O ‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

AL - XORAZMIY NOMIDAGI URGANCH DAVLAT UNIVERSITETI

Qo‘l yozma huquqida

UDK. .517. 55

Xaytboyeva O‘g‘iloy Xaytboyevna

Kuchsiz $m-$subgarmonik funksiyalarning $m-$subgarmoniklik xususiyatlari

5A130101 - Matematika (Matematik analiz)

Magistr akademik darajasini olish uchun yozilgan dissertatsiya

Ilmiy rahbar: **f.-m.f.d. B.I.Abdullayev**

URGANCH - 2016

MUNDARIJA

Kirish...3

I BOB. Funksiyalarning ba'zi muhim sinflari...8

1.1. Garmonik funksiyalar va ularning xossalari...8

1.2. Subgarmonik funksiyalar...11

1.3. Plyurisubgarmonik funksiyalar...17

I bob bo'yicha xulosa...22

II BOB Musbat aniqlangan differensial formalar...23

2.1. Musbat aniqlangan differensial forma va oqimlar...................................23

2.2. Musbat gessianlar bilan differensial formalar orasidagi bog'lanish.......29

II BOB bo'yicha xulosa...36

III BOB. Kuchsiz m - subgarmonik funksiyalarning m - subgarmoniklik xususiytlari...38

3.1. m - subgarmonik funksiyalar va ularning xossalari...................................38

3.2. Kuchsiz m - subgarmonik funksiyalar va ularning xossalari.....................43

3.3. Kuchsiz m - subgarmonik funksiyalarning m - subgarmoniklik xususiyatlari...48

III bob bo'yicha xulosa...60

Xulosa...61

Adabiyotlar ro'yxati...62

KIRISH

1. Magistrlik dissertatsiyasi mavzusining asoslanishi va uning dolzarbligi.

Klassik potensiallar nazariyasi Laplas operatori va subgarmonik funksiyalar sinfiga asoslanib, plyuripotensiallar nazariyasi nochiziqli Monj-Amper tenglamasi va plyurisubgarmonik funksiyalari bilan bog‘langan. Plyuripotensiallar nazariyasi hozirda jadal suratlarda rivojlanib kelmoqda va geometrik ko‘pxilliklar, Eynshteynning nisbiylik nazariyasida, jumladan, Eynshteyn metrikasini mavjudligini isbotlashda, shuningdek, hususiy hosilali differensial tenglamalarning bir qancha masalalarida o‘z tatbiqini topgan. Plyurisubgarmonik funksiyalar sinfining o‘ziga xos kengaytmalarini o‘rganish va bu kengaytmalar uchun potensiallar nazariyasini yaratish bilan bog‘liq tatqiqotlar hozirgi paytda kompleks analizning ustivor yo‘nalishlaridan hisoblanadi. $m-$subgarmonik va kuchsiz $m-$subgarmonik funksiyalar sinflari subgarmonik va plyurisubgarmonik funksiyalar sinflari orasidagi sinflar bo‘lgani bois bu sohalar kompleks potensiallar nazariyasida alohida obyekt sifatida mutaxassislar e’tiborida bo‘lib kelmoqda.Bu sinfar orasida $m-sh \subset m-wsh$ munosabat o‘rinli

Shu sababdan kuchsiz m-subgarmonik funksiyalarning $m-$subgarmoniklik xususiyatlarini o‘rganish, dolzarb muammolardan hisoblanadi.

2. Tadqiqot obyekti va predmeti. Kuchsiz $m-sh$ funksiyalar, $m-sh$

funksiyalar. $\left(dd^c u\right)^n -$ Monje –Ampere operatori hamda gessianlar, oqimlar to‘g‘risidagi masalalar.

3. Ishning maqsadi va vazifalari. Kuchsiz $m-$subgarmonik funksiyalarning $m-$subgarmoniklik xususiyatlarini o‘rganish.

— Agar $u(z)$ funksiya bir vaqtda garmonik va $m-$subgarmonik bo‘lsa, u holda bu funksiya plyurigarmonik bo‘lishini ko‘rsatish.

$$u = \ln\left(x_1^2 + x_2^2 + x_3^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + (z_3 + \overline{z}_3)^2] - 2\ln 2,$$

funksiyaning $2-sh$ funksiya bo‘lishi yoki bo‘la olmasligini ko‘rsatish.

$$u = \ln\left(x_1^2 + x_2^2 + ... + x_n^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + ... + (z_n + \overline{z}_n)^2] - 2\ln 2$$

funksiyani $m-$subgarmoniklikka tekshirish.

4. Ilmiy yangiligi. Mazkur ishning asosiy natijalari yangi.

5. Tadqiqotning asosiy masalalari va farazlari. Asosiy masala kuchsiz $m-$subgarmonik funksiyalarning $m-$subgarmoniklik xususiyatlarini aniqlash.

6. Tadqiqotda qo‘llangan metodikaning tavsifi. Ko‘p kompleks o‘zgaruvchining funksiyalar nazariyasi, kompleks potensiallar nazariyasi va chiziqli algebra usullari.

7. Tadqiqot natijalarining nazariy va amaliy ahamiyati. Olingan natijalar va usullar funksiyalar nazariyasining keyingi rivojlanishida va kompleks analizning tadbiqlarida qo‘llanilishi mumkin.

8. Ish tuzilmasining tavsifi.

Dissertatsiya ishi kirish, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yxatidan iborat. Birinchi bob uchta paragrafdan iborat bo'lib, unda garmonik, subgarmonik va plyurisubgarmonik funksiyalar haqida zarur ma'lumotlar hamda ularning xossalari keltirilgan.

Ikkinchi bob ikki paragrafdan iborat. Bu bobda musbat aniqlangan differensial forma va oqimlar hamda musbat gessianlar bilan differensial formalar orasidagi bog'lanishlar o'rganilgan.

Ikki marta silliq ixtiyoriy $u \in C^2\left(D\right)$ funksiya uchun uning 2-tartibli differensiali

$$dd^c u = \frac{i}{2}\sum_{j,k} u_{j,\bar{k}} dz_j \wedge d\bar{z}_k \tag{1}$$

(tayinlangan $z^0 \in D$ nuqtada) ermit kvadratik formasidan iborat bo'ladi.

Bu yerda odatdagidek, $d = \partial + \overline{\partial}$, $d^c = \dfrac{\partial - \overline{\partial}}{4i}$ bo'lib,

$$du = \partial u + \overline{\partial} u = \sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z}_k} d\overline{z}_k$$

$$d^c u = \frac{1}{4i}\left(\sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k - \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z}_k} d\overline{z}_k\right), u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}$$

(1) tenglik unitar ([4] ga qarang) almashtirishlar yordamida quyidagi diagonal formaga keltiriladi:

$$dd^c u = \frac{i}{2}\left(\lambda_1 dz_1 \wedge d\overline{z}_1 + \ldots + \lambda_n dz_n \wedge d\overline{z}_n\right) \tag{2}$$

Bunda $\lambda_1,...,\lambda_n$ lar ermit $\left(u_{j\bar{k}}\right)$ matrisasining xos qiymatlari bo‘lib, $\lambda=\left(\lambda_1,\lambda_2,...,\lambda_n\right)\in\mathbb{R}^n$ haqiqiy qiymatlardan iborat bo‘ladi.

1-ta’rif. Ikki marta silliq $u\in C^2\left(D\right)$ $\left(D\subset\mathbb{C}^n\right)$ funksiya uchun $\forall z^\circ\in D$ da

$$\left(dd^c u\right)^k\wedge\left(dd^c\left|z\right|^2\right)^{n-k}\geq 0,\quad \forall k=1,2...,m \tag{3}$$

tengsizlik bajarilsa, u holda bu funksiya D sohada $m-sh$ $(1\leq m\leq n)$ funksiya deyiladi.

1- ta’rif. Agar $u(z)$ funksiya bir vaqtda garmonik va $m-$subgarmonik bo‘lsa, u holda bu funksiya plyurigarmonik bo‘ladi.

2-ta’rif. *Agarda* $D\subset\mathbb{C}^n$ *sohada aniqlangan* $u\left(z\right)\in L^1_{loc}\left(D\right)$ *funksiya:*

1) D *sohada yuqoridan yarim uzluksiz, ya’ni*

$$\overline{\lim_{z\to z^0}}u\left(z\right)=\lim_{\varepsilon\to 0}\sup_{B\left(z^0,\varepsilon\right)}u\left(z\right)\leq u\left(z^0\right);$$

2) D *sohada* $dd^c u\wedge\left(dd^c\left|z\right|^2\right)^{n-m}\geq 0,\quad 1\leq m\leq n,$ *ya’ni*

$$\forall\,\omega\in F^{(m-1,m-1)}\ \textit{uchun}\qquad dd^c u\wedge\left(dd^c\left|z\right|^2\right)^{n-m}\left(\omega\right)=u\left(dd^c\left|z\right|^2\right)^{n-m}\wedge dd^c\omega\geq 0,$$

bo‘lsa, bu funksiyaga D *sohada kuchsiz* $m-sh$ *funksiya (*$\left(n-m+1\right)$ *o‘lchamli kompleks tekisliklarda subgarmonik funksiya) deyiladi.*

Bunday funksiyalar sinfini $m-wsh\left(D\right)$ orqali belgilaymiz. Bu yerda “w” xarfi bu sinfni $m-sh$ funksiyalar sinfidan farqlash uchun ishlatilgan.

$D\subset\mathbb{C}^n$ sohada $m-wsh$ funksiyalar ham ayni paytda $D\subset\mathbb{R}^{2n}$ da ham subgarmonik bo‘ladi.

Binobarin subgarmonik funksiyalarning barcha xossalari $m-wsh$ funksiyalar uchun ham o‘rinli.

Bizga ma’lumki,

$$u=\ln\left(x_1^2+x_2^2+x_3^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+(z_3+\overline{z}_3)^2]-2\ln 2, \tag{4}$$

(bunda $z_j = x_j + iy_j, j = 1,2,3,$) funksiya $1 - wsh$ va $2 - wsh$ sinflariga tegishli. Ushbu dissertatsiyada bu funksiyaning $2 -$subgarmonik funksiya emasligi ko‘rsatib o‘tilgan, hamda quyidagi teorema isbotlanadi:

2-teorema.

$$u = \ln\left(x_1^2 + x_2^2 + ... + x_n^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + ... + (z_n + \overline{z}_n)^2] - 2\ln 2$$

funksiya $1 \le m \le \left[\dfrac{n}{2}\right], n \ge 2$ bo‘lganda $m - sh$ bo‘ladi.

Muallif o‘z ilmiy rahbari fizika-matematika fanlari doktori B.I.Abdullayevga qo‘yilgan masalalarni hal etishda ko‘rsatgan doimiy e’tibori uchun chuqur minnatdorchiligini izhor etadi. Shuningdek, dots. A.A.Atamuratov, dots. M.Vaisova va D.A. Qalandarovalarga hamda kafedrani‘qituvchilariga dissertatsiya ishini tayyorlashda bergan qimmatli maslahatlari hamda dissertatsiya ishi masalalarini muhokama qilishda ko‘rsatgan yordamlari uchun o‘zining minnatdorchiligini bildiradi.

I BOB. FUNKSIYALARNING BA'ZI MUHIM SINFLARI

Subgarmonik va plyurisubgarmonik funksiyalar sinflari potensiallar va plyuripotensiallar nazariyalarining asosiy obyektlaridan bo‘lib, bu nazariyalarning barcha metodlari asosida ushbu tushunchalar yotadi. Masalan, Laplas tenglamasi uchun Dirixle masalasi, kompleks Monj Amper tenglamasi uchun Dirixle masalasi, ko‘p argumentli golomorf funksiyalarni analitik davom ettirish masalalari, ko‘phad va ratsional funksiyalar bilan yaqinlashtirishlar masalalarida ekstremal plyurisubgarmonik funksiyalar muhim ro‘l o‘ynaydi([2],[6],[7],[8],[13],[16] ga qarang).

1.1. Garmonik funksiyalar.

$D \subset \mathbf{R}^n$ sohada ushbu

$$\Delta u = 0$$

tenglamani qanoatlantiruvchi $u \in C^2\left(D\right)$ funksiya *garmonik funksiya* deyiladi, bunda

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

Laplas operatori.

Barcha garmonik funksiyalar to‘plami $H(D)$ kabi belgilanadi.

1.1.1 – teorema. Agar $u(x)$ funksiya $D \subset R^n$ sohada garmonik bo‘lsa, u holda ixtiyoriy $x^o \in D$ nuqta va $S(x^o, r) \subset D$ sfera uchun

$$u(x^o) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^o,r)} u(x)\, d\sigma \qquad (1.1.1)$$

bo‘ladi, bunda σ_n - birlik sferaning yuzi.

Aksincha, agar D dan olingan har bir $x^o \in D$ nuqta va yetarlicha kichik r $(0 < r \le r_0(x^o))$ lar uchun (1.1.1) formula o‘rinli bo‘lsa, u holda $u(x) \in H(D)$ bo‘ladi.

Endi garmonik funksiyalarning ba’zi bir xossalarini keltiramiz.

$a)$ Agar $u,v \in H(D)$ bo‘lib, $\lambda,\mu \in R$ bo‘lsa, u holda

$$\lambda u + \mu v \in H\left(D\right)$$

bo‘ladi;

$b)$ agar umumlashgan funksiya f uchun $\Delta f = 0$, ya’ni ixtiyoriy $\phi \in F\left(D\right)$ da $f(\Delta\phi) = 0$ tenglik bajarilsa, u holda f garmonik funksiya bo‘ladi, aniqroq qilib aytganda, shunday $v(x) \in C^2(D)$, $\Delta v = 0$ bo‘lgan funksiya mavjudki,

$$f(\phi) = v(\phi) = \int v(x)\phi(x)\,dV$$

bo‘ladi;

$v)$ agar $u(x)$ funksiya $B(x_o, r)$ sharda garmonik, uning yopig’ida esa uzluksiz, ya’ni

$$u(x) \in H(B(x_o, r)) \bigcap C(\overline{B(x_o, r)})$$

bo‘lsa, u holda

$$u(x) = \int\limits_{S(x^o,r)} u(y)P(x,y)d\sigma(y)$$

formula o‘rinli bo‘ladi, bunda

$$P(x,y)=\frac{r^2-\left|x-x^0\right|^2}{\sigma_n r\left|x-y\right|^n}, \qquad n\geq 2.$$

Puasson yadrosi, $\sigma - R^n$dagi birlik sfera yuzi, $n\geq 2$.Ikkinchi tomondan, agar $\phi(y)$ funksiya $S(x^o,r)$ sferada uzluksiz bo‘lsa, u holda

$$u(x)=\int\limits_{S(x^o,r)}\varphi(y)P(x,y)d\sigma(y)$$

funksiya $B(x^0,r)$ sharda Dirixle masalasi $\Delta u=0$, $u\Big|_{S(x^o,\ r)}=\phi(y)$ning yechimi bo‘ladi.

Dirixle masalasini chegarasi silliq bo‘lgan ixtiyoriy $D\subset\mathbf{R}^n$ sohada ham qarash mumkin: $\phi(y)\in C(\partial D)$ uchun yagona $u(x)\in C(\overline{D})$ mavjudki, $\Delta u=0$, $u\Big|_{\partial D}=\phi\ (y)$ bajariladi;

$g)$ kompleks tekislik $\mathbf{C}\approx\mathbf{R}^2$ da garmonik funksiyalar golomorf funksiyalarga bevosita bog'langandir.

Agar $z=x+iy$ deyilsa, unda

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial\overline{z}}$$

bo‘ladi va har bir golomorf $f\in\mathcal{O}\left(D\right)$ funksiya uchun

$$\operatorname{Re}f=\frac{f+\overline{f}}{2}, \quad \operatorname{Im}f=\frac{f-\overline{f}}{2i}$$

tengliklardan

$$\Delta\operatorname{Re}f=0, \quad \Delta\operatorname{Im}f=0\ ,$$

ya’ni $\operatorname{Re}f$ va $\operatorname{Im}f$ larning garmonik funksiyalar ekanligi kelib chiqadi.

Shuningdek, agar $u(z) \in H\left(D\right)$ bo‘lsa, u holda har bir $z^o \in D$ nuqta va uning atrofi $B(z^o, r) \subset D$ uchun shunday $f \in \mathcal{O}\left(D\right)$ funksiya topiladiki, $\operatorname{Re} f(z) = u(z), \quad z \in B(z^o, r)$ bo‘ladi.

Haqiqatan ham, $z^0 = 0$ deb $B = B(0, r)$ doirada Puasson formulasini ushbu

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{r^2 - |z|^2}{|\xi - z|^2} dt = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \operatorname{Re} \frac{\xi + z}{\xi - z} dt ,$$

bunda $\xi = re^{it}$ ko‘rinishda ifodalaymiz. Bu formuladan $B(o, r)$ da golomorf

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} dt \qquad (1.1.2)$$

funksiyasi uchun $\operatorname{Re} f(z) = u(z)$ ekanligi kelib chiqadi.

(1.1.2) formula golomorf funksiyani uning haqiqiy qismi orqali ifodalashda ham ishlatiladi: ixtiyoriy $f(z) \in \mathcal{O}(B) \bigcap C(\bar{B})$ uchun ushbu

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} dt + i \operatorname{Im} f(0)$$

formula o‘rinlidir.

1.2. Subgarmonik funksiyalar

Faraz qilaylik, $D \subset \mathbf{R}^n$ sohada aniqlangan lokal integrallanuvchi

$$u: \quad D \to [-\infty, +\infty)$$

funksiya berilgan bo‘lsin: $u \in L^1_{loc}(D)$.

Agar bu funksiya quyidagi ikki shartni bajarsa:

1) $u(x)$ yuqoridan yarim uzluksiz:

ya’ni $\forall x^o \in D$ uchun $\overline{\lim\limits_{x \to x^o}} u(x) \leq u(x^o)$

(bundan $u(x)$ funksiyaning D ga tegishli ixtiyoriy kompaktda yuqoridan chegaralanganligi kelib chiqadi);

2) har bir $x^o \in D$ nuqta uchun shunday $r(x^o) > 0$ topiladiki, $r \leq r(x^o)$ uchun

$$u(x^o) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(x^o,r)} u(x) d\sigma \tag{1.2.1}$$

u holda $u(x)$ funksiya D sohada *subgarmonik funksiya* deyiladi.

Subgarmonik funksiyalar sinfi $Sh\left(D\right)$ kabi belgilanadi. Kelgusida qulaylik uchun biz trivial $u(z) \equiv -\infty$ funksiyasi ham D da $Sh(D)$ ga tegishli deb qaraymiz. Endi subgarmonik funksiyalarning xossalarini keltiramiz.

a) subgarmonik funksiyalarning musbat koeffisentli chiziqli kombinatiyasi subgarmonik funksiya bo‘ladi:

$$\begin{aligned} &u_k(x) \in Sh(D), \quad a_k \in R_+ \quad (k = 1,2,\dots,m) \Rightarrow \\ &\Rightarrow a_1 u_1(x) + a_2 u_2(x) + \dots + a_m u_m(x) \in Sh(D)\ ; \end{aligned}$$

b) chekli sondagi subgarmonik funksiyalarning maksimumi subgarmonik funksiya bo‘ladi:

$$\begin{aligned} &u_1(x), u_2(x), \dots, u_m(x) \in Sh(D) \Rightarrow \\ &\Rightarrow \max\left\{ u_1(x), u_2(x), \dots, u_m(x) \right\} \in Sh(D)\ ; \end{aligned}$$

v) monoton kamayuvchi subgarmonik funksiyalar ketma – ketligining limiti subgarmonik funksiya bo‘ladi:

$$u_j(x) \in Sh(D)\ , u_j(x) \geq u_{j+1}(x) \quad (j = 1,2,...) \Rightarrow$$
$$\Rightarrow \lim_{j\to\infty} u_j(x) \in Sh(D)\ ;$$

g) tekis yaqinlashuvchi subgarmonik funksiyalar ketma–ketligi subgarmonik funksiyaga yaqinlashadi:

$$u_j(x) \in Sh(D)\ \left(k = 1,2,3,...\right), u_j\left(x\right) \to u\left(x\right) \Rightarrow u\left(x\right) \in Sh\left(D\right).$$

a) – *g*) xossalarning isbot bevosita yuqorida keltirilgan ta'rifdan kelib chiqadi.

d) *Maksimumlar prinsipi.* Agar $u(x) \in Sh(D)$ funksiya biror $x^o \in D$ nuqtada o'zining maksimumiga erishsa, ya'ni

$$u(x^o) = \sup_{x\in D} u(x) \tag{1.2.2}$$

bo'lsa, u holda $u(x) \equiv const$ bo'ladi.

Ushbu

$$M = \left\{ x \in D:\ u(x) = u(x^o) \right\}$$

to'plamni qaraylik. $u(x)$ funksiyaning yuqoridan yarim uzluksiz bo'lganligidan va (1.2.2) shartdan M to'plamning D da yopiqligi kelib chiqadi.

Ikkinchi tomondan, ixtiyoriy $p \in M$ uchun (1.2.1) formulaga ko'ra

$$u(x^o) = u(p) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(p,r)} u(x) d\sigma \leq u(x^o)\ , \qquad r \leq r(p)\ ,$$

bo'ladi. Bu munosabatdan $u\Big|_{S(p,r)} \equiv u(x^o)$, ya'ni p ning $B(p, r(p))$ atrofida $u(x) = u(x^o)$ bo'lishini ko'ramiz. Bu esa M to'plamning D da ochiq ekanligini ko‘rsatadi. Demak, $M = D$.

$e)$ Agar $\vartheta(x) \in Sh(D)$, $u(x) \in H(D)$ funksiyalar uchun $S(x^o, r) \subset\subset D$ sferada

$$\vartheta\Big|_S \le u\Big|_S$$

bo‘lsa, u holda $B(x^o, r)$ sharda $\vartheta(x) \le u(x)$ tengsizlik o‘rinli bo‘ladi.

Bu xossaning isboti yuqoridagi $d)$ xossadan kelib chiqadi;

$e)$ – xossadan quyidagi muhim xulosaga kelamiz. Faraz qilaylik,

$$\vartheta(x) \in Sh(D) \bigcap C(D)$$

berilgan bo‘lib, $B(x^o, r) \subset\subset D$ bo‘lsin. Ushbu

$$u(x) = \int\limits_{S(x^o, r)} \vartheta(y) P(x, y) d\sigma(y)\ , \qquad x \in B(x^o, r)\ ,$$

Puasson integralini qaraylik. $u(x) \in h(B)$, $u\Big|_S = \vartheta\Big|_S$ bo‘lib, $e)$ – xossaga ko‘ra $B = B(x^o, r)$ da $\vartheta(x) \le u(x)$ bo‘ladi. Uzluksiz funksiyalar uchun isbotlangan bu xossa ixtiyoriy $\vartheta \in Sh\big(D\big)$ uchun ham o‘rinlidir: $\vartheta(x) \le u(x)$ va deyarli barcha $x \in S$ lar uchun $\vartheta(x) = u(x)$ o‘rinlidir;

j) aytaylik, $u(x) \in Sh(D)$ funksiya va $x^o \in D$ nuqta berilgan bo‘lsin.

Ushbu

$$m(x^o, r) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^0, r)} u(x) d\sigma$$

va

$$n(x^o, r) = \frac{1}{V_n r^n} \int\limits_{B(x^o, r)} u(x) dV\ ,$$

bunda $V_n r^n$ miqdor $B(x^o, r)$ ning hajmi, integrallarni (o‘rta qiymatlarni) qaraylik.

1.2.1 – teorema. Faraz qilaylik, $u(x) \in Sh(D)$ va $u(x) \not\equiv -\infty$ bo‘lsin. U holda ixtiyoriy $x^o \in D$ uchun

$$u(x^0) \le n(x^o, r) \le m(x^o, r) \ , \ n(x^o, r) > -\infty \ , \ 0 < r < \rho(x^o, \partial D) \ ,$$

tengsizliklar o‘rinli bo‘ladi. Bundan tashqari, agar r monoton kamayib nolga intilsa, unda $n(x^o, r)$ va $m(x^o, r)$ o‘rta qiymatlar monoton kamayib, $u(x^o)$ ga intiladi.

1.2.1 – teoremadan ushbu muhim natijaga ega bo‘lamiz: $D \subset \mathbf{R}^n$ sohada subgarmonik $u(x) \ne -\infty$ funksiya D da lokal integrallanuvchidir, ya’ni ixtiyoriy $B \subset\subset D$ uchun $\int\limits_B u(x)dV$ integral mavjuddir;

$z)$ subgarmonik funksiya D sohaning har bir nuqtasida uzilishga ega bo‘lishi mumkin. Ayni paytda, quyidagi teorema o‘rinli bo‘ladi.

1.2.2 – teorema. Agar $u(z) \in Sh(D)$ bo‘lsa, u holda shunday monoton to‘plamlar

$$D_j \subset D_{j+1} \subset D \ , \quad \bigcup_{j=1}^{\infty} D_j = D$$

ketma – ketligi va shunday monoton kamayuvchi funksiyalar

$$u_j(x) \in Sh(D_j) \cap C^{\infty}(D_j) \ , \qquad j = 1,2,3,...$$

ketma – ketligi mavjudki,

$$u(x) = \lim_{j \to \infty} u_j(x) \qquad (x \in D)$$

bo'ladi.

Isboti.Ushbu

$$K(x) = \begin{cases} ce^{-\frac{1}{1-|x|^2}}, & agar \ |x| \le 1 \ \ bo'lsa, \\ 0, & agar \ |x| > 1 \ \ bo'lsa. \end{cases}$$

funksiyani qaraylik. Bu funksiya $C^\infty(\mathbf{R}^n)$ sinfga tegishli bo'lib, uning havzasi $\operatorname{supp} K(x) = B(0,1)$ dir. Endi $K(x)$ ning ifodasidagi o'zgarmas c ni shunday tanlab olamizki,

$$\int_{\mathbf{R}^n} K(x)dV = 1$$

bo'lsin. $u(x) \not\equiv -\infty$ deb, bu yadro yordamida ushbu

$$u_\delta(x) = \frac{1}{\delta^n} \int_{|y-x| \le \delta} u(y) K\left(\frac{y-x}{\delta}\right) dV \ , \quad \delta > 0 \ , \tag{1.2.3}$$

integralni qaraylik. $u_\delta(x)$ funksiyasi

$$D_\delta = \left\{ x \in D: \ \rho(x, \partial D) < \delta \right\}$$

ochiq to'plamda aniqlangan bo'lib, $x \in D_\delta$ da

$$u_\delta(x) = \frac{1}{\delta^n} \int_{\mathbf{R}^n} u(y) K\left(\frac{y-x}{\delta}\right) dV = \int_{\mathbf{R}^n} u(x + \delta y) K(y) dV$$

bo'ladi. Keyingi tenglikdagi birinchi integral $C^\infty(D_\delta)$ sinfga, ikkinchi integral esa $Sh(D_\delta)$ sinfga tegishli bo'ladi. Binobarin,

$$u_\delta(x) \in Sh(D_\delta) \bigcap C^\infty(D_\delta).$$

$u(x)$ ning subgarmonik funksiya ekanligidan, u_δ ning $\delta \downarrow 0$ da, monoton kamayuvchi bo‘lishligi va

$$\int_{\mathbf{R}^n} K(x)dV = 1$$

tenglikdan, $u_\delta(x)$ ning $u(x)$ ga intilishini topamiz.

Teorema isbotlandi.

i) Subgarmonik funksiyalar umumiy holda C^2 sinfga tegishli bo‘lmasligidan, Δu Laplas operatorini faqat umumlashgan funksiya sifatida qarash mumkin bo‘ladi:

$$\Delta u(\varphi) = \int u \Delta\varphi, \qquad \varphi \in F(D).$$

1.2.3-teorema. Har qanday subgarmonik funksiya $u(x) \in Sh(D)$, $u \not\equiv -\infty$, uchun umumlashgan ma’noda $\Delta u \geq 0$ bo‘ladi, ya’ni ixtiyoriy $\phi \in F(D)$, $\phi \geq 0$ uchun $\int u \Delta \varphi \geq 0$ munosabat o‘rinlidir.

Jumladan, $u(x)$ funksiya C^2 sinfga tegishli bo‘lib, u subgarmonik bo‘lsa, $u \in Sh(D) \bigcap C^2(D)$, u holda $\Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + ... + \dfrac{\partial^2 u}{\partial x_n^2} \geq 0$ dir.

Yuqorida keltirilgan teoremaning aksi ham o‘rinli: agar umumlashgan funksiya u uchun

$$\Delta u(\varphi) = \int u \Delta\varphi \geq 0 , \quad \varphi \in F(D) , \quad \varphi \geq 0 ,$$

munosabat bajarilsa, u holda u subgarmonik funksiya bo‘ladi.

1.3. Plyurisubgarmonik funksiyalar

1.3.1. Plyurigarmonik funksiyalar

$\mathbf{C}^n$ kompleks fazodagi $D \subset \mathbf{C}^n$ sohada $u(z)$ funksiya berilgan bo‘lib, $u(z) \in C^2(D)$ bo‘lsin.

Agar ixtiyoriy $z^0 \in D$ va bu nuqtadan o‘tuvchi ixtiyoriy kompleks to‘g‘ri chiziq $l : z = z^0 + w\xi$, $w \in \mathbf{C}^n$, $\xi \in \mathbf{C}$ uchun ushbu $\varphi(\xi) = u / l = u(z^0 + w\xi)$ kesim–funksiya $\xi = 0$ nuqtada garmonik, ya’ni $\xi = 0$ da $\dfrac{\partial^2 \phi}{\partial \xi \partial \overline{\xi}} = 0$ bo‘lsa, $u(z)$ funksiya D da *plyurigarmonik funksiya* deyiladi.

Plyurigarmonik funksiyalar sinfi $Ph(D)$ kabi belgilanadi. Aytaylik, $u(z) \in Ph(D)$ bo‘lsin. Unda

$$\frac{\partial^2 u(z^0 + w\xi)}{\partial \xi \, \partial \overline{\xi}} = 0$$

bo‘lib, murakkab funksiyaning differensiallash qoidasidan $\sum_{j,k=1}^{n} \dfrac{\partial^2 u}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k} = 0$, $w \in \mathbf{C}^n$, ekanligini ko‘ramiz. Bundan va $w \in \mathbf{C}^n$ vektorning ixtiyoriyligidan

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = 0 \qquad (k, j = 1,2,\dots,n) \tag{1.3.1}$$

bo‘lishi kelib chiqadi.

Demak, plyurigarmonik funksiyalar sinfi (1.3.1) tenglamalar sistemasi bilan aniqlanadi va ba'zan bu sistema plyurigarmonik funksiyaning ta‘rifi sifatida ham qaraladi.

1.3.1–teorema. Agar $f(z) = u(z) + iv(z)$ funksiya $D \subset \mathbf{C}^n$ sohada golomorf bo‘lsa, u holda

$$\operatorname{Re} f(z) = u(z), \quad \operatorname{Im} f(z) = v(z)$$

funksiyalar D sohada plyurigarmonik bo‘ladi va, aksincha, agar $u(z)$ funksiya D sohada plyurigarmonik bo‘lsa, u holda ixtiyoriy $B(z^0, r) \subset D$ atrofda $u(z)$ funksiya biror $f(z) \in \mathcal{O}(B(z^0, r))$ golomorf funksiyaning haqiqiy qismi bo‘ladi: $u(z) \equiv \operatorname{Re} f(z)$.

1.3.2. Plyurisubgarmonik funksiyalar

Agar $D \subset \mathbf{C}^n$ sohada aniqlangan $u(z):\ D \to [-\infty, \infty)$ funksiya quyidagi ikki shartni bajarsa:

1) $u(z)$ yuqoridagi yarim uzluksiz;

2) ixtiyoriy l kompleks to‘g'ri chiziq uchun u/l funksiya $l \bigcap D$ da subgarmonik bo‘lsa, $u(z)$ funksiya D sohada *plyurisubgarmonik* funksiya deyiladi.

Plyurisubgarmonik funksiyalar to‘plami $Psh(D)$ kabi belgilanadi. $D \subset \mathbf{C}^n$ sohada plyurisubgarmonik bo‘lgan funksiya $D \subset \mathbf{R}^{2n}$ da aniqlangan funksiya sifatida subgarmonik funksiya bo‘ladi. Uni subgarmonik funksiyalarning 2 – shartini bajarishini ko‘rish qiyin emas. Binobarin, subgarmonik funksiyalarning xossalari plyurisubgarmonik funksiyalar uchun ham saqlanadi.

Endi plyurisubgarmonik funksiyalarning xossalarini keltiramiz:

a) plyurisubgarmonik funksiyalarning musbat koeffitientli chiziqli kombinatiyasi plyurisubgarmonik funksiya bo‘ladi;

a) agar $u(z) \in Psh(D)$ bo‘lsa, u holda $u_\delta(z)$ funksiya $Psh(D_\delta) \bigcap C^\infty(D_\delta)$ sinfga tegishli bo‘lib, $\delta \downarrow 0$ da $u_\delta(z) \downarrow u(z)$ bo‘ladi;

b) agar $u(z) \in Psh(D)$ funksiya $z^0 \in D$ nuqtada maksimumga erishsa, ya’ni $u(z^0) \geq u(z) \quad (z \in D)$ bo‘lsa, u holda $u(z) \equiv const$ bo‘ladi;

v) monoton kamayuvchi yoki tekis yaqinlashuvchi plyurisubgarmonik funksiyalar ketma – ketligining limiti plyurisubgarmonik funksiya bo‘ladi;

g) agar $u(z) \in Psh(D)$ bo‘lsa, u holda $u_\delta(z)$ funksiya $Psh(D_\delta) \bigcap C^\infty(D_\delta)$ sinfga tegishli bo‘lib, $\delta \downarrow 0$ da $u_\delta(z) \downarrow u(z)$ bo‘ladi;

d) D sohada plyurisubgarmonik funksiyalar sinfi $\{u_\alpha\}_{\alpha \in \wedge}$ berilgan bo‘lib, $u = \sup\limits_{\alpha \in \wedge} u_\alpha$ yuqoridan yarim uzluksiz funksiya bo‘lsin. U holda $u(z)$ plyurisubgarmonik funksiya bo‘ladi, $u \in Psh(D)$

Jumladan, $u_j(z) \in Psh(D) \qquad (j = 1, 2, ..., k)$ bo‘lsa,

$$\sup\{u_1(z),\, u_2(z), ..., u_k(z)\} \in Psh(D)$$

bo‘ladi;

e) agar $f(z) \in \mathrm{O}(D)$ bo‘lsa, $\alpha \cdot \ln\left| f(z) \right| \in Psh(D)$ bo‘ladi, bunda $\alpha \geq 0$;

$j)$ agar $u(z) \in Psh(D)$ bo'lsa, u holda $e^{u(z)} \in Psh(D)$ dir, agar $u(z) \in Psh(D)$, $u\big|_D < 0$ bo'lsa, u holda $-\ln[-u(z)] \in Psh(D)$ dir.

$z)$ D sohada aniqlangan $u(z) \in C^2(D)$ funksiyaning D da plyurisubgarmonik bo'lishi uchun ushbu

$$L(u,w) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \tag{1.3.2}$$

kvadratik formaning (Levi formasining) ixtiyoriy $z \in D$ da musbat aniqlangan bo'lishi zarur va yetarlidir. (1.3.2) munosabatda $w = (w_1, w_2, ..., w_n) \in \mathbf{C}^n$ bo'lib, kvadratik formaning musbat aniqlanganligi $\forall w \in \mathbf{C}^n$ da $L(u,w) \geq 0$ bo'lishini anglatadi.

Agar $\forall w \in \mathbf{C}^n$, $w \neq 0$ da $L(u,w) > 0$ bo'lsa, $u(z)$ funksiya z nuqtada ***qat'iy plyurisubgarmonik*** funksiya deyiladi; agar $u(z)$ funksiya D sohaning har bir nuqtasida qat'iy plyurisubgarmonik bo'lsa, $u(z)$ ***sohada qat'iy plyurisubgarmonik*** funksiya deyiladi.

I BOB bo‘yicha xulosa

Ushbu bobda garmonik, subgarmonik va plyurisubgarmonik funksiyalar, hamda ularning xossalari keltirib o‘tilgan. subgarmonik va plyurisubgarmonik funksiyalar sinflari potensiallar va plyuripotensiallar nazariyalarining asosiy obyektlaridan bo‘lib, bu nazariyalarning barcha metodlari asosida ushbu tushunchalar yotadi. Masalan, Laplas tenglamasi uchun Dirixle masalasi, kompleks Monj Amper tenglamasi uchun Dirixle masalasi, ko‘p argumentli golomorf funksiyalarni analitik davom ettirish masalalari, ko‘phad va ratsional funksiyalar bilan yaqinlashtirishlar masalalarida ekstremal plyurisubgarmonik funksiyalar muhim ro‘l o‘ynaydi.

2.1. Musbat aniqlangan differensial forma va oqimlar

2.1.1. Differensial formalar.

2.1.1-ta'rif. Ushbu

$$\omega = \sum_{0 \le j_1 < j_2 < \ldots < j_p \le n} a_{j_1 j_2 \cdots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p}$$

ko'rinishdagi ifodaga p-darajali differensial forma(bunda $a_{j_1 j_2 \cdots j_p}$ -G sohada aniqlangan funksiya) deyiladi va $\deg \omega = p$ kabi belgilanadi. p-darajali differensial formalar to'plami $F^p(G)$ orqali belgilanadi. $\omega \in F^p(G)$ differensial formani quyidagi ko'rinishda ham yozishimiz mumkin:

$$\omega = \sum_{0 \le j_1 < j_2 < \ldots < j_p \le n} a_{j_1 j_2 \cdots j_p} dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p} = \sum_J a_J dx_J$$

bunda $dx_J = dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_p}$, $J = (j_1, j_2, \ldots, j_p)$, $1 \le j_1, j_2, \ldots, j_p \le n$ — multiindekslar. Ravshanki, 0-darajali differensial forma bu- $a(x)$ skalyar funksiya.

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$ kompleks fazoda differensial formani dz_j *va* $d\overline{z_j}$ lardan foydalanib yozish mumkin. $z = (z_1, z_2, \ldots, z_n)$ koordinatalar $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_j = x_j + ix_{n+j}$, $j = 1,2,$ dagi kompleks koordinatalar bo'lsin. Unda

$$dx_j = \frac{dz_j + d\overline{z_j}}{2}, \quad dx_{n+j} = \frac{dz - d\overline{z_j}}{2i}$$

bo'lganligidan $\omega = \sum_J a_J dx_J$ differensial formani dz_j *va* $d\overline{z_k}$ larning chiziqli kombinatsiyasi ko'rinishida yozish mumkin. Har bir hadda dz_j *va* $d\overline{z_k}$ lar turlicha bo'lishi mumkin. Masalan, $\mathbb{C}^2 \simeq \mathbb{R}^4$ da

$$dx_1 \wedge dx_2 \wedge dx_3 = \frac{i}{4}\left(-dz_1 \wedge dz_2 \wedge d\overline{z_1} + dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}\right)$$

Ammo ba'zi hollarda ω dagi barcha qo'shiluvchilarda dz_j larning darajasi p ga, $d\overline{z_k}$ larniki esa q ga teng bo'lib qolishi mumkin. Bunday hollarda biz ω differensial formani (p,q) bidarajali differensial forma deb aytamiz va $\omega \in F^{p,q}$ deb belgilaymiz.

2.1.2. Musbat aniqlangan differensial formalar

2.1.2-ta'rif. Ushbu $\omega = \bigwedge_{j=1}^{p} \frac{i}{2} dl_j \wedge d\overline{l_j}$

ko'rinishdagi differensial forma (p,p) bidarajali bosh kuchli musbat differensial forma deyiladi, bu yerda $l_j = a_{j_1} dz_1 + ... + a_{j_n} dz_n - (dz_1,...,dz_n)$ bazis bo'yicha chiziqli kombinatsiya $a_{jk} \in \mathbb{C},\ j = 1,2,...,p, k = 1,2,...,n$

Bunday formalarning ixtiyoriy chiziqli kombinatsiyasi kuchli musbat differensial forma deb ataladi, ya'ni kuchli musbat differensial forma bu-

$$\omega = \sum_s \lambda_s(z) \bigwedge_{j=1}^{p} \frac{i}{2} dl_{j_s} \wedge d\bar{l}_{j_k}$$

ko'rinishdagi formadir, bu yerda $\lambda_s(z)$ qaralayotgan $G \subset \mathbb{C}^n$ sohadagi manfiy bo'lmagan funksiya.

Agar $z_j = x_j + ix_{n+j}$ kompleks koordinatalardan $x_j,\ x_{n+j},\ j = 1,2,...,n$ haqiqiy koordinatalarga o'tsak, unda ushbu

$$\left(\frac{i}{2}\right)^p dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge dz_p \wedge d\overline{z_p} = dx_1 \wedge dx_{n+1} \wedge \ldots \wedge dx_p \wedge dx_{n+p}$$

ifoda $\mathbb{R}^{2n}$ fazoda $2p$ darajali musbat forma bo‘ladi.

(n,n) bidarajali differensial forma (kuchli) musbat bo‘ladi, faqat va faqat shu holdaki, qachonki u

$$\omega = \lambda(z) \mathop{\wedge}_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z_j} = \lambda(z) dx_1 \wedge dx_{n+1} \wedge \ldots \wedge dx_n \wedge dx_{2n}$$

ko‘rinishga ega bo‘lsa,bu yerda $\lambda(z)-$ musbat funksiya.

(n,n) bidarajali $\Omega = \mathop{\wedge}_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z_j} = dV$ differensial forma $\mathbb{C}^n$ da hajm formasi rolini o‘ynaydi.

$p=1$ da $\omega = \sum \lambda_{jk} dz_j \wedge d\overline{z_k} -$ differensial forma musbat bo‘ladi, faqat va faqat shu holdaki, qachonki $\sum_{jk} \lambda_{jk} \xi_j \xi_k -$ ermit kvadratik formasi musbat bo‘lsa. Shuningdek, ixtiyoriy $z^o \in G$ tayinlangan nuqtada unitar almashtirish yordamida uni

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \mu_j dz_j \wedge d\overline{z_j},\ \mu_j \geq 0$$

ko‘rinishda yozish mumkin.

2.1.3-ta’rif. $\omega \in F^{p,p}$ differensial forma kuchsiz musbat yoki musbat deyiladi, agar ixtiyoriy kuchli musbat $\nu \in F^{n-p,n-p}$ forma uchun $\omega \wedge \nu \in F^{n,n}$ forma musbat bo‘lsa.

$p = 0,1,n-1,n$ da kuchli musbatlik kuchsiz musbatlikka ekvivalent. Biroq boshqa p lar uchun bu sinflar ustma-ust tushmaydi. Buni aniq tushunish uchun kuchli musbat formani quyidagi ko‘rinishda yozamiz:

$$\omega = \sum_s \lambda_s(z) \overset{p}{\underset{j=1}{\wedge}} \frac{i}{2} dl_{j_s} \wedge d\bar{l}_{j_s} = \frac{i^{p^2}}{2^p} \sum_s \lambda_s(z) \overset{p}{\underset{j=1}{\wedge}} \frac{i}{2} dl_{j_s} \overset{p}{\underset{j=1}{\wedge}} d\bar{l}_{j_s} = \frac{i^{p^2}}{2^p} \sum_s \lambda_s(z) dl_{j_s} \wedge d\bar{l}_{j_s} \quad (1.3.3)$$

Agar $\theta = \sum_s \mu_s(z) dl_s$ formani qarasak , bunda $\mu_s -$ kompleks qiymatli funksiya, unda $\frac{i^{p^2}}{2^p} \theta \wedge \overline{\theta}$ ko‘rinishdagi forma (kuchsiz) musbat bo‘ladi, biroq $2 \leq p \leq n-2$ da u har doim ham $(1.3.3)$ ko‘rinishga ega bo‘lavermaydi.

2.1.3. Oqimlar. M ko‘pxillikdagi oqimni koeffitsiyentlari umumlashgan funksiyalardan iborat differensial forma deb qarash mumkin.

Tayinlangan $G \subset \mathbb{R}^n$ soha uchun asosiy diffferensial formalar fazosi quyidagicha aniqlanadi:

$$F^p = F^p(G) = \left\{\omega \in F^p(G) \cap C^\infty(G) : \operatorname{supp} \omega \subset\subset G\right\}$$

F^p da topologiya asosiy funksiyalar fazosidagi topologiya kabi aniqlanadi:

2.1.4-ta’rif. F^p fazodagi uzluksiz chiziqli (kompleks qiymatli) T funksional $n-p=q$ darajali oqim deyiladi.Barcha $q-$ darajali oqimlar fazosini $\Im^q$ orqali belgilaymiz. $\Im^q$ oqimlar fazosi umumlashgan funksiyalarning barcha xossalarini qanoatlantiradi.Xususan,ular uchun kuchsiz yaqinlashish, kuchsiz chegaralanganlik tushunchalarini kiritish mumkin va $T_\delta \in \Im^q \cap C^\infty$ o‘ramani aniqlash mumkinki, $\delta \to 0$ da oqim sifatida T da yaqinlashadi, ya’ni $T_\delta \circ \omega \to T \circ \omega, \quad \forall \omega \in \Im^p$ .Undan tashqari T oqimni differensial forma kabi differensiallashimiz mumkin: $dT \circ \omega = \left(-1\right)^p T \circ d\omega$

2.1.5-ta'rif. Agar $F^p = \left\{\omega \in \Im^p\left(G\right) \cap C^\infty\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$ fazodan

olingan T oqim (funksional) $\left\{\omega \in \Im^p\left(G\right) \cap C^m\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$fazoga

uzluksiz davom qilsa, unda T oqim m dan ko‘p bo‘lmagan singulyarlikka

ega deyiladi, bunda $0 \le m \le \infty$. Agar undan tashqari T oqim

$\left\{w \in F^p\left(G\right) \cap C^{m-1}\left(G\right) : \operatorname{supp} w \subset\subset G\right\}$

fazoga davom qilmasa, unda T oqim aniq m singulyarlikka ega deyiladi.

$T = \sum_{|k|=q} \beta_k dx_k$ koeffitsiyentlari umumlashgan funksiya bo‘lgan differensial

forma oqimni ifodalaydi.

2.1.1-teorema. Ixtiyoriy $q-$darajali oqim koeffitsiyentlari umumlashgan funksiya bo‘lgan differensial forma bo‘ladi, shu bilan birga singulyarligi nolga teng bo‘lgan oqim o‘lchov tipidagi oqim bo‘ladi va uni barcha koeffitsiyentlari kompleks qiymatli o‘lchov bo‘ladi.O‘lchov tipidagi oqimlar fazosini $\Im_0^q$ orqali belgilaymiz. $\Im_0^q$ nafaqat $G \subset \mathbb{R}^n$ soha uchun, balki ixtiyoriy $E \subset \mathbb{R}^n$ borel to‘plami uchun ham aniqlangan. Xususan, $K \subset \mathbb{R}^n$ kompakt uchun differensial forma koeffitsiyentlaridan tashkil topgan vektorlarni $F_0^q\left(K\right) = F_0^q\left(K\right) \cap C\left(K\right)$ fazoga qo‘shma fazo bo‘lgani uchun u Banax fazosi bo‘ladi. $T \in \Im_0^q$ oqim uchun o‘lchov tipidagi oqim

$T \circ \omega = \mu\left(z\right)\Omega$ bo‘ladi,bu yerda $\mu-$ kompleks qiymatli o‘lchov bo‘lib, u $\omega \in \Im_0^p$ ga bog’liq va $\Omega = dx_1 \wedge \ldots \wedge dx_n$,shu bilan birga funksional qiymati

$T \circ \omega = \int\limits_G d\mu$ bo‘ladi.

Xuddi o‘lchovlar fazosi kabi quyidagi xossalar o‘rinli:

a) $\{T_j\} \subset \Im_0^{\,q}$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, ya’ni $\lim\limits_{j\to\infty} T_j \circ \omega = T \circ \omega,\ \forall \omega \in \Im_0^{\,p}$ faqat va faqat shu holdaki,qachon unga mos T_j dan olingan koeffitsiyentlardan tuzilgan o‘lchovlar ketma-ketligi yaqinlashuvchi bo‘lsa.

b) $\{T_j\} \subset \Im_0^{\,q}$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, faqat va faqat shu holdaki,qachon T_j norma bo‘yicha chegaralangan bo‘lsa $\|T_j\|_k \le c,\ \forall K \subset\subset G,\ j = 1,2,...$ va $j \to \infty$ da $T_j \circ \omega \to T \circ \omega$ limitik munosabat hamma yerda zich biror $E \subset \Im_0^{\,p}(K)$ to‘plamdan olingan barcha ω lar uchun o‘rinli bo‘lsa, masalan barcha cheksiz differensiallanuvchi $\omega \in F^p \bigcap C^\infty(K)$ lar uchun

d)ixtiyoriy kuchsiz chegaralangan $E \subset \Im_0^{\,q}(K)$ qiymatlar to‘plami ham kuchsiz kompakt ya’ni uni ixtiyoriy ketma-ketligi kuchsiz yaqinlashuvchi qismiy ketma-ketligi $\{T_j\} \subset E$ ni o‘zida saqlaydi.

2.1.6-ta’rif. $q-$ darajali differensial forma (oqim) yopiq deyiladi, agar uni differensiali nolga teng bo‘lsa;u aniq deyiladi, agar u biror $q-1$ darajali formani(oqimni) differensialini ifodalash. Aniq formalar(oqimlar) yopiq bo‘ladi, teskari tasdiq uchun G sohaga qo‘shimcha geometrik shartlarni qo‘yish zarur.

Endi $\mathbb{C}^n$ da asosiy differensial formalar fazosini qaraymiz:

$$\Im^{p,p} = \Im^{p,p}\left(G\right)\left\{\omega \in \Im^{p,p}\left(G\right) \cap C^{\infty}\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$$

2.1.7-ta'rif. $F^{p,p}$ fazoda uzluksiz chiziqli(kompleks qiymatli) T funksionalga $\left(n-p, n-p\right) = \left(q,q\right)$ bidarajali oqim deyiladi.

2.1.2-teorema.Musbat oqimlar o'lchov tipidagi oqimlar bo'ladi.

2.1.1-misol. Koeffitsiyentlari L^1_{loc} sinfdan olingan ixtiyoriy $T \in F^{q,q}$ differensial forma $\left(q,q\right)$ bidarajali oqim bo'ladi.

$T \circ \omega$ funksional $T \circ \omega = \int\limits_G T \wedge \omega, \ \ \forall \omega \in F^{p,p}$ kabi aniqlanadi. Bunday oqimlar regulyar oqim deyiladi.

2.1.2-misol. $A \subset \mathbb{C}^n - p$ o'lchamli analitik to'plam bo'lsin.U holda

$\left[A\right] \circ \omega = \int\limits_A \omega$ integral $\left(q,q\right)$ bidarajali oqimni aniqlaydi. Ko'rish qiyin emaski,$\left[A\right]$ musbat oqim bo'ladi.

2.2. Musbat gessianalar bilan differensial formalar orasidagi bog'lanish

2.2.1.Gessianlar. 2-marta silliq $u \in C^2(D)$ funksiya uchun uning ikkinchi tartibli differensiali

$$dd^c u = \frac{i}{2}\sum_{j,k} u_{j\bar{k}} dz_j \wedge d\overline{z_k} \quad \left(bunda \ u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) \tag{2.2.1}$$

(tayinlangan $z^0 \in D$ nuqtada) ermit kvadratik formasidan iborat bo'ladi.

$\left(dd^c u\right)^n$ -operator Monj-Amper operatori deyiladi.

Ko‘rish qiyin emaski, $\left(dd^c u\right)^m \wedge \beta^{n-m} = m!\left(n-m\right)!H_m\left(u\right)\beta^n$ (2.2.2)

Bunda $H_m(u) = \sum\limits_{1\le j_1<\ldots<j_m\le n} \lambda_{j_1},\lambda_{j_2},\ldots\lambda_{j_m}$ – gessian vektori.Shunday qilib,

2 marta silliq u funksiya uchun $\left(dd^c u\right)^m \wedge \beta^{n-m}$ operator gessian vektori bilan bog‘liq ekan.

2.2.2. Musbat gessianlar.

Berilgan $\lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) \in \mathbb{R}^n$ uchun

$$H_m(\lambda) = \sum_{1\le j_1<\ldots<j_m\le n} \lambda_{j_1},\lambda_{j_2},\ldots\lambda_{j_m} \qquad (2.2.3)$$

gessianni tushunish uchun quyidagi yordamchi ifodadan foydalanamiz:

$$\left(t+\lambda_1\right)\left(t+\lambda_2\right)\ldots\left(t+\lambda_n\right) = t^n + H_1\left(\lambda\right)t^{n-1} + \ldots + H_n\left(\lambda\right), \quad t\in\mathbb{R} \qquad (2.2.4)$$

(2.2.8) ni $n=2$ uchun yozib ko‘ramiz:

$$\left(t+\lambda_1\right)\left(t+\lambda_2\right) = t^2 + \left(\lambda_1+\lambda_2\right)t + \lambda_1\lambda_2 \Rightarrow H_1\left(\lambda\right) = \left(\lambda_1+\lambda_2\right), \qquad H_2\left(\lambda\right) = \lambda_1\lambda_2$$

(2.2.7) formula bo‘yicha yozadigan bo‘lsak,

$$H_1(\lambda) = \sum_{1\le j_1\le 2} \lambda_{j_1} = \lambda_1+\lambda_2, \;\; H_2(\lambda) = \sum_{1\le j_1<j_2\le 2} \lambda_{j_1}\lambda_{j_2} = \lambda_1\lambda_2$$

n=3 uchun:

$$\left(t+\lambda_1\right)\left(t+\lambda_2\right)\left(t+\lambda_3\right) = t^3 + \left(\lambda_1+\lambda_2+\lambda_3\right)t^2 + \left(\lambda_1\lambda_2+\lambda_3\lambda_2+\lambda_1\lambda_3\right)t + \lambda_1\lambda_2\lambda_3.$$

(2.2.7) formula bo‘yicha yozadigan bo‘lsak,

$$H_1(\lambda) = \sum_{1\le j_1\le 3} \lambda_{j_1} = \lambda_1+\lambda_2+\lambda_3,$$
$$H_2(\lambda) = \sum_{1\le j_1<j_2\le 3} \lambda_{j_1}\lambda_{j_2} = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$$
$$H_3(\lambda) = \sum_{1\le j_1<j_2<j_3\le 3} \lambda_{j_1}\lambda_{j_2}\lambda_{j_3}$$

Endi biz $H_m(\lambda) \ge 0$ bo‘ladigan $\lambda \in \mathbb{R}^n$ vektorlar bilan qiziqamiz va shunday vektorlarni birlashtiramiz, ya’ni $S_m = \{\lambda \in \mathbb{R}^n,\ H_m(\lambda) \ge 0\}$.

$m > 1$ da bu to‘plam qavariq konusni hosil qilmaydi, ya’ni $\lambda', \lambda'' \in S_m$ ekanligidan $\lambda' + \lambda'' \in S_m$ ekanligi kelib chiqmaydi. Shuning uchun bu to‘plamni toraytiramiz, ya’ni $\lambda \in S_m$ vektorlar uchun shunday $\Gamma_m \subset S_m$ to‘plamni olamizki, $\{\lambda + tI = (\lambda_1 + t_1, ..., \lambda_n + t_n),\ t \in \mathbb{R}^n_+\} \subset S_m$ bo‘ladi. Unda Γ_m ning ko‘rinishi $\Gamma_m = \{H_1(\lambda) \ge 0, ..., H_m(\lambda) \ge 0\}$ bo‘ladi. Undan tashqari Γ_m $(1,1,...1)$ vektorni o‘z ichiga oluvchi S_m to‘plamni $S_m^0 = \{\lambda \in \mathbb{R}^n : H_m(\lambda) > 0\}$ ochiq yadrosining bog‘lamli komponentasini yopig‘i bilan ustma-ust tushadi.

Γ_m **quyidagi xossalarga ega:**

1) $\Gamma_n \subset \Gamma_{n-1} \subset ... \subset \Gamma_1, \Gamma_n = \{\lambda_1 \ge 0, ...\lambda_n \ge 0\} = \mathbb{R}^n_+$

Biz bu xossani $n = 3$ bo‘lgan holda qaraymiz va $\Gamma_3 = \{H_1(\lambda) \ge 0, H_2(\lambda) \ge 0, H_3(\lambda) \ge 0\}$ ekanligidan

$$H_1(\lambda) = \sum_{1 \le j_1 \le 3} \lambda_{j_1} = \lambda_1 + \lambda_2 + \lambda_3 \ge 0$$
$$H_2(\lambda) = \sum_{1 \le j_1 < j_2 \le 3} \lambda_{j_1}\lambda_{j_2} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \ge 0 \ \Rightarrow \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0$$
$$H_3(\lambda) = \sum_{1 \le j_1 < j_2 < j_3 \le 3} \lambda_{j_1}\lambda_{j_2}\lambda_{j_3} = \lambda_1\lambda_2\lambda_3 \ge 0$$

bo‘lishi kelib chiqadi.

$\Gamma_2 = \{H_1(\lambda) \ge 0, H_2(\lambda) \ge 0\}$ esa

$$H_1(\lambda)=\sum_{1\le j_1\le 3}\lambda_{j_1}=\lambda_1+\lambda_2+\lambda_3\ge 0$$
$$H_2(\lambda)=\sum_{1\le j_1<j_2\le 3}\lambda_{j_1}\lambda_{j_2}=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\ge 0$$

shartlarni qanoatlantiradi.

$\Gamma_1=\{H_1(\lambda)\ge 0\}$ ekanligidan esa $H_1(\lambda)=\sum_{1\le j_1\le 3}\lambda_{j_1}=\lambda_1+\lambda_2+\lambda_3\ge 0$

tengsizlikning bajarilishi yetarli ekan.

2) Γ_m qavariq konusdan iborat, ya'ni agar $\lambda,\mu\in\Gamma_m$ bo'lsa, u holda $\forall a,b\in\mathbb{R}^+$ uchun $a\lambda+b\mu\in\Gamma_m$ o'rinli.

3) $\left\{H_m^{1/m}(\lambda),\lambda\in\Gamma_m\right\}$ o'suvchi va botiq konusdan iborat, ya'ni

$$H_m^{1/m}(\mu)\le H_m^{1/m}(\lambda)+\sum_j(\mu_j-\lambda_j)\frac{\partial}{\partial\lambda_j}H_m^{1/m}(\lambda)\forall\lambda,\mu\in\Gamma_m$$

Soddalik uchun m=1, n=1 bo'lgan holni qarab chiqamiz:

$$H_1(\mu)\le H_1(\lambda)+(\mu-\lambda)\frac{\partial}{\partial\lambda}H_1(\lambda)\quad\forall\lambda,\mu\in\Gamma_m\Rightarrow\mu\le\lambda+(\mu-\lambda)\Rightarrow\mu=\mu$$

Endi $m=1,\ n=2$ bo'lgan holni qaraymiz:

$\mu_1+\mu_2\le\lambda_1+\lambda_2+(\mu_1-\lambda_1)+(\mu_2-\lambda_2)$ demak, $n=k,m=1$ uchun hamma vaqt tenglik bajarilar ekan. $m=2,n=2$ bo'lgan holda esa quyidagicha ko'rinishga o'tadi:

$$\sqrt{\mu_1\mu_2} \le \sqrt{\lambda_1\lambda_2} + (\mu_1 - \lambda_1)\frac{\lambda_2}{2\sqrt{\lambda_1\lambda_2}} + (\mu_2 - \lambda_2)\frac{\lambda_1}{2\sqrt{\lambda_1\lambda_2}}$$

$$\sqrt{\mu_1\mu_2} \le \frac{2\lambda_1\lambda_2 + \mu_1\lambda_2 - 2\lambda_1\lambda_2 + \mu_2\lambda_1}{2\sqrt{\lambda_1\lambda_2}} \Rightarrow$$

$$2\sqrt{\lambda_1\lambda_2\mu_1\mu_2} \le \mu_1\lambda_2 + \mu_2\lambda_1,\ \ a = \mu_1\lambda_2, b = \mu_2\lambda_1$$

$$\Rightarrow a + b \ge 2\sqrt{ab}$$

4) $\sum_j \mu_j \frac{\partial}{\partial\lambda_j} H_m(\lambda) \ge m H_m^{1/m}(\mu) H_m^{1-1/m}(\lambda)\ \forall\lambda,\mu \in \Gamma_m$

$m = 1, n = 1$ bo'lsa, $\mu \cdot 1 \ge \mu, \Rightarrow \mu = \mu$

$m = 1, n = 2 \Rightarrow \mu_1 + \mu_2 \ge \mu_1 + \mu_2$

$m = 2,\ n = 2 \Rightarrow \mu_1\lambda_2 + \mu_2\lambda_1 \ge 2\sqrt{\mu_1\mu_2}\sqrt{\lambda_1\lambda_2}$

$$m = 2,\ n = 3\ \Rightarrow \mu_1 \frac{\partial}{\partial\lambda_1}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3\lambda_2) +$$
$$+\mu_2 \frac{\partial}{\partial\lambda_2}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3\lambda_2) + \mu_3 \frac{\partial}{\partial\lambda_3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3\lambda_2) \ge$$

$$\ge 2\sqrt{\mu_1\mu_2 + \mu_1\mu_3 + \mu_3\mu_2} \cdot \sqrt{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3\lambda_2}$$

$$\mu_1(\lambda_2 + \lambda_3) + \mu_2(\lambda_1 + \lambda_3) + \mu_3(\lambda_1 + \lambda_2) \ge 2\sqrt{\mu_1\mu_2 + \mu_1\mu_3 + \mu_3\mu_2} \cdot \sqrt{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3\lambda_2}$$

5) $\left[\frac{H_m}{\binom{n}{m}}\right]^{1/m} \le \left[\frac{H_m}{\binom{n}{k}}\right]^{1/k}\ \ \forall\ 1 \le k \le m \le n$ Bu tengsizlik Makloren tengsizligi deb yuritiladi.

2.2.3. Musbat gessianlar bilan differensial formalar

Bizga $(1,1)$ - bidarajali $\alpha = \frac{i}{2}\sum_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k$ haqiqiy differensial forma berilgan bo‘lsin. Bunda $(a_{j\bar{k}})$ – ermit matrisa. Agar $\lambda(\alpha) = (\lambda_1(\alpha),...,\lambda_n(\alpha))$ – bu matrisaning xos qiymati bo‘lsa, u holda $H_m(\alpha) = H_m(\lambda(\alpha))$ gessian aniqlash mumkin.Bu esa (1,1) – ermit formasi va $\lambda \in \mathbb{R}^n$ vektor orasidagi bog‘liklikni aniqlashga yordam beradi:

$$\hat{\Gamma}_m = \{\alpha : \lambda(\alpha) \in \Gamma_m\} = \left\{ \begin{matrix} H_m\left(\alpha + t_1 dd^c |z_1|^2 + ... + t_n dd^c |z_n|^2\right) \geq 0, \\ \forall t = (t_1,...,t_n) \in \mathbb{R}^n_+ \end{matrix} \right\} =$$

$$= \{\alpha \wedge \beta^{n-1} \geq 0,...,\alpha^m \wedge \beta^{n-m} \geq 0\}. \tag{2.2.9}$$

Gessianning 2-xossasiga ko‘ra agar $\lambda', \lambda'' \in \Gamma_m$ bo‘lsa, u holda $a,b \in \mathbb{R}_+$ uchun $a\lambda' + b\lambda'' \in \Gamma_m$ bo‘ladi. Biz λ' sifatida α differensial formaning xos qiymatlarini, λ'' sifatida $\lambda'' = (t_1, t_2,...,t_n) \in \mathbb{R}^n_+$ ni belgilasak va $a,b = 1$ xususiy holni qarasak

$$H_m\left(\alpha + t_1 dd^c |z_1|^2 + ... + t_n dd^c |z_n|^2\right) \geq 0 \ \forall t = (t_1,...,t_n) \in \mathbb{R}^n_+$$

Ekanligiga ishonch hosil qilishimiz mumkin. Chunki Γ_m ni $\Gamma_m = \{H_1(\lambda) \geq 0,...,H_m(\lambda) \geq 0\}$ ko‘rinishda aniqlagan edik.

Endi boshqa holatlarni ko‘rib chiqamiz, ya’ni $\lambda' \notin \Gamma_m$ bo‘lsin. Buni misollar yordamida tushunib olishimiz osonroq bo‘ladi:

2.2.1-misol $\alpha = \frac{i}{2}\left(-7dz_1 \wedge d\overline{z_1} - 11dz_2 \wedge d\overline{z_2} + 3dz_3 \wedge d\overline{z_3}\right)$ bo‘lsin**.**

$\lambda(\alpha)=(\lambda_1,\lambda_2,\lambda_3)=(-7,-11,3)\notin\Gamma_2=\{H_1(\lambda)\geq 0, H_2(\lambda)\geq 0\}$ Chunki

$$H_1(\lambda(\alpha))=\lambda_1+\lambda_2+\lambda_3=-7-11+3=-15<0,$$

$$H_2(\lambda(\alpha))=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=77-21-33=23>0$$

$$H_3(\lambda(\alpha))=\lambda_1\lambda_2\lambda_3=231>0$$

Biz $t=(t_1,t_2,t_3)\in\mathbb{R}^3_+$ ning ixtiyoriyligidan foydalanib $t=(6,1,1)\in\mathbb{R}^3_+$ deb tanlab olamiz va

$$H_2\left(\frac{i}{2}\left(-7dz_1\wedge d\overline{z_1}-11dz_2\wedge d\overline{z_2}+3dz_3\wedge d\overline{z_3}\right)+6dd^c|z_1|^2+dd^c|z_2|^2+dd^c|z_3|^2\right)=$$

$$=H_2(-1,-10,4)=-44+10=-34<0,$$

bo‘lishini ko‘ramiz. Xulosa qilishimiz uchun yana bir misolni ko‘rib o‘taylik:

2.2.2-misol. $\alpha=\frac{i}{2}\left(-2dz_1\wedge d\overline{z_1}+dz_2\wedge d\overline{z_2}+3dz_3\wedge d\overline{z_3}\right)$ bo‘lsin.

$\lambda(\alpha)=(\lambda_1,\lambda_2,\lambda_3)=(-2,1,3)\notin\Gamma_2,\Gamma_3$. Chunki

$$H_1(\lambda(\alpha))=\lambda_1+\lambda_2+\lambda_3=-2+1+3=2>0$$

$$H_2(\lambda(\alpha))=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=-2-6+3=-5<0$$

$$H_3(\lambda(\alpha))=\lambda_1\lambda_2\lambda_3=-6<0$$

Biz $t=(t_1,t_2,t_3)\in\mathbb{R}^3_+$ ning ixtiyoriyligidan foydalanib $t=(0,1,1)\in\mathbb{R}^3_+$ deb tanlab olamiz va

$$H_2\left(\frac{i}{2}\left(-2dz_1\wedge d\overline{z_1}+1dz_2\wedge d\overline{z_2}+3dz_3\wedge d\overline{z_3}\right)+dd^c\left|z_2\right|^2+dd^c\left|z_3\right|^2\right)=$$

$$=H_2\left(-2,2,4\right)=-4-8+8=-4<0$$

tengsizlik o‘rinli bo‘ladi. Agar $t=\left(1,3,5\right)\in\mathbb{R}_+^3$ deb olsak, unda ushbu

$$H_3\left(\frac{i}{2}\left(-2dz_1\wedge d\overline{z_1}+1dz_2\wedge d\overline{z_2}+3dz_3\wedge d\overline{z_3}\right)+dd^c\left|z_1\right|^2+3dd^c\left|z_2\right|^2+5dd^c\left|z_3\right|^2\right)=$$

$$=H_3\left(-1,4,8\right)=-32<0$$

ifodaga ega bo‘lamiz. Shunday qilib, biz quyidagi xulosaga keldik: $\lambda\left(\alpha\right)\in\Gamma_m$ bo‘lgandagina $H_m\left(\alpha+t_1dd^c\left|z_1\right|^2+...+t_ndd^c\left|z_n\right|^2\right)\geq 0\ \forall t=\left(t_1,...,t_n\right)\in\mathbb{R}_+^n$ tengsizlik o‘rinli bo‘lar ekan. Boshqa hollarda bu tengsizlik bajarilmas ekan.

2.2.1-teorema.[18] Agar $\alpha_1,...,\alpha_k\in\hat{\Gamma}_m$, $1\leq k\leq m$ bo‘lsa, u holda $\alpha_1\wedge...\wedge\alpha_k\wedge\beta^{n-m}\geq 0$ bo‘ladi.

II BOB bo‘yicha xulosa

Ushbu bobda differensial forma va oqimlar, musbat aniqlangan differensial formalar o‘rganildi. Xususan, musbat gessianlar bilan differensial formalar orasidagi bog’lanish o‘rganildi. Musbat gessianlardan biz ikki marta silliq $m-sh$ funksiyalarga ta’rif berishda va ularning xossalarini isbotlashda hamda qat’iy $m-sh$(ikki marta silliq funksiyalar uchun) funksiyalarni ta’riflashda foydalanamiz.

III BOB.KUCHSIZ $m-$SUBGARMONIK FUNKSIYALARNING $m-$SUBGARMONIKLIK XUSUSIYATLARI.

3.1. $m-sh$ funksiyalar va ularning xossalari

3.1.1-ta'rif. Ikki marta silliq $u\in C^2(D)$ $(D\subset\mathbb{C}^n)$ funksiya $z^0\in D$ nuqtada $m-sh$ $(1\le m\le n)$ deyiladi, agar $\left(u_{j\bar{k}}\right)|_{z=z^0}$ matrisaning $\lambda(u)=\left(\lambda_1(u),...,\lambda_n(u)\right)$ xos qiymatlari Γ_m ga tegishli bo'lsa yoki $dd^cu\in\hat{\Gamma}_m$ bo'lsa. $u\in C^2(D)$ funksiya $D\subset\mathbb{C}^n$ sohada $m-sh$, $(1\le m\le n)$ deyiladi, agar u har bir $z^0\in D$ nuqtada $m-sh$ bo'lsa.

Boshqacha aytganda, $u\in C^2(D)$ funksiya $m-sh$ $(1\le m\le n)$ deyiladi, agar $dd^cu\in\hat{\Gamma}_m$ bo'lsa yoki unga quyidagi tengsizlik teng kuchli:

$$\left(dd^cu\right)^k\wedge\beta^{n-k}\ge 0,\quad\forall\ k=1,2,...,m \tag{3.1.1}$$

(3.1.1) ga quyidagi tengsizlik ham ekvivalent:

$$\left[\left(dd^cu+t_1dd^c\left|z_1\right|^2+...+t_ndd^c\left|z_n\right|^2\right)\right]^m\wedge\beta^{n-m}\ge 0\ \forall\ t=\left(t_1,...,t_n\right)\in\mathbb{R}^n_+]$$

Undan tashqari quyidagi tasdiq ham o'rinli:

$$dd^cu\wedge\alpha_1\wedge...\wedge\alpha_{k-1}\wedge\beta^{n-m}\ge 0\quad\forall\ \alpha_1,...,\alpha_{k-1}\in\hat{\Gamma}_m\ ,\ 1\le k\le m \tag{3.1.2}$$

Bu tasdiq ikki xil xarakterga ega: agar u ikki marta silliq bo'lib, $\alpha_1,...,\alpha_{m-1}\in\hat{\Gamma}_m$ uchun (3.1.2) shartni qanoatlantirsa, u holda u funksiya $m-sh$ bo'ladi.

u funksiyani $m-sh$ ligini tekshirish uchun (3.1.2) da faqat $\alpha_j = dd^c u_j$, $u_j \in C^2\left(D\right)$, $j = 1,2,\ldots,m-1$ differensial formani chegaralangan bo‘lishi yoki $m-sh$ ko‘phad bo‘lishi yetarli.

Bu esa bizga L^1_{loc} funksiyalar sinfida $m-sh$ funksiyalarni aniqlashga yordam beradi.

3.1.2-ta’rif. $u \in L^1_{loc}\left(D\right)$ funksiya $D \subset \mathbb{C}^n$ sohada $m-sh$ deyiladi, agar u yuqoridan yarim uzluksiz va ixtiyoriy ikki marta silliq $v_1,\ldots,v_{m-1}$ $m-sh$ funksiyalar uchun $dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{m-1} \wedge \beta^{n-m}$ oqim

$$\begin{gathered}\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{m-1} \wedge \beta^{n-m}\right]\left(\omega\right) = \\ u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \wedge dd^c \omega,\ \omega \in F^{0,0}\end{gathered} \tag{3.1.3}$$

musbat aniqlangan bo‘lsa.

Biz bilamizki, $\left(p,p\right)-$ finit differensial formalar fazosidagi chiziqli, uzluksiz T funksional (kompleks qiymatli)

$F^{p,p} = F^{p,p}\left(D\right) = \left\{\omega \in^{p,p}\left(D\right) \cap C^\infty\left(D\right) : \operatorname{supp}\omega \subset\subset D\ \right\}$,

$\left(n-p, n-p\right) = \left(q,q\right)$ bidarajali oqim deyilar edi.

$m-sh$ funksiyalarning xossalari:

1) $u \in C^2\left(D\right)$ funksiya $m-sh$ $(1 \le m \le n)$ bo‘ladi, faqat va faqat shu holdaki, qachonki $dd^c u \in \overset{\wedge}{\Gamma}$ $\forall z^0 \in D$ bo‘lsa.

2) agar $u \in m-sh\left(D\right)$ bo‘lsa, u holda $u_j\left(z\right) = u * K_j\left(z-w\right)$ standart approksimatsiya (o‘rama) ham $m-sh$ bo‘ladi, ($j \to \infty$ da $u_j \downarrow u$ bo‘ladi)

3) agar $u,v \in m-sh$ bo‘lsa, u holda ixtiyoriy $a,b \geq 0$ uchun $au+bv \in m-sh(D)$.

$$au+bv \in m-sh(D) \quad \Rightarrow \lambda(u),\lambda(v) \in \Gamma_m \quad a,b \geq 0$$

ekanligidan

$a\lambda(u)+b\lambda(v) \in \Gamma_m \Rightarrow au+bv \in m-sh(D)$ bo‘lishi kelib chiqadi.

4) $psh = n-sh \subset ... \subset 1-sh = sh$

5)agar $\gamma(t)-$ $t \in \mathbb{R}_+$ parametrning o‘suvchi qavariq funksiyasi va $u \in m-sh$ bo‘lsa, u holda $\gamma \circ u \in m-sh$ bo‘ladi.

6)tekis yaqinlashuvchi yoki kamayuvchi $m-sh$ funksiyalar ketma-ketligining limiti yana $m-sh$ bo‘ladi.

7)chekli sondagi $m-sh$ funksiyalarning maksimumi $m-sh$ funksiyadan iborat bo‘ladi; ixtiyoriy lokal tekis chegaralangan $\{u_\theta\} \subset m-sh$ m-subgarmonik funksiyalar oilasining supremumi $u(z) = \left\{\sup\limits_\theta u_\theta(z)\right\}$ regulyarizatsiyasi $u^*(z)$ ham $m-sh$ bo‘ladi. $m-sh \subset sh$ ekanligidan $\{u(z) < u^*(z)\}-$to‘plam $\mathbb{C}^n \approx \mathbb{R}^{2n}$ da polyar to‘plam bo‘ladi. Qisqacha aytganda uning Lebeg o‘lchovi nolga teng. Ixtiyoriy lokal tekis chegaralangan $\{u_j\} \subset m-sh$ funksiyalar ketma-ketligining limiti $u(z) = \overline{\lim\limits_{j\to\infty}} u_j(z)$ regulyarizatsiyasi $u^*(z)$ ham $m-sh$ va $\{u(z) < u^*(z)\}-$polyar to‘plam bo‘ladi.

8)agar $u \in m-sh$ bo‘lsa, u holda ixtiyoriy $P \subset \mathbb{C}^n$ kompleks gipertekislik uchun $u|_{\mathrm{P}} \in sh_{m-1}$ bo‘ladi.

Haqiqatdan, gessianning 4-xossasidan foydalanib, $\lambda, \mu \in \Gamma_m$ vektor uchun

$$\sum_j \mu_j \frac{\partial}{\partial \lambda_j} H_m(\lambda) \geq m H_m^{1/m}(\mu) H_m^{1-1/m}(\lambda)$$

$\mu = (0,...,0,1)$ deb oladigan bo‘lsak, $\frac{\partial}{\partial \lambda_n} H_m(\lambda) = H_{m-1}('\lambda) \geq 0$. Bu shuni anglatadiki, agar $\lambda = (\lambda_1,...,\lambda_{n-1},\lambda_n) \in \Gamma_m$ bo‘lsa, u holda $'\lambda = (\lambda_1,...,\lambda_{n-1}) \in \Gamma_{m-1}$

bo‘ladi. Endi $u \in m-sh(D)$ va $P \subset \mathbb{C}^n$ gipertekislik bo‘lsin.Umumiylikka zid ish qilmagan holda biz $u \in C^2(D)$, $\mathrm{P} = \{z_n = 0\}$ deb faraz qilamiz va $u|_{\mathrm{P}} = u('z,0) \in sh_m$ ekanini isbotlaymiz.

$\lambda = (\lambda_1,...,\lambda_n) - U = (u_{j\bar{k}})$, $j,k = 1,2,...,n$ matrisani xos qiymatlari bo‘lsin. $\mu = (\mu_1,...,\mu_{n-1}) -$ esa $'U = (u_{j\bar{k}})$, $j,k = 1,2,...,n-1$ matrisani xos qiymatlari bo‘lsin. Unitar almashtirishlardan keyin (tayinlangan $('z^0,0) \in D$ nuqtada) U quyidagi ko‘rinishga keladi:

$$U = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 & u_{1\bar{n}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu_{n-1} & u_{n-1\bar{n}} \\ u_{n\bar{1}} & u_{n\bar{2}} & \cdots & u_{n\overline{n-1}} & u_{n\bar{n}} \end{pmatrix}$$

$u(z) \in m-sh$ ekan, u holda (ixtiyoriy tayinlangan $t \in \mathbb{R}_+$ parametr uchun) $u_t(z) = u(z) + t|z_n|^2 \in m-sh$ bo‘ladi.

Natijada $\lambda(t)=\lambda(u_t)\in\Gamma_m$ vektor yuqorida isbotlanganiga

ko‘ra' $\lambda(t)=(\lambda_1(t),...,\lambda_{n-1}(t))\in\Gamma_{m-1}$. Boshqa tomondan $\lambda(t)$ ushbu

$$\begin{vmatrix} \lambda-\mu_1 & 0 & ... & 0 & u_{1\bar{n}} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \lambda-\mu_{n-1} & u_{n-1\bar{n}} \\ u_{n\bar{1}} & u_{n\bar{2}} & ... & u_{n\overline{n-1}} & \lambda-t-u_{n\bar{n}} \end{vmatrix}=0$$

tenglamaning ildizi.Yana almashtirishlar bajarsak,

$$\begin{vmatrix} \lambda-\mu_1 & 0 & ... & 0 & \frac{u_{1\bar{n}}}{t} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \lambda-\mu_{n-1} & \frac{u_{n-1\bar{n}}}{t} \\ u_{n\bar{1}} & u_{n\bar{2}} & ... & u_{n\overline{n-1}} & \frac{\lambda-u_{n\bar{n}}}{t}-1 \end{vmatrix}=0$$

$t\to\infty$ da $\lambda_j(t)\to\mu_j$, $'\lambda(t)\in\Gamma_{m-1}$ ekan, u holda Γ_{m-1} ning yopiqligidan $\mu\in\Gamma_{m-1}$ ya’ni $u|_{\mathrm{P}}=u('z,0)\in(m-1)-sh$ ekanligi kelib chiqadi.

8-xossa isbotlandi.

Garmonik va $m-sh$ funksiyalar kesishmasi plyurigarmonik funksiya bo‘lishi B.Abdullayev va M.Vaisovalar [23 ga qarang] tomonidan isbotlangan. Biz bu haqdagi teoremani quyida keltirib o’tamiz hamda bu teoremani $m-sh$ funksiya ta’rifi va gessianning xossalaridan foydalanib isbotlaymiz:

3.1.1-teorema. Agar *$u(z)$ funksiya bir vaqtda garmonik va $m-$subgarmonik bo‘lsa, u holda bu funksiya plyurigarmonik bo‘ladi.*

Isboti. Teoremani $k = 1{,}2$ holda ko'ramiz

$$\begin{cases} \left(dd^c u\right) \wedge \left(dd^c \left|z\right|^2\right)^{n-1} = 0 \;\Rightarrow\quad H_1(\lambda) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0 \\ \left(dd^c u\right)^2 \wedge \left(dd^c \left|z\right|^2\right)^{n-2} \geq 0 \;\Rightarrow\; H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n \geq 0 \end{cases}$$

Tenglamalar sistemasini hosil qilamiz.Bu yerda oddiy arifmetik va mantiqiy mulohazalardan

$$\begin{cases} H_1^{\,2}(\lambda) = \left(\lambda_1 + \lambda_2 + \ldots + \lambda_n\right)^2 = 0 \\ H_2(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n \geq 0 \end{cases}$$ ni hosil qilamiz.

$H_2(\lambda) \geq 0$ ligidan foydalanib sistemaning 1-tenglamasini quyidagicha yozib olamiz

$$\left\{ H_1^{\,2}(\lambda) = \lambda_1^{\,2} + \lambda_2^{\,2} + \ldots + \lambda_n^{\,2} + 2H_2\left(\lambda\right) = 0 \right.$$

Bundan esa

$$\left\{ \lambda_1^{\,2} + \lambda_2^{\,2} + \ldots + \lambda_n^{\,2} = -2H_2\left(\lambda\right) \leq 0 \right.$$

tengsizlikni olamiz. Bu tengsizlik faqat $\lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0$ yechimlarga ega. Demak $u(z)$ D sohada plyurigarmonik funksiya bo'ladi.

Teorema isbotlandi.

3.2. Kuchsiz $m-$subgarmonik funksiyalar va ularning xossalari

3.2.1-ta'rif *Agarda* $D \subset \mathbb{C}^n$ *sohada aniqlangan* $u\left(z\right) \in L^1_{loc}\left(D\right)$ *funksiya:*

1) D sohada yuqoridan yarim uzluksiz, ya'ni

$$\overline{\lim_{z \to z^0}} u\left(z\right) = \lim_{\varepsilon \to 0} \sup_{B\left(z^0, \varepsilon\right)} u\left(z\right) \leq u\left(z^0\right);$$

2) D sohada $dd^c u \wedge \left(dd^c |z|^2\right)^{n-m} \geq 0, \quad 1 \leq m \leq n,$ *ya'ni*

$\forall\, \omega \in F^{(m-1,m-1)}$ *uchun*

$dd^c u \wedge \left(dd^c |z|^2\right)^{n-m}(\omega) = u\left(dd^c |z|^2\right)^{n-m} \wedge dd^c\omega \geq 0\,,$ *bo'lsa, bu funksiyaga D sohada $m-wsh$ funksiya ($(n-m+1)$ o'lchamli kompleks tekisliklarda subgarmonik funksiya) deyiladi.*

Bunday funksiyalar sinfini $m-wsh(D)$ orqali belgilaymiz. Bu yerda "w" harfi bu sinfni $m-sh$ sinfdan ajratish uchun ishlatilgan.

$D \subset \mathbb{C}^n$ sohada $m-wsh$ funksiyalar ham ayni paytda $D \subset \mathbb{R}^{2n}$ da ham subgarmonik bo'ladi.

Binobarin, subgarmonik funksiyalarning barcha xossalari $m-wsh$ funksiyalar uchun ham o'rinli.

Quyida $m-wsh$ funksiyalarni ba`zi muhim xossalarini keltiramiz:

1) $m-wsh$ funksiyalarning nomanfiy koeffitsiyentli chiziqli kombinatsiyani $m-wsh$ funksiya bo'ladi ya`ni

$u_j(z) \in m-wsh(D) \qquad a_j \in R_+ \qquad (j=1,2,\dots,N) =>$

$a_1u_1(z)+a_2u_2(z)+\dots+a_Nu_N(z) \in m-wsh(D);$

2) Monoton kamayuvchi $m-wsh$ funksiyalar ketma-ketligining limiti $m-wsh$ funksiya bo'ladi ya`ni $u_j(z) \in m-wsh(D)$

$$u_j(z) \geq u_{j+1}(z)\ ,\ (j=1,2,\dots) \Rightarrow \lim_{j\to\infty} u_j(z) \in m-wsh(D);$$

3) Tekis yaqinlashuvchi $m-wsh$ funksiyalar ketma-ketligi $m-wsh$ funksiyaga yaqinlashadi ya`ni agar

$u_j(z) \in m-wsh(D),\ (j=1,2,\dots)\,, u_j(z) \rightrightarrows u(z)$

bo'lsa u holda $u(z) \in m-wsh(D)$ bo'ladi.

4) (Maxsimum prinsipi.) Agar $u(z) \in m-wsh(D)$ funksiya va biror $z^0 \in D$ nuqtada maksimumga erishsin,ya'ni

$$u(z^0) = \sup_{z\in D} u(z) \qquad (3.2.1)$$

Unda $u(z) \equiv const$ bo'ladi.

5) Agar $u(z) \in m-wsh(D)$ bo'lsa,u holda $u_j(z) = u * K_{1/j}(z-w)$ ham

$m-wsh(D)$ bo‘ladi,hamda $j\to\infty$ $u_j(z)\downarrow u(z)$;

Bunda $K_{1/j}(z)=j^nK(jz)$, $K-$ standart cheksiz silliq yadro $suppK\subset B(0,1)$,

$$\int_{R^n}K(z)dV=\int_{B(0,1)}K(z)dV=1.$$

Quyidagi teorema bizga $m-wsh$ funksiyalarni geometrik xarakteristikasini tushinishga imkon beradi.

3.2.1-teorema. *$D\subset\mathbb{C}^n$ sohada aniqlangan va yuqoridan yarimuzluksiz u funksiya $m-wsh$ bo‘lishi uchun, ixtiyoriy $(n-m+1)$-o‘lchamli $\Pi\subset\mathbb{C}^n$ kompleks tekislikda*

$$u\big|_{\Pi}\in sh(\Pi\cap D) \qquad (3.2.2)$$

bo‘lishi zarur va yetarli.

Isbot.(Zaruriyligi) $u\in m-wsh(D)$ bo‘lsin. 5) xossaga ko‘ra u funksiya cheksiz silliq u_j funksiyalar ketma- ketligi bilan yaqinlashishi mumkin. $u_j\downarrow u,\ \ u_j\in m-sh(D)\cap\mathbb{C}^\infty(D)$. $\Pi\subset\mathbb{C}^n$ kompleks tekislikni fiksirlaymiz, bunda $\dim_c\Pi=n-m+1$ $u(z)\in m-wsh(D)$va Π da ortonormal $\xi_1,\dots,\xi_{n-m+1}$ bazis olamiz. U holda $\left(dd^c|z|^2\right)^{n-m}\Big|_{\Pi}=\left(dd^c|\xi|^2\right)^{n-m}$ bo‘lib,

$$dd^cu_j\wedge\left(dd^c|z|^2\right)^{n-m}\Big|_{\Pi}=dd^cu_j\Big|_{\Pi}\wedge\left(dd^c|\xi|^2\right)^{n-m}$$

bo‘ladi. Shuning uchun $dd^cu_j\wedge\left(dd^c|z|^2\right)^{n-m}$ forma $(n-m+1,n-m+1)$ bidarajali musbat differensial formadan iborat bo‘lishidan $dd^cu_j\wedge\left(dd^c|z|^2\right)^{n-m}\Big|_{\Pi}\ge 0$ bo‘lib, bundan $dd^cu_j\Big|_{\Pi}\wedge\left(dd^c|\xi|^2\right)^{n-m}\ge 0$ kelib chiqadi. Bu esa $u_j\in sh(\Pi\cap D)$ ni bildiradi. Shuningdek $j\to\infty$ da $u_j\big|_{\Pi}\downarrow u\big|_{\Pi}$, u holda $u\big|_{\Pi}\in sh(D)$ bo‘ladi.

(*Yetarliligi.*) Dastlab yuqoridan yarim uzluksiz $u(z)$ funksiyalarni

(3.2.2) shartni qanoatlantiruvchi xossalarini ko‘rib chiqamiz. (bu xossalar yordamida teoremani yetarliligini isbotlaymiz).

1) $\alpha_1,...,\alpha_k \geq 0$ musbat koeffisentli $\alpha_1 u_1 + ... + \alpha_k u_k$ cheklita yig’indi (3.2.2) shartni qanoatlantiradi agar faqat $u_1,...,u_k$ lar ham bu shartni qanoatlantirsa.

2) (3.2.2) shartni qanoatlantiruvchi kamayuvchi yoki tekis yaqinlashuvchi $\{u_j\}$ funksiyalar ketma-ketligi (4) tipdagi funksiyaga yaqinlashadi.

3) (3.2.2) shartni qanoatlantiruvchi u funksiya $u \equiv -\infty$ yoki $u \in L'_{loc}(D)$

Haqiqatdan, u-yuqoridan yarim uzluksiz va u yuqoridan lokal chegaralangan. Bu bo‘yicha biz umumiylikni buzmasdan D da $u < 0$ deb qabul qilish mumkin. u funksiya $z^0 = 0$ nuqtada $u(0) \neq -\infty$ bo‘lsin. U holda ixtiyoriy fiksirlangan $\Pi \ni 0$, $\dim \Pi = n - m + 1$ tekislikda $u\big|_\Pi$ funksiya $D \cap \Pi$ da subgarmonik bo‘ladi. Binobarin,

$$u(0) \leq \frac{1}{V_{n-m+1} r^{n-m+1}} {}_{B(0,r)\cap\Pi} u\big|_\Pi dV\,|_\Pi \tag{3.2.3}$$

bu yerda $B(0,r) = \{\|z\| < r\}$-shar, $dV\,|_\Pi$ – da hajm elementi va V_{n-m+1} birlik shar hajmi. Bunda, ixtiyoriy $\Pi \ni 0, \dim \Pi = n - m + 1$. $u\big|_\Pi$ funksiya $\Pi \cap B(0,r)$ da tekis chegaralangan bo‘lib. Fubini teoremasiga ko‘ra (3.2.3) tengsizlikdan

$$-\infty < {}_{B(0,r)} u(z)\,dz < 0$$

ekanligi kelib chiqadi.

Bu esa u funksiya 0 nuqta atrofida lokal integrallanuvchiligini bildiradi va bundan ixtiyoriy $B(z^0,r), z^0 \in D, r > 0$ shar bo‘yicha u funksiya integrallanuvchi.

Izoh: *Bu yerda Fubini teoremasini biz 0 nuqtadan o‘tuvchi kompleks tekisliklarni yig’ish bo‘yicha olamiz.*

Ma’lumki bu Grassman ko‘pxilligi. $M_{n,n-m+1}$ $u \in L'_{loc}(D)$ ligini isbotlash uchun Fubini teoremasini qo‘llash mumkin. ($M_{n,n-m+1}$-ko‘pxillik bo‘yicha emas) biror $L \ni 0, \dim L = n-m$ o‘lchamli tekisliklarda o‘tuvchi barcha Π kompleks tekisliklar bo‘yicha qo‘llash mumkin.Bunday P tekisliklar proektiv P^{m-1} fazoni ifodalaydi va $u(z) \in L'_{loc}(D)$ ni isbotlash uchun biz quyidagi Fubini formulasidan foydalanishimiz mumkin.

$$\int\limits_{B(0,r)} u(z)dv = \int\limits_{\Pi \in \mathrm{P}^{m-1}} \omega^{m-1} \int\limits_{B(0,r)\cap\Pi} u|_{\Pi}(z)dV|_{\Pi} \qquad (3.2.4)$$

bu yerda $\omega -$Fubini-Shtudi proektiv fazosida standart forma.

4) Agar u (4) shartni qanoatlantirsa u holda $u_j(z) = u * k_{\frac{1}{j}}(z-w)$ o‘rama ham bu shartni qanoatlantiradi. $j \to \infty$ da $u_j(z) \downarrow u(z)$. Bu munosabatdan

$$u_j = u * k_{\frac{1}{j}}(z-w) = j^n \int\limits_{R^n} u(w)k(j(z-w))dw = \int\limits_{R^n} u(z+\frac{w}{j})k(w)dw$$

(3.2.5)

kelib chiqadi.

Birinchi integral cheksiz silliq funksiyani ifodalaydi ikkinchi integral esa $m-wsh$ funksiyani ifodalaydi $u_j(z) \downarrow u(z)$ yaqinlashishi (3.2.4) formuladan kelib chiqadi.

4) xossaga ko‘ra $u_j(z) \downarrow u(z)$ mumkin. Shuningdek $u_j \in \mathbb{C}^{\infty}$ va $u_j|_{\Pi}, \dim_C \Pi = n-m+1-$ o‘lchamli ixtiyoriy Π kompleks tekislikda subgarmonik, u holda kesimda subgarmonik funksiya

$$dd^c u_j \wedge \left(dd^c |z|^2\right)^{n-m}|_{\Pi} \geq 0$$

bo‘ladi.

Bu esa $dd^c u_j \wedge \left(dd^c |z|^2\right)^{n-m}|_{\Pi} \geq 0$ ligini bildiradi. $u_j(z) \downarrow u(z)$ yaqinlashtirishdan Potok ma’nosida $dd^c u \wedge \left(dd^c |z|^2\right)^{n-m} \geq 0$ va $u \in m-wsh(D)$ kelib chiqadi.

3.3. KUCHSIZ $m-$ SUBGARMONIK FUNKSIYALARNING $m-$ SUBGARMONIKLIK XUSUSIYATLARI

Ushbu funksiyani qaraymiz:

$$u=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+(z_3+\overline{z}_3)^2]=\ln\left\|z+\overline{z}\right\|^2=\ln\left(x_1^2+x_2^2+x_3^2\right)+\ln 2$$

bunda $z_j=x_j+iy_j, j=1,2,3.$

Ko'rinib turibdiki, bu funksiya D sohada $3-wsh$ funksiya emas, ya'ni sohada plyurisubgarmonik funksiya emas. Bu funksiya subgarmonik funksiya, ya'ni $\Delta u\geq 0$.Endi biz bu funksiyani $\mathbb{C}^3$da 2-wsh funksiyadan iborat ekanligini ko'rsatamiz:

Dastlab quyidagi hisoblashlarni bajaramiz:

$$\frac{\partial u}{\partial z_1}=\frac{1}{\left|z+\overline{z}\right|^2}2\left(z_1+\overline{z_1}\right)=2\frac{z_1+\overline{z_1}}{\left|z+\overline{z}\right|^2};$$

$$\frac{\partial u}{\partial z_2}=\frac{1}{\left|z+\overline{z}\right|^2}\cdot 2(z_2+\overline{z}_2)=2\frac{z_2+\overline{z}_2}{\left|z+\overline{z}\right|^2};$$

$$\frac{\partial u}{\partial z_3}=\frac{1}{\left|z+\overline{z}\right|^2}\cdot 2(z_3+\overline{z}_3)=2\frac{z_3+\overline{z_3}}{\left|z+\overline{z}\right|^2};$$

....

$$\frac{\partial u}{\partial z_n}=\frac{1}{\left|z+\overline{z}\right|^2}\cdot 2(z_n+\overline{z}_n)=2\frac{z_n+\overline{z_n}}{\left|z+\overline{z}\right|^2};$$

1) $$u_{1\bar{1}}=\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}=\frac{2\left|z+\overline{z}\right|^2}{\left|z+\overline{z}\right|^4}-\frac{2(z_1+\overline{z}_1)\cdot 2(z_1+\overline{z}_1)}{\left|z+\overline{z}\right|^4}=\frac{2}{\left|z+\overline{z}\right|^2}-\frac{4(z_1+\overline{z}_1)^2}{\left|z+\overline{z}\right|^4};$$

2) $$u_{1\bar{2}}=\frac{\partial^2 u}{\partial z_1\partial\overline{z}_2}=-\frac{2(z_1+\overline{z}_1)\cdot 2(z_2+\overline{z}_2)}{\left|z+\overline{z}\right|^4}=-4\frac{(z_1+\overline{z}_1)(z_2+\overline{z}_2)}{\left|z+\overline{z}\right|^4};$$

3) $$u_{1\bar{3}}=\frac{\partial^2 u}{\partial z_1\partial\overline{z}_3}=-4\frac{(z_1+\overline{z}_1)(z_3+\overline{z}_3)}{\left|z+\overline{z}\right|^4};$$

4) $u_{2\bar{1}} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} = -4\dfrac{(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)}{\left|z + \bar{z}\right|^4}$;

5) $u_{2\bar{2}} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} = \dfrac{2}{\left|z + \bar{z}\right|^2} - \dfrac{4(z_2 + \bar{z}_2)^2}{\left|z + \bar{z}\right|^4}$;

6) $u_{2\bar{3}} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} = -4\dfrac{(z_2 + \bar{z}_2)(z_3 + \bar{z}_3)}{\left|z + \bar{z}\right|^4}$;

7) $u_{3\bar{1}} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} = -4\dfrac{(z_1 + \bar{z}_1)(z_3 + \bar{z}_3)}{\left|z + \bar{z}\right|^4}$;

8) $u_{3\bar{2}} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} = -4\dfrac{(z_2 + \bar{z}_2)(z_3 + \bar{z}_3)}{\left|z + \bar{z}\right|^4}$ $u_{3\bar{3}} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} = \dfrac{2}{\left|z + \bar{z}\right|^2} - \dfrac{4(z_3 + \bar{z}_3)^2}{\left|z + \bar{z}\right|^4}$;

$$\Rightarrow\ u_{s\bar{t}} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_t} = -4\frac{(z_s + \bar{z}_s)(z_t + \bar{z}_t)}{\left|z + \bar{z}\right|^4},\ \ s, t = \overline{1, n};\ s \neq t$$

$$u_{s\bar{s}} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_s} = \frac{2}{\left|z + \bar{z}\right|^2} - \frac{4(z_s + \bar{z}_s)^2}{\left|z + \bar{z}\right|^4},\ \ s = \overline{1, n}$$

$$w = \left(dd^c u\right) \wedge dd^c \left|z\right|^2 = \frac{i}{2}\left[\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} dz_1 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} dz_1 \wedge d\bar{z}_2 + \right.$$

$$\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_3} dz_1 \wedge d\bar{z}_3 + \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} dz_2 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} dz_2 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} dz_2 \wedge d\bar{z}_3 +$$

$$\left.\frac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} dz_3 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} dz_3 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} dz_3 \wedge d\bar{z}_3\right] \wedge$$

$$\wedge \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) =$$

$$= -\frac{1}{4}\left[\left(\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2}\right) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 + \right.$$

$$+ \left(\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3}\right) dz_1 \wedge d\bar{z}_1 \wedge dz_3 \wedge d\bar{z}_3 +$$

$$+ \left(\frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3}\right) dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 +$$

$$+\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_3} dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_3 + \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_2} dz_1 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_2 +$$

$$+\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_3} dz_1 \wedge d\overline{z}_3 \wedge dz_2 \wedge d\overline{z}_2 + \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_1} dz_3 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 +$$

$$\left. +\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_2} dz_1 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 + \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_1} dz_2 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3 \right];$$

$(1,1)$ bidarajali ixtiyoriy $\nu = \frac{i}{2} d\ell \wedge d\overline{\ell}$ forma uchun (bunda $d\ell = a_1 dz_1 + a_2 dz_2 + a_3 dz_3$)

$$v = \frac{i}{2} d\ell \wedge d\overline{\ell} = \frac{i}{2}\Big(\left|a_1\right|^2 dz_1 \wedge d\overline{z}_1 + a_1\overline{a}_2 dz_1 \wedge d\overline{z}_2 + a_1\overline{a}_3 dz_1 \wedge d\overline{z}_3 + a_2\overline{a}_1 dz_2 \wedge d\overline{z}_1 +$$

$$+\left|a_2\right|^2 dz_2 \wedge d\overline{z}_2 + a_2\overline{a}_3 dz_2 \wedge d\overline{z}_3 + a_3\overline{a}_1 dz_3 \wedge d\overline{z}_1 + a_3\overline{a}_2 dz_3 \wedge d\overline{z}_2 + \left|a_3\right|^2 dz_3 \wedge d\overline{z}_3 \Big)$$

dan

$$\nu \wedge \omega = -\frac{i}{8}\left[\left|a_1\right|^2 \left(\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} + \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} \right) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 + \right.$$

$$+a_1\overline{a}_2 \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_1} dz_1 \wedge d\overline{z}_2 \wedge dz_2 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3 +$$

$$a_1\overline{a}_3 \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_1} dz_1 \wedge d\overline{z}_3 \wedge dz_3 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 +$$

$$a_2\overline{a}_1 \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_2} dz_2 \wedge d\overline{z}_1 \wedge dz_1 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 +$$

$$\left|a_1\right|^2 \left[\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} \right] dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 +$$

$$a_2\overline{a}_3 \frac{\partial^2 u}{\partial z_3 \partial \overline{z}_2} dz_2 \wedge d\overline{z}_3 \wedge dz_1 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3 +$$

$$+a_3\overline{a}_1 \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_3} dz_3 \wedge d\overline{z}_1 \wedge dz_1 \wedge d\overline{z}_2 \wedge dz_2 \wedge d\overline{z}_3 +$$

$$+a_3\overline{a}_2 \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_3} dz_3 \wedge d\overline{z}_2 \wedge dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_3 +$$

$$\left|a_3\right|^2 \left[\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} \right] dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge dz_3 \wedge d\overline{z}_3 =$$

$$=\left[\left|a_1\right|^2\left(\frac{\partial^2 u}{\partial z_2\partial \overline{z}_2}+\frac{\partial^2 u}{\partial z_3\partial \overline{z}_3}\right)+\left|a_2\right|^2\left(\frac{\partial^2 u}{\partial z_1\partial \overline{z}_1}+\frac{\partial^2 u}{\partial z_3\partial \overline{z}_3}\right)+\left|a_3\right|^2\left(\frac{\partial^2 u}{\partial z_1\partial \overline{z}_1}+\frac{\partial^2 u}{\partial z_2\partial \overline{z}_2}\right)\right.$$

$$-a_1\overline{a}_2\frac{\partial^2 u}{\partial z_2\partial \overline{z}_1}-a_1\overline{a}_3\frac{\partial^2 u}{\partial z_3\partial \overline{z}_1}-a_2\overline{a}_1\frac{\partial^2 u}{\partial z_1\partial \overline{z}_2}-a_2\overline{a}_3\frac{\partial^2 u}{\partial z_3\partial \overline{z}_2}-a_3\overline{a}_1\frac{\partial^2 u}{\partial z_1\partial \overline{z}_3}$$

$$\left.-a_3\overline{a}_2\frac{\partial^2 u}{\partial z_2\partial \overline{z}_3}\right]\cdot\frac{i}{2}dz_1\wedge d\overline{z}_1\wedge\frac{i}{2}dz_2\wedge d\overline{z}_2\wedge\frac{i}{2}dz_3\wedge d\overline{z}_3=$$

$$=\alpha(z)\frac{i}{2}dz_1\wedge d\overline{z}_1\wedge\frac{i}{2}dz_2\wedge d\overline{z}_2\wedge\frac{i}{2}dz_3\wedge d\overline{z}_3,$$

Ifodani olamiz, bunda

$$\alpha(z)=\left|a_1\right|^2\left(\frac{2\left|z+\overline{z}\right|^2-4(z_2+\overline{z}_2)^2}{\left|z+\overline{z}\right|^4}+\frac{2\left|z+\overline{z}\right|^2-4(z_3+\overline{z}_3)^2}{\left|z+\overline{z}\right|^4}\right)+$$

$$+\left|a_2\right|^2\left(\frac{2\left|z+\overline{z}\right|^2-4(z_1+\overline{z}_1)^2}{\left|z+\overline{z}\right|^4}+\frac{2\left|z+\overline{z}\right|^2-4(z_3+\overline{z}_3)^2}{\left|z+\overline{z}\right|^4}\right)+$$

$$+\left|a_3\right|^2\left(\frac{2\left|z+\overline{z}\right|^2-4(z_1+\overline{z}_1)^2}{\left|z+\overline{z}\right|^4}+\frac{2\left|z+\overline{z}\right|^2-4(z_2+\overline{z}_2)^2}{\left|z+\overline{z}\right|^4}\right)+$$

$$+a_1\overline{a}_2\frac{4(z_1+\overline{z}_1)(z_2+\overline{z}_2)}{\left|z+\overline{z}\right|^4}+a_1\overline{a}_3\frac{4(z_1+\overline{z}_1)(z_3+\overline{z}_3)}{\left|z+\overline{z}\right|^4}+$$

$$+a_2\overline{a}_1\frac{4(z_1+\overline{z}_1)(z_2+\overline{z}_2)}{\left|z+\overline{z}\right|^4}+a_2\overline{a}_3\frac{4(z_2+\overline{z}_2)(z_3+\overline{z}_3)}{\left|z+\overline{z}\right|^4}+$$

$$+a_3\overline{a}_1\frac{4(z_1+\overline{z}_1)(z_3+\overline{z}_3)}{\left|z+\overline{z}\right|^4}+a_3\overline{a}_2\frac{4(z_2+\overline{z}_2)(z_3+\overline{z}_3)}{\left|z+\overline{z}\right|^4}=$$

$$=\frac{4}{\left|z+\overline{z}\right|^4}\Big(\left|a_1\right|^2(z_1+\overline{z}_1)^2+\left|a_2\right|^2(z_2+\overline{z}_2)^2+\left|a_3\right|^2(z_3+\overline{z}_3)^2+a_1\overline{a}_2(z_1+\overline{z}_1)(z_2+\overline{z}_2)+$$

$$+a_1\overline{a}_3(z_1+\overline{z}_1)(z_3+\overline{z}_3)+a_2\overline{a}_1(z_2+\overline{z}_2)(z_1+\overline{z}_1)+a_2\overline{a}_3(z_2+\overline{z}_2)(z_3+\overline{z}_3)+$$

$$+a_3\overline{a}_1(z_3+\overline{z}_3)(z_1+\overline{z}_1)+a_3\overline{a}_2(z_2+\overline{z}_2)(z_3+\overline{z}_3)\Big)=$$

$$=\frac{4}{\left|z+\overline{z}\right|^4}\left|a_1(z_1+\overline{z}_1)+a_2(z_2+\overline{z}_2)+a_3(z_3+\overline{z}_3)\right|^2\geq 0.$$

ℓ-ixtiyoriy chiziqli funksiya ekan, u holda $\mathbb{C}^3\setminus\mathbb{R}^3(x)$da

$dd^c u \wedge dd^c |z|^2 \geq 0$, ya'ni u funksiya $\mathbb{R}^3(x)$ nuqtalardan tashqarida 2-wsh funksiyadan iborat bo'ladi. $\mathbb{R}^3(x)$ nuqtalarda $u\Big|_{\mathbb{R}^3(x)} = -\infty$ bo'ladi. Demak, bu funksiya $\mathbb{R}^3(x)$ nuqtalarda ham 2-wsh funksiyadan iborat ekan.

Endi

$$u = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + (z_3 + \overline{z}_3)^2] = \ln\left\|z + \overline{z}\right\|^2 = \ln\left(x_1^2 + x_2^2 + x_3^2\right) + \ln 2$$

funksiyani $2 - sh$ likka tekshirib ko'ramiz:

Yuqoridagi ta'rifga ko'ra bu funksiya $2 - sh$ bo'lishi uchun

$$\begin{cases} \left(dd^c u\right) \wedge \left(dd^c |z|^2\right)^2 \geq 0 \\ \left(dd^c u\right)^2 \wedge \left(dd^c |z|^2\right) \geq 0 \end{cases} \tag{3.3.1}$$

tengsizliklar bir vaqtda o'rinli bo'lishi kerak edi.

Endi (3.3.1) tengsizliklar sistemasidagi 1-tengsizlikni tekshirib ko'ramiz:

$$\left(dd^c u\right) \wedge \left(dd^c |z|^2\right)^2 = \left(\frac{i}{2}\right)^3 \left(u_{1\bar{1}} + u_{2\bar{2}} + u_{3\bar{3}}\right) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3} =$$

$$\left(\frac{i}{2}\right)^3 \frac{2}{\left|z + \overline{z}\right|^2} dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3} > 0$$

2-tengsizlikni ham tekshirib ko'ramiz:

$$\left(dd^c u\right) \wedge \left(dd^c |z|^2\right)^2 = 2\left(\frac{i}{2}\right)^3 (u_{1\bar{1}} \cdot u_{2\bar{2}} + u_{1\bar{1}} \cdot u_{3\bar{3}} + u_{2\bar{2}} \cdot u_{3\bar{3}} -$$

$$-u_{1\bar{2}} \cdot u_{2\bar{1}} - u_{1\bar{3}} \cdot u_{3\bar{1}} - u_{2\bar{3}} \cdot u_{3\bar{2}}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3} =$$

$$=\left(\frac{4}{\left|z+\overline{z}\right|^4}-\frac{8\left(z_2+\overline{z_2}\right)^2}{\left|z+\overline{z}\right|^6}-\frac{8\left(z_1+\overline{z_1}\right)^2}{\left|z+\overline{z}\right|^6}+\frac{16\left(z_1+\overline{z_1}\right)^2\left(z_2+\overline{z_2}\right)^2}{\left|z+\overline{z}\right|^8}+\right.$$

$$+\frac{4}{\left|z+\overline{z}\right|^4}-\frac{8\left(z_2+\overline{z_2}\right)^2}{\left|z+\overline{z}\right|^6}-\frac{8\left(z_3+\overline{z_3}\right)^2}{\left|z+\overline{z}\right|^6}+\frac{16\left(z_3+\overline{z_3}\right)^2\left(z_2+\overline{z_2}\right)^2}{\left|z+\overline{z}\right|^8}+$$

$$+\frac{4}{\left|z+\overline{z}\right|^4}-\frac{8\left(z_3+\overline{z_3}\right)^2}{\left|z+\overline{z}\right|^6}-\frac{8\left(z_1+\overline{z_1}\right)^2}{\left|z+\overline{z}\right|^6}+\frac{16\left(z_1+\overline{z_1}\right)^2\left(z_3+\overline{z_3}\right)^2}{\left|z+\overline{z}\right|^8}-$$

$$\left.-\frac{16\left(z_1+\overline{z_1}\right)^2\left(z_2+\overline{z_2}\right)^2}{\left|z+\overline{z}\right|^8}-\frac{16\left(z_1+\overline{z_1}\right)^2\left(z_3+\overline{z_3}\right)^2}{\left|z+\overline{z}\right|^8}-\frac{16\left(z_2+\overline{z_2}\right)^2\left(z_3+\overline{z_3}\right)^2}{\left|z+\overline{z}\right|^8}\right)dV=$$

$$=\frac{12}{\left|z+\overline{z}\right|^4}-\frac{16\left[\left(z_1+\overline{z_1}\right)^2+\left(z_2+\overline{z_2}\right)^2+\left(z_3+\overline{z_3}\right)^2\right]}{\left|z+\overline{z}\right|^8}dV=\frac{12-16}{\left|z+\overline{z}\right|^4}dV=-\frac{4}{\left|z+\overline{z}\right|^4}dV<0$$

bunda $dV=2\left(\frac{i}{2}\right)^3 dz_1\wedge d\overline{z_1}\wedge dz_2\wedge d\overline{z_2}\wedge dz_3\wedge d\overline{z_3}$. Demak,

$$\begin{cases}\left(dd^c u\right)\wedge\left(dd^c\left|z\right|^2\right)^2>0\\ \left(dd^c u\right)\wedge\left(dd^c\left|z\right|^2\right)^2<0\end{cases}$$

Shunday qilib, (1) funksiya $2-sh$ funksiya emas ekan.

Endi $u=\ln\left(x_1^2+x_2^2+...+x_n^2\right)$ funksiyani $m-$subgarmoniklikka tekshirishda $dd^c u=\frac{i}{2}\left[\lambda_1 dz_1\wedge d\overline{z_1}+...+\lambda_n dz_n\wedge d\overline{z_n}\right]$ diagonal formadan foydalanamiz. $\left(dd^c u\right)^m\wedge\beta^{n-m}=m!\left(n-m\right)!H_m\left(u\right)\beta^n$ bo‘lgani uchun

$H_m(u) = \sum_{1\le j_1<\dots<j_m\le n} \lambda_{j_1}, \lambda_{j_2}, \dots \lambda_{j_m}$ – gessian vektori musbat aniqlangan

bo‘lsa, u holda $\left(dd^c u\right)^m \wedge \left(dd^c |z|^2\right)^{n-m} \ge 0$ bo‘ladi.

Dastlab (2) funksiyani ermit $\left(u_{j\bar{k}}\right)$ matritsasining $\lambda_1, \lambda_2, \dots, \lambda_n$ xos

qiymatlarni topamiz:

$$\begin{vmatrix} u_{1\bar{1}} - \lambda & \dots & u_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ u_{n\bar{1}} & \cdots & u_{n\bar{n}} - \lambda \end{vmatrix} = \begin{vmatrix} \frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4} - \lambda & \dots & -4\frac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} \\ \vdots & \ddots & \vdots \\ -4\frac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} & \cdots & \frac{2}{|z+\bar{z}|^2} - \frac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4} - \lambda \end{vmatrix} =$$

$$\left(-4\frac{(z_1+\bar{z}_1)}{|z+\bar{z}|^4}\right) \cdots \left(-4\frac{(z_n+\bar{z}_n)}{|z+\bar{z}|^4}\right)$$

$$\begin{vmatrix} \frac{|z+\bar{z}|^4}{-4(z_1+\bar{z}_1)}\left(\frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4} - \lambda\right) & \dots & (z_n+\bar{z}_n) \\ \vdots & \ddots & \vdots \\ (z_1+\bar{z}_1) & \cdots & \frac{|z+\bar{z}|^4}{-4(z_n+\bar{z}_n)}\left(\frac{2}{|z+\bar{z}|^2} - \frac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4} - \lambda\right) \end{vmatrix} =$$

Yuqoridagi determinantni hisoblash qulay bo‘lishi uchun quyidagicha

almashtirish bajaramiz:

$$x_k = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_k+\bar{z}_k)^2}{|z+\bar{z}|^4} - \lambda, \quad k = \overline{1,n}$$

$$a_k = z_k + \bar{z}_k, \; k = \overline{1,n}$$

Shunday qilib, biz quyidagi [4 ga qarang] determinantni hisoblashga keldik:

$$\begin{vmatrix} x_1 & a_2 \dots & a_n \\ a_1 & \ddots & \vdots \\ \vdots & & \\ a_1 & a_2 \cdots & x_n \end{vmatrix} = (x_1 - a_1)(x_2 - a_2)\dots(x_n - a_n)\left(1 + \frac{a_1}{x_1 - a_1} + \frac{a_2}{x_2 - a_2} + \dots + \frac{a_n}{x_n - a_n}\right);$$

$$x_1 - a_1 = \frac{|z+\overline{z}|^4}{-4(z_1+\overline{z}_1)}\left(\frac{2}{|z+\overline{z}|^2} - \frac{4(z_1+\overline{z}_1)^2}{|z+\overline{z}|^4} - \lambda\right) - (z_1+\overline{z}_1) = \frac{2|z+\overline{z}|^2 - |z+\overline{z}|^4\lambda}{-4(z_1+\overline{z}_1)}$$

$$\frac{a_1}{x_1 - a_1} = \frac{-4(z_1+\overline{z}_1)^2}{2|z+\overline{z}|^2 - |z+\overline{z}|^4\lambda}, \dots, \quad \frac{a_n}{x_n - a_n} = \frac{-4(z_n+\overline{z}_n)^2}{2|z+\overline{z}|^2 - |z+\overline{z}|^4\lambda}$$

$$1 + \frac{a_1}{x_1 - a_1} + \frac{a_2}{x_2 - a_2} + \dots + \frac{a_n}{x_n - a_n} = 1 - \frac{4}{2 - |z+\overline{z}|^2\lambda} = \frac{-2 - |z+\overline{z}|^2\lambda}{2 - |z+\overline{z}|^2\lambda} \quad \Rightarrow$$

$$\begin{vmatrix} u_{1\bar{1}} - \lambda & \dots & u_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ u_{n\bar{1}} & \cdots & u_{n\bar{n}} - \lambda \end{vmatrix} = \frac{\left(2 - |z+\overline{z}|^2\lambda\right)^{n-1}\left(-2 - |z+\overline{z}|^2\lambda\right)}{|z+\overline{z}|^{2n}} =$$

$$= -\left(\frac{2}{|z+\overline{z}|^2} - \lambda\right)^{n-1}\left(\frac{2}{|z+\overline{z}|^2} - \lambda\right) = 0 \Rightarrow \lambda_{1,2,3,\dots,n-1} = \frac{2}{|z+\overline{z}|^2}, \quad \lambda_n = -\frac{2}{|z+\overline{z}|^2}$$

$n = 2$ da (2) funksiyani ermit $\left(u_{j\bar{k}}\right)$ matritsasining λ_1, λ_2 xos qiymatlari

$\lambda_{1,2} = \pm\dfrac{2}{|z+\overline{z}|^2}$ ga teng va

$$\begin{cases} H_1\left(\lambda\left(u\right)\right) = \lambda_1 + \lambda_2 = \dfrac{2}{\left|z+\overline{z}\right|^2} - \dfrac{2}{\left|z+\overline{z}\right|^2} = 0 \\ H_2\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2 = -\dfrac{4}{\left|z+\overline{z}\right|^4} < 0 \end{cases}$$

bo'ladi.

Demak, $u = \ln\left(x_1^2 + x_2^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2] - 2\ln 2$ funksiya $1-sh$ bo'la oladi, lekin $2-sh$ bo'la olmas ekan.

v) $n = 3$ da: $\lambda_{1,2} = \dfrac{2}{\left|z+\overline{z}\right|^2}$, $\lambda_3 = -\dfrac{2}{\left|z+\overline{z}\right|^2}$

$$\begin{cases} H_1\left(\lambda\left(u\right)\right) = \lambda_1 + \lambda_2 + \lambda_3 = \dfrac{2}{\left|z+\overline{z}\right|^2} > 0 \\ H_2\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -\dfrac{4}{\left|z+\overline{z}\right|^4} < 0 \end{cases}$$

Demak, $u = \ln\left(x_1^2 + x_2^2 + x_3{}^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + (z_3 + \overline{z}_3)^2] - 2\ln 2$ funksiya ham $1-sh$ bo'la oladi, lekin $2-sh$ bo'la olmas ekan.

g) $n = 4$ da esa (2) funksiya $2-sh$ bo'la oladi, ammo $3-sh$ bo'la olmaydi.

$n = 5$ da ham (2) funksiya $2-sh$ bo'ladi :

$$\lambda_{1,2,3,4} = \frac{2}{\left|z+\overline{z}\right|^2},\ \lambda_5 = -\frac{2}{\left|z+\overline{z}\right|^2}$$

$$H_1\left(\lambda\left(u\right)\right) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = \frac{6}{\left|z+\overline{z}\right|^2} > 0$$

$$H_2\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_4 +$$

$$+\lambda_3\lambda_5 + \lambda_4\lambda_5 = \frac{12}{\left|z+\overline{z}\right|^4} - \frac{8}{\left|z+\overline{z}\right|^4} > 0$$

$$H_3\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_4\lambda_5 + +\lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 +$$

$$+\lambda_2\lambda_4\lambda_5 + \lambda_3\lambda_4\lambda_5 = \frac{32}{\left|z+\overline{z}\right|^6} - \frac{48}{\left|z+\overline{z}\right|^6} < 0$$

e) $n = 6$ da esa (2) funksiya $3-sh$ bo‘ladi:

$$\lambda_{1,2,3,4,5} = \frac{2}{\left|z+\overline{z}\right|^2},\ \lambda_6 = -\frac{2}{\left|z+\overline{z}\right|^2}$$

$$H_1\left(\lambda\left(u\right)\right) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \frac{8}{\left|z+\overline{z}\right|^2} > 0$$

$$H_2\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_1\lambda_6 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_2\lambda_6 +$$

$$\lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6 = \frac{20}{\left|z+\overline{z}\right|^4} - \frac{10}{\left|z+\overline{z}\right|^4} > 0$$

$$H_3\left(\lambda\left(u\right)\right) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_2\lambda_6 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_3\lambda_6 + \lambda_1\lambda_4\lambda_5 +$$

$$+\lambda_1\lambda_4\lambda_6 + \lambda_1\lambda_5\lambda_6 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_2\lambda_3\lambda_6 + \lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_4\lambda_6 + \lambda_2\lambda_5\lambda_6 + \lambda_3\lambda_4\lambda_5 +$$

$$\lambda_3\lambda_4\lambda_6 + \lambda_3\lambda_5\lambda_6 + \lambda_4\lambda_5\lambda_6 = \frac{20}{\left|z+\overline{z}\right|^6} - \frac{20}{\left|z+\overline{z}\right|^6} = 0 \Rightarrow \left(dd^c u\right)^3 \wedge \left(dd^c\left|z\right|^2\right)^3 = 0$$

$H_4\left(\lambda\left(u\right)\right) < 0$ ekanligi bizga ma'lum. Demak, $n = 2$ va $n = 3$ bo'lgan xususiy holda (2) funksiya subgarmonik funksiya, $n = 4$ *va* $n = 5$ da $2 - sh$ funksiya $n = 6$ *va* $n = 7$ da $3 - sh$ funksiya bo'ladi.

Haqiqatdan, $n = 2k$ *va* $n = 2k + 1$ da

$H_1(\lambda), H_2(\lambda), ..., H_k(\lambda) \geq 0, \quad H_{k+1}(\lambda) < 0$ bo'ladi. Buni $n = 2k + 1$ uchun tekshirib ko'ramiz: Bizga ma'lumki,

$$\lambda_{1,2,...,2k-1,2k} = \frac{2}{\left|z + \bar{z}\right|^2}, \quad \lambda_{2k+1} = -\frac{2}{\left|z + \bar{z}\right|^2} \; .$$

$H_m\left(\lambda\right)$ yig'indida C_{2k+1}^m ta had ishtirok etadi va shulardan C_{2k}^m ta had musbat bo'ladi, chunki bu hadlarda λ_{2k+1} ishtirok etmaydi. λ_{2k+1} ishtirok etganlari

$C_{2k+1}^m - C_{2k}^m = C_{2k}^{m-1}$ ga teng, ya'ni $H_m\left(\lambda\right)$ yig'indida $\dfrac{\left(2k\right)!}{\left(m-1\right)!\left(2k-m+1\right)!}$ ta had manfiy, $\dfrac{\left(2k\right)!}{m!\left(2k-m\right)!}$ ta had musbat va barcha hadlar bir xil qiymatga ega. $m \leq k + \dfrac{1}{2}$ bo'lganda

$$\frac{\left(2k\right)!}{\left(m-1\right)!\left(2k-m+1\right)!} \leq \frac{\left(2k\right)!}{m!\left(2k-m\right)!}$$

tengsizlik o'rinli ekanini ko'rish qiyin emas. Demak, $m = \overline{1,k}$ da $H_m\left(\lambda\right) \geq 0$ va $m = \overline{\left(k+1\right), n}$ da esa $H_m\left(\lambda\right) < 0$ bo'lar ekan. $n = 2k$ uchun ham yuqoridagi kabi mulohazalarni yuritib

$H_1(\lambda), H_2(\lambda), ..., H_k(\lambda) \geq 0, \quad H_{k+1}(\lambda) < 0$ ekanini ko‘rish mumkin.

Shunday qilib, biz quyidagi xulosaga keldik:

Teorema.

$$u = \ln\left(x_1^2 + x_2^2 + ... + x_n^2\right) = \ln[(z_1 + \overline{z}_1)^2 + (z_2 + \overline{z}_2)^2 + ... + (z_n + \overline{z}_n)^2] - 2\ln 2$$

funksiya $1 \leq m \leq \left[\dfrac{n}{2}\right]$, $n \geq 2$ bo‘lganda $m-sh$ bo‘ladi.

III BOB bo‘yicha xulosa

Ushbu bobda Kuchsiz $m-sh$ funksiyalarning $m-sh$lik xususiyatlari misol yordamida o‘rganildi. Bu misolni kuchsiz $m-sh$likka tekshirishda Differensial formaning xossalaridan foydalanildi. $m-sh$likka tekshrishda esa chiziqli algebra usullaridan keng foydalanildi. Olingan natijalar va usullar funksiyalar nazariyasining keyingi rivojlanishida va kompleks analizning tadbiqlarida qo‘llanilishi mumkin.

XULOSA

Dissertatsiya kuchsiz $m-$subgarmonik funksiyalarning $m-$subgarmoniklik xususiyatlarini o'rganishga bag'ishlangan.

Dissertatsiyadan olingan asosiy natijalar yuzasidan quyidagi xulosaga kelindi:

Agar $u(z)$ *funksiya bir vaqtda garmonik va* $m-$*subgarmonik bo'lsa, u holda bu funksiya plyurigarmonik bo'lishi isbotlandi.*

$m-$subgarmonik funksiyalarni o'rganish jarayonida B.I. Abdullayev tomonidan

$u=\ln\left(x_1^2+x_2^2+x_3^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+(z_3+\overline{z}_3)^2]-2\ln 2$, (bunda $z_j=x_j+iy_j, j=1,2,3,$) funksiyaning $1-wsh$ va $2-wsh$ sinflariga tegishli ekanligi isbot qilingan edi. Shuningdek, bu funksiyaning $2-$subgarmoniklikka, ya'ni $2-sh$ sinfiga tegishliligi masalasi ko'tarilgan edi. Ushbu dissertatsiya ishida ushbu funksiyaning $\mathbb{C}^3$da $2-sh$ funksiya emasligi va $u=\ln\left(x_1^2+x_2^2+...+x_n^2\right)$ funksiyaning $1\le m\le\left[\frac{n}{2}\right], n\ge 2$ shartda $m-$subgarmonik bo'lishi isbotlandi.

Umuman, dissertatsiya ishidagi olingan asosiy natijalar yangi bo'lib, kompleks o'zgaruvchining funksiyalar nazariyasining keyingi rivojida qo'llanilishi mumkin.

ADABIYOTLAR RO‘YXATI

Adabiyotlar:

1. Садуллаев А. С. Кўп аргументли голоморф функциялар Урганч 2004-йил, 193 бет.

2. Садуллаев А., Теория плюрипотенциала. Применения. Palmarium. placecountry-region Germany, 2012.

3.И.М.Гелфанд, Лекции по линейной алгебре, М., НАУКА, 1971,с.

4. I.V.Proskuryakov, Sbornik zadach po lineynoy algebre, M., NAUKA, 1966, s.32-40.

5.А.Г.Курош, Олий алгебра курси, Тошкент, 1976.

6. Шабат Б.В., Введение в комплексный анализ. Т.1.2, М.: Наука, 1976.

7.Владимиров В.С. Методы теории функций многих комплексных переменных. М.: Наука 1964.

8. Ганнинг Р., Росси Х., Аналитические функции многих комплексных переменных , М.: Мир, 1969.

9. Привалов И.И., Субгармонические функции. М., Гостехиздат,1937.

10. Ронкин Л.И., Введение в теорию целых функций многих переменных. М., НАУКА,1971.

11.Уолш Дж. Л. «Интерполяция и аппроксимация рациональными функциями в комплексной области» , М.: ИЛ, 1961.

12. Хёрмандер Л., Введение в теорию функций нескольких комплексных

переменных, М.: Мир, 1968.

13. Хейман У.,Кеннеди П., Субгармонические функции. М., МИР, 1980.

Ilmiy jurnallardagi maqolalar:

14. А.Садуллаев, Б.Абдуллаев. Теория потенциалов на $m-$ субгармонических функциях. //Труди Московского Математического Института им. В.А.Стеклова, РАН, 2012, № 279, 166-192.

15.B.I.Abdullayev, Subharmonic Functions on Complex Hyperplanes of $\mathbb{C}^n$ //Journal of Siberian Federal University Mathematics and Physics, 2013 6(4) 409-416.

16.C1.Садуллаев А., Оператор $(dd^c u)^n$ и емкости конденсаторов, ДАН СССР, Т.251:1, (1980), 44-57.

17. Blocki Z., The domain of definition of the complex Monge-Ampère operator. Amer. J. Math., 2006, V.128:2, 519-530.

18. Blocki Z., Weak solutions to the complex Hessian equation. Ann.Inst.Fourier (Grenoble),2005,V.55:5,1735-1756.

19.DK. Dinev S., Kolodziej S., A priori estimates for the complex Hessian equations. arXive math.,1112.3063V1,1-18.

20.Ko. Kolodziej S., The complex Monge-Ampére equation and pluripotential theory, Mem. Amer. Math. Soc.,2005,V.178.64p.

21.T. Trudinger N.S., On the Dirichlet problem for Hessian equation, Acta Math., 1995, V.175, 151-164.

22.TW. Trudinger N.S., Wang X.J., Hessian measures II. Ann.of Math.,1999,V.150, 579-604.

23.Qalandarova D. Xaytboyeva O‘. $m-sh$ funksiyalarning plyurigarmonikligi. // Ilm sarchashmalari. 2016.№2.

24.Qalandarova D. Xaytboyeva O‘.Kuchsiz $m-sh$ funksiyalarning $m-sh$ lik xususiyatlari. // Ilm sarchashmalari. 2016.№5

Internet manbalari:

1. www.mathnet.ru (Rossiya matematika sayti)

2. www.arxiv.org (yangi maqolalar sayti)

3. www.ziyonet.uz (ta’lim portal)

4. www.rutracker.org/forum/viewtopic.php?t=3653871