Teylor va Makloren qatorlari.

Teylor va Makloren qatorlarining tadbiqi.

Ba'zi elementar funksiyalarni Makloren qatoriga yoyish.

$y = f(x)$ funksiya ***a*** nuqtada va uning biror atrofida uzluksiz va ***a*** nuqtada istalgan tartibli hosilalarga ega bo‘lsin.

Ushbu masalani qo‘yamiz: $y = f(x)$ funksiyani darajali qator ko‘rinishida tasvirlash mumkin va hamma vaqt hosil bo‘lgan darajali qator berilgan $y = f(x)$ funksiyani darajali qator ko‘rinishda tasvirlash mumkin, ya’ni

$$f(x) = c_0 + c_1 \cdot (x-a) + c_2 \cdot (x-a)^2 + \cdots + $$
$$+ c_n \cdot (x-a)^n + \cdots \qquad (\mathbf{1})$$

endi $y = f(x)$ funktsiyaning darajali qator koeffitsiyentlari bilan qanday bog‘langanligini topamiz.

(1) - da $\boldsymbol{x = a}$ deb $f(a) = c_0$ ekanligini topamiz. Faraz qilaylik $y = f(x)$ funksiyani qator yaqinlashish intervaliga tegishli $\boldsymbol{a}$ nuqtaning biror atrofida uzluksiz bo‘lsin. U holda qatorni bu atrofda hadma-had differensiallash mumkin. **(1)**-tenglikni differensiallaymiz:

$$f'(x) = c_1 + 2c_2 \cdot (x - a) + \\ + \cdots + nc_n \cdot (x - a)^{n-1} + \cdots \qquad (2)$$

(2)-tenglikda $\boldsymbol{x = a}$ deb $f'(a) = c_1$ ni hosil qilamiz. **(2)**-tenglikni differensiallab

$$f''(x) = 2c_2 + 2 \cdot 3 \cdot c_3 \cdot (x - a) + \cdots + \\ + n \cdot (n - 1) \cdot c_n \cdot (x - a)^{n-2} + \cdots \qquad (3)$$

ga kelamiz va **(3)** – tenglikda $\boldsymbol{x = a}$ desak

$$f''(a) = 2c_2 \quad \Longrightarrow \quad c_2 = \frac{f''(a)}{2!}$$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4 \cdot c_4 \cdot (x - a) + \cdots +$$
$$+n \cdot (n - 1) \cdot (n - 2)c_n \cdot (x - a)^{n-3} + \cdots \quad (4)$$

(**4**) da $\boldsymbol{x = a}$ desak

$$f'''(a) = 2 \cdot 3c_3 \quad \Longrightarrow \quad c_3 = \frac{f'''(a)}{3!}$$

va hakazo.

$$c_n = \frac{f^{(n)}(a)}{n!} \qquad \textbf{(5)}$$

Teyler koeffitsiyenti .

(5) Teyler koeffitsiyentlarini **(1)** ga qo‘yamiz.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \qquad (6)$$

Agar $y = f(x)$ funksiya $\boldsymbol{x = a}$ nuqtada istalgan tartibli hosilasiga ega bo‘lsa, u holda Teylor formulasida $\boldsymbol{n}$ sonini istalgancha katta qilib olish mumkin. Qaralayotgan atrofda

$$\lim_{n \to \infty} R_n(x) = 0$$

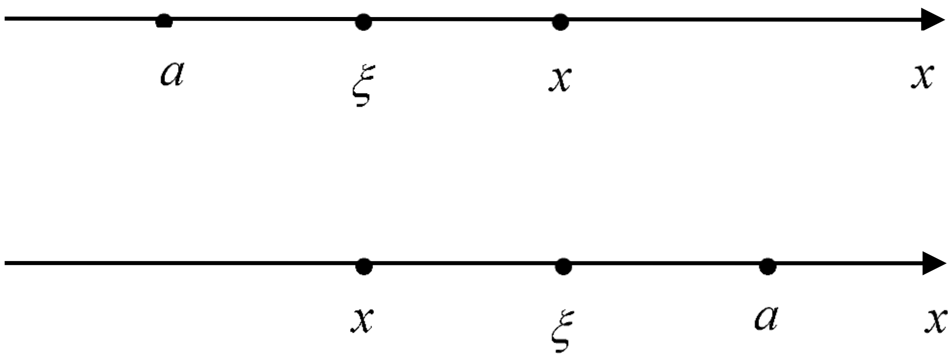
deb faraz qilsak.U holda **(6)**-formulada $n \to \infty$ da limitga o‘tib, o‘ngda cheksiz qatorga ega bo‘lamiz,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \qquad \textbf{(7)}$$

(7) formula $y = f(x)$ funksiya uchun $\boldsymbol{a}$ nuqtaning atrofidagi Teylor qatori deyiladi.

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} \cdot f^{(n+1)}(\xi)$$

bu yerda $\boldsymbol{a < \xi < x}$.



Teorema: (*Teylor teoremasi*)

$y = f(x)$ funksiyani $(x - a)$ ning darajasi bo‘yicha darajali qatorga yoyish uchun $y = f(x)$ funksiya a nuqtada aniqlangan va bu nuqtadaning atrofida absolyut qiymati bo‘yicha aynan bir sonning o‘zi bilan chegaralangan yuqori tartibli hosilalarga ega bo‘lsa, u holda bu funksiya ko‘rsatilgan $x = a$ nuqta atrofida Teylor qatoriga yoyish mumkin.

(**7**)-qatorning aniqlanish sohasidagi $\forall x$ uchun qatorning yig‘indisi $y = f(x)$ funksiyaning bu nuqtadagi qiymatiga teng va bu yoyilma yagonadir.

Teylor qatorining xususiy holi ***Makloren qatoridir***. Agar (**7**)-yoyilmada $a = 0$ bo‘lsa (**7**)-dan, Makloren qatori deb ataluvchi qatorga ega bo‘lamiz.

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \cdots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \cdots \qquad (8)$$

Ba’zi elementarr funksiyalarini Makloren qatoriga yoyish.

1) ***sinx*** funksiyani ***x*** ning darajalari bo‘yicha yoyish.

$f(0) = Sin0 = 0$,

$f'(x) = cosx = sin\left(x + \frac{\pi}{2}\right)$, $f'(0) = 1$,

$f''(x) = -sinx = sin\left(x + 2 \cdot \frac{\pi}{2}\right)$, $f''(0) = 0$,

$f'''(x) = -cosx = sin\left(x + 3 \cdot \frac{\pi}{2}\right)$, $f'''(0) = -1$,

....,

$f^{(n)}(x) = sin\left(x + n \cdot \frac{\pi}{2}\right)$, $f^{(n)}(0) = sin\left(n \cdot \frac{\pi}{2}\right)$,

Hosilalarning qiymatlari:

$$0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \dots$$

bularni **(7)** formulaga qo‘ysak.

1) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n+1} \cdot \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

Yaqinlashish sohasi: $x \in (-\infty; +\infty)$.

Bundan hosila olsak.

2) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{n+1} \cdot \frac{x^{2n-2}}{(2n-2)!} + \cdots$$

Yaqinlashish sohasi: $x \in (-\infty; +\infty)$.

3) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Yaqinlashish sohasi: $x \in (-\infty; +\infty)$.

4) $f(x) = (1 + x)^m$ funksiyani x ning darayalari bo‘yicha yoyish.

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} \cdot x^2 + \cdots + \frac{m(m-1)(m-2)\cdot \cdots \cdot (m-n)}{n!} \cdot x^n + \cdots$$

Yaqinlashish sohasi: $x \in (-1; 1)$.

4) da $m = -1$ desak

5) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^{n-1} \cdot x^{n-1} + \cdots$$

Yaqinlashish sohasi: $x \in (-1; 1)$.

5) da $\boldsymbol{x}$ ning o‘rniga $-\boldsymbol{x}$ qo‘ysak.

6) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^{n-1} + \cdots$$

Yaqinlashish sohasi: $x \in (-1; 1)$.

5) da $\boldsymbol{x}$ ning o‘rniga $\boldsymbol{x^2}$ qo‘ysak.

(7) $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots + (-1)^{n-1} \cdot x^{2n-2} + \cdots$$

5) ni $x \in (-1; 1)$ oraliqda **0** dan $\boldsymbol{x}$ gacha integrallaymiz. Chunki darajali qatorni absolyut yaqinlashish sohasiga tegishli ixtiyoriy intervalda integrallash mumkin.

$$\int_0^x \frac{dx}{1+x} = \int_0^x (1 - x + x^2 - x^3 + x^4 + \cdots)\, dx$$

8) $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$$

Yaqinlashish sohasi: $x \in (-1; 1)$.

8) da $\boldsymbol{x}$ ning o‘rniga $-\boldsymbol{x}$ qo‘ysak.

9) $$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + + \frac{x^4}{4} + \ldots + \frac{x^n}{n} + \cdots\right)$$

Yaqinlashish sohasi: $x \in (-1; 1)$.

(8 - 9) dan

$$10)\ \ln\left|\left(\frac{1+x}{1-x}\right)\right| = 2\cdot\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots+\frac{x^{2n-1}}{2n-1}+\cdots\right)$$

7) ni $(0; x)$ da integrallab:

$$11)\ arctgx = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^{n-1}\cdot\frac{x^{2n-1}}{2n-1}+\cdots$$

1-misol. $\int_0^{\boldsymbol{a}} e^{-x^2}\,dx$ integralni qator yordamida hisoblang:

Echish. e^{-x^2} ning boshlang‘ich funksiyasi elementar funksiya bo‘lmaydi. Bu funksiyani hisoblash uchun e^x ning yoyilmasida x ni $-x^2$ ga almashtirib, integral ostidagi funksiyani qatorga yoyamiz:

$$e^{-x^2} = 1 - \frac{x^2}{1} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots + (-1)^n \cdot \frac{x^n}{n!} + \cdots,$$

Bu tenglikning ikkala tomonini 0 dan a gacha chegarada integrallaymiz:

$$\int_0^a e^{-x^2} dx = \int_0^a \left(1 - \frac{x^2}{1} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right) dx =$$

$$= \left(\frac{x}{1} - \frac{x^3}{1! \cdot 3} + \frac{x^5}{2! \cdot 5} - \frac{x^7}{3! \cdot 7} + \frac{x^9}{4! \cdot 9} - \cdots\right)\Bigg|_0^a =$$

$$= \frac{a}{1} - \frac{a^3}{1! \cdot 3} + \frac{a^5}{2! \cdot 5} - \frac{a^7}{3! \cdot 7} + \frac{a^9}{4! \cdot 9} - \cdots$$

Bu tenglik yordamida ***a*** ning istalgan qiymatida berilgan integralni ixtiyoriy darajadagi aniqlik bilan hisoblash mumkin.