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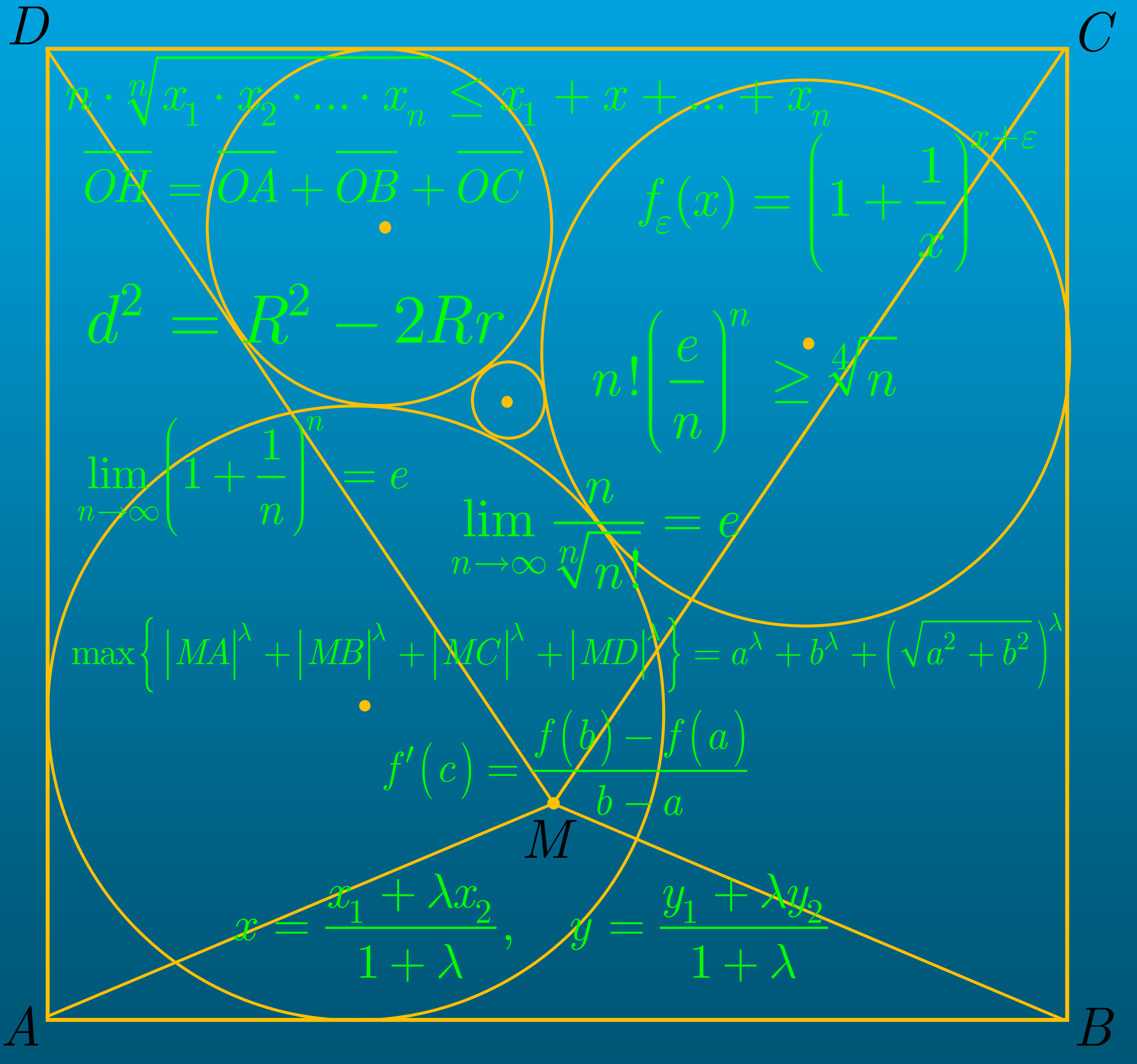

O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI
“MATEMATIKA” kafedrasi

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MATEMATIKADAN OLIMPIADA MASALALARI

Akademik litsey, kasb-hunar kollejlari uchun
uslubiy qo‘llanma

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Matematikadan olimpiada masalalari. Uslubiy qo‘llanma.

Uslubiy qo‘llanma akademik litsey va kasb-hunar kollejlari talabalari, umumiy o‘rta ta’lim maktablarining yuqori sinf o‘quvchilari uchun mo‘ljallangan bo‘lib, matematika fani bo‘yicha nostandart masalalar va murakkab masalalarni yechishga mo‘ljallangan masalalardan tashkil topgan. Har bir masalaga izox berilib yechimlari yoritilgan.

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So‘z boshi

Mustaqil fikrlaydigan, zamonaviy ilm-fan va kasb-xunarlarni puxta egallay oladigan, xalqimizga fidoyi yoshlarni tarbiyalash va qo‘llab quvvatlash barchamizni vazifamiz deb xisoblaymiz.

Xalqaro olimpiadalarda yoshlarimiz muvoffaqiyati yildan-yilga yaxshilanib bormoqda. O‘zbekiston yoshlari 2015- yil Taylantda bo‘lib o‘tgan matematika bo‘uyicha xalqaro olimpiadada bitta kumush va bitta bronza medalni qo‘lga kiritishdi. Biz yoshlarimizni bundan ham yuqori natijalarga erishib, davlatimiz obru e’tiborini ya’nada oshiradi degan umiddamiz. Agar xar bir o‘qituvchi o‘z vazifasini sidqidildan bajarib, yoshlarga ta’lim va tarbiya bersa, albatta yoshlarimiz xalqaro olimpiada g‘oliblari bo‘lishiga ishonamiz.

Bizning maqsadimiz yosh avlodni layoqati, qobiliyati, iqtidorini aniqlash, ochish va ularnig rivojlanishi uchun imkoniyat yaratishdan iboratdir. Agar ta’lim dargoxida to‘garaklar tashkil qilinsa, o‘quvchilar o‘rtasida matematik bellashuvlar o‘tkazilsa, fanlar bo‘yicha sirtqi olimpiadalarni o‘tkazish yo‘lga qo‘yilsa, o‘quvchilarni qiziqishi oshadi, qobiliyati shakllanadi va o‘ziga bo‘lgan ishonch oshadi.

Olimpiada masalalari elementar matematikaning eng jozibador masalalar to‘plamidir. Olimpiada masalalari o‘quvchini chuqur fikrlashga, o‘z ustida ishlab, iqtidorini-malakasini takomillashtirishga, boy fantaziyaga ega bo‘lishga, qa’tiyatli inson bo‘lishga va qaror qabul qila olishga o‘rgatadi.

Ma’lumki, geometrik masalalar o‘quvchilarni mantiqiy fikrlash bilan birga o‘z xulosalarini asoslashga undaydi. Masalalarni yechish davomida o‘quvchilar nazariy bilimlarni takrorlaydi va uni amaliy jihatdan qo‘llash ko‘nikmasiga ega bo‘ladilar. Qo‘llanmada Dekart koordinatalar sistemasidan foydalanib, uni turli geometrik va algebraik masalalarga, jumladan uchburchak yuzini hisoblash, tengliklarni isbotlash, tenglamalarni yechishga tadbiq qilishga doir misollar yechimi uslubiyati ko‘rsatilgan hamda aylanalar bilan bog‘liq murakkab masalalar yechimlari o‘rganilgan.

Ushbu ulubiy qo‘llanma ham yoshlarning matematikaga qiziqishlarini uyg‘otishga ozgina bo‘lsa ham xizmat qilsa, mualliflar o‘z maqsadlariga erishgan bo‘lar edilar.

1-§. Koshi tengsizligi va uning tadbiqlari

$x_1, x_2, ..., x_n$ – nomanfiy sonlar berilgan bo‘lsin. $\frac{x_1 + x + ... + x_n}{n}$ va $\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}$ sonlar mos ravishda berilgan *n* sonning o‘rta arifmetigi va o‘rta geometrigi deyiladi. Tabiiy savol tug‘iladi. O’rta arifmetik va o‘rta geometrik orasida qanday munosabat o‘rinli? Bu masala dastlab 1821-yilda fransuz matematigi Ogyusten-Lui Koshi tomonidan hal qilingan.

Teorema 1.1. (Koshi). Agar $x_1, x_2, ..., x_n$ – ixtiyoriy nomanfiy sonlar bo‘lsa, u holda ularning o‘rta geometrigi o‘rta arifmetigidan oshmaydi, ya’ni

$$\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n} \leq \frac{x_1 + x + ... + x_n}{n} \tag{1.1}$$

tengsizlik o‘rinli bo‘ladi.

Koshi tengsizligi asosan tengsizliklarni o‘rinli ekani isbotlashda keng qo‘llaniladi. Bu paragrafda biz Koshi tengsizligini soda isbotini va bir qancha murakkab masalalarga tadbig‘ini keltirdik.

Tenglik sharti faqat $x_1 = x_2 = ... = x_n$ bo‘lganda bajariladi. (1.1) tengsizlikka Koshi tengsizligi deyiladi. Hozirgi vaqtda Koshi tengsizligining o‘ndan ortiq isbotlari mavjud.

Agar $\sqrt[n]{x_1} = a_1,\ \sqrt[n]{x_2} = a_2, ... , \sqrt[n]{x_n} = a_n$ deb belgilash kiritsak, (1.1) tengsizlik

$$a_1 \cdot a_2 \cdots\ a_n \leq \frac{a_1{}^n + a_2{}^n + \cdots + a_n{}^n}{n} \tag{1.2}$$

ko‘rinishga keladi.

Ushbu uslubiy qo‘llanmada dastlab $n = 2$, $n = 3$ holatlarda Koshi teoremasi isbotlarini keltiramiz. Undan keyin umumiy holda isbotini keltiramiz.

1. $n = 2$ bo‘lganda (1.1) tengsizlik

$$\sqrt{x_1 \cdot x_2} \leq \frac{x_1 + x_2}{2}$$

ko‘rinishga keladi. Bu tengsizlikni chap tomonidagi ifodani o‘ng tomoniga o‘tkazib shakl almashtiramiz:

$$\frac{x_1 + x_2}{2} - \sqrt{x_1 \cdot x_2} = \frac{x_1 + x_2 - 2\sqrt{x_1 \cdot x_2}}{2} = \frac{\left(\sqrt{x_1} - \sqrt{x_2}\right)^2}{2}$$

$\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0$, bo‘lganligi uchun tengsizlik o‘rinli. Tenglik sharti $x_1 = x_2$ bo‘lganda bajariladi.

2. $n = 3$ bo‘lganda (1.1) tengsizlik

$$\sqrt[3]{x_1 \cdot x_2 \cdot x_3} \leq \frac{x_1 + x_2 + x_3}{3} \tag{1.3}$$

ko‘rinishga keladi. Belgilash kiritamiz $\sqrt[3]{x_1} = a,\ \sqrt[3]{x_2} = b,\ \sqrt[3]{x_3} = c$. U holda (1.3) tengsizlik $a \cdot b \cdot c \le \dfrac{a^3 + b^3 + c^3}{3}$ ko‘rinishga keladi.

Endi quyidagi ayniyatdan foydalanamiz:

$$a^3 + b^3 + c^3 - 3abc = \frac{\left(a+b+c\right)\left(\left(a-b\right)^2 + \left(b-c\right)^2 + \left(c-a\right)^2\right)}{2}$$

(bu ayniyatni to‘g‘riligini tekshirishni o‘quvchilarning o‘ziga xavola qilamiz). $\dfrac{a+b+c}{2} \ge 0$ va $\left(a-b\right)^2 \ge 0,\ \left(b-c\right)^2 \ge 0,\ \left(c-a\right)^2 \ge 0$ ekanligini e’tiborga olsak, $a^3 + b^3 + c^3 - 3abc \ge 0$ tengsizlikka ega bo‘lamiz. Bundan esa $abc \le \dfrac{a^3 + b^3 + c^3}{3}$ tengsizlik kelib chiqadi. Tenglik sharti $a = b = c$ bo‘lganda bajariladi.

Endi teoremani umumiy holda isbotlash uchun zarur bo‘ladigan lemmani keltiarmiz.

Lemma. Agar $x_1, x_2, ..., x_n -$ ixtiyoriy musbat sonlar bo‘lib, $x_1 \cdot x_2 \cdot ... \cdot x_n = 1$ bo‘lsa, u holda

$$x_1 + x_2 + ... + x_n \ge n \qquad (1.4)$$

tengsizlik o‘rinli bo‘ladi.

Tenglik sharti $x_1 = x_2 = ... = x_n = 1$ bo‘lganda bajariladi.

Isboti. (1.4) tengsizlikni matematik induksiya metodi yordamida isbotlaymiz. $n = 1$ da tengsizlikni to‘g‘riligi ravshan, $n = k$ da (1.4) tengsizlik to‘g‘ri deb faraz qilamiz. $n = k + 1$ da ham (1.4) tengsizlik o‘rinli bo‘lishini ko‘rsatamiz, ya’ni $x_1 \cdot x_2 \cdot ... \cdot x_k \cdot x_{k+1} = 1$ bo‘lsa, $x_1 + x_2 + ... + x_k + x_{k+1} \ge k + 1$ ekanligini isbotlaymiz.

a) $x_1, x_2, ..., x_{k+1} -$ sonlarining hammasi birga teng bo‘lsin. U holda $x_1 + x_2 + ... x_k + x_{k+1} = k + 1$ bo‘ladi.

b) $x_1, x_2, ..., x_{k+1} -$ sonlarining ichida shunday ikkitasi borki, ulardan bittasi birdan qat’iy katta, bittasi birdan kat’iy kichik, chunki $x_1 \cdot x_2 \cdot ... \cdot x_k \cdot x_{k+1} = 1$. Umumiylikka zid ish qilmagan holda, faraz qilishimiz mumkinki,

$$x_k < 1,\ x_{k+1} > 1,$$

$$\left(x_{k+1} - 1\right)\left(1 - x_k\right) > 0 \Rightarrow 1 + x_k \cdot x_{k+1} \le x_1 + x_2.$$

$$x_1 \cdot x_2 \cdot .... \cdot x_{k+1} \cdot \left(x_k \cdot x_{k+1}\right) = 1$$

bo‘lgani uchun farazimizga ko‘ra $x_1 + x_2 + ... + x_{k-1} + \left(x_k \cdot x_{k+1}\right) \geq k$

$$x_1 + x_2 + ... + x_{k-1} + \left(x_k + x_{k+1}\right) > x_1 + x_2 + ... + x_{k-1} + 1 + x_k \cdot x_{k+1} =$$
$$= \left(x_1 + x_2 + ... + x_{k-1} + x_k \cdot x_{k+1}\right) + 1 \geq k + 1$$

Demak, $x_1 + x_2 + ... + x_{k+1} \geq k + 1$.

tenglik sharti faqat $x_1 = x_2 = ... = x_k = x_{k+1}$ bo‘lganda bajariladi.

Koshi teoremasini isboti. Agar $x_1, x_2, ..., x_n -$ sonlaridan birortasi nolga teng bo‘lsa, u holda (1.1) bir tengsizlikning chap tomoni nolga aylanib, to‘g‘ri tengsizlik hosil bo‘ladi. Shuning uchun $x_1 > 0,\ x_2 > 0, x_3 > 0,\ ...x_n > 0$ deb hisoblashimiz mumkin. Belgilash kiritamiz.

$$a_1 = \frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}},\ a_2 = \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}}, ..., a_n = \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}}.$$

Ravshanki, $a_1,\ a_2, ..., a_n -$ sonlar musbat va $a_1 \cdot a_2 \cdot ... \cdot a_n = 1$. U xolda isbotlangan lemmaga ko‘ra $a_1 + a_2 + ... + a_n \geq n$ bo‘ladi.

Demak, $\frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} + \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} + ... + \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} \geq n$ ya’ni

$$\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n} \leq \frac{x_1 + x_2 + ... + x_n}{n}$$

bo‘ladi. Teorema to‘liq isbotlandi.

1-misol. Agar *a, b, c* – sonlar uchburchak tomonlari bo‘lib, p-uchburchakning yarim perimetri bo‘lsa, $\left(p - a\right)\left(p - b\right)\left(p - c\right) \leq \frac{abc}{8}$ tengsizlik o‘rinli bo‘lishini isbotlang.

Isboti. Agar $x \geq 0,\ y \geq 0,$ bo‘lsa, Koshi tengsizligiga ko‘ra $2\sqrt{xy} \leq x + y$ tegsizlik o‘rinli bo‘ladi. Tenglik sharti $x = y$ da bajariladi. Shu tengsizlikdan foydalanamiz.

$$\left(p-a\right)\left(p-b\right)\left(p-c\right) = \frac{2\sqrt{\left(p-a\right)\left(p-b\right)} \cdot 2\sqrt{\left(p-b\right)\left(p-c\right)} \cdot 2\sqrt{\left(p-c\right)\left(p-a\right)}}{8} \leq$$
$$\leq \frac{\left(p-a+p-b\right)\left(p-b+p-c\right)\left(p-c+p-a\right)}{8} =$$
$$= \frac{\left(2p-a-b\right)\left(2p-b-c\right)\left(2p-c-a\right)}{8} = \frac{abc}{8}$$

Tenglik sharti $p - a = p - b = p - c$, ya’ni $a = b = c$ bo‘lganda bajariladi.

2-misol . Ixtiyoriy uchburchak uchun $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ bo‘lishini isbotlang. Bunda a, b, c uchburchak tomonlari, S-esa uchburchak yuzasi.

Isboti. Belgilash kiritamiz: $x = p - a,\ y = p - b,\ y = p - c$. Koshi tengsizligiga ko‘ra $\sqrt[3]{x \cdot y \cdot z} \le \dfrac{x + y + z}{3}$. Bundan

$$\sqrt[3]{(p-a)(p-b)(p-c)} \le \frac{p-a+p-b+p-c}{3} = \frac{p}{3}$$

ya’ni $(p-a)(p-b)(p-c) \le \dfrac{p^3}{27}$ tengsizlikka ega bo‘lamiz. Geron formulasiga ko‘ra

$$S = \sqrt{p(p-a)(p-b)(x-c)} \le \sqrt{p \cdot \frac{p^3}{27}} = \frac{p^2}{3\sqrt{3}} = \frac{(a+b+c)^2}{12\sqrt{3}} =$$
$$= \frac{a^2+b^2+c^2+2ab+2bc+2ac}{12\sqrt{3}}$$

Tengsizlikka ko‘ra $2ab \le a^2 + b^2;\ 2bc \le b^2 + c^2;\ 2ac \le a^2 + c^2;$ Shuning uchun

$$S \le \frac{3\left(a^2+b^2+c^2\right)}{12\sqrt{3}} = \frac{a^2+b^2+c^2}{4\sqrt{3}}.$$

Demak, $a^2 + b^2 + c^2 \ge 4\sqrt{3}\,S$ tenglik sharti $a = b = c$ bo‘lganda bajariladi.

3-misol. Agar $a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ ...a_n > 0$ bo‘lsa,

$$\left(a_1 + a_2 + ... + a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \ge n^2$$

bo‘lishini isbotlang.

Isboti. Koshi tengsizligiga ko‘ra $\dfrac{a_1 + a_2 + ... + a_n}{n} \ge \sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n}$

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}{n} \ge \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdot ... \cdot \frac{1}{a_n}}$$

shuning uchun

$$\left(a_1 + a_2 + ... + a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \ge$$
$$n \cdot \sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n} \cdot n \cdot \frac{1}{\sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n}} = n^2$$

Demak, $\left(a_1 + a_2 + ... + a_n\right)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}\right) \ge n^2$.

Tenglik sharti $a_1 = a_2 = ... = a_n$ holatda o‘rinli bo‘ladi.

4-misol. Agar a, b, c - ixtiyoriy nomanfiy sonlar bo‘lib yig‘indisi 3 ga teng bo‘lsa, u holda $\sqrt{a}+\sqrt{b}+\sqrt{c} \geq a \cdot b+b \cdot c+c \cdot a$ tengsizlik o‘rinli bo‘lishini isbotlang.

Isboti.

$$a^2+b^2+c^2+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=\left(a^2+2\sqrt{a}\right)+\left(b^2+2\sqrt{b}\right)+\left(c^2+2\sqrt{c}\right)=$$

$$=\left(a^2+\sqrt{a}+\sqrt{a}\right)+\left(b^2+\sqrt{b}+\sqrt{b}\right)+\left(c^2+\sqrt{c}+\sqrt{c}\right)\geq$$

$$\geq 3\sqrt[3]{a^2\sqrt{a}\sqrt{a}}+3\sqrt[3]{b^2\sqrt{b}\sqrt{b}}+3\sqrt[3]{c^2\sqrt{c}\sqrt{c}}=$$

$$=\left(a+b+c\right)^2=a^2+b^2+c^2+2ab+2bc+2ac.$$

$$a^2+b^2+c^2+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\geq a^2+b^2+c^2+2ab+2bc+2ac$$

$$\Rightarrow 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\geq 2\left(ab+bc+ac\right)\Rightarrow \sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ac.$$

Tenglik sharti $a=b=c$ bo‘lganda bajariladi.

5-misol. Agar $a \geq 0,\ b \geq 0,\ c \geq 0$ bo‘lsa, ushbu

$$a^3+b^3+c^3 \geq a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}$$

tengsizlikni isbotlang.

Isboti.

$$a^3+b^3+c^3=\frac{1}{2}\left(a^3+abc+b^3+abc+c^2+abc+\left(a^3+b^3+c^3-3abc\right)\right)\geq$$

$$\geq \frac{1}{2}\left(2\sqrt{a^3abc}+2\sqrt{b^3abc}+2\sqrt{c^3abc}+3\sqrt{a^3b^3c^3}-3abc\right)=$$

$$=\frac{1}{2}\left(2a^2\sqrt{bc}+2b^2\sqrt{ac}+2c^2\sqrt{ab}\right)=$$

$$=a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}.$$

Demak, $a^3+b^3+c^3 \geq a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}.$

Tenglik sharti $a=b=c$ da bajariladi.

6-misol. Agar R, r $-$mos ravishda ABC uchburchakka tashqi va ichki chizilgan aylanalar radiuslari bo‘lsa, u holda $R \geq 2r$ bo‘lishini isbotlang.

Isboti. Aytaylik ABC uchburchakning tomonlari a, b, c bo‘lsin. 1-misolga ko‘ra $\left(p-a\right)\left(p-b\right)\left(p-c\right)\leq \frac{abc}{8}$ tengsizlik o‘rinli. Geron formulasiga ko‘ra $S=\sqrt{p\left(p-a\right)\left(p-b\right)\left(p-c\right)}$ bundan

$$S^2=p\left(p-a\right)\left(p-b\right)\left(p-c\right)\leq p\cdot\frac{abc}{8}\Rightarrow$$

$$S\cdot S\leq p\cdot\frac{abc}{8}\Rightarrow \frac{abc}{4R}\cdot p\cdot r\leq p\cdot\frac{abc}{8}\Rightarrow R\geq 2r$$

tenglik sharti $a=b=c$ bo‘lganda bajariladi.

7-misol. Agar a, b, c – uchburchak tomonlari, S - esa uning yuzasi bo'lsa,

$$\left(p-a\right)\left(p-b\right)+\left(p-b\right)\left(p-c\right)+\left(p-a\right)\left(p-c\right)\geq\sqrt{3}S \qquad (1.5)$$

tengsizlik o'rinli bo'lishini isbotlang.

Isboti. Belgilash kiritamiz: $p-a=x,\ p-b=y,\ p-c=z.$ Bundan $x+y+z=p$ tenglik kelib chiqadi. Geron formulasiga ko'ra (1.5) tengsizlik

$$xy+yz+zx\geq\sqrt{3\left(xyz\right)\left(x+y+z\right)} \qquad (1.6)$$

ko'rinishga o'tadi. (1.6) tengsizlikni har ikkala tomoni kvadratga oshiramiz.

$$\left(xy\right)^2+\left(yz\right)^2+\left(zx\right)^2+2xzy^2+2xyz^2+2yzx^2\geq 3x^2yz+3xzy^2+3xyz^2\Rightarrow$$
$$\Rightarrow\left(xy\right)^2+\left(yz\right)^2+\left(zx\right)^2\geq x^2yz+xzy^2+xyz^2$$

Shu tengsizlikni o'rinli bo'lishini ko'rsatamiz.

$$\left(xy\right)^2+\left(yz\right)^2+\left(zx\right)^2=\frac{1}{2}\left(\left(x^2y^2+y^2z^2\right)+\left(y^2z^2+z^2x^2\right)+\left(z^2x^2+x^2y^2\right)\right)\geq$$
$$\geq\frac{1}{2}\left(2\sqrt{x^2z^2y^4}+2\sqrt{y^2x^2z^4}+2\sqrt{z^2y^2x^4}\right)=$$
$$=xzy^2+yxz^2+zyx^2=x^2yz+y^2xz+z^2yz$$

Tenglik sharti $x=y=z$ holatda bajariladi.

Demak, $\left(p-a\right)\left(p-b\right)+\left(p-b\right)\left(p-c\right)+\left(p-a\right)\left(p-c\right)\geq\sqrt{3}S$ tengsizlik o'rinli. Tenglik sharti $a=b=c$ bo'lganda bajariladi.

8-misol. Uchburchakning yuzi S, tashqi va ichki chizilgan aylanalar radiuslari mos ravishda $R,\ r$ - bo'lsa,

$$S\geq\sqrt{\frac{27}{2}R\cdot r^3}$$

tengsizlikni isbotlang.

Isboti. Aytaylik uchburchakning tomonlari a, b, c – bo'lsin. U holda uchta son uchun Koshi tengsizligiga ko'ra $abc\leq\left(\frac{a+b+c}{3}\right)^3$ tengsizlik o'rinli. Yuza hisoblash formulasiga ko'ra $S=\frac{abc}{4R}\leq\frac{\left(\frac{a+b+c}{3}\right)^3}{4R}=\frac{8p^3}{27\cdot 4R}=\frac{2}{27R}\cdot\left(\frac{S}{r}\right)^3.$

Bundan $S^2\geq\frac{27}{2}R\cdot r^3$, ya'ni $S\geq\sqrt{\frac{27}{2}R\cdot r^3}$ tengsizlik kelib chiqadi.

Tenglik sharti $a=b=c$ da bo'ladi.

9-misol. Agar $a>0,\, b>0,\, c>0$ bo‘lsa,

$\sqrt{\dfrac{a}{b+c}}+\sqrt{\dfrac{b}{a+c}}+\sqrt{\dfrac{c}{a+b}}>2$ tengsizlikni isbotlang.

Isboti. Ikkita son uchun Koshi tengsizligiga ko‘ra

$2\sqrt{a}\sqrt{b+c}<a+b+c,\, 2\sqrt{b}\sqrt{a+c}<a+b+c,\;\; 2\sqrt{c}\sqrt{b+c}<a+b+c$.

Shu tengsizliklardan foydalanamiz.

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+c}}+\sqrt{\frac{c}{a+b}}=$$

$$=\frac{2a}{2\sqrt{a}\sqrt{b+c}}+\frac{2b}{2\sqrt{b}\sqrt{a+c}}>\frac{2a}{a+b+c}+\frac{2b}{a+b+c}+\frac{2c}{a+b+c}=2$$

Demak, $\sqrt{\dfrac{a}{b+c}}+\sqrt{\dfrac{b}{a+c}}+\sqrt{\dfrac{c}{a+b}}>2$ tengsizzlik o‘rinli.

10-misol. Agar $a,\, b,\, c-$ uchburchak tomonlari, R-tashqi chizilgan aylananing radiusi bo‘lsa, $R\geq\sqrt{\dfrac{abc}{a+b+c}}$ tengsizlikni isbotlang.

Isboti. Yuqoridagi 1-misolga ko‘ra, $\left(p-a\right)\left(p-b\right)\left(p-c\right)\leq\dfrac{abc}{8}$ tengsizlik o‘rinli. Shuning uchun

$$R=\frac{abc}{4S}=\sqrt{\frac{\left(abc\right)^2}{16\cdot p\left(p-a\right)\left(p-b\right)\left(p-c\right)}}\geq\sqrt{\frac{\left(abc\right)^2}{16\cdot p\cdot\dfrac{abc}{8}}}=\sqrt{\frac{abc}{a+b+c}}\,.$$

Demak, $R\geq\sqrt{\dfrac{abc}{a+b+c}}$ tengsizlik o‘rinli.

Tenglik sharti $a=b=c$ bo‘lganda bajariladi.

Mustaqil yechish uchun masalar

1. Agar $a,\, b,\, c-$uchburchak tomonlari bo‘lsa, $a^3+b^3+c^3+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\geq 4p$ bo‘lishini isbotlang.

2. Agar $\alpha,\, \beta,\, \gamma-ABC$ uchburchakning burchaklari bo‘lsa,

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\leq\frac{1}{8}$$

ekanini ko‘rsating.

3. Agar $a>0,\, b>0,\, c>0,\, d>0$ bo‘lsa, $\dfrac{a}{b+c}+\dfrac{b}{c+d}+\dfrac{c}{d+a}+\dfrac{d}{a+b}>2$

tengsizlik o‘rinli bo‘lishini isbotlang.

4. Agar $x_1 > 0,\ x_2 > 0, \ldots,\ x_n > 0$ bo‘lsa,

$$\sqrt{\frac{x_1}{x_2 + x_3 + \ldots + x_n}} + \sqrt{\frac{x_2}{x_1 + x_3 + \ldots + x_n}} + \ldots + \sqrt{\frac{x_n}{x_1 + x_2 + \ldots + x_{n-1}}} > 2$$

bo‘lishini isbotlang.

5. Agar $x \cdot y \cdot z = 1$ bo‘lsa, $xy + yz + zx + x + y + z \geq 6$ bo‘lishini ko‘rsating.

6. Agar $a,\ b,\ c$ $-$uchburchak tomonlari bo‘lsa,

$\dfrac{1}{p-a} + \dfrac{1}{p-b} + \dfrac{1}{p-c} \geq 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$ tengsizlikni isbotlang.

7. Agar $a^2 + b^2 + c^2 = 1$ bo‘lsa, $ab + bc + ca \leq 1$ bo‘lishini isbotlang.

8. Agar $a,\ b,\ c$ $-$uchburchak tomonlari, r-uning yarim perimetri bo‘lsa,

$a)\quad 2\sqrt{(p-a)(p-b)} \leq c$

$b)\quad \sqrt{(p-a)(p-b)} + \sqrt{(p-b)(p-c)} + \sqrt{(p-c)(p-a)} \leq p$

bo‘lishini isbotlang.

9. Ixtiyoriy $a,\ b,\ c$ $-$sonlari uchun $(a+b+c)^2 \geq 3(ab+bc+ac)$ tengsizlikni isbotlang.

10. Ixtiyoriy nomanfiy $a,\ b,\ c$ $-$sonlari uchun $(a+b+c)(ab+bc+ac) \geq 9abc$ bo‘lishini ko‘rsating.

11. Agar $a_1 > 0,\ a_2 > 0,\ \ldots,\ a_n > 0$ va $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$ bo‘lsa, $(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n$ tengsizlikni isbotlang.

12. Agar $a > 0,\ b > 0,\ c > 0$ bo‘lsa, $\dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b} \geq a + b + c$ bo‘lishini isbotlang.

13. Agar $a \geq 0,\ b \geq 0,$ bo‘lsa, $3a^3 + 7b^3 \geq 9ab^2$ bo‘lishini isbotlang.

14. Agar $a \geq 0,\ b \geq 0,\ c \geq 0$ bo‘lsa, $a(1-b) + b(1+c) + c(1-a) \geq 6\sqrt{abc}$ tengsizlikni isbotlang.

15. Ixtiyoriy uchburchak uchun

1) $h_a \leq \sqrt{p(p-a)}$

2) $h_a^{\,2} + h_b^{\,2} + h_c^{\,2} \leq p^2$

tengsizlikni isbotlang. Bunda $h_a, h_b,\ h_c$ $-$uchburchak balandliklari.

16. Agar $x_1,\ x_2,\ ,\ldots, x_n \in [a,\ b],\ 0 < a < b$ bo‘lsa,

$$n^2 \le \left(x_1 + x_2 + ... + x_n\right)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \le \frac{\left(a+b\right)^2}{4ab} n^2$$

tengsizlikni isbotlang.

17. Agar $a_1 > 0, a_2 > 0, ... ,a_n > 0$ bo‘lsa,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + ... + \frac{a_{n-2}}{a_{n-1} + a_n} + \frac{a_{n-1}}{a_n + a_1} + \frac{a_n}{a_1 + a_2} > \frac{n}{2}$$

tengsizlikni isbotlang.

18. Agar $a > 0, b > 0, c > 0$ bo‘lsa,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

tengsizlikni isbotlang.

19. Ixtiyoriy uchburchak uchun $p\sqrt{3} \le r_a + r_b + r_c \le \frac{9}{2}R$ bo‘lishini isbotlang. Bunda r_a, r_b, r_c $-$lar mos ravishda a, b, c tomonlarga va qolgan tomonlarning davomlariga urinuvchi aylanalar radiuslari.

20. Ixtiyoriy uchburchak uchun

1) $9r \le p\sqrt{3}$

2) $r^2 \le \frac{\sqrt{3}}{9}S$

3) $p^2 \le \frac{27}{4} \cdot R^2$

tengsizliklarni isbotlang.

2-§. Sonli ketma- ketliklar va ularning limiti

Akademik litsey va kasb-hunar kollejlariga sonli ketma-ketlik tushunchasi, monoton o‘suvchi (kamayuvchi) ketma-ketliklar, ketma-ketlikning limiti tushunchasi, yaqinlashuvchi ketma-ketliklar haqidagi teoremalar organilganligi uchun biz bu paragrafda qisqa ma’lumotlar keltirdik. Bu paragrafdagi masalalar talabani ketma-ketlikni limitini toppish bo‘yicha malakasini oshirishga qaratilgan.

Bizga $\{x_n\}$-ketma-ketlik berilgan bo‘lsin.

1-Ta‘rif. Ketma-ketlik yuqoridan (quyidan) chegaralangan deyiladi, agar shunday $M \in \mathbb{R}$ soni topilib, barcha $n \in \mathbb{N}$ larda $x_n \le M$ $(x_n \ge M)$ tengsizlik bajarilsa.

Quyidan va yuqoridan chegaralangan ketma-ketlik chegaralangan ketma-ketlik deyiladi.

1-misol. Ushbu

$$x_n = \frac{2015^n}{n!}$$

ketma-ketlikni chegaralanganligini isbotlang.

Isbot. Ma‘lumki, $\frac{x_{n+1}}{x_n} = \frac{2015^{n+1}}{(n+1)!} \cdot \frac{n!}{2015^n} = \frac{2015}{n+1}$ tenglik o‘rinli.

Agar $n \geq 2014\,(1 \leq n \leq 2013)$ bo‘lsa, u holda $x_{n+1} \leq x_n\,(x_{n+1} \geq x_n)$ bo‘lib, $\{x_n\}$ ketma-ketlik kamayuvchi (o‘suvchi) bo‘ladi. Bu esa istalgan $n \in \mathbb{N}$ larda $0 \leq x_n \leq x_{2014} = \frac{2015^{2014}}{2014!}$ tengsizlik o‘rinli bo‘lishini bildiradi, ya’ni $\{x_n\}$ ketma-ketlik chegaralangan.

Endi ketma - ketlikning limiti tushinchasini keltiramiz.

2-Ta‘rif. $a \in \mathbb{R}$ soni $\{x_n\}$ ketma-ketlikning limiti deyiladi, agar har qanday $\varepsilon > 0$ son olinganda ham shunday $n_0(\varepsilon) \in N$ son topilib, barcha $n > n_0(\varepsilon)$ larda

$$|x_n - a| < \varepsilon$$

tengsizlik bajarilsa.

$\{x_n\}$-ketma-ketlikning limiti odatda $\lim\limits_{n\to\infty} x_n = a$ kabi belgilanadi.

Agar ketma-ketlik chekli limitga ega bo‘lsa, u yaqinlashuvchi ketma-ketlik deyiladi.

3-**Ta‘rif.** Agar ixtiyoriy a son va ixtiyoriy natural n_0 son olganda ham shunday musbat ε_0 soni va shunday natural $n > n_0$ son topilsaki,

$$|x_n - a| \geq \varepsilon_0$$

bo‘lsa, $\{x_n\}$-ketma-ketlik limitga ega emas deb ataladi.

Agar ketma-ketlik limitga ega bo‘lmasa, u uzoqlashuvchi ketma-ketlik deyiladi.

2-misol. Ushbu $x_n = \frac{a}{n}\,(n \in N,\, a \in R)$ ketma-ketlikning limiti 0 ga teng bo‘lishini ko‘rsating.

Ixtiyoriy $\varepsilon > 0$ soniga ko‘ra $n_0 = \left[\frac{|a|}{\varepsilon}\right] + 1$ ni tanlaymiz.

U holda barcha $n > n_0$ larda

$$|x_n - a| = \left|\frac{a}{n} - 0\right| = \frac{|a|}{n} < \frac{|a|}{n_0} = \frac{|a|}{\left[\frac{|a|}{\varepsilon}\right] + 1} < \varepsilon$$

tengsizlik o‘rinli bo‘ladi. Demak, $\lim_{n\to\infty} x_n = 0$ bo‘ladi.

3-misol. Ushbu $x_n = (-1)^n$ ketma-ketlikning limiti mavjud emasligini isbotlang.

Isbot. Faraz qilamiz $\{x_n\}$-ketma-ketlik chekli a limitga ega bo’lsin. U holda ixtiyoriy $\varepsilon\,(0 < \varepsilon < 1)$ soni uchun shunday $n_0(\varepsilon) \in N$ son mavjudki, barcha $n > n_0(\varepsilon)$ larda $\left|x_n - a\right| < \varepsilon$ tengsizlik bajariladi. $x_n = (-1)^n$ bo‘lgani uchun $n > n_0(\varepsilon)$ da $\left|1 - a\right| < \varepsilon$, $\left|1 + a\right| < \varepsilon$ tengsizlik o‘rinli va bundan

$$2 = \left|1 - a + 1 + a\right| \le \left|1 - a\right| + \left|1 + a\right| < 2\varepsilon$$

tengsizlikni hosil qilamiz. Bu esa mumkin emas. Demak, $\{(-1)^n\}$ ketma-ketlik limitga ega emas.

Endi yaqinlashuvchi ketma-ketliklarning ba‘zi xossalarini keltiramiz.

1^0. Agar $\left\{x_n\right\}$ ketma-ketlik yaqinlashuvchi va $\lim_{n\to\infty} x_n = a$ bo‘lib, $a > p$ ($a < q$) bo‘lsa, u holda ketma-ketlikning biror hadidan keyingi barcha hadlari ham p sondan katta (q sondan kichik) bo‘ladi.

2^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, u chegaralangan bo‘ladi.

3^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, uning limiti yagona bo‘ladi.

4^0. Agar $\{x_n\}$ ketma-ketlik o‘suvchi (kamayuvchi) bo‘lib, yuqoridan (quyidan) chegaralangan bo‘lsa, u yaqinlashuvchi bo‘ladi.

5^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lib, $\lim_{n\to\infty} x_n = a$ bo‘lsa, u holda istalgan $k \in N$ uchun $\lim_{n\to\infty} x_{n+k} = a$ bo‘ladi.

Teorema 2.1. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lsa, $\{x_n \pm y_n\}$, $\{x_n \cdot y_n\}$ ketma-ketliklar ham yaqinlashuvchi va

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

formula o‘rinli bo‘ladi.

Teorema 2.2. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lib, $\forall\, n \in N$ uchun $y_n \ne 0$ va $\lim_{n\to\infty} y_n \ne 0$ bo‘lsa, $\left\{\dfrac{x_n}{y_n}\right\}$ ketma-ketlik ham yaqinlashuvchi hamda

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

formula o‘rinli bo‘ladi

Keltirilgan teoremalarni isbotini keltirmadik. Uni mustaqil ravishda [1], [11] adabiyotlardan o‘rganishingizni tavsiya qilamiz.

Endi ketma-ketliklar nazariyasining eng ajoyib misollarini isbotlari bilan keltiramiz.

4-misol. $x_n = \dfrac{a^n}{n!}$ $\hat{o}(a > 0)$ ketma-ketlikni limiti 0 ga teng bo‘lishini isbotlang.

Isbot. $\dfrac{x_{n+1}}{x_n} = \dfrac{a^{n+1}}{(n+1)!} \cdot \dfrac{n!}{a^n} = \dfrac{a}{n+1}$. Agar $n \geq [a]+1$ bo‘lsa, u holda $x_{n+1} \leq x_n$ bo‘lib, $\{x_n\}$ ketma-ketlik monoton kamayuvchi va quyidan $(x_n > 0)$ chegaralangan. Yuqorida keltirilgan 4^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_n = c$ mavjud va chekli. 5^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_{n+1} = c$ o‘rinli bo‘ladi. Bundan

$$c = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n \cdot \frac{a}{n+1}\right).$$

1-teoremaga asosan

$$\lim_{n\to\infty}\left(x_n \cdot \frac{a}{n+1}\right) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty}\frac{a}{n+1} = c\cdot 0 = 0$$

Demak, $\lim\limits_{n\to\infty}\dfrac{a^n}{n!} = 0$ bo‘ladi.

5-misol ([10]). Ushbu

$$x_n = \sqrt{a+\sqrt{a+\sqrt{a+...+\sqrt{a}}}} \quad (a>0)$$

ketma-ketlikning limitini toping.

Yechish: Har qanday $a>0$ soni uchun $\sqrt{a+\sqrt{a}} > \sqrt{a}$ tengsizlikni o‘rinli bo‘lishini e’tiborga olsak, istalgan $n \in N$ larda $x_{n+1} > x_n$ tengsizlik bajariladi, ya’ni $\{x_n\}$ ketma-ketlik monoton o‘suvchi. $\sqrt{a+\sqrt{a}} < \sqrt{a}+1$ va $\sqrt{a+\sqrt{a}+1} < \sqrt{a}+1$ tengsizliklardan foydalansak, istalgan $n \in N$ larda $x_n < \sqrt{a}+1$ tengsizlikka ega bo‘lamiz. 4^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_n = c$ mavjud va chekli. 5^0 xossaga ko‘ra,

$$c = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\sqrt{a+x_n} = \sqrt{a+c}.$$

$c = \sqrt{a+c}$ tenglamadan c ni topsak, $c = \dfrac{1+\sqrt{1+4a}}{2}$ bo‘ladi.

6-misol. Agar $\{x_n\}$ ketma-ketlik $x_{n+1} - 2x_n + x_{n-1} = A$ $(n \geq 2,\ A \in \mathbb{R})$

rekurrent munosabat orqali berilgan bo‘lsa, $\lim\limits_{n\to\infty}\frac{x_n}{n^2}$ ni toping.

Yechish. Belgilash kiritamiz: $x_n - x_{n-1} = a_n$. Bundan $a_{n+1} - a_n = x_{n+1} - x_n - (x_n - x_{n-1}) = A$ bo‘ladi. Bu esa $\{a_n\}$ ketma-ketlik arifmetik progressiya ekanini bildiradi. Arifmetik progressiyani umumiy hadi formulasiga asosan $a_n = a_2 + (n-2)A$ bo‘ladi.

$$x_n - x_{n-1} = a_2 + (n-2)A$$
$$x_{n-1} - x_{n-2} = a_2 + (n-3)A$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_3 - x_2 = a_2 + A$$

Bu tengliklar hadlab qo‘shsak,

$$x_n - x_2 = (n-2)a_2 + \frac{(n-1)(n-2)}{2}A$$

tenglikni hosil qilamiz. Bu tenglikka asosan $\lim\limits_{n\to\infty}\frac{x_n}{n^2} = \frac{1}{2}A$ bo‘ladi.

7-misol. Istalgan $0 \le x \le 1$ uchun

$$\ln(1+x) \ge x - \frac{1}{2}x^2$$

tengsizlik o‘rinli bo‘lishini isbotlang.

Isbot. Funksiya kiritamiz:

$$f(x) = \ln(1+x) - x + \frac{1}{2}x^2.$$

Uni hosilasini hisoblaymiz.

$$f'(x) = \frac{1}{1+x} - 1 + x = \frac{1-(1-x^2)}{1+x} = \frac{x^2}{1+x}.$$

$f'(x) = \dfrac{x^2}{1+x^2} \ge 0$ bo‘lgani uchun $f(x)$ funksiya $0 \le x \le 1$ da o‘suvchi va

$$f(x) \ge f(0) = 0$$

Bundan esa $\ln(1+x) \ge x - \frac{1}{2}x^2$ tengsizlik kelib chiqadi.

8-misol. Istalgan $0 \le \varepsilon \le \frac{1}{4}$ da

$$f_\varepsilon(x) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon}, \ x \ge 2$$

funksiyani monoton o‘suvchi bo‘lishini isbotlang.

Isbot. Funksiya hosilasini nomanfiy qiymatlar qabul qilishini isbotlaymiz.

$$f'_{\varepsilon}(x)=\left(1+\frac{1}{x}\right)^{x+\varepsilon}\left(\ln\left(1+\frac{1}{x}\right)-\left(\frac{\varepsilon}{x}+\frac{1-\varepsilon}{x+1}\right)\right)$$

Agar $\ln\left(1+\frac{1}{x}\right)\geq\frac{1}{x}-\frac{1}{2x^2},\quad(x\geq 2)$ tengsizlikdan foydalansak,

$$f'_{\varepsilon}(x)=\left(1+\frac{1}{x}\right)^{x+\varepsilon}\left(\frac{1}{x}-\frac{1}{2x^2}-\frac{\varepsilon}{x}-\frac{1-\varepsilon}{x+1}\right)=\left(1+\frac{1}{x}\right)^{x+\varepsilon}\left(\frac{2x(1-\varepsilon)-x-1}{2x^2(x+1)}\right)=$$

$$=\left(1+\frac{1}{x}\right)^{x+\varepsilon}\left(\frac{(1-2\varepsilon)x-1}{2x^2(1+x)}\right)\geq 0\,.$$

bo‘ladi. Bu esa $f_{\varepsilon}(x)$ funksiyani o‘suvchi ekanini bildiradi.

1 – natija. $x=n\in N$, $n\geq 2$ bo‘lsa,

$$x_n=\left(1+\frac{1}{n}\right)^{n+\varepsilon},\ 0\leq\varepsilon<\frac{1}{4}$$

ketma – ketlik monoton o‘suvchi .

2 – natija. Istalgan $0\leq\varepsilon<\frac{1}{2}$ da

$$f_{\varepsilon}(x)=\left(1+\frac{1}{x}\right)^{x+\varepsilon},\ \ x\geq\frac{1}{1-2\varepsilon}$$

funksiya monoton o‘suvchi bo‘ladi.

3 – natija. Istalgan $\frac{1}{2}\leq\varepsilon\leq 1$ da

$$f_{\varepsilon}(x)=\left(1+\frac{1}{x}\right)^{x+\varepsilon},\ \ x\geq 1$$

funksiya monoton kamayuvchi bo‘ladi.

Agar $\varepsilon=1$, $x=n$ $(n\in N)$ qilib tanlasak,

$$x_n=f_1(n)=\left(1+\frac{1}{n}\right)^{n+1}$$

ketma – ketlik monoton kamayuvchi va quyidan chegaralangan $(x_n>0,\ n\in N)$.

Agar $\varepsilon=0$, $x=n$ $(n\in N)$ qilib tanlasak,

$$x_n=f_0(n)=\left(1+\frac{1}{n}\right)^{n}$$

ketma – ketlik monoton o‘suvchi va yuqoridan chegaralangan (Istalgan $n\in N$ larda $\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n}\right)^{n+1}<\left(1+\frac{1}{1}\right)^{2}=4$).

Shuning uchun 4^0 hossaga ko'ra $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ mavjud va chekli.

4-Ta'rif. Ushbu $\left\{x_n\right\}=\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ ketma – ketlikning limiti e soni deb ataladi:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\,.$$

9 – misol. Ixtiyoriy $n\in N$ larda

$$n!\left(\frac{e}{n}\right)^n\geq\sqrt[4]{n} \qquad (2.1)$$

tengsizlik o'rinli bo'lishini isbotlang.

Isbot. (2.1) tengsizlikni matematik induksiya metodi yordamida isbotlaymiz.

1. $n=1$ va $n=2$ holatlarda (2.1) tengsizlikni o'rinli ekanini tekshirib ko'rish mumkin.
2. (2.1) tengsizlik $n=k$ da $(k\geq 2)$ da to'g'ri deb faraz qilamiz, ya'ni

$$k!\left(\frac{e}{k}\right)^k\geq\sqrt[4]{k},\ \ k\geq 2$$

tengsizlik o'rinli.

3. (2.1) tengsizlik $n=k+1$ da ham o'rinli bo'linishini ko'rsatamiz, ya'ni

$$\left(k+1\right)!\left(\frac{e}{k+1}\right)^{k+1}\geq\sqrt[4]{k+1}.$$

Isbot.

$$\left(k+1\right)!\left(\frac{e}{k+1}\right)^{k+1}=k!\left(\frac{e}{k}\right)^k\cdot\frac{e}{\left(1+\frac{1}{k}\right)^k}=\frac{k!\left(\frac{e}{k}\right)^k}{\sqrt[4]{k}}\cdot\frac{e}{\left(1+\frac{1}{k}\right)^{k+\frac{1}{4}}}\cdot\left(1+\frac{1}{k}\right)^{\frac{1}{4}}\cdot\sqrt[4]{k}=$$

$$=\frac{k!\left(\frac{e}{k}\right)^k}{\sqrt[4]{k}}\cdot\frac{e}{\left(1+\frac{1}{k}\right)^{k+\frac{1}{4}}}\cdot\sqrt[4]{k+1}$$

Farazimizga ko'ra istalgan $k\geq 2$ larda $k!\left(\dfrac{e}{k}\right)^k\geq\sqrt[4]{k}$ va 1-natijaga ko'ra

$$\left(1+\frac{1}{k}\right)^{k+\frac{1}{4}} < e$$

tengsizlik o‘rinli.
Demak,

$$(k+1)!\left(\frac{e}{k+1}\right)^{k+1} \geq \sqrt[4]{k+1}.$$

To‘la induksiya metodiga asosan (1) tengsizlik o‘rinli.

4 – natija. Agar $x_n = n!\left(\frac{e}{n}\right)^n$ bo‘lsa, $\lim\limits_{n\to\infty} n!\left(\frac{e}{n}\right)^n = +\infty$ bo‘ladi.

10-Misol. Agar $x_n = n!\cdot\left(\frac{a}{n}\right)^n$, $a > 0$ bo‘lsa, $\lim\limits_{n\to\infty} x_n$ ni toping.

Yechish: Misolni bir nechta holatlarga ajratib yechamiz.
1) $0 < a < e$ bo‘lsin.

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}(n+1)!\cdot\left(\frac{a}{n+1}\right)^{n+1}\cdot\frac{1}{n!\cdot\left(\frac{a}{n}\right)^n} = \lim_{n\to\infty}\frac{a}{\left(1+\frac{1}{n}\right)^n} = \frac{a}{e} < 1$$

1^0 - xossaga ko‘ra $\exists n_0 \in N$ son mavjudki, istalgan $n > n_0$ larda $\frac{x_n}{x_{n-1}} < \frac{a}{2}$.

Bundan $x_n < \left(\frac{a}{2}\right)^{n-n_0}\cdot x_{n_0}$ va $\lim\limits_{n\to\infty} x_n = 0$.

2) $a > e$ bo‘lsin. U holda $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n} = \frac{a}{e} > 1$ bo‘lib, $\lim\limits_{n\to\infty} x_n = \infty$ bo‘ladi.

3) $a = e$ holatda $n!\left(\frac{e}{n}\right)^n > \sqrt[4]{n}$ tengsizlikdan foydalansak

$$\lim_{n\to\infty} n!\left(\frac{e}{n}\right)^n = +\infty$$

bo‘ladi.

Demak, agar $0 < a < e (a \geq e)$ bo‘lsa, $\lim\limits_{n\to\infty} n!\left(\frac{a}{n}\right)^n = 0$

$\left(\lim\limits_{n\to\infty} n!\left(\frac{a}{n}\right)^n = +\infty\right)$ bo‘ladi.

Mustaqil yechish uchun masalar

1. $x_n = \underbrace{\sqrt[m]{a + \sqrt[m]{a + ... + \sqrt[m]{a}}}}_{n\ \ ta}$ $\quad \left(a > 0,\ m \in N,\ \ m \geq 2\right)$ ketma-ketlikni yaqinlashuvchi ekanini isbotlang.

2 ([10]). $x_n = \sqrt{2\sqrt{3\sqrt{4....\sqrt{n}}}}$ $(n \geq 2)$ ketma ketlikni yaqinlashuvchi bo‘lishini isbotlang.

3 ([10]). Agar $x_1 = 0,\quad x_2 = 1,\quad x_n = \dfrac{1}{2}\ (x_{n-1} + x_{n-2}),\quad (n = 2,3,...)$ bo‘lsa, $\lim\limits_{n \to \infty} x_n$ ni toping.

4. $\{x_n\}$ ketma-ketlik berilgan bo‘lib, $x_1 = 2015,\quad x_{n+1} = \dfrac{1}{4 - 3x_n}\ (n \in N)$ bo‘lsa, u holda $\lim\limits_{n \to \infty} x_n$ ni toping.

5. $\{x_n\}$ ketma-ketlik uchun

$$x_0 = 2015,\quad x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n})\ ,\quad (n = 0,1,2,...)$$

bo‘lsa, $\{x_n\}$ ketma-ketlikni yaqinlashuvchiligini isbotlang.

6. $\{x_n\}$ ketma-ketlikda $x_1 = a,\ x_2 = b$ va $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ $(n \geq 3)$ bo‘lsa, $\lim\limits_{n \to \infty} x_n$ ni toping.

7. Musbat hadli $\{x_n\}, \{y_n\}$ ketma – ketlik berilgan bo‘lib,

$$x_1 = a,\ y_1 = b,\ x_{n+1} = \sqrt{x_n y_n},\ y_{n+1} = \frac{x_n + y_n}{2}$$

bo‘lsa, $\lim\limits_{n \to \infty} x_n$ va $\lim\limits_{n \to \infty} y_n$ ni toping.

8. Agar $x_n = \dfrac{n}{\sqrt[n]{n!}}$ bo‘lsa, $\lim\limits_{n \to \infty} x_n = e$ bo‘lishini isbotlang.

9 ([10]). Aytaylik, $x_{\delta_{n+1}} = \lambda \cdot x_n + y_n,\quad y_{n+1} = \lambda \cdot y_n - x_n\ (\lambda \neq 0)$ bo‘lsin. Agar

$$x_0^2 + y_0^2 \neq 0, \text{ bo‘lsa, } \lim_{n \to \infty} x_n \text{ ni toping.}$$

3-§. Differensial hisobning asosiy teoremalari.

Biz ushbu paragrafda funksiya hosilasi va differensiali tushunchalarini, hosilaning geometrik va mexanik ma'nolarini keltirmadik. Ushbu paragrafda differensial hisobning asosiy teoremalari bo'lgan Ferma, Roll, Lagranj va Koshi teoremalari isbotlari bilan keltirildi. Funksiya hosilasini bir nechta sodda masalalarga tadbiqi keltirildi.

Teorema3.1 (Ferma) $f(x)$ funksiya $x = (a,b)$ oraliqda aniqlangan va $c \in \left(0,b\right)$ nuqtada uzining eng katta (kichik) qiymatga erishsin. Agar $f(x)$ funksiya c nuqtada chekli $f'\left(c\right)$ hosilaga ega bo'lsa, u holda $f'\left(c\right) = 0$ bo'ladi.

Isboti. $f(x)$ funksiya $c \in \left(0,b\right)$ nuqtada eng katta qiymatga ega bo'lgani uchun $\forall x \in \left(a,b\right)$ larda $f\left(x\right) \le f\left(c\right)$ bo'ladi. Hosila ta'rifiga ko'ra,

$$f'\left(c\right) = \lim_{x \to c} \frac{f\left(x\right) - f\left(c\right)}{x - c} = \lim_{x \to c+0} \frac{f\left(x\right) - f\left(c\right)}{x - c} \le 0$$

$$f'\left(c\right) = \lim_{x \to c} \frac{f\left(x\right) - f\left(c\right)}{x - c} = \lim_{x \to c-0} \frac{f\left(x\right) - f\left(c\right)}{x - c} \ge 0$$

Demak, $f'\left(c\right) = 0$ bo'ladi.

Teorema3.2 (Roll). $f(x)$ funksiya $[a,b]$ segmentda aniqlangan, uzluksiz va $f\left(a\right) = f\left(b\right)$ bo'lsin. Agar bu funksiya (a,b) oraliqda chekli $f'\left(x\right)$ hosilaga ega bo'lsa, uchun holda shunday $c \in \left(a,b\right)$ nuqta topiladiki $f'\left(c\right) = 0$ bo'ladi.

Isboti. $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo'lgani uchun shunday $c_1, c_2 \in \left[a,b\right]$ nuqtalar mavjudki,

$$f\left(c_1\right) = \max_{x \in [a,b]} f\left(x\right)$$

$$f\left(c_2\right) = \min_{x \in [a,b]} f\left(x\right)$$

bo'ladi.

1). Agar $f\left(c_1\right) = f\left(c_2\right)$ bo'lsa, u xolda $f\left(x\right) = const$ bo'lib. $f'\left(c\right) = 0$ $\forall x \in \left(a,b\right)$ bo'ladi.

2). Agar $f\left(c_1\right) > f\left(c_2\right)$ bo'lsa, u xolda $c_1 \in \left(a,b\right)$ yoki $c_2 \in \left(a,b\right)$yoki $c_1, c_2 \in \left(a,b\right)$ bo'ladi. U holda teorema.5.1 ga ko'ra $f'\left(c_1\right) = 0,$ yoki $f'\left(c_2\right) = 0,$ ya'ni shunday $c \in \left(a,b\right)$ mavjudki, $f'\left(c\right) = 0$ bo'ladi.

Teorema3.3 (Lagranj). $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo'lsin. Agar bu funksiya (a,b) oraliqda $f'\left(x\right)$ hosilaga ega bo'lsa, uchun holda shunday $c \in (a,b)$ nuqta mavjudki, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ bo‘ladi.

Isboti. $f(x)$ funksiya yordamida yangi funksiya kiritamiz. $F(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.

Bu funksiya $[a,b]$ segmentda uzluksiz va (a,b) oraliqda $F'(x)$ hosilaga ega, chunki $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va $f'(x)$ hosilaga ega.

1. $F(a) = f(a) - f(a) - \dfrac{f(b) - f(a)}{b - a}(a - a) = 0$

$F(b) = f(b) - f(a) - \dfrac{f(b) - f(a)}{b - a}(b - a) = 0$

Demak, $F(a) - F(b)$

2. $F'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$

3. Roll teoremasiga ko‘ra shunday $c \in (a,b)$ nuqta mavjudki, $F'(c) = 0$.

Bundan $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ bo‘ladi.

Teorema3.4. $f(x)$ funksiya (a,b) oraliqda chekli hosilaga ega bo‘lsin. Bu funksiya shu oraliqda o‘suvchi (kamayuvchi) bo‘lishi uchun $f'(c) \geq 0$ $\left(f'(c) \leq 0\right)$ bo‘lishi zarur va yetarli.

Endi yuqorida isbotlangan teoremalarni ba’zi masalalarga tadbiq qilamiz.

1-masala. $\forall x \in R$ uchun $e^x \geq x + 1$ bo‘lishi isbotlang.

Isboti. Funksiya tuzamiz. $f(x) = e^x - x - 1, \ \ x \in R$

Ravshanki, $f'(x) = e^x - 1, \ f(0) = f'(0) = 0$

1). $x \geq 0$ bo‘lsin. U holda $f'(x) = e^x - 1 \geq e^0 - 1 = 0$. Bundan esa $f(x)$ funksiyani o‘suvchi ekanligi kelib chiqadi.

Shuning uchun $f(x) \geq f(0) \Rightarrow e^x \geq x + 1,$

2). $x \leq 0$ bo‘lsin. U holda $f'(x) = e^x - 1 \leq e^0 - 1 = 0$. Bundan esa $f(x)$ funksiyani kamayuuvchi ekanligini bildiradi.

Shuning uchun $f(x) \geq f(0) \Rightarrow e^x \geq x + 1.$

Demak, $\forall x \in R$ uchun $e^x \geq x + 1$ bo‘ladi.

2-masala. Agar $a \geq 0, \ b \geq 0, \ p \geq 1$ bo‘lsa, $\left(\dfrac{a + b}{2}\right)^p \leq \dfrac{a^p + b^p}{2}$ bo‘lishini isbotlang.

Isboti. Agar $a=0$ yoki $b=0$ bo‘lsa. Tengsizlikni tenglik sharti bajariladi. Shuning uchun $a\geq b>0$ faqat holatni qaraymiz.

$$\left(\frac{a+b}{2}\right)^p\leq\frac{a^p+b^p}{2}\Rightarrow\left(\frac{\frac{a}{b}+1}{2}\right)^p\leq\frac{\left(\frac{a}{b}\right)^p+1}{2}$$

Funksiya tuzamiz.

$$f(x)=\frac{x^p+1}{2}-\left(\frac{x+1}{2}\right)^p.\ x\geq 1,\ p\geq 1.$$

U holda $f(0)=0$, $f'(x)=\frac{p}{2}\left(x^{p-1}-\left(\frac{x+1}{2}\right)^{p-1}\right)\geq 0$ bo‘ladi.

$f'(x)\geq 0$ bo‘lgani uchun $f(x)$ funksiya o‘suvchi. Shuning uchun $f(x)\geq f(0)$;

Bundan $\frac{x^p+1}{2}\geq\left(\frac{x+1}{2}\right)^p$ bo‘ladi. $x=\frac{a}{b}$ qilib tanlasak $\left(\frac{a+b}{2}\right)^p\leq\frac{a^p+b^p}{2}$ tengsizlikka ega bo‘lamiz.

3-masala. Agar $x\in\left[0;\frac{\pi}{2}\right)$ bo‘lsa, $2^{\mathrm{tgx}}+2^{\sin x}\geq 2^{x+1}$ bo‘lishini isbotlang.

Isboti. Koshi tengsizligiga ko‘ra $2^{\mathrm{tgx}}+2^{\sin x}\geq 2\sqrt{2^{\mathrm{tgx}}\cdot 2^{\sin x}}=2\cdot 2^{\frac{tgx+\sin x}{2}}$
Endi $tgx+\sin x\geq 2x$ tengsizlikni o‘rinli bo‘lishini ko‘rsatamiz.

Funksiya tuzamiz. $f(x)=tgx+\sin x-2x$, $x\in\left[0;\frac{\pi}{2}\right)$

Bundan, $f(0)=0$, $f'(x)=\frac{1}{\cos^2 x}+\cos x-2$. Koshi tengsizligiga ko‘ra,

$$f'(x)=\frac{1}{\cos^2 x}+\cos x-2\geq\frac{1}{\cos x}+\cos x-2\geq 2\sqrt{\frac{1}{\cos x}\cdot\cos x}-2=2-2=0$$

, ya’ni $f'(c)\geq 0$. Bu esa $f(x)$ funksiyani $x\in\left[0;\frac{\pi}{2}\right)$ oraliqda o‘suvchi ekanligini bildiradi.

Bundan, $f(x)\geq f(0)$, ya’ni $tgx+\sin x\geq 2x$ bo‘ladi.

Demak, ixtiyoriy $x\in\left[0;\frac{\pi}{2}\right)$ uchun $2^{\mathrm{tgx}}+2^{\sin x}\geq 2^{x+1}$ tengsizlik o‘rinli.

4-masala. c ning xech bir qiymatida $x\left(x^2-1\right)\left(x^2-10\right)=c$ tenglama beshta butun yechimga ega bo‘la olmasligini isbotlang.

Isboti. Funksiya tuzamiz. $f\left(x\right)=x\left(x^2-1\right)\left(x^2-10\right)$ bu funksiya butun sonlar o‘qida aniqlangan va $f'\left(x\right)=5x^2-3x^2+10$ ga teng.

1). $f'\left(x\right)=0$ tenglamani yechamiz.

$5x^4-3x^2+10=0$, $x^2=t$ belgilash kiritamiz.

$5t^2-33t+10=0$,

$$D=33^2-4\cdot5\cdot10=889,\quad t_1=\frac{33+\sqrt{889}}{10},\quad t_2=\frac{33-\sqrt{889}}{10}$$

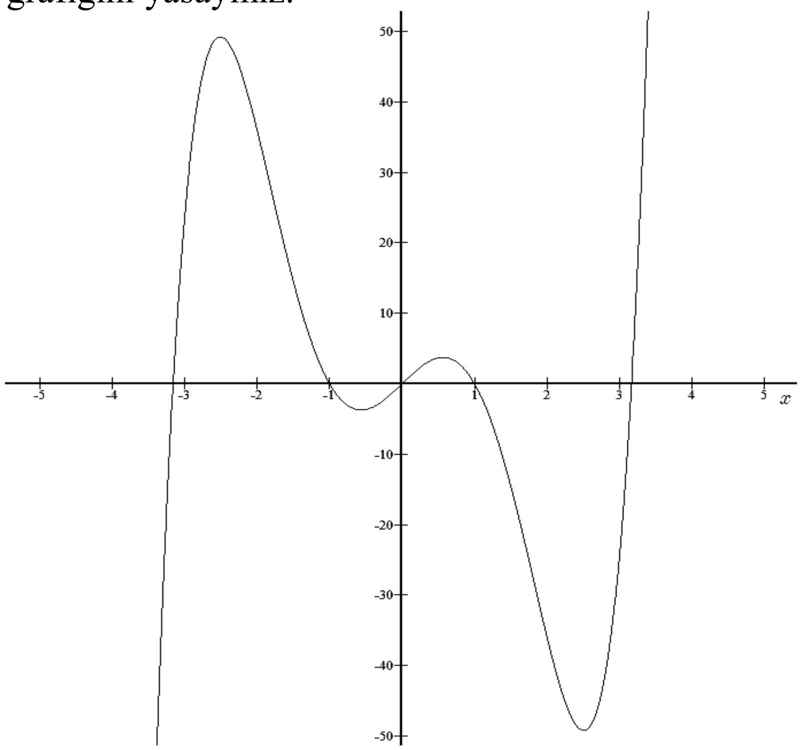
Bundan.

$x_1,x_4=\pm\sqrt{\dfrac{33+\sqrt{889}}{10}},\quad x_2,x_3=\pm\sqrt{\dfrac{33-\sqrt{889}}{10}}$,kelib chiqadi.

2). $f\left(x\right)=0$ tenglamani yechamiz.

$x\left(x^2-1\right)\left(x^2-10\right)=0\Rightarrow x=0,x=\pm1,x=\pm\sqrt{10}$

3). Funksiya grafigini yasaymiz.



1-chizma

Funksiyaning monotonlik oraliqlari beshta.

$1)(-\infty;x_1],\ \ 2)[x_1;x_2],\ \ 3)[x_2;x_3],\ \ 4)[x_3;x_4],\ \ 5)[x_4;+\infty)$

Shuning uchun $y=c$ to‘g‘ri chiziq $f(x)$ funksiyani ko‘pi bilan beshta nuqta kesishi mumkin. $[x_2,x_3]$ $\left([x_2,x_3]\subset\left(-1;1\right)\right)$ monotonlik oralig‘ida yagona $x=0$

butun son bor.

Demak, $c = 0$ bo‘lgandagi $f\left(x\right) = c$ tenglama ko‘pi bilan beshta butun yechimga ega bo‘lishi mumkin.

$f\left(x\right) = 0$ tenglama esa $x = 0, x = 1, \ x = -1$ butun yechimlarga ega. Bu holat $f\left(x\right) = c$ tenglama beshta butun yechimga ega emasligini bildiradi.

4-masala. Agar $a > 0\left(a \neq 1\right), \ p > q > 0$ bo‘lsa, u holda

$$\frac{a^p - 1}{p} > \frac{a^q - 1}{q} \tag{3.1}$$

tengsizlik o‘rinli bo‘ladi ([9], 134-masala).

Isbot. Ushbu funksiyani qaraymiz:

$$f\left(x\right) = \frac{a^x - 1}{x}, \ x > 0$$

Bu funksiyani $\left(0;\infty\right)$ oraliqda o‘suvchi ekanini ko‘rsatamiz. Shu maqsadda uning hosilasini hisoblaymiz:

$$f'\left(x\right) = \frac{a^x\left(\ln a\right)x - \left(a^x - 1\right)}{x^2}.$$

Bu hosilani musbatligini ko‘rsatish uchun quyidagi

$$g\left(x\right) = xa^x \ln a - a^x + 1, \ x \geq 0$$

funksiyaning nomanfiyligini ko‘rsatamiz. Buning uchun avvalo uning hosilasini hisoblaymiz:

$$g'\left(x\right) = a^x \ln a + xa^x\left(\ln a\right)^2 - a^x \ln a = xa^x\left(\ln a\right)^2 \geq 0.$$

$g\left(x\right)$ funksiya hosilasi nomanfiy bulgani uchun u o‘suvchi bo‘ladi. Bunga ko‘ra $g\left(x\right) \geq g\left(0\right)$, yani $g\left(x\right) \geq 0$ bo‘ladi. Bu esa $x > 0$ bo‘lganda $f'\left(x\right) > 0$ tengsizlik o‘rinli ekanini bildiradi, yani $f\left(x\right)$ funksiya o‘suvchi. Demak, $p > q > 0$ bo‘lgani uchun (3.1) tengsizlik o‘rinli.

Mustaqil yechish uchun masalar

1. $f\left(x\right) = \sqrt{x - 2} + \sqrt{16 - x}$ funksiyaning eng katta qiymatini toping.
2. $f\left(x\right) = \left|x - 1\right| + \left|x - 2\right| + \left|x - 3\right| + \left|x - 4\right|$ funksiyaning eng kichik qiymatini toping.
3. To‘g‘ri burchakli parallelepipedning balandligi asosining dioganaliga teng va asosining yuzi $4m^2$. Asosining tomonlari va balandligi qanday uzunlikda tanlansa, parallelepipedning hajmi eng kichik bo‘ladi.
4. R radiusli sharga ichki chizilgan to‘g‘ri burchakli parallelepipedlardan hajmi eng kattasini toping.
5. tomoni *a* ga teng kvadrat shaklidagi tunukaning burchaklaridan shunday bir

xil kvadratlar kesib olinishi kerakki, natijada chekkalari bukilsa, hajmi eng katta bo‘lgan quti hosil bo‘lsin. Kvadratlarning tomoni qanday bo‘lishi kerak?

6. *ABC* uchburchakda *AB* asosga parallel qilib KL to‘g‘ri chiziq o‘tkazingki, hosil bo‘ladigan KLMN to‘g‘ri to‘rtburchakning yuzi eng katta bo‘lsin.

7. Radiusi R ga teng bo‘lgan yarim shar atrofiga hajmi eng kichik bo‘ladigan konus chizing.

8. Balandligi *h* ga asosining radiusi *r* ga teng bo‘lgan konus ichiga hajmi eng katta bo‘lgan silindr chizing.

9. Agar $x \in \left[0; \frac{\pi}{2}\right)$ bo‘lsa, $3^{\mathrm{tgx}} + 3^{x} + 3^{\sin x} \geq 3^{x+1}$ bo‘lishini isbotlang.

10.* Agar $x_1, x_2, ..., x_n$ sonlar $[a,b]$ $\left(0 < a < b\right)$ kesmaga tegishli bo‘lsa, $\left(x_1, x_2, ..., x_n\right)\left(\frac{1}{x_1} +, ..., + \frac{1}{x_{n\,n}}\right) \leq \frac{\left(a+b\right)^2}{4ab} n^2$ bo‘lishini isbotlang.

11. tenglamani nomanfiy sonlar to‘plamida yeching. $2^{\sqrt[12]{x}} + 2^{\sqrt[4]{x}} = 2 \cdot 2^{\sqrt[6]{x}}$

12.* Agar musbat a_1, a_2, b_1, b_2 sonlar quyidagi shartlarni qanoatlantirsa

1) $a_1 + a_2 = b_1 + b_2$

2) $\max\left(a_1; a_2\right) \geq \max\left(b_1; b_2\right)$

U holda ixtiyoriy nomanfiy x, y sonlar uchun $x^{a_1} \cdot y^{a_2} + x^{a_2} \cdot y^{a_1} \geq x^{b_1} \cdot y^{b_2} + x^{b_2} \cdot y^{b_1}$ tengsizlik o‘rinli bo‘ladi.

13.* $e^{\pi} > \pi^{e}$ ekanini isbotlang.

14. Agar $a > b > c$ bo‘lsa quyidagi tengsizlikni isbotlang.

$a^2\left(b-c\right) + b^2\left(c-a\right) + c^2\left(a-b\right) > 0.$

15.* Agar *a, b, c* lar musbat sonlar bo‘lsa, $\frac{a^3}{b^2+c^3} + \frac{b^3}{c^3+a^3} + \frac{c^3}{a^3+b^3} \geq \frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}$ tengsizlikni isbotlang.

16. Tenglamani yeching. $\sqrt{x-5} + 3\sqrt{x+3} = 10$

17. $f(x)$ funksiya $(-1;1)$ oraliqda berilgan bo‘lib, $f'\left(x\right)$ hosilaga ega. Agar $f\left(0\right) = 0$, $\left|f'\left(x\right)\right| \leq 1$ bo‘lsa, u holda $\left|f\left(x\right)\right| \leq \left|x\right|$ bo‘lishini isbotlang.

18. Agar $a > 0$, $b > 0, m \in Z$ bo‘lsa, $\left(1 + \frac{a}{b}\right)^m + \left(1 + \frac{b}{a}\right)^m \geq 2^{m+1}$ bo‘lishini isbotlang.

19. $a_1, a_2, ..., a_n$ $-$xaqiqiy sonlar berilgan. X ning qanday qiymatida $f\left(x\right) = \left(x - a_1\right)^2 + \left(x - a_2\right)^2 + ... + \left(x - a_n\right)^2$ funksiya eng kichik qiymatga erishadi.

20. Musbat a, b sonlar uchun $\frac{1}{a^3}+\frac{1}{3a^2b}+\frac{1}{3ab^2}+\frac{1}{b^3}\geq\frac{64}{3\left(a+b\right)^3}$ tengsizlikni isbotlang.

4-§. Funksiya hosilasini ba'zi murakkab masalalariga tadbiqlari

Ushbu paragrafda funksiya hosilasi ba'zi murakkab masalalarni yechishga tadbiq qilingan.

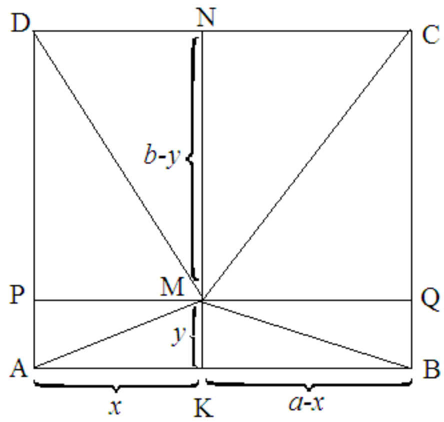
Teorema 4.1. $ABCD$ to'g'ri to'rtburchakda ixtiyoriy M nuqta olingan bo'lib, $\left|AB\right|=a$, $\left|AD\right|=b$, $\lambda\geq 1$ bo'lsa, u holda

a) $\max\left\{\left|MA\right|^\lambda+\left|MB\right|^\lambda+\left|MC\right|^\lambda+\left|MD\right|^\lambda\right\}=a^\lambda+b^\lambda+\left(\sqrt{a^2+b^2}\right)^\lambda$;

b) $\min\left\{\left|MA\right|^\lambda+\left|MB\right|^\lambda+\left|MC\right|^\lambda+\left|MD\right|^\lambda\right\}=4\left(\frac{a^2+b^2}{4}\right)^{\frac{\lambda}{2}}$

tengliklar o'rinli bo'ladi.

Isbot. M nuqtadan $KN\perp AB$, $PQ\perp AD$ kesmalarni o'tkazamiz (1-chizma). Aytaylik, $\left|AK\right|=x$, $\left|MK\right|=y$ bo'lsin. M nuqta to'rtburchakda bo'lganligi uchun $0\leq x\leq a$, $0\leq y\leq b$ bo'ladi. Pifagor teoremasiga ko'ra



1-chizma

$$\left|MA\right|^\lambda+\left|MB\right|^\lambda+\left|MC\right|^\lambda+\left|MD\right|^\lambda=(x^2+y^2)^{\frac{\lambda}{2}}+$$
$$+((x-a)^2+y^2)^{\frac{\lambda}{2}}+((x-a)^2+(y-b)^2)^{\frac{\lambda}{2}}+(x^2+(y-b)^2)^{\frac{\lambda}{2}}$$

bo'ladi. Ushbu

$$P(x,y)=(x^2+y^2)^{\frac{\lambda}{2}}+((x-a)^2+y^2)^{\frac{\lambda}{2}}+((x-a)^2+$$
$$+(y-b)^2)^{\frac{\lambda}{2}}+(x^2+(y-b)^2)^{\frac{\lambda}{2}}$$

belgilashni kiritib olib, quyidagi ikkita holni ko'rib chiqamiz.

1-hol. $y=0$ yoki $y=b$ bo'lsin. Bu holda

$$P(x,0)=P(x,b)=x^{\lambda}+(a-x)^{\lambda}+((x-a)^2+b^2)^{\frac{\lambda}{2}}+(x^2+b^2)^{\frac{\lambda}{2}}$$

bo‘lganligi sababli faqat $R(x,0)$ ni o‘rganish yetarli. $R(x,0)$ funksiyani $(0,a)$ oraliqda x bo‘yicha birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$P'(x,0)=\lambda(x^{\lambda-1}-(a-x)^{\lambda-1})+\lambda(x-a)((x-a)^2+b^2)^{\frac{\lambda}{2}-1}+\lambda x(x^2+b^2)^{\frac{\lambda}{2}-1},$$

$$P''(x,0)=\lambda(\lambda-1)(x^{\lambda-2}+(a-x)^{\lambda-2})+$$

$$+\lambda((x-a)^2+b^2)^{\frac{\lambda}{2}-2}((\lambda-1)(x-a)^2+b^2)+\lambda(x^2+b^2)^{\frac{\lambda}{2}-2}((\lambda-1)x^2+b^2).$$

$P''(x,0)\geq 0$ bo‘lgani uchun $P'(x,0)$ funksiya $(0,a)$ oraliqda o‘suvchi. Shuning uchun $P'(x,0)=0$ tenglama $(0,a)$ oraliqda ko‘pi bilan bitta yechimga ega bo‘lishi mumkin. $P'(\frac{a}{2},0)=0$ bo‘lgani sababli, yagona yechim $x=\frac{a}{2}$ dan iborat bo‘ladi. Demak, *P(x,0)* funksiya $\left[0;\frac{a}{2}\right]$ kesmada kamayuvchi, $\left[\frac{a}{2};a\right]$ kesmada esa o‘suvchi bo‘ladi. Bunga asosan kuyidagi tengliklarga ega bulamiz:

$$\max_{x\in[0;a]}P(x,0)=P(0,0)=P(a,0)=a^{\lambda}+b^{\lambda}+(\sqrt{a^2+b^2})^{\lambda},$$

$$\min_{x\in[0;a]}P(x,0)=P\left(\frac{a}{2},0\right)=2\left(\frac{a}{2}\right)^{\lambda}+2\left(\left(\frac{a}{2}\right)^2+b^2\right)^{\frac{\lambda}{2}}.$$

2-hol. $0<y<b$ bo‘lsin. Bu holda y o‘zgaruvchini fiksirlab, *P(x,y)* funksiyani x bo‘yicha birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$P'(x,y)=\lambda x\left(x^2+y^2\right)^{\frac{\lambda}{2}-1}+\lambda\left(x-a\right)((x-a)^2+y^2)^{\frac{\lambda}{2}-1}+$$

$$+\lambda\left(x-a\right)\left(\left(x-a\right)^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-1}+\lambda x\left(x^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-1},$$

$$P''(x,y)=\lambda\left(x^2+y^2\right)^{\frac{\lambda}{2}-2}\left(x^2\left(\lambda-1\right)+y^2\right)+$$

$$+\lambda((x-a)^2+y^2)^{\frac{\lambda}{2}-2}\left(\left(x-a\right)^2\left(\lambda-1\right)+y^2\right)+$$

$$+\lambda\left(\left(x-a\right)^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-2}\left(\left(x-a\right)^2\left(\lambda-1\right)+\left(y-b\right)^2\right)+$$

$$+\lambda\left(x^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-2}\left(x^2\left(\lambda-1\right)+\left(y-b\right)^2\right).$$

Yuqorida qilingan ishlarni takrorlab, ushbu

$$\max_{x\in[0;a]} P(x,y) = P(0,y) = P(a,y) = y^\lambda + (a^2+y^2)^{\frac{\lambda}{2}} + (a^2+(y-b)^2)^{\frac{\lambda}{2}} + \left(b-y\right)^\lambda,$$

$$\min_{x\in[0;a]} P(x,y) = P\left(\frac{a}{2},y\right) = 2\left(\left(\frac{a}{2}\right)^2 + y^2\right)^{\frac{\lambda}{2}} + 2\left(\left(\frac{a}{2}\right)^2 + (y-b)^2\right)^{\frac{\lambda}{2}}$$

tengliklarni hosil qilamiz.

Endi ushbu

$$f(y) = y^\lambda + (a^2+y^2)^{\frac{\lambda}{2}} + (a^2+(y-b)^2)^{\frac{\lambda}{2}} + \left(b-y\right)^\lambda,$$

$$g(y) = 2\left(\left(\frac{a}{2}\right)^2 + y^2\right)^{\frac{\lambda}{2}} + 2\left(\left(\frac{a}{2}\right)^2 + (y-b)^2\right)^{\frac{\lambda}{2}}$$

yordamchi funksiyalarni kiritib olamiz. Bu funksiyalar uchun $\max\limits_{y\in[0;b]} f(y)$ va $\min\limits_{y\in[0;b]} g(y)$ larni hisoblaymiz. Shu maqsadda, $\left(0;b\right)$ oralikda, $f(y)$ va $g(y)$ funksiyalarning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$f'(y) = \lambda y^{\lambda-1} + \lambda y\left(a^2+y^2\right)^{\frac{\lambda}{2}-1} + \lambda\left(y-b\right)\left(a^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-1} - \lambda\left(b-y\right)^{\lambda-1},$$

$$f''(y) = \lambda\left(\lambda-1\right)y^{\lambda-2} + \lambda\left(a^2+y^2\right)^{\frac{\lambda}{2}-2}\left(a^2+\left(\lambda-1\right)y^2\right) +$$
$$+\lambda\left(a^2+\left(y-b\right)^2\right)^{\frac{\lambda}{2}-2}\left(a^2+\left(\lambda-1\right)\left(y-b\right)^2\right) + \lambda\left(\lambda-1\right)\left(b-y\right)^{\lambda-2}$$

$$g'\left(y\right) = 2\lambda y\left(\left(\frac{a}{2}\right)^2 + y^2\right)^{\frac{\lambda}{2}-1} + 2\lambda(y-b)\left(\left(\frac{a}{2}\right)^2 + (y-b)^2\right)^{\frac{\lambda}{2}-1},$$

$$g''\left(y\right) = 2\lambda\left(\left(\frac{a}{2}\right)^2 + y^2\right)^{\frac{\lambda}{2}-2}\left(\left(\frac{a}{2}\right)^2 + (\lambda-1)y^2\right) +$$
$$+2\lambda\left(\left(\frac{a}{2}\right)^2 + (y-b)^2\right)^{\frac{\lambda}{2}-2}\left(\left(\frac{a}{2}\right)^2 + \left(\lambda-1\right)\left(y-b\right)^2\right).$$

Bu ifodalardan $0 \le y \le b$ bo‘lganida $f''(y) \ge 0,\ g''\left(y\right) \ge 0$ va $f'\left(\frac{b}{2}\right) = 0,\ g'\left(\frac{b}{2}\right) = 0$ kelib chiqadi. Demak, $f\left(y\right)$ va $g\left(y\right)$ funksiyalar $y = \frac{b}{2}$ nuqtada o‘zining eng kichik qiymatlariga erishadi, $\left[0;b\right]$ kesmaning chetki nuqtalarida esa eng katta qiymatlarga erishadi, xususan

$$\max_{y\in[0;b]} f(y) = f(0) = f(b) = a^{\lambda} + b^{\lambda} + \left(\sqrt{a^2+b^2}\right)^{\lambda},$$

$$\min_{y\in[0;b]} g(y) = g\left(\frac{b}{2}\right) = 4\left(\frac{a^2+b^2}{4}\right)^{\frac{\lambda}{2}}$$

bo‘ladi. Birinchi va ikkinchi hollarni hisobga olsak, a) va b) tengliklar kelib chiqadi. *Teorema isbotlandi*.

Natija. Agar $\lambda = 1$ bo‘lsa, ushbu

$$\max\{|MA|+|MB|+|MC|+|MD|\} = a+b+\sqrt{a^2+b^2},$$

$$\min\{|MA|+|MB|+|MC|+|MD|\} = 2\sqrt{a^2+b^2}$$

tengliklar, $\lambda = 2$ bo‘lsa, quyidagi

$$\max\left\{|MA|^2+|MB|^2+|MC|^2+|MD|^2\right\} = 2\left(a^2+b^2\right),$$

$$\min\left\{|MA|^2+|MB|^2+|MC|^2+|MD|^2\right\} = a^2+b^2$$

tengliklar o‘rinli bo‘ladi.

1-masala. Yuzasi S bo‘lgan $ABCD$-qavariq to‘rtburchakning AB, BC, CD, DA tomonlarida mos ravishda M, N, P, Q nuqtalar shunday olinganki, bunda $\frac{|AM|}{|MB|} = \frac{|BN|}{|NC|} = \frac{|CP|}{|PD|} = \frac{|DQ|}{|QA|}$. $MNPQ$ to‘rtburchak yuzasining eng kichik qiymatini toping.

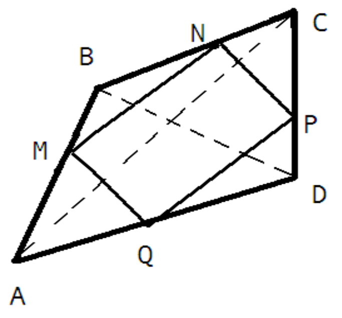
Yechish. Aytaylik, $\frac{|AM|}{|MB|} = \lambda$ bo‘lsin. U holda

$$|AB| = |AM|+|MB| = (1+\lambda)|MB|,$$

$$|BC| = |BN|+|NC| = \frac{\lambda+1}{\lambda}|BN|,$$

$$|AD| = |AQ|+|QD| = \frac{\lambda+1}{\lambda}|QD|,$$

$$|CD| = |CP|+|PD| = (1+\lambda)|PD|$$



2-chizma

bo‘ladi. Bularga asosan quyidagilarga ega bo‘lamiz (2-chizma).

$$S_{PDQ} = \frac{\lambda}{\left(1+\lambda\right)^2} \cdot S_{ACD}, \quad S_{MBN} = \frac{\lambda}{(1+\lambda)^2} \cdot S_{ABC}.$$

Bundan ushbu $S_{MBN} + S_{PDQ} = \frac{\lambda}{(1+\lambda)^2} \cdot S$ tenglik kelib chiqadi.

Xuddi shuningdek, $S_{QAM} + \quad S_{NCP} = \frac{\lambda}{\left(1+\lambda\right)^2} \cdot S$ tenglikka ega bo‘lamiz.

Demak,

$$S_{MNPQ} = S_{ABCD} - \left(S_{QAM} + S_{NCP} + S_{MBN} + S_{PDQ}\right) = S - \frac{2\lambda}{\left(1+\lambda\right)^2} S = \frac{\lambda^2+1}{\left(1+\lambda\right)^2} S.$$

Oxirgi ifodaning eng kichik qiymatini topish maqsadida ushbu

$$f\left(\lambda\right) = \frac{\lambda^2+1}{\left(1+\lambda\right)^2}, \qquad \lambda > 0$$

funksiyani kiritib olamiz va uning eng kichik qiymatini topamiz. Buning uchun avvalo uning hosilasini hisoblaymiz:

$$f'\left(\lambda\right) = \frac{2\lambda\left(\lambda+1\right)^2 - (\lambda^2+1)2(\lambda+1)}{(\lambda+1)^4} = \frac{2(\lambda-1)}{(\lambda+1)^3}.$$

Bu ifodaga binoan $f'\left(\lambda\right) = 0$ tenglamaning yagona yechimi $\lambda = 1$ bo‘lib, $(0;1)$ oraliqda $f'\left(\lambda\right) < 0$ va $(1;+\infty)$ oraliqda $f'\left(\lambda\right) > 0$. bo‘ladi. Bu esa $f(\lambda)$. funksiyaning eng kichik qiymatida $f\left(1\right) = \frac{1}{2}$ ga teng ekanini bildiradi. Demak, $MNPQ$ to‘rtburchak yuzasining eng kichik qiymati $\frac{1}{2}S$ ga teng bo‘lib, bu qiymat yuqoridagi nisbat 1 ga teng bo‘lganda, yani M, N, P, Q nuqtalar to‘rtburchak tomonlarining o‘rtalarida bo‘lganida erishiladi.

2-masala. Agar a, b, c – musbat sonlar bo‘lsa, u holda ixtiyoriy $\lambda, \mu(\lambda \geq \mu \geq 0)$, sonlari uchun ushbu

$$\frac{a^\lambda}{b^\lambda + c^\lambda} + \frac{b^\lambda}{c^\lambda + a^\lambda} + \frac{c^\lambda}{a^\lambda + b^\lambda} \geq \frac{a^\mu}{b^\mu + c^\mu} + \frac{b^\mu}{c^\mu + a^\mu} + \frac{c^\mu}{a^\mu + b^\mu} \qquad (4.1)$$

tengsizlik o‘rinli bo‘ladi.

Isbot. Quyidagi funksiyani qaraymiz:

$$f\left(\lambda\right) = \frac{p^\lambda}{q^\lambda + 1} + \frac{q^\lambda}{p^\lambda + 1} + \frac{1}{p^\lambda + q^\lambda}, \qquad \lambda \geq 0.$$

Bunda p, q $(p \geq q \geq 1)$ – o‘zgarmas sonlar. Bu funksiyaninghosilasini hisoblab, uni quyidagi tarzda yozib olamiz:

$$f'(\lambda)=\frac{(p^{\lambda}\ln p)\left(q^{\lambda}+1\right)-p^{\lambda}q^{\lambda}\ln q}{(q^{\lambda}+1)^2}+\frac{(q^{\lambda}\ln q)\left(p^{\lambda}+1\right)-q^{\lambda}p^{\lambda}\ln p}{(p^{\lambda}+1)^2}-$$

$$-\frac{p^{\lambda}\ln p+q^{\lambda}\ln q}{(p^{\lambda}+q^{\lambda})^2}=(pq)^{\lambda}(\ln\frac{p}{q})\left(\frac{1}{(q^{\lambda}+1)^2}-\frac{1}{(p^{\lambda}+1)^2}\right)+$$

$$+p^{\lambda}(\ln p)\left(\frac{1}{(q^{\lambda}+1)^2}-\frac{1}{(p^{\lambda}+q^{\lambda})^2}\right)+(q^{\lambda}\ln q)\left(\frac{1}{(p^{\lambda}+1)^2}-\frac{1}{(p^{\lambda}+q^{\lambda})^2}\right)$$

Bu ifodaga muvofiq $p\geq q\geq 1$ bo‘lgani uchun $f'\left(\lambda\right)\geq 0$, ya’ni $f(\lambda)$ funksiya o‘suvchi bo‘ladi. Demak, har qanday λ ,μ $(\lambda\geq\mu\geq 0)$- nomanfiy sonlar uchun $f\left(\lambda\right)\geq f(\mu)$, ya’ni

$$\frac{p^{\lambda}}{q^{\lambda}+1}+\frac{q^{\lambda}}{p^{\lambda}+1}+\frac{1}{p^{\lambda}+q^{\lambda}}\geq\frac{p^{\mu}}{q^{\mu}+1}+\frac{q^{\mu}}{p^{\mu}+1}+\frac{1}{p^{\mu}+q^{\mu}} \tag{4.2}$$

tengsizliko‘rinli. Umumiylikka zid ishqilmagan holda $a\geq b\geq c$ deb olib, $p=\frac{a}{c}$, $q=\frac{b}{c}$ deb tanlasak, uholda (4.2) tengsizlik quyidagi ko‘rinishga keladi:

$$\frac{\left(\frac{a}{c}\right)^{\lambda}}{\left(\frac{b}{c}\right)^{\lambda}+1}+\frac{\left(\frac{b}{c}\right)^{\lambda}}{\left(\frac{a}{c}\right)^{\lambda}+1}+\frac{1}{\left(\frac{a}{c}\right)^{\lambda}+\left(\frac{b}{c}\right)^{\lambda}}\geq\frac{\left(\frac{a}{c}\right)^{\mu}}{\left(\frac{b}{c}\right)^{\mu}+1}+\frac{\left(\frac{b}{c}\right)^{\mu}}{\left(\frac{a}{c}\right)^{\mu}+1}+\frac{1}{\left(\frac{a}{c}\right)^{\mu}+\left(\frac{b}{c}\right)^{\mu}}.$$

Bu tengsizlikning chap va o‘ng tomonlarida shakl almashtirishlar bajarib, (4.1) tengsizlikni hosil qilamiz.

Endi (4.1) tengsizlikni ba’zi xususiy xollarni keltirib o‘tamiz.

1. Agar $\lambda=1$, $\mu=0$ bo‘lsa, ushbu

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$$

tengsizlik hosil bo‘ladi ([6], 3.18-masala).

2. Agar $\lambda\geq 0$, $\mu=0$ bo‘lsa, ushbu

$$\frac{a^{\lambda}}{b^{\lambda}+c^{\lambda}}+\frac{b^{\lambda}}{a^{\lambda}+c^{\lambda}}+\frac{c^{\lambda}}{a^{\lambda}+b^{\lambda}}\geq\frac{3}{2}$$

tengsizlik hosil bo‘ladi.

3. Agar $\lambda=2$, $\mu=1$ bo‘lsa, ushbu

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}\geq\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}$$

tengsizlik hosil bo‘ladi.

3-masala. Agar $a\geq 0$, $b\geq 0$, $p\geq 1$ bo‘lsa, kuyidagi

$$\left(\frac{a+b}{2}\right)^p \le \frac{a^p+b^p}{2} \tag{4.3}$$

tengsizliko‘rinlibo‘ladi ([7] , 144-masala).

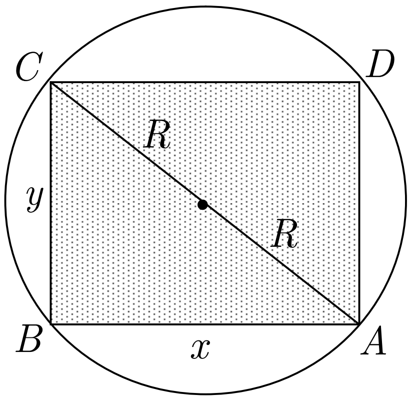
Isbot. $a=0$ yoki $b=0$ bo‘lsa, bu tengsizliko‘rinli bo‘lishi ravshan.Umumiylikka zid ish qilmagan holda $a \ge b > 0$ deb qarashimiz mumkin. Quyidagifunksiyaniqaraymiz: $f\left(x\right)=\frac{x^p+1}{2}-\left(\frac{x+1}{2}\right)^p \quad x\ge 1,\ p\ge 1$

Bu funksiyaning hosilasini hisoblaymiz: $f'\left(x\right)=\frac{p}{2}\left(x^{p-1}-\left(\frac{x+1}{2}\right)^{p-1}\right)$.

Bu ifodadan $x \ge 1$ ekanini etiborga olib, $f'\left(x\right)\ge 0$ tengsizlikka ega bo‘lamiz. Demak, $f\left(x\right)$ funksiya $\left[1;+\infty\right)$ oraliqda o‘suvchi. Shuning uchun $f\left(x\right)\ge f\left(1\right), f\left(x\right)\ge f\left(0\right)$, ya’ni $\frac{x^p+1}{2}\ge\left(\frac{x+1}{2}\right)^p$. Agar $x=\frac{a}{b}$ qilibtanlasak, (4.3) tengsizlikkaegabo‘lamiz.

4-masala. R radiusli doirada ichki chizilgan to‘g‘ri to‘rtburchaklardan yuzasi eng kattasini toping.

Yechish. Doiraga $ABCD$ to‘g‘ri to‘rtburchak ichki chizilgan bo‘lib, $AB=x, BC=y$ bo‘lsin. Ravshanki, $x^2+y^2=4R^2$ va $S_{ABCD}=S=x\cdot y$



3-chizma

$$S_{ABCD}=x\cdot\sqrt{4R^2-x^2}=S\left(x\right).\quad 0\le x\le 2R.$$

$$S'\left(x\right)=\sqrt{4R^2-x^2}+x\cdot\frac{-2x}{2\sqrt{4R^2-x^2}}=\frac{4R^2-2x^2}{\sqrt{4R^2-x^2}}$$

$$S'\left(x\right)=0\Rightarrow x=\sqrt{2}R.$$

Funksiya xosilasi $\left(0,\sqrt{2}R\right]$ da musbat, $\left[\sqrt{2}R,2R\right)$ da manfiy bo‘lgani uchun $x=\sqrt{2}R$ –nuqta funksiya eng katta qiymatiga ega bo‘ladi.

$$S\left(\sqrt{2}R\right) = \sqrt{2}R\sqrt{4\mathrm{R}^2 - 2R^2} = 2R^2.$$

Demak, $AB = BC = \sqrt{2}R,\ S_{ABCD} = 2R^2$.

Mustaqil yechish uchun masalalar

1. Yuzasi S ga teng bo‘lgan ABC uchburchakning AB, BC, AC tomonlarida mos ravishda M, N, P nuqtalar shunday olinganki, bunda $\dfrac{AM}{MB} = \dfrac{BN}{NC} = \dfrac{CP}{PA}$. MNP uchburchak yuzasining eng kichik qiymatini toping.

2. ABC to‘g‘ri burchakli ($\angle C = 90^o$) uchburchakda ixtiyoriy M nuqta olingan. Agar $\left|AC\right| = b$, $\left|BC\right| = a$ bo‘lsa, $\max\left\{\left|MA\right| + \left|MB\right| + \left|MC\right|\right\}$ va $\min\left\{\left|MA\right| + \left|MB\right| + \left|MC\right|\right\}$ ni toping.

3. $ABCD$ parallelogrammda ixtiyoriy M nuqta olingan. Agar $\left|AD\right| = a$, $\left|AB\right| = b$, $\angle A = \alpha\ (\alpha < 90^o)$ bo‘lsa, $\max\left\{\left|MA\right| + \left|MB\right| + \left|MC\right| + \left|MD\right|\right\}$ ni toping.

4. Agar $0 \le x < \dfrac{\pi}{2}$ bo‘lsa, u xolda $3^{\mathrm{tg}x} + 3^x + 3^{\sin x} \ge 3^{x+1}$ tengsizlik o‘rinli bo‘lishini isbotlang.

5. Agar a, b, c - musbat sonlar bo‘lsa, $a^x + b^x = c^x$ tenglama ko‘pi bilan bitta yechimga ega bo‘lishini isbotlang.

5-§. Pifagor teoremasining tadbiqlari.

Pifagor teoremasini 10 dan ortiq turli xil isbotlari mavjud. Shu sabab biz uni isbotini keltirmadik ([] ga qarang). Ushbu paragrafda Pifagor teoremasini qo‘llanilishi sohasi naqadar kengligini ko‘rish mumkin.

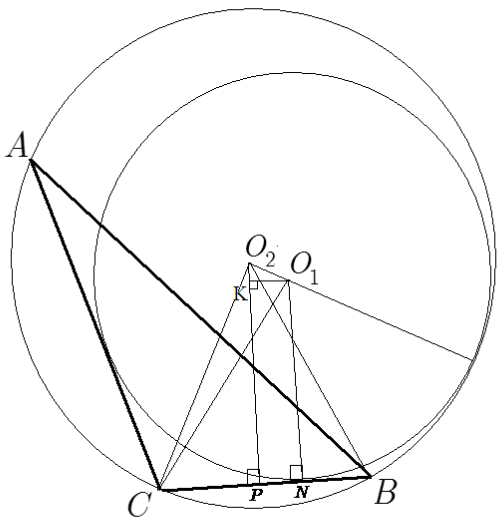
Teorema 5.1. ABC uchburchakning CA, CB tomonlariga va unga tashqi chizilgan aylanaga urinuvchi aylananing radiusi r^* uchun

$$r^* = \frac{ab}{p}\sqrt{\frac{(p-a)(p-b)}{p(p-c)}} \tag{5.1}$$

tenglik o‘rinli bo‘ladi.

Bunda $a = \left|BC\right|, b = \left|AC\right|, c = \left|AB\right|$ bo‘lib, p - uchburchakning yarim perimetri.

Isbot: ABC uchburchak o‘tmas burchakli ($\angle C > 90^\circ$) bo‘lgan holda isbotlaymiz. ABC uchburchakka tashqi



chizilgan aylana markazi O_2, CA, CB va uchburchakka tashqi chizilgan aylanaga urinuvchi aylana markazi O_1 bo‘lsin. Bu aylana AC tomonga N nuqtada urinsin. $O_2P \perp CA$, $O_1K \perp O_2P$ kesmalarni o‘tkazamiz (6-chizmaga qarang).

$\angle A = \alpha$, $\angle C = \gamma$ bo‘lsin. U holda $\mathrm{tg}\dfrac{\gamma}{2} = \dfrac{O_1N}{CN} = \dfrac{r^*}{CN}$. Bundan

6-chizma

$$CN = r^*\mathrm{ctg}\frac{\gamma}{2} = r^*\frac{1+\cos\gamma}{\sin\gamma} = r^*\frac{1+\dfrac{a^2+b^2-c^2}{2ab}}{\sin\gamma} =$$

$$= r^*\frac{(a+b-c)(a+b+c)}{2ab\sin\gamma} = r^*\frac{4(p-c)p}{4S_{\Delta ABC}} = r^*\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}.$$

O_1KO_2 uchburchakda $O_1O_2 = R - r^*$,

$$O_1K = CN - CP = r^*\sqrt{\frac{p(p-c)}{(p-a)(p-b)}} - \frac{a}{2},$$

$O_2K = O_2P - r^* = \sqrt{R^2 - \left(\dfrac{a}{2}\right)^2} - r^*$. Bunda R - uchburchakka tashqi chizilgan aylana radiusi. Endi ΔO_1KO_2 uchun Pifagor teoremasini qo‘llaymiz. $O_1K^2 + O_2K^2 = O_1O_2{}^2$, ya’ni

$$\left(\sqrt{R^2 - \left(\frac{a}{2}\right)^2} - r^*\right)^2 + \left(r^*\sqrt{\frac{p(p-c)}{(p-a)(p-b)}} - \frac{a}{2}\right)^2 = (R - r^*)^2. \qquad (5.2)$$

Endi ba’zi shakl almashtirishlarni bajaramiz:

1. $\sqrt{R^2 - \left(\dfrac{a}{2}\right)^2} = \sqrt{R^2 - R^2\sin^2\alpha} = R\left|\cos\alpha\right| = R\cos\alpha = R\dfrac{b^2+c^2-a^2}{2bc} =$

$$= R\left(\frac{(b-c)^2-a^2}{2bc} + 1\right) = R\left(1 - \frac{2(p-b)(p-c)}{bc}\right).$$

2. $\dfrac{4R(p-b)(p-c)}{bc} = \dfrac{abc}{S_{\Delta ABC}}\cdot\dfrac{(p-b)(p-c)}{bc} = a\cdot\sqrt{\dfrac{(p-b)(p-c)}{p(p-a)}}.$

(2) tenglikdan

$$-2\sqrt{R^2 - \frac{a^2}{4}} + r^*\frac{p(p-c)}{(p-a)(p-b)} - a\sqrt{\frac{p(p-c)}{(p-a)(p-b)}} = -2R$$

tenglik hosil bo‘ladi. Bundan

$$-2R\left(1 - \frac{2(p-b)(p-c)}{bc}\right) + r^*\frac{p(p-c)}{(p-a)(p-b)} - a\sqrt{\frac{p(p-c)}{(p-a)(p-b)}} = -2R,$$

$$\frac{4R(p-b)(p-c)}{bc}+r^*\frac{p(p-c)}{(p-a)(p-b)}-a\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}=0,$$

$$a\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}+r^*\frac{p(p-c)}{(p-a)(p-b)}-a\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}=0,$$

$$r^*\frac{p(p-c)}{(p-a)(p-b)}=a\sqrt{\frac{p-c}{p-a}}\left(\sqrt{\frac{p}{p-b}}-\sqrt{\frac{p-b}{p}}\right),$$

$$r^*=\frac{ab}{p}\sqrt{\frac{(p-a)(p-b)}{p(p-c)}}.$$

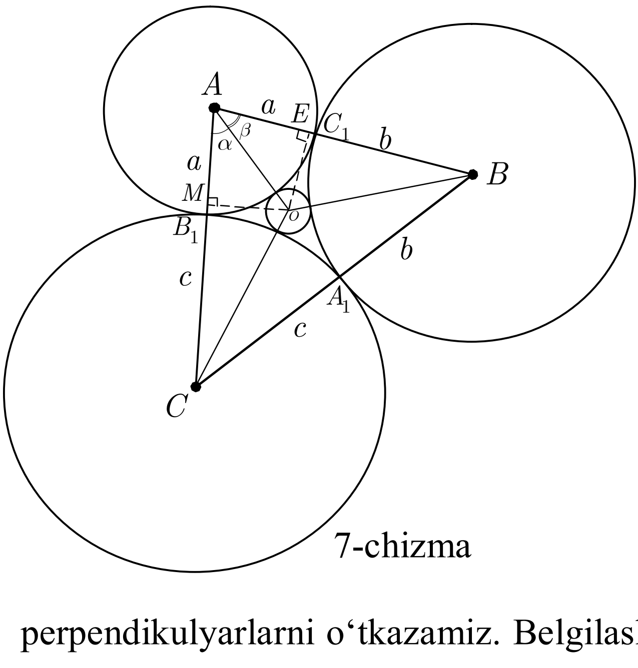
Demak, (5.1) tenglik o‘rinli.

ABC uchburchak to‘g‘ri burchakli yoki o‘tkir burchakli bo‘lsa ham (5.1) formula o‘rinli bo‘ladi (bu holatlarni tekshirib ko‘rishni o‘quvchilarga havola qilamiz). *Teorema isbotlandi*

1-izoh: Agar ABC uchburchak to‘g‘ri burchakli bo‘lsa, (5.1) tenglikdan $r^*=a+b-c$ kelib chiqadi.

2-izoh: ABC uchburchakka ichki chizilgan aylananing radiusi r bo‘lsa, (5.1) tenglik $r^*=\dfrac{abr}{p(p-c)}$ ko‘rinishni oladi.

Teorema 5.2. Markazi A,B,C nuqtalarda va radiuslari mos ravishda a,b,c ga teng bo‘lgan aylanalar tashqi tomondan o‘zaro urinadi. Bu aylanalar bilan chegaralangan shaklga eng katta r radiusli aylana ichki chizilgan bo‘lsa,



7-chizma

$$r=\frac{abc}{ab+bc+ac+2\sqrt{abc(a+b+c)}}$$

formula o‘rinli bo‘ladi.

Isboti Umumiylikka zid ish qilmagan holda $a\le b, a\le c$ deb hisoblaymiz.

Bu aylanalarning o‘zaro urinish nuqtalari A_1,B_1,C_1 bo‘lib, biz izlayotgan aylana markazi O bo‘lsin (7-chizmaga qarang). OA,OB,OC kesmalarni va OM,OE ($OM\perp AC$, $OE\perp AB$) perpendikulyarlarni o‘tkazamiz. Belgilash kiritamiz: $MB_1=z$, $EC_1=y$ bo‘lsin. AOM va COM uchburchaklar to‘g‘ri burchakli bo‘lgani uchun Pifagor teoremasidan foydalanib, $AO^2-AM^2=CO^2-CM^2$ tenglikka ega bo‘lamiz, ya’ni $(a+r)^2-(a-z)^2=(c+r)^2-(c+z)^2$. Bu tenglikdan $z=\dfrac{c-a}{c+a}\cdot r$

kelib chiqadi. Xuddi shunga o‘xshash jarayonni takrorlab, $y=\dfrac{b-a}{b+a}\cdot r$ tenglikka ega bo‘lamiz.

Belgilashlar kiritamiz: $\angle MAO=\alpha$, $\angle EAO=\beta$ va $n=\dfrac{c}{c+a}$, $m=\dfrac{b}{b+a}$, $x=\dfrac{r}{r+a}$. AMO va AEO uchburchaklar to‘g‘ri burchakli bo‘lgani uchun

$$\cos\alpha=\frac{a-z}{a+r}=1-\frac{z+r}{a+r}=1-\frac{2c}{c+a}\cdot\frac{r}{r+a}=1-2xn,$$
$$\cos\beta=\frac{a-y}{a+r}=1-\frac{y+r}{a+r}=1-\frac{2b}{b+a}\cdot\frac{r}{r+a}=1-2xm$$

bo‘ladi.

ABC uchburchak uchun kosinuslar teoremasini qo‘llab,

$$\cos(\alpha+\beta)=\frac{(a+b)^2+(a+c)^2-(b+c)^2}{2(a+b)(a+c)}=1-2mn$$

tenglikka ega bo‘lamiz.

Ushbu $(1-\cos^2\alpha)(1-\cos^2\beta)=\big((\cos\alpha\cdot\cos\beta-\cos(\alpha+\beta)\big)^2$ ayniyatni e’tiborga olsak,

$$\Big(1-(1-2nx)^2\Big)\Big(1-(1-2mx)^2\Big)=\Big((1-2nx)(1-2mx)-(1-2mn)\Big)^2$$

tenglik hosil bo‘ladi. Bu tenglikni soddalashtirsak, x ga nisbatan quyidagi kvadrat tenglama kelib chiqadi.

$$(4(mn)^2+(m-n)^2)x^2-2mn(m+n)x+(mn)^2=0.$$

bundan

$$x_1=\frac{m+n-2\sqrt{mn\left(1-mn\right)}}{4\left(mn\right)^2+\left(m-n\right)^2}\cdot mn$$
$$x_2=\frac{m+n+2\sqrt{mn\left(1-mn\right)}}{4\left(mn\right)^2+\left(m-n\right)^2}\cdot mn$$

yechimlarga ega bo‘lamiz.

Biz izlayotgan r radius uchun faqat x_1 qiymat to‘g‘ri keladi (x_2 qiymat to‘g‘ri kelmasligini tekshirib ko‘rish mumkin).

Belgilashimizga ko‘ra

$$n=\frac{c}{c+a},\quad m=\frac{b}{b+a},\quad x=\frac{r}{r+a}$$

ekanini e’tiborga olsak, x_1 ning ifodasidan

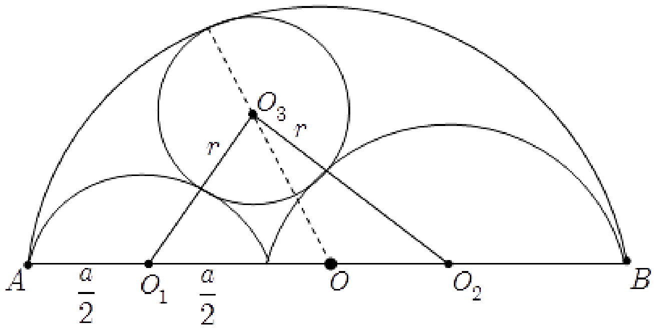
$$r = \frac{a \cdot b \cdot c}{ab + bc + ac + 2\sqrt{abc\left(a + b + c\right)}}$$

bo‘lishi kelib chiqadi. *Teorema isbotlandi.*

Teorema 5.3. Diametri $AB = a + b\ \left(a > 0,\, b > 0\right)$ ga teng bo‘lgan yarim doiraga markazi AB da yotgan, diametri a va b ga teng bo‘lgan yarim doiralar ichki chizilgan (8-chizmaga qarang). Uchta yarim aylana bilan chegaralangan shaklga diametri d ga teng aylana ichki chizilgan bo‘lsa,

$$d = \frac{ab\left(a + b\right)}{a^2 + ab + b^2}$$

tenglik o‘rinli bo‘ladi.



8-chizma

Isbot. Diametri $a + b$, a, b bo‘lgan doiralarning markazlari mos ravishda O, O_1, O_2 bo‘lsin. Aytaylik biz izlayotgan doira markazi O_3 va radiusi r bo‘lib, $\angle O_1OO_3 = \alpha$ bo‘lsin. U holda

$$OO_1 = AO - AO_1 = \frac{a + b}{2} - \frac{a}{2} = \frac{b}{2}\,, \tag{5.3}$$

$$OO_2 = BO - BO_2 = \frac{a + b}{2} - \frac{b}{2} = \frac{a}{2}\,, \tag{5.4}$$

$$OO_3 = \frac{a + b}{2} - r \tag{5.5}$$

tengliklar o‘rinli.

Endi O_1OO_3 va O_2OO_3 uchburchaklarga kosinuslar teoremasini ko‘llaymiz:

$$O_1O_3^2 = OO_1^2 + OO_3^2 - 2OO_1 \cdot OO_3 \cdot \cos\alpha\,, \tag{5.6}$$

$$O_2O_3^2 = OO_2^2 + OO_3^2 - 2OO_2 \cdot OO_3 \cdot \cos(\pi - \alpha)\,. \tag{5.7}$$

(5.3), (5.4), (5.5) tengliklardan foydalansak, (5.6), (5.7) tenglamalar quyidagi ko‘rinishga keladi:

$$\left(\frac{a}{2} + r\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{a + b}{2} - r\right)^2 - b\left(\frac{a + b}{2} - r\right)\cos\alpha\,, \tag{5.8}$$

$$\left(\frac{b}{2}+r\right)^2=\left(\frac{a}{2}\right)^2+\left(\frac{a+b}{2}-r\right)^2-a\left(\frac{a+b}{2}-r\right)\cos\alpha\,. \qquad (5.9)$$

(5.8) tenglikni a ga, (5.9) tenlikni esa b ga ko‘paytirib, hosil bo‘ladigan tengliklarni qo‘shamiz. U holda quyidagi tenglikka ega bo‘lamiz:

$$a\left(\frac{a}{2}+r\right)^2+b\left(\frac{b}{2}+r\right)^2=a\left(\frac{b}{2}\right)^2+b\left(\frac{a}{2}\right)^2+(a+b)\left(\frac{a+b}{2}-r\right)^2$$

Bu tenglikni soddalashtirib, undan r ni topamiz

$$r=\frac{ab(a+b)}{2(a^2+ab+b^2)}.$$

Bu esa teorema o‘rinli ekanini bildiradi. *Teorema isbotlandi.*

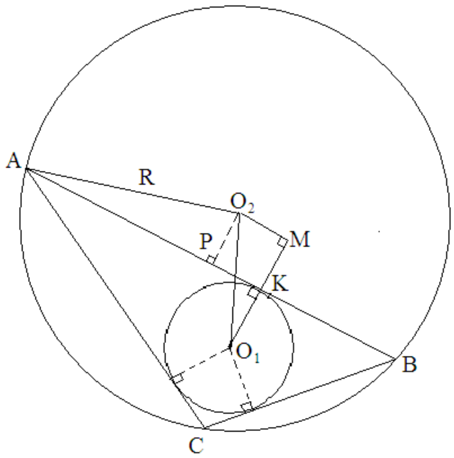
Pifagor teoremasini qo‘llab, Eylerning mashhur teoremasini sodda isbotini keltiramiz.

Teorema 5.4. (Eyler). Agar ABC uchburchakka ichki va tashqi chizilgan aylanalarning radiuslari r, R bo‘lsa, ularning markazlari orasidagi masofa d uchun

$$d^2=R^2-2Rr \qquad (5.10)$$

tenglik o‘rinli bo‘ladi.

Isbot: ABC uchburchak o‘tmas burchakli ($\angle C=\gamma>90^\circ$) deb hisoblaymiz. Bu holda tashqi chizilgan aylana markazi uchburchakdan tashqarida bo‘ladi. $AB=c$, $BC=a$, $CA=b$ va ichki chizilgan aylana markazi O_1, tashqi chizilgan aylana markazi O_2 bo‘lsin. Ichki chizilgan aylana AB tomonga K nuqtada urinsin. O_1K ning davomiga perpendikulyar bo‘lgan O_2M kesmani va $O_2P\perp AB$ kesmani o‘tkazamiz (9-chizmaga qarang).



9-chizma

O_1MO_2 uchburchakda $O_1M=r+\sqrt{R^2-\left(\frac{c}{2}\right)^2}$, $O_2M=PK=\frac{|a-b|}{2}$,

$O_1O_2=d$ bo‘lgani uchun

$$d^2=\left(r+\sqrt{R^2-\left(\frac{c}{2}\right)^2}\right)^2+\left(\frac{a-b}{2}\right)^2=R^2-2Rr\cos\gamma-(p-a)(p-b)+r^2.$$

Endi ba’zi shakl almashtirishlarni bajaramiz:

1. $\cos\gamma=\frac{a^2+b^2-c^2}{2ab}=\frac{(a-b)^2-c^2+2ab}{2ab}=-\frac{2(p-a)(p-b)}{ab}+1;$

2. $\dfrac{Rr(p-a)(p-b)}{ab}=\dfrac{\dfrac{abc}{4S_{\Delta ABC}}\cdot\dfrac{S_{\Delta ABC}(p-a)(p-b)}{p}}{ab}=\dfrac{c(p-a)(p-b)}{4p}$;

3. $r^2=\left(\dfrac{S_{\Delta ABC}}{p}\right)^2=\dfrac{(p-a)(p-b)(p-c)}{p}=(p-a)(p-b)-c\dfrac{(p-a)(p-b)}{p}$.

Bularga ko‘ra $d^2=R^2-2Rr\left(-\dfrac{2(p-a)(p-b)}{ab}+1\right)-\dfrac{c(p-a)(p-b)}{p}=$

$=R^2-2Rr+4\dfrac{Rr(p-a)(p-b)}{ab}-\dfrac{c(p-a)(p-b)}{p}=$

$=R^2-2Rr+4\dfrac{c(p-a)(p-b)}{4p}-\dfrac{c(p-a)(p-b)}{p}=R^2-2Rr$.

Demak, (5.10) tenglik o‘rinli. ABC uchburchak to‘g‘ri burchakli va o‘tkir burchakli bo‘lgan hollarda Eyler formulasini isbotlashni o‘quvchiga hovola qilamiz. *Teorema isbotlandi*.

Mustaqil yechish uchun masalalar

1. Tomonlari a,b,c bo‘lgan o‘tkir burchakli uchburchak berilgan bo‘lib, unga tashqi chizilgan aylana markazidan mos tomonlarga k_a,k_b,k_c perpendikulyarlar tushurilgan bo‘lsa, u holda

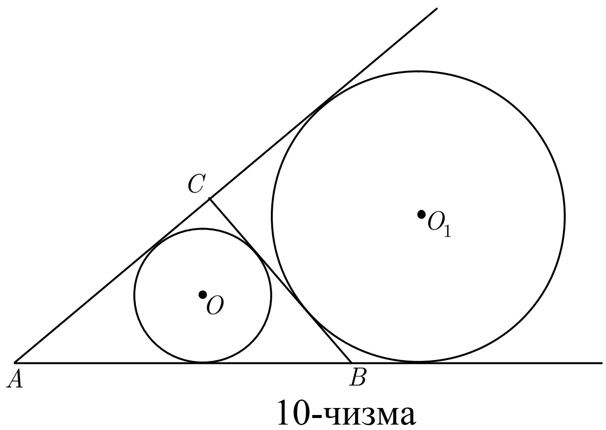
$$k_a+k_b+k_c=R+r$$

tenglik o‘rinli bo‘lishini isbotlang. Bunda R,r - uchburchakka tashqi va ichki chizilgan aylanalar radiuslari.

2. Muntazam uchburchakka tashqi chizilgan aylananing ihtiyoriy nuqtasidan uchburchak uchlarigacha bo‘lgan masofalar kvadratlarining yig‘indisi o‘zgarmas ekanini isbotlang.

3. ABC uchburchakda $AB=c$, $BC=a$, $CA=b$ bo‘lib, unga ichki chizilgan aylana markazi O bo‘lsin. (10-chizmaga qarang).

AB va AC tomonlarning davomlariga hamda BC tomonga urinuvchi aylana markazi O_1 va $d_a=OO_1$ bo‘lsin. Huddi shunga o‘xshash d_b,d_c uzunliklarni ham kiritamiz. U holda $d_a\cdot d_b\cdot d_c=16R^2\cdot r$ tenglik o‘rinli bo‘lishini isbotlang. Bunda R,r - uchburchakka tashqi va ichki chizilgan aylanalar radiuslari.



10-чизма

4. Radiuslarining nisbatlari 1:3 bo‘lgan ikkita aylana tashqi urinadi va ularning umumiy tashqi urinmasi uzunligi $6\sqrt{3}$ ga teng. Aylanalarning tashqi urinmalari va tashqi yoylari hosil qilgan figuraning perimetrini toping.

5. Doiraga ichki chizilgan to‘rtburchakda qarama-qarshi tomonlari ko‘paytmalarning yig‘indisi diagonallari ko‘paytmasiga tengligini isbotlang (Ptolomey teoremasi).
6. ([2]). $ABCD$ qavariq to‘rtburchakning diagonallari O nuqtada kesishadi. Agar AOB, BOC va COD uchburchaklarga ichki chizilgan aylanalarning radiuslari 3, 4 va 6 ga teng bo‘lsa, DOA uchburchakka ichki chizilgan aylana radiusini toping.
7. ([3]). ABC uchburchak ichidagi O nuqtadan bil xil radiusli uchta aylana o‘tkazilgan. Bu aylanalar uchburchak ichida yotadi va uchburchakning ikkita tomoniga urinadi. U holda O nuqta ABC uchburchakka tashqi chizilgan aylana markazi ekanini isbotlang.
8. Teorema 5.2. ning fazoviy analogini aniqlang.
9. Teorema 5.4. ning fazoviy analogini aniqlang.

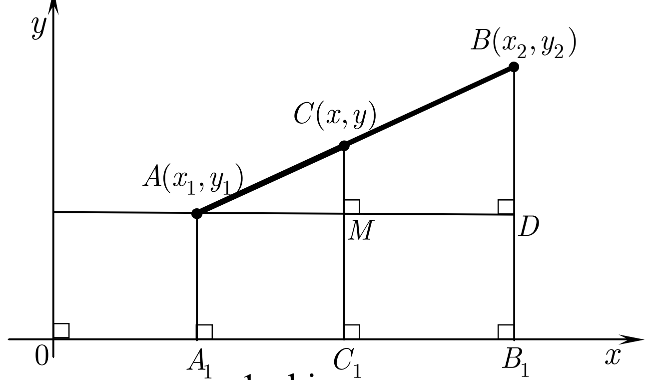
6-§. Dekart koordinatalar sistemasining ba’zi masalalarga tadbiqi

Dekart koordinatalar sistemasiga oid nazariy tushunchalarni [4], [5], [8] adabiyotlardan o‘rganishni tavsiya qilamiz.

xOy koordinatalar sistemasida $A\left(x_1, y_1\right)$, $B\left(x_2, y_2\right)$ nuqtalar berilgan bo‘lsin. AB kesmada $C\left(x,\, y\right)$ nuqta shunday olinganki, $\dfrac{AC}{CB} = \lambda \ \left(0 < \lambda < \infty\right)$ bo‘lsin. U holda $C\left(x,\, y\right)$ nuqtaning koordinatalari uchun

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda} \tag{6.1}$$

tenglik o‘rinli bo‘ladi. Haqiqatan, A, B va C nuqtalardan Ox o‘qiga perpendikulyar to‘g‘ri chiziqlar o‘tkazib, ularning Ox o‘q bilan kesishgan nuqtalarini mos ravishda A_1, B_1 va C_1 bilan belgilaymiz (1-chizmaga qarang).



1-chizma

$AD \perp BB_1$ bo‘ladigan qilib AD perpendikulyarni o‘tkazamiz. Bu AD va CC_1 perpendikulyarlar M nuqtada kesishsin. ΔACM va ΔABD uchburchaklar o‘xshash bo‘lgani uchun $\dfrac{AC}{AB}=\dfrac{AM}{AD}=\dfrac{CM}{BD}$ bo‘ladi. Bundan

$$\frac{\lambda}{1+\lambda}=\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

tenglikka ega bo‘lamiz. Bu tenglikda shakl almashtirib,

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\quad y=\frac{y_1+\lambda y_2}{1+\lambda}$$

formulaga ega bo‘lamiz.

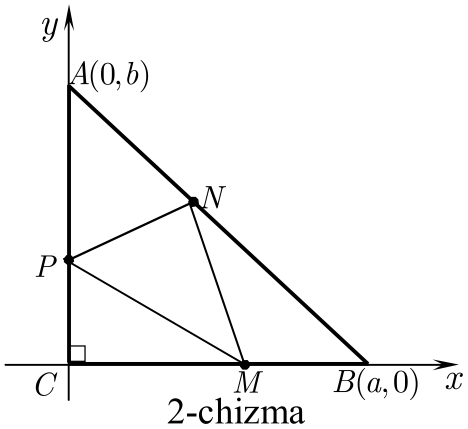
Tekislikda $A\left(x_1,y_1\right)$, $B\left(x_2,y_2\right)$, $C\left(x_3,y_3\right)$ nuqtalar berilgan bo‘lsa, ABC uchburchakning yuzini hisoblash uchun

$$S_{\Delta ABC}=\frac{1}{2}\operatorname{mod}\left(\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}+\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}+\begin{vmatrix}x_3 & y_3\\ x_1 & y_1\end{vmatrix}\right) \tag{6.2}$$

formula o‘rinli bo‘ladi, bu erda mod - haqiqiy sonning absolyut qiymati. Uning isbotini [5] adabiyotdan o‘rganishni tavsiya qilamiz. Yuqoridagi (6.1) va (6.2) formulalardan quyidagi masalani yechishda foydalanamiz.

1-masala. ABC to‘g‘ri burchakli $\left(\angle C=90^0\right)$ uchburchakning CB, BA, AC tomonlarida mos ravishda M, N, P nuqtalar shunday olinganki, $\dfrac{MB}{CM}=\dfrac{NA}{BN}=\dfrac{PC}{AP}$ tengliklar o‘rinli. Agar $S_{\Delta ABC}=S$ bo‘lsa, $S_{\Delta MNP}$ ning eng kichik qiymatini toping.

Yechish. Uchburchakning C to‘g‘ri burchagidan xOy koordinatalar sistemasini kiritamiz.



2-chizma

Aytaylik, $CB=a, CA=b, \dfrac{MB}{CM}=\lambda$ bo‘lsin. Masala shartidan va (6.1) formuladan foydalanib, $M(x_1,y_1)$, $N(x_2,y_2)$, $P(x_3,y_3)$ nuqtalarning koordinatalari uchun quyidagi tengliklarga ega bo‘lamiz (2-chizmaga qarang):

$$x_1=\frac{a}{1+\lambda}, y_1=0,\quad x_2=\frac{\lambda a}{1+\lambda}, y_2=\frac{b}{1+\lambda},\quad x_3=0, y_3=\frac{\lambda b}{1+\lambda}.$$

Yuqoridagi (6.2) formuladan foydalanib MNP uchburchak yuzini hisoblaymiz:

$$S_{\Delta MNP} = \frac{1}{2}\operatorname{mod}\left(\begin{vmatrix} \frac{a}{1+\lambda} & 0 \\ \frac{\lambda a}{1+\lambda} & \frac{b}{1+\lambda} \end{vmatrix} + \begin{vmatrix} \frac{\lambda a}{1+\lambda} & \frac{b}{1+\lambda} \\ 0 & \frac{\lambda b}{1+\lambda} \end{vmatrix} + \begin{vmatrix} 0 & \frac{\lambda b}{1+\lambda} \\ \frac{a}{1+\lambda} & 0 \end{vmatrix} \right) =$$

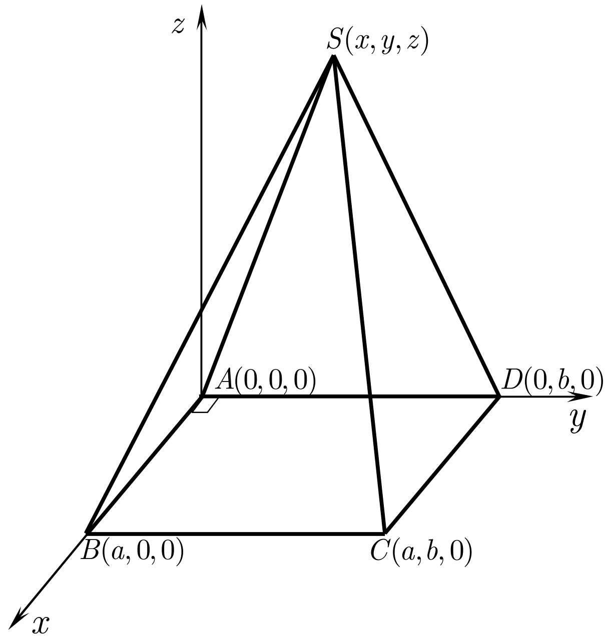
$$= \frac{1}{2}\left(\frac{ab + \lambda^2 ab - \lambda ab}{\left(1+\lambda\right)^2} \right) = \frac{1}{2}ab\left(\frac{\lambda^2 - \lambda + 1}{\left(\lambda+1\right)^2} \right) = \left(\frac{\left(\lambda+1\right)^2 - 3\left(\lambda+1\right) + 3}{\left(\lambda+1\right)^2} \right) S =$$

$$= \left(1 - 3\cdot\frac{1}{\lambda+1} + 3\left(\frac{1}{\lambda+1}\right)^2 \right) S = \left(3t^2 - 3t + 1\right) S, \text{ bu yerda } t = \frac{1}{1+\lambda}.$$

$\varphi(t) = 3t^2 - 3t + 1$ parabola eng kichik qiymatiga $t_0 = \frac{1}{2}$ nuqtada, ya'ni $\lambda = 1$ da erishganligi uchun MNP uchburchak yuzasining eng kichik qiymati $\frac{1}{4}S$ ga teng bo‘ladi.

2-masala. $ABCD$ to‘g‘ri to‘rtburchak va fazoda S nuqta berilgan. U holda $AS^2 + SC^2 = BS^2 + SD^2$ tenglik o‘rinli bo‘lishini isbotlang.

Yechish. $ABCD$ to‘g‘ri to‘rtburchakning A uchidan Dekart koordinatalar sistemasini kiritamiz. AB to‘g‘ri chiziq Ox o‘q, AD to‘g‘ri chiziq Oy o‘q bo‘lsin (3-chizmaga qarang). Aytaylik



3-chizma

$AB = a$, $AD = b$ va $S(x, y, z)$ bo‘lsin. U holda $A(0,0,0)$, $B(a,0,0)$, $C(a,b,0)$, $D(0,b,0)$ bo‘ladi. Ikki nuqta orasidagi masofani hisoblash formulasiga ko‘ra

$$SA^2 = x^2 + y^2 + z^2, \quad SC^2 = (x-a)^2 + (y-b)^2 + z^2,$$
$$SB^2 = (x-a)^2 + y^2 + z^2, \quad SD^2 = x^2 + (y-b)^2 + z^2.$$

Bundan $AS^2 + SC^2 = BS^2 + SD^2$ tenglik kelib chiqadi.

Natija. Agar $AS = a$, $CS = b$, $BS = \sqrt{a^2 + b^2}$ bo‘lsa, u holda $S_{ABCD} = a \cdot b$ bo‘ladi.

3-masala. Tenglamani yeching:

$$\sqrt{x^2 + a^2 - 4x - 6a + 13} + \sqrt{x^2 + a^2 - 10x - 14a + 74} = 5, \quad (6.3)$$

bunda $a \in R$.

Yechish. (3) tenglamada shakl almashtirib, uni

$$\sqrt{(x-2)^2 + (a-3)^2} + \sqrt{(x-5)^2 + (a-7)^2} = 5 \quad (6.4)$$

ko‘rinishida yozib olamiz. Quyidagicha belgilash kiritamiz: $A(x,a)$, $B(2,3)$, $C(5,7)$ bo‘lsin. U holda (1.4) tenglikka asosan $BA + AC = BC$ bo‘ladi. Bu esa $\overrightarrow{BA}(x-2, a-3)$ va $\overrightarrow{BC}(3,4)$ vektorlarning kollinear ekanligini bildiradi. Kollinearlik shartlariga asosan

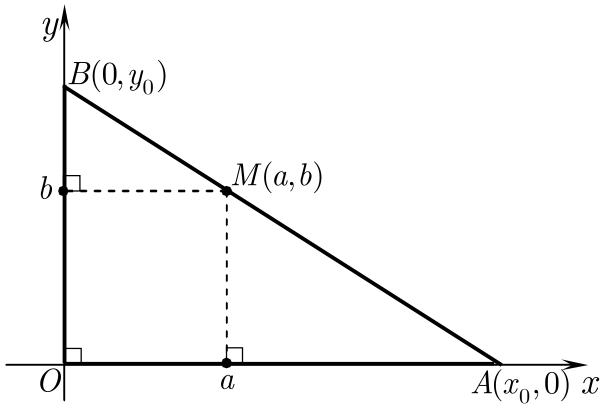
$$\frac{x-2}{3} = \frac{a-3}{4} \text{yoki } x = \frac{3a}{4} - \frac{1}{4} \text{ bo‘ladi.}$$

Endi quyidagi

$$\begin{aligned} &\sqrt{x^2 + a^2 - 2\lambda x - 2\mu a + \lambda^2 + \mu^2} + \sqrt{x^2 + a^2 - 2kx - 2ma + k^2 + m^2} = \\ &= \sqrt{(k-\lambda)^2 + (m-\mu)^2} \end{aligned} \quad (6.5)$$

tenglamani qaraymiz, bunda a, λ, μ, k, m- haqiqiy sonlar. (6.5) tenglama (6.3) tenglamaning umumlashgan holatidir. Bu tenglamani yechish uchun belgilash kiritamiz: $A(x,a)$, $B(\lambda,\mu)$, $C(k,m)$. U holda (6.5) tenglamaga ko‘ra $BA + AC = BC$ bo‘ladi. Bu esa $\overrightarrow{BA}(x-\lambda, a-\mu)$ va $\overrightarrow{BC}(k-\lambda, m-\mu)$ vektorlarning kollinearligini bildiradi. Kollinearlik shartiga asosan $\frac{x-\lambda}{k-\lambda} = \frac{a-\mu}{m-\mu}$ bo‘ladi. Bundan $x = \frac{k-\lambda}{m-\lambda}a - \frac{k-\lambda}{m-\mu}\mu + \lambda$ bo‘ladi.

4-masala. To‘g‘ri burchakli koordinatalar sistemasida $M(a,b)$ nuqta berilgan $(a > 0,\ b > 0)$. Bu nuqta orqali AB to‘g‘ri chiziqni shunday o‘tkazingki, bu to‘g‘ri chiziq Ox va Oy o‘qlarini musbat yo‘nalishini kesib o‘tsin va AOB uchburchakning yuzi eng kichik bo‘lsin (4-chizmaga qarang).



4-chizma

Yechish. Aytaylik $A(x_0,0)$ va $B(0,y_0)$ bo‘lsin. U holda AB to‘g‘ri chiziq tenglamasi $\frac{x-x_0}{a-x_0}=\frac{y-0}{b-0}$ ko‘rinishda bo‘lib, $y_0=\frac{b}{a-x_0}\cdot(-x_0)=\frac{bx_0}{x_0-a}$ ga teng bo‘ladi. To‘g‘ri burchakli uchburchakning yuzi formulasiga ko‘ra $S=\frac{1}{2}x_0y_0=\frac{1}{2}\frac{bx_0^2}{x_0-a}$ ni hosil qilamiz. Ravshanki, u x_0 ning funksiyasiga aylanadi. Endi $S_{\Delta AOB}$ ning eng kichik qiymatini topish uchun $S(t)=\frac{1}{2}\frac{bt^2}{t-a}$ funksiyani $\left\{t\in R,\ t>0\right\}$ oraliqda eng kichik qiymatini hosila yordamida topamiz:

$$S(t)=\frac{1}{2}\frac{bt^2}{t-a}=\frac{1}{2}\frac{b((t-a)+a)^2}{t-a}=\frac{1}{2}\left(b(t-a)+2ab+\frac{ba^2}{t-a}\right),$$

$$S'(t)=\frac{1}{2}\left(b-\frac{ba^2}{(t-a)^2}\right)=0.$$

Natijada $b\frac{(t-a)^2-a^2}{(t-a)^2}=0$ tenglamani yechib, funksiya ekstremumining yetarli shartidan foydalanib, $t=2a$ nuqtada $S(t)$ funksiyaning eng kichik qiymatga erishishini topamiz. Demak, AB to‘g‘ri chiziq tenglamasi $y=-\frac{b}{a}\left(x-2a\right)$ ko‘rinishida bo‘ladi.

5-masala. xyz fazoda $M\left(x_0,y_0,z_0\right)$, $\left(x_0>0,\ y_0>0,\ z_0>0\right)$ nuqta berilgan. Bu nuqtadan Ox,Oy,Oz o‘qlarni A, B, C (koordinatalar boshidan farqli) nuqtalarda kesib o‘tuvchi tekislik o‘tkazilgan. A,B,C,O nuqtalarni tutashtirishdan piramida hosil bo‘ladi. Piramida hajmi V_{ABCO} ning eng kichik qiymatini toping.

Yechish. $A\left(x_1,0,0\right)$, $B\left(0,y_1,\ 0\right)$, $C\left(0,0,\ z_1\right)$ bo‘lsin. Bizga ma’lumki, ([4], [5]) ga qarang) $M\left(x_0, y_0, z_0\right)$ nuqtadan o‘tuvchi tekislik tenglamasi

$$\begin{aligned} &a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0, \\ &a^2+b^2+c^2=1,\ a\cdot b\cdot c>0 \end{aligned} \tag{6.6}$$

ko‘rinishda bo‘ladi. Undan x_1, y_1, z_1 larni topamiz:

1) Agar (6.6) tenglikda $y=0, z=0$ qilib tanlasak, $x_1=\dfrac{ax_0+by_0+cz_0}{a}$ bo‘ladi;

2) Agar (6.6) tenglikda $x=0, z=0$ qilib tanlasak, $y_1=\dfrac{ax_0+by_0+cz_0}{b}$ bo‘ladi;

3) Agar (6.6) tenglikda $y=0, x=0$ qilib tanlasak, $z_1=\dfrac{ax_0+by_0+cz_0}{c}$ bo‘ladi.

Piramida hajmini hisoblash formulasi va Koshi tengsizligiga ko‘ra

$$V_{ABCO}=\frac{1}{6}x_1y_1z_1=\frac{1}{6abc}\cdot\left(ax_0+by_0+cz_0\right)^3\geq\frac{1}{6abc}\cdot\left(3\sqrt[3]{abc\cdot x_0\cdot y_0\cdot z_0}\right)^3$$
$$=\frac{9}{2}x_0y_0z_0=V_{\min}$$

bo‘ladi. Tenglik sharti $ax_0=by_0=cz_0$ bo‘lganda bajariladi. Demak, V_{ABCO} ning eng kichik qiymati $\frac{9}{2}x_0y_0z_0$ ga teng.

6-masala. Tekislikda $ABCD$ qavariq to‘rtburchak berilgan bo‘lib, M va N nuqtalar AC va BD diagonallarning o‘rtalari bo‘lsa, u holda

$$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2-4MN^2$$

tenglik o‘rinli bo‘lishini isbotlang.

Yechish. Tekislikda koordinatalar sistemasini kiritamiz. Bu sistemaga nisbatan $A\left(x_1, y_1\right)$, $B\left(x_2, y_2\right)$, $C\left(x_3, y_3\right)$, $D\left(x_4, y_4\right)$ bo‘lsin. U holda $M\left(\dfrac{x_1+x_3}{2};\dfrac{y_1+y_3}{2}\right)$ va $N\left(\dfrac{x_2+x_4}{2};\dfrac{y_2+y_4}{2}\right)$ bo‘ladi.

$$AC^2+BD^2=\left(x_3-x_1\right)^2+\left(y_3-y_1\right)^2+\left(x_4-x_2\right)^2+\left(y_4-y_2\right)^2;$$

$$\begin{aligned} &AB^2+BC^2+CD^2+DA^2-4MN^2=\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2+\left(x_2-x_3\right)^2+ \\ &+\left(y_2-y_3\right)^2+\left(x_3-x_4\right)^2+\left(y_3-y_4\right)^2+\left(x_4-x_1\right)^2+\left(y_4-y_1\right)^2- \\ &-\left(\left(\left(x_1-x_2\right)+\left(x_3-x_4\right)\right)^2+\left(\left(y_1-y_2\right)+\left(y_3-y_4\right)\right)^2\right)=\left(x_2-x_3\right)^2+\left(x_4-x_1\right)^2+ \\ &+\left(y_2-y_3\right)^2+\left(y_4-y_1\right)^2-2\left(x_1-x_2\right)\left(x_3-x_4\right)-2\left(y_1-y_2\right)\left(y_3-y_4\right)= \\ &=\left(x_1^2+x_3^2\right)+\left(x_2^2+x_4^2\right)-2x_2x_3-2x_4x_1+\left(y_1^2+y_3^2\right)+\left(y_2^2+y_4^2\right)- \\ &-2y_2y_3-2y_4y_1-2\left(x_1x_3-x_1x_4-x_2x_3+x_2x_4\right)-2\left(y_1y_3-y_1y_4-y_2y_3+y_2y_4\right)= \end{aligned}$$

$$=\left(x_1-x_3\right)^2+\left(x_2-x_4\right)^2+\left(y_1-y_3\right)^2+\left(y_2-y_4\right)^2=$$

$$=\left(\left(x_3-x_1\right)^2+\left(y_3-y_1\right)^2\right)+\left(\left(x_4-x_2\right)^2+\left(y_4-y_2\right)^2\right)=AC^2+BD^2.$$

Yuqoridagi teorema *Eyler teoremasi* deb yuritiladi.

7-masala. $l-$ fazodagi kesmaning uzunligi, a,b,c lar esa uning Oxy, Oyz, Oxz koordinata tekisliklaridagi proeksiyalari bo‘lsin. U holda $\dfrac{a+b+c}{l}$ nisbatning eng katta qiymatini toping.

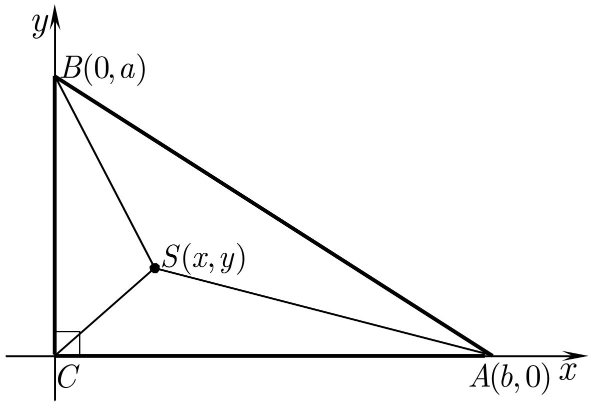
Yechish: Bu kesmaning Ox, Oy, Oz o‘qdagi proeksiyalari mos ravishda x,y,z bo‘lsin. U holda $a^2=x^2+y^2$, $b^2=y^2+z^2$, $c^2=z^2+x^2$ va $l^2=x^2+y^2+z^2=\dfrac{1}{2}(a^2+b^2+c^2)$ bo‘ladi. Bundan esa

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac\le a^2+b^2+c^2+(a^2+b^2)+$$
$$+(b^2+c^2)+(a^2+c^2)=3(a^2+b^2+c^2)=6l^2$$

tengsizlikka ega bo‘lamiz. Demak, $\dfrac{a+b+c}{l}\le\sqrt{6}$ ekan. Tenglik sharti $a=b=c$ bo‘lganda bajariladi.

8-masala. To‘g‘ri burchakli ABC $(\angle C=90^0)$ uchburchakda $CA=b$, $CB=a$ bo‘lib, uchburchak ichida S nuqta olingan. $SA^2+SB^2+SC^2$ yig‘indining eng kichik qiymatini toping.

Yechish: Uchburchakning C uchidan Dekart koordinatalar sistemasini kiritamiz. Aytaylik, CA va CB to‘g‘ri chiziqlar mos ravishda Ox va Oy o‘qlari bo‘lsin (5-chizmaga qarang).



5-chizma

U holda $C(0,0)$, $A(b,0)$ va $B(0,a)$ bo‘ladi. Agar $S(x,y)$ desak, u holda

$$SA^2+SB^2+SC^2=(x-b)^2+y^2+x^2+(y-a)^2+x^2+y^2=$$

$$= 3x^2 + 3y^2 - 2bx - 2ay + a^2 + b^2 =$$

$$= 3(x - \frac{b}{3})^2 + 3(y - \frac{a}{3})^2 + \frac{2}{3}(a^2 + b^2) \geq \frac{2}{3}(a^2 + b^2)$$

bo‘ladi. Demak, $x = \frac{b}{3}$, $y = \frac{a}{3}$ bo‘lsa, $SA^2 + SB^2 + SC^2$ yig‘indi o‘zining eng kichik qiymati $\frac{2}{3}(a^2 + b^2)$ ga teng bo‘ladi.

Mustaqil yechish uchun masalalar

1. $ABCD$ tetraedrning qarama-qarshi qirralarining o‘rtalarini tutashtiruvchi kesmalar bir nuqtada kesishishini isbotlang.
2. To‘g‘ri burchakli parallelepipedning barcha qirralari yig‘indisi l ga teng. Uning to‘la sirtining eng katta qiymatini toping.
3. To‘g‘ri burchakli parallelepipedning to‘la sirti S ga teng. Uning barcha qirralari yig‘indisining eng kichik qiymatini toping.
4. ABC teng yonli uchburchak berilgan. Uchburchak ichida shunday M nuqtani topingki, undan uchburchak tomonlarigacha bo‘lgan masofalar yig‘indisi eng kichik bo‘lsin.
5. ABC uchburchakning yuzasi S ga teng. Uning ichida shunday M nuqtani topingki, undan tomonlargacha bo‘lgan masofalar ko‘paytmasi eng katta bo‘lsin.

6.* ABC to‘g‘ri burchakli uchburchak berilgan. Uning ichida shunday M nuqtani topingki, $MA + MB + MC$ – eng katta bo‘lsin.

7.** ABC to‘g‘ri burchakli uchburchak berilgan. Uning ichida shunday M nuqtani topingki, $MA + MB + MC$ – eng kichik bo‘lsin.

7-§. Ba’zi masalalarga vektorlarning tadbiqlari

Maktab o‘quvchilari vektorlar mavzusini qiyin o‘zlashtirishi, vektorlarga ishonchsizlik bilan qarashlari yaxshi ma’lum. Agar murakkab masalalarga vektorlarni tadbiq qilishni o‘rgansalar, bu holat o‘z o‘zidan o‘tib ketadi deb o‘ylaymiz. Shu maqsadda, quyidagi masalalarni vektorlar yordamida yechib ko‘rsatamiz.

1-masala. Agar α, β va γ ixtiyoriy uchburchakning ichki burchaklari bo‘lsa, u holda quyidagi

$$\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2} \tag{7.1}$$

tengsizlikni isbotlang.

Isbot. Tekislikda bitta nuqtadan chiquvchi $\overline{r_1}$, $\overline{r_2}$, $\overline{r_3}$ birlik vektorlarni olamiz. Ular oralaridagi burchaklar $180^0 - \alpha$, $180^0 - \beta$ va $180^0 - \gamma$ bo‘lsin.

Ushbu $(\overline{r_1}+\overline{r_2}+\overline{r_3})^2\geq 0$ tengsizlik o‘rinli bo‘lishi ayon. Bunga ko‘ra

$$\overline{r_1}^2+\overline{r_2}^2+\overline{r_3}^2+2\overline{r_1}\overline{r_2}+2\overline{r_1}\overline{r_3}+2\overline{r_2}\overline{r_3}\geq 0,$$

$$3+2\cos(180^0-\alpha)+2\cos(180^0-\beta)+2\cos(180^0-\gamma)\geq 0,$$

$$3-2[\cos\alpha+\cos\beta+\cos\gamma]\geq 0.$$

Oxirgi tengsizlikdan (7.1) tengsizlik kelib chiqadi.

2-masala. Agar α, β va γ ixtiyoriy uchburchakning ichki burchaklari bo‘lsa, u holda quyidagi

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma\leq\frac{9}{4}\tag{7.2}$$

tengsizlikni isbotlang.

Isbot. Tekislikda bitta nuqtadan chiquvchi $\overline{r_1}$, $\overline{r_2}$, $\overline{r_3}$ birlik vektorlarni olamiz. Ular oralaridagi burchaklar 2α, 2β va 2γ bo‘lsin. U holda ushbu $(\overline{r_1}+\overline{r_2}+\overline{r_3})^2\geq 0$ tengsizlikka ko‘ra

$$3+2\cos 2\alpha+2\cos 2\beta+2\cos 2\gamma\geq 0,$$

$$3+2[3-2\sin^2\alpha-2\sin^2\beta-2\sin^2\gamma]\geq 0.$$

Oxirgi tengsizlikdan (7.2) kelib chiqadi.

1-natija. Agar (7.2) tengsizlikka ushbu $\sin\alpha=\dfrac{a}{2R}$, $\sin\beta=\dfrac{b}{2R}$, $\sin\gamma=\dfrac{c}{2R}$ ifodalarni qo‘ysak, u holda $a^2+b^2+c^2\leq 9R^2$ tengsizlik hosil bo‘ladi. Bu yerda a, b va c lar uchburchak tomonlarining uzunliklari bo‘lib, R - bu uchburchakka tashqi chizilgan aylana radiusi.

3-masala. Agar ABC uchburchakka tashqi chizilgan aylana markazi O va bu uchburchak medianalari kesishgan nuqtasi M bo‘lsa, ushbu

$$\overline{OM}=\frac{1}{3}(\overline{OA}+\overline{OB}+\overline{OC})\tag{7.3}$$

tenglikni isbotlang.

Isbot. Ushbu $\overline{OM}=\overline{OA}+\overline{AM}$, $\overline{OM}=\overline{OB}+\overline{BM}$, $\overline{OM}=\overline{OC}+\overline{CM}$ tengliklarni qo‘shsak, $3\overline{OM}=(\overline{OA}+\overline{OB}+\overline{OC})+(\overline{AM}+\overline{BM}+\overline{CM})$ kelib chiqadi. Endi, uchburchak medianalari kesishish nuqtasida $2:1$ nisbatda bo‘linishini inobatga olib, ushbu $\overline{AM}+\overline{BM}+\overline{CM}$ yig’indi nolga tengligini ko‘rsatamiz:

$$\overline{AM}+\overline{BM}+\overline{CM}=\frac{2}{3}\overline{AA_1}+\frac{2}{3}\overline{BB_1}+\frac{2}{3}\overline{CC_1}=$$

$$=\frac{1}{3}(\overline{AB}+\overline{AC})+\frac{1}{3}(\overline{BA}+\overline{BC})+\frac{1}{3}(\overline{CA}+\overline{CB})=0.$$

Bularga muvofiq, (7.3) tenglik o‘rinli bo‘ladi.

2-natija. Agar (7.3) tenglikni kvadratga ko‘tarsak, ushbu

$$\left|OM\right|^2 = \frac{1}{9}[3R^2 + 2R^2(\cos 2\alpha + \cos 2\beta + \cos 2\gamma)] =$$

$$= \frac{1}{9}[3R^2 + 2R^2(3 - 2\sin^2\alpha - 2\sin^2\beta - 2\sin^2\gamma)] = \frac{1}{9}[9R^2 - (a^2 + b^2 + c^2)]$$

tenglik kelib chiqadi. Bu tenglikdan, xususan 1-natijadagi tengsizlik kelib chiqadi.

4-masala. Agar ABC uchburchakka tashqi chizilgan aylana markazi O va bu uchburchak balandliklari kesishgan nuqtasi H bo‘lsa, ushbu

$$\overline{OH} = \overline{OA} + \overline{OB} + \overline{OC} \tag{7.4}$$

tenglikni isbotlang.

Isbot. Ushbu $\overline{OH} = \overline{OC} + \overline{CH}$ tenglik o‘rinli. Demak, $\overline{CH} = \overline{OA} + \overline{OB}$ tenglikni isbotlash kifoya. $\overline{CH}$ vektor ham, $\overline{OA} + \overline{OB} = 2\overline{OK}$ vektor ham $\overline{AB}$ vektorga perpendikular bo‘lgani uchun ular o‘zaro parallel bo‘ladi. Demak, bu vektorlar qollinear ekan $\overline{CH} = \lambda \cdot (\overline{OA} + \overline{OB})$. Bu vektorlar bir tomonga yo‘nalganligi uchun $\lambda \geq 0$ bo‘ladi. λ ning qiymatini topish uchun $\overline{CH}$ va $\overline{OA} + \overline{OB}$ vektorlar uzunliklarini topamiz. Quyidagi ifodalarni topish qiyin emas $\left|\overline{OA} + \overline{OB}\right|^2 = 4R^2\cos^2\gamma$, $\left|\overline{CH}\right|^2 = 4R^2\cos^2\gamma$. Bu erda R orqali ABC uchburchakka tashqi chizilgan aylana radiusi va γ orqali AB tomon qarshisidagi burchak belgilangan. Demak, $\lambda = 1$ ekan, ya’ni ushbu $\overline{CH} = \overline{OA} + \overline{OB}$ tenglik o‘rinli ekan.

3-natija. (7.3) va (7.4) tengliklardan ushbu $\overline{OH} = 3\overline{OM}$ muxim tenglik kelib chiqadi. Bu tenglikka asosan ixtiyoriy uchburchakda tashqi chizilgan aylana markazi O, medianalar kesishgan nuqta M va balandliklar kesishgan nuqta H bir to‘g’ri chiziqda yotadi. Bu to‘g’ri chiziqqa Eyler to‘g’ri chizig’i deyiladi. Bundan tashqari, hamisha $\left|OM\right| : \left|MH\right| = 1:2$ nisbat o‘rinli bo‘ladi.

4-natija. Agar (4) tenglikni kvadratga ko‘tarsak, ushbu

$$\left|OH\right|^2 = 9R^2 - (a^2 + b^2 + c^2)$$

tenglik kelib chiqadi.

5-masala. ABC uchburchakning CA va CB tomonlarida mos ravishda A_1 va B_1 nuqtalar shunday olinganki, bunda $\frac{\left|CA_1\right|}{\left|CA\right|} = \lambda$ va $\frac{\left|CB_1\right|}{\left|CB\right|} = \mu$. Agar AB_1 va BA_1 kesmalarning kesishish nuqtasi K bo‘lsa, $\frac{\left|AK\right|}{\left|AB_1\right|}$ va $\frac{\left|BK\right|}{\left|BA_1\right|}$ nisbatlarni toping.

Echish. $\frac{\left|AK\right|}{\left|AB_1\right|} = x$ va $\frac{\left|BK\right|}{\left|BA_1\right|} = y$ deb belgilaymiz hamda $\overline{AB}$ vektorni ikki xil usulda topamiz. Birinchidan

$$\overline{AB} = \overline{AC} + \overline{CB} = \overline{CB} - \overline{CA}. \tag{7.5}$$

Ikkinchidan

$$\begin{aligned} \overline{AB} &= \overline{AK} + \overline{KB} = \overline{AK} - \overline{BK} = x\overline{AB_1} - y\overline{BA_1} = \\ &= x(\overline{AC} + \overline{CB_1}) - y(\overline{BC} + \overline{CA_1}) = \\ &= x(-\overline{CA} + \mu\overline{CB}) - y(-\overline{CB} + \lambda\overline{CA}) = \\ &= (y + \mu x)\overline{CB} - (x + \lambda y)\overline{CA} \end{aligned} \tag{7.6}$$

Agar (5) va (6) ifodalarni o‘zaro tenglasak, ushbu

$$(y + \mu x)\overline{CB} - (x + \lambda y)\overline{CA} = \overline{CB} - \overline{CA},$$

$$(y + \mu x - 1)\overline{CB} = (x + \lambda y - 1)\overline{CA} \tag{7.7}$$

tenglik kelib chiqadi. $\overline{CA}$ va $\overline{CB}$ vektorlar qollinear bo‘lmaganligi uchun (7.7) tenglik faqat $y + \mu x - 1 = 0$ va $x + \lambda y - 1 = 0$ bo‘lgandagina bajariladi. Bunga ko‘ra

$$x = \frac{1-\lambda}{1-\lambda\mu} \text{ va } y = \frac{1-\mu}{1-\lambda\mu}.$$

Izox. 5-masalada to‘rtta nisbatdan ikkitasini berib, qolgan ikkitasini topishni talab qilsak, yangi masala kelib chiqadi. Aslida bu masalalarning echimi ham 5-masala echimida mujassamlashgan.

6-masala. $ABCD$ qavariq to‘rtburchakning BC va DA qarama-qarshi tomonlarida M va N nuqtalar shunday olinganki, bunda ushbu

$$\frac{|BM|}{|MC|} = \frac{|AN|}{|ND|} = \frac{|AB|}{|CD|}$$

tenglik o‘rinli. MN to‘g’ri chiziq AB va CD tomonlar yordamida hosil qilingan burchak bissektrissasiga parallel bo‘lishini isbotlang.

Isbot. $\dfrac{|BM|}{|MC|} = \dfrac{|AN|}{|ND|} = \dfrac{|AB|}{|CD|} = \lambda$ deb olamiz. U holda $\overline{BM} = \lambda\overline{MC}$, $\overline{AN} = \lambda\overline{ND}$ lardan $\overline{BM} = \dfrac{\lambda}{\lambda+1}\overline{BC}$ va $\overline{AN} = \dfrac{\lambda}{\lambda+1}\overline{AD}$ kelib chiqadi. Bularga ko‘ra

$$\begin{aligned} \overline{MN} &= \overline{MB} + \overline{BA} + \overline{AN} = -\frac{\lambda}{\lambda+1}\overline{BC} + \overline{BA} + \frac{\lambda}{\lambda+1}\overline{AD} = \\ &= \frac{\lambda}{\lambda+1}(\overline{AD} - \overline{BC}) + \overline{BA} \end{aligned}.$$

Ushbu $|BA| \cdot \overline{CD}$ va $|CD| \cdot \overline{BA}$ vektorlarning uzunliklari o‘zaro teng bo‘lgani uchun ularning yig’indisi, ya’ni

$$\overline{p} = |BA| \cdot \overline{CD} + |CD| \cdot \overline{BA} = |CD| \cdot (\lambda\overline{CD} + \overline{BA})$$

vektor BA va CD tomonlar yordamida hosil qilingan burchak bissektrissasi bo‘yicha yo‘naladi. $\overline{CD} = -\overline{BC} + \overline{BA} + \overline{AD}$ bo‘lgani uchun

$$\overline{p} = \left|CD\right| \cdot [\lambda(\overline{AD} - \overline{BC}) + (\lambda + 1)\overline{BA}] =$$

$$= \left|CD\right| \cdot (\lambda + 1)[\frac{\lambda}{\lambda + 1}(\overline{AD} - \overline{BC}) + \overline{BA}] = \left|CD\right| \cdot (\lambda + 1) \cdot \overline{MN}$$

Demak, $\overline{p}$ va $\overline{MN}$ o‘zaro parallel ekan.

7-masala. Agar ABC uchburchakka tashqi chizilgan aylana markazi O va ichki chizilgan aylana markazi J nuqtada bo‘lsa, u holda ushbu

$$\left|OJ\right|^2 = R^2 - 2Rr$$

Eyler formulasi o‘rinli bo‘lishini isbotlang. Bu erda R va r orqali mos ravishda ABC uchburchakka tashqi va ichki chizilgan aylanalar radiuslari belgilangan.

Isbot. Ushbu $\overline{CB} = \overline{a}$, $\overline{CA} = \overline{b}$, $\left|CB\right| = a$, $\left|CA\right| = b$, $\left|AB\right| = c$ belgilashlarni kiritib olamiz. $\overline{CJ}$ vektor ham, $b\overline{a} + a\overline{b}$ vektor ham ACB burchak bisektrissasi bo‘yicha yo‘nalganligi uchun ular o‘zaro qollinear, ya’ni $\overline{CJ} = \lambda(b\overline{a} + a\overline{b})$ bo‘ladi. Bunga ko‘ra $\overline{AJ} = \overline{AC} + \overline{CJ} = \lambda b\overline{a} + (\lambda a - 1)\overline{b}$. Ikkinchi tomondan $\overline{AJ}$ vektor ham, $c \cdot \overline{AC} + b \cdot \overline{AB}$ vektor ham BAC burchak bisektrissasi bo‘yicha yo‘nalganligi uchun ular o‘zaro qollinear, ya’ni $\overline{AJ} = \mu(c \cdot \overline{AC} + b \cdot \overline{AB}) = \mu b\overline{a} - (\mu b + \mu c)\overline{b}$ bo‘ladi. $\overline{AJ}$ uchun olingan ikkita tenglikni bir biridan ayirsak, ushbu

$$(\lambda - \mu)b\overline{a} + (\lambda a + \mu b + \mu c - 1)\overline{b} = 0$$

tenglik hosil bo‘ladi. Bu erda $\overline{a}$ va $\overline{b}$ vektorlar qollinear emasligini hisobga olsak, $\lambda = \mu$ va $\lambda a + \mu b + \mu c = 1$ tengliklarni olamiz. Bundan ushbu

$$\overline{CJ} = \frac{1}{a + b + c}(b\overline{a} + a\overline{b})$$

tenglik kelib chiqadi.

Agar A_1 orqali CB tomonning o‘rtasini va B_1 orqali CA tomonning o‘rtasini belgilasak, u holda $\overline{CO} + \overline{OA_1} = \frac{1}{2}\overline{a}$ va $\overline{CO} + \overline{OB_1} = \frac{1}{2}\overline{b}$ tengliklarni mos ravishda $\overline{a}$ va $\overline{b}$ vektorlarga skalyar ko‘paytirib, ushbu

$$\overline{CO} \cdot \overline{a} = \frac{1}{2}a^2 \text{ va } \overline{CO} \cdot \overline{b} = \frac{1}{2}b^2 \tag{7.8}$$

tengliklarni olamiz.

Endi ushbu $\overline{OJ} = \overline{CJ} - \overline{CO} = \frac{1}{a + b + c}(b\overline{a} + a\overline{b}) - \overline{CO}$ tenglikning kvadratini (7.8) formulalardan foydalanib hisoblaymiz:

$$\overline{OJ}^2 = \frac{1}{(a+b+c)^2}(b\overline{a} + a\overline{b})^2 - \frac{2}{a+b+c}(b\overline{a} + a\overline{b})\overline{CO} + R^2 =$$

$$= \frac{2a^2b^2(1+\cos\gamma)}{(a+b+c)^2} - \frac{ab(a+b)}{a+b+c} + R^2.$$

Bu erda ushbu

$$1+\cos\gamma = 1 + \frac{a^2+b^2-c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab}$$

tenglikdan foydalansak,

$$\overline{OJ}^2 = R^2 - \frac{abc}{a+b+c} = R^2 - 2\cdot\frac{abc}{4S}\cdot\frac{2S}{a+b+c} = R^2 - 2Rr$$

Eyler formulasi kelib chiqadi.

Mustakil yechish uchun masalalar

1. $ABCDA_1B_1C_1D_1$ to‘gri burchakli parallelepipedda $\angle BAB_1 = \alpha$, $\angle CBC_1 = \beta$ bo‘lsa, qo‘shni yon yoqlarning AB_1 va BC_1 diagonallari orasidagi γ burchakni toping.
2. Agar $ABCD$ qavariq to‘rtburchakda $AB = a, BC = b, CD = c$, $\angle ABC = \alpha$, $\angle BCD = \beta$ bo‘lsa, u holda

$$AD^2 = a^2 + b^2 + c^2 - 2ab\cos\alpha - 2bc\cos\beta + 2ac\cos(\alpha+\beta)$$

tenglik o‘rinli bo‘lishini isbotlang.

3. Tekislikda bir xil yo‘nalishli ikkita $ABCD$ va $A_1B_1C_1D_1$ kvadratlar berilgan. AA_1, BB_1, CC_1 va DD_1 kesmalarning o‘rtalari uchinchi kvadratning uchlari bo‘lishini isbotlang.
4. Radiusi 1 ga teng bo‘lgan aylanaga $ABCDE$ beshburchak tashki chizilgan bo‘lib, $AB = a$, $BC = b$, $CD = c$, $DE = d$, $AE = 2$ bo‘lsa,

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$$

bo‘lishini isbotlang.

5. Agar ABC uchburchakda m_a, m_b, m_c - medianalar, R - unga tashqi chizilgan aylananing radiusi bo‘lsa, u holda $m_a^2 + m_b^2 + m_c^2 \leq \frac{27R^2}{4}$. tengsizlik o‘rinli bo‘lishini isbotlang.

6. $ABCDA_1B_1C_1D_1$ parallelepipedda AC_1 diagonal BDA_1 tekislikni K nuqtada kesib o‘tadi. Ushbu $\frac{|AK|}{|AC_1|}$ niatni toping

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