TOSHKENT AXBOROT TEXNOLOGIYALARI UNIVERSITETI

DASTURIY INJINIRING FAKULTETI

OLIY MATEMATIKA KAFEDRASI

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Mavzu: Aniq integralning tadbiqi. Aniq integral yordamia yoy uzunligi va soha yuzlarini hisoblash.

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Aniq integralning tatbig'i va yoy uzunligi, soha yuzining integral ifodasi.

Reja:

I. ANIQ INTEGRALNING TA`RIFI VA UNING GEOMETRIK MA`NOSI.

II. ANIQ INTEGRALNING XOSSALARI.

III. SOHANING YUZINI ANIQ INTEGRAL YORDAMIDA TOPISH.

IV. EGRI CHIZIQ YOYINING UZUNLIGINI HISOBLASH.

1.Aniq integralning ta`rifi va uning geometrik ma`nosi

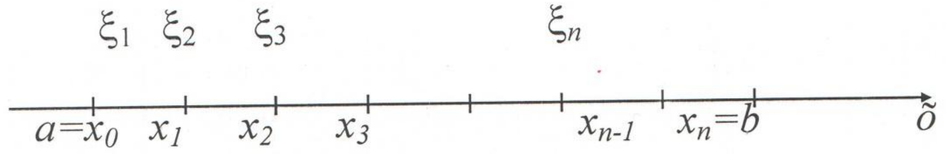
Aniq integral- matematik analizning asosiy tushunchalaridan biridir. Egri chiziqlar bilan chegaralangan yuzalarni, egri chiziq yoylari uzunliklarini, hajmlarini, ishlarni, tezliklarni, yo'llarni, inersiya momentlarini hisoblash masalasi u bilan bogliq.

$[a,b]$ kesmada $y=f(x)$ uzluksiz funksiya berilgan bo'lsin. Quyidagi amallarni bajaramiz.

1) $[a,b]$ kesmani $a= x_0,x_1,x_2,\ldots,x_{n-1},x_n=b$ nuqtalar bilan n ta qismga ajratamiz va ular quyidagicha joylashgan bo'lsin.

$$a= x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$$

Bularni qismiy intervallar deymiz.



2) Qismiy intervallarning uzunliklarini quyidagicha belgilaymiz:

$$\Delta x_1=x_1-x_0\,;\ \Delta x_2=x_2-x_1\,;\ \Delta x_3=x_3-x_2\,;\ldots\ldots\ \Delta x_i=x_i-x_{i-1}\,;\ldots\ \Delta x_n=x_n-x_{n-1}\,;$$

3) Har bir qismiy intervalning ichidan bittadan ixtiyoriy nuqta olamiz:

$$\xi_1,\ \xi_2,\ \xi_3,\ldots\ldots\ \xi_{n-1},\ \xi_n$$

4) Olingan ξ nuqtalarda funksiyaning qiymatini topamiz:

$$f(\xi_1);\ f(\xi_2);f(\xi_3),\ldots\ldots\ f(\xi_{n-1});\ f(\xi_n)$$

5) Har bir funksiyaning hisoblangan qiymatini tegishli qismiy intervalning uzunligiga ko’paytiramiz:

$$f(\xi_1)\ \Delta x_1;\ f(\xi_2)\ \Delta x_2\,;\ f(\xi_3)\ \Delta x_3,\ldots\ldots\ f(\xi_n)\ \Delta x_n$$

6) Hosil bo’lgan ko’paytmalarni qo’shamiz va σ deb belgilaymiz.

$$\sigma=f(\xi_1)\ \Delta x_1+f(\xi_2)\ \Delta x_2+f(\xi_3)\ \Delta x_3+\ldots\ +f(\xi_{n-1})\ \Delta x_{n-1}+f(\xi_n)\ \Delta x_n\,;$$

Shunday qilib, hosil bo’lgan σ yig’indi $f(x)$ funksiya uchun $[a,b]$ kesmada tuzilgan integral yig’indi deb ataladi va quyidagicha belgilanadi.

$$\sigma=\sum_{i-1}^{n} f(\xi_i)\Delta x_i \quad (1)$$

Bu integral yig’indining geometrik ma`nosi, agar $f(x)\geq 0$ bo’lsa, u holda asoslari Δx_1, Δx_2 ,... Δx_n va balandliklari $f(\xi_1)$, $f(\xi_2)$,... $f(\xi_n)$ bo’lgan to’g’ri to’rtburchak yuzlarining yig’indisidan iborat.

Agarda bo'lishlar sonini, n ni orttira borsak ($n \to \infty$)da u holda eng katta intervalning uzunligi nolga intiladi, ya`ni $\max \Delta x_i \to 0$ bo'ladi.

Ta`rif: Agar *S* integral yig'indi *[a,b]* kesmani qismiy $[x_{i-1}, x_i]$ kesmalarga ajratish usuliga va ularning har biridan ξ_I nuqtasini tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son *[a,b]* kesmada *f(x)* funksiyadan olingan aniq integral deyiladi va quyidagicha belgilanadi.

$$\int_a^b f(x)dx$$

f(x) dan *x* bo'yicha *a* dan *b* gacha olingan aniq integral deb o'qiladi.

Bu yerda *f(x)* integral ostidagi funksiya *[a,b]* kesma-integrallash oralig'i; *a* son integralning quyi chegarasi, *b* son integralning yuqori chegarasi;

Shunday qilib, aniq integralning ta`rifidan quyidagini yozish mumkin.

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

Aniq integral hamma vaqt mavjud bo'lavermas ekan. Aniq integralning mavjudlik teoremasini quyida keltiramiz. (Isbotsiz).

Teorema: Agar *f(x)* funksiya *[a,b]* kesmada uzluksiz bo'lsa, u integrallanuvchidir, ya`ni bunday funksiyaning aniq integrali mavjuddir.

Shunday qilib, $\int_a^b f(x)dx$ aniq integralning qiymati $y=f(x)$ funksiyaning grafigi bilan va $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuziga son jihatdan teng bo'ladi.

1- Izoh: Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2-Izoh. Agar aniq integralning chegaralari teng bo'lsa, har qanday funksiya uchun quyidagi tenglik o'rinli ;

$$\int_a^a f(x)dx = 0$$

haqiqatdan ham, geometrik nuqtai nazardan egri chiziqli trapetsiya asosining uzunligi nolga teng bo'lsa, uning yuzi ham nolga teng bo'ladi.

2.Aniq integralning asosiy xossalari

1- xossa: O'zgarmas ko'paytuvchini aniq integral belgisining tashqarisiga chiqarish mumkin.

$$\int_a^b Af(x)dx = A\int_a^b f(x)dx$$

Isbot: $\int_a^b Af(x)dx = \lim_{\max\Delta x_i \to 0} \sum_{0=1}^{n} Af(\xi_i)\Delta x_i = A \lim_{\max\Delta x_i \to 0} \sum_{0=1}^{n} f(\xi_i)\Delta x_i = A\int_a^b f(x)dx$

2-xossa: Bir necha funksiyalar algebraik yig'indisining aniq integrali qo'shiluvchilar aniq integrallarning algebraik yig'indisiga teng.

Masalan: $\int_a^b [f_1(x) + f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$

3-xossa. Agar *[a, b]* kesmada *f(x)* va φ *(x)* funksiyalar uchun *f(x)* ≥ φ *(x)* shart bajarilsa, u holda $\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$ bo'ladi.

4-xossa: Agar *[a,b]* kesma bir necha qismga bo'linsa, u holda *[a,b]* kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng.

Masalan: $a<c<b$ bo'lsa, u holda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

5-xossa: Aniq integralning qiymati funksiyaning ko'rinishiga va integrallash chegaralariga bog'liq, lekin integral ostidagi ifodaning harflariga bog'liq emas.

$$\int_a^b f(x)dx = \int_a^c f(t)dt = \int_a^b f(z)dz$$

3.Soha yuzini aniq integral yordamida topish.

Figuralar yuzlarini Dekart koordinatalar sistemasida hisoblashni ko'rib chiqamiz.

Agar *[a,b]* kesmada $f(x)\geq 0$ bo'lsa, u holda $y=f(x)$ egri chiziq *OX* o'qi va $x=a$ va $x=b$ egri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzi quyidagiga teng edi.

$$S=\int_a^b f(x)dx$$

$y_1=f_1(x)$ va $y_2=f_2(x)$ egri chiziqlar hamda $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan figuraning yuzini hisoblash kerak bo'lsin.

U holda $f_1(x)>f_2(x)$ shart bajarilgan figuraning yuzi quyidagiga teng bo'ladi.

$$S=\int_a^b f_1(x)dx-\int_a^b f_2(x)dx=\int_a^b\big(f_1(x)-f_2(x)\big)dx \qquad (1)$$

1-Misol : $y=x^2+1$ va $y=3-x$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechish: Buning uchun chiziqlarning kesishish nuqtalarini topamiz.

$x^2+1=3-x\Rightarrow x^2+x-2=0$

$\underline{x_1=1}$ $\underline{x_2=-2}$

Demak,

$$S=\int_{-2}^{1}(3-x)dx-\int_{-2}^{1}(x^2+1)dx=\int_{-2}^{1}(3-x-x^2-1)dx=\int_{-2}^{1}(2-x-x^2-1)dx=\left(2x-\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-2}^{1}=$$

$$=\left(2\cdot 1-\frac{1}{2}-\frac{1}{3}\right)-\left(2\cdot(-2)-\frac{4}{2}-\frac{(-8)}{3}\right)=2-\frac{1}{2}-\frac{1}{3}+4+2-\frac{8}{3}=8-\frac{1}{2}-3=5-\frac{1}{2}=4{,}5 \text{ кв. бирлик}$$

Agar egri chiziqli trapetsiyaning yuzi $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$ parametrik shaklda berilgan chiziq bilan chegaralangan bo'lsa, sohaning yuzi quyidagi formula bilan hisoblanadi.

$$S=\int_{a}^{b} ydx = S=\int_{\alpha}^{\beta}\psi(t)\varphi'(t)dt$$

1-Misol .

$\left.\begin{matrix} x=a\cdot\cos t \\ y=b\sin t\end{matrix}\right\}$ Ellips bilan chegaralangan sohani yuzini hisoblang.

Yechish : Ellipsning yuqori yarim yuzini hisoblab, uni ikkiga ko'paytiramiz.

Demak,

$x=a\cdot\cos t;\, dx=-a\sin tdt$

$x\to -a \;\; da \;\; t=\pi; \;\; x\to a \;\; da \;\; t=0$

$$S=2\cdot\int_{\pi}^{b} ydx=2\int_{\pi}^{0} b\sin t\cdot a\sin tdt=2ab\int_{\pi}^{0}\sin^2 tdt=2ab\int_{\pi}^{0}\frac{1-\cos 2t}{2}dt=2ab\left[\int_{\pi}^{0}\frac{1}{2}dt-\int_{\pi}^{0}\frac{\cos 2t}{2}\right]dt=$$

$$=\frac{1}{2}2ab\left[t\Big|_0^{\pi}-\frac{\sin 2t}{2}\Big|_0^{\pi}\right]=ab\left(\pi-0-\frac{\sin 2\pi}{2}+\frac{\sin 2\cdot 0}{2}\right)=ab(\pi)=\pi ab$$

Agar funksiya qutb koordinatalar sistemasida berilgan bo'lsa, u holda sohaning yuzi quyidagi formula bilan hisoblanadi.

$$S=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 du$$

4.Egri chiziq yoyining uzunligini hisoblash

y=f(x) egri chiziqning *AB* yoyining uzunligi *l* bo'lsin. Bu yoy uzunligining *dl* differensiali quyidagi formula bilan ifodalanadi:

$$dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx \qquad (3)$$

Demak, AB yoyning uzunligi quyidagi formuła bilan ifodalanadi.

$$L_{AB} = \int_a^b dl = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx \qquad (4)$$

bunda a va b erkli o'zgaruvchi bo'lib, X ning A va B nuqtalardagi qiymatlari.

Agarda egri chiziq $x=f(y)$ tenglama bilan berilgan bo'lsa, u holda yoy uzunligi:

$$L_{AB} = \int_m^n \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

formula bilan ifodalanadi.

1-Misol : $x^2+y^2=r^2$ aylananing uzunligini toping.

Yechish: Aylananing tenglamasini differensiallab, quyidagini topamiz.

$$2x + 2y \cdot \frac{dy}{dx} = 0;$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Integrallash chegaralarini O dan r gacha olib, aylana yoyining choragi uzunligini (4) formula bo'yicha hisoblaymiz:

$$L = \int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2} \cdot dx = \int_0^r \sqrt{1+\frac{x^2}{y^2}} dx = \int_0^r \sqrt{\frac{y^2+x^2}{y^2}} dx = \int_0^r \sqrt{\frac{r^2}{y^2}} dx = \int_0^r \frac{r}{y} dx = r\int_0^r \frac{dx}{\sqrt{z^2-x^2}} =$$

$$= r \cdot \arcsin\frac{x}{r}\Big|_0^r = r \cdot \arcsin\frac{r}{r} - r \cdot \arcsin\frac{0}{r} = r \cdot \arcsin 1 - r \cdot \arcsin 0 = r \cdot \frac{\pi}{2} - 0 = \frac{r\pi}{2};$$

Bundan aylananing uzunligi $C = 4L = L \cdot \frac{r\pi}{2} = 2\pi r$