymiz:

$$\iint\limits_{\Delta_\varepsilon}\left[G_1Lu-MG_1\right]d\eta d\theta=0\,, \qquad (2.13)$$

bundan

$$0=\iint_{\Delta_\varepsilon}\left[G_1(\eta u_{\eta\eta}+\alpha_1 u_\eta-u_\theta)-u(\eta G_{1\eta\eta}+(2-\alpha_1)G_{1\eta}+G_{1\theta})\right]d\eta d\theta=$$

$$=\iint_{\Delta_\varepsilon}\left[(\eta G_1 u_\eta)_\eta-(uG_1)_\eta-(\eta uG_{1\eta})_\eta+\alpha_1(uG_1)_\eta-(uG_1)_\theta\right]d\eta d\theta$$

bo'lib, Grin formulasiga asosan,

$$0=\iint_{\Delta_\varepsilon}\left[(\eta G_1 u_\eta)_\eta-(1-\alpha_1)(uG_1)_\eta-(\eta uG_{1\eta})_\eta-(uG_1)_\theta\right]d\eta d\theta=$$

$$=\int_{OA_1\cup A_1B_{1\varepsilon}\cup B_{1\varepsilon}B_\varepsilon\cup B_\varepsilon O}\left[\eta G_1 u_\eta-u((1-\alpha_1)G_1+\eta G_{1\eta})\right]d\theta+uGd\eta \tag{2.14}$$

hosil bo'ladi. $OA_1:\ t=0,\ \ A_1B_{1\varepsilon}:\ s=1,\ \ B_{1\varepsilon}B_\varepsilon:\ t=1-\varepsilon$ va $B_\varepsilon O: s=0$.

1) $OA_1$, $B_{1\varepsilon}B_\varepsilon$ da $d\theta=0$, shuning uchun

$$\int_{\partial\Delta_\varepsilon}\left[\eta G_1 U_\eta-u((1-\alpha_1)G_1+\eta G_{1\eta})\right]d\theta=$$
$$=\int_{A_1B_{1\varepsilon}}\left[\eta G_1 U_\eta-u((1-\alpha_1)G_1+\eta G_{1\eta})\right]d\theta+$$
$$+\int_{B_\varepsilon O}\left[\eta G_1 U_\eta-u[(1-\alpha_1)G_1+\eta G_{1\eta}]\right]d\theta$$

o'rinli. (2.11) ga ni hisobga olgan holda ushbu

$$\eta G_1(s,\eta,t-\theta;\alpha_1)=$$
$$=\sum_{k=0}^{\infty}e^{-\frac{\beta_k^2(t-\theta)}{4}}s^{\frac{1-\alpha_1}{2}}\frac{J_{1-\alpha_1}(\beta_k\sqrt{s})}{J_{2-\alpha_1}^2(\beta_k)}\eta^{\frac{1+\alpha_1}{2}}J_{1-\alpha_1}(\beta_k\sqrt{\eta})$$

va

$\eta G_1|_{A_1B_1}=\lim\limits_{\eta\to1}\eta G_1$ dan $\eta G_1|_{A_1B_{1\varepsilon}}=0$, $\eta G_1|_{BO}=\lim\limits_{\eta\to0}\eta G_1=0$ dan $G_1|_{B_\varepsilon O}=0$

tengliklarga egamiz. Bular asosida esa quyidagilarni olamiz:

$$\int_{A_1B_{1\varepsilon}} \left[\eta G_1 u_\eta - u((1-\alpha_1)G_1 + \eta G_{1\eta})\right] d\theta =$$

$$= -\int_0^{t-\varepsilon} u[(1-\alpha_1)G_1 + \eta G_{1\eta}]\Big|_{\eta=1} d\theta \qquad ,$$

$$\int_{B_{2\varepsilon}A_1} \left[\eta G_1 u_\eta - u((1-\alpha_1)G_1 + \eta G_{1\eta})\right] d\theta =$$

$$= \int_0^{t-\varepsilon} u[(1-\alpha_1)G_1 + \eta G_{1\eta}]\Big|_{\eta=0} d\theta;$$

2) $A_1B_{1\varepsilon}$, $B_\varepsilon O$ da $d\eta = 0$ bo'lgani uchun quyidagi tenglikni yozish mumkin:

$$\int_{d\Delta_\varepsilon} uG_1 d\eta = \int_{OA_1} uG_1 d\eta + \int_{B_{1\varepsilon}B_\varepsilon} uG_1 d\eta =$$

$$= \int_0^1 u(\eta,0)G_1(s,\eta,t-0;\alpha_1)d\eta -$$

$$- \int_0^1 u(\eta,t+\varepsilon)G_1(s,\eta,t-(t+\varepsilon);\alpha_1)d\eta =$$

$$= \int_0^1 u(\eta\ ,t+\varepsilon)G_1(s,\eta,t-(t+\varepsilon);\alpha_1)d\eta,$$

chunki, $u(\eta,0) = 0.$

Shunday qilib, (2.14) dan quyidagi tenglikni hosil qilamiz

$$0 = \int_0^{t-\varepsilon} u[(1-\alpha_1)G_1 + \eta G_{1\eta}]\Big|_{\eta=0} d\theta - \int_0^{t-\varepsilon} u[(1-\alpha_1)G_1 + \eta G_{1\eta}]\Big|_{\eta=1} d\theta +$$

$$+ \int_0^1 \overline{\varphi}(\eta)G_1(s,\eta,t;\alpha_1)d\eta - \int_0^1 u(\eta\ ,t+\varepsilon)G_1(s,\eta,t-(t+\varepsilon);\alpha_1)d\eta\ ,$$

bundan $\varepsilon \to 0$ da limitga o'tamiz va (2.12) ga ko'ra quyidagini olamiz

$$u(s,t) = \int_0^t \tau(\theta)\left[(1-\alpha_1)G_1 + \eta G_1\right]\Big|_{\eta=0} d\theta -$$
$$- \int_0^t \varphi_1(\theta)\left[(1-\alpha_1)G_1 + \eta G_1 + \eta G_{1\eta}\right]\Big|_{\eta=1} d\theta, \tag{2.15}$$

bu yerda

$$\lim_{\eta\to 0}\left[(1-\alpha_1)G_1+\eta G_{1\eta}\right]=$$

$$=\sum_{k=0}^{\infty} e^{\frac{-\beta_k^2(t-\theta)}{4}} s^{\frac{1-\alpha_1}{2}} \frac{J_{1-\alpha_1}(\beta_k\sqrt{s})}{J_{2-\alpha_1}^2(\beta_k)}\cdot\left(\frac{\beta_k}{2}\right)^{1-\alpha_1}\frac{1}{\Gamma(1-\alpha_1)},$$

$$\lim_{\eta\to 1}\left[(1-\alpha_1)G_1+\eta G_{1\eta}\right]=$$

$$=\sum_{k=0}^{\infty} e^{\frac{-\beta_k^2(t-\theta)}{4}} s^{\frac{1-\alpha_1}{2}} \frac{J_{1-\alpha_1}(\beta_k\sqrt{s})}{J_{2-\alpha_1}(\beta_k)}\cdot\left(-\frac{\beta_k}{2}\right).$$

(2.6) almashtirishga asosan (2.15) dan, $(2.1_1)$ tenglama uchun 1-chegaraviy masalaning yechimini hosil qilamiz:

$$u(x,t)=$$

$$=\frac{\partial}{\partial t}\int_0^t G^{(1)}(x,t-\theta;\alpha_1)\tau(\theta)d\theta+ \quad , \qquad (2.16)$$

$$+\frac{\partial}{\partial t}\int_0^t G^{(2)}(x,t-\theta;\alpha_1)\varphi_1(\theta)d\theta.$$

bu yerda

$$G(x,\xi,t-\theta;\alpha_1)=$$

$$=\sum_{k=0}^{\infty} e^{-\frac{\beta_k^2(t-\theta)}{4}}(1-\alpha_1)\sqrt{x\xi}\times \qquad (2.17)$$

$$\times\frac{J_{1-\alpha_1}(\beta_k(1-\alpha_1)\sqrt{x}^{\frac{1}{1-\alpha_1}})J_{1-\alpha_1}(\beta_k(1-\alpha_1)\sqrt{\xi}^{\frac{1}{1-\alpha_1}})}{J_{2-\alpha_1}^2(\beta_k)}$$

\- $(2.1_1)$ tenglama uchun 1-chegaraviy masalaning Grin funksiyasi,

$$G_1^{(1)}(x,t;\alpha_1)=1-(1-\alpha_1)^{2(1-\alpha_1)}x-$$

$$-\int_0^{h_0}\overline{G_1}(x,\xi,t;\alpha_1)\left[1-(1-\alpha_1)^{2(1-\alpha_1)}\xi\right]\xi^{\lambda_0}d\xi \quad , \qquad (2.18)$$

$$G_1^{(2)}(x,t;\alpha_1) = (1-\alpha_1)^{2(1-\alpha_1)}x - \\ -\int_0^{h_0}\overline{G_1}(x,\xi,t;\alpha_1)(1-\alpha_1)^{2(1-\alpha_1)}\xi^{\lambda_0}d\xi. \tag{2.19}$$

Xuddi shunday, $D^-$ sohada (2.1) tenglama

$$u_{xx} = (-x)^{\lambda_1}u_t \tag{$2.1_2$}$$

ko'rinishga ega bo'lib, bu tenglama uchun 1-chegaraviy masalani qo'yamiz:

**1-chegraviy masala.** Quyidagi xossalrga ega $u(x,t)$ funksiya aniqlansin:

1) $u(x,t) \in C\left(\overline{D^-}\right) \cap C^{2,1}\left(D^-\right)$;

2) $u(x,t)$ funksiya $(2.1_2)$ tenglamani $D^-$ sohada qanoatlantiradi;

3) $u(x,t)$ quyidagi chegaraviy shartlarni qanoatlantiradi:

$$u|_{x=0} = \tau(t), \quad 0 \le t \le 1, \tag{2.2}$$

$$u\big|_{t=0} = 0, \quad -a_2 \le x \le 0, \tag{2.3'}$$

$$u\big|_{x=a_1} = \varphi_2(t), \quad 0 \le t \le 1, \tag{2.4'}$$

$$\tau(0) = \varphi_2(0) = 0,$$

bu yerda $\tau(t), \varphi_1(t)$ - berilgan funksiyalar bo'lib,

$$\tau(t) \in C[0,1] \cap C^2(0,1) \tag{2.5'}$$

$$\varphi_2(t) \in C[0,1] \cap C^2(0,1).$$

Bu maslaning yechimini xuddi yuqoridagidek, quyidagi ko'rinishda aniqlash mumkin:

$$u(x,t)=$$

$$=\frac{\partial}{\partial t}\int_0^t G_1^{(1)}(-x,t-\theta;\alpha_1)\tau(\theta)d\theta- \qquad (2.20)$$

$$-\frac{\partial}{\partial t}\int_0^t G_1^{(2)}(-x,t-\theta;\alpha_1)\varphi_2(\theta)d\theta.$$

## 2.2-§. 2-chegaraviy masala

Ushbu paragrafda (2.1) tenglama uchun $D^+$ va $D^-$ - sohalarda 2- chegaraviy masaalni qo'yamiz.

**2-chegaraviy masala.** $(2.1_1)$ tenglamaning $D^+$ sohadagi quyidagi shartlarni qanoatlantiruvchi $u(x,t)\in C\left(\overline{D^+}\right)\cap C^{2,1}\left(D^+\right)$ yechimi topilsin:

$$\frac{\partial u}{\partial x}|_{x=0}=\nu(t),\quad 0<t\le T, \qquad (2.21)$$

$$u|_{t=0}=0,\quad 0\le x\le a_1 \qquad (2.3)$$

$$u|_{x=a_1}=\varphi_1(t),\quad 0\le t\le T, \qquad (2.4)$$

$$\varphi_1(0)=0,$$

bu yerda $\nu(t),\varphi_1(t)$ - berilgan funksiyalar bo'lib,

$$\nu(t)\in C^1(0,T] \qquad (2.21)$$

$$\varphi_1(t)\in C[0,T]\cap C^2(0,T].$$

$(1)$ tenglamaga mos qo'shma tenglama

$$u_{xx}=-x^{\lambda_1}u_t \qquad (2.22)$$

ko'rinishda bo'ladi.

Belgilash kiritamiz:

$$Lu = u_{xx} - x^{\lambda_1} u_t = 0, \tag{2.23}$$

$$L^* v = v_{xx} - x^{\lambda_1} v_t = 0. \tag{2.24}$$

Ta'rif. $(2.1_1)$ tenglama uchun $D^+$ sohada 2-chegaraviy masalaning Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi $G_2(x,\xi,t-\theta;\alpha_1)$ funksiyani ataymiz:

(2.23) va (2.24) dan ko'rish qiyin emaski,

$$\bar{G}_2\, Lu - uL^*\, \bar{G}_2 =$$

$$= \xi G_2 \left( \frac{\partial^2 u}{\partial \xi^2} - \xi^{\lambda_1} \frac{\partial u}{\partial \theta} \right) - u \left( \frac{\partial^2 (\xi G_2)}{\partial \xi^2} + \xi^{\lambda_1} \frac{\partial (\xi G_2)}{\partial \theta} \right). \tag{2.25}$$

(2.25) ni $D_\varepsilon^+$ soha bo'yicha integrallab, quyidagiga ega bo'lamiz:

$$0 = \iint\limits_{D_\varepsilon^+} \left[ \bar{G}_2\, Lu - uL^*\, \bar{G}_2 \right] d\xi d\theta =$$

$$= \iint\limits_{D_\varepsilon^+} \xi G_2 \left( \frac{\partial^2 u}{\partial \xi^2} - \xi^{\lambda_1} \frac{\partial u}{\partial \theta} \right) - u \left( \frac{\partial^2 (\xi G_2)}{\partial \xi^2} + \xi^{\lambda_1} \frac{\partial (\xi G_2)}{\partial \theta} \right) d\xi d\theta =$$

$$= \iint\limits_{D_\varepsilon^+} \left( \frac{\partial (\xi G_2 u_\xi)}{\partial \xi} - \frac{\partial (\xi u (\xi G_2)_\xi)}{\partial \xi} - \frac{\partial (\xi^{\lambda_1+1} u G_2)}{\partial \theta} \right) d\xi d\theta.$$

Endi Grin formulasini qo'llasak,

$$0 = \iint\limits_{D_\varepsilon^+} \left( \frac{\partial (\xi G_2 u_\xi)}{\partial \xi} - \frac{\partial (\xi u (\xi G_2)_\xi)}{\partial \xi} - \frac{\partial (\xi^{\lambda_1+1} u G_2)}{\partial \theta} \right) d\xi d\theta =$$

$$= \int\limits_{\partial D_\varepsilon^+} \left[\xi G_2 u_\xi - u\left(G_2 + \xi G_{2\xi}\right)\right] d\theta + \xi^{\lambda_1+1} u G_2 d\xi =$$

$$= \int\limits_{OA_1 \cup A_1B_{1\varepsilon} \cup B_{1\varepsilon}B_\varepsilon \cup B_\varepsilon O} \left[\xi G_2 u_\xi - u\left(G_2 + \xi G_{2\xi}\right)\right] d\theta + \xi^{\lambda_1+1} u G_2 d\xi .$$

Bu yerdan (2.2), (2.3'), (2,4') shartlar va Grin funksiyasi xossasiga asosan, $(2.1_1)$ tenglama uchun 2-chegaraviy masalaning yechimini hosil qilamiz:

$$u(x,t) =$$

$$= \frac{\partial}{\partial t}\int\limits_0^t G_2^{(1)}\left(x, t-\theta; \alpha_1\right)\nu(\theta) d\theta - \tag{2.26}$$

$$- \frac{\partial}{\partial t}\int\limits_0^t G_2^{(2)}\left(x, t-\theta; \alpha_1\right)\varphi_1(\theta) d\theta,$$

bu yerda

$$G_2^{(1)}(x,t;\alpha_1) = \frac{(1-\alpha_1)^{2(1-\alpha_1)} x - 1}{1-\alpha_1} -$$

$$- \int\limits_0^{a_1} \overline{G}_2(x,\xi,t;\alpha_1) \frac{(1-\alpha_1)^{2(1-\alpha_1)} \xi - 1}{1-\alpha_1} \xi^{\lambda_0} d\xi,$$

$$G_2^{(2)}(x,t;\alpha_1) = (1-\alpha_1)^{2(1-\alpha_1)-1} x -$$

$$- \int\limits_0^{h_0} \overline{G}_2(x,\xi,t;\alpha_1)(1-\alpha_1)^{2(1-\alpha_1)-1} \xi^{\lambda_0} d\xi.$$

## **II-bob** bo’yicha xulosa

Aralash tipdagi tenglama uchun qo’yilgan chegaraviy masalalarni yechimini mavjudligi va yagonaligini isbotlashda, tenglamani tipi saqlanadigan har bir sohada o’rganish talab qilinadi.

Mazkur bob ikkita paragrafga ajratilgan bo’lib, unda buziluvchan parabolik tipdagi tenglamalar uchun o’zi qaralayotagan har bir sohada 1- va 2-chegaraviy masalalar qo’yilgan. Bu masalalar uchun Grin funksiyalari va ularning xossalari o’rganilgan. Grin funksiyalari yordamida qo’yilgan chegaraviy masalalarning yechimlari aniqlangan.

Chegaraviy masalalarning yechimlarini topishda matematik analiz kursidan ma’lum bo’lgan soha bo’yicha oingan ikki karrali integralni soha chrgarasi bo’yicha olingan egri chiziqli integral bilan bog’lovchi Grin formulasi keng qo’llanilgan.

## III-bob. Aralash tipdagi buziluvchan parabolik tenglama uchun chegaraviy masalalar

Mazkur bobda aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar o’rganilgan bo’lib, bu masalalar yechimlarining mavjudigi va yagonaligi isbotlangan.

### 3.1-§. Aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalarning qo’yilishi

Quyidagi

$$u_{xx} = k(x)u_t , \tag{3.1}$$

tenglamani qaraymiz, bu yerda $x > 0$ da $k(x) = x^{\lambda_1}$ , $x < 0$ da $k(x) = (-x)^{\lambda_2}$ , $\lambda_1, \lambda_2 > 0.$

$D$ - soha $x > 0$ da $t = 0,\ x = a_1, t = T$ va $x < 0$ da esa $t = T$ , $x = -a_2,\ t = 0,$ to’g’ri chiziqlarning mos ravishda $OA_1, A_1B_1, B_1B$ va $BB_2, B_2A_2, A_2O$ kesmalari bilan chegaralangan bo’lsin. Bunda $a_i = (2q_i)^{1/2q_i}$ , $2q_i = \lambda_i + 2(i = 1,2).$

Bu tenglama $D$- sohada aralash tipdagi buzuluvchan parabolic tenglamadir[1-2].

$D^+$ va $D^-$ orqali $D$- sohaning mos ravishda $x > 0$ va $x < 0$ bo’lgan holdagi qismlarini belgilaymiz:

$$D^+ = \{(x,t) : 0 < x < a_1, 0 < t \le T\},$$

$$D^- = \{(x,t) : -a_2 < x < 0, 0 < t \le T\},$$

$$I=\{(x,t): x=0, 0<t\le T\},$$

$$J^{+}=\{(x,t): 0<x<a_1, t=0\},\quad J^{-}=\{(x,t): -a_2<x<0, t=0\},$$

$$L^{+}=\{(x,t): x=a_1, 0<t\le T\},\quad L^{-}=\{(x,t): x=-a_2, 0<t\le T\}.$$

Demak, $D=D^{+}\cup I\cup D^{-}$.

**T a’ r i f**. (3.1) tenglamaning $D$ - sohadagi regulyar yechimi deb, quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani ataymiz:

1) $u(x,t)\in C^{2,1}(D)$;

2) $u(x,t)$ - $D^{+}\cup D^{-}$ sohada (1) tenglamani qanoatlantiradi.

$D$ - sohada (1) tenglama uchun quyidagi masalani o’rganamiz.

**M a s a l a I**. Quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani aniqlang:

1) $u(x,t)\in C(\overline{D})$; ;

2) $u(x,t)$ - (3.1) tenglamaning $D$ - sohadagi regulyar yechimi;

3) $u(x,t)$ - chegaraviy shartlarni qanoatlantiradi:

$$u\big|_{J^{+}}=0,\ u\big|_{J^{-}}=0; \tag{3.2}$$

$$u(a_1,t)=\varphi_1(t),\qquad 0\le t\le T, \tag{3.3}$$

$$u(-a_2,t)=\varphi_2(t),\qquad 0\le t\le T; \tag{3.4}$$

4) $\lim\limits_{x\to+0}u_x(x,t_0)=\lim\limits_{x\to-0}u_x(x,t_0)$ , (3.5)

bu yerda $\varphi_i(t)(i=1,2)$ - berilgan funksiyalar bo'lib,

$$\varphi_i(t)\in C[0,T]\cap C^2(0,T). \qquad (3.6_i)$$

**M a s a l a II**. Quyidagi xossalarga ega $u(x,t)$ funksiyani aniqlang:

1) $u(x,t)\in C(\overline{D})$;

2) $u(x,t)$ - (3.1) tenglamaning $D$ - sohadagi regulyar yechimi;

3) $u(x,t)$ - (3.2), (3.5) va quyidagi

$$u(a_1,t)+\mu_1(t)u(\alpha_1(t),t)=\chi_1(t), \qquad 0\le t\le T,$$

(3.7)

$$u(-a_2,t)+\mu_2(t)u(\alpha_2(t),t)=\chi_2(t), \qquad 0\le t\le T; \qquad (3.8)$$

nolokal chegaraviy shartlalarni qanoatlantiradi, bu yerda $\mu_i(t),\chi_i(t),\alpha_i(t)(i=1,2)$ - berilgan funksiyalar bo'lib,

$$\mu_i(t),\chi_i(t),\alpha_i(t)\in C[0,T]\cap C^2(0,T). \qquad (3.9_i)$$

**M a s a l a III**. Quyidagi xossalarga ega $u(x,t)$ funksiya topilsin:

1) $u(x,t)\in C(\overline{D})$;

2) $u(x,t)$ - (3.1) tenglamaning $D$ - sohadagi regulyar yechimi;

3) $u(x,t)$ - (3.2), (3.5) va ushbu nolokal shartlarni qanoatlantiradi:

$$u(a_1,t)+\mu_{01}(t)u(0,t)+\mu_1(t)u(\alpha_1(t),t)=\chi_1(t),\ \ 0\le t\le T, \qquad (3.10)$$

$$u(-a_2,t)+\mu_{02}(t)u(0,t)+\mu_2(t)u(\alpha_2(t),t)=\chi_2(t),\ \ 0\le t\le T; \qquad (3.11)$$

bu yerda $\mu_i(t),\mu_{0i}(t),\chi_i(t),\alpha_i(t)(i=1,2)$ - berilgan funksiyalar bo'lib,

$$\mu_i(t), \mu_{0i}(t), \chi_i(t), \alpha_i(t) \in C[0,1] \cap C^2(0,1). \qquad (3.12_i)$$

## 3.2-§.Yechimining yagonaligi

1. Masala I yechimining yagonaligi. Quyidagi teorema o'rinli.

**Teorema 1(Ekstremum prinsipi).** (3.1) tenglamaning $\overline{D}$ yopiq sohada uzluksiz $u(x,t)$ yechimi, bu sohada o'zining maksimum va minimum qiymatlariga $L^+ \cup L^- \cup J^+ \cup J^-$ da erishishi mumkin.

Bu teoremani isbotlashda quyidagi lemmalardan foydalanamiz[2]:

**L e m m a 1**. Agar (3.1) tenglamaning $u(x,t)$ yechim $\overline{D^+}$ yopiq sohada musbat maksimumga (manfiy minimumga) $(+0, t_0) \in I$ nuqtada erishsa, u holda

$$\lim_{x \to +0} u_x(x,t_0) < 0 \left( \lim_{x \to +0} u_x(x,t_0) > 0 \right) \qquad (3.13)$$

**L e m m a 2**. Agar (3.1) tenglamaning $u(x,t)$ yechim $\overline{D^-}$ yopiq sohada musbat maksimumga (manfiy minimumga) $(-0, t_0) \in I$ nuqtada erishsa, u holda

$$\lim_{x \to -0} u_x(x,t_0) > 0 \left( \lim_{x \to -0} u_x(x,t_0) < 0 \right) \qquad (3.14)$$

Teoremaning isboti. Parabolik tipdagi tenglamalar uchun ekstremum prinsipiga asosan, $D^+$ va $D^-$ sohalarni ichida musbat maksimum va manfiy minimumga erishmaydi.

Shuni ko'rsatamizki, $u(x,t)$ funksiya ekstremum qiymatalrga $I$ oraliqda erishmaydi.

Teskarisini faraz qilamiz. $u(x,t)$ funksiya biror $(0,t_0) \in I$ nuqtada musbat maksimum va manfiy minimumga erishsin. U holda lemma 1ga asosan, $D^+$ sohada musbat maksimum( manfiy minimum)ga erishadigan nuqtada

$$u_x(0,t_0) < 0 \; \left(u_x(0,t_0) > 0\right) \tag{3.15}$$

Ikkinchi tomondan, lemma 2ga asosan $D^-$ sohadan esa quyidagini olamiz

$$u_x(0,t_0) > 0 \; \left(u_x(0,t_0) < 0\right)$$

Bu tengsizlik esa (3.15) ga zid., demak, $I$ oraliqda musbat maksimum va manfiy minimumga erishilmaydi.

Ko'rsatish mm,kinki, (0,T) nuqtada ham musbat maksimum va manfiy minimum mavjud emas.

Shunday qilib, $u(x,t)$ yechim ekstremumga faqat $L^+ \cup L^- \cup J^+ \cup J^-$ da erishadi. Teorema isbotlandi.

$D^+$ va $D^-$ sohalarda birinchi chegaraviy masalani yechimlari ifodasidan shuni ko'rsatish mumkinki, (0,0) nuqtada $u(x,t)$ funksiyaning qiymati 0 ga teng bo'ladi, ya'ni $u(0,0) = 0$.

Endi isbot etilgan ekstremum prinsipi yordamida masala I yechimining yagonaligini ko'rsatamiz.

$u(x,t)$ bir jinsli masala I ning yechimi bo'lsin, ya'ni quyidagi chegaraviy shartlar bajariladi: (3.2) va

$$u(a_1,t) = 0, \quad u(-a_2,t) = 0, \qquad 0 \le t \le T. \tag{3.16}$$

Ekstremum prinsipiga ko'ra, masala I ning $u(x,t)$ yechimi musbat maksimum va manfiy minimumga $L^{+} \cup L^{-} \cup J^{+} \cup J^{-}$ da erishadi. (3.2) va (3.16) ni hisobga olgan holda, $u(x,t)$ funksiyaning uzluksizligiga asosan,

$\overline{D}$ sohada $u(x,t) \equiv 0$ ni olamiz. Shu bilan masala I yechimining yagonaligi isboylanidi.

**T e o r e m a 2**. Agar

$$|\mu_i(t)| \leq 1 \quad (i = 1,2) \tag{3.17}$$

tengsizliklar bajarilsa, u holda (3.1) tenglama uchun qo'yilgan masala II yagona yechimga ega.

I s b o t. Yechimning yagonaligi. Faraz qilamiz, $u(x,t)$ - bir jinsli masalaning yechimi bo'lsin. U holda, parabolik tipdagi tenglamalar uchun ekstremum prinsipiga va lemmaga ko'ra, $u(x,t)$ funksiya o'zining musbat maksimumiga va manfiy minimumiga $L^{+} \cup J^{+} \cup J^{-} \cup L^{-}$ da erishadi[2]. $D^{+}, D^{-}$ -sohalarning ichida va $I$ oraliqda ekstremumlarga erishishi mumkin emas.

Biror $(a_1, t_0) \in L^{+}$ musbat maksimumga erishilsin. Belgilash kiritamiz:

$$M = u(a_1, t_0) = \max_{\overline{D}} |u(x,t)|. \tag{3.18}$$

Ravshanki, $\forall (x,t) \in D$ uchun $|u(x,t)| < M$ o'rinli.

(3.7) shartga ko'ra,

$$u(a_1, t_0) = -\mu_1(t) u(\alpha_1(t_0), t_0),$$

bu yerdan (3.18) tengsizlikka asosan, quyidagini olamiz

$$|u(a_1,t_0)| \le |\mu_1(t_0)||u(\alpha_1(t_0),t_0)| \le M|\mu_1(t_0)| \le M.$$

Bu esa ziddiyat.

Shunday qilib, $u(x,t)$ funksiya $L^+$ da musbat maksimum va manfiy minimumga erishmaydi.

Xuddi shunday, $L^-$ da ham ekstremumlarga erishilmasligini ko'rsatish mumkin.

Demak, $u(x,t)$ funksiya o'zining ekstremumlariga $J^+ \cup J^-$ da erishadi. (3.2) ga asosan $D$ sohada $u(x,t) \equiv 0$ ekanligini olamiz.

**T e o r e m a 3**. Agar

$$|\mu_{0i}(t)| + |\mu_i(t)| \le 1 \quad (i = 1,2) \tag{3.19}$$

tengsizliklar o'rinli bo'lsa, u holda (3.1) tenglama uchun qo'yilgan masala III yagona yechimga ega.

Isbot. Xuddi teorema 2 ning isboti kabi, (3.10) shartga ko'ra, (3.19) ni hisobga olgan holda, quyidagiga ega bo'lamiz

$$|u(a_1,t_0)| \le |\mu_{01}(t)||u(0,t)| + |\mu_1(t_0)||u(\alpha_1(t_0),t_0)| \le$$

$$\le M\left(|\mu_{01}(t)| + |\mu_1(t_0)|\right) \le M,$$

zidlikka duch keldik. Demak, $L^+$ da $u(x,t)$ ekstremumga erishmaydi, shu bilan birga $L^-$ da ham ekstremum qiymatlar mavjud emasligi xuddi shunday ko'rsatiladi.

Shunday qilib, $u(x,t)$ musbat maksimum va manfiy minumumga faqat $J^{+}\cup J^{-}$ da erishadi va (3.2) ga ni hisobga olib, $D$ sohada $u(x,t)\equiv 0$ o'rinli ekanligiga ishonch hosil qilamiz.

## 3.3-§.Yechimining mavjudligi

**T e o r e m a 4.** Agar $(3.6_1)$, $(3.6_2)$ shartlar bajarilsa, u holda $D$ sohada masala I ning yagona yechimi mavjud.

Isbot. Belgilashlar kiritamiz:

$$\tau(t)=u(\pm 0,t),\ \varphi_1(t)=u(a_1,t),\ \varphi_2(t)=u(-a_2,t),\ 0\le t\le T \tag{3.20}$$

va bu yerda $\tau(0)=\tau(T)=\varphi_1(0)=\varphi_2(T)=0.$

Bizga ma'lumki, (3.1) tenglamaning $D^{+}$ va $D^{-}$ sohalarda 1- chegaraviy masalaning yechimlari quyidagi ko'rinishda bo'ladi(II-bob):

$$u^{+}(x,t)=\frac{\partial}{\partial t}\int_0^t G^{(1)}(x,t-\theta;\alpha_1)\tau(\theta)d\theta+$$

$$+\frac{\partial}{\partial t}\int_0^t G^{(2)}(x,t-\theta;\alpha_1)\varphi_1(\theta)d\theta \tag{*}$$

va

$$u^{-}(x,t)=\frac{\partial}{\partial t}\int_0^t G_1^{(1)}(-x,t-\theta;\alpha_1)\tau(\theta)d\theta-$$

$$-\frac{\partial}{\partial t}\int_0^t G_1^{(2)}(-x,t-\theta;\alpha_1)\varphi_2(\theta)d\theta. \tag{**}$$

Bu tengliklardan $x$ - bo'yicha differensiallaymiz va $x=0$ ni qo'yamiz, u holda

$$\nu^{+}(t)=\frac{\partial}{\partial t}\int_{0}^{t} z(t-\theta)\tau(\theta)d\theta+f^{+}(t), \tag{3.21}$$

$$\nu^{-}(t)=-\frac{\partial}{\partial t}\int_{0}^{t} z(t-\theta)\tau(\theta)d\theta+f^{-}(t), \tag{3.22}$$

bu yerda

$$\nu^{+}(t)=\lim_{x\to+0}u_x(x,t_0)\ ,\quad \nu^{-}(t)=\lim_{x\to-0}u_x(x,t_0)\ ,$$

$$z(t-\theta)=\frac{\partial}{\partial x}G^{(1)}(x,t-\theta;\alpha_1),$$

$$f^{+}(t)=\frac{\partial}{\partial t}\int_{0}^{t}\frac{\partial}{\partial x}G^{(2)}(x,t-\theta;\alpha_1)\varphi_1(\theta)d\theta,$$

$$f^{-}(t)=\frac{\partial}{\partial t}\int_{0}^{t}\frac{\partial}{\partial x}G^{(2)}(x,t-\theta;\alpha_1)\varphi_2(\theta)d\theta.$$

(3.5) shartga asosan, $\nu^{+}(t)=\nu^{-}(t)=\nu(t)$. Buni hisobga olgan holda, (3.21) va (3.22) dan $\nu(t)$ funksiyani yo’qotib, quyidagi tenglikka ega bo’lamiz:

$$\frac{\partial}{\partial t}\int_{0}^{t} z(t-\theta)\tau(\theta)d\theta+f^{+}(t)=-\frac{\partial}{\partial t}\int_{0}^{t} z(t-\theta)\tau(\theta)d\theta-f^{-}(t),$$

yoki

$$\frac{\partial}{\partial t}\int_{0}^{t} z(t-\theta)\tau(\theta)d\theta=f(t), \tag{3.23}$$

bu yerda

$$f(t)=-\frac{1}{2}\left[f^{+}(t)+f^{-}(t)\right]. \tag{3.24}$$

(3.23) dan quyidagini olamiz:

$$\tau(t) = \int_0^t K(t,\theta)\tau(\theta)d\theta + f(t), \qquad (3.25)$$

bu yerda

$$|K(t,\theta)| \le C(t-\theta)^{-\alpha_1},$$

$$|f(t)| \le const,$$

$$f(t) \in C^1(\overline{I}).$$

(3.25) integral tenglama Volterra 2-tur integral tenglamasi bo'lib, u $C^1(\overline{I})$ sinfda yotuvchi yagona yechimga ega.

(3.25) dan $\tau(t)$ ni topib, uning qiymatini (3.21) ga qo'yib, $\nu(t)$ funksiyani topamiz. Masala I ning yechimi esa $D^+$ va $D^-$ sohalarda 1- yoki 2- chegaraviy masalalarning yechimlari sifatida aniqlanadi.

**T e o r e m a 5.** Agar $(3.9_1)$, $(3.9_2)$ shartlar bajarilsa, u holda $D$ sohada masala II ning yagona yechimi mavjud.

Isbot. (3.20) belgilashlarga asosida (*) va (**) yechimlarni mos ravishda (3.7) va (3.8) shartlarga qo'yib, quyidagi munosabatlarni olamiz:

$$\varphi_1(t) + \mu_1(t)\frac{\partial}{\partial t}\int_0^t G^{(1)}(\alpha_1(t), t-\theta; \alpha_1)\tau(\theta)d\theta +$$

$$+\mu_1(t)\frac{\partial}{\partial t}\int_0^t G^{(2)}(\alpha_1(t), t-\theta; \alpha_1)\varphi_1(\theta)d\theta = \chi_1(t),$$

va

$$\varphi_2(t) + \mu_2(t)\frac{\partial}{\partial t}\int_0^t G^{(1)}(-\alpha_2(t), t-\theta; \alpha_1)\tau(\theta)d\theta -$$

$$-\mu_2(t)\frac{\partial}{\partial t}\int_0^t G^{(2)}(-\alpha_2(t), t-\theta; \alpha_1)\varphi_2(\theta)d\theta = \chi_2(t)$$

yoki

$$\varphi_1(t) + \int_0^t K_{11}(t,\theta)\tau(\theta)d\theta + \\ + \int_0^t K_{12}(t,\theta)\varphi_1(\theta)d\theta = \chi_1(t), \qquad (3.26_1),$$

$$\varphi_2(t) + \int_0^t K_{21}(t,\theta)\tau(\theta)d\theta + \\ + \int_0^t K_{22}(t,\theta)\varphi_2(\theta)d\theta = \chi_2(t), \qquad (3.26_2),$$

bu yerda

$$K_{11}(t,\theta) = \mu_1(t)\frac{\partial}{\partial t}G^{(1)}(\alpha_1(t), t-\theta; \alpha_1),$$

$$K_{12}(t,\theta) = \mu_1(t)\frac{\partial}{\partial t}G^{(2)}(\alpha_1(t), t-\theta; \alpha_1),$$

$$K_{21}(t,\theta) = \mu_2(t)\frac{\partial}{\partial t}G^{(1)}(-\alpha_2(t), t-\theta; \alpha_1),$$

$$K_{22}(t,\theta) = -\mu_2(t)\frac{\partial}{\partial t}G^{(2)}(-\alpha_2(t), t-\theta; \alpha_1),$$

bo'lib, bu yadrolar $(3.9_1)$, $(3.9_2)$ shartlar va (3.1) tenglama uchun $D^+$, $D^-$ sohalarda qo'yilgan 1-chegaraviy masalalar Grin funksiyasi xossalariga asosan(II-bob), quyidagi baholarga ega bo'ladi:

$$|K_{ij}(t,\theta)| \le const,\ i,j=1,2.$$

Shunday qilib, (3.1) tenglama uchun $D$ sohada qo'yilgan masala II yechimining mavjudligi (3.25), $(3.26_1)$, $(3.26_2)$ - uzluksiz yadroli integral tenglamalar sistemasiga keltirildi. Bu sistemaning integral tenglamalar nazariyasiga asosan, yagona yechimi mavjud bo'lib, ekvivalentlikka ko'ra,

masala II ni yagona yechimi mavjud. Teorema isbotlandi.

Xuddi shunday, quyidagi teorema o'rinli ekanligi isbotlanadi:

**T e o r e m a 6.** $(3.12_1)$, $(3.12_2)$ shartlar o'rinli bo'lsin. U holda $D$ sohada (3.1) tenglama uchun qo'yilgan masala III ning yagona yechimi mavjud.

Masala III yechimining mavjudligi ham ekvivalent ravishda Volterra integral tenglamalar sistemasiga keltiriladi. Integral tenglamalar nazariyasi asosida[5-6] va ekvivalentlikka ko'ra, olingan sistemaning yagona yechimi mavjud bo'ladi.

(3.1) tenglama uchun qo'yilgan masalalarning $D^+$ va $D^-$ sohadagi yechimlari

$$u_{xx} = k(x)u_t$$

-parabolik tenglama uchun 1- chegaraviy masalaning yoki 2- chegaraviy masalaning yechimi sifatida topiladi.

## **III-bob** bo'yicha xulosa

Uchbu bob uchta paragrafdan iborat bo'lib, unda aralash tipdagi buziluvchan parabolik tenglama uchun lokal va nolokal chegaraviy masalalar tadqiq etilgan.

3.1-§-da aralash tipdagi buziluvchan parabolik tenglama uchun chekli sohada lokal va nolokal chegaraviy masalalar qo'yilgan.

3.2-§-da esa lokal va nolokal chegaraviy masalalar yechimining yagonaligi haqidagi teorema ekstremum prinsipi yordamida isbotlangan.

3.3-§.da aralash tipdagi buziluvchan parabolik tenglama uchun chekli sohada qo'yilgan lokal va nolokal chegaraviy masalalar yechimining mavjudligi ko'rsatilgan.

Yechimning mavjudligi integral tenglamalar nazariyasi asosida isbotlangan bo'lib, integral tenglamalarning yadrosi va o'ng tomonlari maxsus funksiyalar nazariyasi asosida o'rganilgan.

## XULOSA

Magistrlik dissertatsiyasi aralash tipdagi buziluvchan parabolik tenglama uchun chekli sohada lokal va nolokal nolokal chegaraviy masalalarni tadqiq etishga bag'ishlangan.

Chiziqli parabolik tipdagi tenglamalar uchun ekstremum prinsipi asosida lokal va nolokal nolokal chegaraviy masalalar yechimining yagonaligi haqidagi teorema isbotlandi.

Buziluvchan parabolik tipdagi tenglamalar uchun 1- va 2- chegaraviy masalalar Grin funksiyalari va ularning xossalarini qo'llagan holda, bu masalalar yechimlari topildi.

Lokal va nolokal chegaraviy masalalar yechimining mavjudligi integral tenglamalar nazariyasi asosida isbotlandi.

Mazkur dissertatsiya kirish qismi, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yhatidan iborat.

Dissertasiyaning I-bobida xususiy hosialali differensial tenglamalar nazaruyasidan ba'zi ma'lumotlar keltirilgan bo'lib,, bunda xarakeristik forma, xarakteristik sirt, tenglamani tiplarga ajratish bo'yicha tushunchalar, ikkinchi tartibli ikki o'zgaruvchili, yuqori taribli hosilalarga nisbatan chiziqli xususiy hosilali tenglamalar bo'yicha boshlang'ich ma'lumotlar yoritilgan. Ikkinchi tartibli chiziqli parabolik tipdagi tenglamalar uchun ekstremum prinsipi va undan kelib chiqadigan natijalar o]rganilgan. Shu bilan birga muhim maxsus funksiyalardan Bessel funksiyalari va ularning xossalari o'rganilgan.

II-bobda buziluvchan parabolik tipdagi tenglamalar uchun o'zi qaralayotagan har bir sohada 1- va 2-chegaraviy masalalar uchun Grin funksiyalari va ularning xossalari o'rganilgan. Grin funksiyalari yordamida qo'yilgan chegaraviy masalalarning yechimlari aniqlangan.

III-bobda aralash tipdagi buziluvchan parabolik tenglama uchun chekli sohada lokal va nolokal chegaraviy masalalar qo'yilgano'rganilgan. Bu masalalar yechimining mavjudligi va yagonaligi haqidagi teorema isbotlangan.

Ushbu magistrlik ishida olingan natijalar fizika, texnika va boshqa sohalarda uchraydigan amaliy masalalarni hal etishda katta ahamiyatga egadir. Bu natijalarni aralash tipdagi tenglamalar nazariyasidagi va bunday tenglamalarga keladigan amaliy masalalarni hal etishda qo'llanilishi mumkin.

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