TOSHKENT AXBOROT TEXNOLOGIYALARI UNIVERSITETI

DASTURIY INJINIRING FAKULTETI

OLIY MATEMATIKA KAFEDRASI

MUSTAQIL ISH

Mavzu: Ixtiyoriy davriy funksiya uchun Furye qatori.
Furyening sinus va kosinus almashtirishlari

Toshkent 2015

20 – MA'RUZA.
IXTIYORIY DAVRLI FUNKSIYA UCHUN FURYE QATORI.
TOQ VA JUFT FUNKSIYALARNI FURYE QATORIGA YOYISH.

Reja.

1. Toq va juft funksiyalarni Furye qatoriga yoyish.
2. Ixtiyoriy davrli funksiya uchun Furye qatori.

Tayanch ibora va tushunchalar

Toq funksiyalarning Furye qatori, juft funksiyalarning Furye qatori, ixtiyoriy davrli funksiya uchun Furye qatori, davom ettirish, sinuslar bo'yicha yoyish, kosinuslar bo'yicha yoyish.

1. Toq va juft funksiyalarni Furye qatori

Bizga davri $T = 2\pi$ bo'lgan funksiya berilgan bo'lsin, ya'ni $f(x+2\pi) = f(x)$. Berilgan funksiyaning Furye qatori va koeffitsiyentlari quyidagicha edi:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cdot \cos nx + b_n \cdot \sin nx) \tag{1}$$

$$a_0 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x)\,dx\,, \tag{2}$$

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos nx\,dx\,, \tag{3}$$

$$b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \sin nx\,dx \tag{4}$$

Quyida biz juft va toq funksiyalarning Furye qatori koeffitsientlarini hisoblash formulalarini keltirib chiqaramiz.

Agar $f(x)$ funksiya $[-a;\ a]$ da integrallanuvchi bo'lsa, u holda

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx. \tag{5}$$

Ikkinchi integralda *x* ni -*x* ga almashtirish bajarib, (5)ga qo'yamiz:

$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-x)d(-x) = -\int_{a}^{0} f(-x)dx = \int_{0}^{a} f(-x)dx,$$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]dx,$$

$f(x)$ funksiya toq bo'lsa, ya'ni $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) - f(x)]dx = 0\,, \tag{6}$$

$f(x)$ funksiya juft bo'lsa, ya'ni $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(x)]dx = 2 \cdot \int_{0}^{a} f(x)dx \tag{7}$$

Ikkita juft funksiyalarning yoki ikkita toq funksiyalarning ko'paytmasi juft funksiya, juft va toq funksiyalarning ko'paytmasi toq funksiya ekanligini va (7) ni e'toborga olgan holda juft va toq funksiyalarning Furye qatori koeffitsientlarini hisoblaymiz.

1) $f(x)$ funksiya davri $T = 2\pi$ bo'lgan, $[-\pi, \pi\,]$ da Dirixle shartlarini qanoatlantiradigan ***juft funksiya*** bo'lsin.

$$a_0 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi} \cdot \int_{0}^{\pi} f(x)\,dx, \tag{8}$$

$$a_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos kx\,dx = \frac{2}{\pi} \cdot \int_{0}^{\pi} f(x) \cdot \cos kx\,dx \tag{9}$$

$$b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \sin kx\,dx = 0 \tag{10}$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{11}$$

Juft funksiya uchun Furye qatori faqat kosinuslardan iborat, $b_k = 0$.

2) $f(x)$ funksiya davri $T = 2\pi$ bo'lgan, $[-\pi, \pi\,]$ da Dirixle shartlarini qanoatlantiruvchi ***toq funksiya*** bo'lsin.

$$a_0 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x)\,dx = 0\,, \tag{12}$$

$$a_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos kx\,dx = 0\ , \tag{13}$$

$$b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \sin kx\,dx = \frac{2}{\pi} \cdot \int_{0}^{\pi} f(x) \cdot \sin kx\,dx \tag{14}$$

$$f(x) = \sum_{k=1}^{\infty} b_k \cdot \sin kx \tag{15}$$

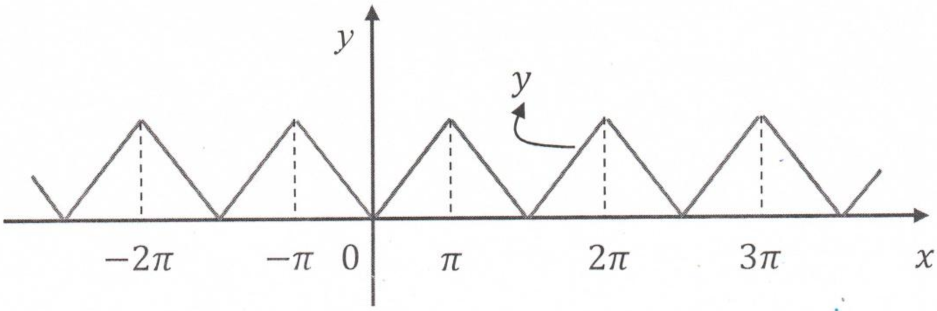
Toq funksiya uchun Furye qatori faqat sinuslardan iborat ekan, $a_0 = 0,\ a_k = 0$

Misol. Davri $T = 2\pi$ ga teng bo'lgan

$$f(x) = \begin{cases} -x\,, & agar \quad x \in (-\pi\,; 0) \\ \ \ x\,, & agar \quad x \in [\,0\,;\ \ \pi\,) \end{cases}$$

funksiyaning Furye qatoriga yoying.

Yechish. Juft funksiya $(-\pi\ ,\ \pi\,)$ intervalda Dirixle shartlarini qanoatlantiradi(1-shakl).



1 – shakl

$$a_0 = \frac{2}{\pi} \cdot \int_0^{\pi} x dx = \frac{2}{\pi} \cdot \frac{x^2}{2}\bigg|_0^{\pi} = \pi \,,$$

$$a_k = \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot coskxdx = \frac{2}{\pi} \cdot \left(\frac{x \cdot sinkx}{k} + \frac{coskx}{k^2}\right)\bigg|_0^{\pi} =$$

$$= \frac{2}{\pi} \cdot \frac{1}{k^2} \cdot (cosk\pi - cos0) = \frac{2}{\pi} \cdot \frac{1}{k^2} \cdot [(-1)^k - 1] =$$

$$= \begin{cases} -\dfrac{4}{k^2 \cdot \pi}, & k \quad toq \;\; bo\text{‘}lsa \,, \\ 0 \qquad\quad , & k \quad juft \;\; bo\text{‘}lsa. \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cdot \sum_{k=1}^{\infty} \frac{\cos(2k-1)\,x}{(2k-1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \cdot \left(cosx + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \cdots\right)$$

$$x = 0\,, \qquad 0 = \frac{\pi}{2} - \frac{4}{\pi} \cdot \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots\right)$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \frac{\pi^2}{8}$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = ?$$

$$S = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2} + \cdots\right)$$

$$S = \frac{\pi^2}{8} + \frac{1}{2^2} \cdot \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots\right)$$

$$S = \frac{\pi^2}{8} + \frac{1}{4} \cdot S\,, \qquad \frac{3 \cdot S}{4} = \frac{\pi^2}{8}\,, \quad S = \frac{\pi^2}{6}\,.$$

2. Ixtiyoriy davrli funksiya uchun Furye qatori.

Endi ixtiyoriy $2l$ davrli, Dirixle shartlarini qanoatlantiruvchi $f(x)$ funksiyani qaraymiz. $x = \frac{1}{\pi}t$ o'rniga qo'yish bizni 2π davrli $f(x) = f\left(\frac{1}{\pi}\,t\right)$ funksiyaga olib keladi, bu funksiyani

Furye qatoriga yoyamiz:

$$f\left(\frac{l}{\pi}t\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos kt + b_k \sin kt\right),$$

bu yerda

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{l}{\pi}t\right)dt,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{l}{\pi}t\right)\cos kt\,dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{l}{\pi}t\right)\sin kt\,dt,$$

Qatorda va Furye koeffitsentlari formulalarida yangi t o'zgaruvchidan eski x o'zgaruvchiga qaytib va $t = \frac{\pi}{l}x,\ \ dt = \frac{\pi}{l}dx$ ekanini hisobga olib, quyidagiga ega bo'lamiz:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cdot \cos\frac{k\pi x}{l} + b_n \cdot \sin\frac{k\pi x}{l}\right) \tag{1}$$

bu yerda

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, \quad a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}dx, \quad b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,dx \tag{2}$$

Koeffitsentlari (2) formulalari bilan aniqlanadigan (1) qator ixtiyoriy $2l$ davrli $f(x)$ funksiya uchun Furye qatori deyiladi.

$2l$ davrli juft funksiya uchun hamma $b_k = 0$ bo'ladi, demak Furye qatori faqat kosinuslarni o'z ichiga oladi:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{l}, \tag{3}$$

bu yerda

$$a_0 = \frac{1}{l}\int_{0}^{l} f(x)dx, \qquad a_k = \frac{1}{l}\int_{0}^{l} f(x)\cos\frac{k\pi x}{l}dx \quad .$$

$2l$ davrli toq funksiya uchun esa hamma $a_k = 0$ va $a_0 = 0$ bo'ladi, demak, Furye qatori faqat sinuslarni o'z ichiga oladi:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l}, \tag{4}$$

bu yerda

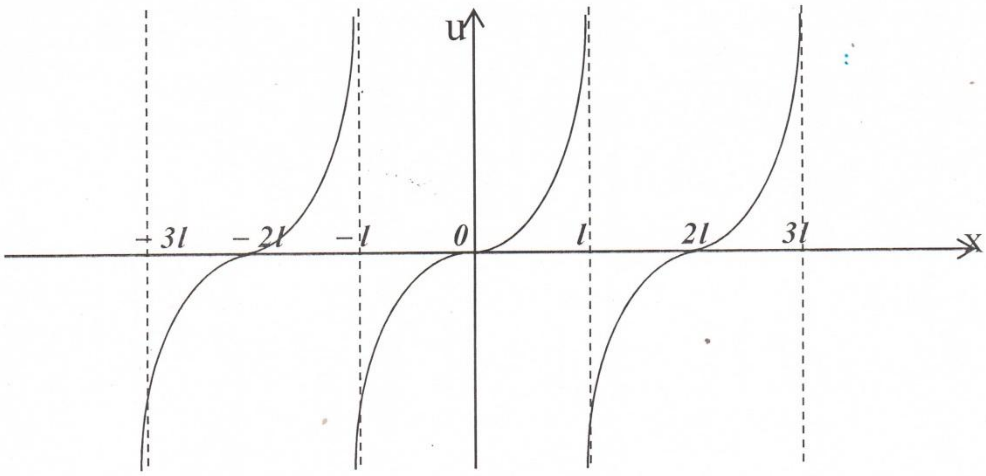
$$b_k = \frac{1}{l}\int_{0}^{l} f(x)\sin\frac{k\pi x}{l}dx.$$

Ko'pincha $[0,l]$ kesmada(yarim davrda) berilgan $f(x)$ funksiyani sinuslar bo'yicha yoki kosinuslar bo'yicha yoyish masalasi talab etiladi.

$f(x)$ funksiyani kosinuslar bo'yicha qatorga yoyish uchun funksiya juftligicha $[0,l]$ kesmadan $[-l,0]$ kesmaga davom ettiriladi. U holda «davom ettirilgan» juft funksiya uchun Furye qatori faqat kosinuslarni o'z ichiga oladi. Agar $f(x)$ funksiyani qatoriga sinuslar bo'yicha yoyishni istasak, u holda funksiyani toqligicha $[0,l]$ kesmadan $[-l,0]$ kesmagacha davom ettiramiz, bunda $f(x)=0$ deb olishimiz kerak. «Davom ettirilgan» toq funksiya uchun Furye qatori faqat sinuslarni o'z ichiga oladi.

Aslida kesmadan-kesmaga davom ettirishni amalga oshirmasa ham bo'ladi, chunki Furye koeffisentlarini hisoblash formulalaridan juft yoki toq funksiya holida $f(x)$ funksiyaning $[0,l]$ kesmadagi qiymatlari qatnashadi.

1-misol. $f(x)=x^2$ funksiyani $[0,l]$ kesmada sinuslar bo'yicha qatorga yoying.



2-shakl.

$f(x)$ funksiyaning $[-l,0]$ kesmaga toq davom ettirish va undan keyingi davriy davom ettirish grafigi yuqoridagi 2-shaklda ko'rsatilgan.

$f(x)$ funksiya toq va Dirixle shartlarini qanoatlantiradi.

Demak,

$$b_k=\frac{2}{l}\int_0^l x^2\sin\frac{\pi kx}{l}dx=\frac{2}{l}\left(-\frac{l}{\pi k}x^2\cos\frac{\pi kx}{l}\Big|_0^l+\frac{2l^2}{(\pi k)^2}x\sin\frac{\pi kx}{l}\Big|_0^l+\frac{2l^3}{(\pi k)^3}\cos\frac{\pi kx}{l}\Big|_0^l\right)=$$

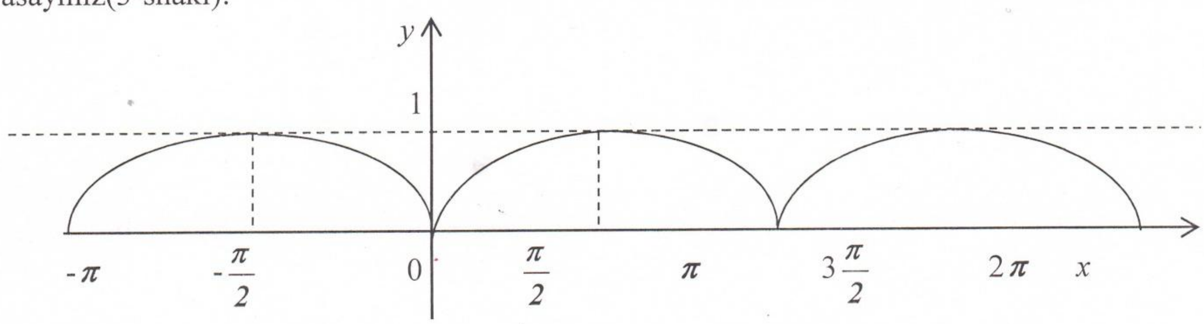
$$=\frac{2}{l}\left(-\frac{l^2}{\pi k}\cos\pi k+\frac{2l^3}{(\pi k)^3}(\cos\pi k-1)\right)=\frac{2}{l}\left((-1)^{k+1}\frac{l^3}{\pi k}+\frac{2l^3}{(\pi k)^3}\left((-1)^k-1\right)\right)$$

Izlanayotgan yoyilma quyidagi ko'rinishga ega:

$$f(x)=\frac{2l^2}{\pi}\sum_{k=1}^{\infty}\left((-1)^{k+1}\frac{1}{k}+\frac{2}{\pi^2k^2}\left((-1)^k-1\right)\right)$$

2-misol. $f(x)=\sin x$ funksiyani $\left[0,\frac{\pi}{2}\right]$ kesmada kosinuslar bo'yicha qatorga yoying.

Yechish. Juft davom ettirish va undan keyingi davriy davom ettirish bo'yicha grafikni yasaymiz(3-shakl):



3-shakl.

Funksiya juft funksiya. Shu sababli,

$$a_0 = 2\cdot\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin x dx = \frac{4}{\pi}(-\cos x)\Bigg|_0^{\frac{\pi}{2}} = \frac{4}{\pi}$$

$$a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\cos 2x dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{2}(\sin(2k+1)x - \sin(2k-1)x)dx =$$

$$= \frac{4}{\pi}\frac{1}{2}\left(-\frac{1}{2k+1}\cos(2k+1)x + \frac{1}{2k-1}\cos(2k-1)x\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right) = -\frac{4}{\pi}\cdot\frac{1}{4k^2-1}$$

Demak,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{36}\cos 6x + \ldots\right)$$

$\boldsymbol{x = 0}$ deb, quyidagiga ega bo'lamiz:

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$$

Bundan

$$\frac{1}{2} = \sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots$$

Mustaqil yechish uchun misollar

1. Berilgan funksiyani Furye qatoriga yoying($T = 1$):

$$f(x) = x^2,\ x\in\left[-\frac{1}{2};\frac{1}{2}\right].$$

2. $\boldsymbol{f(x)}$ funksiyaning sinuslar bo'yicha Furye qatoriga yoying:

$$f(x) = \begin{cases} x, & agar \quad x\in[0;1[\\ 2-x, & agar \quad x\in[1;2]. \end{cases}$$

3. Davri T = 8 bo'lgan $\boldsymbol{f(x)}$ davriy funksiyani Furye qatoriga yoying:

$$f(x) = \begin{cases} x, & agar \quad x\in[0;1[, \\ 1, & agar \quad x\in[1;2[, \\ 3-x, & agar \quad x\in[2;3[, \\ 0, & agar \quad x\in[3;4[. \end{cases}$$

Mustahkamlash uchun savollar

1. $[-\pi,\pi]$ kesmada juft funksiyalar uchun Furye koeffitsentlari qanday boladi?
2. $[-\pi,\pi]$ kesmada toq funksiyalar uchun Furye koeffitsentini keltirib chiqaring.
3. Ixtiyoriy davrli funksiya uchun Furye qatori va koeffitsientlari qanday hisoblanadi?
4. Toq davom ettirish deganda nimani tushunasiz?
5. Yarim davrda berilgan funksiyalarni sinuslar bo'yicha qatorga qanday yoyiladi?
6. Juft davom ettirish deganda nimani tushunasiz?
7. Yarim davrda berilgan funksiyalarni kosinuslar bo'yicha qatorga qanday yoyiladi?

21 – MA'RUZA.

KOMPLEKS SHAKLDAGI FURYE QATORI. PARSEVAL TENGLIGI. FURYE INTEGRALI. FURYENING SINUS VA KOSINUS ALMASHTIRISHLARI.

Reja.

3. Kompleks shakldagi Furye qatori
4. Furye integrali.
5. Furye integralining kompleks shakli.
6. Furyening kosinus va sinus almashtirishlari.

Tayanch ibora va tushunchalar

Kompleks shakldagi Furye qatori, Furye integrali, juft funksiyalarning Furye integrali , toq funksiyalarning furye integrali, Furye intagralining kompleks shakli, kosinus almashtirishi, sinus almashtirishi.

1. Kompleks shakldagi Furye qatori.

Davri $T = 2\pi$ bo'lgan davriy funksiya uchun Furye qatoriga ega bo'laylik:

$$f(x) == \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nx + b_n \cdot \sin nx) \qquad (1)$$

$\cos nx$ va $\sin nx$larni ko'rsatkichli funksiyalar bilan ifodalaymiz:

$$cosnx = \frac{e^{inx} + e^{-inx}}{2} \quad \text{va} \quad sinnx = \frac{e^{inx} - e^{-inx}}{2i} = -i \cdot \frac{e^{inx} - e^{-inx}}{2}.$$

Bularni (1)-formulaga qo'yamiz va tegishli almashtirishlar bajaramiz,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \frac{e^{inx} + e^{-inx}}{2} - i \cdot b_n \cdot \frac{e^{inx} - e^{-inx}}{2} \right) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - i \cdot b_n}{2} \cdot e^{inx} + \frac{a_n + i \cdot b_n}{2} \cdot e^{-inx} \right), \qquad (2)$$

$$\frac{a_0}{2} = c_0 , \qquad \frac{a_n - i \cdot b_n}{2} = c_n , \qquad \frac{a_n + i \cdot b_n}{2} = c_{-n} \qquad (3)$$

Bu belgilashlarni (2) ga qo'yib ushbuga kelamiz:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n \cdot e^{inx} + c_{-n} \cdot e^{-inx} \right). \qquad (4)$$

(4) tenglikni yanada qisqaroq yozamiz,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx} . \qquad (5)$$

Bu ***Furye qatorining kompleks shakli***dir.

2. Furye integrali.

Ushbu xosmas integralni qaraylik

$$\int_{-\infty}^{\infty} f(x)\,dx \quad . \tag{1}$$

Agar

$$Q = \int_{-\infty}^{\infty} |f(x)|\,dx \tag{2}$$

xosmas integral yaqinlashuvchi bo'lsa, (1) integral ***absolyut yaqinlashuvchi*** deyiladi, $f(x)$ funksiya esa ***absolyut integrallanuvchi*** funksiya deyiladi.

Faraz qilaylik, (2) integral mavjud bo'lsin, hamda $f(x)$ funksiya $[-l, l]$ kesmada Furye qatoriga yoyilsin, ya'ni

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x\right), \tag{3}$$

bu yerdagi a_0, a_k va b_k koeffitsientlar quyidagi formulalardan topiladi:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad ,$$

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi}{l}x\,dx \quad , \tag{4}$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}x\,dx \; .$$

(4) ni (3) qatorga qo'yib, quyidagi ifodani hosil qilamiz:

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\frac{k\pi}{l}tdt\right)\cos\frac{k\pi}{l}x +$$

$$+\frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\sin\frac{k\pi}{l}tdt\right)\sin\frac{k\pi}{l}x = \frac{1}{2l}\int_{-l}^{l} f(t)dt +$$

$$+\frac{1}{2l}\sum_{k=1}^{\infty}\int_{-l}^{l} f(t)\left[\cos\frac{k\pi}{l}t\cos\frac{k\pi}{l}x + \sin\frac{k\pi}{l}t\sin\frac{k\pi}{l}x\right]dt$$

yoki

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{k=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{k\pi(t-x)}{l}dt \, . \tag{5}$$

Endi $l \to \infty$ da (5) qator qanday ifodaga ega bo'lishini tekshiramiz. Buning uchun quyidagi belgilashlarni kiritamiz:

$$\alpha_1 = \frac{\pi}{l},\ \alpha_2 = \frac{2\pi}{l},\ \ldots,\ \alpha_k = \frac{k\pi}{l} \text{ va } \Delta\alpha_k = \frac{\pi}{l}. \tag{6}$$

(6) ni (5) ga qo'yamiz,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\alpha_k(t-x)dt\right)\Delta\alpha_k \ , \tag{7}$$

bu yerda $\Delta\alpha_k = \dfrac{\pi}{l}$ dan $\dfrac{1}{l} = \dfrac{\Delta\alpha_k}{\pi}$ deb olingan.

(7) ning o'ng tomonidagi birinchi had $l \to \infty$ da nolga intiladi, chunki

$$\left|\frac{1}{2l}\int_{-l}^{l} f(t)dt\right| \le \frac{1}{2l}\int_{-l}^{l}|f(t)|dt \le \frac{1}{2l}\int_{-\infty}^{\infty}|f(t)|dt = \frac{1}{2l}\cdot Q = 0 \, .$$

O'zgarmas l sonining har qanday qiymatida qavs ichida turgan ifodalar $\frac{\pi}{l}$ dan ∞ gacha qiymatlar qabul qiluvchi α_k ning funksiyasidir ((6) formulaga qarang).

Agar $f(x)$ funksiya cheksiz oraliqda chegaralangan, har bir chekli oraliqda bo'lakli monoton, hamda (1) shartni qanoatlantirsa, u holda $l \to +\infty$ da (7) formulani quyidagi ko'rinishda isbotsiz keltiramiz:

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)dt \right) d\alpha . \qquad (8)$$

(8) formulaning o'ng tomonidagi ifoda $f(x)$ funksiya uchun ***Furye integrali*** deyiladi.

(8) formula $f(x)$ funksiyaning uzluksiz bo'lgan barcha nuqtalarida o'rinlidir.

Uzilish nuqtalarda quyidagi tenglik o'rinli bo'ladi:

$$\frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)dt \right) d\alpha = \frac{f(x+0)+f(x-0)}{2} . \qquad (9)$$

(8) tenglikning o'ng tomonidagi $\cos\alpha(t-x)$ ni ochib, integralni almashtiramiz.

$$\cos\alpha(t-x) = \cos\alpha t \cos\alpha x + \sin\alpha t \sin\alpha x .$$

Bu formulani (8) ga qo'yib, $\cos\alpha x$ va $\sin\alpha x$ larni integral belgisidan tashqariga chiqaramiz. Natijada integrallash t o'zgaruvchi bo'yicha integrallanadigan quyidagi tenglikni hosil qilamiz:

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(t)\cos\alpha t dt \right)\cos\alpha x d\alpha + $$

$$+ \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t dt \right)\sin\alpha x d\alpha \qquad (10)$$

$f(t)$ funksiya $[-\infty, \infty]$ kesmada absolyut integrallanuvchi bo'lgani uchun, qavslar ichidagi t bo'yicha olingan integrallarning har biri mavjud bo'ladi. $f(t)\cos\alpha t$ va $f(t)\sin\alpha t$ funksiyalar ham absolyut integrallanuvchidir.

Juft va toq funksiyalar uchun Furye integralining xususiy hollarini ko'rib chiqamiz.

Faraz qilaylik, $f(x)$ funksiya juft bo'lsin. U holda $f(t)$ va $\cos\alpha t$ funksiyalar juft bo'lgani uchun $f(t)\cos\alpha t$ ko'paytma ham juft bo'ladi. Natijada,

$$\int_{-\infty}^{\infty} f(t)\cos\alpha t dt = 2\int_0^\infty f(t)\cos\alpha t dt$$

bo'ladi.

$\sin\alpha t$ toq bo'lgani uchun $f(t)\sin\alpha t$ ko'paytma toq funksiya bo'ladi. Natijada,

$$\int_{-\infty}^{\infty} f(t)\sin\alpha t dt = 0$$

tenglik hosil bo'ladi.

U holda (10) formula quyidagicha bo'ladi:

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos\alpha t dt \right)\cos\alpha x d\alpha . \qquad (11)$$

Endi faraz qilaylik, $f(x)$ funksiya toq bo'lsin. U holda $f(t)\cos\alpha t$ ko'paytma toq funksiya bo'ladi. Natijada

$$\int_{-\infty}^{\infty} f(t)\cos\alpha t dt = 0$$

bo'ladi.

$\sin\alpha t$ toq bo'lgani uchun $f(t)\sin\alpha t$ ko'paytma juft funksiya bo'ladi. Demak,

$$\int_{-\infty}^{\infty} f(t)\sin\alpha t dt = 2\int_{0}^{\infty} f(t)\sin\alpha t dt$$

tenglik o'rinli bo'ladi.

Natijada (10) formula quyidagicha bo'ladi:

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(t)\sin\alpha t dt\right)\sin\alpha x d\alpha \cdot \tag{12}$$

Uzilish nuqtalarida (11) va (12) tengliklarda chapdagi $f(x)$ o'rniga

$$\frac{f(x+0)+f(x-0)}{2}$$

ifodani yozish mumkinligini yana eslatib o'tamiz.

3. Furye integralining kompleks shakli

Butun sonlar o'qida aniqlangan va absolyut integrallanuvchi$f(x)$funksiyaning Furye integrali

$$f(x) = \int_0^{\infty}[a(\lambda)cos\lambda x + b(\lambda)sin\lambda x]d\lambda. \tag{1}$$

Bu yerda

$$a(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)cos\lambda tdt \quad , \quad b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)sin\lambda tdt \tag{2}$$

Eyler formulasi $e^{i\varphi} = cos\varphi + isin\varphi$ dan ma'lumki,

$$cos\lambda x = \frac{e^{i\lambda x}+e^{-i\lambda x}}{2} \text{va} \quad sin\lambda x = \frac{e^{i\lambda x}-e^{-i\lambda x}}{2i}$$

iborat.

Bularni (1) ga qo'ysak,

$$f(x) = \int_0^{\infty}\left[a(\lambda)\frac{e^{i\lambda x}+e^{-i\lambda x}}{2} + b(\lambda)\frac{e^{i\lambda x}-e^{-i\lambda x}}{2i}\right]d\lambda =$$

$$= \tfrac{1}{2}\textstyle\int_0^{\infty}\left[\big(a(\lambda)-ib(\lambda)\big)\cdot e^{i\lambda x} + \big(a(\lambda)+ib(\lambda)\big)e^{-i\lambda x}\right]d\lambda\,. \tag{3}$$

$$c(\lambda) = \pi(a(\lambda) - i\cdot b(\lambda))$$

belgilash kiritib, (2) formulalardan.

$$c(\lambda) = \int_{-\infty}^{\infty} f(t)\,(\cos\lambda t - i\cdot\sin\lambda t)d\lambda = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\,dt$$

$$c(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\,dt \tag{4}$$

$$c(-\lambda) = \int_{-\infty}^{\infty} f(t)e^{i\lambda t}\,dt \tag{5}$$

(4) va (5) ni (3) ga qo'ysak,

$$f(x)=\frac{1}{2\pi}\int_0^{\infty}\left[c(\lambda)e^{i\lambda x}+\ c(-\lambda)e^{-i\lambda x}\right]d\lambda=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+$$
$$+\frac{1}{2\pi}\int_0^{\infty}c(-\lambda)e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+\frac{1}{2\pi}\int_0^{-\infty}c(\lambda)e^{i\lambda x}d(-\lambda)=$$
$$=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+\frac{1}{2\pi}\int_{-\infty}^{0}c(\lambda)e^{i\lambda x}d=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\lambda)e^{i\lambda x}$$

Shunday qilib,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\lambda)e^{i\lambda x}\ , \tag{6}$$

bunda

$$c(\lambda)=\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}\,dt$$

(6) formula ***Furye integralining kompleks shakli*** deyiladi. Furye integralining kompleks shaklini ikki karrali integral ko‘rinishda quyidagicha ifodalanadi:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}f(t)\cdot e^{i\lambda(x-t)}\,dt \tag{7}$$

4. Furyening kosinus va sinus almashtirishlari.

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\alpha t dt\right)\cos\alpha x\,d\alpha+\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\sin\alpha t dt\right)\sin\alpha x\,d\alpha \tag{1}$$

formuladagi qavs ichida turgan integrallar α ning funksiyalaridan iborat.

Quyidagi almashtirishlarni kiritamiz:

$$A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos\alpha t dt$$
$$B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin\alpha t dt \tag{2}$$

(2) ni e’tiborga olib, (1) ni quyidagi ko’rinishda yozish mumkin:

$$f(x)=\int_0^{\infty}\left[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x\right]d\alpha\ . \tag{3}$$

(3) formulaning qisqacha fizik ma’nosini talqin qilamiz.
(3) formula $f(x)$ funksiyani 0 dan ∞ gacha uzluksiz o’zgaruvchi α chastotali garmonikaga yoyish imkoniyatini beradi. Amplitudalarning taqsimlanish qonuni va boshlang’ich fazalar α chastotaga bog’lanishida $A(\alpha)$ va $B(\alpha)$ lar bilan ifodalanadi.

Agar

$$F(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\cos\alpha t dt$$

deb olsak,

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty}f(t)\cos\alpha t dt\right)\cos\alpha x\,d\alpha \tag{4}$$

formula quyidagi ko’rinishni oladi:

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F(\alpha)\cos\alpha t dt\ . \tag{5}$$

$F(\alpha)$ funksiyani $f(x)$ funksiya uchun Furyening ***kosinus almashtirishi*** deyiladi.

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty}f(t)\sin\alpha t dt\right)\sin\alpha x\, d\alpha \qquad (6)$$

formulaga asosan quyidagi tenglikni yozish mumkin:

$$\Phi(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\sin\alpha t dt\ ,$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\Phi(\alpha)\sin\alpha t dt\ . \qquad (7)$$

$\Phi(\alpha)$ funksiya Furyening ***sinus almashtirishi*** deyiladi.

Misol. Quyida berilgan funksiyani Furye integrali ko'rinishida yozing.

$$f(x)=\begin{cases}1, & 0<x<1 \text{ да}\\ \frac{1}{2}, & x=0,\ x=1 \text{ да}\\ 0, & x<0, x>1 \text{ да}\end{cases}\cdot$$

Yechish. Bu funksiya silliq qismlardan iborat bo'lgani uchun bo'lakli silliq bo'ladi, $y=0$, $(-\infty;0)$, $y=1$, $(0;1)$, $y=0$, $(1;\infty)$ va $x=0$, $x=1$ nuqtalarda birinchi tur uzilish ega.

Bu funksiya sonlar o'qining barcha nuqtalarida absolyut integrallanuvchidir, $[0;1]$ kesmadan tashqarida uning qiymati nolga teng. Demak, bu funksiyaning butun sonlar o'qi bo'yicha integrali $[0;1]$ kesma bo'yicha integraliga keladi. Natijada bu funksiyani Furye integrali ko'rinishda ifodalash mumkin bo'ladi.

$$f(x)=\frac{1}{\pi}\int_0^{\infty}d\alpha\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)dt=$$

$$=\frac{1}{\pi}\int_0^{\infty}d\alpha\int_0^1 1\cdot\cos\alpha(t-x)dt=\frac{1}{\pi}\int_0^{\infty}\left.\frac{\sin\alpha(t-x)}{\alpha}\right|_0^1 d\alpha=$$

$$=\frac{1}{\pi}\int_0^{\infty}\left.\frac{\sin\alpha(1-x)+\sin\alpha x}{\alpha}\right|_0^1 d\alpha=\frac{1}{\pi}\int_0^{\infty}\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha(1-2x)}{2}}{\alpha}d\alpha\cdot$$

Funksiya $x=0$, $x=1$ nuqtalarda uzilishga ega bo'lgani uchun, bu integral uzilish nuqtalarida o'zgarmaydi. Uzilish nuqtalarida

$$f(x)=\frac{f(x+0)+f(x-0)}{2}=\frac{1}{2}$$

bo'ladi.

Xususiy holda $x=0$ bo'lganda

$$f(x)=\frac{1}{2}=\frac{1}{\pi}\int_0^{\infty}\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\alpha}d\alpha=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha$$

bo'ladi. Bundan

$$\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha=\frac{\pi}{2}$$

kelib chiqadi.

Mustahkamlash uchun savollar

1. Furye qatorining kompleks sakli qanday?
2. Furye integrali formulasi qanday?
3. Furye integrali koeffitsientlarining formulalari qanday?
4. Furye integrali qaysi nuqtalarda o'rinli?
5. Juft funksiyalar uchun Furye integrali qanday bo'ladi?
6. Toq funksiyalar uchun Furye integrali qanday bo'ladi?
7. Furye integralining kompleks shakli qanday bo'ladi?
8. Furyening kosinus almashtirishi deganda nimani tushunasiz?
9. Furyening sinus almashtirishi nima?