ANDIJON MASHINASOZLIK INSTITUTI

“MASHINASOZLIK” FAKULTETI

“OLIY MATEMATIKA” KAFEDRASI

“Oliy matematika” fanidan

Kompleks sonlar va ular ustida amallar

mavzusida yozilgan

REFERAT

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R E J A:

1. Kompleks sonlar va ular ustida amallar.

2. 2.1. Kompleks sonning logarifmi.

2.2. Soha tushunchasi.

2.3. Jordan chizig‘i.

2.4. Stereografik proyeksiya.

3. Kompleks o‘zgaruvchining funksiyalari va ularning aniqlanish sohasi.

4. Funksiyaning limiti va uzluksizligi.

5. Asosiy elementar funksiyalar.

6. Kompleks o‘zgaruvchili funksiyasining hosilasi.

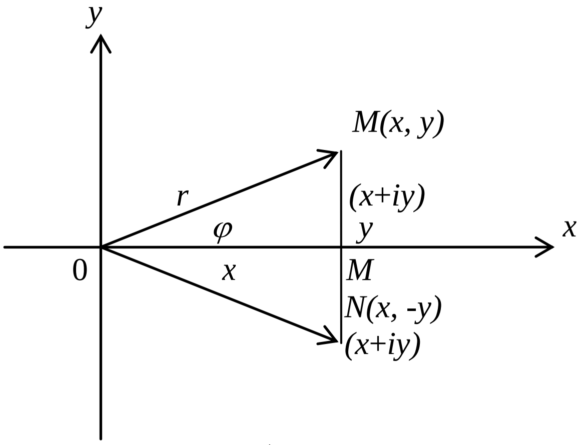
I. **<u>1-ta’rif</u>. Kompleks son** deb *x+iy* ko‘rinishdagi ifodaga aytiladi, bunda *x* va *y* – haqiqiy sonlar, *i* – mavhum birlik; $i=\sqrt{-1}$ kompleks sonlarni z harfi bilan belgilaymiz, ya’ni $z=x+iy, x$ – kompleks sonning haqiqiy qismi, $i\cdot y$- kompleks sonning mavhum qismi, *y* – mavhum qismining koeffitsiyenti deyiladi. *x* va *y* lar quyidagicha belgilanadi:

$$x=Re\,z, \quad y=Jm\,z$$

<u>2-ta’rif</u>. Agar $x_1=x_2,\ y_1=y_2$ bo‘lsa, $z_1=x_1+iy_1, z_2=x_2+iy_2$- ikki kompleks son o‘zaro teng, ya’ni $z_1=z_2$ deyiladi.

<u>3-ta’rif</u>. $z=x+iy$ va $\bar{z}=x-iy$ kompleks sonlar **qo‘shma kompleks sonlar** deyiladi.

Kompleks sonlarning geometrik tasviri va trigonometric formasini ko‘ramiz. To‘g‘ri burchakli Dekart koordinatalar sistemasidagi har bir (*x, y*) nuqtaga bitta $x+iy$ kompleks sonni mos keltiraylik. Umuman shu usulda har bir kompleks songa tekislikda bitta nuqta mos keladi va aksincha tekislikdagi har bir nuqtaga bitta kompleks son mos keladi. Absssissa o‘qi haqiqiy sonlarning geometrik o‘rni, ordinata o‘qi mavhum *iy* sonlarning geometric o‘rni bo‘ladi. Shuning uchun absississalar o‘qi haqiqiy o‘q, ordinatalar o‘qi **mavhum o‘q** deyiladi.



1-rasm

Tekislikning har bir (*x, y*) nuqtasiga koordinatalar boshidan chiqqan, oxiri shu nuqtada bo‘lgan vektorni mos keltirish mumkin. Shuningdek, har bir *(x+iy)* kompleks songa koordinatalar *x* va *y* bo‘lgan $\overrightarrow{OM}$ vektor mos keltiriladi.

1-rasmgaga asosan: $r=\sqrt{x^2+y^2}$, $tg\,\varphi=\dfrac{y}{x}$, $\varphi=arctg\,\dfrac{y}{x}$, $x=r\cos\varphi$, $y=r\sin\varphi$.

Unda $z=x+iy=r\cos\varphi+ir\sin\varphi=r(\cos\varphi+i\sin\varphi)$, yoki

$$z=r(\cos\varphi+i\sin\varphi) \qquad (1)$$

bunda *r* – kompleks sonning moduli, ya’ni $r=|z|$, φ - uning argumenti $\varphi=Arg\,z$. (Agar $-\pi<Arg\,z\le\pi$ bo‘lsa, unda *Argz=argz* bo‘ladi *argz* – bosh argument deyiladi). (1) formula – kompleks sonning **trigonometrik formasi** deyiladi. Agar Eyler formulasini $e^{i\varphi}=\cos\varphi+i\sin\varphi$ e’tiborga olsak, unda

$$z=re^{i\varphi}$$

(2) kompleks sonning **ko‘rsatkichli formasi** deyiladi.

1-misol. $z=l+i$ trigonometrik formaga keltirilsin.

Yechish. $x=1,\ y=1,\ r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}\ \ tg\,\varphi=1\Rightarrow\varphi=\pi/4$. Demak

$$z=\sqrt{2}\left(cos\frac{\pi}{4}+i\,sin\frac{\pi}{4}\right)$$

2-misol. $z=-1$ son trigonometrik formaga keltirilsin.

Yechish. $x=-1,\ \ y=0,\ \ r=\sqrt{x^2+y^2}=1,\ \ tg\,\varphi=0,\ \varphi=\pi,\ z=cos\,\pi+i\,sin\,\pi$.

Kompleks sonlar ustidagi amallar.

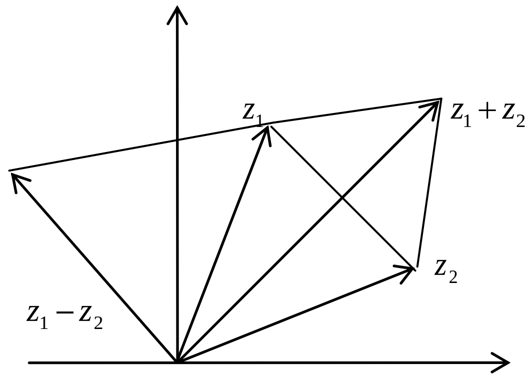
1) qo‘shish va ayirish.

$z_1=x_1+i\,y_1,$

$z_2=x_2+i\,y_2$

$$z_1\pm z_2=(x_1+i\,y_1)\pm(x_2+i\,y_2)=(x_1\pm x_2)+i(y_1\pm y_2) \qquad (3)$$

Demak, kompleks sonlar qo‘shilganda (ayrilganda) ularning haqiqiy qismlari alohida va mavhum qismlari alohida qo‘shiladi (ayriladi). Kompleks sonlarni qo‘shish va ayirish vektorlar qo‘shilishi va ayrilishiga mos bo‘ladi (2-rasmga qarang).



2-rasm

$|z_2-z_1|$ - kompleks sonlar ayirmasining moduli.

2) ko‘paytirish va bo‘lish.

$z_1=x_1+i\,y_1,$

$z_2=x_2+i\,y_2$

a) $z_1\cdot z_2=(x_1+i\,y_1)(x_2+i\,y_2)=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$.

Agar kompleks sonlarni trigonometrik formada olsak, unda

$$z_1\cdot z_2=r_1(cos\,\varphi_1+i\,sin\,\varphi_1)\cdot r_2(cos\,\varphi_2+i\,sin\,\varphi_2)=r_1r_2\left[(cos\,\varphi_1\,cos\,\varphi_2-sin\,\varphi_1\,sin\,\varphi_2)+\right.$$
$$\left.+\ i(sin\,\varphi_1\,cos\,\varphi_2+cos\,\varphi_1\,sin\,\varphi_2)\right]=r_1r_2\left[(cos(\varphi_1+\varphi_2)+i\,sin(\varphi_1+\varphi_2))\right],\quad yoki$$
$$z_1\cdot z_2=r_1r_2\left[(cos(\varphi_1+\varphi_2)+i\,sin(\varphi_1+\varphi_2))\right]$$

Demak, kompleks sonlarni ko‘paytirishda modullari ko‘paytiriladi, argumentlari esa qo‘shiladi. $z_1=r_1e^{i\varphi 1},\ \ z_2=r_2e^{i\varphi 2},\ \ z_1*z_2=r_1r_2e^{i\varphi 1}\cdot e^{i\varphi 2}=r_1r_2e^{i(\varphi 1+\varphi 2)};$

$z_1=r_1e^{i\varphi_1},\ \ z_2=r_2e^{i\varphi_2},\ \ z_1\cdot z_2=r_1r_2e^{i\varphi_1}\cdot e^{i\varphi_2}=r_1r_2e^{i(\varphi_1+\varphi_2)}$

b) $\dfrac{z_1}{z_2}=\dfrac{x_1+i\,y_1}{x_2+i\,y_2}=\dfrac{(x_1+i\,y_1)(x_2-i\,y_2)}{(x_2+i\,y_2)(x_2-i\,y_2)}=\dfrac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}=$

$=\dfrac{x_1 x_2+y_1 y_2}{x_2^2+y_2^2}+i\dfrac{x_2 y_1-x_1 y_2}{x_2^2+y_2^2}$. Agar z_1 va z_2 trigonometrik formada berilgan bo‘lsa,

unda $\dfrac{z_1}{z_2}=\dfrac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_{21}}}=\dfrac{r_1}{r_2}e^{i(\varphi_1-\varphi_2)}=\dfrac{r_1}{r_2}\left[cos(\varphi_1-\varphi_2)+i\,sin(\varphi_1-\varphi_2)\right]$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}=\left[cos(\varphi_1-\varphi_2)+i\,sin(\varphi_1-\varphi_2)\right] \qquad (5)$$

Demak, kompleks sonlarni bo‘lishda ularning argumentlari ayriladi, modullari bo‘linadi.

<u>3) darajaga ko‘tarish va ildiz chiqarish.</u>

a) $z=re^{i\varphi}$, kompleks sonini n–darajaga ko‘taraylik $z^n=\left(re^{i\varphi}\right)^n=r^n e^{in\kappa}$, yoki

$$z^n=r^n\left(cos\,n\varphi+i\,sin\,n\varphi\right) \qquad (6)$$

Demak, trigonometrik formada berilgan kompleks sonni darajaga ko‘tarishda modul shu darajaga ko‘tariladi, argument darajaga ko‘paytiriladi.

Agar (6) da r=1 bo‘lsa $\left[r\left(cos\,\varphi+i\,sin\,\varphi\right)\right]^n=cos\,n\varphi+i\,sin\,\varphi$ Muavr formulasi hosil bo‘ladi.

b) $z=re^{i\varphi}$, kompleks sonini n–darajali ildizi w bo‘lsin, ya’ni

$\sqrt[n]{z}=w=\rho e^{i\psi}$, $\quad z=w^n=\rho^n\left(cos\,n\psi+i\,sin\,n\psi\right),\quad r\left(cos\,\varphi+i\,sin\,\varphi\right)=\rho^n\left(cos\,n\psi+i\,sin\,n\psi\right)\Rightarrow$

$r=\rho^n,\quad n\psi=\varphi+2k\pi,\quad \rho=\sqrt[n]{z},\quad \psi=\dfrac{\varphi+2k\pi}{n},\quad ya'ni$

$$\sqrt[n]{z}=\sqrt[n]{r}\left[cos\frac{\varphi+2k\pi}{n}+i\,sin\frac{\varphi+2k\pi}{n}\right] \qquad (7)$$

3-misol. $\sqrt[3]{-8}=?\quad -8=8\left(cos\,\pi+i\,sin\,\pi\right)$, chunki $r=\sqrt{64}=8,\ \varphi=\pi$.

Yechish. $\sqrt[3]{-8}=2\left(cos\dfrac{\varphi+2k\pi}{n}+i\,sin\dfrac{\varphi+2k\pi}{n}\right)k=0,1,2$, bo‘lganda

$$\sqrt[3]{-8}=\begin{cases}1+i\sqrt{3}\\ -2\\ 1-i\sqrt{3}\end{cases}$$

II. 2.1. $z=r\left(cos\,\varphi+i\,sin\,\varphi\right)=re^{i\varphi}$ kompleks son berilgan bo‘lsin. $z=re^{i(\varphi+2k\pi)}=re^{i\varphi}$

$ln\,z=ln\left(re^{i\varphi}\right)=ln\,r+i\varphi\,i\,ne=l\,n\,r+i\varphi$, ya’ni

$$ln\,z=ln\,r+i\varphi \qquad (1)$$

$$ln\,z=l\,n\,r+i\varphi+2\pi i \qquad (2)$$

1-misol. z=-1 ning logarifmini toping.

Yechish. $z=-1=\cos\pi+i\sin\pi;\quad r=1,\quad \varphi=\pi$

$ln\,z=ln1+i\pi=i\pi$

$ln\,z=i\pi+2k\pi i=i\pi\left(1+2k\right),\ k=0,\ \pm1,\ \pm2,...$

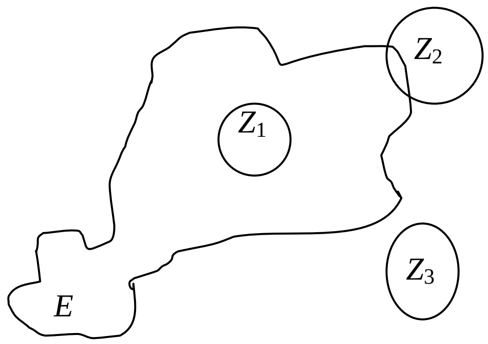
2.2. Kompleks sonlar tekisligi (Z) da biror E to‘plam berilgan bo‘lsin.

<u>1 - ta’rif</u>. z – nuqtaning kichik atrofi deb, markazi z nuqtada bo‘lgan yetarli kichik radiusli doiraga tegishli nuqtalar to‘plamiga aytiladi.

<u>2 - ta'rif</u>. Agar z nuqtaning kichik atrofidagi barcha nuqtalar to'plamga tegishli bo'lsa *z* nuqta *E* to'plamning **ichki nuqtasi** deyiladi.

<u>3 - ta'rif</u>. Agar z nuqtaning kichik atrofidagi nuqtalarning ba'zilari *E* ga tegishli, ba'zilari tegishli bo'lmasa, u *E* ning chegaraviy nuqtasi deyiladi.

3-rasmda z_1 - ichki, z_2 - chegaraviy, z_3 - tashqi nuqtalardir.



3-rasm

2-misol. a) $E: |z|<1,\ \ x^2+y^2<1$ — aylana ichki nuqtalari to'plami.

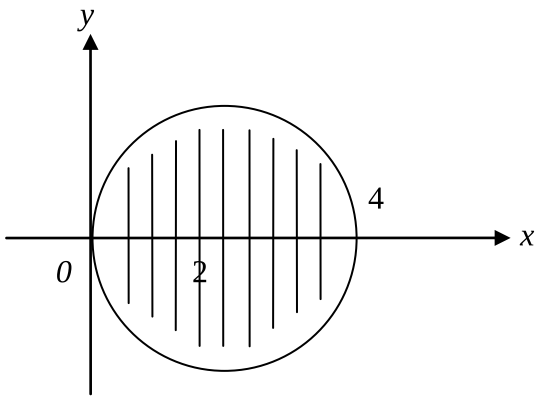
b) $E: |z|=1,\ \ x^2+y^2=1$ — aylana nuqtalari to'plami.

Agar quyidagi ikki shart bajarilsa:

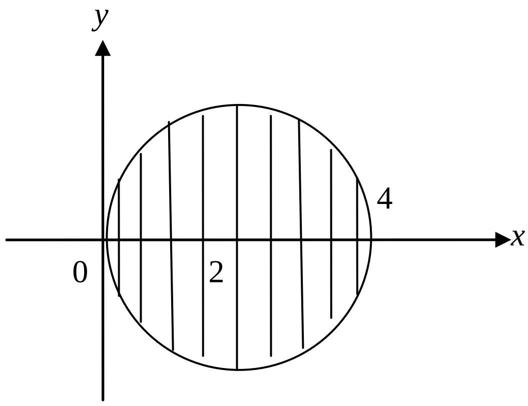
1. *E* – to'plam faqat ichki nuqtalardan iborat bo'lsa;
2. *E* – to'plamning har qanday ikki nuqtasini birlashtiruvchi uzluksiz chiziqning barcha nuqtalari *E* ga tegishli bo'lsa, tekislikdagi nuqtalar to'plami (*E*) — **soha** deyiladi.

Agar soha chegarasidagi har qanday nuqta atrofida shu sohaning hech bo'lmaganda bitta nuqtasi mavjud bo'lsa, shu nuqta **chegaraviy nuqta** deyiladi. Chegaraviy nuqtalari o'ziga tegishli bo'lmagan *E* soha ochiq soha, chegaraviy nuqtalari o'ziga tegishli bo'lgan soha **yopiq soha** deyiladi.

3-misol. a) $E: |z-2|<2,\ \ |x+iy-2|<2,\ \ (x-2)^2+y^2<4$ - ochiq soha (rasm 4).



4-rasm



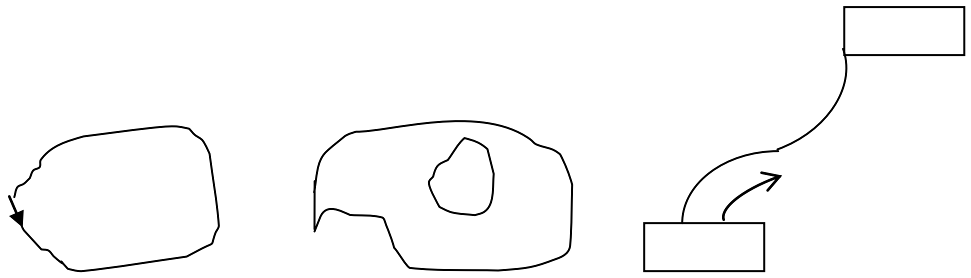
5-rasm

b) $E: |z-2|\le 2$ yopiq soha, (rasm 5).

2.3. Haqiqiy t argumentli $x=x(t),\ \ y=y(t)\ \ (\alpha<t<\beta)$ uzluksiz funksiyalar berilgan bo'lsin. Ular tekislikdagi biror uzluksiz egri chiziqning parametrik tenglamasidan iborat bo'ladi. Agar (bu egri chiziqdagi) t ning ikkita har xil

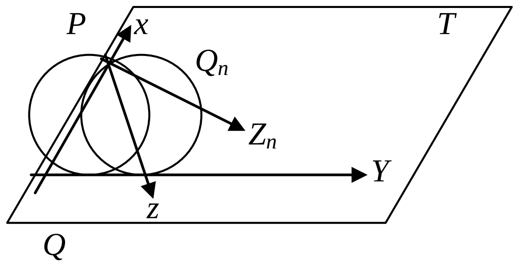
$Z(\beta)$

nuqtalar mos kelsa, ya'ni karrali nuqtalarga ega bo'lmasa bu chiziq **Jordan chizig'i** deyiladi yoki uzluksiz silliq chiziq deyiladi (6 v-rasm).



Agar $z=x+iy$ ga $x=x(t),\ y=y(t)$ ni qo'ysak $z=x(t)+iy(t)=z(t)$ $(\alpha\le t\le\beta)$ egri chiziq tenglamasi hosil bo'ladi. Bunda parametr t α dan β gacha o'zgarganda z nuqta Jordan chizig'ini chizadi. Agar $z(\alpha)=z(\beta)$ bo'lsa, chiziq **yopiq chiziq** deyiladi. Bitta yopiq Jordan chizig'i bilan chegaralangan soha bir bog'lamli (6 a-rasm), aks holda ko'p bog'lamli soha deyiladi (6 b–rasm).

2.4. Berilgan $z=x+iy$ kompleks sonni tekislikda nuqtaga mos keltirish mumkinligini ko'rgan edik. Endi har qanday kompleks sonni sferadagi nuqta bilan tasvirlash ham mumkinligini ko'rsatamiz. Buning uchun sferaning janubiy qutbini *xoy* tekislikning 0 markazi bilan ustma-ust qo'yamiz. Mana shu tekislikdagi $z=x+iy$ nuqtani *P* shimoliy qutb bilan to'g'ri chiziq orqali tutashtirsak, u chiziq sferani biror *Q* nuqta tekislikdagi *z* nuqtaning **sferadagi aksi** deyiladi. Shu usulda *xoy* tekislikning barcha z_n nuqtalarining ham sferadagi aksini topish mumkin, faqat *P* nuqtaning o'ziga tekislikdagi cheksiz uzoqlashgan $z=\infty$ nuqta mos keladi deb qabul qilinadi. *xoy* tekislikning va sferaning nuqtalarini yuqoridagidek bir qiymatli moslash **stereografik proyeksiya** deyiladi.

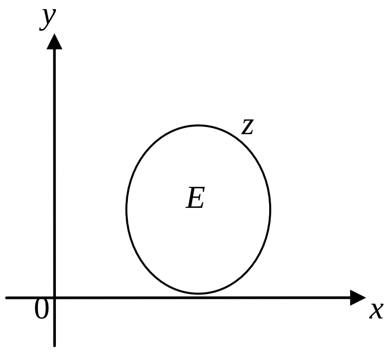


7-rasm

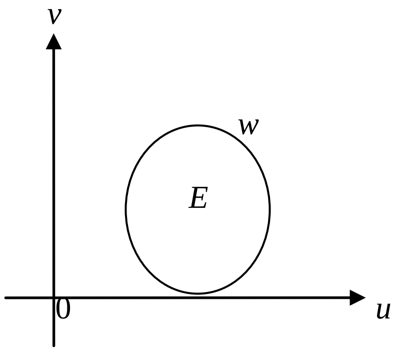
III. Biror (Z) kompleks tekisligida *E* kompleks $z=x+iy$ sonlar to'plami berilgan bo'lsin.

<u>1-ta'rif.</u> Agar *E* to'plamdan olingan har bir $z=x+iy$ songa biror qonun bo'yicha *G* dan olingan aniq bir $w=u+iv$ kompleks son mos kelsa, *E* to'plamda $w=f(z)$ **funksiya berilgan** deyiladi.

Bunda $z=x+iy$ argument, $w=u+iv$ esa funksiyadir. *E* to'plam $f(z)$ funksiyaning **aniqlanish sohasi** deyiladi.



8-rasm



9-rasm

<u>2-ta’rif.</u> Agar $z=x+iy$ ning har bir qiymatiga *w* ning birgina qiymati mos kelsa, $w=f(z)$ bir qiymatli, aks holda ko‘p qiymatli funksiya deyiladi.

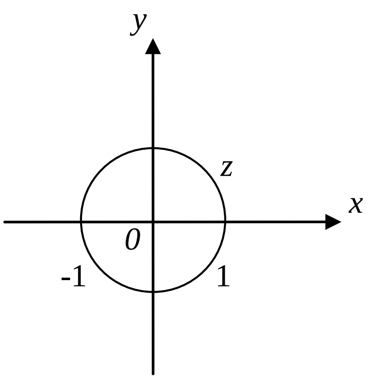
Masalan, $w=z^2$, $w=\frac{1}{2}$, $w=2z^3$,... - bir qiymatli, $w=\sqrt{z}$, $w=\sqrt[4]{z}$, $w=\frac{1}{\sqrt[3]{z-1}}$,… - ko‘p qiymatli funksiyalardir.

Agar *z* ning qiymatlariga tegishli nuqtalarni (*Z*) tekisligida, *w* ning qiymatlariga tegishli nuqtalarni (*W*) tekisligiga joylashtirsak, (*Z*) tekisligidagi *E* to‘plamdan olingan har bir *z* nuqta (*W*) tekisligidagi *w* nuqtaga mos keladi. Natijada *E* to‘plamning aksi (*W*) tekislikka tushib, biror *G* to‘plamni hosil qiladi. Bunga esa, $w=f(z)$ funksiya yordamida to‘plamni *G* to‘plamga **akslantirish** deyiladi.

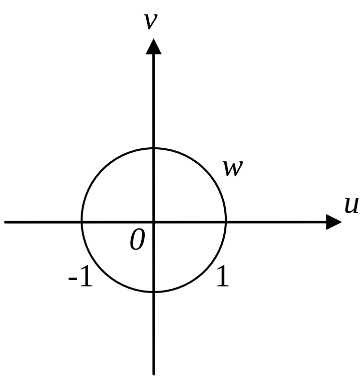
1-misol. $w=z^2$ funksiya yordami bilan (*Z*) tekisligidagi $|z|=1$ chiziqning (*W*) tekisligidagi aksi topilsin.

Yechish. $w=u+iv$, $w=z^2=(x+iy)^2=x^2-y^2+2xyi$

$u=x^2-y^2$, $v=2xy$, $u^2+v^2=(x^2-y^2)^2+(2xy)^2=(x^2+y^2)^2=|z|^4=1$



10-rasm



11-rasm

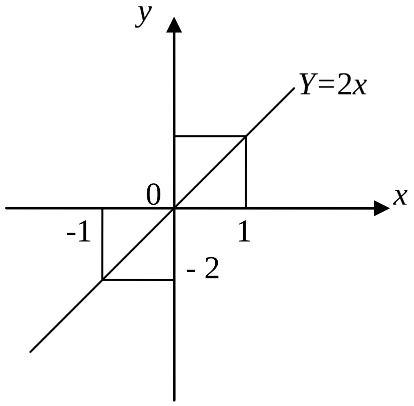
2-misol. $w=z^2$ funksiya yordami bilan (*Z*) tekisligidagi $y=kx$ to‘g‘ri chiziqning (*W*) tekisligidagi aksi topilsin.

Yechish. $y=kx,\ w=z^2$

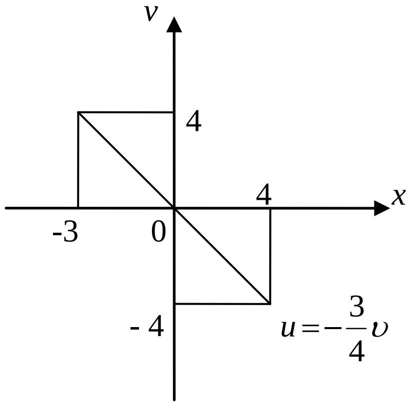
$u=x^2-y^2,\ \ v=2xy,\ \ y=kx$

$$\begin{cases} u=x^2-k^2x^2=x^2\left(1-k^2\right) \\ v=2x\cdot kx=2kx^2 \end{cases} \Rightarrow \frac{u}{v}=\frac{1-k^2}{2k};\quad u=\frac{1-k^2}{2k}$$

Agar $k = 2$ bo‘lsa, u holda 12 va 13 rasmga ega bo‘lamiz.



12-rasm



13-rasm

IV. Biror E – kompleks sohada $w=f(z)$ funksiya berilgan bo‘lib, $z_0\in E$ nuqta berilgan bo‘lsin.

<u>1-ta’rif.</u> Agar oldindan berilgan har qanday kichik $\varepsilon>0$ son uchun shunday musbat $\delta(\varepsilon)>0$ sonni topish mumkin bo‘lsaki, $|z-z_0|<\delta\ (\varepsilon)$ bo‘lganda $|f(z)-A|<\varepsilon$ o‘rinli bo‘lsa, $f(z)$ funksiya A o‘zgarmas songa intiladi deyiladi va quyidagicha yoziladi:

$$\lim_{z\to z_0} f(z)=A \tag{1}$$

<u>2-ta’rif.</u> Agar oldindan berilgan har qanday kichik musbat $\varepsilon>0$ son uchun shunday musbat $\delta>0$ sonni topish mumkin bo‘lsaki, bunda $|z-z_0|<\delta$ o‘rinli bo‘lganda, $|f(z)-f(z_0)|<\varepsilon$ tengsizlik o‘rinli bo‘lsa, $f(z)$ funksiya z_0 nuqtada **uzluksiz** deyiladi va quyidagicha yoziladi:

$$\lim_{z\to z_0} f(z)=f(z_0) \tag{2}$$

Bu geometrik jihatdan $w=f(z)$ funksiya z_0 nuqtada uzluksiz bo‘lsa, (Z) tekisligidagi markazi z_0 nuqtada, radiusi $\delta(\varepsilon)$ ga teng bo‘lgan doira nuqtalari, (w) tekislikdagi markazi w_0 nuqtada, radiusi ε bo‘lgan doira nuqtalariga o‘tishini ko‘rsatadi.

<u>3-ta’rif.</u> Sohaning har bir nuqtasida uzluksiz bo‘lgan funksiyalar shu sohada **uzluksiz** deyiladi.

Kompleks o‘zgaruvchili funksiyaning limiti va uzluksizligi ta’riflari haqiqiy o‘zgaruvchining limiti va uzluksizligi ta’rifiga o‘xshash bo‘lgani uchun uzluksiz funksiyaning xossalari, ular bilan bajariladigan amallar, ular haqidagi teoremalar va ularning isboti ham haqiqiy o‘zgaruvchili funksiyalar isbotlari kabi bo‘ladi.

Uzluksizlikni quyidagicha ham ta'riflash mumkin: $w(z)=f(z)$, $w_0=f(z)$, $z=x+iy$, $z_0=x_0+iy_0$, $\Delta x=x-x_0$, $\Delta y=y-y_0$, bo'lsa, $\Delta z=z-z_0=\Delta x+i\Delta y$ va $\Delta w=f(z)-f(z_0)$ funksiya orttirmasi bo'ladi.

4-ta'rif. Agar haqiqiy kichik musbat $\varepsilon>0$ uchun shunday $\delta(\varepsilon)>0$ son topish mumkin bo'lsa, $|\Delta z|<\delta(\varepsilon)$ bo'lganda $|\Delta W|<\varepsilon$ o'rinli bo'lsa, $f(z)$ funksiya z_0 nuqtada uzluksiz deyiladi va quyidagicha yoziladi

$$\lim_{\Delta z\to 0}\Delta W=0 \tag{3}$$

Agar $z=x_0+iy_0$ $f(z)-f(z_0)=[u(x,y)-u(x_0,y_0)]+i[v(x,y)-v(x_0,y_0)]$ bo'ladi va

$$|f(z)-f(z_0)|=\sqrt{[u(x,y)-u(x_0,y_0)]^2+[v(x,y)-v(x_0,y_0)]^2} \tag{4}$$

4-ta'rifdan quyidagi tengsizliklar kelib chiqadi

$$\left.\begin{array}{l} |[u(x,y)-u(x_0,y_0)]|\prec\varepsilon \\ |[v(x,y)-v(x_0,y_0)]|\prec\varepsilon \end{array}\right\} \tag{5}$$

Demak, $u(x,y)$ va $v(x,y)$ funksiyalar $(x_0,\ y_0)$ nuqtada uzluksiz ekan.

1-misol. $w=z^2$ funksiya ixtiyoriy z_0 nuqtada uzluksizmi?

Yechish. $\Delta w=(z_0+\Delta z)^2-z_0^2=2z_0\,\Delta z+(\Delta z)^2$

$$\lim_{\Delta z\to 0}\Delta W=\lim_{\Delta z\to 0}\left[2z_0\,\Delta z+(\Delta z)^2\right]=2z_0\lim_{\Delta z\to 0}\Delta z+\lim_{\Delta z\to 0}(\Delta z)^2=0$$

V. 1. Darajali funksiya: $w=z^n$

a) n – natural son bo'lsa, $n\in N$, $w=z^n=r^ne^{in\varphi}$;

b) $n=\dfrac{1}{q}$ - kasr son bo'lsa

$$w=\sqrt[q]{z}=\sqrt[q]{r}\left[\cos\frac{\varphi+2\pi k}{q}+i\sin\frac{\varphi+2\pi k}{q}\right],\ \ k=0,\ \pm1,\ \pm2,...$$

q ta ildizga ega.

2. Ko'rsatkichli funksiya:

Biz $w=z^n$ bo'lgan hol bilan ko'proq ish ko'ramiz, ya'ni

$w=e^z=e^{x+iy}=e^x(\cos y+i\sin y)$ bundan

a) $e^{z+2\pi i}=e^{x+iy+2\pi i}=e^x[\cos(y+2\pi)+i\sin(y+2\pi)]=e^x\cdot e^{iy}=e^z$, ya'ni $w=e^z$ funksiya $2\pi i$ sof mavhum davrli. Bu haqiqiy sonlar nazariyasidagi ko'rsatkichli funksiyadan farqli demakdir.

b) $e^{z1+z2}=e^{z1}\cdot e^{z2}$; v) $e^{z1-z2}=e^{z1}/e^{z2}$; g) $(e^z)^m=e^{zm}$ mos bo'ladi.

3. Logarifmik funksiya: $w=\ln z$.

Logarifmik funksiya deb, ko'rsatkichli funksiyaga teskari bo'lgan $w=\ln z$ ushbu ko'rinishdagi funksiyaga aytiladi. Agar $z=e^w$ bo'lsa, $w=\ln z$ bo'ladi.

$$w=\ln z=\ln\left(re^{i\varphi}\right)=\ln r+i\varphi+2k\pi i\ \ (k=0,\pm1,\pm2,...)$$

Bunda $\ln r+i\varphi$ - **logarifmik funksiyaning bosh qismi** deyiladi. Bulardan ko'rinadiki, kompleks o'zgaruvchining logarifmik funksiyasi ko'p qiymatli ekan. Kompleks o'zgaruvchining logarifmik funksiyasi ham haqiqiy o'zgaruvchining logarifmik funksiyasining ko'pgina xossalariga bo'ysunadi.

Masalan: 1) $\ln(z_1\ z_2)=\ln z_1+\ln z_2$ 3) $\ln(z^n)=n\ln z+2k\pi i$

2) $\ln(z_1/z_2)=\ln z_1-\ln z_2$ 4) $\ln\sqrt[n]{Z}=\frac{1}{n}\ln Z$

2-misol. $3+4i$ ning logarifmini toping.

Yechish. $|z|=|3+4i|=\sqrt{9+16}=5,\quad \arg z=arctg\frac{4}{3}$

$$\ln(3+4i)=\ln 5+i\,arctg\frac{4}{3}$$

$$\ln(3+4i)=\ln 5+i\,arctg\frac{4}{3}+2k\pi i\ \ (k=0,\ \pm1,\ \pm2,...)$$

4. Kompleks o‘zgaruvchilarning trigonometrik funksiyalari.

Ushbu $e^{iz}=\cos z+i\sin z$ va $e^{-iz}=\cos z-i\sin z$ Eyler formulalari berilgan bo‘lsin. Bu formulalarni hadlab qo‘shib va ayirib, quyidagi funksiyaning trigonometrik funksiyalarini aniqlaymiz.

$$w=\cos z=\frac{e^{iz}+e^{-iz}}{2},\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad tg\,z=\frac{\sin z}{\cos z}=i\frac{e^{-iz}-e^{iz}}{e^{iz}+e^{-iz}},\quad ctg\,z=i\frac{e^{iz}+e^{-iz}}{e^{-iz}-e^{iz}}$$

Kompleks o‘zgaruvchilarning trigonometric funksiyalari ham haqiqiy o‘zgaruvchili funksiyalarning ko‘pgina xossalariga bo‘ysunadi. Bunda faqat kompleks son $\cos z$ va $\sin z$ funksiyalarining modullari birdan katta ham bo‘lishi mumkin.

Masalan: $\sin i=\frac{e^{ii}-e^{-ii}}{2i}=\frac{e^{-1}-e}{2i}=i\frac{e^2-1}{2e}\approx 1{,}17i$

$$\cos i=\frac{e^{ii}+e^{-ii}}{2}=\frac{e^{-1}+e}{2}=\frac{e^2+1}{2e}\approx 1{,}54$$

$$|\sin i|>1,\quad |\cos i|>1$$

5. Teskari trigonometrik funksiyalar.

Agar $z=\sin w$ trigonometrik funksiya berilgan bo‘lsa, w – o‘zgaruvchi unga teskari funksiya bo‘lib, u z ning **arksinusi** deyiladi va bunday yoziladi $w\,Arc\sin z$. Xuddi shuningdek, $w\,Arc\cos z,\ \ w\,Arctg\,z,\ \ w\,Arcctg\,z$.

a) $z=\sin w=\frac{e^{iw}-e^{-iw}}{2i}=\frac{e^{2iw}-1}{2ie^{iw}}\Rightarrow e^{2iw}-2ize^{iw}-1=0$

$e^{iw}=y$ desak, unda $y^2-(2iz)y-1=0$

$$y_{1,2}=iz\pm\sqrt{(iz)^2+1}=iz+\sqrt{1-z^2}$$

$$e^{iw}=iz\pm\sqrt{1-z^2};\quad iw\ln e=\ln\left(iz\pm\sqrt{1-z^2}\right)\Rightarrow w=Arc\sin z=$$

$$=\frac{1}{i}\ln\left(iz\pm\sqrt{1-z^2}\right)\Rightarrow w=Arc\sin z=-i\ln\left(iz\pm\sqrt{1-z^2}\right)\qquad(6)$$

Xuddi shuningdek

$$w=Arc\cos z=-i\ln\left(z\pm\sqrt{z^2-1}\right) \qquad (7)$$

$$w=Arcctg\ z=-\frac{i}{2}\ln\frac{1+iz}{1-iz} \qquad (8)$$

$$w=Arcctg\ z=\frac{i}{2}\ln\frac{z-i}{z+i} \qquad (9)$$

Teskari trigonometric funksiyalar *ln* ga bog‘liq bo‘lganligi uchun ular ham ko‘p qiymatli funksiyalardir.

3-misol. *Arcsin* 2 ning barcha qiymatlarini hisoblang.

Yechish.

$$Arc\sin 2=-i\ln\left(2i\pm i\sqrt{3}\right)=-i\ln\left[\left(2\pm\sqrt{3}\right)+i\frac{\pi}{2}+2k\pi i\right]=$$

$$=\frac{\pi}{2}-i\ln\left(2\pm i\sqrt{3}\right)+2k\pi \qquad (k=0,\ \ \pm1,\ \pm2,...)$$

4-misol. $Arctg(1+2i)$ning barcha qiymatlarini toping.

Yechish. $Arctg\,(1+2i)=\frac{1}{2i}ln\frac{1+i(1+2i)}{1-i(1+2i)}=\frac{1}{2i}\ln\frac{i-1}{3-i}$ kasrning maxrajini komplekslikdan ozod qilib, uning moduli va argumentini topaylik:

$$\frac{i-1}{3-i}=-\frac{2}{5}+\frac{i}{5};\ \left|\frac{i-1}{3-i}\right|=\left|-\frac{2}{5}+\frac{i}{5}\right|=\sqrt{\frac{4}{5}+\frac{1}{25}}=\frac{1}{\sqrt{5}}$$

$$\arg\left(-\frac{2}{5}+\frac{i}{5}\right)=\varphi\,tg=\frac{1}{5}\left(-\frac{2}{5}\right)=-\frac{1}{2};\quad \varphi=arctg\left(-\frac{1}{2}\right)=-arctg\frac{1}{2}=-arcctg\ 2$$

$$\ln\left(-\frac{2}{5}+\frac{i}{5}\right)=\ln\frac{1}{\sqrt{5}}-i\,arcctg\ 2+2k\pi$$

U holda $Arctg\,(1+2i)=k\pi-\frac{1}{2}arcctg\ 2+\frac{i}{4}\ln 5$

6. Giperbolik funksiyalar.

Kompleks o‘zgaruvchilarning giperbolik funksiyalari ham haqida o‘zgaruvchilarning giperbolik funksiyalari kabi aniqlanadi.

$$sh\,z=\frac{e^z-e^{-z}}{2};\ \ ch\,z=\frac{e^z+e^{-z}}{2};\ \ th\,z=\frac{sh\,z}{ch\,z}=\frac{e^z-e^{-z}}{e^z+e^{-z}};\ \ cthz=\frac{e^z+e^{-z}}{e^z-e^{-z}}$$

Bunda *shz, chz* lar $2\pi i$ davrli, *shz, chz* lar πi davrli funksiyalar. Kompleks o‘zgaruvchining giperbolik va trigonometrik funksiyalar orasida quyidagi bog‘lanish mavjud.

$$sh\,z=-i\sin iz,\quad ch\,z=\cos iz,\quad th\,z=itgiz,\quad cthz=-ictgiz.$$

Isboti. $-i\sin iz=-i\frac{e^{i(iz)}-e^{-i(iz)}}{2i}=-\frac{e^{-z}-e^{z}}{2}=\frac{e^{-z}-e^{z}}{2}=shz$

5-misol. $\cos(1+i)$ ning qiymatini hisoblang.

Yechish.

$$\cos(1+i)=ch[-i(1+i)]=ch(1-i)=\frac{e^{1-i}+e^{-(1-i)}}{2}=\frac{1}{2}\left(e^{1-i}+e^{i-1}\right)=\frac{1}{2}\left[e^{1}(\cos 1-i\sin 1)+\right.$$

$$\left.+e^{-1}(\cos 1+i\sin 1)\right]=\cos 1\frac{e+e^{-1}}{2}-i\sin 1\frac{e-e^{-1}}{2}$$

VI. Agar E kompleks sohada $w=f(z)$ funksiya berilgan bo‘lib va bu sohaning biror z_0 nuqtasidagi argument va funksiya orttirmalari quyidagicha bo‘lsin: $\Delta z=z-z_0,\quad \Delta w=f(z_0+\Delta z)-f(z_0)$.

<u>1-ta’rif.</u> Agar Δz har qanday yo‘l bilan nolga intilganda $\frac{\Delta w}{\Delta z}$ nisbat faqat birgina aniq limitga intilsa, u limitning qiymati $f(z)$ funksiyaning z_0 **nuqtadagi hosilasi** deyiladi va $w^1,\ f^1(z_0),\ \frac{dw}{dz}$ yoki $\frac{df}{dz}$ kabi belgilanadi, demak

$$f'(z_0)=\lim\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \qquad (1)$$

yoki $w=f(z_0)=u(x,y)+iv(x,y);\ \Delta w=\Delta u+i\Delta v$ bo‘lgani uchun (1) ni quyidagicha yozish mumkin:

$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\substack{\Delta z\to 0\\ \Delta y\to 0}}\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y} \qquad \mathbf{(2)}$$

chunki

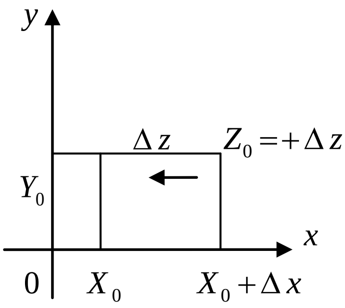
$$\Delta w=f(z+\Delta z)-f(z)=[u(x+\Delta x,y+\Delta y-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]=\Delta u+i\Delta v$$

bunda

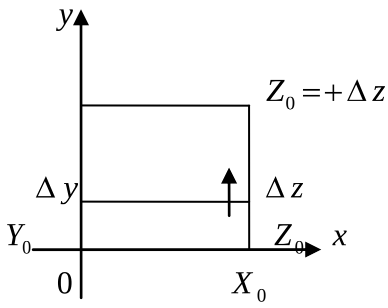
$$\Delta u=u(x+\Delta x,y+\Delta y)-u(x,y)$$
$$\Delta \upsilon=\upsilon(x+\Delta x,y+\Delta y)-\upsilon(x,y)$$

<u>2-ta’rif.</u> Agar $w=f(z)$ funksiya z_0 nuqtada hosilaga ega bo‘lsa, uni bu nuqtada differensiallanuvchi yoki **monogen funksiya** deyiladi.

1-ta’rifdan ko‘rinadiki, agar $f(z)$ z_0 nuqtada hosilaga ega bo‘lsa, (1) limitining qiymati Δz nolga qaysi yo‘l bilan intilishiga bog‘liq emas. Demak, biz $z_0+\Delta z$ nuqtani z_0 nuqtaga parallel holda ham intiltirishimiz mumkin. Bu holda $\Delta z=\Delta x,\quad \Delta y=0$ bo‘ladi (1 a) rasm).



a)



b)

1-rasm

$$f'(z_0)=\frac{\Delta u}{\Delta x}+i\frac{\Delta y}{\Delta x} \qquad (3)$$

Xuddi shuningdek $z_0+\Delta z$ nuqta z_0 ga *Oy* ga parallel holda intiltirsak $\Delta x=0, \quad \Delta z=i\Delta y$ bo‘ladi va (2) dan ushbu kelib chiqadi (1 b –rasm).

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}+\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta y}{i\Delta y}=\lim_{\Delta y\to 0}\left(\frac{\Delta y}{\Delta y}-i\frac{\Delta u}{\Delta y}\right)=\frac{\partial \upsilon}{\partial y}-i\frac{\partial u}{\partial y}$$

$$f'(z_0)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \qquad (4)$$

(3) va (4) lardan ushbu tenglamalarni hosil qilish mumkin

$$\frac{\partial u}{\partial x}+i\frac{\partial \upsilon}{\partial x}=\frac{\partial \upsilon}{\partial y}-i\frac{\partial u}{\partial y}\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial \upsilon}{\partial y};\quad \frac{\partial \upsilon}{\partial x}=\frac{\partial u}{\partial y} \qquad (5)$$

(5) Koshi-Riman shartlari.

__Teorema__. $f(z)$ funksiya z_0 nuqtada differensiallanuvchi bo‘lishi uchun $u(x,y),\ v(x,y)$ funksiyalar z_0 da differensiallanuvchi va Koshi – Riman shartlarining bajarilishi zarur va yetarlidir.

1-misol. $w=(x^2-y^2)+2xyi$ hosilaga egaligi yoki ega emasligini toping.

Yechish. $\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=-2y,\quad \frac{\partial v}{\partial x}=2y,\quad \frac{\partial v}{\partial y}=2x$ Koshi-Riman shartlarini $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$ tekshiramiz.

$2x=2x;\quad -2y=-(2y)$. Demak, bu funksiya hosilaga ega.

$$f'(z)=w'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+i2y=2(x+iy)=2z \quad \text{yoki}$$

$$f(z)=(x^2-y^2)+i2xy=(x+iy)^2=z^2$$

$$f'(z)=(z^2)'=2z$$

2-misol. $w=y+ix$ hosilaga ega ekanligini tekshiring.

Yechish. $u=y,\quad v=x,\quad \frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial y}=1,\quad \frac{\partial v}{\partial x}=1,\quad \frac{\partial v}{\partial y}=0.$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0 \quad \frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}(1\neq-1)$ bitta shart bajarilmagani uchun bu funksiya hosilaga ega emas.

Biz ko‘rdikki, agar funksiyaning nuqtadagi hosilasini topish kerak bo‘lsa, quyidagi 4 ta formulaning biridan foydalanish mumkin.

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y},\quad f'(z)=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x},\quad f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad (6)$$

Lekin $f(z)$ funksiyaning haqiqiy va mavhum qismlari ajralmagan holda bo‘lsa, bu formulalardan foydalanish noqulay bo‘ladi. $f(z)$ ning hosilasiga matematik analizdagi haqiqiy o‘zgaruvchili funksiyaning hosilasi qoidalarini qo‘llash mumkin, ya’ni:

$$1.\ c'=0 \qquad 2.\ z'=1 \qquad 3.\ \left[f_1(z)\pm f_2(z)\right]'=f_1'(z)\pm f_2'(z)$$

$$4.\ \left[c\,f(z)\right]'=c\,f'(z) \qquad 5.\ \left[\ f^{(n)}(z)'=n\,f^{\,n-1}(z)\,f'(z)\right]$$

<u>3-ta’rif</u>. Agar $f(z)$ funksiya E sohaning z_0 nuqtasida va uning atrofida ham differensiallanuvchi bo‘lsa, u funksiya shu nuqtada **analitik** deyiladi.

<u>4-ta’rif</u>. Agar $f(z)$ funksiya E sohaning barcha nuqtalarida hosilaga ega bo‘lsa, u funksiya E da **analitik** deyiladi.

<u>5-ta’rif</u>. $f(z)$ funksiya analitik bo‘lgan nuqtalar uning to‘g‘ri nuqtasi, analitik bo‘lmagan nuqtalari esa **maxsus nuqtalari** deyiladi.

3-misol. $w=x^2+y^2+i\,x\,y^3$ funksiyaning analitik yoki analitik emasligi tekshirilsin.

Yechish. $u=x^2+y^2,\quad v=x\,y^3,\quad \dfrac{\partial u}{\partial x}=2x,\quad \dfrac{\partial v}{\partial y}=2y,\quad \dfrac{\partial v}{\partial x}=y^3,\quad \dfrac{\partial v}{\partial y}=3x\,y^2$

$$2x=3x\,y^2\Rightarrow x\left(2-3y^2\right)=0;\quad x=0,\ y=\sqrt{\frac{2}{3}}$$

$y^3=-2y\Rightarrow y\left(y^2+2\right)=0;\ \ y=0,\ \ y\in\varnothing\Rightarrow(0,0)$ - shu nuqtadagina hosila mavjud, boshqa nuqtada hosila yo‘q, ya’ni funksiya analitik emas.

Foydalanilgan adabiyotlar

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