THE MINISTRY OF HIGHER AND SECONDARY SPECIAL EDUCATION OF THE REPUBLIC OF UZBEKISTAN

SAMARKAND STATE UNIVERSITY NAMED AFTER ALISHER NAVOI

THE FACULTY OF MECHANICS AND MATHEMATICS

THE DEPARTMENT OF PROBABILITY THEORY AND MATHEMATICAL STATISTICS

Based on the manuscript rights

UDK 519.21

DJALILOV ALISHER

DUALITY RELATIONS FOR THE PRIORITY QUEUE SYSTEMS AND TRANSITIONAL POSITIONS

GRADUATING MASTER DEGREE'S DISSERTATION
SPECIALTY —5A130102 – PROBABILITY THEORY AND MATHEMATICAL STATISTICS

Dean of the faculty: **Ass.Prof. X.X Ruzimuradov**

The head of the chair: **Prof.J.I.Abdullayev**

Research supervisor: **Ass.Prof. H. Kurbanov**

SAMARQAND - 2016

Contents

Introduction……………………………………………………………………3

Chapter I. The review and some known results on the duality relations

§-1.1. The brief review of articles on some duality results in the theory of queues……………………..………………………………………………….6

§-1.2. The duality relation between distributions of the queue sizes of the dual systems $M|G|1|N$ and $G|M|1|N-1$……………………………......10

§-1.3. On inverse queuing processes……………………………………13

Chapter II. The duality relations in the systems $M|G|1|N$ and $G|M|1|N-1$

§-2.1. Relation between the stationary queue sizes of the service systems $M|G|1$ and $M|G|1|N$…………………………………………………………...26

§-2.2. Duality relations for the distributions of non-stationary queue sizes of the models queue systems $M|G|1|N$ and $GJ|M|1|N$………………………………………………………………………30

Chapter III. The duality relations in the priority systems

§-3.1. The relationship between distributions of the queue sizes of ordinary customers in the priority queue systems $\vec{M}_2|\vec{G}_2|1|N_1,N_2$ and $\vec{M}_2|\vec{G}_2|1|N_1,\infty$…………………………………..…………………………35

§-3.2. Duality relations between distributions of the queue sizes non-stationary pressing customers in the systems $M_1M_2|G_1G_2|1|N_1,N_2$ and $GJ_1M_2|M_1,G_2|1|N_1-1,N_2$ ………………………………...………..…....38

§-3.3. Duality relation of the means busy period for the service systems $M|G|1|N$ and $GJ|M|1|N-1$ ……………………………………..................43

Conclusion……………………………………………………………….48

References……………………………………………………………….…51

Introduction

1. Statement and urgency of the problem. Consider two single-server queuing systems $M|G|1$ and $G|M|1$ which are characterized as follows: in the system $M|G|1$, let the customers arrive in a Poisson process with parameter λt, in the system $G|M|1$ this means that service – time distribution is negative exponential with mean λ^{-1}. Similarly, let the service times in the queue $M|G|1$ and the interarrival times in the system $G|M|1$ be independent and identically distributed random variables with the distribution function $B(x)$. The systems obtained in this manner are called "dual systems".

"Duality" in queuing theory refers to interchanging the interarrival time and service time distributions of a queue. The new system obtained in this manner is considered the dual of the original system, and conversely. The value of the concept is that the distribution of certain random variables for one system can often be expressed in terms of distributions of variables in the dual; knowing the solution for one system leads to the solution for the other.

The authors well-known scientific works have established and have used duality relationship for the systems $M|G|1$ and $G|M|1$ or for the systems $M|G|1|N$ and $G|M|1|N$ where arrive customers the same type. Statement of a question consists in the establishing duality relationship between distributions of the characteristics of the dual priority queue systems where arrive customers of two types.

Duality in the queuing theory has been discussed by Finch, Foster,Ghocal, Gordon, Newell, Makino, Daley, Bhat, Takacs, Shanbhag, Shahbazov, Azlarov, Tashmanov, Kurbanov([1] – [3], [5] – [9], [13] – [20]) and others for the characteristics of the queue systems $M|G|1$, $G|M|1$, $G|G|1$, $M|G|1|N$, $G|M|1|N-1$.

In this dissertation work several duality relations obtained for the queue size distributions of the priority queue systems $\overrightarrow{MM}\big|\overrightarrow{GG}\big|1|N_1N_2$ and

$\overrightarrow{GM}\left|\overrightarrow{MG}\right|1|N_1N_2$. Established some relations between distributions of the stationary queue sizes of the queue systems $M|G|1$ and $M|G|1|N$, also, between distributions of the stationary queue sizes of the priority queue systems $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|\infty,\infty$ and $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|N,\infty$.

2. Object and subject of investigations.

The object of investigation - the priority queue systems $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|\infty,\infty$, $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|N_1,N_2$, $\overrightarrow{GM}\left|\overrightarrow{MG}\right|1|N_1,N_2$.

Subject of investigations – the distribution of stationary queue sizes and non-stationary queue sizes.

3. Purpose and tasks of the investigations. Purpose of this work is establishment the duality relations between queue size distributions for the priority systems which have been obtained between queue size distributions for the systems $M|G|1|N$ and $G|M|1|N-1$, also, determination the methods of using duality relationship for the investigation priority systems. With this purpose are defined the following tasks: study the works on the duality relations and methods of their using, application these methods for obtaining new results for the dual priority queue systems

4. The basic problems and conjectures of investigations:

a) the establishing a relationship between stationary queue size distributions in the system $M|G|1$ and $M|G|1|N$;

b) the establishing a relationship between stationary queue size distributions in the priority queue systems $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|\infty,\infty$ and $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|N,\infty$;

c) the establishing a duality relation between stationary queue size distributions in the dual priority queue systems $\overrightarrow{MM}\left|\overrightarrow{GG}\right|1|N_1N_2$ and $\overrightarrow{GM}\left|\overrightarrow{MG}\right|1|N_1-1,N_2$

5. Research methods. In this work were used common research methods of the probability theory, methods of the renewal process, method of introducing the concept of “service cycle”, method of introducing an additional variable, method of birth and death process.

6. Theoretical and practical significance of this work. It is known that connection with the technical difficulties the priority queue systems of type $\overrightarrow{GG}|\overrightarrow{MM}|1$ are little studied. The relations established in this work give possibility transforming some results defined by $\overrightarrow{MM}|\overrightarrow{GG}|1$ into the queue system $\overrightarrow{GG}|\overrightarrow{MM}|1$ and conversely. Also, by applying these relations it is possible to obtain some results considerably easily. Knowing the solution for one system leads to the solution for the other.

7. Scientific novelty. It is now known results on the duality relations between distributions of the characteristics of dual priority queue systems. In this dissertation work for the first time established duality relation between distributions of the stationary queue sizes of the priority queue systems $\overrightarrow{MM}|\overrightarrow{GG}|1|N_1 N_2$ and $\overrightarrow{GM}|\overrightarrow{MG}|1|N_1 - 1, N_2$

8. Composition of the dissertation work. The dissertation work consists of the introduction, three chapters(the six paragraphs) and conclusion. Volume of this work the 50 pages.

In chapter I is given a brief review of basic scientific works of the duality relations, also, is proved the result obtained by Kurbanov [_] for the dual queue system $M|G|1|N$ and $GJ|M|1|N-1$. In chapter II are considered the relations established between stationary queue sizes distributions of the systems $M|G|1$ and $M|G|1|N$, also, between stationary queue size distributions in the priority queue systems $\overrightarrow{MM}|\overrightarrow{GG}|1|\infty, \infty$ and $\overrightarrow{MM}|\overrightarrow{GG}|1|N, \infty$. In chapter III is established duality relation between stationary queue size distributions of the priority queue systems $\overrightarrow{MM}|\overrightarrow{GG}|1|N_1, N_2$ and $\overrightarrow{GM}|\overrightarrow{MG}|1|N_1, N_2$.

Chapter I. The review and some known results on the duality relations.

§ – 1.1. The brief review of articles on some duality results in the theory of queues

"Duality" in queuing theory refers to interchanging the interarrival time and service time distributions of a queue. The new system obtained in this manner is considered the dual of the original system, and conversely. The value of the concept is that the distribution of certain random variables for one system can often be expressed in terms of distributions of variables in the dual; knowing the solution for one system leads to the solution for the other.

Several authors have used duality relationships in this manner:

1.Takacs L. [13] derived some duality relationships both for system $M|G|1$ and $G|M|1$ variables at a specified time and at specified epochs of arrival and departure.

Consider two independent families of random variables $\{\sigma_j, j = 1,2,...\}$ and $\{\tau_j, j = 1,2,...\}$. The σ_j are independently and identically distributed with distribution function $A(\sigma)$ and τ_j are independently and identically distributed with distribution $B(\tau)$.

The queue discussed thus far can be denoted by $G_A|G_B|1$. If the interarrival and service time distributions are interchanged, the resulting $G_B|G_A|1$ queue has independently and identically distributed interarrival times $\{\sigma_1', \sigma_2', ...\}$ with distribution function $B(\sigma)$ and independently and identically distributed service time$\{\tau_1', \tau_2', ...\}$ with distribution function $A(\tau)$.

Let

$$k(t) = \sum_{i=1}^{r(t)} \tau_i \text{ , where } r(t) = \max\{i: \sum_{j=1}^{i} \sigma_j < t$$

$$k'(t) = \sum_{i=1}^{r'(t)} \tau_i' \text{ where } r'(t) = \max\{i: \sum_{j=1}^{i} \sigma_j' < t\}.$$

Takacs has defined the dual of a queue in terms of the $k(t) or\ k'(t)$ process. In accordance with his definition the dual of the $k'(t)$ process is

$$k^{''}(t) = \sup_{0<y<\infty} \{k^{'}(y) < t\}.$$

The duality is a result of identity of the events

$$\{k^{''}(t) < x\} \Leftrightarrow \{k^{'}(x) > t\}.$$

2. Shanbhag D.N.[14] applied duality techniques to $M|G|1$ queues with batch service and finite waiting space and their duals – $G|M|1$ queues with batch arrivals and finite waiting space. In this paper, an application is given in which the joint distribution at the number server in a busy period and the length of the period are related to those same variables in the dual system.

3. Greenberg J. [15] Consider two dual systems $G_A|G_B|1$ and $G_B|G_A|1$ obtained above. Let $\nu(0)$ be a random variable independent of the σ_j and τ_j with $c(\nu) = P(\nu(0) < \nu)$. Define

$\rho_0 = \min_j\{j: \sum_{i=1}^{j} \sigma_i > \nu(0) + \sum_{i=1}^{j-1} \tau_i\}$, if there exists such a $j = \infty$, otherwise;

$\rho_0^{'} = \min_j\{j: \sum_{i=1}^{j} \sigma_i^{'} > \nu^{'}(0) + \sum_{i=1}^{j-1} \tau_i^{'}\}$, if there exists such a$j = \infty$, otherwise;

$\theta_0 = \inf\{t: k(t) + \nu(0) = t\}$, if there exists such a $t = \infty$, otherwise;

$\theta_0^{'} = \inf_{t>0}\{t: k^{'}(t) + \nu^{'}(0) = t\}$, if there exists such a t$t = \infty$, otherwise

Obtained the following relations:

$$\{\theta_0 < t, \theta_0 < \infty\} \Leftrightarrow \{\theta_0^{'} < t, \theta_0^{'} < \infty\},$$

$$\{\rho_0 < t, \theta_0 < \infty\} \Leftrightarrow \{\rho_0^{'} < t, \theta_0^{'} < \infty\},$$

4. Daley D.J. [16] Noted that in a single server queue with impatient customers who depart at time k if their service has not been completed, the waiting time of the n-th customer is the dual of k minus waiting time of the n-th customer in the dual system; the distribution of one is conveniently represented in terms of the distribution function of the other.

5. Bhat N. [17] Considered queues with a maximum of C waiting spaces and used duality principles to relate the time of the first “overflow” for the

$M|G|1$ and $G|M|1$ queue, the time measured in terms of the number of arriving or departing customers before overflow.

6. Foster F.G. [18] considered queues with maximum of N waiting spaces and derived duality principles to relate the number of customers to the number of waiting spaces, since the number of customers to the number of waiting spaces decreased by one as the number of customers increased by one. In relationship between the number of customers and the number of waiting spaces, the arrival and service processes are simply reversed. This is the reason why the dual system in a single–server queuing system has commonly been defined by interchanging the arrival process and the service process in the primary system. On the other hand, duality in tandem queuing systems has been applied only to the following special cases.

7. Yamazaki G and Sakasegawa H. [19] considered cyclic queuing systems where N customers advanced sequentially in clockwise. They studied the relationship between a customers and a waiting space as follows: when a customer completed his service at the i-th stage and advanced to the $(i+1)$-th stage, waiting spaces at the $(i+1)$-th stage were decreased by one and those at the i-th stage were increased by one just like one of waiting spaces at the$(i+1)$-th stage completed her “service”, that is, waiting spaces advanced sequentially in counterclockwise. Therefore they defined the dual system as reversing the order of service in the primary system.

8. Makino S. [20] analyzed the reversibility for some two and three stage tandem queuing system where he compared the ordinary system with the reversed order-of-service system.

9. Shahbazov A. [9] Considered the dual systems $M|G|1|N$ and $G|M|1|N-1$. Are denoted by $\xi_1(t)$ and $\xi_2(t)$ the queue size at time t of the systems $M|G|1|N$ and $G|M|1|N-1$, respectively.

Let $t_1, t_2, \dots t_n$ are moments of the service termination of the customers in the system $M|G|1|N$ and $t_1^{'}, t_2^{'}, \dots t_n^{'}, \dots$ are moments of the arrival of the customers in the system $G|M|1|N-1$.

Denote

$$\xi_n = \xi_1(t_n + 0) \quad , \qquad \xi_n^{'} = \xi_2(t_n^{'} - 0) \ .$$

In the article proved the following relation:

$$\lim_{n\to\infty} P(\xi_n = k) = \lim_{n\to\infty} P(\xi_n^{'} = N - k), k \geq 0$$

10. Azlarov T.A. and Kurbanov H. [2] In their article is generalized the result of Shahbazov and obtained following equality:

$$P\left(\xi_n = k \big/ \xi_1(0) = j\right) = P\left(\xi_n^{'} = N - k \big/ \xi_2(0) = N - j + 1\right), \quad k = \overline{0, N} \ .$$

§-1.2. The duality relation between distributions of the queue sizes of the dual systems $M|G|1|N$ and $G|M|1|N-1$

We shall consider two single-server queuing systems $F_1 = M|G|1|N$ and $F_2 = G|M|1|N-1$ as duals of each other.

In F_1, let customers arrive in a Poisson process with parameter λt; in the dual queue F_2 this means that service time distribution is exponential with mean λ^{-1}. Similarly, let the service times in the queue F_1 and interarrival times in the queue F_2 be independent and identically distributed random variables with the distribution function $B(x)[B(x+0)=0]$ and with mean λ^{-1}. Suppose that there is a waiting room of size $N(N \geq 1)$ in the system F_1 (a waiting room size $N-1$ in the system F_2), that is, the number of customers in the system F_1 (in the system F_2) is at most $N+1$(at most N).

We shall denote by $\xi_1(t)$ and $\xi_2(t)$ the queue size at time t in the systems F_1 and F_2, respectively.

Let $t_1, t_2, \dots, t_n, \dots$ are service termination moments of the customers in the system F_1 and $t_1^{'}, t_2^{'}, \dots, t_n^{'}, \dots$ are arrival moments (after the time $t=0$) in the system. We use the notations:

$$\xi_n = \xi_1(t_n + 0)\,, \xi_n^{'} = \xi_2(t_n^{'} - 0)$$

$$\Pi_{n,k}^{(j)} = P\left(\xi_n = k \Big/ \xi_1(0) = j\right), j = \overline{2,N}\,, k = \overline{0,N}, \qquad n \geq 1\,,$$

$$Q_{n,k}^{(j)} = P\left(\xi_n^{'} = k \Big/ \xi_2(0) = j\right), j = \overline{1,N-1}\,, k = \overline{0,N}\,, n \geq 1\,.$$

Theorem 1.2.1. For all $j = \overline{2,N}$ holds the following relation:

$$\Pi_{n,k}^{(j)} = Q_{n,N-k}^{(N-j+1)}\,, k = \overline{0,N}\,, n \geq 1. \tag{1.2.1}$$

From this relation it follows that

$$M\left(\xi_n / \xi_1(0) = j\right) = N - M\left(\xi_n^{'} / \xi_2(0) = N - j + 1\right), n \geq 1 \ .$$

Now we shall suppose that $\xi_1(0)$ and $\xi_2(0)$ are random variables with distributions

$$r_j = P(\xi_1(0) = j)\ , j = \overline{2, N}\ ;$$

$$l_j = P(\xi_2(0) = j)\ , j = \overline{1, N-1}\ ;$$

Theorem 1.2.2. If for all $= \overline{2, N}$ $r_j = l_{N-j+1}$, then holds the following equality

$$P(\xi_n = k) = P(\xi_n^{'} = N - k), k = \overline{1, N}, n \geq 1\ .$$

Proof of the Theorem 1.2.2. The sequences of random variables $\xi_1, \xi_2, \dots, \xi_n, \dots$ and $\xi_1^{'}, \xi_2^{'}, \dots, \xi_n^{'}, \dots$ are Markov's chain. We introduce the following notations:

$$\gamma_r = P(\nu_r = r) = \frac{\lambda^r}{r!}\int_0^\infty t^r e^{-\lambda t} dB(t)\ , r \geq 0, n \geq 1$$

$$q_{k-m} = \begin{cases} 0\ , k < m\ , \\ \gamma_{k-m}\ , m \leq k \leq N\ , \\ \gamma_{N-m} + \gamma_{N-m+1} + \cdots \quad , \qquad k = N \end{cases} .$$

Since for all $n \geq 2$

$$P\left(\xi_n = k / \xi_{n-1} = m + 1\right) = \gamma_{k-m}\ , \ \ k = \overline{0, N}, \ \ m \leq k$$

then by formula of total probabilities we obtain the following recurrence relations for $\Pi_{n,k}^{(j)}$:

$$\begin{cases} \Pi_{n,k}^{(j)} = q_{k-j+1}, n = 1 \\ \Pi_{n,k}^{(j)} = \sum_{m=j-n}^{k} \Pi_{n-1,m+1}^{(j)} \cdot q_{k-m}, 1 < n \le j \\ \Pi_{n,k}^{(j)} = q_k \Pi_{n-1,0}^{(j)} + \sum_{m=0}^{k} \Pi_{n-1,m+1}^{(j)} \cdot q_{k-m} \quad , n > j \end{cases} \tag{1.2.2}$$

Let now

$$\mathrm{D}_{n,k}^{(j)} = \mathcal{Q}_{n,N-k}^{(N-j+1)}, k = \overline{0,N}, j = \overline{1,N-1}; n \ge 1 \ .$$

Since for $m \ge k$, $k = \overline{0,N}$

$$P\left(N - \xi_n' = k \middle/ N - \xi_{n-1}' = m + 1\right) = \gamma_{k-m} \ ,$$

then for $\mathrm{D}_{n,k}^{(j)}$ we obtain

$$\begin{cases} \mathrm{D}_{n,k}^{(j)} = q_{k-j+1}, n = 1 \\ \mathrm{D}_{n,k}^{(j)} = \sum_{m=j-n}^{k} \mathrm{D}_{n-1,m+1}^{(j)} \cdot q_{k-m}, 1 < n \le j \\ \mathrm{D}_{n,k}^{(j)} = q_k \mathrm{D}_{n-1,0}^{(j)} + \sum_{m=0}^{k} \mathrm{D}_{n-1,m+1}^{(j)} \cdot q_{k-m} \quad , n > j \end{cases} \tag{1.2.3}$$

From the relations (1.2.2) and (1.2.3) it follows that for $\Pi_{n,k}^{(j)}$ and $\mathrm{D}_{n,k}^{(j)}$ are obtained one and the same recurrence relations and

$$\Pi_{1,k}^{(j)} = \mathrm{D}_{1,k}^{(j)} = q_{k-j+1} \ .$$

From here it follows that for all $k = \overline{0,N}$ and $n \ge 1$

$$\Pi_{n,k}^{(j)} = \mathrm{D}_{n,k}^{(j)} = \mathcal{Q}_{n,N-k}^{(N-j+1)} \ .$$

The theorem 1.2.2. It follows from (1.2.1) by using the formula of total probabilities.

§-1.3. On inverse queuing processes

Suppose that in the time interval $(0,\infty)$ customers arrive at a counter at times $\tau_1, \tau_2, \dots, \tau_n, \dots$ where $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ are random variables. The customers are served by a single server who starts working time $t = 0$. The successive service times, $\chi_1, \chi_2, \dots, \chi_n, \dots$, are positive random variables. The order of serving is not specified. We suppose only that the server is busy if there is at least one customer in the system.

We shall denote by $\eta(t)$ the waiting time at time t, that is, the time that a customer would have to wait if he arrived at time t and if the customers are served in the order of their arrivals, $\eta(0)$ is the initial waiting time, that is, the occupation time of the server at time $t = 0$.

Denote by ϱ_0 the number of customers served during the initial busy period, and by θ_0 the length of the initial busy period. If $\xi(0) = 0$, then $\varrho_0 = 0$, and if $\eta(0)$, then $\theta_0 = 0$.

For $0 \le i \le k$ define

$$P(k|i) = P\{\sup_{0 \le t \le \theta_0} \xi(t) \le k | \xi(0) = i\} \tag{1.3.1}$$

as the probability that the maximal queue size during the initial busy period is $\le k$ given that the initial queue size is i.

For $0 \le c \le x$ define

$$G(x|c) = P\{\sup_{0 \le t \le \theta_0} \eta(t) \le x | \eta(0) = c\} \tag{1.3.2}$$

as the probability that the maximal waiting time during the initial busy period is $\le x$ given that the initial waiting time is c.

Now we shall define the inverse process of the queuing process defined before. Suppose that in the time interval $[0,\infty)$ customers arrive at a counter at times $\tau_0^*, \tau_1^*, \dots, \tau_n^*, \dots$ where $\tau_0^* = 0$ and $\tau_n^* = \chi_1 + \cdots + \chi_n$ $(n = 1,2,\dots)$. The customers are served by a single server who starts working at time $t = 0$. The successive service times are $\chi_1^*, \chi_2^*, \dots, \chi_n^*, \dots$ where $\chi_n^* = \tau_n - \tau_{n-1}$ $(n = 1,2,\dots;\ \tau_0 = 0)$. We suppose that the server is busy if there is at least one customer in the system. This queuing process is called the inverse process of a queuing process if we interchange the interarrival times and service times.

For the inverse queuing process denote by $\xi^*(t)$ the queue size at time t. $\xi^*(0)$ is the initial queue size, that is, the queue size immediately before $t = 0$. The customer arriving at the time $t = 0$ is not counted in the initial queue size. Further denote by $\eta^*(t)$ the waiting time at time t in the inverse queuing process. $\eta^*(0)$ is the initial waiting time (immediately before $t = 0$) is not included in the initial waiting time.

Denote by ϱ_0^* the number if customers served in the initial busy period, and by θ_0^* the length of the initial busy period.

For the inverse process, we use the same notation as for the original process, except that an asterisk is added. Thus, we use the notation

$$P^*(k|i) = P\{\sup_{0\le t\le\theta_0} \xi^*(t) \le k|\xi^*(0) = i\} \qquad (1.3.3)$$

$(0 \le i \le k)$ for the probability that the maximal queue size during the initial busy period is $\le k$ if the initial queue size is i, and

$$G^*(x|c) = P\{\sup_{0\le t\le\theta_0} \eta^*(t) \le x|\eta^*(0) = c\} \qquad (1.3.4)$$

$(0 \le c \le x)$ for the probability that the maximal waiting time during the initial busy period is $\le x$ if the initial waiting time is c.

Dual theorems. We shall show that there are simple relations between the distributions (1.3.1) and (1.3.3) as well as between (1.3.2) and (1.3.4) We shall always suppose that

$$P\{\sup_{0\le t\le\theta_0} |\chi_1 + \cdots + \chi_n - \tau_n| = \infty\} = 1 \qquad (1.3.5)$$

Theorem 1.3.1. If $0 < i \leq k$, then

$$P^*(k|k-i) = 1 - P(k|i) \tag{1.3.6}$$

Proof of the Theorem 1.3.1. Define a stochastic process $\{\delta(t), 0 \leq t \leq \infty\}$ in the following way: $\delta(0) = 0$ and $\delta(t)$ changes only in jumps. A jump of magnitude $+1$ occurs at times $t = \tau_1, \tau_2, \ldots, \tau_n, \ldots$ and a jump of magnitude -1 occurs at times $t = \tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots$ Then evidently $P(k|i)$ is the probability that $\delta(t), 0 < t < \infty$, reaches the line $z = -i$ first without touching the line $z = k + 1 - i$ in the meantime. $P(k|k—i)$ is the probability that $\delta(t), 0 \leq t < \infty$, reaches the line $z = k + 1 - i$ first without touching the line $z = -i$ in the meantime. If (5) is satisfied, then $\delta(t), 0 \leq t < \infty$, will sooner or later reach either $z = -i$ or $z = k + 1 - i$ with probability 1. Hence $P(k|i) + P^*(k|k-i) = 1$ which was to be proved.

Theorem 1.3.2. If $0 < c \leq x$, then we have

$$G^*(x|x-c) = 1 - G(x|c) \tag{1.3.7}$$

Proof of the Theorem 1.3.2. Define a stochastic process $\{\chi(t), 0 \leq t < \infty\}$, in the following way: $\chi(0) = 0$ and

$$\chi(t) = \sum_{0<\tau_n\leq t} \chi_n \tag{1.3.8}$$

for , $0 \leq t < \infty$. Then $G(x|c)$ can be interpreted as the probability that $\chi(t), 0 \leq t < \infty$, intersects the line $z = t - c$ first without intersecting the line $z = t + x - c$ in the meantime. $G^*(x|x—c)$ is the probability that $\chi(t), 0 \leq t < \infty$, intersects the line $z = t + x - c$ first without intersecting the line $z = t - c$ in the meantime. If (1.3.5) is satisfied, then $\chi(t), 0 \leq t < \infty$), will sooner or later intersect either $z = t - c$ or $z = t + x - c$ with probability 1. Hence $G(x|c) + G^*(x|x-c) = 1$ which was to be proved.

In the next two sections we shall give some examples for the applications of the above theorems. We shall suppose that $\tau_n - \tau_{n-1}$ $(n = 1,2,\dots;\ \tau_0 = 0)$ and $\chi_n (n = 1,2,\dots)$ are independent sequences of mutually independent and identically distributed positive random variables.

If

$$P\{\chi_n = \tau_n - \tau_{n-1}\} < 1, \tag{1.3.9}$$

then (1.3.5) is satisfied and Theorem 1.3.1 and Theorem 1.3.2 are applicable. We shall use the following notation:

$$P\{\tau_n - \tau_{n-1} \leq x\} = F(x)\,, \tag{1.3.10}$$

$$P\{\chi_n \leq x\} = H(x)\,, \tag{1.3.11}$$

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x)\ , \tag{1.3.12}$$

$$a = \int_0^\infty x dH(x)\ . \tag{1.3.13}$$

Queues with Poisson input. In this section we shall give direct proofs for some theorems which have been found by the author [10]. (See also [11] and [12])

Consider the single-server queuing process defined in the introduction, in the case when the inter arrival times $\tau_n - \tau_{n-1}$ $(n = 1,2,\dots;\ \tau_0 = 0)$ are mutually independent random variables having the common distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0\,, \end{cases} \tag{1.3.14}$$

and the service times $\chi_1, \chi_2, \dots, \chi_n, \dots$ are also mutually independent random variables having the common distribution function$H(x)$. Further, the two sequences $\tau_n - \tau_{n-1}$ are also independent.

In this process the arrivals form a Poisson process of density λ, and the probability that exactly j $(j = 0,1,2,\dots)$ customers arrive during a service time is given by

$$\pi_i = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dH(x). \tag{1.3.15}$$

The generating function of π_i $(j = 0,1,2,\dots)$ is given by

$$\pi(z) = \sum_{j=0}^{\infty} \pi_i z^j = \int_0^\infty e^{-\lambda(1-z)x} dH(x) = \psi\big(\lambda(1-z)\big) \tag{1.3.16}$$

and (1.3.16) is necessarily convergent for $|z| \leq 1$.

Theorem 1.3.3. For $0 \leq i \leq k$ we have

$$P(k|i) = \frac{Q_{k-i}}{Q_k} \tag{1.3.17}$$

where

$$Q(z) = \sum_{k=0}^{\infty} Q_k z^k = \frac{Q_0 \pi(z)}{\pi(z) - z} \tag{1.3.18}$$

for $|z| < \delta$ and δ is the smallest nonnegative real root of

$$\pi(z) = z \tag{1.3.19}$$

If $\lambda a \leq 1$, then $\delta = 1$ and if $\lambda a > 1$, then $\delta < 1$. Q_0 is an arbitrary nonnull constant.

Proof of the Theorem 1.3.3. Define the process $\{\delta(t), 0 \leq t < \infty\}$ in the same way as in the proof of Theorem 1.3.1. If we measure time from a transition $i + 1 \rightarrow i$, then independently of the past, the increments of the process

$\{\delta(t), 0 \leq t < \infty\}$ have the same stochastic behavior as the original process $\{\delta(t), 0 \leq t < \infty\}$. This implies that

$$P(k|k-i) = P(j|j-1)P(k|k-i)$$

for $0 \leq i \leq j \leq k$. Since $0 < P(k|i) \leq 1$ if $0 \leq i \leq k$, it follows that

$$P(k|i) = \frac{Q_{k-i}}{Q_k} \tag{1.3.21}$$

for $0 \leq i \leq k$ where $Q_0 \neq 0$ and Q_k/Q_0 $(k = 0,1 \ldots)$ is non-decreasing sequence.

If we take into consideration that during the first service time in the initial busy period the number of arrivals may be $j = 0,1,2, \ldots$ then we can write that

$$P(k+i/i) = \sum_{j=0}^{k} \pi_i P(k+i/i+j-1) \tag{1.3.22}$$

for $i \geq 0, k \geq 0$. If we multiply (1.3.22) by Q_{k+i} and use (1.3.21), then we get the following recurrence formula for the determination of Q_k $(k = 0,1,2, \ldots)$:

$$Q_k = \sum_{j=0}^{k} \pi_i Q_{k+1-j} \quad (k = 0,1,2, \ldots) \tag{1.3.23}$$

If we introduce generating functions, we get (1.3.18)

We have explicitly $Q_1 = {}^{Q_0}/_{\pi_0}$ and for $k = 1,2, \ldots$

$$Q_{k+1} = Q_0 \sum_{\nu=1}^{k} \frac{(-1)^\nu \nu!}{\pi_0^{\nu+1}} \sum_{\substack{i_1+i_2+\cdots i_n=\nu \\ i_1+2i_2+\cdots k i_k=k}} \frac{(\pi_i - 1)^{i_1} \pi_2^{i_2} \ldots \pi_k^{i_k}}{i_1! \ldots i_k!} \tag{1.3.24}$$

We remark that if $\lambda a < 1$, then

$$\lim_{k\to\infty} Q_k = \frac{Q_0}{1-\lambda a}, \tag{1.3.25}$$

that is, by choosing $Q_0 = 1 - \lambda a$ we have

$\lim_{k\to\infty} Q_k = 1$. If $\lambda a \geq 1$, then $\lim_{k\to\infty} {Q_k}/{Q_0} = \infty$

If we consider any busy period other than initial one, then

$$P(k/1) = \frac{Q_{k-1}}{Q_k} \quad (k = 1,2,\dots) \qquad (1.3.26)$$

is the probability that the maximal queue size during the busy periods $\leq k$.

Example. If, in particular, $H(x) = 1 - e^{-\mu x}$ for $x \geq 0$, that is, the

$$Q_k = \frac{Q_0}{1 - \frac{\lambda}{\mu}}\left(1 - \left(\frac{\lambda}{\mu}\right)^{k+1}\right) \qquad (3.1.27)$$

for $\lambda \neq \mu$ and $Q_k = Q_0(k+1)$ for $\lambda = \mu$.

In this case

$$P(k/1) = \frac{1 - \left(\frac{\lambda}{\mu}\right)^{k}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \qquad (1.3.28)$$

if $\lambda \neq \mu$ and $P(k/1) = k/k + 1)$ if $\lambda = \mu$.

Formula (1.3.28) has been found by S. Karlin and J.Me.Gregor.

Note. First, we note that is $\lambda a < 1$, then

$$\lim_{t\to\infty} P(\xi(t) \leq k) = Q_k \qquad (k = 0,1,2,\dots.) \qquad (1.3.29)$$

where $Q_0 = 1 - \lambda a$. The limit (1.3.29) is independent of the initial state.

If $\lambda a > 1$ then $\lim_{t\to\infty} P(\xi(t) \leq k) = 0$ for all k.

Second, consider the queuing process studied in this section with the modification that there is a waiting room of size m, that is, the number of customers in the system is at most $m + 1$.

If an arriving customer findings $m+1$ customers in the system, then he departs without being served. In this modified process denote by ζ_n the queue size immediately after the n-th departure. Then $\{\zeta_n\}$ is an irreducible and a periodic Markov chain with state space $J = \{0,1, \dots m\}$. Consequently the limiting distribution

$$\lim_{n\to\infty} P(\zeta_n \leq k) = P_k \quad (k = 0,1, \dots, m)$$

exists, is independent of the initial state and can be obtained as the unique solution of the following system of linear equations

$$P_k = \sum_{j=0}^{k} \pi_j P_{k+1-j} \quad (k = 0,1, \dots, m-1) \qquad (1.3.30)$$

and $P_m = 1$. A comparison of (1.3.23) and (1.3.30) shows that

$$P_k = \frac{Q_k}{Q_m} \qquad (k = 0,1, \dots m) \qquad (1.3.31)$$

Where Q_k $(k = 0,1, \dots)$ is defined in Theorem 1.3.3.

If $\xi(t)$ denotes the queue sized at time for the modified process, then it can easily be proved that

$$\lim_{t\to\infty} P(\xi(t) \leq k) = P_k^* \qquad (k = 0,1, \dots, m+1)$$

exists, is independent of the initial state, and

$$P_k = \frac{Q_k}{Q_0 + a\lambda Q_m} \qquad (1.3.32)$$

for $k = 0,1, \dots, m$. Obviously P_{m+1}^*. If we take into consideration that the number of transitions $i \to i+1$ and $i+1 \to i$ $(i = 0,1, \dots, m)$ in any interval $(0,1)$ may differ by at most 1, then we obtain that $P_k^* = P_k P_m^*$ for $k = 0,1,,,,m$ and evidently

$$P_0^* = P_0/(\lambda a + P_0) \, .$$

Thus we get (1.3.32)

Theorem 1.3.4. For $0 \le c \le x$ we have

$$G(x/c) = \frac{W(x-c)}{W(x)} \tag{1.3.33}$$

where

$$\Omega(s) = \int_0^\infty e^{-sx} dW(x) = \frac{W(0) \cdot s}{s + \lambda(1 - \psi(s))} \tag{1.3.34}$$

For $Re(s) > \omega$ and ω is the largest non-negative real root of

$$\lambda[1 - \psi(s)] = s \tag{1.3.35}$$

If $\lambda a \le 1$, then $\omega = 0$, where as if $\lambda a > 1$, then $\omega > 0$. $W(0)$ is an arbitrary nonnull constant.

Proof of the Theorem 1.3.4. In this case the process $\{\chi(t), 0 \le t < \infty\}$ defined by (1.3.8) has non-negative stationary independent increments. This implies that for $0 \le c \le y \le x$ we have

$$G(x/x-c) = G\left(y/y-c\right) G(x/x-c). \tag{1.3.36}$$

Since $0 < G(x/y) \le 1$ if $0 \le y \le x$, it follows that $G(x/y)$ can be represented in the following form

$$G(x/c) = \frac{W(x-c)}{W(x)} \tag{1.3.37}$$

or $0 \le c \le x$ where $W(0) \ne 0$ and $W(x)/W(0)$ $(0 \le x \le \infty)$ is a non-decreasing function of x.

If we take into consideration that in the time interval $(0, n)$ one customer arrives with probability $\lambda n + o(n)$, and more than one customer arrives with probability $o(n)$, then we can write that for $x \ge 0$ and $y \ge 0$

$$G\left({x+y}/_{y}\right) =$$

$$= (1-\lambda u) G\left({x+y}/_{y-u}\right) + \lambda u \int_0^x G\left({x+y}/_{y+z}\right) dH(z) + o(u) \qquad (1.3.38)$$

If we multiply this equation by $W(x+y)$, then we obtain

$$W(x) = (1-\lambda u) W(x+u) + \lambda u \int_0^x W(x-z)\, dH(z) + o(u) \qquad (1.3.39)$$

for $x \geq 0$. Hence

$$W'(x) = \lambda W(x) - \lambda \int_0^x W(x-z) dH(z) \qquad (1.3.40)$$

for $\boldsymbol{x} > 0$. Forming the Laplace–Stieltjes transform of (1.3.40) we get

$$S\big(\Omega(s) - W(0)\big) = \lambda\Omega(s) - \lambda\Omega(s)\psi(s), \qquad (1.3.41)$$

whence

$$\Omega(s) = \frac{W(0)s}{s - \lambda[1-\psi(s)]} \quad \text{for} \quad Re(s) > \omega. \qquad (1.3.42)$$

This proves Theorem 1.3.4.

If we suppose that a is finite positive number and introduce a distribution function $H^*(x)$ for which

$$\frac{dH^*(x)}{dx} = \frac{1-H(x)}{a} \qquad (1.3.43)$$

if $x > 0$ and $H^*(x) = 0$ if $x \leq 0$, then by inverting (1.3.42) we obtain

$$W(x) = W(0) \sum_{n=0}^{\infty} (\lambda a)^n\, H_n^*(x) \qquad (1.3.44)$$

where $H_n^*(x)$ is the n-th iterated convolution of $H^*(x)$ with itself;

$H_0^*(x) = 1$ if $x \geq 0$ and $H_0^*(x) = 0$ if $x < 0$.

We remark that if $\lambda a < 1$, then

$$\lim_{a\to\infty} W(x) = \frac{W(0)}{1-\lambda a} \tag{1.3.45}$$

that is, by choosing $W(0) = 1 - \lambda a$ we have $\lim_{x\to\infty} W(x) = 1$. If $\lambda a \geq 1$, then $\lim_{x\to\infty} {W(x)}/{W(0)} = \infty$.

If we consider any busy period of her than the initial one, then the probability that the maximal waiting during the busy period is $\leq x$ is given by

$$G(x) = \frac{1}{W(x)} \int_0^x W(x-z)dH(z) = 1 - \frac{W'(x)}{\lambda W(x)} \tag{1.3.46}$$

for $x > 0$ which follows from (1.3.40)

Example. If, in particular, $H(x) = 1 - e^{-\mu x}$ for $x \geq 0$, then

$$W(x) = \frac{W(0)}{\mu - \lambda}\left[\mu - \lambda e^{(\lambda-\mu)x}\right] \tag{1.3.47}$$

for $\lambda \neq \mu$ and $W(x) = W(0)(1 + \mu x)$ for $\lambda = \mu$.

In this case

$$G(x) = \frac{\mu - \lambda e^{(\lambda-\mu)x}}{\mu - \mu e^{(\lambda-\mu)x}} \tag{1.3.48}$$

if $\lambda \neq \mu$ and $G(x) = \mu x/(1 + \mu x)$ if $\lambda = \mu$.

Note. First, we note that if $\lambda a < 1$, then

$$\lim_{t\to\infty} P\{\eta(t) \leq x\} = W(x) \quad (0 \leq x < \infty) \tag{1.3.49}$$

where $W(0) = 1 - \lambda a$. The limit (1.3.49) is independent of the initial state. If $\lambda a > 1$, then $\lim_{t\to\infty} P\{\eta(t) \leq x\} = 0$ for all x.

Second, consider the queuing process studied in this section with the modification that the total time spent in the system by a customer cannot increase above $\boldsymbol{m}$ where $\boldsymbol{m}$ is a positive number. That is, if a customer already spent time $\boldsymbol{m}$ in the system, then he departs even if his serving has not yet been completed. (This model can be used also in the theory of dams where there is an overflow if the dam is full.)

If $\boldsymbol{\eta(t)}$ denotes the waiting time at time $\boldsymbol{t}$ also for the modified process, then we have

$$\lim_{t\to\infty} P\{\eta(t) \leq x\} = \frac{W(x)}{W(m)} \tag{1.3.50}$$

for $0 \leq x \leq m$, where $W(x)$ is defined in Theorem 1.3.4. If $\eta_n = \eta(\tau_n - 0)$, that is, η_n is the waiting time immediately before the arrival of the n-th customer, then we have also

$$\lim_{n\to\infty} P\{\eta_n \leq x\} = \frac{W(x)}{W(m)} \tag{1.3.51}$$

for $0 \leq x \leq m$.

Queues with exponentially distributed service times. Now consider the inverse process of the queuing process discussed in the previous section. In this case the service times have the distribution function (1.3.14).

Theorem 1.3.5. If $0 \leq i \leq k$, then we have

$$P^*(k|i) = 1 - \frac{Q_i}{Q_k} \tag{1.3.52}$$

where Q_k $(k = 0,1,2, \dots)$ is defined in Theorem 1.3.3.

Proof of the Theorem 1.3.5. (1.3.52) follows immediately from Theorem 1.3.1 and Theorem 1.3.3. The probability that in any busy period other than the initial one the maximal queue size is $\leq k$ is given by

$$P^*(k|0) = 1 - \frac{Q_0}{Q_k} \qquad (1.3.52)$$

for $k = 0,1,2,\ldots$

Note. First, let us remark that if $\lambda a < 1$, then

$$P\{\varrho_0^* < \infty |\, \boldsymbol{\xi}^*(\mathbf{0}) = i\} = \mathbf{1} - \boldsymbol{Q_i} \qquad (\mathbf{1.3.53})$$

for $i = 0,1,2,\ldots$ where $\boldsymbol{Q_0} = 1 — \lambda a$.

Second, consider the queuing process investigated in this section with the modification that there is a waiting room of size $m — 1$, that is, the number of customers in the system is at most m. If an arriving customer finds m customers in the system, then he departs without being served. In this modified process denote by ξ_n the queue size immediately before the arrival of the n-th customer, and by $\xi(t)$ the queue size at time t. Then independently of the initial state we have

$$\lim_{n\to\infty} P\{m - \xi_n \leq k\} = \frac{Q_k}{Q_m} \qquad (1.3.55)$$

for $k = 0,1,\ldots,m$ and if $H(x)$ is not a lattice distribution function, then

$$\lim_{t\to\infty} P\{m - \xi(t) \leq k\} = \frac{Q_{k+1} - Q_0}{\lambda a Q_m} \qquad (1.3.56)$$

for $\boldsymbol{k = 0, 1, \ldots, m - 1}$.

Chapter II. The duality relations in the systems $M|G|1|N$ and $G|M|1|N-1$

§-2.1. Relation between the stationary queue sizes of the service systems M|G|1 and M|G|1|N

Let us consider a single-server queue systems $F - M|G|1$ and $F_N - M|G|1|N$ which are characterized as follows: the customers arriving to the system constitute a Poisson flow with the rate λ, that is, interarrival times have the exponential distribution with a parameter λ, customers are served in the order of their arrivals, the length of service times are mutually independent identically distributed random variable with common distribution $B(x)$ and with the mean μ^{-1}, the number of the positions in the system F_n is bounded by N $(N \geq 1)$, that is, may be at most $N+1$ customers in the system (with the served customers). A server working starts at time $t = 0$ with the arrival of the first customer to the system.

Let denote the following notation:

$\xi(t)$ and $\xi_N(t)$ — the number of customers in the system F and F_N at the moment t

ξ and ξ_N — the number of customers in the system F and F_N at the arbitrary moment.

ζ_k − busy period of the system $F_k - M|G|1|K$ $(1 \leq k \leq N)$, $\rho = \lambda\mu^{-1}$ loading of the system.

It is shown in [4] the existence of the followings limits on the condition $\rho < 1$.

$$P(k) = \lim_{t\to\infty} P(\xi(t) = k)\,,\ k \geq 0\,,$$

$$P_N(k) = \lim_{t\to\infty} P(\xi_N(t) = k)\,,\quad 0 \leq k \leq N+1.$$

In the considered paper is established realitonship between $P_N(k)$ and $M\zeta_k$, as well as $P(k)$ and $P_N(k)$.

Theorem 2.1.1. The following relations are hold:

$$P_N(0) = (1 + \lambda M\zeta_N)^{-1} ,$$

$$P_N(k) = \frac{\mu(M\zeta_k - M\zeta_{k-1})}{1 + \lambda M\zeta_N} , \qquad 1 \le k \le N , \;\; M\zeta_0 = 0 , \qquad (2.1.1)$$

$$P_N(N+1) = \frac{1 - (\mu - \lambda)M\zeta_N}{1 + \lambda M\zeta_N} .$$

Theorem 2.1.2. By $\rho < 1$ are hold the following equalities:

$$P_N(k) = P(k)(1 - \rho P(\xi > N))^{-1} , 0 \le k \le N \;\; , \qquad (2.1.2)$$

$$P_N(N+1) = (1 - \rho)P(\xi > N)\big(1 - \rho P(\xi > N)\big)^{-1} .$$

Proof of the Theorem 2.2.1. In [1] are proved the following relations for the probabilities $P_N(k)$:

$$P_N(0) = [1 + \rho(1 + \lambda f_N)]^{-1} ,$$

$$P_N(1) = \lambda f_1 P_N(0) , \qquad (2.1.3)$$

$$P_N(k) = \lambda(f_k - f_{k-1})P_N(0), \quad k = \overline{2, N}$$

where f_k is defined by the following equation:

$$\sum_{k=0}^{\infty} \vartheta^k f_k = \frac{\vartheta}{\lambda - \lambda\vartheta} \cdot \frac{1 - \bar{b}(\lambda - \lambda\vartheta)}{\bar{b}(\lambda - \lambda\vartheta) - \vartheta} , \qquad (2.1.4)$$

$$\bar{b}(s) = \int_0^{\infty} e^{-Sx} dB(x) , \qquad ReS \ge 0 .$$

The equality for the function $\bar{g}_N(S) = \int_0^{\infty} e^{-Sx} dP(\zeta_N < x)$ was shown by Harris [12]:

$$\bar{g}_N(s) = \frac{\Delta_{N-1}(S)}{\Delta(\mathrm{S})} , \qquad (2.1.5)$$

$$\sum_{k=0}^{\infty} \vartheta^k \Delta_k(\mathrm{S}) = \frac{\vartheta \bar{b}(s) - \bar{b}(S + \lambda - \lambda\vartheta)}{(1-\vartheta)[\vartheta - \bar{b}(S + \lambda - \lambda\vartheta)]}$$

Granting the equality $\bar{g}_N'(0) = -M\zeta_N$, from (2.1.5) we have

$$\sum_{k=0}^{\infty} \vartheta^k M\zeta_k = \frac{\vartheta\mu^{-1}}{\vartheta - \bar{b}(\lambda - \lambda\vartheta)} \tag{2.1.6}$$

By (2.1.4) and (2.1.6) we have

$$f_k = \frac{1}{\rho} M\zeta_N - \frac{1}{\lambda}\,. \tag{2.1.7}$$

According to (2.1.7) from equality (2.1.3) it follows (2.1.1).

Proof of the Theorem 2.1.2. It is known ([11], 62 page) that for $\rho < 1$

$$\lim_{N\to\infty} M\zeta_N = \frac{1}{\mu(1-\rho)} \text{ and for } \rho > 1 \ \lim_{N\to\infty} M\zeta_N = \infty \qquad (*)$$

If we take into account this relation from equality (2.1.1) for $\rho < 1$ we have the following expression for the probabilities $P(k)$:

$$P(0) = 1 - \rho\,,$$

$$P(k) = \mu(1-\rho)(M\zeta_k - M\zeta_{k-1})\,, \qquad k \geq 1 \tag{2.1.8}$$

By (2.1.1) and (2.1.8) relations we have

$$P_N(k) = P(k)(1 + \lambda M\zeta_N)^{-1}(1-\rho)^{-1} \qquad 1 \leq k \leq N \tag{2.1.9}$$

$$f_N = P(\xi > N) = \sum_{k=N+1}^{\infty} P(k)$$

Since, on the basis of (2.1.8) hold the equalities

$$\sum_{k=1}^{N} P(k) = \mu(1-\rho)M\zeta_N\,,$$

(2.1.9) takes the following form

$$M\zeta_N = \frac{1}{\mu(1-\rho)} P(\xi \le N)$$

$$P_N(k) = P(k)(1-\rho f_N)\,,\quad 1 \le k \le N \qquad (2.1.10)$$

Also, on the basis of (2.1.1) and (2.1.10) we have

$$P_N(N+1) = 1 = \sum_{k=1}^{N} P_N(k) = \frac{(1-\rho)f_N}{1-\rho f_N} \qquad (2.1.11)$$

From the equalities (2.1.10) and (2.1.11) it follows the relation (2.1.2).

Remark. In [8] are shown the following relations: for $\rho > 1$

$$\lim_{N\to\infty} P_N(N+1) = 1-\rho^{-1} \qquad (2.1.12)$$

$$\lim_{N\to\infty} P_N(N-k) = \rho^{-1}\sigma^k(1-\sigma)\,,\quad k \ge 0,$$

where is the unique solution of the equation $\vartheta = \bar{b}(\lambda - \lambda\vartheta)$ lying at $(0,1)$. If we take into consideration of the equality (2.1.6), from (2.1.1) we can easily get these results.

It is known that for $\rho > 1$ $\lim_{N\to\infty} M\zeta_N = \infty$ and the first equality of (2.1.12) it follows from the third equation of (2.1.1). In [7] proved the following relation:

$$\lim_{N\to\infty} \frac{M\zeta_{N-k}}{M\zeta_N} = \begin{cases} \sigma^k, & \rho \ge 1 \\ 1, & \rho < 1 \end{cases}.$$

On the basis of this from the second equation of (2.1.12) it follows the second relation of (2.1.11).

§-2.2. Duality relations for the distributions of non-stationary queue sizes of the models queue systems $\mathbf{M|G|1|N}$ and $\mathbf{GJ|M|1|N}$

Consider mathematical model of the single-server queue system which is characterized as follows: the customers arriving to the system constitute a Poisson flow with parameter λ that interarrival times have the common distribution

$$A(x) = 1 - e^{-\lambda x}, x > 0, \qquad \lambda > 0 .$$

Customers are served in the order or their arrivals, the lengths of the service time are independent and identically distributed random variables with the distribution function $B(x)$ $[B(+0) = 0]$ and with mean μ^{-1}, there is waiting room of size N, that is, the number of customers in the system may be at most $N + 1$. This system is usually denoted as $\mathrm{F}_1 - \mathrm{M|G|1|N}$.

Interchanging the distributions $A(x)$ and $B(x)$ in the system $\mathrm{F}_1 - \mathrm{M|G|1|N}$ we obtain second service system the mathematical model which is denoted by $\mathrm{F}_2 - \mathrm{GJ|M|1|N}$. Thus, is supposed that the interarrival times in the system F_1 and the service times in the system F_2 have the common distribution $A(x)$, also the service times in the system F_1 and the interarrival times in the system F_2 have the common distribution $B(x)$. The systems such defined are called dual of each other.

We shall denote by $\xi_1(t)$ and $\xi_2(t)$ the queue sizes at time t, that is, the total number of customers at time t in the system F_1 and F_2, respectively

Theorem 2.2.1. For $k = \overline{0, N+1}$ and $j = \overline{1, N}$ hold the following relations:

$$P(\xi_1(t) = k/\xi_1(0) = j) =$$

$$= P(\xi_2(t) = (N - k + 1)/\xi_2(0) = N - j + 1)) \qquad (2.2.1)$$

$$M(\xi_1(t)/\xi_1(0) = j) = N + 1 - M\left(\frac{\xi_2(t)}{\xi_2(0)} = N - j + 1\right) \qquad (2.2.2)$$

Theorem 2.2.2. For arbitrary $k = \overline{0, N+1}$ holds the following equality:

$$\lim_{t\to\infty} P(\xi_1(t) = k/\xi_1(0) =$$
$$= j) = \lim_{t\to\infty} P(\xi_2(t) = N - k + 1/\xi_2(0)$$
$$= N - j + 1) \qquad (2.2.3)$$

Remark. It is known that there is the relations $\rho_1 = {\rho_2}^{-1}$ between the loads $\rho_1 = \lambda\mu^{-1}$ and $\rho_2 = \mu\lambda^{-1}$ of the systems F_1 va F_2. For $\rho_1 > 1$ and $\rho_2 > 1$ investigation in the both system the distributions of the characteristics, all the more their asymptotical states depended with the great technical difficulties.

With the help of the relations (2.2.1) and (2.2.3) the results obtained for the systems F_1 and F_2 when loads less than one can be transferred to the dual system when the loads greater than one.

Relations similar to the equality (2.2.1) were established in [14] on the conditions $1 \le \xi_1(t) \le N$, $0 \le \xi_1(t) \le N - 1$ and in [7] in the general case for the regeneration moments (for the values t which constitute the Markov chain). Also, in [9] the relation (2.2.3) was proved for the case when limit is taken by regeneration moments. In this paper we show that the results obtained in [7] and [9] hold for arbitrary t.

Proof of the Theorem 2.2.1. Let us consider system F_1. We introduce the following notations:

$$P^{(j)}(k,t) = P\left((\xi_1(t) = \frac{K}{\xi_j(t)} = j\right), j \ge 1, \qquad k = \overline{0, N+1}$$

$$\overline{P}^{(j)}(k,s) = \int_0^\infty e^{-st}(P^{(j)}(k,t))dt \;, s \ge 0, k = \overline{0, N+1}$$

$$\overline{B}(s) = \int_0^\infty e^{-sx} dB(x)$$

In [7] obtained the following result: for $Res \geq 0$ and $j = \overline{1, N}$

$$\overline{P}^{(j)}(0, s) = \frac{\Delta_{N-j}}{(S+\lambda)\Delta_N - \lambda\Delta_{N-1}},$$

$$\overline{P}^{(j)}(1, s) = (S+\lambda)\overline{P}^{(j)}(0, s) f_1,$$

$$\overline{P}^{(j)}(k, s) = \overline{P}^{(j)}(0, s)[(S+\lambda) f_k - \lambda f_{k-1}], 2 \leq k \leq j \qquad (2.2.4)$$

$$\overline{P}^{(j)}(k, s) = \overline{P}^{(j)}(0, s)[(S+\lambda) f_k - \lambda f_{k-1}] - f_{k-j}, \; j < k \leq N$$

$$\overline{P}^{(j)}(N+1, s) = \overline{P}^{(j)}(0, s)\left[(S+\lambda)\varphi_N - \lambda\varphi_{N-j}\right] - \varphi_{N-j} + \frac{1 - S\overline{P}^{(j)}(0, s)}{S},$$

where

$\varphi_N = \sum_{k=1}^{n} f_k$, $f_k = f_k(s)$ and $\Delta_k = \Delta_k(s)$ are the coefficients of v^k in the expansion in the series by v accordingly functions

$$f(v, s) = \frac{v}{s+\lambda-\lambda v} \cdot \frac{1 - \overline{b}(s+\lambda-\lambda v)}{\overline{b}(s+\lambda-\lambda v) - v}, \qquad (2.2.4')$$

$$\Delta(\mathrm{v}, \mathrm{s}) = \frac{v\overline{b}(s) - \overline{b}(s+\lambda-\lambda v)}{(1-v)(v - \overline{b}(s+\lambda-\lambda v)}$$

Suppose now that $\eta(t)$ is the number of the free positions in the system F_2 at the moment t. We introduce the following notations:

$$P^{(j)}(k, t) = P\left(\eta(t) = \frac{k}{\eta(0)} = j\right), \; j \geq 1, \; k = \overline{0, N+1},$$

$$\overline{P}^{(j)}(k, s) = \int_0^\infty e^{-st} P^{(j)}(k, t) dt, \qquad ReS \geq 0, k = \overline{0, N+1}$$

Completely repeating proof given in [7] can be obtained the same relation as (2.2.4) for the functions $\overline{P}^{(j)}(k, s)$, $k = \overline{0, N+1}$, that is, the distribution of the number of customers in the system F_1 at the moment t coincides with the distribution of the number of free positions in the system F_2 at the moment t:

$$\Pi^{(j)}(k,t) = P^{(j)}(k,t) \tag{2.2.5}$$

If we take into account the equality $\eta(t) - \xi_2(t) = N + 1$, then from the relation (2.2.5) we have the equality (2.2.1). (2.2.2) it follows immediately from (2.2.1).

Proof of the Theorem 2.2.2. In [4] and [3] are given existence proof of the limits

$$\lim_{t\to\infty} P(\xi_1(t) = k/\xi_1(0) = 1)$$

$$\lim_{t\to\infty} P(\xi_2(t) = k/\xi_2(0) = 1)\,.$$

in accordingly for the systems F_1 and F_2.

Let ζ_N – be the initial busy period of the system F_1 and

$$\overline{g}^{(j)}(s) = \int_0^\infty e^{-st} dP(\zeta_n < \frac{x}{\xi_1(0)} = j)\ , j = \overline{1,N}\,, \qquad Res \geq 0\ . \tag{2.2.6}$$

In [6] is shown the equality

$$\overline{g}^{(j)}(s) = \frac{\Delta_{N-j}}{\Delta_N} \quad . \tag{2.2.7}$$

Where $\Delta_k = \Delta_k(s)$ is the function defined in (2.2.4′). By (2.2.6) the first equality of the relation (2.2.4) take the following form:

$$\Pi^{(j)}(0,s) = \frac{\overline{g}^{(j)}(s)}{s + \lambda - \lambda\overline{g}^{(1)}(s)}\ .$$

Multiplying the both sides of this equality and giving over to the limit when $s \to 0$, we obtain

$$\lim_{S\to 0} s\overline{\Pi}^{(j)}(0,s) = \frac{1}{1+\lambda M(\zeta_N/\xi_1(0)=1)}\,.$$

So, for any arbitrary finite $N \geq 1$ for $j = \overline{1,N}$ there exists a limit which of j independent, that is, there exists the stationary regime. If we introduce the notations

$$\overline{\Pi}(k) = \lim_{S\to 0} s\overline{\Pi}^{(j)}(k,s) \quad ,k = \overline{0,N+1}\,, \qquad j = \overline{1,N}$$

Then from (2.2.4) we have the following equalities:

$$\overline{\Pi}(0) = \frac{1}{1+\lambda M(\zeta_N/\xi_1(0)=1)}$$

$$\overline{\Pi}(1) = \lambda\Pi(0)\overline{f_1}$$

$$\overline{\Pi}(k) = \lambda\Pi(0)\left(\overline{f_k} - \overline{f_{k-1}}\right), 1 < k \leq N\,,$$

$$\overline{\Pi}(N+1) = \lambda\Pi(0)f_N - \overline{\Pi}(0) + 1 \quad ,$$

where $\overline{f_k}$ the coefficient of v^k in the expansion of the function $f(v,0)$ in series by v. Thus, for any arbitrary $N \geq 1$ and $j = \overline{1,N}$ there exists a limit is independent of j:

$$\lim_{S\to 0} s\overline{\Pi}^{(j)}(k,s) \quad , \qquad k = \overline{0,N+1}\ .$$

By the properties of the Laplace transformation

$$\lim_{t\to\infty} P(\xi_1(t) = k/\xi_1(0) = j) = \lim_{s\to 0} S\overline{\Pi}^{(j)}(k,s) \quad .$$

So, in the both sides of the equality (2.2.1) we can go over to the limit when $t \to \infty$.

Chapter III. The duality relations in the priority systems.

§-3.1. The relationship between distributions of the queue sizes of ordinary customers in the priority queue systems $\vec{M}_2|\vec{G}_2|1|N_1,N_2$ and $\vec{M}_2|\vec{G}_2|1|N_1,\infty$

Let consider a single-server priority queue system. $F(N_1,N_2)-\vec{M}_2|\vec{G}_2|1|N_1,N_2$ which is characterized as follows: to the system arrive the customers of the two types (pressing and ordinary) which constitute the Poisson flow by parameter λ_1 and λ_2, respectively. The distributed random variable with common distribution $B_1(x)$ (for pressing customers) and $B_2(x)$ (for ordinary customers) and with means ${\mu_1}^{-1}$ and ${\mu_2}^{-1}$. Suppose that there are waiting rooms of size N_1(for pressing customers) and N_2(for the ordinary customers), that is, the number of pressing and ordinary customers in the queue system are at most N_1+1 and N_2 (if in the system are pressing customers) or N_2+1 (if in the system are not pressing customers).

We introduce the following notations:

$\xi_{N_1,N_2}-$ the stationary queue size of pressing customers in the queue system $F(N_1,N_2)$;

$\zeta_{N_{1,k}}-$ the busy period of the queue system $F(N_1,k)$ (the server is busy if there is at least one customer in the system), $k=\overline{1,N_2}$

$\zeta_{N_1}-$ the busy period of the system $F(N_1,N_2)$ by pressing customers;

$$\rho_1=\lambda_1{\mu_1}^{-1}, \qquad \rho_2=\lambda_2{\mu_2}^{-1}, \qquad \rho=\rho_2(1+\lambda_1 M\zeta_{N_1})$$

Theorem 3.1.1. For $\rho<1$ holds the following equalities:

$$P\big(\xi_{N_1,N_2}=k\big)=\frac{P(\xi_{N_{1,\infty}}=k)}{(1-\rho)^2+\rho P(\xi_{N_1}\le N_2)}\quad,\qquad k=\overline{0,N_2}$$

$$P\big(\xi_{N_1,N_2}=N_2+1\big)=\frac{(1-\rho)P\big(\xi_{N_{1,\infty}}=N_2+1\big)-\rho(1-\rho)}{(1-\rho)^2+\rho P(\xi_{N_{1,\infty}}\le N_2)}\quad.$$

The analogues relations were established in the work [5] for the queue systems $M|G|1|N$ and $M|G|1|\infty$.

Proof of the Theorem 3.1.1. To prove of this theorem we introduce the concept of "service cycle". Let C be the service cycle of ordinary customer, that is, the length of the time beginning of service of an ordinary customer until its completion. Introduce the following notations:

$$\overline{b}_2(s) = \int_0^\infty e^{-sx} dB_2(x) \quad , \; Re\, s \geq 0 \; ,$$

$$\overline{g}_{N_1}(s) = \int_0^\infty e^{-sx}\, dP\left(\xi_{N_1} < x\right) \; , \; Re\, s \geq 0 \; ,$$

$$\overline{c}(s) = \int_0^\infty e^{-sx} dP(c < x) \; , \; Re\, s \geq 0 \; .$$

In [10] obtained relation for the $\overline{c}(s)$ when $N_1 = \infty$ and $N_2 = \infty$, that is for the system $\vec{M}_2\left|\vec{G}_2\right|1|\infty,\infty$.

Repeating the proof of this relation we can obtain the following equality:

$$\overline{c}(s) = \bar{b}_2\left(s + \lambda_1 - \lambda_1 \bar{g}_{N_1}(s)\right) \tag{3.1.1}$$

from here we have

$$c(s) = \overline{b}_2^{'}(0)[1 - \lambda_1 \overline{g}_{N_1}{}'(0)] \; . \tag{3.1.2}$$

It is known that

$$\overline{c}^{'}(0) = -Mc \, , \qquad \overline{b}_2^{'}(0) = -\mu_2{}^{-1} \; , \; \overline{g}_{N_1}^{'}(0) = -M\zeta_{N_1} \; .$$

By these equalities from (3.1.2) we obtain

$$Mc = \mu_2{}^{-1}\left(1 + \lambda_1 M\zeta_{N_1}\right) = \rho\lambda_2{}^{-1}$$

Now we can consider the queue system $F(N_1, N_2)$ as system where arrive only ordinary customers the service times of which is the c and the load is

$$\rho = \lambda_2 Mc = \rho_2(1 + \lambda_1 M\zeta_{N_1})$$

By relations established in [8] we have following equalities:

$$P\left(\xi_{N_1N_2}=k\right)=\frac{\lambda_2\rho^{-1}(M\zeta_{N_1,k}-MM\zeta_{N_1,k-1})}{1+\lambda_2 M\zeta_{N_1N_2}}\ , \qquad k=\overline{1,N_2} \qquad (3.1.3)$$

$$P\left(\xi_{N_1N_2}=0\right)=\frac{1}{1+\lambda_2 M\zeta_{N_1N_2}}\ .$$

It is known ([11], page 62) that for $\rho<1$

$$\lim_{N_2\to\infty} M\zeta_{N_1N_2}=M\zeta_{N_1,\infty}=\frac{\rho}{\lambda_2(1-\rho)}\ .$$

By this equality from (3.1.3) on $N\to\infty$ we have

$$P\left(\xi_{N_1N_2}=k\right)\to P\left(\xi_{N_1,\infty}=k\right)=\lambda_2\rho^{-1}(1-\rho)\left(M\zeta_{N_1,k}-M\zeta_{N_1,k-1}\right),\ \ k\ge 1\ ,$$
(3.1.4)

$$P\left(\xi_{N_1N_2}=0\right)\to P\left(\xi_{N_1,\infty}=0\right)=1-\rho$$

here $M\zeta_{N_{1,0}}=0$

By (3.1.3) and (3.1.4) for $k=\overline{0,N_2}$ we obtain

$$P\left(\xi_{N_1N_2}=k\right)=P\left(\xi_{N_1\infty}=k\right)\cdot\frac{1}{(1-\rho)\left(1+\lambda_2 M\zeta_{N_1N_2}\right)}\ . \qquad (3.1.5)$$

From (3.1.4) we have the following relation:

$$P\left(\xi_{N_1,\infty}\le N_2\right)=\sum_{k=0}^{N_2}P\left(\xi_{N_1,\infty}=k\right)=\lambda_2\rho^{-1}(1-\rho)M\zeta_{N_1N_2}\ .$$

By this the equality (3.1.5) takes the following form:

$$P\left(\xi_{N_1N_2}=k\right)=\frac{P(\xi_{N_1,\infty}=k)}{(1-\rho)^2+\rho P(\xi_{N_1,\infty}\le N_2)}\quad ,\qquad k=\overline{0,N_2} \qquad (3.1.6)$$

If we take into consideration that

$$P\left(\xi_{N_1N_2}=N_2+1\right)=1-\sum_{k=0}^{N_2}P(\xi_{N_1N_2}=k)$$

than from (3.1.6) it follows the second equality of the theorem.

§-3.2 Duality relations between distributions of the queue sizes non-stationary pressing customers in the systems $M_1M_2|G_1G_2|1|N_1, N_2$ and $GJ_1M_2|M_1, G_2|1|N_1 - 1, N_2$

We consider the following priority queue system: suppose that pressing (class 1) and ordinary (class 2) customers arrive at a single-server queue system. Take $t_{0i} = 0$, and let $0 < t_{1i} < t_{2i} < \cdots$ be the arrival times of customers of the i-th class ($i = 1; 2$). We assume that the interarrival times $u_{ni} = t_{ni} - t_{(n-1)i}$, and the service times v_{ni} of successive class i customers form for independent renewal processes determined by the distribution functions

$$A_i(x) = \begin{cases} 0\,, & x \le 0 \\ 1 - e^{-\lambda_i x}\,, & x \ge 0\,, \lambda_i > 0 \end{cases},$$

(for interarrival times) and $B_i(x)$ (for service times). We assume only that $B_i(x)$ has finite mean μ^{-1} ($i = 1; 2$). The number of the positions in the system is bounded by N_i ($N_i \ge 1$), that is, may be at most $N_i + 1$ customers of the class i in the system (with the served customer). The order of service is "absolute priority", that is, an ordinary customer (class 2) will be served in the case only if there are no pressing customers (class 1) in the system. This system is denoted by $F_1 - M_1M_2|G_1G_2|1|N_1, N_2$.

By interchanging in this system the distributions $A_1(x)$ and $B_1(x)$ and assuming that may be at most N_1 customers of the class 1 in the system we obtain the dual priority queue system which is denoted by $F_2 - GJ_1M_2|M_1G_2|1|N_1 - 1, N_2$.

We suppose that the customers are served by a single-server who starts working at time $t = t_{0i} = 0$ in the presence j_1 customers of the class 1 and j_2 customers of the class 2 ($j_1 \ge 1, j_2 \ge 1$).

Let t_n be the completion moment of service of the n-th pressing customer in the system F_1 and T_n be the arrival moment of the n-th customer (after the time $t = 0$) in the system F_2.

We introduce the following notations:

$\xi_i^{(j_i)}(t) -$ the number of pressing customer at time t in the system F_i $(i = 1;2)$

$\eta_i^{(j_i)}(t) -$ the number of ordinary customer at time t in the system F_i $(i = 1;2)$

Here and in what follows upper index in the bracket denotes the number of customers at time t in the considered system.

Denote

$$\xi_{1,n}{}^{(j_1)} = \xi_1{}^{(j_1)}(t_n + 0) \ ,$$

$$\eta_{1,n}{}^{(j_2)} = \eta_1{}^{(j_2)}(t_n + 0) \,,$$

$$\xi_{2,n}{}^{(j_1)} = \xi_2{}^{(j_1)}(T_n - 0) \ ,$$

$$\eta_{2,n}{}^{(j_2)} = \eta_2{}^{(j_2)}(T_n - 0) \,,$$

$$\Pi_n{}^{(j_1,j_2)}(k,m) = P\left(\xi_{1n}{}^{(j_1)} = k, \eta_{1n}{}^{(j_2)} = m\right), \quad k = \overline{0, N_1}, \ \ m = \overline{0, N_2 + 1} \,,$$

$$\mathrm{Q}_n{}^{(j_1,j_2)}(k,m) = P\left(\xi_{2n}{}^{(j_1)} = k, \eta_{2n}{}^{(j_2)} = m\right) \ , \ \overline{k = 0, N_1} \ ,$$
$$m = \overline{0, N_2 + 1} \,.$$

Theorem 3.2.1. For arbitrary $j_1 = \overline{2, N_1}$, $j_2 = \overline{0, N_2}$ and $n \geq 1$ holds the following equality:

$$\Pi_n{}^{(j_1,j_2)}(k,m) = \mathrm{Q}_n{}^{(N_1 - j_1 + 1, j_2)}(N_1 - k, m), \qquad k = \overline{0, N_1},$$
$$m = \overline{0, N_2 + 1} \,, \qquad n \geq 1 \tag{3.2.1}$$

The similar relation has been obtained in [2] and in [9] on $n \to \infty$ for the system $M|G|1|N$ and $GJ|M|1|N - 1$

Proof of the Theorem 3.2.1. It is known that $\{t_n; n \geq 1\}$ and $T_n; n \geq 1$ are regeneration moment of the random processes $\xi_1{}^{(j)}(t)$ and $\xi_2{}^{(j)}(t)$

respectively, that is, $\{\xi_{1n}{}^{(j)}; n \geq 1\}$ and $\{\xi_{2n}{}^{(j)}; n \geq 1\}$ constitute the homogenous Markov chain with possible states $(0,1,\ldots N)$.

First we consider the system F_1. Let ν_n and θ_n be the number of pressing and ordinary customers who arrive during the service time of the n-th pressing customer. Then $\nu_1, \nu_2, \ldots$ and $\theta_1, \theta_2, \ldots$ are independent random variables with distributions

$$\alpha_r = P(\nu_r = r) = \frac{\lambda_1{}^r}{r!}\int_0^\infty t^r e^{-\lambda_1 t} dB_1(t) \quad , \qquad r \geq 0, \qquad n \geq 1, \qquad (3.2.2)$$

$$\beta_r = P(\theta_r = r) = \frac{\lambda_2{}^r}{r!}\int_0^\infty t^r e^{-\lambda_2 t} dB_1(t) \quad , \qquad r \geq 0, \qquad n \geq 1, \qquad (3.3.3)$$

Then is hold the following recurrence relations:

$$\xi_{1,n+1}{}^{(j_1)} = \min\{(\xi_{1n}{}^{(j_1)} - 1)^+ + \nu_{n+1}, N_1\}$$

$$\eta_{1,n+1}{}^{(j_2)} = \min\{\eta_{1n}{}^{(j_2)} + \theta_{n+1}, N_2\} \quad ,$$

here $x^+ = \max(0; x)$. Introduce the following notations:

$$\gamma_{r_1,r_2} = P(\nu_n = r_1, \theta_n = r_2) = \alpha_{r_1} \cdot \beta_{r_2} , \qquad (3.2.4)$$

$$q_{k-m,r} = \begin{cases} 0 \,, k < m \\ \gamma_{k-m,r} \,, m \leq k \leq N_1 \\ \gamma_{N_1-m,r} + \gamma_{N_1-m+1,r} + \cdots, k = N_1 \end{cases} \qquad (3.2.5)$$

If we take into consideration that for $n \geq 2$

$$P(\xi_{1,n}{}^{(j_1)} = k, \eta_{1,n}{}^{(j_2)} = i / \xi_{1,n-1}{}^{(j_1)} = m+1 \,, \eta_{1,n-1}{}^{(j_2)} = r) =$$
$\gamma_{k-m,i-r}$,

where $\leq k, k = \overline{0,N}$, we have the following recurrence relations:

$$\begin{cases} \Pi_1{}^{(j_1,j_2)}(k,m) = q_{k-j_1+1,m-j_2} \ ,j_1 - 1 \le k < N_1 \ ,j_2 \le m \le N_2 \ , \\ \Pi_n{}^{(j_1,j_2)}(k,m) = \sum_{i=j_1-n}^{k} \sum_{r=j_2}^{m} \Pi_{n-1}{}^{(j_1,j_2)}(i+1,r)\, q_{k-i,m-r} \\ 1 \le n \le j_1, j_1 - n \le k < N_1, j_2 \le m \le N_2, \\ \Pi^{(j_1,j_2)}(k,m) = \sum_{r=0}^{m} \Pi_{n-1}{}^{(j_1,j_2)}\,(0,m-r) q_{k,r} + \sum_{r=0}^{m} \sum_{m}^{k} \Pi_{n-1}{}^{(j_1,j_2)}(i+1,m-r) q_{k-i,m-r} \\ n > j_1, j_2 \le m \le N_2, 0 \le k \le N_1 \end{cases} \tag{3.2.6}$$

We consider now the random variables $\xi_{2,n}{}^{(j_1)}$ and $\eta_{2,n}{}^{(j_2)}$. Set y_n and $z_n$be the number of customers who abandon the system during (T_{n-1}, T_n). The random variables y_n and z_n have the same distribution as (3.2.4) and (3.2.5).

For $\xi_{2,n}{}^{(j_1)}$ and $\eta_{2,n}{}^{(j_2)}$ we obtain the following recurrence relations:

$$\xi_{2,n}{}^{(j_1)} = \left\{\min\left(\xi_{2,n-1}{}^{(j_1)} + 1, N\right) - y_n\right\}^{+}, n \ge 1,$$

$$\eta_{2,n}{}^{(j_2)} = \min\left\{\eta_{1,n}{}^{(j_2)} + z_n, N_2\right\}$$

We introduce the following notations:

$$\xi_n{}^{(j_1)} = N_1 - \xi_{2,n}{}^{(j)}$$

$$D_n^{(j_1,j_2)}(k,m) = P\left(\xi_n{}^{(N_1-j_1+1)} = k, \eta_{2,n}{}^{(j_2)} = m\right) = Q_n^{(N_1-j_1+1,j_2)}(N-k,m),$$

$$k = \overline{0,N};\ \ j_1 = \overline{1,N-1}\,, n \ge 1\,.$$

It is clear that,

$$P\left(\xi_n{}^{(N_1-j_1+1)} = k, \eta_{2,n}^{(j_2)} = i / \xi_n{}^{(N_1-j_1+1)} = m+1, \eta_{2,n-1}^{(j_2)} = r\right) = \ \gamma_{k-m,i-r},$$

$$m \ge k;\ k = \overline{0,N_1}\,.$$

If we take into consideration this equality then by formula of total probability obtain the following system of recurrence equations:

$$\begin{cases} D_1^{(j_1,j_2)}(k,m) = q_{k-j_1+1,m-j_2}, j_1 - 1 \le k < N_1, \quad j_2 \le m \le N_2 \\ D_1^{(j_1,j_2)}(k,m) = \sum_{i=j_1-n}^{k} \sum_{r=j_2}^{m} D_{n-1}^{(j_1,j_2)}(i+1,r) q_{k-i,m-r}, \\ 1 \le n \le j_1, j_1 - n \le k < N_1, j_2 \le m \le N_2 \\ D_n^{(j_1,j_2)}(k,m) = \sum_{r=0}^{m} D_{n-1}^{(j_1,j_2)}(0,m-r) q_{k,r} + \sum_{r=0}^{m} \sum_{i=0}^{k} D_{n-1}^{(j_1,j_2)}(i+1,m-r) q_{k-i,m-r}, \\ n > j_1, j_2 \le m \le N_2, 0 \le k \le N_1 \end{cases} \quad (3.2.7)$$

here $q_{k,r}$ is defined by equality (3.2.5). As is obvious from (3.2.6) and (3.2.7) for the functions, $\Pi_n{}^{(j_1,j_2)}(k,m)$ and $\mathrm{D}_n{}^{(j_1,j_2)}(k,m)$ are obtained the same recurrence equations and

$$\Pi_n{}^{(j_1,j_2)}(k,m) = \mathrm{D}_1{}^{(j_1,j_2)}(k,m) .$$

From here

$$\Pi_n{}^{(j_1,j_2)}(k,m) = \mathrm{D}_n{}^{(j_1,j_2)}(k,m) ,$$

or

$$\Pi_n{}^{(j_1,j_2)}(k,m) = Q_n{}^{(N_1-j_1+1,j_2)}(N-k,m) .$$

§–3.3 Duality relation of the means busy period for the service systems $\mathbf{M|G|1|N}$ and $\mathbf{GJ|M|1|N-1}$

Let consider following queue systems $F_{1N} - \mathrm{M|G|1|N}$ and $F_{2N} - \mathrm{GJ|M|1|N-1}$:

Service time of the queue systems $F_{1N} - \mathrm{M|G|1|N}$ and $F_{2N} - \mathrm{GJ|M|1|N-1}$ have the same distribution function

$$A(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-x}, & x > 0, \qquad \lambda > 0 \end{cases}$$

The service times in the queue F_{1N} and interarrival times in the queue F_{2N} be independent and identically distributed random variables with the distribution function $B(x)[B(x+0)=0]$ and with mean μ^{-1}.

There is single-channel queue system and customers are served in the order of their arrivals. There are waiting room of size $N(N \geq 1)$ in the system F_{1N} and $N-1$ in the system F_2. We call this kinds of service systems duality queue systems.

Let denote the following notation:

ζ_{1k} –busy period of the system F_{1k} , $k = 1,2, \ldots$;

ζ_{2N} – busy period of the system F_{2N};

$\rho_1 = \lambda\mu^{-1}$ and $\rho_2 = \mu\lambda^{-1}$ are respectively loadings of the systems;

$$\bar{b}(s) = \int_0^\infty e^{-sx} dB(x) \ , \qquad Re\, s \geq 0 \ ,$$

$$\bar{g}_N(s) = \int_0^\infty e^{-sx} dP(\zeta_{1N} < x) \ , \qquad Re\, s \geq 0$$

$$\bar{f}_N(s) = \int_0^\infty e^{-sx} dP(\zeta_{2N} < x) \ , \qquad Re\, s \geq 0$$

In the considered work are established duality relationship between $M\zeta_{1N}$ and $M\zeta_{2N}$. Here we use equalities are given in [12] and [2] determines the function $\bar{g}_N(s)$ and $\bar{f}_N(s)$.

Theorem 3.3.1. By $N \geq 1$ the following equality holds:

$$M\zeta_{2N} = \frac{M\zeta_{1N} - \mu^{-1}}{\lambda(M\zeta_{1N} - M\zeta_{1N-1})} \ . \tag{3.3.1}$$

Theorem 3.3.2. The following equality holds:

$$\lim_{N\to\infty} M\zeta_{2N} = \begin{cases} \dfrac{1}{\lambda(1-r)} \ , \rho_2 < 1 \ , \\ \infty \ , \rho_2 \geq 1 \end{cases} \tag{3.3.2}$$

here $\nu = \bar{b}(\lambda - \lambda v)$ is the unique valid the least root of the equation is different from one.

The equality (3.3.2) are given in [6]. In the considered work it is shown as result of the equality (3.3.1).

Proof of the Theorem 3.3.1. In [4] are given the following equality for $\bar{g}_N(s)$:

$$\bar{g}_N(s) = \frac{\Delta_{N-1}(s)}{\Delta_N(s)} \ , \tag{3.3.3}$$

here $\Delta_k(s)$ is defined by following equality:

$$\sum_{k=0}^{\infty} v^k \Delta_k(s) = \frac{v\bar{b}(s) - \bar{b}(s + \lambda - \lambda v)}{(1-v)[v - \bar{b}(s + \lambda - \lambda v)]} \ \ , \tag{3.3.4}$$

If we take into account $\Delta_k(0) = 1$ by (3.3.4), then directly determine by (3.3.3)

$$\frac{1 - \bar{g}_N(s)}{s} = \frac{1}{\Delta_N(s)} \cdot \frac{\Delta_N(s) - \Delta_{N-1}(s)}{s}$$

multiplying the both sides of this equality and giving over to the limit when $s \to 0$, we obtain

$$M\zeta_{1N} = \lim_{s\to 0}\frac{\Delta_N(s) - \Delta_{N-1}(s)}{s} \tag{3.3.5}$$

By (3.3.4) holds the following equality:

$$(1-v)\sum_{k=0}^{\infty} v^k \Delta_k(s) = \frac{v(1-\bar{b}(s))}{\bar{b}(s+\lambda-\lambda v) - v} + 1$$

or

$$\sum_{k=0}^{\infty} v^k (\Delta_k(s) - \Delta_{k-1}(s)) = \frac{1-\bar{b}(s)}{\bar{b}(s+\lambda-\lambda v) - v} \quad . \tag{3.3.6}$$

Dividing both sides of the equality by s and giving over to the limit when $s \to 0$ by (3.3.5) we obtain the following equality:

$$\sum_{k=0}^{\infty} v^k M\zeta_{1k} = \frac{v}{\mu[b(\lambda-\lambda v) - v]} \quad . \tag{3.3.7}$$

In [2] proved following equality:

$$\bar{f}_N(s) = 1 - \frac{sP_{N-1}(s)}{Q_{N-1}(s) - Q_{N-2}(s)} \quad , N \geq 2 \, . \tag{3.3.8}$$

here $P_k(s)$ and $Q_k(s)$ are defined by following equalities:

$$\sum_{k=0}^{\infty} v^k P_k(s) = \frac{1}{\bar{b}(s+\lambda-\lambda v) - v} \cdot \frac{1-\bar{b}(s+\lambda-\lambda v)}{s+\lambda-\lambda v} \quad ,$$

$$\sum_{k=0}^{\infty} v^k Q_k(s) = \frac{1}{\bar{b}(s+\lambda-\lambda v) - v} \quad . \tag{3.3.9}$$

By (3.3.8)

$$\frac{1-\bar{f}_N(s)}{s}=\frac{P_{N-1}}{Q_{N-1}-Q_{N-2}} \quad , \tag{3.3.10}$$

here $P_k = P_k(0)$ and $Q_k = Q_k(0)$ defines by (3.3.9) with derivative functions

$$\sum_{k=0}^{\infty} v^k P_k = \frac{1}{\bar{b}(\lambda-\lambda v)-v}\cdot\frac{1-\bar{b}(\lambda-\lambda v)}{\lambda(1-v)} \ , \tag{3.3.11}$$

$$\sum_{k=0}^{\infty} v^k Q_k = \frac{1}{\bar{b}(\lambda-\lambda v)-v} \tag{3.3.12}$$

By this equalities

$$\lambda\sum_{k=1}^{\infty} v^k \left(P_k - P_{k-1}\right) = \frac{1-v}{\bar{b}(\lambda-\lambda v)-v} \ ,$$

$$\sum_{k=1}^{\infty} v^k \left(Q_k - Q_{k-1}\right) = \frac{1-v}{\bar{b}(\lambda-\lambda v)-v} \ .$$

Hence

$$(Q_{N-1}-Q_{N-2}) = \lambda(P_{N-1}-P_{N-2})$$

or based on (3.3.10) we obtain the following equality:

$$M\zeta_{2N} = \frac{P_{N-1}}{\lambda(P_{N-1}-P_{N-2})} \ . \tag{3.3.13}$$

By (3.3.11) we obtain the following equality:

$$v^k P_k = \frac{1}{\lambda[\bar{b}(\lambda-\lambda v)-v]}-\frac{1}{\lambda(1-v)}=\frac{1}{\rho}\left(\frac{1}{\mu[\bar{b}(\lambda-\lambda v)-v]}-\frac{1}{\mu(1-v)}\right).$$

Since by (3.3.7) and $|v|<1$ if we take into consideration that following expansion takes place

$$\frac{1}{1-v} = \sum_{k=0}^{\infty} v^k$$

we obtain the following equality:

$$P_{N-1} = M\zeta_{1N} - \mu^{-1} \quad .$$

Herein by equality (3.3.13) follows equality (3.3.1).

Proof of the Theorem 3.3.2. In [6] proved the following equality

$$\lim_{N\to\infty} \frac{M\zeta_{1N-k}}{M\zeta_{1N}} = \begin{cases} r\ , \rho_1 \geq 1\ , \\ 1\ , \rho_1 < 1 \end{cases} \quad . \tag{3.3.14}$$

And in [11] are given the following equality

$$\lim_{N\to\infty} M\zeta_{1N} = \begin{cases} \dfrac{1}{\mu(1-\rho_1)}\ , \rho_1 < 1 \\ \infty\ , \qquad \rho_1 \geq 1 \end{cases} \quad . \tag{3.3.15}$$

If we take into consideration that by $\rho_2 < 1$ follows $\rho_1 = {\rho_2}^{-1} > 1$, proof of the theorem follows based on (3.3.14) and (3.3.15)

Conclusion

In this dissertation work considered the dual single–server queue systems with one classes of customers, also, the single–server priority systems with two priority classes (pressing and ordinary) of customers.

"Duality" in queuing there means to interchanging the interarrival time and service time distributions of a queue. The new system obtained in this manner is considered the dual of the original system, and conversely. When the interarrival and service time distributions of a queue are interchanged a new queue is obtained which can be considered as the dual of the original. Another dual system can also be associated with the original queue. Events defined for the original system can be transformed into events defined the duals and conversely, and hence, probabilities obtained for one system can be extended to the others.

Several authors have used duality relationships in this manner (Daley, Bhat, Takacs, Shanbhag). Our objective in this work is to study the duality relations between queue size distribution in some ordinary queue systems, also, in some priority queue systems. This problem has been solved for the systems $M|G|1$ and $G|M|1|N$ by Shahbazov for the case when limit is given by the regeneration points. By author of this dissertation work established the duality relation between stationary queue size distributions for these systems when limit is given by the continuous time t.

Further, in this work considered the following priority system: suppose that low(class 0) and high priority(class 1) customers arrive at a single–server queue system. Take $t_{0i} = 0$, and let $0 < t_{1i} < t_{2i} < \cdots$ be the arrival times of customers of the i-th class ($i = 0,1$). We assume that the interarrival times $u_{ni} = t_{ni} - t_{(n-1)i}$, and the service times v_{ni} of successive class i customers form for independent renewal processes determined by the distribution functions

$$A_i(x) = \begin{cases} 0\,, & x \le 0 \\ 1 - e^{-\lambda_i x}\,, & x \ge 0\,, \lambda_i > 0 \end{cases},$$

(for interarrival times) and $B_i(x)$ (for service times). We assume only that $B_i(x)$ has finite mean λ^{-1} $(i = 0,1)$. The number of the positions in the system are bounded by N_i $(N_i \geq 1)$, that is, may be at most $N_i + 1$ customers of the class i in the system (with the served customers). This system is denoted by $\overrightarrow{MM}|\overrightarrow{GG}|1|N_0, N_1$. By interchanging in this system the distributions $A_0(x)$ and $B_0(x)$ we obtain the dual priority system which is defined as $\overrightarrow{GM}|\overrightarrow{GM}|1|N_0, N_1$.

In this work obtained the duality relationship between stationary queue size distributions for the dual queue system $\overrightarrow{MM}|\overrightarrow{GG}|1|N_0, N_1$ and $\overrightarrow{GM}|\overrightarrow{GM}|1|N_0, N_1$.

It is not known us duality relations for the priority queue systems. Evidently, such relations obtained for the first time.

The results of the §-2.1, §-2.2, §-3.1, §-3.2 and §-3.3 belongs immediately to the author of this work. Some of them are results of joint operation author and his scientific supervisor.

Investigations on the subjects of this work can be continual as follows:

1. Establishment a duality relation between non-stationary queue size distributions in the queue systems $M|G|1|N$ and $GJ|M|1|N$.

2. Establishment a relationship between distributions of the stationary queu sizes of the systems $G|G|1$ and $G|G|1|N$.

3. Development the methods of using duality relations.

4. Determination a duality relation between stationary queue sizes distributions for the priority queue systems $\overrightarrow{MM}|\overrightarrow{GG}|1|N_1, N_2$ and $\overrightarrow{MM}|\overrightarrow{MM}|1|N_1, N_2$.

5. Investigation of the asymptotic behavior of the queue size distributions by using duality relations.

6. Establishment a duality relations for the systems with relative priority

7. Establishment a duality relation between service time distribution for the dual queue systems.

8. Establishment a duality relation between busy period distributions for the dual queue systems.

References

1. Азларов Т.А., Курбанов Х. Изв. АН УзССР, серия физ.мат.наук,1980, №1,3-8.
2. Азларов Т.А., Тахиров А. Случайные процессы и статистические выводы. Ташкент, “Фан”, 1974,10-14
3. Азларов Т.А., Курбанов Х. Соотношение двойственности для нестационарных распределений систем $M|G|1|N$ и $GJ|M|1|N-1$. Доклады АН УзССР, 1980, №10, 3-4.
4. Висков О. В., Исмаилов А.И. Система массового обслуживания с ограниченной очередью. Исслед - по мат.стат. и смежные вопросы. Научн. труды. ТашГУ, вып. 402, 1972, 17-29.
5. Курбанов Х., Худойбердиев З. Взаимоотношение между распределениями стационарных длин очередей системы с ожиданием и системы с ограниченной очередью. Сб.труд. СамГУ, 1997,20-22.
6. Курбанов Х. Некоторые предельные теоремы, связанные со средним значением периода занятости системы с ограниченной очередью, Сб.науч.тр. СамГУ,1990,61-64
7. Курбонов Х. Соотношение двойственности и переходные явления в системах массового обслуживания. Диссертация на соис.уч.степ.канд.физ.-мат. наук, Ташкент, 1987, 121 стр.
8. Ташманов Х, Курбанов Х. Исследование длин очередей двойственных систем, Сб. тр. ТашГУ, 1987, 79-82 стр.
9. Шахбазов А.А. Об одном соотношении двойственности в теории массового обслуживания. Изв. АзССР, 1975, №3,17-21
10. Джейсуон Н. Очереди с приоритетами, М.: Мир, 1973, 279 стр.
11. Климов Г.Б. Стохастические системы обслуживания, М.: Наука, 1966, 62-67 стр.
12. Harris T. J. The remaining busy period of finite queue. Oper.Res. V19, 1971, 79-82.

13. Takacs L. Combinatorical methods in the theory of queues., Rev.Inst. internat. Statist., 1964, 32, №3, 207-219.
14. Shanbhag D.N. On a duality principle in the theory of queues. Operat. Res.,1966, 14, №5, 947-949.
15. Greenberg I. Some duality results in the theory of queues. J.Appl.Prob., 6, 1969, 99-121.
16. Dalley D. J. On inverse queuing processes.. Zasta-sow. mat., 1969, 10, 213-224.
17. Bhat N. Some remarks about duality relation in queues. Acta math. Acad.Scient.hung., 1968, 19, №1-2, 163-170/
18. Foster F.G. A unified theory of stock, storage and queue control. Operat.Res.Quart., 10, 121-130.
19. Yamazaki G. and Sakasegawa H. Properties of duality in tandem queuing system, Received Dec. 19, 1974, 201-211.
20. Makino S. Emptiness in the finite dam. Ann.Math.Statist., 1960, 31, 803-808.
21. Qurbonov H., Djalilov A. $\mathrm{M}|\mathrm{G}|1$ va $\mathrm{M}|\mathrm{G}|1|\mathrm{N}$ modelli xizmat ko‘rsatish tarmoqlari statsionar navbat uzunliklari o‘rtalaridagi munosabatlar, Matematika va uni zamonaviy pedagogik texnologiyalar yordamida o‘qitish muammolari., Respublika ilmiy-amaliy konferensiya materiallari, II qism, 2015, 113-115.
22. Qurbonov H., Djalilov A. Xizmat ko'rsatish tarmoqlarining $\mathrm{M}|\mathrm{G}|1|\mathrm{N}$ va $\mathrm{GJ}|\mathrm{M}|1|\mathrm{N}$ modellari nostatsionar navbat uzunliklari taqsimoti uchun ikkilanmalik munosabatlari, Предельные теоремы теории вероятностей и их приложения., Материалы VII Ферганской конференции, 2015, 130-133.
23. Qurbonov H., Djalilov A. $\mathrm{M}_1\mathrm{M}_2|\mathrm{G}_1\mathrm{G}_2|1|\mathrm{N}_1,\mathrm{N}_2$ va $\mathrm{GJ}_1\mathrm{M}_2|\mathrm{M}_1,\mathrm{G}_2|1|\mathrm{N}_1-1,\mathrm{N}_2$ xizmat ko'rsatish tarmoqlarida imtiyozli talabalar nostatsionar navbat uzunliklari taqsimotlari o'rtasida ikkilanma munosabatlar., Statistics and its

applications., Materials of republican scientific–applied conference, 230-232.

24. Qurbonov H., Djalilov A. $\overrightarrow{M}_2|\overrightarrow{G}_2|1|N_1,N_2$ va $\overrightarrow{M}_2|\overrightarrow{G}_2|1|N_1,\infty$ imtiyozli xizmat ko'rsatish tarmoqlari oddiy talablarning statsionar navbat uzunliklari taqsimotlari o'rtasidagi munosabat haqida., Mathematical physics and related problems of the modern analysis., Abstracts of the scientific conference.,

25. Qurbonov H., Djalilov A. $M|G|1|N$ VA $GJ|M|1|N-1$ xizmat ko'rsatish tarmoqlari bandlik davrlari o'rta qiymatlari uchun ikkilanmalik munosabati., Magistrantlarning XVI ilmiy konferensiyasi materiallari, 44-47.