O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

FIZIKA-MATEMATIKA FAKULTETI

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BITIRUV MALAKAVIY ISHI

Mavzu: Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi xos qiymatlarining xossalari**.**

Ilmiy rahbar: **f.-m.f.d., dots. Yaxshimuratov A.B.**

Urganch 2016 yil

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Bajaruvchi: Esomurodova R.J.

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1. Talaba Esomurodova Risolat Jumaniyoz qizi Universitet rektorining № 199-T sonli buyrug‘i bilan bitiruv malakaviy ish bajarish uchun “Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi xos qiymatlarining xossalari” mavzusi tasdiqlangan.

2. Kafedra majlisining qaroriga binoan f.-m.f.d., dots.Yaxshimuratov A.B. bitiruv malakaviy ishini bajarishga rahbar qilib tayinlangan.

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1. Левитан Б.М., Саргсян И.С. Операторы Штурма-Лиувилля и Дирака. М.: Наука, 1988.
2. Hasanov A.B. Shturm-Liuvill chegaraviy masalalar nazariyasiga kirish. T.: Fan, 2011.
3. Юрко В.А. Введение в теорию обратных спектральных задач. М.: Физматлит, 2007.
4. Binding P.A., Browne P.J., Watson B.A. Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, I.

4. Ishning maqsadi: Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi xos qiymatlarining va xos funksiyalarining xossalarini o‘rganish.

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Bo‘limlar	Maslahatchi F.I.SH.	Imzo, sana	
		Topshiriq berdi	Topshiriq qabul qildi
Kirish	Yaxshimuratov A.B.	06.11.2015	12.11.2015
Masalaning qo‘yilishi va Herlots-Nevanlinna funksiyalarining xossalari	Yaxshimuratov A.B.	12.11.2015	15.01.2016
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M U N D A R I J A

Kirish

Tadqiqot mavzusining dolzarbligi. Matematik fizikaning bir qator masalalari Shturm-Liuvill operatorining xos qiymatlarini va ortonormallangan xos funksiyalarini topishga keltiriladi. Jumladan, klassik matematik fizikaning asosiy tenglamalari hisoblangan tor tebranishi va issiqlik o‘tkazuvchanlik tenglamalarini Furye usuli bilan yechishda Shturm-Liuvill chegaraviy masalasining xos qiymatlarini, ortonormallangan xos funksiyalarini aniqlashga va ixtiyoriy funksiyani ular yordamida Furye qatoriga yoyishga to‘g‘ri keladi. Bu yo‘nalishdagi ilk natijalar D.Bernulli, J.Dalamber, L.Eyler, Liuvill va Shturmlar tomonidan olingan. Shturm-Liuvill operatori spektral nazariyasining asosiy g‘oyalari XX-asrda G.D.Birkgoff, G.Veyl, D.Gilbert, V.A.Steklov, E.Ch.Titchmarsh, N.Levinson, B.M.Levitan va boshqa olimlar tomonidan rivojlantirildi.

Spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasi koeffitsientini va chegaraviy shartlarni aniqlash masalasiga spektral analizning teskari masalasi deyiladi.

Teskari masalalar nazariyasining rivojiga muhim turtki bo‘lgan ilk natija 1929 yilda taniqli astronom V.A.Ambartsumyan tomonidan olingan: agar ushbu

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$y'(0) = 0, \ y'(\pi) = 0$$

Shturm-Liuvill chegaraviy masalasining xos qiymatlari $\lambda_n = n^2, \quad n = 0, 1, 2, ...$ bo‘lsa, u holda $q(x) \equiv 0$ bo‘ladi.

Ambarsumyanning bu natijasi muhim ekanliliga birinchi bo‘lib, 1946 yilda Shved matematigi G.Borg e’tibor berdi, u Ambartsumyanning natijasi umumiy qoidadan istisno ekanligini ko‘rsatib berdi, ya’ni faqat bitta spektr orqali Shturm-Liuvill chegaraviy masalasini bir qiymatli tiklab bo‘lmasligini ko‘rsatdi. G.Borg, faqat bitta chegaraviy sharti bilan farq qiluvchi ikkita Shturm-Liuvill masalasining spektrlari ma’lum bo‘lsa, bu spektrlar orqali dastlabki masalalar bir qiymatli tiklanishini isbot qildi. Hozirda bu tasdiq, G.Borgning yagonalik teoremasi deb yuritiladi.

1949 yilda A.N.Tixonov yarim o‘qda berilgan Shturm-Liuvill operatorini $I(\lambda)$ “impedans” funksiyasi (ya’ni Veyl-Titchmarshning $m(\lambda)$ funksiyasi) yordamida yagona tarzda qurish mumkinligi haqidagi teoremani isbotlashga muvaffaq bo‘lgan. A.N.Tixonovning bu teoremasi, yer ichki qatlamlari elektrik hossalarini o‘rganish masalalarini matematik asoslashda muhim ahamiyatga ega. Veyl-Titchmarshning $m(\lambda)$ funksiyasi bo‘yicha chiziqli oddiy differensial operatorni qurish algoritmi V.A.Yurko tomonidan batafsil o‘rganilgan.

1950 yilda V.A.Marchenko Shturm-Liuvill operatori o‘zining spektral funksiyasi orqali yagona aniqlanishini ko‘rsatib bergan. V.A.Marchenko teoremasidan, hususan, λ_n xos qiymatlar va α_n normallovchi o‘zgarmaslar ketma-ketligi yordamida Shturm-Liuvill chegaraviy masalasi bir qiymatli aniqlashi kelib chiqadi.

Hozirgi kunda teskari masalalar nochiziqli evolyutsion tenglamalar uchun qo‘yilgan Koshi masalasini yechishga tatbiq qilinayotganligi tufayli teskari spektral masalalar sohasiga bo‘lgan qiziqish ortib bormoqda. Jumladan, chegaraviy sharti spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill operatorlari uchun teskari masalalarni o‘rganish dolzarb masalalardan biridir.

Ilmiy izlanishning maqsad va vazifalari. Mazkur ishning asosiy maqsadi - chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun xos qiymatlarning asimptotikasi va xos funksiyalarning ossillyatsiya xossalarini o‘rganishdan iborat. Bu maqsadlarni amalga oshirish uchun Herlots-Nevanlinna funksiyalar sinfini hamda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun Krum-Kreyn almashtirishini o‘rganish vazifalari belgilangan.

Mavzuning o‘rganilish darajasining qiyosiy tahlili. Klassik Shturm-Liuvill masalasi V.A.Ambarsumyan, G.Borg, V.A.Marchenko, B.M.Levitan, I.M.Gelfand, M.G.Kreyn, V.A.Yurko va boshqalar tomonidan batafsil o‘rganilgan.

Chegaraviy shartida $f(\lambda)=a\lambda+b$ yoki $f(\lambda)=a\lambda^2+b\lambda+c$ ko‘rinishdagi funksiyalar bo‘lgan hollardagi masalalar bir qancha mualliflar tomonidan, masalan C.T.Fulton, J.Walter tomonidan o‘rganilgan. [14], [16], [17] ishlarda esa chegaraviy sharti spektral parametrga bog‘liq bo‘lgan singulyar Shturm-Liuvill masalasi o‘rganilgan. [9], [12], [22] ishlarda chegaraviy sharti spektral parametrga bog‘liq bo‘lgan chegaraviy masala yuqori tartibli tenglama uchun o‘rganilgan. [6], [19], [27] ishlarda esa chegaraviy shartida spektral parametri bo‘lgan chegaraviy masala hususiy hosilali tenglamalar uchun qaralgan. $f(\lambda)$ funksiya λ parametrga murakkabroq bog‘liq bo‘lgan holda bu masalaga oid ayrim natijalar [3], [21], [23], [25] ishlarda olingan.

Tadqiqotning ilmiy yangiligi. Mazkur ishda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun xos qiymatlarning asimptotikasi va xos funksiyalarning ossillyatsiya xossalari o‘rganilgan.

Tadqiqot predmeti va ob’yekti. Tadqiqot obyekti chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi bo‘lib, uning predmeti teskari spektral masalani o‘rganishdir.

Tadqiqotning ilmiy ahamiyati. Tadqiqotning ilmiy ahamiyati olingan natijalar matematik fizikaning nochiziqli evolyutsion tenglamalarini yechishda qo‘llanilishi bilan asoslanadi.

1-§. Masalaning qo‘yilishi va
Herlots-Nevanlinna funksiyalarining xossalari

Quyidagi masalani ko‘rib chiqamiz

$$-y''+q(x)y=\lambda y, \quad x\in[0,1] \tag{1}$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha\in[0,\pi) \tag{2}$$

$$\frac{y'(1)}{y(1)}=f(\lambda). \tag{3}$$

Bu yerda $q(x)\in C[0,1]$ - haqiqiy funksiya bo‘lib, $f(\lambda)$ quyidagi tasvirga ega:

$$f(\lambda)=a\lambda+b-\sum_{k=1}^{N}\frac{b_k}{\lambda-c_k} \tag{4}$$

$$a\geq 0, \;\; b_k\geq 0, \;\; b\in R, \quad c_1<c_2<...<c_N, \;\; N\geq 0. \tag{5}$$

Maskur ishda (1)-(3) masala xos qiymatlarining xossalari, xususan asimptotikasi, xos funksiyalar uchun Krum-Kreyn tipidagi almashtirish o‘rganiladi.

Berilgan N uchun (4) ko‘rinishdagi (5) shartlarni qanoatlantiruvchi barcha ratsional funksiyalarni R_N orqali belgilaymiz.

Ta’rif. Agar $f:C\to C$ kompleks o‘zgaruvchili funksiya uchun ushbu

1. $\overline{f(z)}=f(\overline{z})$
2. $\operatorname{Im} z\geq 0 \Rightarrow \operatorname{Im} f(z)\geq 0$

shartlar bajarilsa, $f(z)$ funksiyaga Herlots-Nevanlinna funksiyasi deyiladi.

Lemma1. Qutb maxsus nuqtalari haqiqiy va oddiy bo‘lgan ratsional funksiya Herlots-Nevanlinna funksiyasi bo‘lishi uchun biror N topilib, $f(z)\in R_N$ bo‘lishi zarur va yetarli.

Isbot. (Yetarliligi). Agar $f(z)\in R_N$ bo‘lsa, u holda ushbu

$$\overline{f(z)}=\overline{a}\cdot\overline{z}+\overline{b}-\sum_{k=1}^{N}\frac{b_k}{\overline{z}-\overline{c_k}}=a\overline{z}+b-\sum_{k=1}^{N}\frac{b_k}{\overline{z}-c_k}=f(\overline{z})$$

tenglik bajariladi. $\operatorname{Im} z\geq 0$ bo‘lsin, ya’ni $z=u+iv,\; v\geq 0$ bo`lsin. U holda

$$f(z)=au+iav+b-\sum_{k=1}^{N}\frac{b_k}{(u-c_k)-iv}=$$

$$=au+b-\sum_{k=1}^{N}\frac{b_k(u-c_k)}{(u-c_k)^2+(v)^2}+i\left\{av+\sum_{k=1}^{N}\frac{b_k v}{(u-c_k)^2+v^2}\right\},$$

$$\operatorname{Im} f(z)=a+\sum_{k=1}^{N}\frac{b_k}{(u-c_k)^2+v^2},\ v\geq 0.$$

(Zarurligi). $f(z)$ ratsional funksiya bo‘lgani uchun $f(z)=\dfrac{Q(z)}{P(z)}$ bo‘ladi. Bu yerda $Q(z)$ va $P(z)$ ko‘phadlar. Agar qutblarni oddiyligini hisobga olsak va bu funksiyani sodda kasrlar yig‘indisiga yoysak, ushbu

$$f(z)=P(z)+az+b-\sum_{k=1}^{N}\frac{b_k}{z-c_k}$$

tenglikni olamiz. Bu yerda $P(z)$ ko‘phad bo‘lib, undagi hadlar darajasi $k\geq 2$ bo‘ladi. $P(z)\equiv 0$ ekanligini isbotlaymiz. $\operatorname{Im} z\geq 0$ bo‘lgani uchun $z=re^{i\varphi}$, $0\leq\varphi\leq\pi$ bo‘ladi. Bunga ko`ra $P(z)$ ko‘phadning hadlari uchun quyidagilar o‘rinli:

$$a_k z^k=r_k e^{i\alpha_k} r^k e^{ik\varphi}=r_k r^k[\cos(k\varphi+\alpha_k)+i\sin(k\varphi+\alpha_k)],$$

$$\operatorname{Im}\{a_k z^k\}=r_k r^k\sin(k\varphi+\alpha_k).$$

Agar $0\leq\alpha_k<\pi$ bo‘lsa, u holda

$$0<\frac{\pi-\alpha_k}{k}<\varphi<\frac{2\pi-\alpha_k}{k}\leq\frac{2\pi-\alpha_k}{2}=\frac{\pi-\alpha_k}{2}<\pi$$

bo‘ladi. Bundan esa $\operatorname{Im}\{a_k z^k\}<0$ ziddiyat kelib chiqadi. Qolgan hollar ham shu tarzda qaraladi. Demak, $P(z)\equiv 0$ ekan.

Agar $a<0$ bo‘lsa, $z=iv$ deb, $v\to+\infty$ desak, $\operatorname{Im} f(z)\to-\infty$ bo‘ladi. Demak, $a\geq 0$ ekan.

$b_1>0$ ekanligini isbotlaymiz. Agar $z=c_1+iv,\ v\to 0\ (v>0)$ desak, u holda

$$av+\frac{b_1 v}{v^2}+\frac{b_2 v}{(c_1+c_2)^2+v^2}+\cdots+\frac{b_N v}{(c_1+c_N)^2+v^2}\to b_1\cdot\infty\,.$$

Bu limitning qiymati aslida $+\infty$ bo‘lishi lozim. Bunga ko‘ra $b_1 > 0$ bo‘ladi. $b_2 > 0$, …, $b_N > 0$ tengsizliklar ham shu tarzda ko‘rsatiladi. **Lemma isbotlandi.**

Quyidagi belgilashlarni kiritib olamiz:

$$R_N^+ = \{ f(z) \in R_N, a > 0\}, \quad R_N^0 = \{ f(z) \in R_N, a = 0\}.$$

Lemma 2. $f(z) \in R_N$ bo‘lsin. U holda

1) $f(\lambda)$ chekli bo‘lsa, $f'(\lambda) > 0$ bo‘ladi;

2) $\lim\limits_{\lambda \to c_k+0} f(\lambda) = -\infty$, $\lim\limits_{\lambda \to c_k-0} f(\lambda) = +\infty$;

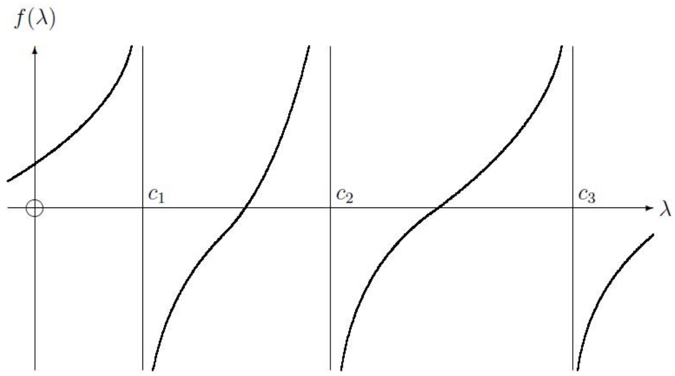
3) $f(z) \in R_N^+$ bo‘lsa, u holda $\lim\limits_{\lambda \to +\infty} f(\lambda) = +\infty$, $\lim\limits_{\lambda \to -\infty} f(\lambda) = -\infty$;

4) $f(z) \in R_N^0$ bo‘lsa, $f(\lambda) \uparrow b,\ (\lambda \to \infty)$ va $f(\lambda) \downarrow b,\ (\lambda \to -\infty)$.

Misol. $f(z) \in R_3^0$ bo‘lsa, u holda

$$f(\lambda) = b - \frac{b_1}{\lambda - c_1} - \frac{b_2}{\lambda - c_2} - \frac{b_3}{\lambda - c_3}$$

bo‘ladi. Bu yerda $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $c_1 < c_2 < c_3$. Bu funksiya grafigi quyidagi ko‘rinishda bo‘ladi:



1-rasm

$f(\lambda) \in R_N$ funksiya va $\mu < c_1$ son berilgan bo‘lsin. Ushbu

$$F(\lambda) = -\frac{\lambda - \mu}{f(\lambda) - f(\mu)} - f(\mu) \tag{6}$$

funksiyani kiritib olamiz. Bu funksiyani uzluksizligi bo‘yicha davom qildiramiz:

$$F(c_k) = -\frac{c_k - \mu}{f(c_k) - f(\mu)} - f(\mu) = -f(\mu), \quad 1 \le k \le N,$$

$$F(\mu)=-\frac{1}{f'(\mu)}-f(\mu).$$

Izoh. Agar $f(\lambda)\in R_N$ bo‘lsa, u holda $F(\lambda)\in R_M$, ya’ni

$$F(\lambda)=A\lambda+B-\sum_{k=1}^{M}\frac{B_k}{\lambda-C_k}, \tag{7}$$

$A\geq 0,\ B_k\geq 0,\ B\in R,\ C_1<C_2<...<C_M$ ekanligini isbotlaymiz. Bu yerda M soni a ga bog‘liq ravishda N yoki $N-1$ bo‘ladi, aniqrog‘i quyidagi teorema o‘rinli.

Teorema 1. 1) Agar $f(\lambda)\in R_N^+$ bo‘lsa, u holda $F(\lambda)\in R_N^0$ bo‘lib,

$$\mu<c_1<C_1<c_2<...<c_N<C_N$$

bo‘ladi;

2) Agar $f(\lambda)\in R_N^0$ bo‘lsa, u holda $F(\lambda)\in R_{N-1}^+$ bo‘lib,

$$\mu<c_1<C_1<c_2<...<C_{N-1}<c_N$$

bo‘ladi.

Isbot. Quydagi tengliklarni bir-biridan ayiramiz:

$$-\begin{cases} f(\lambda)=a\lambda+b-\sum\limits_{k=1}^{N}\dfrac{b_k}{\lambda-c_k}, \\ f(\mu)=a\mu+b-\sum\limits_{k=1}^{N}\dfrac{b_k}{\mu-c_k}, \end{cases}$$

$$f(\lambda)-f(\mu)=a(\lambda-\mu)-\sum_{k=1}^{N}\frac{b_k(\mu-\lambda)}{(\lambda-c_k)(\mu-c_k)}.$$

Bunga ko‘ra

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu}=a+\sum_{k=1}^{N}\frac{b_k}{(\lambda-c_k)(\mu-c_k)}.$$

Bu yerda

$$d_k=\frac{b_k}{\mu-c_k}<0 \text{ va } P_N(\lambda)=(\lambda-c_1)...(\lambda-c_N)$$

belgilashlarni kiritamiz. U holda

$$P_N(\lambda)\cdot\sum_{k=1}^{N}\frac{d_k}{\lambda-c_k}=\sum_{k=1}^{N}\frac{d_k\cdot P_N(\lambda)}{\lambda-c_k}=\sum_{k=1}^{N}d_k\cdot\prod_{j=1,\ j\neq k}^{N}(\lambda-c_j)=Q_{N-1}(\lambda),$$

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu}=a+\frac{Q_{N-1}(\lambda)}{P_N(\lambda)}=\frac{aP_N(\lambda)+Q_{N-1}(\lambda)}{P_N(\lambda)}.$$

Demak,

$$F(\lambda)=-\frac{P_N(\lambda)}{aP_N(\lambda)+Q_{N-1}(\lambda)}-f(\mu)=$$

$$=\frac{P_N(\lambda)-f(\mu)\cdot a\cdot P_N(\lambda)-f(\mu)\cdot Q_{N-1}(\lambda)}{aP_N(\lambda)+Q_{N-1}(\lambda)}=\frac{p(\lambda)}{r(\lambda)}. \tag{8}$$

Bu yerda

$$r(\lambda)=a\cdot\prod_{j=1}^{N}(\lambda-c_j)+\sum_{k=1}^{N}d_k\cdot\prod_{j=1,\,j\neq k}^{N}(\lambda-c_j). \tag{9}$$

$r(\lambda)$ ko‘phad $a\neq 0$ da N darajali ko‘phad, $a=0$ da $N-1$ chi darajali ko‘phad bo‘ladi. $r(c_n)$ ning ishorasini aniqlaymiz. Ushbu

$$r(c_n)=d_n\prod_{j=1}^{N}(c_n-c_j)=d_n\cdot\prod_{j=1}^{N-1}(c_n-c_j)\cdot\prod_{j=n+1}^{N}(c_n-c_j)$$

tenglikda $d_n<0$ ekanligini hisobga olsak,

$$sign(r(c_n))=(-1)^{N-n+1},\ n=1,2,...,N$$

kelib chiqadi. Demak, $r(\lambda)$ ko‘phad (c_n,c_{n+1}) intervalda μ_n ildizga ega $(n=1,2,...,N-1)$. Agar $a=0$ bo‘lsa, bular $r(\lambda)$ ko‘phadning barcha ildizlari, chunki bu holda $\deg(r(\lambda))=N-1$.

Agar $a>0$ bo‘lsa, u holda $sign(r(c_N))=(-1)^{N-N+1}=-1<0$ bo‘lib, $r(\lambda)\to+\infty,\ \lambda\to+\infty$ bo‘lgani uchun $(c_N,+\infty)$ intervalda ham bitta μ_N ildizga ega.

$r(\lambda)$ karrali ildizga ega bo‘lmagani uchun $F(\lambda)=\dfrac{p(\lambda)}{r(\lambda)}$ funksiyaning qutb maxsus nuqtalari oddiy bo‘ladi. (8) ga ko‘ra

$$F(\lambda)=\frac{(-1-af(\mu)\cdot P_N(\lambda))-f(\mu)\cdot Q_{N-1}(\lambda)}{aP_N(\lambda)+Q_{N-1}(\lambda)}.$$

Agar $a=0$ bo‘lsa, u holda

$$F(\lambda) = \frac{-P_N(\lambda) - f(\mu)\cdot Q_{N-1}(\lambda)}{Q_{N-1}(\lambda)} = A\lambda + B - \sum_{k=1}^{N-1} \frac{B_k}{\lambda - C_k}$$

bo‘ladi. Bu yerda

$$A = \frac{-1}{d_1 + \cdots + d_N} > 0 .$$

Agar $a > 0$ bo‘lsa, $F(\lambda) = B - \sum_{k=1}^{N} \frac{B_k}{\lambda - C_k}$ bo‘ladi. Bu yerda $B = -\frac{1}{a} - f(\mu)$.

Endi $B_k \geq 0$ ekanligini isbotlash kerak. $F(\lambda)$ ning ta’rifiga ko‘ra

$$F(\lambda) = -\frac{\lambda - \mu}{f(\lambda) - f(\mu)} - f(\mu)$$

bo‘lgani uchun $F(\lambda)$ funksiyaning qutb maxsus nuqtalari $f(\lambda) = f(\mu)$ tenglikning μ dan boshqa ildizlari bilan ustma-ust tushadi.

Har-bir (C_k, C_{k+1}) oraliqda ushbu

$$f(\lambda) - f(\mu) \to +0, \ (\lambda \to C_k + 0),$$

$$f(\lambda) - f(\mu) \to -0, \ (\lambda \to C_k - 0)$$

munosabatlar o‘rinli bo‘ladi. Bunga ko‘ra

$$\lim_{\lambda \to C_k \pm 0} F(\lambda) = \mp\infty .$$

Bundan esa $B_k > 0$, $k = 1, 2, ..., M$ bo‘lishi kelib chiqadi. **Teorema isbotlandi.**

Izoh. $f(\lambda)$ dan $F(\lambda)$ ga o‘tish amalini bir qancha marta qo‘llab $R_N \to R_0^0$ akslantirishni ko‘rsatish mumkin. Keyinchalik bu fikrni ishlatamiz.

2-§. Xos qiymatlarning mavjudligi va xos funksiyalarning ossillyatsiya xossalari

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq 1 \tag{10}$$

tenglamaning ushbu

$$* \, y(0)\cos\alpha = y'(0)\sin\alpha \tag{11}$$

shartni qanoatlantiruvchi biror nolmas yechimini $y(x,\lambda)$ orqali belgilaymiz.

Ushbu

$$\begin{cases} \theta' = \cos^2\theta + [\lambda - q(x)]\sin^2\theta, \\ \theta(0,\lambda) = \alpha \end{cases} \tag{12}$$

Koshi masalasining yechimini $\theta(x,\lambda)$ orqali belgilaymiz. Bunga ko‘ra

$$ctg\theta(x,\lambda) = \frac{y'(x,\lambda)}{y(x,\lambda)}. \tag{13}$$

Bu holda

$$\frac{y'(1,\lambda)}{y(1,\lambda)} = f(\lambda) \tag{14}$$

harakteristik tenglama ushbu

$$ctg\theta(1,\lambda) = f(\lambda) \tag{15}$$

ko‘rinishini oladi.

Teorema 2. (1)-(3) masalaning xos qiymatlari haqiqiy.

Isbot. $\lambda = u + iv$, $u, v \in R$, $v \neq 0$ xos qiymat bo‘lsin va bu xos qiymatga $y(x)$ xos funksiya mos kelsin. U holda

$$-y'' + q(x)y = \lambda y,$$

$$\frac{y'(0)}{y(0)} = ctg\alpha,$$

$$\frac{y'(1)}{y(1)} = f(\lambda).$$

tengliklarda kompleks qo‘shma olsak, ushbu

$$-\bar{y}'' + q(x)\bar{y} = \bar{\lambda}\bar{y},$$

$$\frac{\bar{y}'(0)}{\bar{y}(0)} = ctg\alpha,$$

$$\frac{\bar{y}'(1)}{\bar{y}(1)} = f(\bar{\lambda})$$

tengliklar hosil bo‘ladi. Demak, $\overline{\lambda} = u - iv$ ham xos qiymat ekan. Biz $\operatorname{Im}\lambda = v > 0$ deb hisoblaymiz. Endi quydagi tengliklarni ko‘rib chiqamiz:

$$(\lambda - \overline{\lambda}) \cdot \int_0^1 |y(x)|^2 dx = (\lambda - \overline{\lambda}) \cdot \int_0^1 y \cdot \overline{y} dx = \int_0^1 [(\lambda \cdot y)\overline{y} - y(\overline{\lambda} \cdot \overline{y})] dx =$$

$$= \int_0^1 [(-y'' + q(x)y)\overline{y} - y \cdot (\overline{y}'' + q(x)\overline{y})] dx =$$

$$= \int_0^1 (\overline{y}'y - y'\overline{y})' dx = (\overline{y}'y - y'\overline{y})\Big|_0^1 = |y(1)|^2 \cdot \left(f(\overline{\lambda}) - f(\lambda) \right),$$

ya’ni

$$\operatorname{Im}\lambda \cdot \int_0^1 |y(x)|^2 dx = -|y(1)|^2 \cdot \operatorname{Im} f(\lambda)$$

Ziddiyat, chunki $\operatorname{Im}\lambda > 0$ va $\operatorname{Im} f(\lambda) > 0$. **Teorema isbotlandi.**

Teorema 3. (1)-(3) masalaning xos qiymatlari oddiy (karrasiz).

Isbot. $\Delta(\lambda) = \varphi'(1,\lambda) - f(\lambda)\varphi(1,\lambda)$ bo‘lsin. Bu yerda $\varphi(x,\lambda)$ orqali (1) tenglamaning $\varphi(0,\lambda) = \sin\alpha$, $\varphi'(0,\lambda) = \cos\alpha$ boshlang‘ich shartlarni qanoatlantiruvchi yechimi belgilangan. Ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi$$

ayniyatni λ bo‘yichia differensiallasak,

$$-\dot{\varphi}'' + q(x)\dot{\varphi} = \varphi + \lambda\dot{\varphi}$$

kelib chiqadi. Bularga ko‘ra ushbu

$$(\dot{\varphi}' \cdot \varphi - \varphi' \cdot \dot{\varphi})' = -\varphi^2$$

ayniyat o‘rinli bo‘ladi. Bu ayniyatni integrallasak, quyidagi

$$\dot{\varphi}'(1,\lambda) \cdot \varphi(1,\lambda) - \varphi'(1,\lambda) \cdot \dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda) dx \qquad (*)$$

formula hosil bo‘ladi.

Endi $\Delta(\lambda) = \varphi'(1,\lambda) - f(\lambda)\varphi(1,\lambda)$ funksiyani differensiallaymiz:

$$\dot{\Delta}(\lambda) = \dot{\varphi}'(1,\lambda) - \dot{f}(\lambda) \cdot \varphi(1,\lambda) - f(\lambda) \cdot \dot{\varphi}(1,\lambda).$$

Demak, ushbu

$$\dot{\varphi}'(1,\lambda) = \dot{\Delta}(\lambda) + \dot{f}(\lambda)\cdot\varphi(1,\lambda) + f(\lambda)\cdot\dot{\varphi}(1,\lambda),$$

$$\varphi'(1,\lambda) = \Delta(\lambda) + f(\lambda)\cdot\varphi(1,\lambda)$$

tengliklar o'rinli ekan. Bu ifodalarni (*) ayniyatga qo'yamiz:

$$[\dot{\Delta}(\lambda) + \dot{f}(\lambda)\cdot\varphi(1,\lambda) + f(\lambda)\cdot\dot{\varphi}(1,\lambda)]\cdot\varphi(1,\lambda) -$$

$$-[\Delta(\lambda) + f(\lambda)\cdot\varphi(1,\lambda)]\cdot\dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx,$$

$$\dot{\Delta}(\lambda)\cdot\varphi(1,\lambda) + \dot{f}(\lambda)\varphi^2(1,\lambda) + f(\lambda)\varphi(1,\lambda)\dot{\varphi}(1,\lambda) -$$

$$-\Delta(\lambda)\dot{\varphi}(1,\lambda) - f(\lambda)\varphi(1,\lambda)\cdot\dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx,$$

$$\dot{\Delta}(\lambda)\cdot\varphi(1,\lambda) + \dot{f}(\lambda)\varphi^2(1,\lambda) - \Delta(\lambda)\dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx. \qquad (**)$$

Agar $\lambda = \lambda_0$ karrali xos qiymat bo'lsa, u holda $\Delta(\lambda_0) = 0$, $\dot{\Delta}(\lambda_0) = 0$ bo'lgani uchun

$$\dot{f}(\lambda_0)\varphi^2(1,\lambda_0) = -\int_0^1 \varphi^2(x,\lambda_0)dx$$

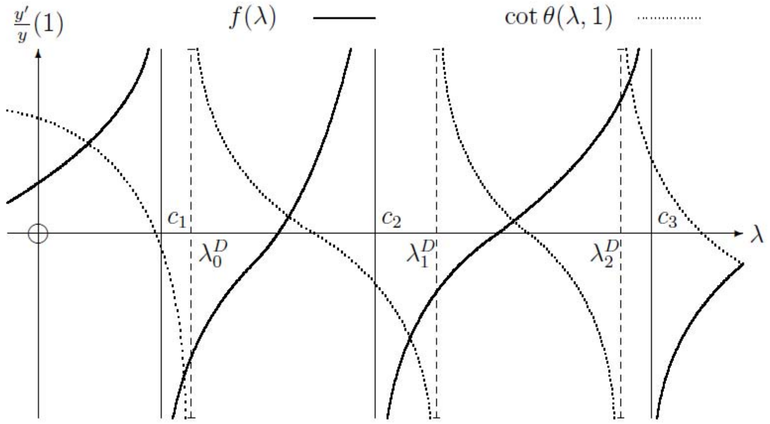
bo'ladi. $f(\lambda_0) > 0$ bo'lgani uchun ziddiyat kelib chiqadi. Agar $\lambda = c_k$ xos qiymat bo'lsa, u holda $y(1,c_k) = 0$, bo'ladi va bu son oddiy Shturm-Liuvill masalasining xos qiymati bo'ladi. Demak, u oddiy xos qiymat ekan. **Teorema isbotlandi.**

Teorema 4. (1)-(3) masala cheksizta xos qiymatga ega. Bu xos qiymatlarni

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

orqali belgilasak, ular ∞ ga intiladi. Bunda $\lambda_0 < c_1$ bo'ladi.

Isbot. (1)-(3) masala cheksizta xos qiymatlari ushbu $ctg\theta(\lambda,1) = f(\lambda)$ harakteristik tenglamadan topilishini biz bilamiz. Bu tenglamani grafik usulda yechamiz. Buning uchun $y = ctg\theta(\lambda,1)$ va $y = f(\lambda)$ funksiyalarning grafiklarini bitta chizmada chizamiz:



Bu chizmadan xususan xos qiymatlar soni cheksizta ekanligi va ular ∞ ga ketishi hamda eng kichik xos qiymat c_1 dan kichik ekanligi kelib chiqadi. **Teorema isbotlandi.**

Teorema 5. (1)-(3) masalada b ning qiymati kamaysa, c_k va $q(x)$ ning qiymatlari ortsa, λ_j larning har biri ortadi. Agar $a > 0$ bo‘lib, a kamaysa va b_k ortsa har bir musbat $\lambda_j > c_k$ ortadi.

Bu teoremaning isboti $f(\lambda)$ ning tuzilishidan kelib chiqadi.

λ_n^D orqali quyidagi masalaning xos qiymatlarini belgilaymiz:

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1],$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha \in [0,\pi),$$

$$y(1) = 0.$$

Endi $\{c_j\}$, $j = 1,2,...,N$, $\{\lambda_n^D\}$, $n = 1,...,\infty$ sonlardan bitta to‘plam tuzamiz va uning elementlarini kamaymaydigan tartibda λ_n^{cD} orqali belgilaymiz. Agar $c_j = \lambda_k^D$ bo‘lsa, λ_n^{cD} ikki karrali deb hisoblaymiz. k_j orqali $c_i \le \lambda_j$ shartni qanoatlantiradigan c_i lar sonini belgilaymiz. $f(\lambda)$ funksiyaning va $ctg\theta(\lambda,1)$ funksiyaning grafiklarini analiz qilsak, quyidagi teorema kelib chiqadi.

Teorema 6. 1) λ_i va λ_j^{cD} ketma-ketliklar quyidagi ma’noda almashinib keladi: $\lambda_0 < \lambda_0^{cD} \le \lambda_1 \le \lambda_1^{cD} \le \cdots$;

2) Ushbu $c_{k_j} \le \lambda_j < c_{k_j+1}$ qo‘sh tengsizliklar bajariladi;

3) Agar $\lambda_{-1}^D = -\infty$ deb olsak, ushbu $\lambda_{j-1-k_j}^D < \lambda_j \le \lambda_{j-k_j}^D$ qo‘sh tengsizliklar o‘rinli bo‘ladi.

Bu teoremaga asoslanib, biz xos funksiyalar uchun ossillyatsiya xossasini keltirib chiqarishimiz mumkin. Agar $\lambda \in (\lambda_{i-1}^D, \lambda_i^D]$ bo‘lsa, u holda (1) tenglamaning (2) shartni qanoatlantiruvchi noldan farqli yechimi $(0,1)$ oraliqda i ta ildizga ega bo‘ladi.

Natija. λ_j ga mos keluvchi xos funksiyaning $(0,1)$ oraliqdagi ildizlar sonini ω_j orqali belgilaymiz. Bu holda $\omega_j = j - k_j$ bo‘ladi, hususan $\omega_0 = 0$ bo‘ladi. Agar $\lambda_j > c_N$ bo‘lsa $\omega_j = j - N$ bo‘ladi.

3-§. Krum-Kreyn almashtirishi

Bu paragrafda (1)-(3) masalani xos qiymatlaridan ko‘pchiligini saqlagan holda boshqa bir Shturm-Liuvill masalasiga o‘tkazamiz. Natijada hosil bo‘ladigan Shturm-Liuvill masalasida $f(\lambda)$ o‘rnida oldingi paragrafda o‘rganilgan $F(\lambda)$ bo‘ladi. Bu almashtirishni bir necha marta qo‘llab, klassik Shturm-Liuvill masalasini hosil qilamiz.

Hosil bo‘ladigan klassik Shturm-Liuvill masalasining xos qiymatlari eng oldingi Shturm-Liuvill masalasining xos qiymatlaridan cheklitaga farq qiladi halos. (1)-(3) masalaning $\lambda_0 < \lambda_1 < \cdots$ xos qiymatlariga mos keluvchi xos funksiyalarini $y_0(x), y_1(x), ...$ orqali belgilaymiz. μ orqali quyidagi shartni qanoatlantiruvchi biror sonni belgilaymiz: agar $\alpha > 0$ bo‘lsa, $\mu = \lambda_0$, agar $\alpha = 0$ bo‘lsa, $\mu < \lambda_0$.

$\omega(x,\mu)$ orqali (1) tenglamaning $\lambda=\mu$ bo‘lgandagi $\omega(1)=1$, $\omega'(1)=f(\mu)$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Quyidagi belgilashlarni kiritib olamiz

$$z=\frac{\omega'(x,\mu)}{\omega(x,\mu)},\quad \hat{q}(x)=q(x)-2z'(x).$$

$F(\lambda)$ oldingi paragrafda aniqlangan funksiya bo‘lsin. $\gamma\in[0,\pi)$ sonni shunday olamizki, bunda agar $\alpha=0$ bo‘lsa $ctg\gamma=-z(0)$, agar $\alpha>0$ bo‘lsa, $\gamma=0$.

Teorema 7. Quyidagi

$$-y''+\hat{q}(x)y=\lambda y, \tag{16}$$

$$\frac{y'(0)}{y(0)}=ctg\gamma, \tag{17}$$

$$\frac{y'(1)}{y(1)}=F(\lambda) \tag{18}$$

chegaraviy masalaning xos qiymatlari λ_j sonlar bilan, xos funksiyalari esa $U_j(x)=y_j'(x)-z(x)y_j(x)$ funksiyalar bilan ustma-ust tushadi. Bu yerda $\alpha>0$ bo‘lsa, $j\ge 1$ deb hisoblanadi, $\alpha=0$ bo‘lsa, $j\ge 0$ deb olinadi.

Isbot. Avvalo $\omega(x)$ funksiya $[0,1]$ kesmada ildizga ega emasligini ko‘rsatamiz. $\alpha\ne 0$ bo‘lganda bu fikr juda oson, chunki bu holda $\omega(x)=y_0(x)$ bo‘lib, $y_0(x)$ funksiya ossillyatsiya teoremasiga ko‘ra $(0,1)$ oraliqda nolga aylanmaydi. Bundan tashqari, $\frac{y_0'(0)}{y_0(0)}=ctg\alpha$ va $\frac{y_0'(1)}{y_0(1)}=f(\lambda_0)$ chekli bo‘lishidan $\omega(x)$ funksiya $[0,1]$ kesmada nolga aylanmasligi kelib chiqadi.

Agar $\alpha=0$ bo‘lsa, u holda $y_0(0)=0$ bo‘lib, $(0,1)$ oraliqda $y_0(x)$ funksiya nolga aylanmaydi. $\lambda_0<\lambda_0^D$ bo‘lgani uchun $y_0(1)\ne 0$ bo‘ladi. Endi quyidagi Prufer tenglamasini ko‘rib chiqamiz:

$$\theta'=\cos'\theta+(\mu-q(x))\sin^2\theta. \tag{19}$$

Bu tenglamani ushbu

$$\theta(\mu,1)=arcctgf(\lambda_0)\in(0,\pi) \tag{20}$$

shart bilan qaraymiz. $\mu < \lambda_0$ bo‘lgani uchun $\theta(\mu,0) \in (0,\pi)$ bo‘ladi.

Agar biz (19) tenglamani $\theta(\mu,1) = arcctgf(\mu)$ shart bilan qarasak, $\theta(\mu,0)$ o‘sadi ammo $(0,\pi)$ intervalda qoladi. Demak, (19) tenglamaning bu shart bilan qaralgandagi yechimi $\psi(x)$ faqat $(0,\pi)$ intervaldan qiymatlar qabul qiladi:

$$x \in [0,1], \ \ \psi(x) \in (0,\pi).$$

Bu fikr esa $\omega(x)$ funksiyaning $[0,1]$ kesmada ildizga ega emas ekanligiga olib keladi. Demak, $z(x)$ funksiya $[0,1]$ da aniqlangan ekan.

Agar $\alpha > 0$ bo‘lsa, $\psi(x) = \theta(\lambda_0, x)$ deb olamiz. Shunday qilib

$$\psi(0) = \begin{cases} \alpha, & \alpha > 0 \\ \pi - \gamma, \alpha = 0 \end{cases}$$

to‘g‘ridan-to‘g‘ri hisoblash ishlarini bajarib, $U_j(x)$ funksiyalar (16) differensial tenglamani va (17) chegaraviy shartni qanoatlantirishini ko‘rsatish mumkin. Bundan tashqari

$$\frac{U_j'(x)}{U_j(x)} = \frac{\mu - \lambda_j}{\frac{y_j'}{y} - z} - z \tag{21}$$

bo‘lgani uchun (18) chegaraviy shart ham bajariladi. Demak, λ_j sonlar haqiqatdan ham (16)-(18) chegaraviy masalaning xos qiymatlari ekan.

(16)-(18) masalaning bulardan boshqa xos qiymati yo‘q ekanligini isbotlash qoldi holos. $\theta(\lambda, x)$ orqali ushbu

$$\theta' = \cos^2\theta + (\lambda - q(x))\sin^2\theta, \ \ \theta(\lambda,0) = \alpha \tag{22}$$

masalaning yechimini belgilaymiz. $\varphi(\lambda, x)$ orqali esa ushbu

$$\varphi' = \cos^2\varphi + (\lambda - \hat{q}(x))\sin^2\varphi, \ \ \varphi(\lambda,0) = \gamma \tag{23}$$

masalaning yechimini belgilaylik. U holda quyidagilar o‘rinli:

$$\frac{y_j'(x)}{y_j(x)} = ctg\theta(\lambda_j, x),$$

$$\frac{U_j'(x)}{U_j(x)} = ctg\varphi(\lambda_j, x)$$

$$z(x) = ctg\,\psi(x).$$

Demak, (21) ayniyatga ko‘ra ushbu

$$ctg\varphi(\lambda_j, x) = \frac{\mu - \lambda_j}{ctg\theta(\lambda_j, x) - ctg\,\psi(x)} - ctg\,\psi(x) \tag{24}$$

tenglik o‘rinli bo‘ladi.

$x \in [0, 1]$ da $\psi(x) \in (0, \pi)$ ekanligini ishlatamiz. Agar $\alpha > 0$ bo‘lsa, u holda $\lambda_j > \mu$, $j \geq 1$ bo‘lgani uchun $\theta(\lambda_j, x) > \psi(x)$, $x \in (0, 1]$ bo‘ladi. Agar $\alpha = 0$ bo‘lsa, u holda $0 = \theta(\lambda_j, 0) < \psi(0)$ o‘rinli bo‘ladi. Bulardan tashqari, agar biror $x \in (0,1)$ uchun $ctg\,\theta(\lambda_j, x) = ctg\,\psi(x)$ bo‘lsa, u holda (22) tenglikka ko‘ra

$$\frac{d}{dx}(\theta(\lambda_j, x) - \psi(x)) > 0$$

bo‘ladi, chunki $\sin^2 \psi(x) > 0$. Demak, $\theta(\lambda_j, x) - \psi(x)$ o‘suvchi funksiya ekan. (21) tenglikka ko‘ra $U_j(x)$ funksiya biror $x \in (0, 1)$ da nolga aylanadi. Bu esa faqat $\theta(\lambda_j, x) = \psi(x) + mx$ bo‘lganida bajariladi. Bu yerda m butun son bo‘lib, $\alpha > 0$ bo‘lganida $m > 0$ va $\alpha = 0$ bo‘lganda $m \geq 0$.

Shunday qilib, $U_j(x)$ funksiyaning [0,1] da n ta ildizi bo‘lishi uchun quyidagi tengsizlik bajarilishi zarur va yetarlidir:

$$\alpha > 0, \text{ bo‘lganida } n\pi < \theta(\lambda_j, 1) - \psi(1) \leq (n+1)\pi,$$

$$\alpha = 0, \text{ bo‘lganida } (n-1)\pi < \theta(\lambda_j, 1) - \psi(1) \leq n\pi.$$

6-teoremaning 3-bandiga ko‘ra $\lambda_j \in (\lambda_{i-1}^D, \lambda_i^D]$. Bu yerda $i = j - k_j$. Shuning uchun $i\pi < \theta(\lambda_j, 1) \leq (i+1)\pi$ bo‘ladi. Bundan esa $(i-1)\pi < \theta(\lambda_j, 1) - \psi(1) < (i+1)\pi$ kelib chiqadi.

Demak, $U_j(x)$ funksiyaning [0,1] kesmada $\alpha > 0$ bo‘lganda $i - 1$ yoki i ta ildizi bor, $\alpha = 0$ bo‘lganida esa i yoki $i + 1$ ta ildizi bor.

Bular $\theta(\lambda_j, 1) = \psi(1) \leq i\pi$ yoki $\theta(\lambda_j, 1) - \psi(1) > i\pi$ bo‘lishiga bog‘liq. $f(\lambda)$ va $ctg\theta(\lambda, 1)$ tilida aytganda $U_j(x)$ funksiyaning [0,1] kesmada $\alpha > 0$ bo‘lganida

$i-1$ yoki i ta ildizi, $\alpha=0$ bo‘lganda i yoki $i+1$ ta ildizi bo‘lishi $f(\mu)\le ctg\theta(\lambda_j,1)$ yoki $f(\mu)>ctg\theta(\lambda_j,1)$ tengsizliklardan qaysi biri bajarilishiga bog‘liq.

$f(\lambda)$ va $F(\lambda)$ funksiyalar ifodasiga ko‘ra

$$f(\lambda)=a\lambda+b-\sum_{k=1}^{N}\frac{b_k}{\lambda-c_k},\qquad F(\lambda)=A\lambda+B-\sum_{k=1}^{M}\frac{B_k}{\lambda-C_k}.$$

Bu yerda agar $a>0$ bo‘lsa, $A=0, M=N$ va $a=0$ bo‘lsa, $A>0,\ M=N-1$ ga teng bo‘ladi. Bundan tashqari C_k sonlar $f(\lambda)=f(\mu)$ tenglamaning $\lambda=\mu$ ildizidan farq qiluvchi ildizlaridir.

$\lambda=\mu$ bo‘lgan holda a ning katta qiymatlarida $\lambda_j>C_N>c_N$ bo‘ladi.

Bundan

$$\frac{y_j'(1)}{y_j(1)}=ctg\theta(\lambda_j,1)=f(\lambda_j)>f(C_N)=f(\mu) \tag{25}$$

kelib chiqadi.

Qisqalik uchun $y(x)$ funksiyaning $(0,1)$ intervaldagi ildizlar sonini $osc(y(x))$ orqali belgilaymiz. U holda (25) tengsizlikka ko‘ra $osc(U_j)=osc(y_j)-1$.

Oldingi paragrafning ohirgi natijasiga ko‘ra $\alpha>0$ bo‘lganida $osc(u_j)=j-N-1$ bo‘ladi. $\alpha=0$ bo‘lganida huddi shu tarzda $osc(u_j)=osc(y_j)=j-N$ ekanligini topamiz. Ikkinchi tomondan λ_j son (16)-(18) masalaning k - xos qiymati bo‘ladi, $(k\ge 0)$.

Demak, oldingi paragrafning oxirgi natijasiga ko‘ra $osc(u_j)=k-N$. Shuning uchun $\alpha>0$ bo‘lsa, $k=j-1$ va $\alpha=0$ bo‘lsa $k=j$ bo‘ladi. Bularga asosan $\alpha>0$ bo‘lganida $\lambda_j,\ j\ge 1$ va $\alpha=0$ bo‘lganida $\lambda_j,\ j\ge 0$ xos qiymatlar (16)-(18) masalaning barcha xos qiymatlarini hosil qiladi.

Endi $\alpha=0$ holni ko‘rib chiqamiz. Bu holda j ning katta qiymatlarida $\lambda_j>C_N$ bo‘lgani uchun

$$\frac{y_j'(1)}{y_j(1)} = ctg\theta(\lambda_j,1) < b < f(\mu)$$

bo‘ladi. Demak, $\alpha > 0$ bo‘lsa, $osc(u_j) = osc(y_j) = j - N$ va $\alpha = 0$ bo‘lsa

$$osc(u_j) = osc(y_j) + 1 = j - N + 1$$

tenglik o‘rinli bo‘ladi. **Teorema isbotlandi.**

Bu teoremani bir necha marta qo‘llab quydagi natijaga kelamiz.

Natija. (1)-(3) Shturm-Liuvill masalasi berilgan bo‘lib, $\lambda_0, \lambda_1, \lambda_2, ...$ bu masalaning xos qiymatlari bo‘lsin. Bu yerda $f(\lambda) \in R_N$. U holda shunday

$$-y'' + \tilde{q}(x)y = \lambda y,$$

$$\frac{y'(0)}{y(0)} = ctg\alpha_0,$$

$$\frac{y'(1)}{y(1)} = ctg\alpha_1$$

Shturm-Liuvill masalasi mavjudki, uning $\mu_0 < \mu_2 < \cdots$ xos qiymatlari uchun quyidagilardan biri bajariladi:

1) $\alpha > 0$, $f \in R_N^+$ bo‘lsa, $\alpha_0 = 0$, $\mu_j = \lambda_{j+N+1}$;

2) $\alpha = 0$, $f \in R_N^0$ bo‘lsa, $\alpha_0 = 0$, $\mu_j = \lambda_{j+N}$

3) boshqa hollarda, $\alpha_0 > 0$, $\mu_j = \lambda_{j+N}$.

4-§. Xos qiymatlarning osimptotikasi

Bu paragrafda klassik Shturm-Liuvill masalasi xos qiymatlarining osimptotikasi yordamida (1)-(3) masala xos qiymatlarining asimptotikasini keltirib chiqaramiz.

Bu osimptotikalar albatta $q(x)$ potensialning silliqligiga bog‘liq [20]. $q(x) \in L_1(0,1)$ bo‘lganida [5] ishda xos qiymatlarning osimptotikasi o‘rganilgan, aniqrog‘i quyidagi tearema isbotlangan.

Teorema 8. Agar $q(x) \in L_1(0,1)$ bo‘lsa, ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \qquad 0 \le x \le 1 \\ \dfrac{y'(0)}{y(0)} = \rho_0, \\ \dfrac{y'(1)}{y(1)} = \rho_1 \end{cases} \tag{26}$$

masalaning λ'_j xos qiymatlari uchun quyidagi

$$\lambda'_j = j^2\pi^2 + o(j^2), \; j \to \infty \tag{27}$$

osimptotik formula o‘rinli bo‘ladi.

Agar (26) masalada chegaraviy shartlardan birinchisi $y(0) = 0$ bo‘lsa, u holda

$$\lambda'_j = \left(j + \frac{1}{2}\right)^2 \pi^2 + \int_0^1 q(x)dx - 2\rho_1 + o\left(\frac{1}{j}\right), \; j \to \infty \tag{28}$$

osimptotika o‘rinli. Agar (26) masalada chegaraviy shartlardan ikkinchisi $y(1) = 0$ bo‘lsa, u holda

$$\lambda'_j = \left(j + \frac{1}{2}\right)^2 \pi^2 + \int_0^1 q(x)dx + 2\rho_0 + o\left(\frac{1}{j}\right), \; j \to \infty \tag{29}$$

osimptotika o‘rinli.

Natija. (1)-(3) masalaning xos qiymatlari uchun ham

$$\lambda_j = j^2\pi^2 + o(j^2) \tag{30}$$

osimptotik formula o‘rinli bo‘ladi. Aniqroq osimptotik formulalar quyidagi teoremada olingan.

Teorema 9. 1) Agar (1)-(3) masalada $f(\lambda) \in R_N^0$ bo‘lsa, u holda quyidagi

$$\lambda_j = \begin{cases} (j - N)^2\pi^2 + \int_0^1 q(x)dx - 2b + 2ctg\alpha + o\left(\frac{1}{j}\right), & \alpha \ne 0 \\ \left(j + \frac{1}{2} - N\right)^2 \pi^2 + \int_0^1 q(x)dx - 2b + o\left(\frac{1}{j}\right), & \alpha = 0 \end{cases} \tag{31}$$

asimptotik formula o‘rinli bo‘ladi.

2) Agar (1)-(3) masalada $f(\lambda)\in R_N^+$ bo‘lsa, u holda quyidagi

$$\lambda_j=\begin{cases}\left(j-\frac{1}{2}-N\right)^2\pi^2+\int\limits_0^1 q(x)dx+2ctg\alpha+\frac{2}{a}+o\left(\frac{1}{j}\right), & \alpha\neq 0\\ (j-N)^2\pi^2+\int\limits_0^1 q(x)dx+\frac{2}{a}+o\left(\frac{1}{j}\right), & \alpha=0\end{cases} \tag{32}$$

asimptotik formula o‘rinli bo‘ladi.

Isbot. $N=0$ bo‘lgan holda bu teorema [5] ishda isbotlangan. Shuning uchun bu teorema $0,1,...,N$ bo‘lgan holda isbotlangan deb olib, $N+1$ uchun isbot qilamiz.

1) $\alpha>0$ va $f(\lambda)\in R_{N+1}^0$ bo‘lsin. U holda $F(\lambda)\in R_N^+$ bo‘ladi. Bu holda Krum-Kreyn almashtirishini qo‘llab, $\hat{q}(x)=q(x)-2z'$ potensialni va $\hat{\alpha}>0$ chegaraviy shartdagi o‘zgarmasni olamiz. Induksiya faraziga ko‘ra (32) formuladan

$$\lambda_{j+1}=(j-N)^2\pi^2+\int\limits_0^1 \hat{q}(x)dx+\frac{2}{\hat{a}}+o\left(\frac{1}{j}\right)$$

formula kelib chiqadi. Shuning uchun

$$\lambda_j=(j-(N+1))^2\pi^2+\int\limits_0^1 q(x)dx-2[f(\lambda_0)-ctg\alpha+\sum c_k]+o\left(\frac{1}{j}\right)=$$

$$=(j-(N+1))^2\pi^2+\int\limits_0^1 q(x)dx-2b+2ctg\alpha+o\left(\frac{1}{j}\right)$$

o‘rinli bo‘ladi. Bu esa (31) formula $N+1$ va $\alpha>0$ holda o‘rinli ekanini ko‘rsatadi. Endi $\alpha=0$ deb hisoblaymiz. Bu holda $\hat{\alpha}>0$ bo‘lgani uchun (32) ga asosan quyidagi

$$\lambda_j=\left(j-\frac{1}{2}-N\right)^2\pi^2+\int\limits_0^1 \hat{q}(x)dx+2ctg\hat{\alpha}+\frac{2}{\hat{a}}+o\left(\frac{1}{j}\right)=$$

$$=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int\limits_0^1 q(x)dx-2[f(\mu)-z(0)+z(0)+\sum c_k]+o\left(\frac{1}{j}\right)=$$

$$=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx-2b+o\left(\frac{1}{j}\right)$$

o‘rinli bo‘ladi. Bu esa (31) formula $N+1$ va $\alpha=0$ holda ham o‘rinli ekanligini ko‘rsatadi.

2) $f(\lambda)\in R_{N+1}^{+}$ holini ko‘rib chiqamiz. Bu holda $F(\lambda)\in R_{N+1}^{0}$ bo‘ladi. Shuning uchun isbotlanayotgan teoremaning birinchi bandidagi natijalardan foydalanamiz. $\alpha>0$ bo‘lganda $z=0$ bo‘ladi. (31) formulaga asosan

$$\lambda_{j+1}=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 \hat{q}(x)dx-2\hat{b}+o\left(\frac{1}{j}\right)$$

tenglik o‘rinli bo‘ladi. Bunga ko‘ra

$$\lambda_j=\left(j-\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx-2[f(\lambda_0)-ctg\alpha-(a^{-1}+f(\lambda_0))]+o\left(\frac{1}{j}\right)=$$

$$=\left(j-\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx+2ctg\alpha+\frac{2}{a}+o\left(\frac{1}{j}\right).$$

Bu esa (32) tenglik $N+1$ va $\alpha>0$ bo‘lgan holda ham o‘rinli ekanini ko‘rsatadi.

Nihoyat, $\alpha=0$ bajarilganda (31) tenglikni ishlatsak

$$\lambda_j=(j-(N+1))^2\pi^2+\int_0^1 \hat{q}(x)dx-a\hat{b}+2ctg\hat{\alpha}+o\left(\frac{1}{j}\right)=$$

$$=(j-(N+1))^2\pi^2+\int_0^1 q(x)dx-2[f(\mu)+z(0)-(a^{-1}+f(\mu)+z(0))]+o\left(\frac{1}{j}\right)=$$

$$=(j-(N+1))^2\pi^2+\int_0^1 q(x)dx+\frac{2}{a}+o\left(\frac{1}{j}\right).$$

kelib chiqadi. **Teorema isbotlandi.**

Xulosa

Mazkur ishda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun xos qiymatlarning asimptotikasi va xos funksiyalarning ossillyatsiya xossalari o‘rganilgan. Shu maqsadda Herlots-Nevanlinna funksiyalar sinfi hamda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun Krum-Kreyn almashtirishi tahlil qilingan.

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