O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

FIZIKA-MATEMATIKA FAKULTETI

Samandarova Nafosat Qodirberdiyevnaning

5130200 – Amaliy matematika va informatika ta’lim yo‘nalishi bo‘yicha bakalavr darajasini olish uchun

BITIRUV MALAKAVIY ISHI

Mavzu: Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyasini spektral berilganlar bo‘yicha tiklash

Ilmiy rahbar: **f.-m.f.d., dots. Yaxshimuratov A.B.**

Urganch 2016-yil

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«AMALIY MATEMATIKA» KAFEDRASI

Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyasini spektral berilganlar bo‘yicha tiklash

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Amaliy matematika kafedrasi

BITIRUV MALAKAVIY ISHNI BAJARISH BOʻYICHA

TOPSHIRIQLAR REJASI:

1. Talaba **Samandarova Nafosat Qodirberdiyevnaga** Universitet rektorining № 199-T sonli buyrugʻi bilan bitiruv malakaviy ish bajarish uchun **“Chegaraviy sharti spektral parametrga ratsional bogʻliq boʻlgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyasini spektral berilganlar boʻyicha tiklash“** mavzusi tasdiqlangan.

2. Kafedra majlisining qaroriga binoan **f.-m.f.d., dots. Yaxshimuratov A.B.** bitiruv malakaviy ishini bajarishga rahbar qilib tayinlangan.

3. **Bitiruv malakaviy ishining tarkibiy tuzilmasi:** Bitiruv malakaviy ishi kirish, asosiy qism, xulosa va adabiyotlar boʻlimlaridan iborat.

4. **Bitiruv malakaviy ish uchun ma’lumotlar:** mavzu boʻyicha ma’lumotlar beruvchi adabiyotlar, mavzuga oid maqolalar va internet saytlaridan olinadi.

5. **Bitiruv malakaviy ishga ilova qilinmaydi.**

Urgansh Davlat universiteti “ Fizika-matematika” fakulteti “Amaliy matematika” kafedrasi “Amaliy matematika va informatika” bakalavr ta’lim yo‘nalishi

“Tasdiqlayman”

Fakultet dekani

__________dots. J.U.Xujamov

«___» ______________2016 y.

BITIRUV MALAKAVIY ISH BO‘YICHA TOPSHIRIQ

Talaba: **Samandarova Nafosat**

1. Ishning mavzusi: «Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyasini spektral berilganlar bo‘yicha tiklash» mavzusi universitet rektorining № 199-T sonli buyrug‘i bilan tasdiqlangan.

2. Ishni topshirish muddati: 28 may 2016 y.

3. Mavzu bo‘yisha dastlabki ma’lumotlar beruvchi adabiyotlar ro‘yxati:

1. Hasanov A.B. Shturm-Liuvill chegaraviy masalalari nazariyasiga kirish. I-qism Toshkent, “Fan” nashriyoti, 2011-yil, 500 bet.
2. Левитан Б.М. Обратные задачи Штурма-Лиувилля. - М.:Наука, 1984.- 240с.
3. Binding P.A., Browne P.G. Watson B.A. Recovery of the m-function from spektral data for generalized Sturm-Liouville problems,
4. www.math-net.ru

4. Ishning maqsadi: Chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyasini spektral berilganlar bo‘yicha tiklashni o‘rganish.

5. Maslahatchi: f.-m.f.d., dots. Yaxshimuratov A.B.

Bo'limlar	**Maslahatchi F.I.SH.**	**Imzo, sana**	
		Topshiriq berdi	**Topshiriq qabul qildi**
Kirish	Yaxshimuratov A.B.	07.01.2016	15.01.2016
Zaruriy ma'lumotlar	Yaxshimuratov A.B.	20.02.2016	05.03.2016
Veyl-Titchmarsh funksiyasi uchun Loran yoyilmasi	Yaxshimuratov A.B.	09.03.2016	24.03.2016
Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o'zgarmaslar orqali tasvirlash	Yaxshimuratov A.B.	26.03.2016	11.04.2016
$\alpha = 0$ bo'lgan holda Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o'zgarmaslar orqali tasvirlash	Yaxshimuratov A.B.	15.04.2016	02.05.2016
Teskari masala yechimining yagonaligi	Yaxshimuratov A.B.	11.05.2016	23.05.2016
Xulosa	Yaxshimuratov A.B.	24.05.2016	27.05.2016
Adabiyotlar	Yaxshimuratov A.B.	07.01.2016	15.01.2016

6. **Ishga taqriz yozuvchining**

F.I.SH., ilmiy darajasi, unvoni: __________ f.-m.f.n., dots. Atamuratov A. A.

7. **Ilmiy rahbar: _____________ f.-m.f.d., dots. Yaxshimuratov A.B.**

8. **BMI bajaruvchi talaba: _____________Samandarova N.Q.**

9. **Kafedra mudiri: _____________ f.-m.f.n., dots. Babajanov B.A.**

M U N D A R I J A

Kirish

Tadqiqot mavzusining dolzarbligi. Matematik fizikaning bir qator masalalari Shturm-Liuvill operatorining xos qiymatlarini va ortonormallangan xos funksiyalarini topishga keltiriladi. Jumladan, klassik matematik fizikaning asosiy tenglamalari hisoblangan tor tebranishi va issiqlik o‘tkazuvchanlik tenglamalarini Furye usuli bilan yechishda Shturm-Liuvill chegaraviy masalasining xos qiymatlarini, ortonormallangan xos funksiyalarini aniqlashga va ixtiyoriy funksiyani ular yordamida Furye qatoriga yoyishga to‘g‘ri keladi. Bu yo‘nalishdagi ilk natijalar D.Bernulli, J.Dalamber, L.Eyler, Liuvill va Shturmlar tomonidan olingan. Shturm-Liuvill operatori spektral nazariyasining asosiy g‘oyalari XX-asrda G.D.Birkgoff, G.Veyl, D.Gilbert, V.A.Steklov, E.Ch.Titchmarsh, N.Levinson, B.M.Levitan va boshqa olimlar tomonidan rivojlantirildi.

Spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasi koeffitsientini va chegaraviy shartlarni aniqlash masalasiga spektral analizning teskari masalasi deyiladi.

Teskari masalalar nazariyasining rivojiga muhim turtki bo‘lgan ilk natija 1929 yilda taniqli astronom V.A.Ambartsumyan tomonidan olingan: *agar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$y'(0) = 0, \; y'(\pi) = 0$$

Shturm-Liuvill chegaraviy masalasining xos qiymatlari $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ *bo‘lsa, u holda* $q(x) \equiv 0$ *bo‘ladi.*

Ambartsumyanning bu natijasi muhim ekanliliga birinchi bo‘lib, 1946 yilda Shved matematigi G.Borg e’tibor berdi, u Ambartsumyanning natijasi umumiy qoidadan istisno ekanligini ko‘rsatib berdi, ya’ni faqat bitta spektr orqali Shturm-Liuvill chegaraviy masalasini bir qiymatli tiklab bo‘lmasligini ko‘rsatdi. G.Borg, faqat bitta chegaraviy sharti bilan farq qiluvchi ikkita Shturm-Liuvill masalasining spektrlari ma’lum bo‘lsa, bu spektrlar orqali dastlabki masalalar bir qiymatli tiklanishini isbot qildi. Hozirda bu tasdiq, G.Borgning yagonalik teoremasi deb yuritiladi.

1949 yilda A.N.Tixonov yarim o‘qda berilgan Shturm-Liuvill operatorini $I(\lambda)$ “impedans” funksiyasi (ya’ni Veyl-Titchmarshning $m(\lambda)$ funksiyasi) yordamida yagona tarzda qurish mumkinligi haqidagi teoremani isbotlashga muvaffaq bo‘lgan. A.N.Tixonovning bu teoremasi, yer ichki qatlamlari elektrik hossalarini o‘rganish masalalarini matematik asoslashda muhim ahamiyatga ega. Veyl-Titchmarshning $m(\lambda)$ funksiyasi bo‘yicha chiziqli oddiy differensial operatorni qurish algoritmi V.A.Yurko tomonidan batafsil o‘rganilgan.

1950 yilda V.A.Marchenko Shturm-Liuvill operatori o‘zining spektral funksiyasi orqali yagona aniqlanishini ko‘rsatib bergan. V.A.Marchenko teoremasidan, hususan, λ_n xos qiymatlar va α_n normallovchi o‘zgarmaslar ketma-ketligi yordamida Shturm-Liuvill chegaraviy masalasi bir qiymatli aniqlashi kelib chiqadi.

Hozirgi kunda teskari masalalar nochiziqli evolyutsion tenglamalar uchun qo‘yilgan Koshi masalasini yechishga tatbiq qilinayotganligi tufayli teskari spektral masalalar sohasiga bo‘lgan qiziqish ortib bormoqda. Jumladan, chegaraviy sharti spektral parametrga bog`liq bo‘lgan Shturm-Liuvill operatorlari uchun teskari masalalarni o‘rganish dolzarb masalalardan biridir.

Ilmiy izlanishning maqsad va vazifalari. Mazkur ishning asosiy maqsadi - chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun umumlashgan normallovchi o‘zgarmaslarni ta’riflash va Veyl-Titmarsh funksiyasini spektral berilganlar orqali ifodalashni o‘rganish. Veyl-Titmarsh funksiyasi uchun Loran yoyilmasini o‘rganish, Veyl-Titmarsh funksiyasini xos qiymatlar va umumlashgan normallovchi o‘zgarmaslar orqali ifodalovchi formulalarga asoslanib, teskari masala uchun yagonalik teoremasini isbotlash.

Mavzuning o‘rganilish darajasining qiyosiy tahlili. G.Borg maqolasidan keyin, Shtrum-Liuvill operatorini spektral funksiya bo‘yicha, Veyl-Titchmarsh funksiyasi bo‘yicha, spektr va normollovchi o‘zgarmaslar bo‘yicha tiklash masalalari o‘rganilgan. Bu yo‘nalishdagi tadqiqotlar N.Levinson, M.G.Kreyn, B.M.Levitan, B.A.Marchenko, B.A.Sadovnichiy, B.A.Yurko va boshqalar tomonidan olib borilgan.

Chegaraviy sharti spektral parametrga bog`liq bo‘lgan Shtrum-Liuvill masalalari bir qancha ishlarda, masalan: [1, 2, 4, 7, 8, 9, 14, 16, 18, 22, 30, 31, 32, 33] ishlarda o‘rganilgan. Bu turdagi chegaraviy masalaning matematik fizika masalalariga va injeneriyaga tatbiqlari [9] ishda keltirilgan. Mazkur bitiruv malakaviy ishda spektral berilganlar orqali Veyl-Titchmarsh funksiyasi uchun formula keltirib chiqariladi.

Tadqiqotning ilmiy yangiligi. Mazkur ishda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun umumlashgan normallovchi o‘zgarmaslarni ta’riflangan va Veyl-Titmarsh funksiyasini spektral berilganlar orqali ifodalovchi formulalar olingan. Veyl-Titmarsh funksiyasini xos qiymatlar va umumlashgan normallovchi o‘zgarmaslar orqali ifodalovchi formulalarga asoslanib, teskari masala uchun yagonalik teoremasi isbotlangan.

Tadqiqot predmeti va ob’yekti. Tadqiqot obyekti chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi bo‘lib, uning predmeti teskari spektral masalani o‘rganishdir.

Tadqiqotning ilmiy ahamiyati. Tadqiqotning ilmiy ahamiyati olingan natijalar matematik fizikaning nochiziqli evolyutsion tenglamalarini yechishda qo‘llanilishi bilan asoslanadi.

1-§. Zaruriy ma'lumotlar

Quyidagi Shturm-Liuvill masalasini ko'rib chiqamiz:

$$l(y) \equiv -y'' + q(x)y = \lambda y, \quad 0 < x < 1 \tag{1}$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \tag{2}$$

$$\frac{y'(1)}{y(1)} = f(\lambda). \tag{3}$$

Bu yerda $q(x)$ haqiqiy differesialanuvchi funksiya bo'lib,

$$f(\lambda) = \frac{h(\lambda)}{g(\lambda)},$$

$h(\lambda)$ va $g(\lambda)$ haqiqiy koeffitsientli umumiy ildizga ega bo'lmagan, ko'phadlardir.

Agar λ ning biror qiymatida $f(\lambda) = \infty$ bo'lsa, (3) chegaraviy shart $y(1) = 0$ bilan almashtiriladi. Bu turdagi chegaraviy masalalar bir qancha ishlarda, masalan: [1, 2, 4, 7, 8, 9, 14, 16, 18, 22, 30, 31, 32, 33] ishlarda o'rganilgan. Bu turdagi chegaraviy masalaning matematik fizika masalalariga va injeneriyaga tatbiqlari [9] ishda keltirilgan.

Bu ishdagi asosiy maqsad, (1)-(3) masalaning spektrial berilganlari orqali Veyl-Titchmarsh funksiyasi uchun formula keltirib chiqarishdir.

$\upsilon(x,\lambda)$ orqali (1) tenglamaning

$$\upsilon(1,\lambda) = g(\lambda), \quad \upsilon'(1,\lambda) = h(\lambda)$$

shartlarni qanoatlantiruvchi noldan farqli yechimini belgilaymiz.

Ta'rif 1. Ushbu

$$m(\lambda) = \frac{\upsilon'(0,\lambda)\cos\alpha + \upsilon(0,\lambda)\sin\alpha}{\upsilon'(0,\lambda)\sin\alpha - \upsilon(0,\lambda)\cos\alpha} \tag{4}$$

funksiyaga Veyl-Titchmarsh funksiyasi deyiladi.

Veyl-Titchmarsh funksiyasining qutb maxsus nuqtalari bilan (1)-(3) masalaning xos qiymatlari ustma-ust tushadi. Bundan tashqari, qutb nuqtaning tartibi bilan xos qiymatning karrasi ustma-ust tushadi. Bu holda spektrial berilganlar xos qiymatlardan va umumlashgan normallovchi o'zgarmaslardan iborat bo‘ladi. Chegaraviy shartda λ qatnashmagan holda normallovchi o‘zgarmaslar quyidagicha aniqlanadi

$$\alpha_n^0 = \int_0^1 \frac{\upsilon^2(x,\lambda_n)}{\upsilon^2(0,\lambda_n)} dx, \quad \alpha \neq 0.$$

Bu holda xos qiymatlar haqiqiy bo‘lmasligi ham mumkin. Bundan tashqari karrali bo‘lishi mumkin. Bu holda umumlashgan normallovchi o‘zgarmaslarni ta’riflash masalasi kelib chiqadi.

Agar $h(\lambda)$ ko‘phadning darajasi $g(\lambda)$ ko‘phadning darajasidan oshmasa, $g(\lambda)$ ko‘phadning darajasini M orqali belgilaymiz va $g(\lambda)$ ko‘phadning ildizlari haqiqiy va karrasiz deb hisoblaymiz.

Agar $h(\lambda)$ ko‘phadning darajasi $g(\lambda)$ ko‘phadning darajasidan katta bo‘lsa, u holda $h(\lambda)$ ko‘phadning darajasini M orqali belgilaymiz va $h(\lambda)$ ko‘phad ildizilari haqiqiy va karrasiz deb hisoblaymiz.

Quyidagi funksiyani kiritib olamiz

$$D(\lambda) = \upsilon'(0,\lambda)\sin\alpha - \upsilon(0,\lambda)\cos\alpha.$$

λ_n xos qiymatlar $D(\lambda)$ funksiyaning nollari bilan ustma-ust tushadi. Bu xos qiymatlarni shunday nomerlaymizki, bunda ularning haqiqiy qismlari kamaymaydigan bo‘lsin:

$$\mathrm{Re}(\lambda_n) \leq \mathrm{Re}(\lambda_{n+1}).$$

Qisqalik uchun (1)-(3) masalani (α, f, q) orqali belgilaymiz.

$h(\lambda)$ va $g(\lambda)$ ko‘phadlarning xossalaridan $f(\lambda)$ uchun $\deg(h(\lambda)) \leq \deg(g(\lambda))$ bo‘lgan holda

$$f(\lambda) = b - \sum_{k=1}^{M} \frac{b_k}{\lambda - c_k} \tag{5}$$

kelib chiqadi. Agar $\deg(h(\lambda)) \geq \deg(g(\lambda))$ bo‘lsa, u holda

$$\frac{1}{f(\lambda)} = b - \sum_{k=1}^{M} \frac{b_k}{\lambda - c_k} \tag{6}$$

o‘rinli bo‘ladi. Bu yerda b, b_k, c_k sonlar haqiqiy.

$H = L^2[0,1] \oplus C^M$ fazoda quyidagi bichiziqli formani kiritib olamiz:

$$< Y, Z >= \int_0^1 y\bar{z}dx + \sum_{k=1}^{M} \frac{y_k \bar{z}_k}{b_k}. \qquad (7)$$

Bu yerda

$$Y = \begin{pmatrix} y(x) \\ y_1 \\ \dots \\ y_M \end{pmatrix}, \qquad Z = \begin{pmatrix} z(x) \\ z_1 \\ \dots \\ z_M \end{pmatrix}.$$

Bu belgilashlardan so‘ng (1)-(3) masala H Pontryagin fazosida quyidagi

$$LY = \begin{pmatrix} l(y) \\ c_1 y_1 - b_1 y(1) \\ \dots \\ c_M y_M - b_M y(1) \end{pmatrix} \qquad (8)$$

operatorning xos qiymatlarini topish masalasiga keltiriladi. Bu operatorning aniqlanish sohasi quyidagicha

$$D(L) = \left\{ Y \in H : y, y' \in AC, l(y) \in L^2, \, y(0)\cos\alpha = y'(0)\sin\alpha, y'(1) - by(1) = \sum_{k=1}^{M} y_k \right\}.$$

λ son (1)-(3) masalaning xos qiymati va $y(x)$ unga mos keluvchi xos funksiya bo‘lishi uchun λ son L operatorning xos qiymati bo‘lib, unga

$$Y = \begin{pmatrix} y(x) \\ \dfrac{b_1}{c_1 - \lambda} y(1) \\ \vdots \\ \dfrac{b_M}{c_M - \lambda} y(1) \end{pmatrix}$$

xos funksiya mos kelishi zarur va yetarlidir. Bundan $\lambda \neq c_j$, $j = 1, ..., M$ kelib chiqadi.

H Pontryagin fazosida L operator o‘z-o‘ziga qo‘shma, quyidan chegaralangan bo‘lib, uning rezolventasi kompakt operator bo‘ladi. Bunga ko‘ra L operatorning spektri diskret bo‘ladi. Hamda L operator $+\infty$ ga intiladigan xos qiymatlarga ega bo‘lib, ulardan faqat cheklitasi kompleks bo‘lishi mumkin. Bu fikrlar [8] ishda to‘liq isbot qilingan.

$\lambda \neq c_j$, $j = 1,...,M$ bo‘lganida

$$V(x,\lambda) = \begin{pmatrix} \upsilon(x,\lambda) \\ \dfrac{b_1}{c_1 - \lambda}\upsilon(1,\lambda) \\ \vdots \\ \dfrac{b_M}{c_M - \lambda}\upsilon(1,\lambda) \end{pmatrix}$$

vektor-funksiyani kiritib olamiz.

$\lambda = c_j$, bo‘lgan holda esa

$$V(x,c_j) = \begin{pmatrix} \upsilon(x,c_j) \\ 0 \\ \vdots \\ 0 \\ \upsilon'(1,c_j) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \upsilon(x,c_j) \\ 0 \\ \vdots \\ 0 \\ -b_j g'(c_j) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

vektor-funksiya kiritib olamiz.

Bu funksiyalar yordamida esa quyidagi vector-funksiyani tuzib olamiz:

$$U(x,\lambda) = \begin{cases} \sin\alpha \dfrac{V(x,\lambda)}{\upsilon(0,\lambda)}, & \alpha \neq 0 \\ \dfrac{V(x,\lambda)}{\upsilon'(0,\lambda)}, & \alpha = 0 \end{cases}$$

$U(x,\lambda)$ vektor funksiya $\alpha \neq 0$ bo‘lgan holda $\upsilon(0,\lambda) = 0$ shartni qanoatlantiradigan λ nuqtada qutb maxsus nuqtaga ega $\alpha = 0$ bo‘lgan holda esa $\upsilon'(0,\lambda) = 0$ tenglamani qanoatlantiradigan λ nuqtalarda qutb maxsus nuqtaga ega. Shuning uchun bunday nuqtalarda $U(x,\lambda)$ aniqlanmagan.

$\{\mu_n\}_{n=0}^{\infty}$ orqali (α, f, q) chegaraviy masalaning xos qiymatlarini belgilaymiz. m_n orqali μ_n xos qiymatning karrasini begilaymiz.

n ning katta qiymatlarida $\mu_n = \lambda_{n+m}$ bo‘ladi, bu yerda

$$m = \sum_{k=0}^{\infty} (m_k - 1). \tag{9}$$

Bu yerda $0 \le m < \infty$, ekanligi L operatorning H Pontryagin fazosida o‘z-o‘ziga qo‘shma bo‘lishidan kelib chiqadi ([30]).

μ_n xos qiymat uchun quyidagi vektor-funksiyalarni kiritib olamiz ([28]):

$$U_n^{(0)}(x) = U(x, \mu_n),$$

$$U_n^{(j)}(x) = \left.\frac{d^j U(x,\mu)}{d\mu^j}\right|_{\mu=\mu_n}, \qquad j = 1,\dots, m_n - 1.$$

μ_n xos qiymatning geometrik karrasi 1 ga teng. $U_n^{(j)}(x)$ vektor-funksiyaning birinchi komponentasini $u_n^{(j)}(x)$ orqali belgilaymiz. Xuddi shunday $V^{(j)}(x,\lambda)$ vektor funksiyaning birinchi komponentasini $\upsilon^{(j)}(x,\lambda)$ orqali belgilaymiz. $u(x,\lambda)$ orqali (1) tenglamaning $\alpha \ne 0$ bo‘lganda $u(0,\lambda) = \sin\alpha$ shartni qanoatlantiruvchi biror yechimini belgilaymiz. Bu holda

$$U^{(j)}(0,\lambda) = \frac{d^j U(0,\lambda)}{d\lambda^j} = 0, \qquad j \ge 1$$

bo‘ladi. $\lambda = \mu_n$ son ushbu

$$u'(0,\lambda)\sin\alpha - u(0,\lambda)\cos\alpha$$

funksiyaning m_n karrali noli bo‘ladi. Shuning uchun

$$U^{(j)}(0,\mu_n) = 0, \qquad j = 1,\dots, m_n - 1,$$

$$U^{(m_n)}(0,\mu_n) \ne 0$$

bo‘ladi.

$\alpha = 0$ bo‘lgan holda $U(x,\lambda)$ yechim $U'(x,\lambda) = 1$ shartni qanoatlantiruvchi yechim bo‘lib,

$$U'^{(j)}(0,\lambda) = 0, \qquad j \ge 1,$$

$$U^{(j)}(0,\mu_n)=0, \qquad j=0,1,\ldots,m_n-1,$$
$$U^{(m_n)}(0,\mu_n)\neq 0.$$

Bundan tashqari

$$\widetilde{L}V(x,\lambda)=\lambda V(x,\lambda)$$

va

$$\widetilde{L}U(x,\lambda)=\lambda U(x,\lambda)$$

bo‘ladi. Bu yerda $\widetilde{L}$ orqali L operatorning ushbu

$$D(\widetilde{L})=\{Y\in H: y,y'\in AC, l(y)\in L^2[0,1]\}$$

sohadagi kengaytmasi belgilangan.

Endi biz umumlashgan normallovchi o‘zgarmaslarni kiritamiz. Buning uchun ushbu

$$[Y,Z]=<Y,\overline{Z}>=\int_0^1 y(x)z(x)dx+\sum_{k=1}^{M}\frac{1}{b_k}y_k z_k$$

belgilashdan foydalanamiz.

Ta’rif 2. Ushbu

$$\alpha_n^{(j)}=\left[U_n^{(j)},U_n^{(m_n-1)}\right], \qquad j=0,1,\ldots,m_n-1$$

sonlarga μ_n xos qiymatlarga mos keluvchi umumlashgan normallovchi o‘zgarmaslar deyiladi.

Izoh. Agar μ_n kompleks xos qiymat bo‘lsa, u holda

$$\overline{\upsilon(x,\mu_n)}=\upsilon(x,\overline{\mu}_n),$$
$$\overline{U^{(j)}(x,\mu_n)}=U^{(j)}(x,\overline{\mu}_n)$$

bo‘ladi.

2-§. Veyl-Titchmarsh funksiyasi uchun Loran yoyilmasi

Lemma1. a nuqta $F(\lambda)$ funksiyaning N karrali noli bo‘lib, $F(\lambda)$ funksiya bu nuqtaning atrofida golomorf funksiya bo‘lsin. U holda a nuqtaning biror atrofida quyidagi tenglik o‘rinli bo‘ladi:

$$\frac{1}{F(\lambda)} = E(\lambda) + \sum_{k=1}^{N} \frac{P_k}{(\lambda - a)^k}.$$

Bu yerda $E(\lambda)$ funksiya $\lambda = a$ nuqtaning biror atrofida golomorf bo‘lib,

$$P_n = \frac{1}{a_N},$$

$$P_{N-j} = -\frac{1}{a_N}\sum_{k=0}^{j-1} P_{N-k}a_{N+j-k}, \qquad j = 1,\dots,N-1,$$

$$a_k = \frac{1}{k!}\frac{d^k F(a)}{d\lambda^k}.$$

Teorema1. Quyidagi tenglik o‘rinli

$$m_n \alpha_n^{(j)} C_{m_n+j}^{m_n} = \begin{cases} U'^{(m_n+j)}(0,\mu_n)\sin\alpha, & \alpha \neq 0 \\ -U^{(m_n+j)}(0,\mu_n), & \alpha = 0. \end{cases}$$

Isbot. Quyidagi hisoblashlarni bajaramiz:

$$(\lambda-\mu)[V(x,\lambda),V(x,\mu)] = [\lambda V(x,\lambda),V(x,\mu)] - [V(x,\lambda),\mu V(x,\mu)] =$$

$$= \left[\tilde{L}V(x,\lambda),V(x,\mu)\right] - \left[V(x,\lambda),\tilde{L}V(x,\mu)\right] =$$

$$= [\upsilon(x,\lambda),\upsilon'(x,\mu) - \upsilon'(x,\lambda),\upsilon(x,\mu)]\Big|_{x=0}^{x=1} +$$

$$+\sum_{k=1}^{M}\frac{\upsilon(1,\lambda)\upsilon(1,\mu)}{b_k}\left[\left(\frac{c_kb_k}{c_k-\lambda}-b_k\right)\frac{b_k}{c_k-\mu} - \left(\frac{c_kb_k}{c_k-\mu}-b_k\right)\frac{b_k}{c_k-\lambda}\right] =$$

$$= [\upsilon(1,\lambda)\upsilon'(1,\mu) - \upsilon'(1,\lambda)\upsilon(1,\mu)] - [\upsilon(0,\lambda)\upsilon'(0,\mu) - \upsilon'(0,\lambda)\upsilon(0,\mu)] +$$

$$+(\lambda-\mu)\upsilon(1,\lambda)\upsilon(1,\mu)\sum_{k=1}^{M}\frac{b_k}{(c_k-\lambda)(c_k-\mu)} =$$

$$= \upsilon(1,\lambda)\upsilon(1,\mu)[f(\mu) - f(\lambda)] - [\upsilon(0,\lambda)\upsilon'(0,\mu) - \upsilon'(0,\lambda)\upsilon(0,\mu)] +$$

$$+\upsilon(1,\lambda)\upsilon(1,\mu)[f(\lambda) - f(\mu)] = \upsilon'(0,\lambda)\upsilon(0,\mu) - \upsilon(0,\lambda)\upsilon'(0,\mu).$$

Bu yerda $[X,Y]$ skalyar ko‘paytmaning ifodasi, $V(x,\lambda)$ vektor-funksiya ta’rifi hamda bo‘laklab integrallash qoidasi ishlatildi. Bunga ko‘ra

$$(\lambda-\mu)[V(x,\lambda),V(x,\mu)]=\upsilon'(0,\lambda)\upsilon(0,\mu)-\upsilon(0,\lambda)\upsilon'(0,\mu) \qquad (10)$$

bo‘ladi. Bu tenglik $\lambda\neq c_j$ va $\mu\neq c_j$ bo‘lgan holda keltirib chiqarildi. Ammo (10) tenglikdagi funksiyalarning uzluksizligidan (10) tenglik $\lambda=c_j$ va $\mu=c_j$ bo‘lgan holda ham o‘rinli bo‘lishi kelib chiqadi.

$\alpha\neq 0$ holini ko‘rib chiqamiz. Bu holda (10) tenglikni $\upsilon(0,\mu)\upsilon(0,\lambda)$ ga bo‘lib va $\sin\alpha$ ga ko‘paytirib quyidagi tenglikni olamiz:

$$\frac{(\lambda-\mu)}{\sin\alpha}[U(x,\lambda),U(x,\mu)]=u'(0,\lambda)-u'(0,\mu).$$

Bu tenglikdan λ bo‘yicha j-tartibli hosila olamiz

$$(\lambda-\mu)\left[U^{(j)}(x,\lambda),U(x,\mu)\right]+j\left[U^{(j-1)}(x,\lambda),U(x,\mu)\right]=\sin\alpha u'^{(j)}(0,\lambda). \qquad (11)$$

Bu yerda $\lambda=\mu_n$, $\mu=\mu_n$ desak, quyidagi tenglik kelib chiqadi.

$$j\left[U_n^{(j-1)}(x),U_n^{(0)}(x)\right]=\sin\alpha u'^{(j)}(0,\mu_n). \qquad (12)$$

Xususan, $j=m_n$ bo‘lganida quyidagi tenglikni olamiz

$$\left[U_n^{(m_n-1)}(x),U_n^{(0)}(x)\right]=\frac{\sin\alpha}{m_n}u'^{(m_n)}(0,\mu_n). \qquad (13)$$

(11) ayniyatdan μ bo‘yicha k marta hosila olamiz

$$(\mu-\lambda)\left[U^{(j)}(x,\lambda),U^{(k)}(x,\mu)\right]+k\left[U^{(j)}(x,\lambda),U^{(k-1)}(x,\mu)\right]=j\left[U^{(j-1)}(x,\lambda),U^{(k)}(x,\mu)\right].$$

Bu yerda $\lambda=\mu_n$ va $\mu=\mu_n$ desak, quyidagi tenglik kelib chiqadi.

$$k\left[U_n^{(j)}(x),U_n^{(k-1)}(x)\right]=j\left[U_n^{(j-1)}(x),U_n^{(k)}(x)\right].$$

Bundan foydalanib, quyidagi tenglikni keltirib chiqaramiz:

$$C_{m_n+k-1}^{k}\left[U_n^{(m_n-1)},U_n^{(k)}\right]=\left[U_n^{(m_n+k-1)},U_n^{(0)}\right] \qquad (14)$$

(12), (13) va (14) tengliklarga ko‘ra

$$\sin\alpha u'^{(j+m_n)}(0,\mu)=(m_n+j)\left[U_n^{(j+m_n-1)},U_n^{(0)}\right]=$$

$$=(m_n+j)\frac{(m_n+j-1)!}{(m_n-1)!\,j!}\left[U_n^{(m_n-1)},U_n^{(j)}\right]=m_n\frac{(m_n+j)!}{m_n!\,j!}\alpha_n^{(j)}.$$

bo‘ladi.

Endi $\alpha = 0$ holni ko‘rib chiqamiz. (10) tenglikni $\upsilon'(0,\mu)\upsilon'(0,\lambda)$ ga bo‘lib quyidagi ayniyatga ega bo‘lamiz:

$$(\lambda-\mu)[U(x,\lambda),U(x,\mu)] = u(0,\mu) - u(0,\lambda).$$

Bu tenglikka yuqoridagi usulni qo‘llasak,

$$-u^{(j+m_n)}(0,\mu_n) = m_n \frac{(m_n + j)!}{m_n!\, j!}\alpha_n^{(j)}$$

formula kelib chiqadi. **Teorema isbotlandi.**

Lemma1 va teorema1 ga asosan $m(\lambda)$ Veyl-Titchmarsh funksiyasi uchun yozilgan Loran yoyilmasining singulyar qismini normallovchi sonlar orqali topishimiz mumkin.

Natija. $m(\lambda)$ Veyl-Titchmarsh funksiyasi μ_n xos qiymat atrofida quyidagi Loran yoyilmasiga ega

$$m(\lambda) = E_n(\lambda) + \sum_{k=1}^{m_n} \frac{p_n^{(k)}}{(\lambda-\mu_n)^k}. \tag{15}$$

Bu yerda $E_n(\lambda)$ funksiya μ_n xos qiymatning biror atrofida analitik funksiya bo‘lib,

$$p_n^{(m_n)} = \frac{(m_n - 1)!}{\alpha_n^{(0)}}, \tag{16}$$

$$p_n^{(m_n - j)} = -\sum_{k=0}^{j-1} p_n^{(m_n-k)} \frac{\alpha_n^{(j-k)}}{(j-k)!\,\alpha_n^{(0)}}, \qquad j = 1,\dots,m_n - 1. \tag{17}$$

Isbot. $\alpha \neq 0$ bo‘lsin. Veyl-Titchmarsh funksiyasini quyidagi tarzda yozib olamiz:

$$m(\lambda) = ctg\alpha + \frac{1}{Z(\lambda)}.$$

Bu yerda

$$Z(\lambda) = \sin\alpha\, u'(0,\lambda) - \cos\alpha \sin\alpha.$$

1-lemmani tatbiq qilsak, quyidagilar kelib chiqadi

$$m(\lambda)=E_n(\lambda)+\sum_{k=1}^{m_n}\frac{p_n^{(k)}}{(\lambda-\mu_n)^k}.$$

Bu yerda $E_n(\lambda)$ funksiya μ_n xos qiymatning biror atrofida analitik funksiya.

$$p_n^{(m_n)}=\frac{1}{a_n^{(m_n)}}, \tag{18}$$

$$p_n^{(m_n-j)}=-\frac{1}{a_n^{(m_n)}}\sum_{k=0}^{j-1}p_n^{(m_n-k)}a_n^{(m_n+j-k)}, \qquad j=1,...,M-1, \tag{19}$$

$$a_n^{(k)}=\frac{1}{k!}\frac{d^kZ}{d\lambda^k}(\mu_n).$$

Ammo 1-teoremaga ko‘ra

$$a_n^{(m_n+j)}=\frac{\sin\alpha}{(j+m_n)!}u'^{(m_n+j)}(0,\mu_n)=\frac{\alpha_n^{(j)}}{(m_n-1)!\,j!}.$$

Buni hisobga olib, (18) va (19) dan (16) va (17) tengliklarni keltirib chiqaramiz.

$\alpha=0$ bo‘lganida quyidagi tenglik o‘rinli:

$$m(\lambda)=-\frac{1}{u(0,\lambda)}.$$

Agar 1-lemmani $Z(\lambda)=-u(0,\lambda)$ ga qo‘llasak quyidagi tenglikni olamiz:

$$a_n^{(m_n+j)}=\frac{-1}{(m_n+j)!}u^{(m_n+j)}(0,\mu_n)=\frac{\alpha_n^{(j)}}{(m_n-1)!\,j!}.$$

Shundan so‘ng $\alpha\neq0$ bo‘lgan holdagi usulni qo‘llasak, (15) tasvir hosil bo‘ladi.

3-§. Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o‘zgarmaslar orqali tasvirlash

Oldingi paragrafda Veyl-Titchmarsh funksiyasi uchun (15) tasvir olingan edi. Bu yerdagi ifodaning singulyar qismi xos qiymatlar va normallovchi o‘zgarmaslar orqali ifodalangan edi. Bu paragrafda (15) formuladagi $E_n(\lambda)$ funksiyani xos qiymatlar va normallovchi o‘zgarmaslar orqali ifodalash bilan shug’ullanamiz. Buning uchun quyidagi teoremalardan foydalanamiz.

Teorema2. Quyidagi asimptotik formulalar o‘rinli

$$m(\lambda) = ctg\alpha + O\left(\frac{1}{\sqrt{|\lambda|}}\right), \ \lambda \to -\infty, \qquad \alpha \neq 0, \tag{20}$$

$$m(\lambda) = \sqrt{|\lambda|} + O(1), \ \lambda \to -\infty, \qquad \alpha = 0. \tag{21}$$

n ning yetarlicha katta qiymatidan boshlab, barcha xos qiymatlar oddiy, ya’ni karrasiz bo‘ladi. Shuning uchun normallovchi o‘zgarmaslar asimptotikasi faqat oddiy xos qiymatlar bilan bog‘liq.

Teorema3. $q(x)$ absolyut uzluksiz funksiya bo‘lsin. U holda quyidagi asimptotik formulalar o‘rinli

$$\alpha_n^{(0)} = \frac{1}{2}\sin^2\alpha + O\left(\frac{1}{n^2}\right), \qquad \alpha \neq 0, \tag{22}$$

$$\alpha_n^{(0)} = \frac{1}{2\mu_n} + O\left(\frac{1}{n^4}\right), \qquad \alpha = 0. \tag{23}$$

Isbot. n ning yetarlicha katta qiymatlarida $m_n = 1$ bo‘ladi.

$$\alpha_n^{(0)} = \left[U_n^{(0)}, U_n^{(0)}\right] = \int_0^1 u_n^{(2)} dx + \sum_{k=1}^{M}\left(\frac{b_k}{c_k - \mu_n}\right)^2 \frac{u_n^{(2)}(1)}{b_k} = \int_0^1 u_n^{(2)} dx + f'(\mu_n) u_n^{(2)}(1). \tag{24}$$

[9] ishga ko‘ra

$$u_n^{(2)}(x,\lambda) = \frac{2\cos\sqrt{\mu_n}x}{\sqrt{\mu_n}}\{\sin\alpha\cos\alpha\sin\sqrt{\mu_n}x +$$

$$+\sin^2\alpha\int_0^x q(t)\cos\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dt\Bigg\}+$$

$$+\sin^2\alpha\cos^2\sqrt{\mu_n}x+O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\mu_n}x\right|}}{\mu_n}\right),\quad \alpha\neq 0,$$

$$u_n^{(2)}(x,\lambda)=\frac{\sin^2\sqrt{\mu_n}x}{\sqrt{\mu_n}}+$$

$$+\frac{2\sin\sqrt{\mu_n}x}{\mu_n\sqrt{\mu_n}}\int_0^x q(t)\sin\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dt+O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\mu_n}x\right|}}{\mu_n^2}\right),\quad \alpha=0. \quad (25)$$

Endi [9] ishdagi xos qiymatlarning asimptotikasidan foydalanamiz:

$$\sqrt{\mu_n}=(m+n-M)\pi+O\left(\frac{1}{n}\right),\quad \alpha\neq 0, \quad (26)$$

$$\sqrt{\mu_n}=\left(m+n+\frac{1}{2}-M\right)\pi+O\left(\frac{1}{n}\right),\quad \alpha=0. \quad (27)$$

(26) va (27) ifodalarni (25) tengliklarga qo‘ysak,

$$u_n^{(2)}(1)=\sin^2\alpha+O\left(\frac{1}{n}\right),\quad \alpha\neq 0$$

$$u_n^{(2)}(1)=\frac{1}{\mu_n}+O\left(\frac{1}{n^3}\right),\quad \alpha=0$$

bo‘ladi. (26) va (27) asimptotikalardan yanada foydalanamiz:

$$\frac{1}{\mu_n}\int_0^1 2\cos\sqrt{\mu_n}x\sin\sqrt{\mu_n}xdx=\frac{1}{\sqrt{\mu_n}}\int_0^1\sin 2\sqrt{\mu_n}xdx=\frac{1-\cos 2\sqrt{\mu_n}}{2\mu_n}=O\left(\frac{1}{n^2}\right).$$

Yuqoridagilarni hisobga olsak, quyidagi tengliklar kelib chiqadi:

$$\int_0^1\frac{2\cos\sqrt{\mu_n}x}{\sqrt{\mu_n}}\int_0^x q(t)\cos\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dtdx=$$

$$=\int_0^1 2q(t)\cos\sqrt{\mu_n}t\int_t^1\frac{\cos\sqrt{\mu_n}x\sin\sqrt{\mu_n}(x-t)}{\sqrt{\mu_n}}dxdt=$$

$$=\int_0^1\frac{q(t)\cos\sqrt{\mu_n}t}{\sqrt{\mu_n}}\int_t^1\left[\sin\sqrt{\mu_n}(2x-t)-\sin\sqrt{\mu_n}t\right]dxdt=$$

$$= -\frac{1}{2\sqrt{\mu_n}}\int_0^1 (1-t)q(t)\sin 2\sqrt{\mu_n}\,t\,dt + O\left(\frac{1}{\mu_n}\right) =$$

$$= -\frac{1}{4\mu_n}\left[q(0) + \int_0^1 \frac{d[(1-t)q(t)]}{dt}\cos 2\sqrt{\mu_n}\,t\,dt\right] + O\left(\frac{1}{\mu_n}\right) = O\left(\frac{1}{\mu_n}\right).$$

Xuddi shunday quyidagi tenglikni olamiz:

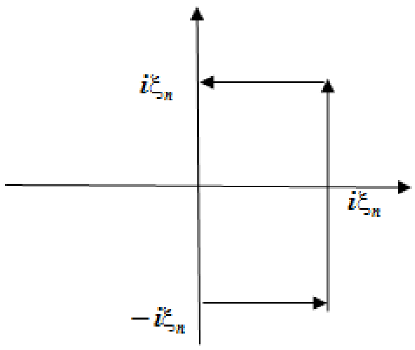
$$\int_0^1 \frac{2\sin\sqrt{\mu_n}\,x}{\sqrt{\mu_n}}\int_0^x q(t)\sin\sqrt{\mu_n}\,t\sin\sqrt{\mu_n}\,(x-t)\,dt\,dx = O\left(\frac{1}{\mu_n}\right).$$

Bulardan (22) va (23) formulalar kelib chiqadi. **Teorema isbotlandi.**

Lemma 2. Agar λ son (1)-(3) masalaning xos qiymati bo‘lmasa, u holda quyidagi tenglik o‘rinli bo‘ladi.

$$m(\lambda) = m(0) + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} p_n^{(j)}\left[\frac{1}{(\lambda-\mu_n)} - \frac{1}{(-\mu_n)^j}\right]. \tag{28}$$

Isbot. $\xi_n = \left(n + \frac{1}{4}\right)\pi$ bo‘lsin. γ_n orqali quyidagi konturni belgilaymiz:



(1-rasm)

$\Gamma_n = \{\xi^2 : \xi \in \gamma_n\}$ bo‘lsin. n ning yetarlicha katta qiymatlarida

$$m(\mu) = O(n), \qquad \mu \in \Gamma$$

bo‘ladi. Agar

$$H(\mu) = \frac{m(\mu)}{\mu(\mu-\lambda)}$$

funksiyani kiritib olsak, u holda

$$H(\mu) = O\left(\frac{1}{n^3}\right), \qquad \mu \in \Gamma_n$$

bo‘ladi va bundan

$$\lim_{n\to\infty}\int_{\Gamma_n} H(\mu)d\mu = 0$$

kelib chiqadi. Shunday qilib, $\lambda \neq 0$ deb va λ xos qiymat emas deb olsak, ushbu

$$0 = \frac{m(\lambda)}{\lambda} - \frac{m(0)}{\lambda} + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(j-1)} \frac{D^{(j-1)}}{D_\mu^{(j-1)}}\left(\frac{1}{\mu} + \frac{1}{\lambda-\mu}\right)\Bigg|_{\mu=\mu_n}$$

tenglik hosil bo‘ladi. Oxirgi tenglikdan esa (28) formula kelib chiqadi. **Lemma 2 isbotlandi.**

Teorema 4. Agar $\alpha \neq 0$ va $\lambda \neq \mu_n$, $n = 0,1,2,...$ bo‘lsa, u holda quyidagi tasvir o‘rinli bo‘ladi

$$m(\lambda) = ctg\alpha + \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} \frac{p_k^{(j)}}{(\lambda-\mu_k)^j}. \tag{29}$$

Isbot. k ning yetarlicha katta qiymatidan boshlab, $m_k = 1$ bo‘ladi. Shuning uchun (15) va (22) formulalarga ko‘ra

$$p_k^{(j)} = p_k^{(1)} = \frac{1}{\alpha_k^{(0)}} = O(1)$$

bo‘ladi. (28) tenglikda $\lambda \to -\infty$ da limitga o‘tamiz:

$$m(\lambda) \to m(0) - \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j}, \tag{30}$$

(20) tenglikka ko‘ra

$$m(\lambda) \to ctg\alpha, \quad \lambda \to -\infty. \tag{31}$$

bo‘ladi. (30) va (31) munosabatlarni taqqoslab,

$$m(0) = ctg\alpha + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} \tag{32}$$

tenglikni olamiz. Bu ifodani (28) tenglikka qo‘yib, (29) formulani keltirib chiqaramiz. **Teorema isbotlandi.**

4-§. $\alpha = 0$ bo‘lgan holda Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o‘zgarmaslar orqali tasvirlash

Lemma 3. $\alpha = 0$ bo‘lgan holda quyidagi formula o‘rinli bo‘ladi.

$$m\left(-s^2\right) = s + O\left(\frac{1}{s}\right), \qquad s \to \infty. \tag{33}$$

Xususan, $m(\lambda) + i\sqrt{\lambda} \to 0$, $\lambda \to -\infty$.

Isbot. [9] ishga asosan $\lambda = is$ desak,

$$\upsilon\left(0,-s^2\right) = \frac{(-1)^M s^{2M} e^s}{4}\left[2 + O\left(\frac{1}{s}\right)\right],$$

$$\upsilon'\left(0,-s^2\right) = \frac{(-1)^{M+1} s^{2M+1} e^s}{4}\left[2 + O\left(\frac{1}{s}\right)\right]$$

tengliklar o‘rinli bo‘ladi. Bundan quyidagi tenglik kelib chiqadi:

$$m\left(-s^2\right) - s = -\frac{\upsilon'\left(0,-s^2\right) + s\upsilon\left(0,-s^2\right)}{\upsilon\left(0,-s^2\right)} = O\left(\frac{1}{s}\right).$$

Lemma isbotlandi.

Teorema 5. Agar $\alpha = 0$ va $M = \deg(h) > \deg(g)$ bo‘lsa, quyidagi tenglik o‘rinli bo‘ladi

$$m(\lambda) = 1 + 2(m - M) + \sum_{n=0}^{\infty}\left[2\sum_{j=1}^{m_n}\frac{p_n^{(j)}}{(\lambda - \mu_n)}\right]. \tag{34}$$

Bu yerdagi $m - M$ o‘zgarmas

$$\sqrt{\mu_n} = (m + n + 1 - M)\pi + O\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{35}$$

asimptotikadan aniqlanadi.

Isbot. (35) ga ko‘ra

$$\mu_n = (n + 1 + m - M)^2\pi^2 + O(1)$$

o‘rinli bo‘ladi. Quyidagi tenglik bajariladi:

$$\sqrt{\lambda}\, ctg\sqrt{\lambda} = -i\sqrt{\lambda}\left[1 + O\left(e^{-\left|\operatorname{Im}\sqrt{\lambda}\right|}\right)\right], \qquad \lambda \to -\infty.$$

Bu tenglik va 3-lemmaga ko‘ra

$$\lim_{\lambda\to-\infty}\left[m(\lambda)-\sqrt{\lambda}ctg\sqrt{\lambda}\right]=0$$

bo‘ladi. (28) tenglikni quyidagicha yozib olamiz:

$$m(\lambda)=m(0)+S(\lambda)+\sum_{n=0}^{\infty}p_n^{(1)}\left(\frac{1}{\lambda-\mu_n}+\frac{1}{\mu_n}\right). \tag{36}$$

Bu yerda

$$S(\lambda)=\sum_{n=0}^{\infty}\sum_{j=2}^{m_n}p_n^{(j)}\left(\frac{1}{(\lambda-\mu_n)^j}-\frac{1}{(-\mu_n)^j}\right). \tag{37}$$

Oxirgi yig’indi aslida chekli yig’indidir, chunki faqat cheklita xos qiymatgina karrali bo‘ladi. Quyidagi belgilashni kiritib olamiz:

$$S_\infty=\lim_{\lambda\to-\infty}S(\lambda)=-\sum_{n=0}^{\infty}\sum_{j=2}^{m_n}\frac{p_n^{(j)}}{(-\mu_n)^j}. \tag{38}$$

Mittag-Leffler yoyilmasiga ko‘ra

$$\sqrt{\lambda}ctg\sqrt{\lambda}=-1-\sum_{n=0}^{\infty}\frac{2\lambda}{n^2\pi^2-\lambda}$$

bo‘ladi. Bularni hisobga olsak.

$$m(\lambda)-\sqrt{\lambda}ctg\sqrt{\lambda}=m(0)+S(\lambda)+1+$$

$$+\sum_{n=0}^{M-m-2}p_n^{(1)}\left(\frac{1}{\lambda-\mu_n}+\frac{1}{\mu_n}\right)+\sum_{n=0}^{\infty}\left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}}\frac{\lambda}{\lambda-\mu_{n+M-m-1}}-2\frac{\lambda}{\lambda-n^2\pi^2}\right]=$$

$$=m(0)+S(\lambda)+1+\sum_{n=0}^{\infty}\left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}}-2\right]\frac{\lambda}{\lambda-\mu_{n+M-m-1}}+$$

$$+\sum_{n=0}^{M-m-2}p_n^{(1)}\left(\frac{1}{\lambda-\mu_n}+\frac{1}{\mu_n}\right)+\sum_{n=0}^{\infty}\left[\frac{2\lambda}{\lambda-\mu_{n+M-m-1}}-\frac{2\lambda}{\lambda-n^2\pi^2}\right]. \tag{39}$$

(16) formuladan va ushbu

$$\alpha_n^{(0)}=\frac{1}{2\mu_n}+O\left(\frac{1}{n^4}\right),\quad n\to\infty$$

tenglikdan foydalansak,

$$\frac{p_n^{(1)}}{\mu_n} = 2 + O\left(\frac{1}{n^2}\right), \quad n \to \infty \tag{40}$$

asimptotik formula kelib chiqadi. Bunga ko'ra

$$\lim_{\lambda\to-\infty} \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right] \frac{\lambda}{\lambda - \mu_{n+M-m-1}} = \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right]$$

bo'ladi. Ushbu

$$\mu_{n+M-m-1} - n^2\pi^2 = O(1)$$

tenglikni inobatga olsak, shunday manfiy, modul jihatdan yetarlicha katta bo'lgan K son topiladiki, bunda $\lambda \to -\infty$ da quyidagi baholash o'rinli bo'ladi

$$\sum_{n=0}^{\infty} \left| \frac{2\lambda}{\lambda - \mu_{n+M-m-1}} - \frac{2\lambda}{\lambda - n^2\pi^2} \right| \le K \sum_{n=0}^{\infty} \left[\frac{|\lambda|}{|\lambda|\left(n^2\pi^2 - \lambda\right)} \right] = -\frac{K\left(\sqrt{\lambda}\,ctg\sqrt{\lambda} + 1\right)}{2|\lambda|} \to 0.$$

Yuqoridagi munosabatlarni hisobga olsak, ushbu

$$m(\lambda) - \sqrt{\lambda}\,ctg\sqrt{\lambda} \to m(0) + S_\infty + 1 + \sum_{n=0}^{M-m-2} p_n^{(1)} \frac{1}{\mu_n} + \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right]$$

munosabat kelib chiqadi. Bunga asosan

$$m(0) = \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} - 1 - 2(M - m - 1) - \sum_{n=0}^{\infty} \left[\frac{p_n^{(1)}}{\mu_n} - 2\right]$$

bo'ladi. Bu ifodani (36) tenglikga qo'ysak, quyidagi formulani olamiz

$$m(\lambda) = 1 - 2M + 2m + \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} - \sum_{n=0}^{\infty} \left[\frac{p_n^{(1)}}{\mu_n} - 2\right] + \sum_{n=0}^{\infty} \sum_{j=1}^{m_n} p_n^{(j)} \left[\frac{1}{(\lambda - \mu_n)^j} - \frac{1}{(-\mu_n)^j}\right] =$$

$$= 1 - 2M + 2m + \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} p_n^{(j)} \frac{1}{(\lambda - \mu_n)} + \sum_{n=0}^{\infty} \left[2 + \frac{p_n^{(1)}}{\lambda - \mu_n}\right].$$

Teorema isbotlandi.

Teorema 6. Agar $\alpha = 0$ bo'lib, $M = \deg(g) \ge \deg(h)$ bo'lsa, u holda quyidagi tenglik o'rinli bo'ladi

$$m(\lambda) = 2(m - M) + \sum_{n=0}^{\infty} \left[2 + \sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(\lambda - \mu_n)^j}\right].$$

Isbot. Bu holda $\mu_n = \left(n + m + \frac{1}{2} - M\right)^2 \pi^2 + O(1)$ bo‘ladi. Ushbu

$$\sqrt{\lambda}tg\sqrt{\lambda} = i\sqrt{\lambda}\left[1 + O\left(e^{-2\left|\operatorname{Im}\sqrt{\lambda}\right|}\right)\right],\ \lambda \to \infty$$

formulaga asosan quyidagi tenglik ([34], 113 bet) kelib chiqadi

$$\frac{tg\sqrt{\lambda}}{\sqrt{\lambda}} = \sum_{k=0}^{\infty} \frac{2}{\left(k + \frac{1}{2}\right)^2 \pi^2 - \lambda}.$$

Buni hisobga olsak, 3-lemmadan $\lim\limits_{\lambda \to -\infty} \left(m(\lambda) + \sqrt{\lambda}tg\sqrt{\lambda}\right) = 0$ tenglik kelib chiqadi.

(36), (37) va (38) tengliklar hamda Mittag-Leffler yoyilmasiga asosan, quyidagi formula o‘rinli:

$$m(\lambda) + \sqrt{\lambda}tg\sqrt{\lambda} = m(0) + S(\lambda) + \sum_{n=0}^{\infty}\left[\frac{p_{n+M-m}^{(1)}}{\mu_{n+M-m}} - 2\right]\frac{\lambda}{\lambda - \mu_{n+M-m}} +$$

$$+ \sum_{n=0}^{M-m-1} p_n^{(1)}\left(\frac{1}{\lambda - \mu_n} + \frac{1}{\mu_n}\right) + \sum_{n=0}^{\infty}\left[\frac{2\lambda}{\lambda - \mu_{n+M-m}} - \frac{2\lambda}{\lambda - \left(n + \frac{1}{2}\right)^2 \pi^2}\right]$$

(40) munosabatdan foydalanib, ushbu tenglikni olamiz:

$$\lim_{\lambda \to -\infty} \sum_{n=0}^{\infty}\left[\frac{p_{n+M-m}^{1}}{\mu_{n+M-m}} - 2\right]\frac{\lambda}{\lambda - \mu_{n+M-m}} = \sum_{n=0}^{\infty}\left[\frac{p_{n+M-m}^{1}}{\mu_{n+M-m}} - 2\right].$$

Ushbu $\mu_{n+M-m} - \left(n + \frac{1}{2}\right)^2 \pi^2 = O(1)$ tenglikni hisobga olsak,

$$\sum_{n=0}^{\infty}\left|\frac{2\lambda}{\lambda - \mu_{n+M-m}} - \frac{2\lambda}{\lambda - \left(n + \frac{1}{2}\right)^2 \pi^2}\right| \le K \sum_{n=0}^{\infty}\left[\frac{|\lambda|}{|\lambda|\left(\left(n + \frac{1}{2}\right)^2 \pi^2 - \lambda\right)}\right] =$$

$$= \frac{Ktg\sqrt{\lambda}}{2\sqrt{\lambda}} \to 0, \qquad \lambda \to -\infty$$

baholash kelib chiqadi. Demak, quyidagi munosabat o‘rinli

$$m(\lambda)+\sqrt{\lambda} tg\sqrt{\lambda} \to m(0)+S_{\infty}+\sum_{n=0}^{M-m-1} p_n^{(1)} \frac{1}{\mu_n}+\sum_{n=0}^{\infty}\left[\frac{p_{n+M-m}^{(1)}}{\mu_{n+M-n}}-2\right].$$

Bunga ko‘ra

$$0=m(0)+2(M-m)+S_{\infty}+\sum_{n=0}^{\infty}\left[\frac{p_n^{(1)}}{\mu_n}-2\right]$$

tenglik o‘rinli bo‘ladi. Bu ifodani (36) tenglikka qo‘ysak, quyidagi formula kelib chiqadi

$$m(\lambda)=-2(M-m)+\sum_{n=0}^{\infty}\sum_{j=2}^{m_n}\frac{p_n^{(j)}}{(\lambda-\mu_n)^j}+\sum_{n=0}^{\infty}\left[2+\frac{p_n^{(1)}}{\lambda-\mu_n}\right].$$

Teorema isbotlandi.

5-§. Teskari masala yechimining yagonaligi

Teorema 7. (1)-(3) chegaraviy masala o‘zining μ_n, $n=\overline{0,\infty}$ xos qiymatlari va umumlashgan normallovchi o‘zgarmaslari $\alpha_n^{(k)}$, $n=\overline{0,\infty}$, $k=\overline{0,m_n}$ orqali bir qiymatli aniqlanadi. Boshqacha qilib aytganda $q(x)$, $f(\lambda)$ funksiyalar va α son $\left\{\mu_n, \alpha_n^{(k)},\ n=\overline{1,\infty},\ k=\overline{0,m_n}\right\}$ spektral berilganlar orqali bir qiymatli aniqlanadi.

Isbot. Beshinchi va oltinchi teoremalarga ko‘ra μ_n, $\alpha_n^{(k)}$, $n=\overline{1,\infty}$, $k=\overline{0,m_n}$ spektral berilganlar bo‘yicha $m(\lambda)$ Veyl-Titchmarsh funksiyasi bir qiymatli aniqlanadi. [9] ishda (1)-(3) masala Veyl-Titchmarsh funksiyasi bo‘yicha bir qiymatli aniqlanishi isbotlangan. **Teorema isbotlandi.**

Xulosa

Mazkur bitiruv malakaviy ishida chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun umumlashgan normallovchi o‘zgarmaslar ta’riflangan va Veyl-Titmarsh funksiyasini spektral berilganlar orqali ifodalovchi formulalar olingan. Jumladan, birinchi chegaraviy shartda qatnashadigan α sonning nolga teng bo‘lmagan va nolga teng bo‘lgan hollari alohida qaralgan. Shu bilan birga ikkinchi chegaraviy shartda ishtirok qiluvchi $h(\lambda)$ va $g(\lambda)$ ko‘phadlar darajalarining katta kichikligiga qarab, formulalar o‘zgarishi hisobga olingan. Asosiy formulani keltirib chiqarishda Veyl-Titmarsh funksiyasi uchun Loran yoyilmasidan foydalanilgan.

Veyl-Titmarsh funksiyasini xos qiymatlar va umumlashgan normallovchi o‘zgarmaslar orqali ifodalovchi formulalarga asoslanib, teskari masala uchun yagonalik teoremasi isbotlangan.

Foydalanilgan adabiyotlar

[1] A. I. Benedek, R. Panzone, On inverse eigenvalue problems for a second-order differential equation with parameter contained in the boundary conditions, Notas de algebra y Analisis, 9 (1980), 1-13.

[2] A. I. Benedek, R. Panzone, On Sturm-Liouville problems with the square-root of the eigenvalue parameter contained in the boundary condition, Notas de Algebra y Analisis, 10 (1981), 1-59.

[3] P. A. Binding, P. J. Browne, Applications of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions, Proc, Roy. Soc. Edinburgh Sect. A, 125 (1995), 1205-1218.

[4] P. A. Binding, P. J. Browne, K. Seddighi, Sturm-Liouville problems with eigenparameter dependent boundary conditions, Proc. Edinburg Math. Soc., 37 (1993), 57-72.

[5] P. A. Binding, P. J. Browne, B. A. Watson, Transformations between Sturm-Liouville problems with eigenvalue dependent and independent boundary conditions, *Bull. London Math. Soc.,* **33** (2001), 749-757.

[6] P. A. Binding, P. J. Browne, B. A. Watson, Spektral isomorphisms between generalized Sturm-Liouville problems, *Operator Theory Adva. Appl.,* **130** (2001), 135-152.

[7] P. A. Binding, P. J. Browne, B. A. Watson, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, I, *Proc. Edinburgh Math. Soc.,* **45** (2002), 631-645.

[8] P. A. Binding, P. J. Browne, B. A. Watson, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, II, *Comp. Appl. Math.,* **148** (2002), 147-168.

[9] P. A. Binding, P. J. Browne, B. A. Watson, Equivalence of inverse Sturm- Liouville problems with boundary conditions rationally dependent on the eigenparameter, submitted.

[10] P. A. Binding, R. Hryniv, H. Langer, B. Najman, Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, *J. Diff. Eq.,* **174** (2001), 30-54.

[11] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, *Acta Math.,* **78** (1946), 1-96.

[12] P. J. Browne, B. D. Sleeman, A uniqueness theorem for inverse inverse eigenparameter dependent Sturm-Liouville problems, *Inverse Problems,* **13** (1997), 1453-1462.

[13] L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen,* Akademische Verlag, Leipzig, 1963.

[14] A. Dijksma, Eigenfunction expansions for a class of J-selfadjoint ordinary differential operators with boundary conditions containing the eigenvalue parameter, *Proc. Roy. Soc. Edinburgh, Series* **A, 87** (1980), 1-27.

[15] A. Dijksma, H. Langer, Operator theory and ordinary differential operators, *ields Inst. Mono.* **3** (1996), 75-139.

[16] A. Dijksma, H. Langer, H. de Snoo, Symmetric Sturm-Liouville operators with eigenvalue depending boundary conditions, *Canadian Math. Soc. Conf. Proc.* **8** (1987), 87-116.

[17] M. S. P. Eastham, Eigenvalue problems with the parameter in the boundary condition, *Quart. J. Math. Oxford Ser. (2),* **14** (1963), 259-272.

[18] A. E. Etkin, Some boundary value problems with a spektral parameter in the boundary conditions, *Amer. Math. Soc. Transl. Series 2,* **136** (1987), 35-41.

[19] C. T. Fulton, Two-point boundary value problems with eigenparameter contained in the the boundary conditions, *proc. Roy. Soc. Edinburgh,* **77A** (1977), 293-308.

[20] G. M. Gubreev, V. N. Pivovarchik, Spektral analysis of the Regge problem with parameters, *Funct. Anal. Appl.,* **31** (1997), 54-57.

[21] E. Hille, *Analytic Function Theory, Vol I, Blaisdell publishing company,Masachusetts,1959.*

[22] D.B.Hinton, Eigenfunction expansions for a singulyar eigenvalue problem with eigenparameter in the boundary conditions, SIAM J. Math. Anal., 12(1981), 572-584.

[23] H. Hochstadt, On inverse problems associated with second-order differential operators, Acta Mathematica,119 (1967), 173-192.

[24] L Ho‘rmander, Lectures on nonlinear hyperbolic differentialequations, Springerverlag, Berlin, 1997.

[25] M. G. Krein ,Solution of the inverse Sturm-Liouville problem, Dokl. Akad. Nauk SSSR,76 (1951),21-24.

[26] B.M. Levitan, M.G. Gasymov, Determination of a differential equation from two of its spectra, Russian Math. Surveys,19 (1964) 1-63.

[27] V. A. Marchenko, Some questions in the theory of one-dimensional linear differential operators of the second order, Part I, *A.M.S. Translations Series 2,* **101** (1973), 1-104.

[28] M. A. Naimark, *Linear Differential Operators I,* Ungar, New York, 1967

[29] V. N. Pivovarchik, Direct and inverse three-point Sturm-Liouville problem with parameter-dependent boundary conditions, *Asymp. Anal.,* **26** (2001), 219-238.

[30] E. M. Russakovskii, Operator treatment of boundary problems with spektral parameters entering via polynomials in the boundary conditions, *Functional Anal. Appl.,* **9** (1975), 358-359.

[31] E. M. Russakovskii, The matrix Sturm-Liouville problem with spektral parameter in the boundary conditions. Algebraic and operator aspects, *Trans. Moscow Math. Soc.,* **57** (1996), 159-184.

[32] A. A. Shkalikov, Boundary problems for ordinary differential equations with parameter in the boundary conditions, *J. Sov. Math.,* **33** (1986), 1311-1342.

[33] A. V. Štraus, On spektral functions of differential operators, *Izvest. Akad. Nauk SSSR Ser. Mat.,* **19** (1955), 201-220.

[34] E. C. Titchmarsh, Theory of Functions, Second Edition, Clarendon Press, Oxford, 1939.

[35] C. Tretter, On λ-nonlinear boundary eigenvalue problems, Mathematics Research **71** (1993), Akademie Verlag.

[36] A. Zugmund, *Trigonometric Series, Vol I, II,* Cambridge University Press, Cambridge, 1979.