Applications of oscillatory integrals to solutions of generalized hyperbolic equations.

A

MASTER THESIS

submitted for
2014-2016 Master degree course
in Mathematics

By

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March 31, 2016

Contents

Introduction

Substantiation and actuality of problems considered in the master thesis. The main problem of the master thesis is investigation of the convolution operators. More precisely, we shall consider the $L^p - L^{p'}$-boundedness of Fourier multipliers of the type

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F, \tag{0.1}$$

where F is the Fourier transform operator

$$\varphi(\xi) \in C^\omega(R^n \setminus 0) \text{ is a real analytic, homogeneous function of order } 1,$$
$$a_k(\xi) \in C^\infty(R^n) \text{ is homogeneous of order } -k \text{ for large } |\xi|$$

Note that solution to the Cauchy problem for the hyperbolic equation can be written as a convolution operator. So, the example which we keep in mind is the solution operator $E_k(t) : g_k \longmapsto u(t)$ of the Cauchy problem

$$P(D_t, D_x)u = 0$$
$$D_t^k u_{|t=0} = g_k,\ D_t^l u_{|t=0} = 0\ (l = 0, 1, ..., m-1, l \neq k). \text{ (C.P.)}$$

Here $P(D_t, D_x)$ is a homogeneous constant coefficient partial differential operator of degree m in the time t and the space $x \in R^n$, which is strictly hyperbolic. $E_k(t)$ is a linear combination of operators of Fourier multipliers of the type (2.1) (modulo a regularizing operator), and the phase function $\varphi(\xi)$ of (2.1) is one of $t\varphi_l(\xi)_{l=1}^m$ if factorize the symbol $p(\tau, \xi)$ as $p(\tau, \xi) = (\tau - \varphi_1(\xi))...(\tau - \varphi_m(\xi))$. The Cauchy problem for hyperbolic systems, such as elastic wave equations and Maxwell equations, is also in our focus since the solution operator can be expressed in a similar way. (The problem (C.P.) itself can be transformed to a hyperbolic system.)

For the sake of simplicity, we assume $\varphi(\xi) > 0$ and set

$$\Sigma := \{\xi \in R^n \setminus 0; \varphi(\xi) = 1\}. \tag{0.2}$$

In the case $\Sigma = S^{n-1}$, which is related to the wave equation, the $L^p - L^{p'}$-boundedness of M_k is obtained by Strichartz. This result has been extended to the case when Σ has non-vanishing Gaussian curvature by Brenner, and the case when Σ is convex by Sugimoto. (Σ with non-vanishing Gaussian curvature is always convex.) In this work, we shall consider the case when Σ is not necessarily convex surface in R^3.

The case when Σ is a non-convex hyper-surface the estimate for the operator is far from being complete. Hence, investigation of the operator for the case when surface is non-convex is an actual problem of the modern analysis. In this thesis, we consider the problem for the case when Σ is non-convex surface in R^3.

Object of the research. The objects of the thesis are convolution operators with oscillatory kernel. The convolution kernel is given by oscillatory integrals. The oscillatory integral is the Fourier transform of measure supported on some surface. The surface is given as level set of the root of the symbol of hyperbolic equation.

Purpose and tasks of the research. The main purpose of the thesis is to find a lower estimate for the order of classical symbol of the convolution operator to be bounded on $L^p \to L^{p'}$. The bound for the order depends on estimates for the Fourier transform of measure supported on hyper-surface. Hence we investigate behavior of the corresponding oscillatory integrals. We get estimate depending on direction of the dual variables.

Scientific results of the thesis. In this thesis we obtain the following results:

1) We get estimate for the order of the amplitude function depending on so-called height of the surface. It is an improvement of the results given by M. Sugimoto [26], which obtain analogical result given by contact index of the hyper-surface;

2) We obtain estimate for oscillatory integrals with A_∞ type singularities. Then we apply the estimates for get a lower bound for the order of amplitude function. It gives a solution to one of the problem posed by M. Sugimoto [26].

3) We get estimates for the oscillatory integrals in terms of Randol type maximal functions.

4) We obtain a solution of the M. Sugimoto problem on estimates for the convolution operator when a zero set of Gaussian curvature is a curve with the first order of contact with tangent line.

Main problems and assumptions of research. The main problem is the following:
For which $k(p)$ the convolution operator M_k is a bounded operator from L^p to $L^{p'}$. The problem is related to the sharp estimates for the kernel function. It can be written as the Fourier transform of measures supported on surface. The behavior of the measures depends on geometric properties of the surface. The geometric properties can be expressed by singularities of the phase function. We investigate classification given by M.Sugimoto [26]. Moreover, we use more precise information for the phase function given in the Monograph [12]. Then the classification given by M. Sugimoto can be expressed in terms of powerful methods of singularities Theory [1]. The methods of the singularities theory allows us to give a solution of Problems posed by M. Sugimoto.

Analysis of references related to research thesis. In the case when $\Sigma = S^{n-1}$ is the unite sphere centered at the origin, which is related to the classical wave equation, the $L^p \to L^{p'}$ boundedness of M_k is obtained by Stricharts [23]. Further, this result has been extended to the case when S has everywhere non-vanishing Gaussian curvature by Brenner [8]. Under these conditions S is a strictly convex hyper-surface. Some estimates are obtained for such kind of convolution operators for the case when S is convex (not-necessarily strictly convex) and also for some classes of non-convex hyper-surfaces in R^3 by M.Sugimoto [27]-[26]. However, as remarked in [26] the problem still is remained open for some classes of hyper- surfaces. In this thesis we have some estimates which cover the open cases indicated by M.Sugimoto in [26].

Description of methodology used in research. In the thesis we use mainly methods of analysis, theory of singularities, methods of Fourier analysis, asymptotic analysis, methods of differential geometry.

Theoretical and practical importance of research results. The results of the thesis have fundamental character. The main results improve estimates by M. Sugimoto. By using powerful methods of singularities theory we give a solution of some problems posed by M. Sugimoto. We get estimate for the order of the amplitude function depending on so-called height of the surface. It is an improvement of the results given by M. Sugimoto [26], which obtain

analogical result given by contact index of the hyper-surface;
We obtain estimate for oscillatory integrals with A_∞ type singularities. Then we apply the estimates to get a lower bound for the order of amplitude function. It gives a solution of one of the problem posed by M. Sugimoto [26].
We get estimates for the oscillatory integrals in terms of Randol type maximal functions.
We obtain a solution of the M. Sugimpoto problem on estimates for the convolution operator when a zero set of the Gaussian curvature is a curve with the first order contact to tangent line.

Description of work structure. Throughout this work, we assume $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

In the case $n = 1$, the boundedness is trivial. Since $F^{-1}e^{i\varphi(\xi)}F$ is nothing but a translation, M_k is essentially a Bessel potential, the boundedness of which is well known.

In the case $n = 2$, we can exactly determine the dependence of the boundedness of M_k on Σ. We define the index $\gamma(\Sigma)$ to be the maximal order of contact of the curve Σ to its tangent line. Then we have

Theorem 0.1 (A). *Suppose $n = 2$. Then M_k is $L^p - L^{p'}$-bounded if $k > (4 - \frac{2}{\gamma(\Sigma)})(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \neq 1$. Furthermore, M_k is not necessarily $L^p - L^{p'}$-bounded if $k < (4 - \frac{2}{\gamma(\Sigma)})(\frac{1}{p} - \frac{1}{2})$.*

We can extend theorem 1 to the case $n \geq 3$ if we use the index

$$\gamma_0(\Sigma) = \sup_p \inf_H \gamma(\Sigma; p, H). \tag{0.3}$$

For a point $p \in \Sigma$ and for a plane H (of dimension2) which contains the normal line of Σ at p, we have defined the index $\gamma(\Sigma; p, H)$ to be the order of contact of the curve $\Sigma \cap H$ to the line $T \cap H$ at p, where T denotes the tangent hyperplane of Σ at p.

By using localization principal we may pay attention in a sufficiently small neighborhood of a particular point and $\gamma_0(\Sigma) = 2$. This means that the surface has at least one non-vanishing principal curvature at every point.

The thesis consists of three chapters. In the chapter one we consider some auxiliary results necessary in the next chapters. We need the inverse function theorem, implicit function theorem. Some results from the classical differential geometry such as Gauss map and Gaussian curvature. In investigation of behavior of oscillatory integrals it is useful notion of Newton polyhedron and adapted coordinates system introduced by A.N. Varchenko [32]. We introduce the necessary notions in the first chapter. In the next chapter two we discus results obtained by M. Sugimoto [26]. In this chapter, it is considered applications of convolution operators to solution of the Cauchy problem for hyperbolic equations.

The main part of the thesis is the chapter 3. In the first section of the chapter 3 we consider estimates for the oscillatory integrals. The main result of this section is reduction of the behavior of the convolution operator near the image of the gradient map. The main results of the section 2 is the following theorem which based on average decaying of the Fourier transform:

Theorem 0.2. *The operator $M_k * f$ is bounded on $L^p \to L^{p'}$ under condition $k > (6 - 2/h)(1/p - 1/2)$.*

In the next section 3 we analyze some condition given in the paper by M. Sugimoto [26]. It is interesting that conditions given by M. Sugimoto can be given easily by using Newton polygon of some function given in the following statement:

Proposition 0.1. *The condition of M.Sugimoto given in the Theorem of the paper* [26] *is equivalent to the condition*

$$\frac{\partial^\mu}{\partial y_1^\mu} \frac{\partial^\nu g(0,0)}{\partial y_2^\nu} = 0,$$

for $\mu = 1, 2, ..., \delta_1 - 1$, $\nu = 0, 1, ...$

In the next section 4 we investigate connection with classification given by M. Sugimoto [26] and theory of singularities which is given in the classical monograph [1]. More precisely we prove the following

Proposition 0.2. *The function h is of type I if and only if when h has singularity of type $A_{\delta_0 - 1}$ at the origin. The function h is of type II or III if and only if h has A_∞ type singularity at the origin.*

The central results of the thesis are the following

Theorem 0.3. *Suppose that $h(x)$ is of type II with $\delta_1 \geqslant 3$ and satisfies*

$$\frac{\partial^\mu}{\partial x_2^\mu}\{\frac{\partial^\nu h(b_1(x_2), x_2)}{\partial x_1^\nu}\}_{x_2=0} = 0 \tag{0.4}$$

For $\mu = 1, 2, 3, ..., \delta_1 - 1$ and $\nu = 2, 3, ...$ then M_k is $L^p \to L^{p'}$ bounded if $k > max\{6(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, (5 - \frac{1}{\delta_1})(\frac{1}{p} - \frac{1}{2})\}$.

The theorem gives a solution ot the problems posed by M. Sugimoto.

The following result gives a quit new estimate for the order of the amplitude function.

Theorem 0.4. *Let $S = \{(x, h(x) + 1)\}$ be the surface and h be a real analytic function with $h(0, 0) = 0$, $\nabla h(0, 0) = 0$. If $K(0, 0) = 0$ and $\nabla K(0, 0) \neq 0$ (where K is the Gaussian curvature of the surface) then M_k is $L^p \to L^{p'}$ bounded if*

$$k > max\{6(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, \frac{14}{3}(\frac{1}{p} - \frac{1}{2})\}$$

provided the cone neighborhood Γ is chosen sufficiently small.

The results of the thesis are published in the papers [].

1. Auxiliary statements.

1.1. **The inverse function theorem.** This theorem states, roughly speaking, that a continuously differentiable mapping f is invertible in a neighborhood of any point x at which the linear transformation $f'(x)$ is invertible:

Theorem 1.1. *Suppose f is a $C'-$ mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$. Then*

(a) there exist open sets U and V in R^n such that $a \in U, b \in V, f$ is one-to-one on U, and $f(U) = V$;

(b) if g is the inverse of f [which exists, by (a)], defined in V by

$$g(f(x)) = x \ (x \in U)$$

then $g \in C'(V)$.

Writing the equation $y = f(x)$ in component form, we arrive at the following interpretation of the conclusion of the theorem: The system of n equations

$$y_i = f_i(x_1, ..., x_n) \ (1 \le i \le n)$$

can be solved for $x_1, ..., x_n$ in terms of $y_1, ..., y_n$, if we restrict x and y to small enough neighborhoods of a and b; the solutions are unique and continuous differentiable.

Proof: Put $f'(a) = A$ and choose λ so that $4\lambda \|A^{-1}\| = 1$. Since $f \in C'(E)$, there is an open ball U, with center at a, such that

$$\|f'(x) - A\| < 2\lambda (x \in U) \tag{1.1}$$

Suppose $x \in U$ and $x + h \in U$, and define

$$F(t) = f(x + th) - tAh \ (0 \le t \le 1).$$

Since U is convex, $x + th \in U$ if $0 \le t \le 1$, and (1.1) implies

$$|F'(t)| = |f'(x + th)h - Ah| \le 2\lambda |h| \le \tfrac{1}{2}|Ah|.$$

The last inequality is true since

$$2\lambda |h| = 2\lambda |A^{-1}Ah| \le 2\lambda \|A^{-1}\| |Ah| = \tfrac{1}{2}|Ah|$$

by our choice of λ. Theorem differentiation of vector-valued functions now implies that

$$|F(1) - F(0)| \le \tfrac{1}{2}|Ah|,$$

or

$$|f(x+h) - f(x) - Ah| \le \frac{1}{2}|Ah|. \tag{1.2}$$

It follows that

$$|f(x+h) - f(x)| \ge \frac{1}{2}|Ah| \ge 2\lambda|h| \tag{1.3}$$

We stress that (1.2) and (1.3) hold whenever $x \in U$ and $x + h \in U$. In particular, (1.3) proves that f is one-to-one on U.

Fix $x_0 \in U$ and let S be an open ball with center at x_0 and radius $r > 0$, whose closure $\bar{S}$ lies in U. We shall prove that $f(S)$ contains the open ball with center at $f(x_0)$ and radius λr.

To do this, fix y so that $|y - f(x_0)| < \lambda r$, and define

$$\Phi(x) = |y - f(x)| \ (x \in \bar{S}).$$

If $|x - x_0| = r$, (1.3) shows that

$$2\lambda r \le |f(x) - f(x_0)| \le \Phi(x) + \Phi(x_0) < \Phi(x) + \lambda r.$$

Thus

$$\Phi(x_0) < \lambda r < \Phi(x)(|x - x_0| = r). \tag{1.4}$$

Since Φ is continuous and $\bar{S}$ is compact, there exist $x^* \in \bar{S}$ such that $\Phi(x^*) \le \Phi(x)$ for all $x \in \bar{S}$. By (1.4), $X^* \in S$.

Put $w = y - f(x^*)$. Since A is invertible, there exist $h \in R^n$ such that $Ah = W$. Choose $t \in (0, 1)$, so small that $x^* + th \in S$. Then

$$|f(x^*) - y + Ath| = (1 - t)|w|, \tag{1.5}$$

and (1.2) shows that

$$|f(x^* + th) - f(x^*) - Ath| \le \frac{1}{2}|tw|. \tag{1.6}$$

Since $\Phi(x^* + th)$ is the norm of the sum of the vectors which appear on the left sides of (1.5) and (1.6), and since $|w| = \Phi(x^*)$, we conclude that

$$\Phi(x^* + th) \le (1 - \frac{t}{2})\Phi(x^*). \tag{1.7}$$

If $\Phi(x^*) > 0$, (1.7) implies that $\Phi(x^* + th) < \Phi(x^*)$, since $t > 0$. But this contradicts the minimal property of x^*.

Thus $\Phi(x^*) = 0$, which says that $f(x^*) = y$.

We have now proved that every point of $f(U)$ has a neighborhood which lies in $f(U)$. Thus $f(U)$ is an open subset of R^n. By setting $V = f(U)$, part (a) of the theorem is established.

To prove part (b), choose $y \in V$, $y + k \in V$, and put $x = g(y)$,

$$h = g(y + k) - g(y).$$

By theorem of invertible linear operators, (1.1) and our choice of λ show that $f'(x)$ has an inverse, say B. Applying B to the equation

$$k = f(x + h) - f(x) = f'(x)h + r(h),$$

where $\frac{|r(h)|}{|h|} \longrightarrow 0$ as $h \longrightarrow 0$, we obtain $Bk = h + Br(h)$, or

$$g(y + k) - g(y) = Bk - B(r(h)). \tag{1.8}$$

By (1.3), $2\lambda|h| \leq |k|$. Thus $h \longrightarrow 0$ if $k \longrightarrow 0$ (which shows, incidentally,that g is continuous at y), and

$$\frac{|B(r(h))|}{|k|} \leq \frac{\|B\|}{2\lambda} \cdot \frac{|r(h)|}{|h|} \longrightarrow 0 \quad as \quad k \longrightarrow 0. \tag{1.9}$$

Comparison of (1.9) and (1.8) shows that g is differentiable at y and that $g'(y) = B$. In other words,

$$g'(y) = f'(g(y))^{-1} \quad (y \in V). \tag{1.10}$$

Also g is a continuous mapping of V onto U, f' is a continuous mapping of U into the set Ω of all invertible elements of $L(R^n)$, and inversion is a continuous mapping of Ω onto Ω, by theorem invertible linear operators. combining these facts with (1.10), we see that $g \in C'(V)$.

This completes the proof [19].

1.2. **The implicit function theorem.** If $x = (x_1, ..., x_n) \in R^n$ and $y = (y_1, ..., y_m) \in R^m$, let us write (x, y) for the point (or vector)

$$(x_1, ..., x_n, y_1, ..., y_m) \in R^{n+m}$$

In this section, the first entry in (x, y) or in a similar symbol will always be a vector in R^n; the second will be a vector in R^m.

Suppose $A \in L(R^{n+m}, R^n)$, and suppose that

$$A(h, 0) = 0 \quad \textit{implies} \quad h = 0 \tag{1.11}$$

Then if $A(h_1, k) = A(h_2, k)$, the linearity of A shows that

$$A(h_1 - h_2, 0) = 0,$$

hence $h_1 = h_2$. Thus, for each $k \in R^m$, the mapping $h \longrightarrow A(h, k)$ is a linear 1-1 mapping of R^n into R^n, hence is onto, by theorem 9.5.

We conclude that the equation

$$A(x, y) = 0 \tag{1.12}$$

has, for each $y \in R^m$, one and only one solution $x \in R^n$ if A satisfies (1.11).

The implicit function theorem asserts that a similar conclusion holds for certain continuously differentiable transformation which are not necessarily linear. Before stating it, let us take another look at (1.11) and observe that if $[A]$ is the n by $n + m$ matrix of A, with respect to the standard bases, then (1.11) says precisely that the first n column vectors of $[A]$ are linearly independent. This is a criterion for determining whether (1.11) holds, but will play no role in the proof which follows.

Theorem 1.2. *Suppose f is a C'-mapping of an open set $E \subset R^{n+m}$ into R^n. Suppose $(a, b) \in E, f(a, b) = 0, A = f'(a, b)$, and A satisfies 1.11. Then there is a neighborhood W of $b (W \subset R^m)$ and a unique function $g \in C'(W)$, with values in R^n, such that $g(b) = a$ and*

$$f(g(y), y) = 0, (y \in W). \tag{1.13}$$

The function g is defined implicitly by (1.13); hence the name of the theorem. It maybe stated in terms of a system of n equations in $n+m$ variables

$$\begin{aligned} &f_1(x_1,...,x_n,y_1,...,y_m)=0\\ &.\\ &.\\ &.\\ &f_n(x_1,...,x_n,y_1,...,y_m)=0. \end{aligned} \tag{1.14}$$

Our assumption on A now means that the n by n matrix which has $(\frac{\partial f_i}{\partial x_i})(a,b)$ in the ith column has independent column vectors. If this holds, and if $x=a, y=b$ satisfies (1.14), then (1.14) can be solved for $x_1,...,x_n$ in terms of $y_1,...,y_m$ for every y near b.

Proof: Let F be the mapping of $(x,y)\in E$ to $(z,w)\in R^{n+m}$, defined by

$$z=f(x,y), w=y. \tag{1.15}$$

Then $F\in C'(E)$. Since $f(a,b)=0$, we have

$$f(a+h,b+k)=A(h,k)+r(h,k)$$

where r is the remainder which appears in the definition of f'. Since

$$F(a+h,b+k)-F(a,b)=(f(a+h,b+k),k),$$

it follows that $F'(a,b)$ is the linear operator on R^{n+m} which maps (h,k) to $(A(h,k),k)$. If this image vector is 0, then $A(h,k)=0$ and $k=0$, hence $A(h,0)=0$, and now (1.11) implies $h=0$. This says that $F'(a,b)$ is one-to-one, and hence it is invertible (Theorem 9.5).

The inverse function theorem therefore holds for F: There are open sets U and V in R^{n+m}, containing (a,b) and $(0,b)$, such that F is a 1-1 mapping of U onto V; by (1.15), the inverse of F is of the form

$$x=\phi(z,x), \quad y=w \quad ((z,w)\in V), \tag{1.16}$$

where $\phi\in c'(v)$. In other words,

$$f(\phi(z,w),w)=z, \quad ((z,w)\in V). \tag{1.17}$$

If we now let W be a neighborhood of b such that $(0, w) \in V$ if $w \in W$ and if we define $g(y) = \phi(0, y)$ for $y \in W$, then, setting $z = 0$ in (1.17), we obtain (1.13).

Since $\phi(0, b) = a$, we see that $g(b) = a$.

The uniqueness of g follows from the fact that F is one-to-one: If $(x, y \in U, (x^*, y) \in U)$ and $f(x, y) = f(x^*, y)$, then $F(x, y) = F(x^*, y)$; hence $x^* = x$ [19].

1.3. **The definition of the Gauss map and its fundamental properties.** We shall begin by briefly reviewing the notion of orientation for surfaces.

As we have seen in the tangent plane; the differential of a map, given a parametrization $x : U \subset R^2 \to S$ of a regular surface S at a point $p \in S$, we can choose a unit normal vector at each point of $x(U)$ by the rule

$$N(q) = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}(q), \quad q \in x(U).$$

Thus, we have a differentiable map $N : x(U) \longrightarrow R^3$ that associates to each $q \in x(U)$ a unit normal vector $N(q)$.

More generally, if $V \subset S$ is an open set in S and $N : V \longrightarrow R^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q, we say that N *is a differentiable field of unit normal vectors on* V.

It is a striking fact that not all surfaces admit a differentiable field of unit normal vectors defined on the whole surface. For instance, on the Möbius strip of Fig.1 one cannot define such a field. This can be seen intuitively by going around once along the figure: After one turn, the vector field N would come back as $-N$, a contradiction to the continuity of N. Intuitively, one cannot, on the Möbius strip, make a consistent choice of a define "side"; moving around the surface, we can go continuously to the other side without leaving the surface.

We shall say that a regular surface is *orientable* if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field N is called an *orientation* of S.

For instance, the Möbius strip referred to above is not an orientable surface. Of course, every surface covered by a single coordinate system (for instance, surfaces represented by graphs of differentiable functions) is trivial orientable.

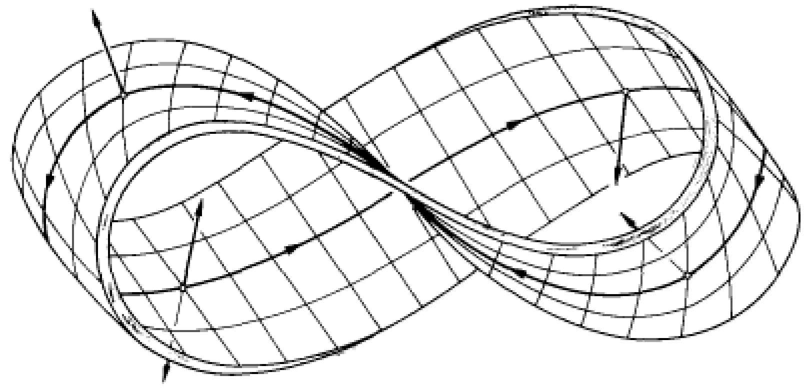

FIGURE 1. The Möbius strip.

Thus, every surface is locally orientable, and orientation is definitely a global property in the sense that it involves the whole surface.

An orientation N on S induces an orientation on each tangent space $T_p(S)$, $p \in S$, as follows. Define a basis $\{v, w\} \in T_p(S)$ to be *positive* if $<v \wedge w, N>$ is positive. It is easily seen that the set of all positive bases of $T_p(S)$ is an orientation for $T_p(S)$.

Throughout this thesis, S will denote a regular orientable surface in which an orientation (i.e., a differentiable field of unit normal vectors N) has been chosen; this will be simply called a surface S with an orientation N.

DEFINITION 1.1. *Let $S \subset R^3$ be a surface with an orientation N. The map $N : S \longrightarrow R^3$ takes its values in the unit sphere*

$$S^2 = \{(x, y, z) \in R^3\};$$

The map $N : S \longrightarrow S^2$, thus defined, is called the Gauss map of S (Fig.2).

It is straightforward to verify that the Gauss map is differentiable. The differential dN_p of N at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel planes, dN_p can be looked upon as a linear map on $T_p(S)$.

The linear map $dN_p : T_p(S) \longrightarrow T_p(S)$ operates as follows. For each parametrized curve $\alpha(t)$ in S with $\alpha(0) = p$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere S^2; this amounts to restricting the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$. It measures the rate of change of the normal vector N, restricted to the curve $\alpha(t)$,

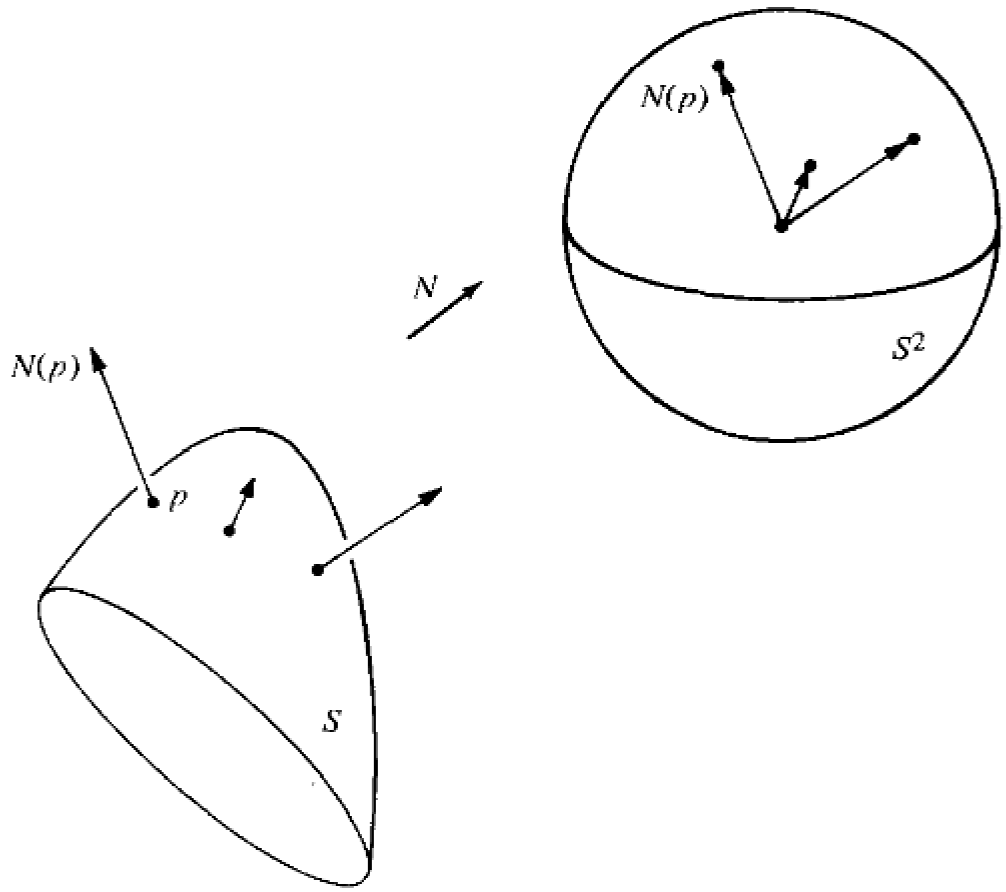

FIGURE 2. The Gauss map

at $t = 0$. Thus, dN_p measures how N pulls away from $N(p)$ in a neighborhood of p. In the case of curves, this measure is given by a number, the curvature. In the case of surfaces, this measure is characterized by a linear map.

Example 1.1. For a plane P given by $ax + by + cz + d = 0$, the unit normal vector $N = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$ is constant, and therefore $dN = 0$.

Example 1.2. Consider the unit sphere

$$S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}.$$

If $\alpha(t) = (x(t), y(t), z(t))$ is a parametrized curve in S^2, then

$$2xx' + 2yy' + 2zz' = 0,$$

which shows that the vector (x, y, z) is normal to the sphere at the point (x, y, z). Thus, $\overline{N} = (x, y, z)$ and $N = (-x, -y, -z)$ are fields of unit normal vectors in

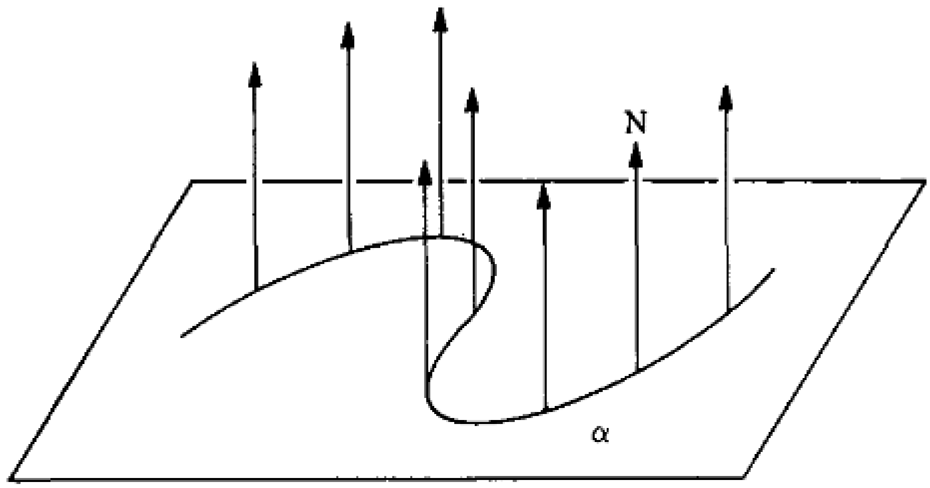

FIGURE 3. Plane: $dN_p = 0$

S^2. We fix an orientation in S^2 by choosing $N = (-x, -y, -z)$ as a normal field. Notice that N points toward the center of the sphere.

Restricted to the curve $\alpha(t)$, the normal vector

$$N(t) = (-x(t), -y(t), -z(t))$$

is a vector function of t, and therefore

$$dN(x'(t), y'(t), z'(t)) = N'(t) = (-x'(t), -y'(t), -z'(t));$$

that is, $dN_p(v) = -v$ for all $p \in S^2$ and all $v \in T(S^2)$. Notice that with the choice of $\bar{N}$ as a normal field (that is, with the opposite orientation) we would have obtained $d\bar{N}_p(v) = v$.

An important fact about dN_p is contained in the following proposition.

Proposition 1.1. *The differential $dN_p : T_p(S) \longrightarrow T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Proof. Since dN_p is linear, it suffices to verify that $< dN_p(w_1), w_2 >=< w_1, dN_p(w_2) >$ for a basis $\{w_1, w_2\}$ of $T_p(S)$. Let $x(u, v)$ be a parametrization of S at p and $\{x_u, x_v\}$ the associated basis of $T_p(S)$. If $\alpha(t) = x(u(t), v(t))$ is a parametrized curve in S, with $\alpha(0) = p$, we have

$$dN_p(\alpha'(0)) = dN_p(x_u u'(0) + x_v v'(0) = \frac{d}{dt} N(u(t), v(t))|_{t=0} = N_u u'(0) + N_v v'(0);$$

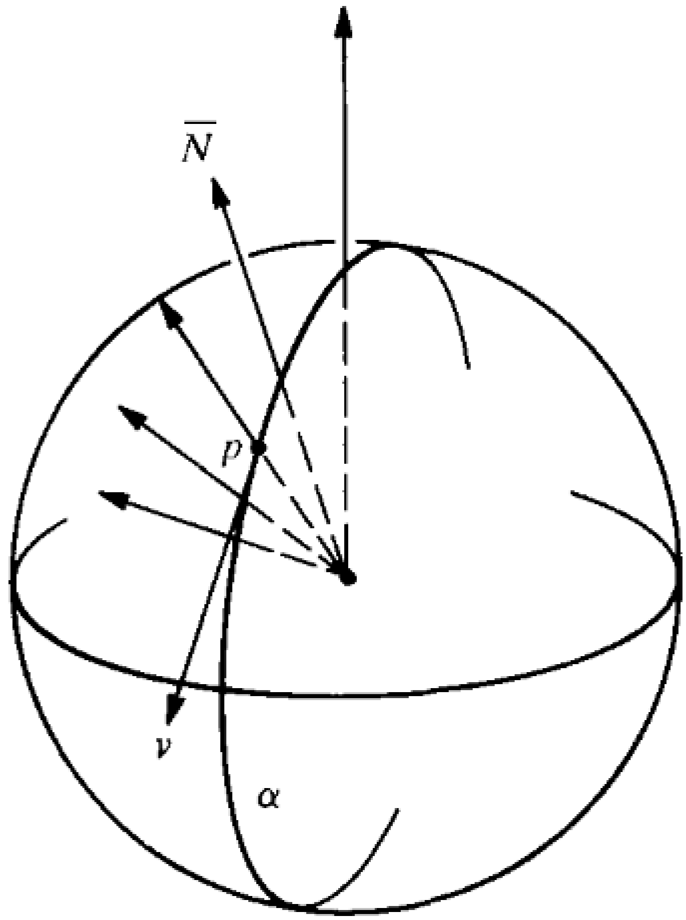

FIGURE 4. Unit sphere $d\bar{N}_p(v) = v$.

in particular, $dN_p(x_u) = N_u$ and $dN_p(x_v) = N_v$. Therefore, to prove that dN_p is self-adjoint, it suffices to show that

$$< N_u, x_v > = < x_u, N_v > .$$

To see this, take the derivatives of $< N, x_u > = 0$ and $< N, x_v > = 0$, relative to v and u, respectively, and obtain

$$< N_v, x_u > + < N, x_{uv} > = 0,$$

$$< N_u, x_v > + < N, x_{vu} > = 0.$$

Thus,

$$(1.18) \qquad < N_v, x_u > = - < N, x_{uv} > = < N_u, x_v >$$

The fact that $dN_p : T_p(S) \longrightarrow T_p(S)$ is a self-adjoint linear map allows us to associate to dN_p a quadratic form Q in $T_p(S)$, given by $Q(v) = < dN_p(v), v >$,

$v \in T_p(S)$. To obtain a geometric interpretation of this quadratic form, we need a few definitions. For reasons that will be clear shortly, we shall use the quadratic form $-Q$.

DEFINITION 1.2. *The quadratic form II_p, defined in $T_p(S)$ by $II_p(v) = -< dN_p(v), v >$ is called the second fundamental form of S at p.*

DEFINITION 1.3. *Let C be a regular curve in S passing through $p \in S, k$ the curvature of C at p, and $cos\theta =< n, N >$, where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = kcos\theta$ is then called the normal curvature of $C \subset S$ at p.*

In other words, k_n is the length of the projection of the vector kn over the normal to the surface at p, with a sign given by the orientation N of S at p.

Remark. The normal curvature of C does not depend on the orientation of C but changes sign with a change of orientation for the surface.

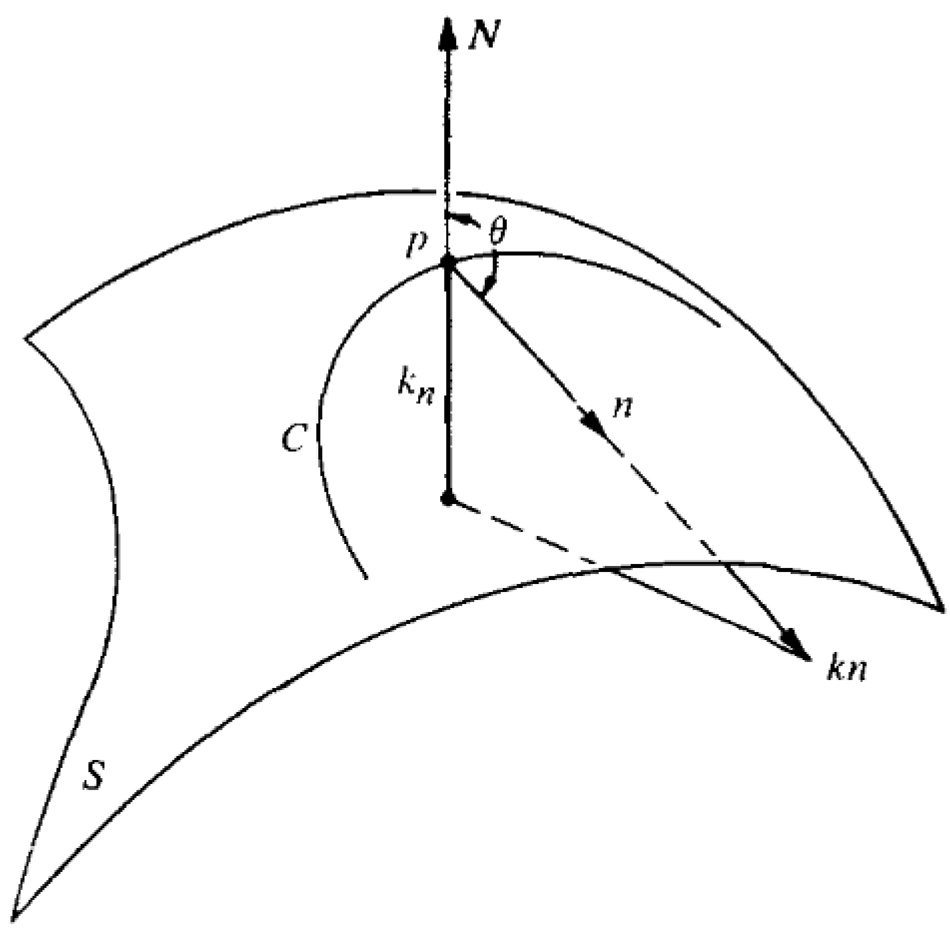

FIGURE 5.

To give an interpretation of the second fundamental form II_p, consider a regular curve $C \subset S$ parametrized by $\alpha(s)$, where s is the arc length of C, and with $\alpha(0) = p$. If we denote by $N(s)$ the restriction of the normal vector N to the curve $\alpha(s)$, we have $< N(s), \alpha'(s) >= 0$. Hence,

$$< N(s), \alpha"(s) >= - < N'(s), \alpha'(s) > .$$

Therefore,

$$II_p(\alpha'(0)) = - < dN_p(\alpha'(0)), \alpha"(0) >= - < N'(0), \alpha(0) >=$$

$$< N(0), \alpha"(0) >=< N, kn > (p) = k_n(p).$$

In other words, the value of the second fundamental form II_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v. In particular we obtained the following result.

Proposition 1.2. *(Meusnier). All curves lying on a surface S and having at a given point $p \in S$ the tangent line have at this point the same normal curvatures.*

The above proposition allows us to speak of the normal curvature along a given direction at p. It is convenient to use the following terminology. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the normal section of S at p along v. In a neighborhood of p, a normal section of S at p is a regular plane curve on S whose normal vector n at $\pm N(p)$ or zero; its curvature is therefore equal to the absolute value of the normal curvature along v at p. With this terminology, the above proposition says that the absolute value of the normal curvature at p of a curve $\alpha(s)$ is equal to the curvature of the normal section of S at p along $\alpha'(0)$.

Let us come back to the linear map dN_p. For each $p \in S$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1e_1, dN_p(e_2) = -k_2e_2$. Moreover, k_1 and $k_2 (k_1 \geq k_2)$ are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

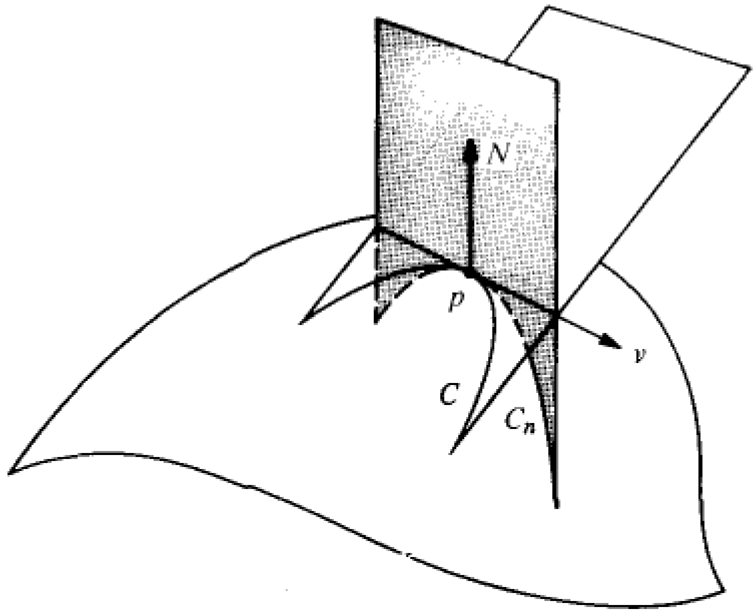

FIGURE 6. Meusnier Theorem: C and C_n have the same normal curvature at p along v.

DEFINITION 1.4. *The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the principal curvatures at p; the corresponding directions, that is, the direction given by the eigenvectors e_1, e_2 are called principal directions at p.*

For instance, in the plane all directions at all points are principal directions. The same happens with a sphere. In both case, this comes from the fact that the second fundamental form each point is constant; thus all directions are externals for the normal curvature.

In the cylinder, the vector v and w give the principal directions at p, corresponding to the principal curvatures 0 and 1, respectively.

DEFINITION 1.5. *If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p, then C is said to be a line of curvatures of C.*

Proposition 1.3. *(Olinde Rodrigues). A nacessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t),$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N \circ \alpha(t)$ and $\lambda(t)$ is a differentiable function of t. In this case, -$\lambda(t)$ is the (principal) curvature along $\alpha'(t)$.

Proof. It suffices to observe that if $\alpha''(t)$ is contained in a direction, then $\alpha'(t)$ is an eigenvector of dN and

$$dN(\alpha'(t)) = N'(t) = \lambda(t)\alpha'(t).$$

The converse is immediate.

The knowledge of the principal curvatures at p allows us to compute easily the normal curvature along a given direction of $T_p(S)$. In fact, let $v \in T_p(S)$ with $|v| = 1$. Since e_1 and e_2 form orthonormal basis of $T_p(S)$, we have

$$v = e_1 cos\theta + e_2 sin\theta,$$

where θ is a angle from e_1 to v in the orientation of $T_p(S)$. The normal curvature k_n along v is given by

$$k_n = II_p(v) = - < dN_p(v), v >= - < dN_p(e_1 cos\theta + e_2 sin\theta), e_1 cos\theta + e_2 sin\theta >=$$

$$< e_1 k_1 cos\theta + e_2 k_2 sin\theta, e_1 cos\theta + e_2 sin\theta >= k_1 cos^2\theta + k_2 sin^2\theta$$

The last expression is known classically as the Euler formula; actually, it is just the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

Given a linear map $A : v \longrightarrow V$ of a vector space of dimension 2 and given a basis $\{v_1, v_2\}$ of V, we recall that

$$determinant \quad of \quad A = a_{11}a_{22} - a_{12}a_{21}, \qquad trace \quad of \quad A = a_{11} + a_{22},$$

where (a_{ij}) is the matrix of A in the basis $\{v_1, v_2\}$. It is known that these numbers do not depend on the choice of the basis $\{v_1, v_2\}$ and are, therefore, attached to the linear map A.

In our case, the determinant of dN is the product $(-k_1)(-k_2) = k_1 k_2$ of the principal curvatures, and the trace of dN is the negative $-(k_1 + k_2)$ of the sum of principal curvatures. If we change the orientation of the surface, the determinant does not change (the fact that the dimension is even is essential here); the trace, however, changes sign.

DEFINITION 1.6. *Let $p \in S$ and let $dN_p : T_p(S) \longrightarrow T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the Gaussian curvature K of S at p. The negative of half of the trace of dN_p is called the mean curvature H of S at p.*

In terms of the principal curvatures we can write

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}.$$

DEFINITION 1.7. *A point of a surface S is called*

1. Elliptic if $det(dN_p) > 0$.
2. Hyperbolic if $det(dN_p) < 0$.
3. Parabolic if $det(dN_p) = 0$, with $dN_p \neq 0$.
4. Planar if $dN_p = 0$.

It is clear that this classification does not depend on the choice of the orientation.

DEFINITION 1.8. *If at $p \in S$, $k_1 = k_2$, then p is called an umbilical point of S; in particular, the planar points $(k_1 = k_2 = 0)$ are umbilical points.*

Proposition 1.4. *If all points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.*

Proof. Let $p \in S$ and let $x(u, v)$ be a parametrization of S at p such that the coordinate neighborhood V is connected.

Since each $q \in V$ is an umbilical point, we have, for any vector $w = a_1 x_u + a_2 x_v$ in $T_q(S)$,

$$dN(w) = \lambda(q) w,$$

where $\lambda = \lambda(q)$ is a real differentiable function in V.

We first show that $\lambda(q)$ is constant in V. For that, we write the above equation as

$$N_u a_1 + N_v a_2 = \lambda(x_u a_1 + x_v a_2);$$

hence, since w is arbitrary,

$$N_u = \lambda x_u,$$

$$N_v = \lambda x_v.$$

Differentiating the first equation in u and the second one in v and subtracting the resulting equations, we obtain

$$\lambda_u x_v - \lambda_v x_u = 0.$$

Since x_u and x_v are linearly independent, we conclude that

$$\lambda_u = \lambda_v = 0$$

for all $q \in V$. Since V is connected, λ is constant in V, as we claimed.

If $\lambda \equiv 0$, $N_u = N_v = 0$ and therefore $N = N_0$ =constant in V. Thus, $< x(u,v), N_0 >_u = < x(u,v), N_0 >_v = 0$; hence,

$$< x(u,v), N_0 > = const,$$

and all points $x(u,v)$ of V belong to a plane.

If $\lambda \neq 0$, then the point $x(u,v) - (1/\lambda)N(u,v) = y(u,v)$ is fixed, because

$$(x(u,v) - \frac{1}{\lambda}N(u,v))_u = (x(u,v) - \frac{1}{\lambda})N(u,v))_v = 0.$$

Since

$$|x(u,v) - y|^2 = \frac{1}{\lambda^2},$$

all points of V are contained in a sphere of center y and radius $1/|\lambda|$.

This proves the proposition locally, that is, for a neighborhood of a point $p \in S$. To complete the proof we observe that, since S is connected, given any other point $r \in S$, there exists a continuous curve $\alpha : [0,1] \longrightarrow S$ with $\alpha(0) = p$, $\alpha(1) = r$. For each point $\alpha(t) \in S$ of this curve there exists a neighborhood V_t in S contained in a sphere or in a plane and such that $\alpha^{-1}(V_t)$ is an open interval of $[0,1]$. The union $\bigcup \lambda^{-1}(V_t), t \in [0,1]$, covers $[0,1]$ and since $[0,1]$ is a closed interval, it is covered by finitely many elements of the family $\{\alpha^{-1}(V_t)\}$. Thus, $c([0,1])$ is covered by a finite number of the neighborhood V_t.

If the points of one of these neighborhoods are on a plane, all the others will be on the same plane. Since r is arbitrary, all the points of S belong to this plane.

If the points of one of these neighborhoods are on a sphere, the same argument shows that all points on S belong to a sphere, and this completes the proof.

DEFINITION 1.9. *Let p be a point in S. An asymptotic direction of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.*

It follows at once from the definition that at an elliptic point there are no asymptotic directions.

A useful geometric interpretation of the asymptotic directions is given by means of the Dupin indicatrix, which we shall now describe.

Let p be a point in S. The Dupin indicatrix at p is the set of vectors w of $T_p(S)$ such that $II_p(w) = \pm 1$.

To write the equations of the Dupin indicatrix in a more convenient form, let (ξ, η) be the Cartesian coordinates of $T_p(S)$ in the orthonormal basis $\{e_1, e_2\}$, where e_1 and e_2 are eigenvectors of dN_p. Given $w \in T_p(S)$, let ρ and θ be polar coordinates defined by $w = \rho v$, with $|v| = 1$ and $v = e_1 cos\theta + e_2 sin\theta$, if $\rho \neq 0$. By Euler's formula,

$$\pm 1 = II_p(w) = \rho^2 II_p(v) = k_1 \rho^2 cos^2\theta + k_2 \rho^2 sin^2\theta = k_1 \xi^2 + k_2 \eta^2,$$

where $w = \xi e_1 + \eta e_2$. Thus, the coordinates (ξ, η) of a point of the Dupin indicatrix satisfy the equation

$$k_1 \xi^2 + k_2 \eta^2 = \pm 1; \tag{1.19}$$

hence, the Dupin indicatrix is a union of conics in $T_p(S)$. We notice that the normal curvature along the direction determined by w is $k_n(v) = II_p(v) = \pm(1/\rho^2)$.

For an elliptic point, the Dupin indicatrix is an ellipse (k_1 and k_2 have the same sign); this ellipse degenerates into a circle if the points is an umbilical nonplanar point ($k_1 = k_2 \neq 0$).

For a hyperbolic point, k_1 and k_2 have opposite signs. The Dupin indicatrix is therefore made up of two hyperbolas with a common pair of asymptotic lines. Along the directions of the asymptotes, the normal curvature is zero; they are therefore asymptotic directions. This justifies the terminology and shows that a hyperbolic point has exactly two asymptotic directions.

For a parabolic point, one of the principal curvatures is zero, and the Dupin indicatrix degenerates into a pair of parallel lines. The common direction of these lines is the only asymptotic direction at the given point.

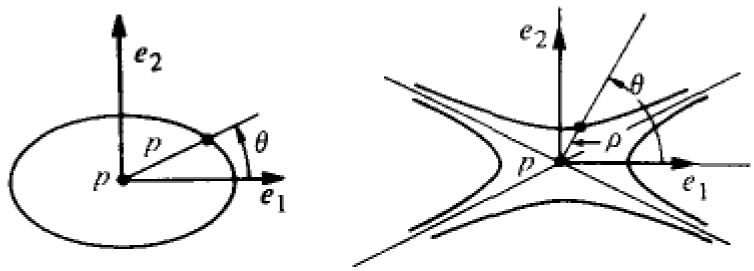

FIGURE 7. The Dupin indicatrix.

Closely related with the concept of asymptotic direction is the concept of conjugate directions, which we shall now define.

DEFINITION 1.10. *Let p be a point on a surface S. Two nonzero vectors $w_1, w_2 \in T_p(S)$ are conjugate if $< dN_p(w_1), w_2 >=< w_1, dN_p(w_2) >= 0$. Two directions r_1, r_2 at p are conjugate if a pair of nonzero vectors w_1, w_2 parallel to r_1 and r_2, respectively, are conjugate.*

It is immediate to check that the definition of conjugate directions does not depend on the choice of the vectors w_1 and w_2 on r_1 and r_2.

It follows from the definition that the principal directions are conjugate and that an asymptotic direction is conjugate to itself. Furthermore, at a nonplanar umbilic, every orthonormal pair of directions is a pair of conjugate directions, and at a planar umbilic each direction is conjugate to any other direction.

Let us assume that $p \in S$ is not an umbilical point, and let $\{e_1, e_2\}$ be the orthonormal basis of $T_p(S)$ determined by $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$. We claim that r_1 and r_2 are conjugate if and only if

$$k_1 cos\theta cos\varphi = -k_2 sin\theta sin\varphi. \tag{1.20}$$

In fact, r_1 and r_2 are conjugate if and only if the vectors

$$w_1 = e_1 cos\theta + e_2 sin\theta, \quad w_2 = e_1 cos\varphi + e_2 sin\varphi$$

are conjugate. Thus,

$$0 =< dN_p(w_1), w_2 >= -k_1 cos\theta cos\varphi - k_2 sin\theta sin\varphi.$$

Hence, condition 1.20 follows.

When both k_1 and k_2 are nonzero (i.e., p is either an elliptic or a hyperbolic point), condition 1.20 leads to a geometric construction of conjugate directions in term of the Dupin indicatrix at p. We shall describe the construction the construction at an elliptic point, the situation at a hyperbolic point being similar. Let r be a straight line through the origin of $T_p(S)$ and consider the intersection points q_1, q_2 of r with the Dupin indicatrix (fig). The tangent lines of the Dupin indicatrix at q_1 and q_2 are parallel, and their common direction r' is conjugate to r [10].

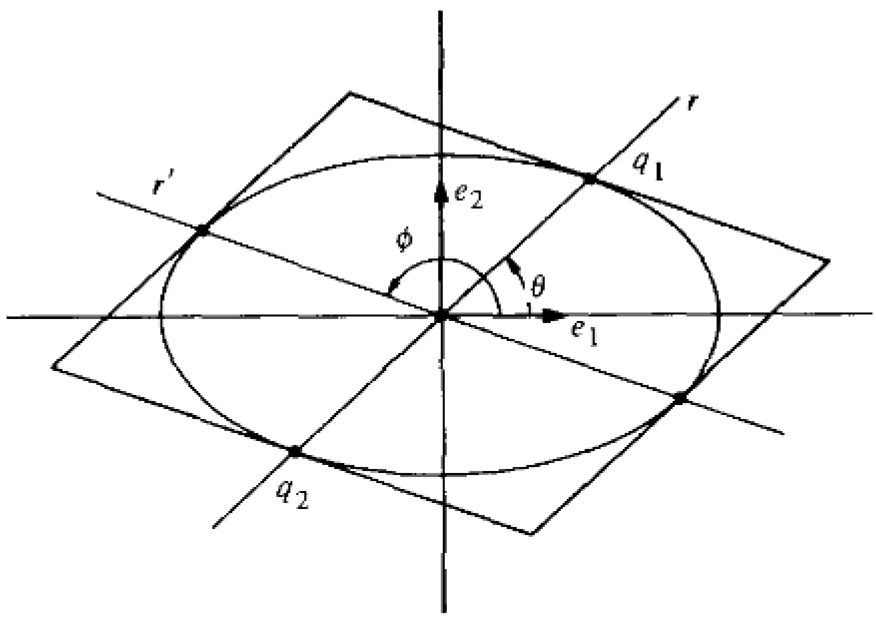

FIGURE 8. Newton polyhedron

1.4. **Newton polyhedra associated to ϕ. Normal forms of ϕ under linear coordinate changes when $h_{lin}(\phi) < 2$.** If ϕ is a smooth function, consider the associated Taylor series

$$\phi(x_1, x_2) \sim \sum_{\alpha_1,\alpha_2=0}^{\infty} c_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_1^{\alpha_2}$$

of ϕ centered at the origin. The set

$$\mathfrak{T}(\phi) := \{(\alpha_1, \alpha_2) \in N^2 : c_{\alpha_1,\alpha_2} = \frac{1}{\alpha_1!\alpha_2!} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \phi(0,0) \neq 0\}$$

will be called the Taylor support of ϕ at $(0,0)$. We shall always assume that the function ϕ is of finite type at every point, i.e., that the associated graph S of ϕ is of finite type. Since we are also assuming that $\phi(0,0)=0$ and $\nabla\phi(0,0)=0$, the finite type assumption at the origin just means that

$$\mathfrak{T}(\phi) \neq \emptyset.$$

The Newton polyhedron $\mathcal{N}(\phi)$ of ϕ at the origin is defined to be the convex hull of the union of all the quadrants $(\alpha_1,\alpha_2)+R_+^2$ in R^2, with $(\alpha_1,\alpha_2)\in\mathfrak{T}(\phi)$. The associated Newton diagram $\mathcal{N}_d(\phi)\mathcal{N}(\phi)$ in the sense of Varchenko [32] is the union of all compact faces of the Newton polyhedron; here, by a face, we shall mean an edge or a vertex.

We shall use coordinates (t_1,t_2) for points in the plane containing the Newton polyhedron, in order to distinguish this plane from the (x_1,x_2) - plane.

The Newton distance in the sense of Varchenko, or shorter distance, $d=d(\phi)$ between the Newton polyhedron and the origin is given by the coordinate d of the point (d,d) at which the bi-sectrix $t_1=t_2$ intersects the boundary of the Newton polyhedron.

The principal face $\pi(\phi)$ of the Newton polyhedron of ϕ is the face of minimal dimension containing the point (d,d). Deviating from the notation in [V76], we shall call the series

$$\phi_{pr}(x_1,x_2) := \sum_{(\alpha_1,\alpha_2)\in\pi(\phi)} c_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_1^{\alpha_2}$$

the principal part of ϕ. In case that $\pi(\phi)$ is compact, ϕ_{pr} is a mixed homogeneous polynomial; otherwise, we shall consider ϕ_{pr} as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which ϕ is expressed. By a local coordinate system (at the origin) we shall mean a smooth coordinate system defined near the origin which preserves 0. The height of the smooth function ϕ is defined by

$$h(\phi) := sup\{d_y\},$$

where the supremum is taken over all local coordinate systems $y = (y_1, y_2)$ at the origin, and where d_y is the distance between the Newton polyhedron and the origin in the coordinates y.

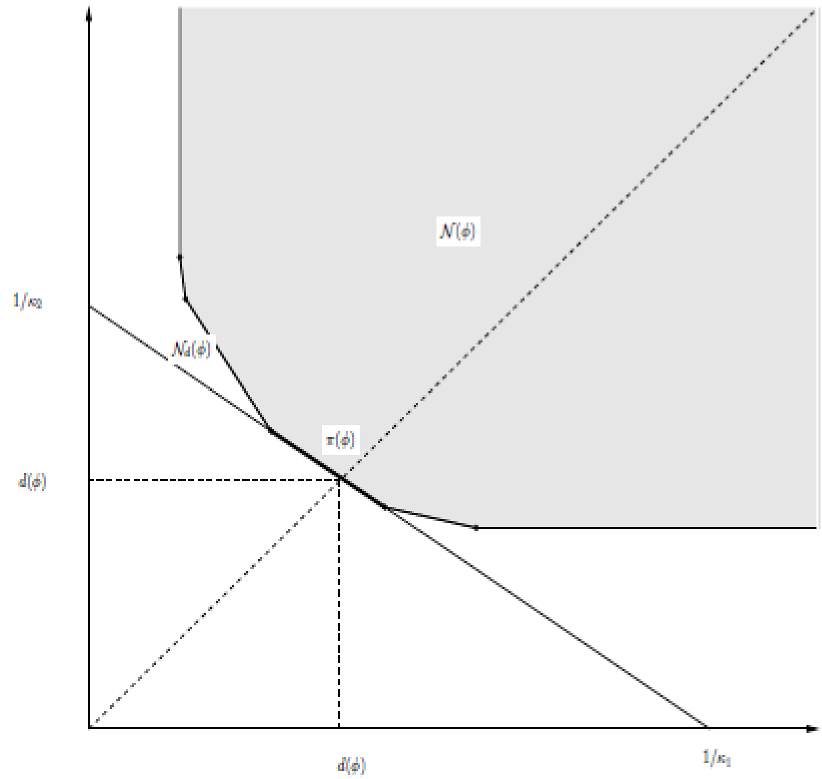

FIGURE 9. Newton polyhedron

A given coordinate system x is said to be adapted to ϕ if $h(\phi) = d_x$. In [14] it is proved that one can always find an adapted local coordinate system in two dimensions, thus generalizing the fundamental work by Varchenko [32] who worked in the setting of real-analytic functions ϕ.

Notice that if the principal face of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact edge, then it lies on a unique principal line

$$L := \{(t_1, t_2) \in R^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

with $\kappa_1, \kappa_2 > 0$. By permuting the coordinates x_1 and x_2, if necessary, we shall always assume that $\kappa_1 \leq \kappa_2$. The weight $\kappa = (\kappa_1, \kappa_2)$ will be called the principal weight associated to ϕ. It induces dilations $\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$, $r > 0$, on R^2, so that the principal part ϕ_{pr} of ϕ is κ-homogeneous of degree one with respect to these dilations, i.e., $\phi_{pr}(\delta_r(x_1, x_2)) = r\phi_{pr}(x_1, x_2)$ for every $r > 0$, and we find that

$$d = \frac{1}{\kappa_1 + \kappa_2} = \frac{1}{|\kappa|}.$$

It can then easily be shown (cf. Proposition 2.2 in [14]) that ϕ_{pr} can be factorized as

$$\phi_{pr}(x_1, x_2) = c x_1^{\nu_1} x_2^{\nu_2} \prod_{l=1}^{M} (x_2^q - \lambda_2 x_1^p)^{n_1}, \tag{1.21}$$

with $M \geq 1$, distinct non-trivial roots $\lambda_1 \in C \backslash \{0\}$ of multiplicities $n_1 \in N \backslash \{0\}$, and trivial roots of multiplicities $\nu_1, \nu_2 \in N$ at the coordinate axes. Here, p and q are positive integers without common divisor, and $\kappa_1/\kappa_2 = p/q$.

More generally, assume that $\kappa = (\kappa_1, \kappa_2)$ is any weight with $0 < \kappa_1 < \kappa_2$ such that the line $L_\kappa := (t_1, t_2) \in R^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1$ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ (recall that a supporting line to a convex set K in the plane is a line such that K is contained in one of the two closed half-planes into which the line divides the plane and such that this line intersects the boundary of K). Then $L_\kappa \bigcap \mathcal{N}(\phi)$ is a face of $\mathcal{N}(\phi)$, i.e., either a compact edge or a vertex, and the κ-principal part of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(\alpha_1, \alpha_2) \in L_\kappa} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_1^{\alpha_2}$$

is a non-trivial polynomial which is κ-homogeneous of degree 1 with respect to the dilations associated to this weight as before, which can be factorized in a similar way as in (1.21). By definition, we then have $\phi(x_1, x_2) = \phi_\kappa(x_1, x_2)+$ terms of higher κ-degree.

Adaptedness of a given coordinate system can be verified by means of the following proposition (see [14]):

If P is any given polynomial which is κ-homogeneous of degree one (such as $P = \phi_{pr}$), then we denote by

$$n(P) := ord_{S^1} P \tag{1.22}$$

the maximal order of vanishing of P along the unit circle S^1. Observe that by homogeneity, the Taylor support $\mathfrak{T}(P)$ of P is contained in the face $L_\kappa \bigcap \mathcal{N}(P)$ of $\mathcal{N}(P)$. We therefore define the homogeneous distance of P by $d_h(P) :=$

$1/(\kappa_1+\kappa_2) = 1/|\kappa|$. Notice that $(d_h(P), d_h(P))$ is just the point of intersection of the line L_κ with the bi-sectrix $t_1 = t_2$, and that $d_h(P) = d(P)$ if and only if $L_\kappa \bigcap \mathcal{N}(P)$ intersects the bi-sectrix. We remark that the height of P can then easily be computed by means of the formula

$$h(P) = max\{n(P), d_h(P)\} \tag{1.23}$$

(see Corollary 3.4 in [14]). Moreover, in [14] (Corollary 4.3 and Corollary 5.3), we also proved the following characterization of adaptedness of a given coordinate system:

Proposition 1.5. *The coordinates x are adapted to ϕ if and only if one of the following conditions is satisfied:*

(a) The principal face $\pi(\phi)$ of the Newton polyhedron is a compact edge, and $n(\phi_{pr}) \le d(\phi)$.

(b) $\pi(\phi)$ is a vertex.

(c) $\pi(\phi)$ is an unbounded edge.

These conditions had already been introduced by Varchenko, who has shown that they are sufficient for adaptedness when ϕ is analytic.

We also note that in case (a) we have $h(\phi) = h(\phi_{pr}) = d_h(\phi_{pr})$. Moreover, it can be shown that we are in case (a) whenever $\pi(\phi)$ is a compact edge and $\kappa_2/\kappa_1 \notin N$; in this case we even have $n(\phi_{pr}) < d(\phi)$ (cf. [14], Corollary 2.3).

Our discussion of the case where $h_{lin}(\phi) < 2$ will be based on normal forms of ϕ under linear coordinate changes. In the analytic setting, such normal forms are due to Siersma, but we shall need them also for smooth, finite type ϕ. The designation of the type of singularity that we list below corresponds to Arnolds classification of singularities in the case of analytic functions (cf. [1]), i.e., in the analytic case, non-linear analytic changes of coordinates would allow to further reduce ϕ to Arnolds normal forms.

Proposition 1.6. *Assume that $h_{lin}(\phi) < 2$, where ϕ satisfies Assumption (NLA).*

Then, after applying a suitable linear change of coordinates, ϕ can be written in the following form on a sufficiently small neighborhood of the origin:

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2\psi(x_1))^2 + b_0(x_1), \tag{1.24}$$

where b, b_0 and ψ are smooth functions, and where $\psi(x_1) = cx_1^m + O(x_1^{m+1})$, with $c \neq 0$ and $m \geq 2$. Moreover, we can distinguish two cases:

Case a. $b(0,0) \neq 0$. Then either

(i) b_0 is flat, (singularity of type A_∞)

or

(ii) $b_0(x_1) = x_1^n\beta(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2m+1$. (singularity of type A_{n1}) In these cases we say that ϕ is of type A.

Case b. $b(0,0) = 0$. Then we may assume that

$$b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2),$$

where b_1 and b_2 are smooth functions, with $b_1(0,0) \neq 0$.

Moreover, either

(i) b_0 is flat, (singularity of type D_∞)

or

(ii) $b_0(x_1) = x_1^n(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2m+2$. (singularity of type D_{n+1}) In these cases we say that ϕ is of type D.

Remarks. (a) It is easy to see that the principal weight κ and the Newton distance $d = d(\phi)$ for these normal forms are given by

$$\kappa = (\frac{1}{2m}, \frac{1}{2}) \quad \textit{and} \quad d = \frac{2m}{m+1}, \quad \textit{if} \quad \phi \quad \textit{is} \quad \textit{of} \quad \textit{type} \quad A,$$

$$\kappa = (\frac{1}{2m+1}, \frac{m}{2m+1}) \quad \textit{and} \quad d = \frac{2m+1}{m+1}, \quad \textit{if} \quad \phi \quad \textit{is} \quad \textit{of} \quad \textit{type} \quad D,$$

and by Proposition 1.7 that $h_{lin}(\phi) = d$, i.e., that the coordinates x are linearly adapted.

(b) Similarly, the coordinates $y_1 := x_1$, $y_2 := x_2 - \psi(x_1)$ are adapted to ϕ, and we can choose as the principal root jet. Comparing this with (1.9), we see that here the leading coefficient b_1 of the principal root jet is given by the constant c.

(c) When ϕ has a singularity of type A_∞ or D_∞ and satisfies Condition (R), then necessarily $b_0 \equiv 0$.

Proof. If $D^2\phi(0,0)$ had full rank 2, then the principal part ϕ_{pr} of ϕ would be a non-degenerate quadratic form, and by Proposition 1.2 one would easily see that the coordinates x would already be adapted to ϕ. This would contradict our assumptions. Therefore rank $D^2\phi(0,0) \leq 1$. Let us denote by P_n the homogeneous part of degree n of the Taylor polynomial of ϕ, i.e., $P_n(x_1,x_2) = \sum j + k = nc_{jk}x_1^j x_2^k$.

1. Case: rank$D^2\phi(0,0) = 1$.

In this case, by passing to a suitable linear coordinate system, we may assume that $P_2(x_1,x_2) = ax_2^2$, where $a \neq 0$. Consider the equation

$$\partial_2\phi(x_1,x_2) = 0.$$

By the implicit function theorem, locally it has a unique, smooth solution $x_2 = \psi(x_1)$,i.e., $\partial_2\phi(x_1,\psi(x_1)) = 0$. A Taylor series expansion of the function $\phi(x_1,x_2)$ with respect to the variable x_2 around $\psi(x_1)$ then shows that

$$\phi(x_1,x_2) = b(x_1,x_2)(x_2\psi(x_1))^2 + b_0(x_1),$$

where b and b_0 are smooth functions and $b(0,0) = \frac{1}{2}\partial_2^2\phi(0,0) = a \neq 0$, whereas $b_0(x_1) = O(x_1^2)$, since $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$ (this is a special instance of what would follow from a classical division theorem, see, e.g., [H90]).

Now, either b_0 is flat, which leads to type A_∞, or otherwise we may write $b_0(x_1) = x_1^n\beta(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2$, which leads to type A_{n1}.

Observe also that the function ψ cannot be flat, for otherwise the Newton polyhedron of ϕ would be the set $(0,2)+R_+^2$, in case that b_0 is flat, or its principal edge would be the compact line segment with vertices $(0,2)$ and $(n,0)$. In the latter case, the principal part of ϕ is given by $\phi_{pr}(x_1,x_2) = ax_2^2 + g(0)x_1^n$, so that the maximal multiplicity $n(\phi_{pr})$ of any real root of ϕ_{pr} along the unit circle is at most 1, whereas the Newton distance is given by $d = 1/(\frac{1}{2} + \frac{1}{n}) \geq 1$. Therefore, in both cases, the coordinates x would already be adapted to ϕ, according to Proposition 1.2. Notice also that the same argument shows that the coordinates y introduced in (1.10) are adapted to ϕ, so that in particular indeed $h = 2$ (in case that b0 is flat) respectively $h = 1/(\frac{1}{2} + \frac{1}{n}) < 2$ (if $b_0(x_1) = x_1^n\beta(x_1)$).

In particular, since $\psi(0) = 0$, we can write $\psi(x_1) = cx_1^m + O(x_1^{m+1})$ for some $m \in N^\times$, where $c \neq 0$. Note that indeed $m \geq 2$, since $P_2(x_1, x_2) = ax_2^2$.

Finally, when $b_0(x_1) = x_1^n \beta(x_1)$, a similar reasoning as before shows that the coordinates x are already adapted if $2m \geq n$, so that under Assumption 1.6 we must have $n \geq 2m + 1$.

2. Case: $D^2\phi(0, 0) = 0$.

Then $P_2 = 0$, and $P_3 \neq 0$, for otherwise we had $h_{lin} \geq d \geq 1/(1/4 + 1/4) = 2$, which would contradict our assumption that $h_{lin} < 2$. Notice also that $P_3 \neq 0$ is homogeneous of odd degree 3, so that necessarily the multiplicity of roots (cf. (1.7)) satisfies $n(P_3) \geq 1$.

Assume first that $n(P_3) = 1$. Then, passing to a suitable linear coordinate system, we may assume that $P_3(x_1, x_2) = x_1(x_2 \alpha x_1)(x_2 \beta x_1)$, where either $\alpha \neq$ are both real, or $\alpha = \bar{\beta}$ are non-real. Then one checks easily that the Newton diagram of P_3 is a compact edge intersecting the bi-sectrix in its interior and contained in the line given by $\frac{1}{3}t_1 + \frac{1}{3}t_2 = 1$. Consequently, it agrees with the principal face $\pi(\phi)$, so that $P_3 = \phi_{pr}$. We thus find that the Newton distance d in this linear coordinate system satisfies $d = 3/2 > n(\phi_{pr})$, so that these coordinates would already be adapted, contradicting our assumptions.

Assume next that $n(P_3) = 3$. Then, in a suitable linear coordinate system, $P_3(x_1, x_2) = x_2^3$. These coordinates are then adapted to P_3, so that $h(P_3) = d(P_3) = 3 > 2$. However, as has been shown in [15], under Assumption 1.6 this implies that the Taylor support of ϕ is contained in the region where $\frac{1}{6}t_1 + \frac{1}{3}t_2 \geq 1$. This in return implies that $h_{lin} \geq d \geq 1/(\frac{1}{6} + \frac{1}{3}) = 2$, in contrast to what we assumed.

We have thus seen that necessarily $n(P_3) = 2$. Then, after applying a suitable linear change of coordinates, we may assume that $P_3(x_1, x_2) = x_1 x_2^2$, i.e.,

$$\phi(x_1, x_2) = x_1 x_2^2 + R(x_1, x_2),$$

where R is a smooth function such that $\partial^\alpha R(0, 0) = 0$ for $|\alpha| \leq 3$. Consider here the equation

$$\partial_2 \phi(x_1, x_2) = 0$$

with respect to x_2. We claim that it has a smooth solution $x_2 = \psi(x_1)$, with $\psi(0) = \psi'(0) = 0$, near the origin. Indeed, we have $\partial_2\phi(x_1, x_2) = 2x_1x_2 + R_1(x_1, x_2)$, where R_1 is a smooth function such that $\partial^\alpha R_1(0,0) = 0$ for $|\alpha| \leq 2$, so that we have to solve the equation

$$2x_1x_2 + R_1(x_1, x_2) = 0. \tag{1.25}$$

To this end, let us write, for $x_1 \neq 0$, $x_2 = x_1 z$. Then 1.25 is equivalent to

$$2x_1^2 z + R_1(x_1, x_1 z) = 0. \tag{1.26}$$

Clearly, by the properties of R_1, we may factorize $R_1(x_1, x_1 z) = x_1^3 g(x_1, z)$, with a smooth function $g(x_1, z)$ defined near the origin, and thus for $x_1 \neq 0$ equation 1.26 is equivalent to

$$2z + x_1 g(x_1, z) = 0.$$

Regard this as an equation near the origin in (x_1, z). We can now apply the implicit function theorem to conclude that locally near the origin this equation has a unique, smooth solution $z = \psi_1(x_1)$. In particular, we find that

$$2x_1^2\psi_1(x_1) + R_1(x_1, x_1\psi_1(x_1)) = 0$$

near the origin in x_1. Setting $\psi(x_1) := x_1\psi_1(x_1)$, we then find that indeed $\psi(0) = \psi'(0) = 0$ and

$$\partial_2\phi(x_1, \psi(x_1)) \equiv 0. \tag{1.27}$$

By means of a Taylor expansion of the function $\phi(x_1, x_2)$ with respect to the variable x_2 around $x_2 = \psi(x_1)$ this implies that

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2\psi(x_1))^2 + b_2(x_1)x_2 + b_0(x_1),$$

where b, b_0 and b_2 are smooth functions. Again, we have that $\psi(x_1) = cx_1^m + O(x_1^{m+1})$, with $m \geq 2$. Observe that 1.27 implies that $b_2 = 0$, hence

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2\psi(x_1))^2 + b_0(x_1).$$

Moreover, since $\partial_2^2\phi(0,0) = 0$, $\partial_1\partial_2^2\phi(0,0) \neq 0$, $\partial_2^3\phi(0,0) = 0$, we have that

$$b(0,0) = 0, \quad \partial_1 b(0,0) \neq 0 \quad \textit{and} \quad \partial_2 b(0,0) = 0.$$

By Taylors formula, this implies that

$$b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2),$$

where b_1 and b_2 are smooth functions, with $b_1(0, 0) \neq 0$.

In a similar way as in Case 1, one can see that the coordinates from (1.10) are adapted to ϕ. Moreover, if b_0 is flat, which leads to case D_∞, then $h = 2$, and if $b_0(x_1) = x_1^n \beta(x_1)$, which leads to case D_{n+1}, then $h = \frac{2n}{n+1} < 2$. Finally, one also checks easily that the coordinates x in (1.10) are already adapted to ϕ, if $2m + 1 \geq n$, so that under our assumption we must have $n \geq 2m + 2$.

This concludes the proof of Proposition 2.11. Q.E.D. Corollary 2.13. Assume that ϕ satisfies Assumption (NLA). By passing to a suitable linear coordinate system, let us also assume that the coordinates x are linearly adapted to ϕ. Then, if $d = d(\phi) < 2$, the critical exponent in Theorem 1.14 is given by $p'_c = 2d + 2$.

Proof. Proposition 1.6 shows that the principal face $\pi(\phi)$ of the Newton polyhedron of ϕ is a compact edge whose upper vertex v is one the following points $(0, 2)$ or $(1, 2)$, which both lie below the line $H := (t_1, t_2) : t_2 = 3$ within the positive quadrant. On the other hand, $m + 1 \geq 3$. It is then clear from the geometry of the lines H, the line L which contains $\pi(\phi)$ and the line $\nabla^{(m)}$, that $\nabla(m)$ will intersect L above the vertex v. Since, by Varchenkos algorithm, the point v will also be a vertex of the Newton polyhedron of ϕ^a, this easily implies that $h^r(\phi) = d$. This proves the claim [12].

2. Estimates for hyperbolic equations of space dimension 3.

Throughout this section, we assume $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We shall consider the $L^p - L^{p'}$-boundedness of Fourier multipliers of the type

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F, \tag{2.1}$$

where

$$\varphi(\xi) \in C^\omega(R^n \setminus 0) \text{ is a real analytic homogeneous function of order } 1$$
$$a_k(\xi) \in C^\infty(R^n) \text{ is homogeneous of order } -k \text{ for large } |\xi|$$

The example which we keep in mind is the solution operator $E_k(t) : g_k \longmapsto u(t)$ of the Cauchy problem

$$P(D_t, D_x)u = 0$$
$$D_t^k u_{|t=0} = g_k,\ D_t^l u_{|t=0} = 0\ (l = 0, 1, ..., m-1, l \neq k). \text{ (C.P.)}$$

Here $P(D_t, D_x)$ is a homogeneous constant coefficient partial differential operator of degree m in the time t and the space $x \in R^n$, which is strictly hyperbolic. $E_k(t)$ is a linear combination of operators of Fourier multipliers of the type (2.1) (modulo a regularizing operator), and the phase function $\varphi(\xi)$ of (2.1) is one of $t\varphi_l(\xi)_{l=1}^m$ if factorize the symbol $p(\tau, \xi)$ as $p(\tau, \xi) = (\tau - \varphi_1(\xi))...(\tau - \varphi_m(\xi))$. The Cauchy problem for hyperbolic systems, such as elastic wave equations and Maxwell equations, is also in our focus since the solution operator can be expressed in a similar way. (The problem (C.P.) itself can be transformed to a hyperbolic system.)

For the sake of simplicity, we assume $\varphi(\xi) > 0$ and set

$$\Sigma = \xi \in R^n \setminus 0; \varphi(\xi) = 1. \tag{2.2}$$

In the case $\Sigma = S^{n-1}$, which is related to the wave equation, the $L^p - L^{p'}$-boundedness of M_k is obtained by Strichartz. This result has been extended to the case when Σ has non-vanishing Gaussian curvature by Brenner, and the case when Σ is convex by Sugimoto. (Σ with non-vanishing Gaussian curvature is always convex.) In this section, we shall consider the case when Σ is not necessarily convex.

In the case $n = 1$, the boundedness is trivial. Since $F^{-1}e^{i\varphi(\xi)}F$ is nothing but a translation, M_k is essentially a Bessel potential, the boundedness of which is well known.

In the case $n = 2$, we can exactly determine the dependence of the boundedness of M_k on Σ. We define the index $\gamma(\Sigma)$ to be the maximal order of contact of the curve Σ to its tangent line. Then we have [11]:

Theorem 2.1 (A)**.** *Suppose $n = 2$. Then M_k is $L^p - L^{p'}$-bounded if $k > (4 - \frac{2}{\gamma(\Sigma)})(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \neq 1$. Furthermore, M_k is not necessarily $L^p - L^{p'}$-bounded if $k < (4 - \frac{2}{\gamma(\Sigma)})(\frac{1}{p} - \frac{1}{2})$.*

We can extend theorem 2.1 to the case $n \geq 3$ if we use the index

$$\gamma_0(\Sigma) = \sup_p \inf_H \gamma(\Sigma; p, H). \tag{2.3}$$

For a point $p \in \Sigma$ and for a plane H (of dimension2) which contains the normal line of Σ at p, we have defined the index $\gamma(\Sigma; p, H)$ to be the order of contact of the curve $\Sigma \cap H$ to the line $T \cap H$ at p, where T denotes the tangent hyperplane of Σ at p. The index $\gamma_0(\Sigma)$ gives a uniform (with respect to directions) of the Fourier transform of measures [28].

Theorem 2.2. *Suppose $n \geq 3$. Then M_k is $L^p - L^{p'}$-bounded if $k > (2n - \frac{2}{\gamma_0(\Sigma)})(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equality if $p \neq 1$. Furthermore, M_k is not necessarily $L^p - L^{p'}$-bounded if $k < (2n - \frac{2}{\gamma_0(\Sigma)})(\frac{1}{p} - \frac{1}{2})$ and $p = 1, 2$.*

But in the case $n \geq 3$, there still remains the problem " What about the optimality of Theorem 2.2 for $1 < p < 2$?" To our surprise, it is not optimal in spite of the optimality for $p = 1, 2$, and the boundedness is dependent the following, we shall exhibit such a strange phenomenon in the simplest case $n = 3$, $\gamma_0(\Sigma) = 2$. We can improve Theorem 2.2 in this special case. It should be noted that $\gamma_0(\Sigma)$ does not give the sharp bound for the Fourier transform of surface carried measures.

2.1. **Microlocalization.** From now on, we shall microlocalize the problem. That is, in (2.1), we shall assume that $a_k(\xi)$ is supported in a sufficiently small conic neighborhood Γ of a particular point $\upsilon \in S^{n-1}$, and $\varphi(\xi) \in C^{\omega}(\Gamma)$. We may assume $\upsilon = e^n = (0, ..., 0, 1) \in S^{n-1}$ without loss of generality. Then in the neighborhood, Σ in (2.2) can be expressed as

$$\Sigma \cap \Gamma = \{\xi \in \Gamma; \varphi(\xi) = 1\} = \{(y, h(y)); y \in U\}, \tag{2.4}$$

where $h(y) \in C^{\omega}(U)$ is a positive real analytic function and $U \subset R^{n-1}$ is an open neighborhood of the origin. Conversely, for a neighborhood $U \subset R^{n-1}$ of the origin and a positive function $h(y) \in C^{\omega}(U)$, we can define a conic neighborhood Γ of the point $\upsilon \in S^{n-1}$ and a homogeneous function $\varphi(\xi) \in C^{\omega}(\Gamma)$ of order 1 by (2.4).

In the section, we shall show a relation between the function $h(y)$ and the boundedness of M_k, in other words, a relation between the oscillatory integral

$$I(\lambda; z) = \int_{R^{n-1}} e^{i\lambda(z\cdot y + h(y))} g(y)dy (\lambda > 0, z \in R^{n-1}, g \in C_0^{\infty}(U)) \tag{2.5}$$

defined by $h(y)$ and the convolution kernel

$$K_k(x) = F^{-1}[e^{i\varphi(\xi)} a_k(\xi)](x) \tag{2.6}$$

of M_k. In this problem, the change of variables

$$\xi = (\lambda y, \lambda h(y)), (\lambda > 0, y \in U) \tag{2.7}$$

makes the relation clear. We remark that the Jacobian is

$$\frac{D\xi}{D(\lambda, y)} = \lambda^{n-1} G(y), G(y) = h(y) - y \cdot \bigtriangledown h(y). \tag{2.8}$$

Proposition 2.1. [26]. *Let $q \geq 2$ and $\alpha \geq 0$. Suppose, for all $g \in C_0^{\infty}(U)$ and $\lambda > 0$,*

$$\|I(\lambda; z)\|_{L^q(R_z^{n-1})} \leq C_g \lambda^{-\alpha}, \tag{2.9}$$

where C_g is independent of λ. Then $K_k(x) \in L^q(R^n)$; hence M_k is $L^p - L^{p'}$-bounded for $p = \frac{2q}{(2q-1)}$, if $k > n - \alpha - \frac{1}{q}$.

Proof. We only have to pay attention to x near the point $-\nabla\varphi(e_n) \in R^n$, by using an integration by parts argument. We write $x-(x', x_n), x' = (x_1, ..., x_{n-1})$. We may assume that the variable x_n is negative and away from the origin since Euler's identity $\varphi(e_n) = e_n \cdot \nabla\varphi(e_n) > 0$ yields $\varphi_n'(e_n) = \varphi(e_n) > 0$.

We can write

$$a_k(\xi) = a_k(\xi)\Psi(|x_n|\varphi(\xi))$$

with a positive function $\Psi(t) \in C^\infty$ which is supported in $t; t > 0$ and equals 1 for large t. We may assume that $a_k(\xi)$ is homogeneous. Then, by the change of variables (2.7) and $\lambda \mapsto |x_n|^{-1}\lambda$, we have

$$\overline{K_k(x)} = \frac{1}{(2\pi)^n} F[e^{-i\varphi(\xi)}\overline{a_k(\xi)}\Psi(|x_n|\varphi(\xi))](x)$$

$$(2.10) \qquad = \frac{|x_n|^{k-n}}{(2\pi)^n} \int_0^\infty \int_U e^{i\lambda(x_n^{-1}x'\cdot y + h(y) + x_n^{-1})} \lambda^{n-1-k}\Psi(\lambda)g(y)d\lambda dy$$

$$= \frac{|x_n|^{k-n}}{(2\pi)^n} F_\lambda[I(\lambda; x_n^{-1}x')\lambda^{n-1-k}\Psi(\lambda)](|x_n|^{-1}).$$

Here $g(y) = a_k(y, \bar{h}(y))|G(y)| \in C_0^\infty(U)$ by (2.8). Hence, we obtain from the estimate (2.9)

$$\|K_k(x)\|_{L^q(|x+\nabla\varphi(e_n)|\ll 1)}$$
$$\leq C\|F_\lambda[I(\lambda; z)\lambda^{n-1-k}\Psi(\lambda)](\tau)\|_{L^q(R_\tau \times R_z^{n-1})}$$
$$\leq C\|\|I(\lambda; z)\lambda^{n-1-k}\Psi(\lambda)\|_{L^{q^*}(R_\lambda)}\|_{L^q(R_z^{n-1})}$$
$$\leq C\|\|I(\lambda; z)\|_{L^q(R_z^{n-1})}\lambda^{n-1-k}\Psi(\lambda)\|_{L^{q^*}(R_\lambda)}$$
$$\leq C\|\lambda^{n-1-\alpha-k}\Psi(\lambda)\|_{L^{q^*}(R_\lambda)}.$$

Here $\frac{1}{q} + \frac{1}{q*} = 1$, and we have used Hausdorff and Young's inequality and Minkowski's inequality for integrals. If $k > n-\alpha-\frac{1}{q}$, we have $(n-1-\alpha-k)q* < -1$, which implies $K_k \in L^q(R^n)$. The boundedness of M_k is trivial if we notice Young's inequality.

In particular, in the case $q = \infty$, we have another version of Proposition 1. Instead of (2.6), we set

$$K_{k,j}(x) = F^{-1}[e^{i\varphi(\xi)}a_k(\xi)\Phi_j(\xi)](x).$$

Here $\Phi_j(\xi)_{j=1}^\infty$ is a Littlewood-Paley partition of unity which is used to define the norm

$$\|\upsilon\|_{B^s_{p,q}} = (\sum_{j=0}^\infty (2^{js}\|F^{-1}\Phi_j(\xi)F\upsilon\|_{L^p})^q)^{\frac{1}{q}}$$

of Besov space $B^s_{p,q}$. For more information about this spaces, see, for example, Berght and Lostrom[7].

Proposition 2.2. *Let $\alpha > 0$. Suppose, for all $g \in C_0^\infty(U)$ and $\lambda > 0$,*

$$\|I(\lambda; z)_{L^\infty(R_z^{n-1})} \leq C_g \lambda^{-\alpha} \tag{2.11}$$

where C_g is independent of λ. Then $K_{k,j}(x)_{j=0}^\infty$ is bounded in $L^\infty(R^n)$ if $k = n - \alpha$. Hence M_k is $L^p - L^{p'}$-bounded if $k > (2n-2\alpha)(\frac{1}{p} - \frac{1}{2})$. This inequality can be replaced by an equation if $p \neq 1$.

Proof. Since we can write $j = 1, 2, ...,$

$$\Phi_j(\xi) = \Phi_j(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)$$

with a positive function $\Psi(t) \in C_0^\infty(t > 0)$, we may replace $\Phi_j(\xi)$ in the definition of $K_{k,j}(x)$ by $\Psi(\frac{\varphi(\xi)}{2^j})$. Then we have similarly to (2.10)

$$K_{k,\bar{j}}(x) = \frac{|x_n|^{k-n}}{(2\pi)^n} F_\lambda[I(\lambda; x_n^{-1}x')\lambda^{n-1-k}\Psi\left(\frac{\lambda}{2^j|x_n|}\right)](|x_n|^{-1})$$

for x near the point $-\nabla\varphi(e_n)$ and large j. Hence we obtain from (2.11)

$$|K_{k,j}(x)| \leq C\int_0^\infty |I(\lambda; x_n^{-1}x')\Psi\left(\frac{\lambda}{2^j|x_n|}\right)\lambda^{n-1-k}|d\lambda$$

$$= C2^{j(n-k)}\int_0^\infty |I(2^j\lambda; x_n^{-1}x')\Psi(\frac{\lambda}{|x_n|})\lambda^{n-1-k}|d\lambda$$

$$\leq C2^{j(n-\alpha-k)},$$

which implies that $K_{k,j}(x)_{j=0}^\infty$ is bounded in $L^\infty(R^n)$ if $k = n - \alpha$'

The boundedness of M_k is trivial if we use the argument of Besov spaces. In fact, we can easily prove the $B^0_{p,2} - B^0_{p',2}$-boundedness if $k = (2n-2\alpha)(\frac{1}{p} - \frac{1}{2})$ by analytic interpolation of the cases $p = 2$ (Plancherel's theorem) and $p = 1$ (Young's inequality), which yields the $L^p - L^{p'}$-boundedness for $p \neq 1$ because of the continuous inclusions $L^p \subset B^0_{p,2}$ and $L^p \supset B^0_{p,2}$.As for $p = 1$, use the inclusion $L^1 \subset B^{-\varepsilon}_{1,2}$ and $B^\varepsilon_{\infty,2}\varepsilon > 0$ instead, whiich are easily obtained from the definition of Besov spaces.

2.2. **Classification.** Hereafter we shall assume $n = 3$ and $\gamma_0(\Sigma) = 2$. A typical example is Σ defined by

$$\varphi(\xi) = \{(\xi_1^2 + \xi_2^2 - \xi_3^2)^2 + \xi_3^4\}^{\frac{1}{4}}, \quad \xi = (\xi_1, \xi_2, \xi_3) \tag{2.12}$$

As is explained in Section 1,Σ can be expressed microlocally by a positive function $h(y) = h(y_1, y_2)$ which is real analytic at the origin.(see (2.4)) We remark that

$$rank h''(0,0) \neq 0$$

which is derived from the assumption $\gamma_0(\Sigma) = 2$. In particular, det $h''(0,0) \neq 0$ if the Gaussian curvature of Σ never vanishes.

In order to show how the boundedness of M_k is dependent on Σ, we shall classify functions $h(y)$. We assume that

$$\bigtriangledown h(0,0) = 0, h''_{11}(0,0) \neq 0 \tag{2.13}$$

otherwise replace $h(y)$ by $h(yT) - yT \bigtriangledown h(0,0)$ with an appropriate orthogonal matrix T. We remark that this change does not affect the boundedness of M_k. Then we define the function $b_0(y_2)$ and $b_1(y_2)$, which are real analytic at the origin, by the equations

$$h'_1(b_1(y_2), y_2) = 0, \qquad b_1(0) = 0 \qquad b_0(y_2) = h(b_1(y_2), y_2). \tag{2.14}$$

They are uniquely determined near the origin by the implicit function theorem. The curve $(b_1(t), t, b_0(t))$ is the ridge of the mountain Σ when we see it parallel to the y_1-axis.

Definition 1. Let $h(y) = h(y_1, y_2)$ be a real analytic function at the origin satisfying (2.13), and $b_j(y_2)$ be defined by (2.14) (j-0,1). Then we define δ_j to be the smallest integer $m \geq 2$ such that $b_j^{(m)}(0) \neq 0$, and we say that $h(y)$ is of type I if $\delta_0 < \infty$, type II if $\delta_0 = \infty$, $\delta_1 < \infty$, and type III if $\delta_0 = \delta_1 = \infty$.

Typical examples are the following:

Example 2.1. $h(y)$ is of type I with $\delta_0 = 2$ if and only if $deth''(0,0) \neq 0$.

Proof. Differentiating Eqs.(2.14), we have

$$h''_{11}(b_1(y_2), y_2)b'_1(y_2) + h''_{12}(b_1(y_2), y_2) = 0.$$

$$b''_0(y_2) = h''_{21}(b_1(y_2), y_2)b'_1(y_2) + h''_{22}(b_1(y_2), y_2)$$

From them, we obtain $b_0''(0) = \frac{deth''(0,0)}{h_{11}''(0,0)}$, which yields the required result.

Example 2.2. Let $N = 2, 3, ...$

$[I]$ $h_I(y) = 1 - (y_1^2 - y_2^N)$ is of type I with $\delta_0 = N$ ($b_0 = 1 + y_2^N, b_1 = 0$).

$[II]$ $h_{II}(y) = 1 - (y_1 - y_2^N)^2$ is of type II with $\delta_1 = N$ ($b_0 = 1, b_1 = y_2^N$).

$[III]$ $h_{III}(y) = 1 - y_1^2$ is of type III ($b_0 = 1, b_1 = 0$). Actually, $h_{III}(y) = 1 - y_1^2$ can not be a solution if $\{\varphi = 1\}$ $\varphi \in C^\omega(R^3 \setminus \{0\})$.

Example 2.3. Σ defined by $\varphi(\xi)$ in (2.12) can be expressed microlocally by a function of type II (or an appropriate replacement of it) at the points $(\omega_1, \omega_2, \pm 1)$, where $\omega_1^2 + \omega_2^2 = 1$, or type I at any other points.

The ridge of a mountain is dependent on the direction in which we see the mountain. But the following proposition says that the direction does not affect the classification.

Proposition 2.3. *Suppose that $\widetilde{h}(y) = h(yT)$ satisfies $\widetilde{h}_{11}''(0,0) \neq 0$ with a 2×2 invertible matrix T. Then $\widetilde{h}$ defines the same δ_0 as h does in Definition 1, and the same δ_1 if $\delta_0 = \infty$.*

Proof. In the following, all functions, which are of one variable or two, are real analytic at the origin. We define the indices $\delta_j (j = 0, 1)$ by using $h(y)$ as in Definition 1, and the functions $\widetilde{b_j}(y_2)(j = 0, 1)$ by

$$\widetilde{h}_1'(\widetilde{b}_1(y_2), y_2) = 0, \qquad \widetilde{b}_1(0) = 0 \qquad \widetilde{b}_0(y_2) = \widetilde{h}(\widetilde{b}_1(y_2), y_2). \tag{2.15}$$

Then all we have to show is

$$\widetilde{b}_j^{(l)}(0) = 0, \qquad 2 \leq l \leq \delta_j - 1 \tag{2.16}$$

for $j = 0$, and for $j = 1$ if $\delta_0 = \infty$. We may assume δ_0, $\delta_1 \leq 3$ or equivalently $b_0'' = b_1'' = 0$.

Because of (2.13) and (2.14), $\eta(y_1, y_2) = h(y_1 + b_1(y_2), y_2)$ satisfies $\eta(0, y_2) = b_0(y_2), \eta_1'(0, y_2) = 0$ and $\eta_{11}''(0,0) = h_{11}''(0,0) \neq 0$. Hence we can write

$$h(y_1, y_2) = b_0(y_2) + (y_1 - b_1(y_2))^2 c(y_1, y_2) \tag{2.17}$$

with a function $c(y_1, y_2)$ such that $c(0,0) \neq 0$. From (2.17), we obtain

$$h''(0,0) = 2c(0,0) \begin{pmatrix} 1 & -b_1'(0) \\ -b_1'(0) & b_1'(0) \end{pmatrix} \tag{2.18}$$

Here we have used $b_0''(0) = b_1(0) = 0$. When we write

$$T = \begin{pmatrix} \alpha & \beta \\ \hat{\alpha} & \hat{\beta} \end{pmatrix}$$

we obtain, from (2.18) and the relation $\tilde{h}''(0,0) = Th''(0,0)^tT$,

$$\tilde{h}_{11}''(0,0) = 2c(0,0)(\alpha - \beta b_1'(0))^2.$$

Accordingly, the assumption $\tilde{h}_{11}'(0,0) \neq 0$ is equivalent to

$$\alpha - \beta b_1'(0) \neq 0 \tag{2.19}$$

On the other hand, by (2.18) and (2.19), $\zeta(y_1, y_2) = \alpha h_1'(y_1, y_2) + \beta h_2'(y_1, y_2)(= \tilde{h}_1'(yT^{-1}))$ satisfies $\zeta(0,0) = 0$ and $\zeta_1'(0,0) = 2c(0,0)(\alpha - \beta b_1'(0)) \neq 0$. Hence we can uniquely define the function $b(y_2)$ by

$$\zeta(b(y_2), y_2) = 0, \qquad b(0) = 0 \tag{2.20}$$

which means

$$\tilde{h}_1'(g_1(y_2), g_2(y_2), \qquad (g_1(y_2), g_2(y_2)) = (b(y_2), y_2)T^{-1}. \tag{2.21}$$

Furthermore we obtain,from(2.17),

$$\begin{aligned} \zeta(y_1, y_2) &= \beta b_0'(y_2) + 2(y_1 - b_1(y_2))c(y_1, y_2)(\alpha - \beta b_1'(y_2)) \\ &\quad + (y_1 - b_1(y_2))^2(\alpha c_1'(y_1, y_2) + \beta c_2'(y_1, y_2)) \\ &= \beta b_0'(y_2) + (y_1 - b_1(y_2))d(y_1, y_2), \end{aligned}$$

where $d(y_1, y_2) = 2c(y_1, y_2)(\alpha - \beta b_1'(y_2)) + (y_1 - b_1(y_2))(\alpha c_1'(y_1, y_2) + \beta c_2'(y_1, y_2))$. Since we have $d(0,0) = 2c(0,0)(\alpha - \beta b_1'(0)) \neq 0$ by (2.19), Eq.(2.20) implies

$$b(y_2) = b_1(y_2) + r_1(y_2) \tag{2.22}$$

Here $r_1(y_2) = -\beta d(b(y_2), y_2)^{-1} b_0'(y_2) = O(|y_2|^{\delta_0 - 1})$ because of

$$b_0'(0) = 0 \tag{2.23}$$

which is obtained from (2.13) and (2.14).

Now, we claim that

$$\widetilde{b_1}(y_2) = g_1 \circ g_2^{-1}(y_2). \tag{2.24}$$

It can be verified since we obtain $g_2'(0) = (\alpha - \beta b_1'(0))|T|^{-1} \neq 0$ from (2.21), (2.22) and (2.19). We have used here $\delta_0 \geq 3$. On the other hand, since we obtain $\widetilde{b_0}(g_2(y_2)) = h(b(y_2), y_2)$ from (2.15), (2.24), and (2.21), we have, by (2.17) and (2.22),

$$\widetilde{b_0}(y_2) = (b_0 + r_0) \circ g_2^{-1}(y_2). \tag{2.25}$$

Here $r_0(y_2) = (r_1(y_2))^2 c(b(y_2), y_2) = O(|y_2|^{2(\delta_0 - 1)})$. If we remark (2.23), $\delta_0 < 2(\delta_0 - 1)$, and $g_2^{-1}(0) = 0$, we obtain (2.16) for $j = 0$ from (2.25).

In particular, in the case $\delta_0 = \infty$, we have $b(y_2) = b_1(y_2)$ by (2.22). Then we obtain, from (2.21), $g_1^{(l)}(0) = g_2^{(l)}(0) = 0$; hence $(g_2^{-1})^{(l)}(0) = 0$ for $2 \leq l \leq \delta_1 - 1$. By using these facts and (2.24), we have (2.16) for $j = 1$.

We shall state our main theorem. Suppose that $n = 3$ and M_k in (2.1) is microlocalized; that is, $a_k(\xi)$ is supported in a sufficiently small conic neighborhood of $\upsilon = (0, 0, 1)$, and associated with $h(y) = h(y_1, y_2)$ by (2.4) which satisfies (2.13). In the following, we define functions $b_0(y_2), b_1(y_2)$ by (2.14) and indices δ_0, δ_1 by Definition 1.

Theorem 2.3. [26]. *Suppose $h(y)$ is of type * ($* = I, II, III$). Then M_k is $L^p - L^{P'}$ -bounded if $k > k_*(p)$. This inequality can be replaced by an equality if $p \neq 1$ and $* \neq II$. Here*

$$k_*(p) = \begin{cases} (5 - \frac{2}{\delta_0})(\frac{1}{p} - \frac{1}{2}) & if \quad * = I, \\ max\{6(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, (5 - \frac{1}{2\delta_1 - 1})(\frac{1}{p} - \frac{1}{2})\} & if \quad * = II, \\ 5(\frac{1}{p} - \frac{1}{2}) & if \quad * = III, \end{cases}$$

Remark 1. We easily see that Theorem 2.3 is an improvement of Theorem 2.2 in the case $n = 3$ and $\gamma_0(\Sigma) = 2$, which is equivalent to Theorem 2.3 with $* = III$. In tact, since Σ defined by (2.2) is a compact analytic hyper surface, it cannot contain any lines. Hence Σ is expressed microlocally by a function of type either I or II, which implies better results than that if type III.

Remark. Theorem 2.3 with $* = I, \delta_0 = 2$, that is, in the case $deth''(0, 0) \neq 0$ (Example 1), corresponds to the result of Brenner[8] which treats Σ with nonvanishing Gaussian curvature.

Proof. We define the oscillatory integral

$$I(\lambda; z) = \int_U e^{i\lambda(z\cdot y + h(y))} g(y) dy \tag{2.26}$$

to be the same one as (2.5) with $n = 3$ and write $y = (y_1, y_2), z = (z_1, z_2)$. We remark that $U \subset R^2$ is a sufficiently small neighborhood of the origin. The following lemma, which is called the scaling principle for oscillatory integrals, is a fundamental tool here[21]:

Lemma 2.4. *Let $\Psi(t) \in C^\infty(R)$ be real-valued and let $\chi(t) \in C_0^\infty(R)$. Suppose $|\Psi^{(\nu)}(t)| \geq d$ on suppχ for some $\nu \geq 2$ and $d > 0$. Then, for $\lambda > 0$,*

$$\left| \int e^{i\lambda\Psi(t)} \chi(t) dt \right| \leq C_{\nu,d}(\|\chi\|_{L^\infty} + \|\chi'\|_{L^1}) \lambda^{\frac{-1}{\nu}},$$

where the constant $C_{\nu,d} > 0$ depends only on ν and d.

For the proof of this lemma, consult Stein. From Lemma 2.4 and (2.13), we obtain

$$\left| \int e^{i\lambda(zy_h(y))} g(y) dy_1 \right| \leq C_g \lambda^{-\frac{1}{2}}$$

for all y_2 and z. Hence we have

$$\|I(\lambda; z)\|_{L^\infty(R_z^2)} \leq C_g \lambda^{-\frac{1}{2}}, \tag{2.27}$$

which implies, by Proposition 2, the boundedness of M_k for $k > k_{III}(p)$ ($k = k_{III}(p)$ if $p \neq 1$).

In particular, in the case $* = I$, we can improve (2.27) as

$$||I(\lambda; z)||_{L^\infty(R_z^2)} \leq C_g \lambda^{-(\frac{1}{2} + \frac{1}{\delta_0})}, \tag{2.28}$$

which implies the desired boundedness by Proposition 2 again. In order to prove (2.28), we shall define a real analytic function $f(y_2, z_1)$ at the origin by

$$z_1 + h_1'(f(y_2, z_2), y_2) = 0 \qquad f(0,0) = 0. \tag{2.29}$$

It is uniquely determined (2.13). We remark that we have

$$b_0(y_2) = h(f(y_2, 0), y_2) \qquad b_1(y_2) = f(y_2, 0) \tag{2.30}$$

by (2.14). then we can rewrite (2.26) as

$$(2.31) \qquad \begin{aligned} I(\lambda; z) &= \int \exp^{i\lambda A(y_2,z_1,z_2)} J(\lambda; y_2, z_1) dy_2; \\ J(\lambda; y_2, z_1) &= \int e^{i\lambda B(y_1,y_2,z_1)} g(y_1 + f(y_2, z_1), y_2) dy_1 \end{aligned}$$

by the change of variable $y_1 \longmapsto y_1 + f(y_2, z_1)$, where

$$A(y_2, z_1, z_2) = z_2 y_2 + G(y_2, z_1),$$

$$G(y_2, z_1) = z_1 f(y_2, z_1) + h(f(y_2, z_1)y_2),$$

$$B(y_1, y_2, z_1) = z_1 y_1 + h(y_1 + f(y_2, z_1), y_2) - h(f(y_2, z_1), y_2).$$

We remark that, we have

$$g(y_2, 0) = b_0(y_2), \quad (\partial G/\partial z_1)(y_2, 0) = b_1(y_2),$$

$$b(0, y_2, z_1) = (\partial B/\partial y_1)(0, y_2, z_1) = 0,$$

$$(\partial^2 B/\partial y_1^2)(0, 0, 0) = h''_{11}(0, 0) \neq 0$$

by (2.29), (2.30), and (2.13) since

$$(2.32) \qquad \partial G/\partial z_1(y_2, z_1) = f(y_2, z_1).$$

Hence we can write

$$(2.33) \qquad A(y_2, z_1, z_2) = z_2 y_2 + b_0(y_2) + b_1(y_2) z_1 + b(y_2, z_1) z_1^2$$

$$(2.34) \qquad B(y_1, y_2, z_1) = c(y_1, y_2, z_1) y_1^2$$

With real analytic functions at the origin

$$(2.35) \qquad b(y_2, z_1) = \sum_{\nu=2}^{\infty} (1/\nu!) b_\nu(y_2) z_1^{\nu-2}, \quad b_\nu(y_2) = (\partial^\nu G/\partial z_1^\nu)(y_2, 0),$$

and $c(y_1, y_2, z_1)$ such that $c(0, 0, 0) \neq 0$.

In order to estimate $J(\lambda; y_2, z_1)$, we shall use the following Van der Corput lemma.

Lemma 2.5. *Let $\Psi(t) \in C^\infty(R)$ be a real-valued function such that $\Psi(0) = \Psi'(0) = 0$, and let $\chi(t) \in C^\infty(R)$. Suppose $|\Psi''(t)| \geq d$ on $[-c, c]$ for some $c, d > 0$. Then, for $\lambda > 0$ and $l = 0, 1, 2, ...$,*

$$\left| \int_{-c}^{c} e^{i\lambda\Psi(t)} t^l \chi(t) dt \right| \leq C_{l,d} (\sum_{j=0}^{l+2} \|\chi^{(j)}\|_{L^\infty}) \lambda^{-1/2-l/2}$$

where $C_{l,d} > 0$ depends only on l and d.

For the proof of this lemma, consult, for example, Stein[21]. From (2.34) and Lemma 2.5, we obtain, for sufficiently small z_1,

$$|J(\lambda; y_2, z_1)|, \left| \frac{\partial J}{\partial y_2}(\lambda; y_2, z_1) \right| \leq C_g \lambda^{-1/2}. \tag{2.36}$$

Since $(\partial^{\delta_0} A / \partial y_2^{\delta_0})(0,0,0) = b_0^{(\delta_0)}(0) \neq 0$ by (2.33), we obtain (2.28) from (2.31), (2.36) and Lemma 2.4. Here we have used an integration by parts argument and (2.13) for large z.

In the case $* = II$, we can have another type of estimate; that is, we have

$$\|I(\lambda; z)\|_{L^{2\delta_1}(R_z^2)} \leq C_g \lambda^{-(1/2+1/\delta_1)+\epsilon} \tag{2.37}$$

for any small $\epsilon > 0$, which implies, by Proposition 1, the boundedness of M_k with $p = 4\delta_1/(4\delta_1 - 1)$ for $k > 5/2 - 3/(2\delta_1) = 6(1/p - 1/2) - 1/2 = (5 - 1/(2\delta_1 - 1))(1/p - 1/2)$. On the other hand, we have already had the boundedness with $p = 1, 2$ because $k_{II}(1) = k_{III}(1), k_{II}(2) = k_{III}(2)$. The boundedness with every other p is given by an interpolation.

We shall prove (2.37). By an integration by parts argument again, we only have to pay attention to z near the origin. By the change of variable $z_2 \mapsto z_1 z_2$, we obtain, from (2.31) and (2.33),

$$\|I(\lambda; z)\|_{L^q(|z| \ll 1)} \leq \left[\int_{|z_1| \ll 1} |z_1| \left(\int_{-\infty}^{\infty} |\widetilde{I}(\lambda; z)|^q dz_2 \right) dz_1 \right]^{1/q}, \tag{2.38}$$

where

$$\begin{aligned} \widetilde{I}(\lambda; z) &= \textstyle\int e^{i(\lambda z_1)\widetilde{A}(y_2, z_1, z_2)} J(\lambda; y_2, z_1) dy_2; \\ \widetilde{A}(y_2, z_1, z_2) &= z_2 y_2 + b_1(y_2) + b(y_2, z_1) z_1. \end{aligned} \tag{2.39}$$

Here we have used the fact that $b_0(y_2)$ is a constant because of $* = II$ and (2.23). We remark that z_2 might be large again while z_1 is still small.

For large z_2, we have, by an integration by part argument and (2.36),

$$|\widetilde{I}(\lambda; z)| \le C_g \lambda^{-1/2}(\lambda|z_1 z_2|)^{-1}.$$

Here we have assumed $b_1'(0) = 0$ (otherwise change variable z_2 to $z_2 - b_1'(0)$). When we interpolate it with a trivial estimate

$$|\widetilde{I}(\lambda; z) \le C_g \int |J(\lambda; y_2, z_1)| dy_2 \le C_g \lambda^{-1/2}, \tag{2.40}$$

we have, for $q \ge 2$

$$|\tilde{I}(\lambda; z)| \le C_g \lambda^{-1/2}(\lambda|z_1 z_2|)^{-2/q+\epsilon}. \tag{2.41}$$

For small z_2, we obtain from Lemma 2.4,

$$|\tilde{I}(\lambda; z)| \le C_g \lambda^{-1/2}(\lambda|z_1|)^{-1/\delta_1}$$

since $(\partial^{\delta_1}\tilde{A}/\partial y_2^{\delta_1})(0,0,0) = b_1^{(\delta_1)}(0) \ne 0$. By interpolation it with (2.40), we have

$$|\tilde{I}(\lambda; z)| \le C_g \lambda^{-1/2}(\lambda|z_1|)^{-1/\delta_1+\epsilon}. \tag{2.42}$$

Combining (2.41) with $q = 2\delta_1$ and (2.42), we have

$$|\tilde{I}(\lambda; z)| \le C_g \lambda^{-(1/2+1/\delta_1)+\epsilon}|z_1|^{-1/\delta_1+\epsilon}(1 + |z_2|)^{-1/\delta_1+\epsilon},$$

which implies (2.37) by (2.38).

In the next section, the optimality of Theorem 2.3 with $* = I, III$ will be shown (Theorem 3). Theorem 2.3 with $* = II$, however, can be improved in a favorable case.

Theorem 2.4. *Suppose that $h(y)$ is of type II with $\delta_1 \ge 3$ and satisfies*

$$\frac{\partial^\mu}{\partial y_2^\mu}\left\{\frac{\partial^\nu h}{\partial y_1^\nu}(b_1(y_2), y_2)\right\}_{|y_2=0} = 0 \tag{2.43}$$

for $\mu = 1, 2, ..., \delta_1 - 1$ and $\nu = 2, 3, ...$ Then M_k is $L^p - L^{p'}$-bounded if $k > \bar{k}_{II}(p) = \max 6(1/p - 1/2) - 1/2, (5 - 1/\delta_1)(1/p - 1/2)$.

Remark. Theorem 2.4 is optimal (Theorem 3). It is still an open problem whether we can remove the assumptions $\delta_1 \geq 3$ and (2.43) or not.

Proof. We shall prove

$$\|I(\lambda; z)\|_{L^{\delta_1+1}(R_z^2)} \leq C_{g,\epsilon}\lambda^{-(1/2+2/(\delta_1+1))+\epsilon} \tag{2.44}$$

for any small $\epsilon > 0$, which implies, by Proposition 1, the boundedness of M_k with $p = 2(\delta_1+1)/(2\delta_1+1)$ for $k > 5/2 - 3/(\delta_1+1) = 6(1/p - 1/2) - 1/2 = (5 - 1/\delta_1)(1/p - 1/2)$. The boundedness with every other p is given by an interpolation.

The proof for (2.44) is carried out by modifying that for (2.37). First we remark that we have

$$\partial^2 G/\partial z_1^2(y_2, z_1) = \partial f/\partial z_1(y_2, z_1) = -1/h_{11}''(f(y_2, z_1), y_2)$$

$$\partial^3 G/\partial z_1^3(y_2, z_1) = \partial^3 h/\partial y_1^3(f(y_2, z_1), y_2)\partial^2 G/\partial z_1^2(y_2, z_1)^3$$

by (2.29) and (2.32). Then we have inductively, by (2.30) and (2.35),

$$b_2(y_2) = -1/h_{11}''(b_1(y_2), y_2),$$

$$b_\nu(y_2) = \sum_{l=3}^{\nu} \frac{\partial^l h}{\partial y_1^l}(b_1(y_2), y_2) g_{l,\nu}(y_2) \quad (\nu \geq 3).$$

Here $g_{l,\nu}(y_2)$ is a leaner combination of $\prod_{m=2}^{\nu-1} b_m(y_2)^{\alpha_m}$ when $\alpha_m{}_{m=2}^{\nu-1}$ entry $\alpha_2 + \alpha_3 + ... + \alpha_{\nu-1} \leq l$. Hence the condition (2.43) implies

$$b_\nu(y_2) = b_\nu(0) + O(y_2^{\delta_1}).$$

Then we may rewrite $\tilde{A}$ in (2.39) as

$$\tilde{A}(y_2, z_1, z_2) = z_2 y_2 + d(y_2, z_1) y_2^{\delta_1}$$

with a real analytic function $d(y_2, z_1)$ at the origin such that $d(0,0) \neq 0$. Here we have used (2.35) and the change of variable $z_2 \mapsto z_2 - b_1'(0)$.

For large z_2, we have (2.41). For small z_2, we change the variable y_2 to $z_2^{1/(\delta_1-1)y_2}$. Then we obtain, from (2.39),

$$
\begin{aligned}
\tilde{I}(\lambda;z) &= z_2^{1/(\delta_1-1)}\tilde{\tilde{I}}(\lambda;z); \\
\tilde{\tilde{I}}(\lambda;z) &= \textstyle\int e^{i(\lambda z_1 z_2^{\delta_1/(\delta_1-1)})\tilde{\tilde{A}}(y_2,z_1,z_2)} J(\lambda; z_2^{1/(\delta_1-1)}y_2, z_1)dy_2, \\
\tilde{\tilde{A}}(y_2,z_1,z_2) &= y_2 + d(z_2^{1/(\delta_1-1)}y_2, z_1)y_2^{\delta_1}.
\end{aligned}
\tag{2.45}
$$

We remark that y_2 might be large again while $z_2^{1/(\delta_1-1)}y_2$ is still small. We split $\tilde{\tilde{I}}(\lambda;z)$ into two parts, that is, $\tilde{\tilde{I}}_1(\lambda;z)$ for small y_2 and $\tilde{\tilde{I}}_2(\lambda;z)$ for large y_2. The estimate for $\tilde{\tilde{I}}_1(\lambda;z)$ is given by an integration by parts argument. In fact, since $(\partial\tilde{\tilde{A}}/\partial y_2)(0,0,0) = 1 \neq 0$, we have

$$|\tilde{\tilde{I}}(\lambda;z)| \leq C_g\lambda^{-1/2}(\lambda|z_1||z_2|^{\delta_1/(\delta_1-1)})^{-1}$$

by (2.36). If we interpolate it with a trivial estimate

$$|\tilde{\tilde{I}}_1(\lambda;z)| \leq C_g\lambda^{-1/2}$$

we have

$$|\tilde{\tilde{I}}_1(\lambda;z)| \leq C_g\lambda^{-1/2}(\lambda|z_1||z_2|^{\delta_1/(\delta_1-1)})^{-1/2} \tag{2.46}$$

The estimate for $\tilde{\tilde{I}}_1(\lambda;z)$ is obtained from Lemma 2.4. In fact, since $(\partial^2\tilde{\tilde{A}}/\partial y_2^2)$ is away from 0 for large y_2 , we have

$$|\tilde{\tilde{I}}_2(\lambda;z)| \leq C_g\lambda^{-1/2}(\lambda|z_1||z_2|^{\delta_1/(\delta_1-1)})^{-1/2} \tag{2.47}$$

by (2.36) again. Combining (2.45), (2.46), and (2.47), we have

$$|\tilde{I}(\lambda;z)| \leq C_g\lambda^{-1/2}(\lambda|z_1||z_2|^{(\delta_1-2)/(\delta_1-1)})^{-1/2}$$

Interpolating it with (2.42), we have

$$|\tilde{I}(\lambda;z)| \leq C_g\lambda^{-1/2}(\lambda|z_1|)^{-2/(\delta_1+1)+\epsilon}|z_2|^{-1/(\delta_1+1)+\epsilon} \tag{2.48}$$

for any small $\epsilon > 0$, where we have used $\delta_1 \geq 3$

Combining (2.41) with $q = \delta_1 + 1$ and (2.48), we have

$$|\tilde{I}(\lambda;z)| \leq C_g\lambda^{-(1/2+2/(\delta_1+1))+\epsilon}|z_1|^{-2/(\delta_1+1)+\epsilon}|z_2|^{-1/(\delta_1+1)+\epsilon}(1+|z_2|)^{-1/(\delta_1+1)},$$

which implies (2.44) by (2.38).

2.3. **Optimality.** We shall show the optimality of the results obtained in Section 2. The following theorem says that Theorem 2.3 with $* = I, III$ and Theorem 2.4 are optimal. Recall that, in (2.1), $a_k(\xi) = a_k(\xi_1, \xi_2, \xi_3)$ is supported in a small conic neighborhood of $v = (0,0,1)$ and $\varphi(\xi) = \varphi(\xi_1, \xi_2, \xi_3)$ is associated with a function $h(y) = h(y_1, y_2)$ microlocally by the relation (2.4).

Theorem 2.5. *Suppose that $a_k(\xi) = |\xi|^{-k}$ for large $|\xi|$ in a conic neighborhood of $v = (0,0,1)$ and $\varphi(\xi)$ is associated with $h_*(y)$ ($* = I, II, III$) defined in Example 2. Then M_k is not $L^p - L^{p'}$-bounded if $k < \overline{k}_*(p)$. Here*

$$k_*(p) = \begin{cases} (5-\frac{2}{N})(\frac{1}{p}-\frac{1}{2}) & if * = I, \\ max\{6(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}, (5-\frac{1}{N})(\frac{1}{p}-\frac{1}{2})\} & if * = II, \\ 5(\frac{1}{p}-\frac{1}{2}) & if * = III, \end{cases}$$

Proof. We shall prove $k \geq \overline{k_*}(p)$ when M_k is $L^p - L^{p'}$-bounded. In the following, we use the change of variables (2.7), that is, $\xi = (\lambda y, \lambda h_*(y))$ without any notation. The Jacobian is

$$\frac{D\xi}{D(\lambda, y)} = \lambda^2 G(y); \quad G(y) = h_*(y) - y \cdot \nabla h_*(y)$$

by (2.8). We also use positive functions $f, g, \Psi \in C_0^\infty(R)$ such that $f(0) = g(0) = \Psi(1) = 1$. We choose their supports to be sufficiently small.

[I] We shall prove the theorem with $* = I$. We set

$$u_j(x) = (2\pi)^3 2^{j(5/2-1/N)(1/p-1)(1/p-1)} F^{-1}[v_j(2^{-j}\xi)](x),$$

where

$$v_j(\xi) = \frac{f(2^{j/2}\xi_1/\varphi(\xi))g(2^{j/N}\xi_2/\varphi(\xi))\Psi(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 |G(\xi_1/\varphi(\xi), \xi_2/\varphi(\xi))|}.$$

Since $F^{-1}[v_j(2^{-j/2}\xi_1, 2^{-j/N}\xi_2, \xi_3)](x)_{j=0}^\infty$ is bounded in L^p, $u_{j}{}_{j=0}^\infty$ is also bounded in L^p. On the other hand, we have, for large numbers j,

$$\|M_k u_j\|_{L^{p'}} = 2^{j(5-2/N)(1/p-1/2)-k} \|A_j(x)\|_{L^{p'}}, \tag{2.49}$$

where

$$A_j(x) = 2^{jk-(5/2-1/N)} M_k u_j \left(\frac{x_1}{2^{j/2}} \frac{x_2}{2^{j(1-1/N)}}, \frac{x_3}{2^j} - 1 \right)$$
$$(x = (x_1, x_2, x_3))$$

$$2^{j(k+1/N-5/2)}\int e^{i(2^{-j/2}x_1\xi_1+2^{-j(1-1/N)}x_2\xi_2+2^{-j}x_3\xi_3-\xi_3+\varphi(\xi))}|\xi|^{-k}v_j(\xi/2^j)d\xi$$
$$2^{j(1/2+1/N)}\int e^{i(2^{j/2}x_1\xi_1+2^{j/N}x_2\xi_2+x_3\xi_3-2/\xi_3+2^j\varphi(\xi))}|\xi|^{-k}v_j(\xi)d\xi$$
$$2^{j(1/2+1/N)}\int\int e^{i\lambda(2^{j/2}x_1\xi_1+2^{j/N}x_2\xi_2+x_3+(2^j-x_3)(y_1^2-y_2^N))}f(2^{j/2}y_1)g(2^{j/N}y_2)\Psi(\lambda)d\lambda dy$$
$$\int\int e^{i\lambda(x_1\xi_1+x_2\xi_2+x_3+(1-2^jx_3)(y_1^2-y_2^N))}f(y_1)g(y_2)\Psi(\lambda)d\lambda dy.$$

Furthermore we have

$$\liminf_{j\to\infty}\|A_j(x)\|_{L^{p'}}\geq\|\int\int\cos(\lambda(x_1y_1+x_2y_2+x_3+y_1^2-y_2^N))$$
$$\times f(y_1)g(y_2)\Psi(\lambda)d\lambda dy\|_{L^{p'}(|x|\ll 1)}\neq 0,$$

Which implies the desired result by (2.49) since $M_ku_j{}_{j=0}^{\infty}$ is bounded in $L^{p'}$.

$[III]$ The proof for $*=III$ is the same as that for $*=I$ when we set $N=\infty$.

$[II]$ We shall prove the theorem with $*=II$. If we replace N by $2N$ in the proof for $*=I$, we have easily $k\geq(5-1/N)(1/p-1/2)$. Hence we shall prove $k\geq 6(1/p-1/2)-1/2$. We set

$$u_j(x)=(2\pi)^3 2^{j(3/p-3)}F^{-1}[v(2^{-j}\xi)](x),$$

where

$$v(\xi)=\frac{f(\xi_1/\varphi(\xi)-(\xi_2/\varphi(\xi))^N)g(\xi_2/\varphi(\xi))\Psi(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2|G(\xi_1/\varphi(\xi),\xi_2/\varphi(\xi))|}.$$

Then we easily see that the set $u_j{}_{j=0}^{\infty}$ is bounded in the space L^p. On the other hand, we have, for large numbers j,

$$\|M_ku_j\|_{L^{p'}}=2^{j6(1/p-1/2)-1/2-k}\|B_j(x)\|_{L^{p'}},\tag{2.50}$$

where

$$B_j(x)=2^{j(k+1/2-3/p)}M_ku_j\left(\frac{x_1}{2^j},\frac{x_2}{2^j},\frac{x_3}{2^j}-1\right)\quad(x=(x_1,x_2,x_3))$$
$$=2^{j(k-5/2)}\int e^{i(2^{-j}x\xi-\xi_3+\varphi(\xi))}|\xi|^{-k}v(\xi/2^j)d\xi$$
$$=2^{j/2}\int e^{i(x\xi-2^j\xi_3+2^j\varphi(\xi))}|\xi|^{-k}v(\xi)d\xi$$
$$=2^{j/2}\int\int e^{i\lambda x_1y_1+x_2y_2+x_3+(2^j-x_3)(y_1-y_2^N)^2}f(y_1-y_2^N)g(y_2)\Psi(\lambda)d\lambda dy$$

$$= \int C_j(\lambda, x) D(\lambda, x) \Psi(\lambda) d\lambda;$$

$$C_j(\lambda, x) = \int e^{i\lambda(2^{-j/2}x_1y_1 + (1-2^{-j}x_3)y_1^2)} f(2^{-j/2}y_1) dy_1,$$

$$D(\lambda, x) = \int e^{i\lambda(x_2y_2 + x_1y_2^N + x_3)} g(y_2) dy_2.$$

We remark that we have used here the change of variable $y_1 \mapsto 2^{-j/2}y_1 + y_2^N$. Noticing

$$C_j(\lambda, x) \to \int e^{i\lambda y_1^2} dy_1 = \sqrt{\frac{\pi i}{\lambda}} \quad (j \longrightarrow \infty)$$

we have

$$\liminf_{j\to\infty} \|B_j(x)\|_{L^{p'}} \geq \left\| \int \frac{\pi i}{\lambda} D(\lambda, x) \varphi(\lambda) d\lambda \right\|_{L^{p'}(|x|\ll 1)}$$

$$\left\| \int \int \cos(\lambda(x_2y_2 + x_1y_2^N + x_3)) g(y_2) \sqrt{\frac{\pi}{\lambda}} \Psi(\lambda) d\lambda dy_2 \right\|_{L^{p'}(|x|\ll 1)} \neq 0$$

which implies the desired result by (2.50) since $M_k u_j{}_{j=0}^{\infty}$ is bounded in $L^{p'}$.

2.4. **Hyperbolic equations.** We shall explain how the $L^p - L^{p'}$-boundedness of Fourier multipliers treated here is related to the analysis of hyperbolic equations. In the following, we always assume $1 \leq p < 2$, $1 < r < 2$, $1/p + 1/p' = 1/r + 1/r' = 1$, and use the notation $L^r(L^p) = L^r(R_t; L^p(R_x^n))$. The argument here is not restricted to the case $n = 3$. Let $M_k = F^{-1}e^{i\varphi(\xi)}a_k(\xi)F$ be the Fourier multiplier defined by (2.1). We assume here that M_k is $L^p - L^{p'}$-bounded with $k > k(p)$ ($k \geq k(p)$ if $p \neq 1$). As we have shown in the previous sections, the bounds $k(p)$ are determined by the geometry of the phase function $\varphi(\xi)$.

We set $\dot{M}_k = F^{-1}e^{i\varphi(\xi)}\dot{a}_k(\xi)F$ and $\dot{M}_k(t) = F^{-1}e^{it\varphi(\xi)}\dot{a}_k(\xi)F$, where $\dot{a}_k \in C^\infty(R^n \setminus \{0\})$ is homogeneous of order $-k$ and coincides with $a_k(\xi)$ for large $|\xi|$. We remark that $M_k - \dot{M}_k$ is $L^p - L^{p'}$-bounded with $k < 2n(1/p - 1/2)$. In fact, with $k < n$, we have $a_k(\xi) - \dot{a}_k(\xi) \in L^1$; hence the convolution kernel is in L^∞, which implies the $L^1 - L^\infty$-boundedness. By Plancherel's theorem, we have the L^2-boundedness with $k = 0$. The result is given by the interpolation. Hence we have the $L^p - L^{p'}$-boundedness of $\dot{M}_k$ and the estimate

$$\|\dot{M}_k(t)g\|_{L^{p'}} \leq Ct^{k-2n(1/p-1/2)}\|g\|_{L^p} \tag{2.51}$$

with $k(p) < k < 2n(1/p - 1/2)$ ($k(p) \leq k < 2n(1/p - 1/2)$ if $p \neq 1$) by the scaling argument $\dot{M}_k(t)g(x) = t^k M_k[g(t\cdot)](t^{-1}x)$. We have also

$$\|\dot{M}_k(t)g\|_{L^{r'}(L^{p'})} \leq C\|g\|_{L^2} \tag{2.52}$$

if $k(p) < 2k = 2n(1/p - 1/2) - 2(1 - 1/r)$. The equality $k(p) = 2k$ is allowed in the case $p \neq 1$. In fact, the estimate (2.52) is equivalent to

$$\|\dot{M}_k(t) * h\|_{L^2} \leq \|h\|_{L^r(L^p)},$$

and it suffices to show

$$\|\dot{M}_k(t)\dot{M}_k(t) * h\|_{L^{r'}(L^{p'})} \leq C\|h\|_{L^r(L^p)}.$$

On the other hand, we have

$$\begin{aligned} \dot{M}_k(t)\dot{M}_k(t) * h &= \int \dot{M}_k(t)\dot{M}_k(-\tau)h(\tau)d\tau \\ &= \int \dot{M}_{2k}(t-\tau)h(\tau)d\tau, \end{aligned}$$

where $\dot{M}_{2k}(t) = F^{-1}e^{it\varphi(\xi)}|\dot{a}_k(\xi)|^2$, and we obtain

$$\|\dot{M}_k(t)\dot{M}_k(t) * h\|_{L^{p'}} \leq Ct^{2k-2n(1/p-1/2)} * \|h(t)\|_{L^p}$$

from (2.51) with k replace by $2k$. By the Hardy-Littlewood-Sobolev inequality, which says that convolutions with $|t|^{-2(1-1/r)}$ are $L^r(R) - L^{r'}(R)$-bounded (see Stein [10, p. 354]), we have the desired estimate.

From the estimates (2.51) and (2.52), we can obtain various a priori estimates for hyperbolic equations. In fact, as is noted in the Introduction, the solution operator $E_k(t)$ to the Cauchy problem (C.P.) is a linear combination of Fourier multipliers of the type M_k (modulo a regularizing operator), more precisely, the type $\dot{M}_k(t)$ with phase functions defined by using the characteristic roots of the operator $P(D_t, D_x)$ in the problem (C.P.). We assume that these phase functions determine the same bounds $k(p)$ for M_k to be $L^p - L^{p'}$-bounded.

For example, we shall consider the problem (C.P.) with $k = m - 1$. We set $E(t) = E_{m-1}(t)$. Then the following energy estimate

$$\|E(t)g\|_{H^s} \leq C\|g\|_{H^{s-(m-1)}} \tag{2.53}$$

is well known. If we use the estimate (5.2), we have a space-time norm estimate, which is useful in the non-linear analysis.

Theorem 2.6. *Let $k(p) < 2(m-1+\alpha) = 2n(1/p-1/2) - 2(1-1/r)$. Then we have*

$$\|E(t)g\|_{L^{r'}(L^{p'})} \leq C\||D|^{\alpha}g\|_{L^2}. \tag{2.54}$$

The inequality $k(p) < 2(m-1+\alpha)$ can be replaced by an equality if $p \neq 1$.

The solution v to the Cauchy problem

$$P(D_t, D_x)v = f D_t^j v_{|t=0} = 0 \quad (j = 0, 1, ..., m-1), \tag{2.55}$$

which is an inhomogeneous version of (C.P.), is expressed as

$$v(t) = \int_0^t E(t-\tau)f(\tau)d\tau.$$

Since we obtain the time decay estimate

$$\|E(t)g\|_{L^{p'}} \leq Ct^{m-1-2n(1/p-1/2)}\|g\|_{L^p}$$

from the estimate (2.51) with $k = m-1$, we have

$$\|v(t)\|_{L^{p'}} \leq C|t|^{m-1-2n(1/p-1/2)} * \|f(t)\|_{L^p}.$$

By the Hardy-Littlewood-Sobolev inequality again, we have

Theorem 2.7. *Let $k(p) < m-1 = 2n(1/p-1/2) - 2(1-1/r)$. Then we have*

$$\|v\|_{L^{r'}(L^{p'})} \leq C\|f\|_{L^r(L^p)}. \tag{2.56}$$

The inequality $k(p) < m-1$ can be replaced by an equality if $p \neq 1$.

In the case of the wave equation, we can take $k(p) = (n+1)(1/p-1/2)$ and $m = 2$ (see Strichartz [23], Littman [17], Sugimoto [27]). Hence we obtain, from the estimate (2.54),

$$\|E(t)g\|_{L^q(L^\infty)} \leq C\||D|^{\alpha}g\|_{L^2} \tag{2.57}$$

for $q = 2/(n-2-2\alpha)$ and $\max\{\frac{n-3}{2}, \frac{n-3}{2}\} < \alpha < (n-2)/2$, and from the estimate (2.56),

$$\|v\|_{L^{p'}(R_t \times R_x^n)} \leq C\|f\|_{L^p(R_t \times R_x^n)} \tag{2.58}$$

for $p = 2(n+1)/(n+3)$ and $n \geq 2$, which was given by Strichartz [25]. We remark that Strichartz[25] proved the estimate

$$\|E(t)g\|_{L^q(R_t \times R_x^n)} \leq C \||D|^\alpha g\|_{L^2}$$

for $q = 2(n+1)/(n-2-2\alpha)$ and $-1/2 \leq \alpha < (n-2)/2$ (see also Stein [21]), which is a special case of Theorem 2.6. Harmse [11] found the estimate

$$\|v\|_{L^q(R_t \times R_x^n)} \leq C \|f\|_{L^p(R_t \times R_x^n)}$$

for $1/p - 1/q = 2/(n+1)$ and $\frac{(n+1)}{2n} - \frac{2}{(n+1)} < 1/q < \frac{(n-1)}{2n}$, which includes the estimate (2.58).

From these types of space-time norm estimates, we can obtain existence and uniqueness theorems for semi-linear equations. For example, we shall consider the equation $(\partial_{tt} - \triangle)v = H(v)$ (with zero Cauchy data). We assume that $H(v) \in L^p$ with $p = 2(n+1)/(n+3)$ for any $v \in L^{p'}$, say $H(v) = v^3$ and $n = 3$. Then the iteration scheme $(\partial_{tt} - \triangle)v_\nu = H(v_{\nu-1})$ works in the space $L^{p'}(R_t \times R_x^3)$ and the solution can be constructed there since we have $\|v_\nu\|_{L^{p'}} \leq C\|H(v_{\nu-1})\|_{L^p}$ by (2.58).(See Strichartz [25]).

We can also have regularity theorems as well. For example, we shall consider the equation $(\partial_{tt} - \triangle)u = u^2$,$u_{|t=0} = 0$, $\partial_t u_{|t=0} = g$ in the case $n = 3$. By applying a classical method to the iteration scheme

$$u_\nu(t) = E(t)g + \int_0^t E(t-\tau)u_{\nu-1}(\tau)^2 d\tau; \quad u_0(t) = E(t)g, \tag{2.59}$$

we can construct the solution in the space $C^0([0,T]; H^s) \bigcap C^1([0,T]; H^{s-1})$ for $g \in H^{s-1}$ if s is sufficiently large ($s > 3/2$) and T is sufficiently small. In fact, we obtain

$$\begin{aligned} &\|u_\nu\|_{C^0([0,T];H^s)} \\ &\leq C \left(\|g\|_{H^{s-1}} + \int_0^T \|u_{\nu-1}(\tau)^2\|_{H^{s-1}} d\tau \right) \\ &\leq C \left(\|g\|_{H^{s-1}} + \|u_{\nu-1}\|_{L^1([0,T];L^\infty)} \cdot \|u_{\nu-1}\|_{C^0([0,T];H^s)} \right) \qquad (2.60) \end{aligned}$$

from (2.53) with $m = 2$. Here we have used the inclusion $H^s \subset H^{s-1}$ and the estimate $\|u^2\|_{H^s} \le C\|u\|_{L^\infty} \cdot \|u\|_{H^s}$. On the other hand, we obtain

$$\|u_{\nu-1}\|_{L^1([0,T];L^\infty)} \le CT\|u_{\nu-1}\|_{C^0([0,T];H^s)} \tag{2.61}$$

from Sobolev's lemma, which means, together with the estimate (2.60), that the scheme (2.59) works in the space $C^0([0,T];H^s)$. Similarly it works in the space $C^1([0,T];H^{s-1})$ as well.

But the same result can be proved for smaller $s(1 < s < 3/2)$ if we the estimate

$$\begin{aligned} &\|u_\nu\|_{L^1([0,T];L^\infty)} \\ &\le T^{1-1/q}\left(\|E(t)g\|_{L^q(L^\infty)} + \int_0^T \|R(t)u_{\nu-1}(\tau)^2\|_{L^q(L^\infty)}d\tau\right) \\ &\le CT^{1-1/q}\left(\|g\|_{H^{s-1}} + \|u_{\nu-1}\|_{L^1([0,T];L^\infty)} \cdot \|u_{\nu-1}\|_{C^0([0,T];H^s)}\right) \end{aligned} \tag{2.62}$$

instead of the estimate (2.61). Here we have used the estimate (2.57) with $\alpha = s - 1$. The estimates (2.60) and (2.62) mean that the scheme (2.59) works in the space $C^0([0,T];H^s) \cap L^1([0,T];L^\infty)$, hence in $C^0([0,T];H^s)$, similarly in $C^1([0,T];H^{s-1})$.

We shall end this section by showing some related problems from physics. The linear elasticity for crystals in R^3 can be described in the form of a 3×3 system,

$$(D_t^2 - A(D_x))U = 0,$$

where

$$A(D_x) = (A_{ij}(D_x)); \quad A_{ij}(D_x) = \sum_{3}^{p,q=1} c_{ijpq}D_{x_p}D_{x_q}$$

is a 3×3 matrix. The constants $\{c_{ijpq}\}$ are not all independent, because we always assume the relations $c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij}$. We assume that this system is hyperbolic in the time direction as well. Solutions to the Cauchy problems for this equation are expressed by using $\dot{M}_k(t)(k = 0, 1)$ as well. The phase function $\varphi(\xi)$ is one of $\lambda_1(\xi)_{l=1,2,3}$, where $\lambda_1(\xi)^2{}_{l=1,2,3}$ are eigenvalues of the matrix $A(\xi)$, and the amplitude functions $a_k(\xi)$ consist of the projections into the eigenspaces. Although they might be singular where the eigenvalues are multiple and the extra argument will be needed there, we can expect that our observation can be applied (at least when Fourier transform of initial data vanishes at the multiple points). The geometry of the surfaces $\Sigma_l = \xi; \lambda_l(\xi) = 1$ is controlled by the constants $\{c_{ijpq}\}$, and some examples are illustrated in Stoth[22] and Racke [18], when the crystal under consideration exhibits some symmetries. It is interesting to classify these surfaces following the argument in section 2 and give the $L^p - L^{p'}$-estimates. We remark that Stoth [22] examines the $L^p - L^{p'}$-estimates in the case of hexagonal symmetry. Information about the multiple point is provided in Duff[9] and Liess [16]. A similar treatment of the Maxwell equations is also expected which can be described in the form of the 6×6 system

$$(D_t - A(D_x))U = 0,$$

where

$$\begin{pmatrix} \alpha & \beta \\ \hat{\alpha} & \hat{\beta} \end{pmatrix}.$$

3. Estimates for the convolution operators

In this section we consider the problem on $L^p \longrightarrow L^{p'}$ boundedness problem for the convolution operator. The convolution kernel of the operator is given by the following relation

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F \tag{3.1}$$

where F is the Fourier transform $\varphi \in C^\omega(\mathbb{R}^n \backslash \{0\})$ is a real analytic homogeneous function of order 1 $a_k \in C^\infty(\mathbb{R}^n)$ is homogeneous of order $-k$ for large $|\xi|$. We generalize the results proved in the paper [26](see chapter 2).

The estimates based on investigation of asymptotic behavior for the Fourier transform on measures supported on the hyper-surface

$$\Sigma = \{\varphi(\xi) = 1\}.$$

It should be noted that behavior of the Fourier transform of measures depends on geometric properties of hyper-surface Σ.

First, we use A.N.Varchenko [32] result on average decay of the Fourier transform and get estimate for the convolution operator in the case of arbitrary smooth hyper-surface Σ. Then we obtain estimates depending on the height of the hyper surface Σ introduced by A.N. Varchenko [31]. This results improve the estimates given by M.Sugimoto [26] for the case $n = 3$. Then we extend the estimates given by M.Sugimoto for arbitrary phase function. This gives a solution of the problem proposed by M.Sugimoto in the paper [26].

In Section 1 we reduce the problem to a sufficiently small neighborhood of the set $-\nabla\phi$. Further, in Section 2 we use the average decaying of the Fourier transform of the corresponding measure supported on hyper-surfaces and get estimate for the convolution operator for arbitrary Σ. Finally in Section 3 we give a solution of the M.Sugimoto problem for arbitrary function with A_∞ type singularity.

3.1. **Estimates for oscillatory integrals when the phase function has no critical points.** Consider the following oscillatory integral:

$$I(\lambda,\, z) := \int_{\underline{\mathbb{R}}^n} g(x) e^{i\lambda\Phi(x,\, z)} dx, \tag{3.2}$$

where $\Phi(x,\, z) := \phi(x) + z \circ x$, with $z \circ x := \sum_{j=1}^{n} z_j x_j$ the usual inner product of vectors z and x, g is a smooth function with compact support.

We assume that ϕ is a real analytic function with $\phi(0) = 0$, and $\nabla\phi(0) = 0$. First, we estimate the oscillatory integral $I(\lambda, z)$ when $|z| \geq 1$ assuming that g is supported in a sufficiently small neighborhood of the origin.

Lemma 3.1. *For any $q \geq 1$, $N \geq 1$ and $\lambda \geq 1$ the following estimate holds true:*

$$\int_{|z|\geq 1} |I(\lambda; z)|^q dz \leq C_N \lambda^{-N} \tag{3.3}$$

provided g is concentrated in a sufficiently small neighborhood of the origin, where C_N is a constant depending on N and C^N-norm of the function g.

Proof. We may assume that $|y| < \delta$ supposing $supp(g) \subset \{|y| < \delta\}$ and $|\nabla h(y)| < \frac{1}{2}$ for any y with $|y| < \delta$. Then

$$\nabla_y \Phi(y, z) = \left(\frac{\partial\phi(y)}{\partial y_1} + z_1, ..., \frac{\partial\phi(y)}{\partial y_n} + z_n \right),$$

so, for $|z| \geq 1$ we have

$$|\nabla_y \Phi(y, z)| \geq |z| - |\nabla h(y)| \geq \frac{|z|}{2}.$$

Consider the following vector field

$$v(y) = \sum_{j=1}^{n} \frac{\frac{\partial\Phi}{\partial y_j}}{i\lambda |\nabla\Phi|^2} \frac{\partial}{\partial y_j}. \tag{3.4}$$

Then $v(e^{i\lambda\Phi}) = e^{i\lambda\Phi}$. Therefore, by using integration by parts formula we get:

$$I(\lambda; z) = \int_{\mathbb{R}^n} e^{i\lambda\Phi} g(y) dy = \int_{\mathbb{R}^n} v(e^{i\lambda\Phi}) g(y) dy = \int_{\mathbb{R}^n} e^{i\lambda\Phi} v^* g(y) dy$$

where v^* is the adjoint operator to the operator generated by the vector field v given by

$$v^* g(y) = \frac{1}{i\lambda} \left[\sum_{j=1}^{n} \frac{\partial}{\partial y_j} \left(\frac{\frac{\partial\Phi}{\partial y_j} g}{|\nabla\Phi|^2} \right) \right].$$

We can use the N-times integration by parts formula and have

$$I(\lambda; z) = \int_{\underline{\mathrm{R}}^n} v^N(e^{i\lambda\Phi(y,z)})g(y)dy = \int_{\underline{\mathrm{R}}^n} e^{i\lambda\Phi(y,z)}(v^*)^N g(y)dy.$$

Note that

$$|(v^*)^N g(y)| \leq \frac{C\|g\|_{C^N}}{\lambda^N |z|^N}.$$

Hence,

$$|I(\lambda; z)| \leq \frac{C\|g\|_{C^N}}{\lambda^N |z|^N}$$

for any $N \geq 1$. Consequently,

$$\|I(\lambda, \cdot)\chi_{\{|z|\geq 1\}}(z)\|_{L^q(\underline{\mathrm{R}}^n)} \leq \frac{C\|g\|_{C^N}}{\lambda^N} \left(\int_{|z|>1} \frac{dz}{|z|^{Nq}} \right)^{\frac{1}{q}} = \frac{CC_{Nq}}{\lambda^N} \|g\|_{C^N},$$

where $\chi_{\{|z|\geq 1\}}$ is the indicator function of the set $\{|z| \geq 1\}$. Lemma 3.1 is proved.

Further, we consider estimate for the function: $\chi_{|z|\leq 1}(z)I(\lambda; z)$. The set $|z| \leq 1$ is a compact set. Let's fixed a point $z = z^0 \in \{|z| \leq 1\}$ and consider the function $\Phi(y, z^0) = (z^0, y) + \phi(y)$.

Lemma 3.2. *If $\nabla\Phi(y, z^0) \neq 0$, then for any $y \in supp(g)$, there exists a $\delta > 0$ such that the following inequality holds true*

$$|I(\lambda; z)| \leq \frac{C\|g\|_{C^N}}{\lambda^N}$$

for any z with $|z - z^0| < \delta$.

Proof. The set $supp(g)$ is a compact set. Hence there exists a positive number $\delta > 0$ such that we have $\nabla\Phi(y, z) \neq 0$ for any z with $|z - z^0| \leq \delta$ and for any $y \in supp(g)$. Therefore for $|z - z^0| \leq \delta$ the vector field (3.4) is well-defined on $supp(g)$.

Hence, we can use integration by parts formula and have

$$I(\lambda; z) = \int_{\underline{\mathrm{R}}^n} v^N(e^{i\lambda\Phi})g(y)dy = \int_{\underline{\mathrm{R}}^n} e^{i\lambda\Phi(y,z)}(v*)^N g(y)dy.$$

Thus as before obtain

$$|v^*(g)| \leq \frac{C\|g\|_{C^N}}{\lambda^N}.$$

Hence, we have a required bound for the integral $I(\lambda; z)$. Lemma 3.2 is proved.

Consider the set $C_\Phi := -\nabla\phi(supp(g))$. It is clear that C_Φ is a compact set. Consider an ε-neighborhood of that set $C_\Phi : U_\varepsilon(C_\Phi) := \{z : dist(z, C_\Phi) < \varepsilon\}$. If $z \notin U_\varepsilon(C_\Phi)$ then for any $y \in supp(g)$ we have $|z - \nabla\phi(y)| > \varepsilon$. Let $K(\lambda; z) := (1 - \chi_{U_\varepsilon(C_\Phi)}(z))I(\lambda; z)$, where $\chi_{U_\varepsilon(C_\Phi)}$ is the indicator function of the set $U_\varepsilon(C_\Phi)$.

Corollary 1. The following estimate holds true

$$\|K(\lambda; t)\|_{L^q(\underline{\mathrm{R}}^n)} \leq \frac{C\|g\|_{C^N}}{\lambda^N}.$$

Thus, we came to the conclusion that main contribution to the oscillatory integral $I(\lambda; z)$ give points from the neighborhood $U_\varepsilon(C_\Phi)$.

3.2. **On average decaying of the Fourier transform.** Since $z \in U_\varepsilon(C_\Phi)$ and $\nabla\phi(0) = 0$ we will assume that *suppg* is concentrated in a sufficiently small neighborhood of the origin and $|z| < \delta$.

The following statement improves the Proposition 1 given in the paper [26].

Proposition 3.3. *For all $g \in C_0^\infty(U)$ and $\lambda > 0$, the following estimate holds true*

$$\|I(\lambda; z)\chi_0(z)\|_{L^2(\underline{R}_z^{n-1})} \leq C_g \lambda^{-\frac{n-1}{2}}, \tag{3.5}$$

where χ_0 is a smooth cutoff function supported in a neighborhood of the origin and C_g is a constant.

Proof. We will use results proved by A.N.Varchenko. Following [32] we compute the L^2 norm of the function $\chi_0(z)I(\lambda; z)$. Then have

$$I(\lambda) := \int_{\underline{\mathrm{R}}^{n-1}} |\chi_0(z)I(\lambda; z)|^2 dz = \int_{\underline{\mathrm{R}}^{n-1}} \chi_0^2(z)I(\lambda; z) \cdot \overline{I(\lambda; z)} dz$$

$$= \int_{\underline{\mathrm{R}}^{n-1}} \chi_0^2(z) \int_{\underline{\mathrm{R}}_y^{n-1}} e^{i\lambda(z\cdot y + \phi(y))} g(y) dy \int_{\underline{\mathrm{R}}_u^{n-1}} e^{-i\lambda(zu + \phi(u))} \overline{g}(u) du dz$$

$$= \int_{\underline{\mathrm{R}}_z^{n-1} \times \underline{\mathrm{R}}_y^{n-1} \times \underline{\mathrm{R}}_u^{n-1}} \chi_0^2(z) g(y) \overline{g}(u) e^{i\lambda(z(y-u) + \phi(y) - \phi(u)} dy du dz.$$

Let's consider the new phase function $\Phi(z,y,u) = z(y-u)+\phi(y)-\phi(u)$. We can investigate asymptotic behavior of the inner integral over $\underline{R}_z^{n-1}\times\underline{R}_y^{n-1}$.

Let's compute the critical points, with respect to (z,y) variables:

$$\frac{\partial\Phi}{\partial z_j}=0,\ j=\overline{1,n-1},\quad \frac{\partial\Phi}{\partial y_j}=0,\ j=\overline{1,n-1}.$$

Then we have $detHess_{z,y}\Phi(z^c(u),y^c(u),u)=1$ and $\Phi(z^c(u),y^c(u),u)=0$ Thus, we can use stationary phase method to the following inner integral [20] and have the uniform asymptotic expansion (for large $|\lambda|$):

$$I_{in}(\lambda;u):=\int_{\underline{R}_z^{n-1}\times\underline{R}_y^{n-1}}\chi_0^2(z)g(y)e^{i\lambda\Phi(z,y,u)}dzdy=\frac{C}{\lambda^{n-1}}+O\left(\frac{1}{\lambda^n}\right).$$

Hence,

$$\|I(\lambda;z)\|_{L^2(\underline{R}^{n-1})}=O(|\lambda|^{\frac{-(n-1)}{2}})(\text{as}\,|\lambda|\to+\infty) \tag{3.6}$$

The last (3.6) asymptotic relation proves the Proposition 3.3.

Corollary 2. If $k>\frac{n}{2}$ and $p=3/4$ then M_k is $L^p\to L^{p'}$ bounded.

Proof. Actually, a proof of Corollary 2 it follows from Proposition 1 by using methods of the paper [26].

Let $\varphi(\xi)\in C^\infty(\underline{R}^3\backslash 0)$ be a smooth homogeneous function of order 1. Consider the hyper-surface Σ given by (2.2). For any point $\xi\in\Sigma$ we can define a notion of height $h_\Sigma(\xi)$ as in the paper. It is [16] proved that $h_\Sigma(\xi)$ is an upper semi continuous function. Thus, there exists a maximal value of the height over surface Σ. We set

$$h_\Sigma:=\max_{\xi\in\Sigma}h_\Sigma(\xi).$$

We call h_Σ a height of the hyper-surface Σ. It is a rational number. The following Theorem holds true.

Theorem 3.1. *If Σ is a smooth finite type surface, then M_k is the $L^p\to L^{p'}$ bounded operator under the condition $k>(6-2/h_\Sigma)(1/p-1/2)$.*

Proof. Since Σ is a compact set, then for any $\psi\in C^\infty(\Sigma)$ we can define the measure supported on the Σ by $d\mu=\psi d\sigma$. Consider the Fourier transform of the measure denoting by $\widehat{d\mu}$. By using partition of unity we may assume that ψ is concentrated in a sufficiently small neighborhood of a fixed point $\xi^0\in\Sigma$.

Then without loss of generality we may assume $\xi^0 = (0,0,1)$ and Σ is given as the graph of a smooth function $(y_1, y_2, 1+\phi(y_1,y_2))$, in a neighborhood of ξ^0. Moreover, we suppose $\phi(0,0) = 0$ and $\nabla\phi(0,0) = 0$. Then the height of the surface Σ at the point ξ^0 is defined by the height of the function ϕ. Hence by the results of the paper [**?**], we can write:

$$\left|\int_U g(y) e^{i\lambda(y_1 z_1 + y_2 z_2 + \phi(y_1,y_2))} dy\right| \leq \frac{C_g}{|\lambda|^{1/h(\phi)}} \log(2+|\lambda|)^\nu,$$

where $\nu = 0$ or 1.

Then by the definition of the height of the hyper-surface we have $h(\phi) \leq h_\Sigma$. Therefore for the oscillatory integral we have the estimate

$$|I(\lambda; z)| \leq \frac{C_g}{\lambda^{\frac{1}{h_\Sigma}}} \log\left(2+|\lambda|\right)^\nu,$$

where $\nu = 0$, or 1. If $\nu = 0$, then from Proposition 1 proved by M. Sugimoto in the paper [26] we have that M_k is $L^p \mapsto L^{p'}$ bounded for $1 \leq p \leq 2$ and $k > (6 - 2/h_\Sigma)(1/p - 1/2)$. If $\nu = 1$ then for any $\varepsilon > 0$ we have

$$|I(\lambda; z)| \leq \frac{C_g}{\lambda^{\frac{1}{h}+\varepsilon}}.$$

Thus for any

$$k > \left(6 - 2\left(\frac{1}{h} + \varepsilon\right)\right)\left(\frac{1}{p} - \frac{1}{2}\right)$$

the operator M_k is bounded from L^p to $L^{p'}$ But, $\varepsilon > 0$ is any fixed positive number. Therefore when $\varepsilon \to +0$ we have $k > (6 - 2/h_\Sigma)(1/p - 1/2)$. Theorem 1 is proved.

From now on, following M. Sugimoto [26] we shall assume that $a_k(\xi)$ is supported in a sufficiently small conic neighborhood Γ of a particular point $v \in S^2$ and $\varphi(\xi) \in C^\omega(\Gamma)$. We may assume $v = (0,\, 0,\, 1) \in S^2$ without loss of generality. Then, in the neighborhood, Σ can be expressed as

$$\Sigma \cap \Gamma := \{\xi \in \Gamma :\, \varphi(\xi) = 1\} = \{(x,\, 1+\phi(x))\},$$

where ϕ is a real analytic function at the origin. And also it satisfies the conditions $\phi(0) = 0$, $\nabla\phi(0) = 0$. The following Theorem gives a solution of the problem posed by M. Sugimoto in the paper [26].

Theorem 3.2. *Suppose that $\phi(x)$ has A_∞ type singularly at the origin and the equation $\frac{\partial\phi(x_1,x_2)}{\partial x_1} = 0$ has the solution $x_1 = x_2^{n+1}\omega(x_2)$ with a real analytic function ω satisfying the conditions $\omega(0) \neq 0$ and $n \geq 2$. Then M_k is $L^p \to L^{p'}$ bounded if*

$$k > \max\{6(1/p - 1/2) - 1/2,\, (5 - 1/\delta_1)(1/p - 1/2)\},$$

where $\delta_1 = n + 1$.

Proof. Note that $\phi(0,0) = 0$ and $\nabla\phi(0,0) = 0$. Also, by assumptions of the Theorem 2 we may suppose that $\frac{\partial^2\phi(0,0)}{\partial x_1} \neq 0$ and $\frac{\partial^2\phi(0,0)}{\partial x_2^2} = 0$ and also $\frac{\partial^2\phi(0,0)}{\partial x_1 \partial x_2} = 0$. Hence by implicit function theorem the equation

$$\frac{\partial\phi(x_1, x_2)}{\partial x_1} = 0$$

has a smooth solution $x_1 = \Psi(x_2)$. Therefore by the division theorem $\phi(x_1, x_2)$ can be written in the form: $\phi(x_1, x_2) = b(x_1, x_2)(x_1 - \Psi(x_2))^2$ and also by the assumptions of the Theorem 2 we have $\Psi(x_2) = x_2^{n+1}\omega(x_2)$. Now, we consider the oscillatory integral given by the following

$$I(\lambda; z) = \int_{R^2} e^{i\lambda\Phi(x,z)} g(x)dx,$$

where

$$\Phi(x, z) = b(x_1, x_2)(x_1 - x_2^{n+1}\omega(x_2))^2 + z_1x_1 + z_2x_2.$$

If $|z| > \varepsilon$ then the phase function has no critical points provided g is supported in a sufficiently small neighborhood of the origin. Hence as before we consider behavior of the oscillatory integral when $z \in U$, where U is a sufficiently small neighborhood of the origin. We prove the following inequality which was proved in the paper [26] in a particular case:

$$\|I(\lambda; z)\|_{L^{n+2}(U)} \leq C_{g,\varepsilon}\lambda^{-(1/2+2/(n+2))+\varepsilon} \tag{3.7}$$

for any $\varepsilon > 0$. From the estimate (3.7) it follows a proof of the Theorem 2.

The rest of our paper is devoted to prove of the inequality (3.7).

First, we use change variables $x_1 - x_2^{n+1}\omega(x_2) \mapsto y_1$ $x_2 \mapsto y_2$ in the integral $I(\lambda,\, z)$ and obtain:

$$I(\lambda,\, z) = \int e^{i\lambda(b(y_1+y_2^{n+1}\omega(y_2),y_2)y_1^2+z_1y_1+z_1y_2^{n+1}\omega(y_2)+z_2y_2)}g(y_1+y_2^{n+1}\omega(y_2),y_2)dy.$$

Consider the following interior integral

$$I_{in}(\lambda,\, z_1,\, y_2) := \int e^{i\lambda\Phi_1(y_1,\, y_2,\, z_1)}g(y_1+y_2^{n+1}\omega(y_2),y_2)dy_1,$$

where $\Phi_1(y_1,\, y_2,\, z_1) := b(y_1+y_2^{n+1}\omega(y_2),y_2)y_1^2+z_1y_1$ is the new phase function. Now, the variables $y_2,\, z_1$ can be considered as parameters. The phase function $\Phi_1(y_1,\, y_2,\, z_1)$ has a unique non-degenerate critical point with respect to y_1. It is a real analytic function of the parameters. The critical point can be written in the form: $y_1^c = z_1B(y_2,z_1)$, where B is a real analytic function with $B(0,0) = -\frac{1}{2b(0,0)}$. Thus by using stationary phase method in the variable y_1 we get:

$$I_{in} = C\frac{e^{i\lambda z_1^2B_1(y_2,z_1)}g(y_1^c(y_2,\, z_1)+y_2^{n+1}\omega(y_2),y_2)}{\lambda^{\frac{1}{2}}} + O\left(\frac{1}{|\lambda|^{\frac{3}{2}}}\right),$$

where C is a constant. Moreover, asymptotic relation in the remainder term holds uniformly with respect to parameters. Therefore it is enough to investigate contribution of the principal part of the interior integral. So, we consider the following integral:

$$I_1(\lambda,\, z) = \frac{C}{\lambda^{\frac{1}{2}}}\int e^{i\lambda(z_1(y_2^{n+1}\omega(y_2)+z_1B_1(y_2,z_1))+z_2y_2)}g(y_1^c+y_2^{n+1}\omega(y_2),y_2)dy_2.$$

Note that

$$|I(\lambda,\, z) - I_1(\lambda,\, z)| \le \frac{C}{|\lambda|^{\frac{3}{2}}}.$$

By using classical van der Corpute Lemma [21] we have the estimate:

$$|I_1(\lambda,\, z)| \le \frac{C}{\lambda^{\frac{1}{2}+\frac{1}{n+1}}|z_1|^{\frac{1}{n+1}}} \tag{3.8}$$

Suppose $|z_1| \le M|z_2|$ (where M is a fixed positive real number) then the phase function of the integral $I_1(\lambda,\, z)$ has no critical point. Then we can use integration by parts arguments and obtain:

$$|I_1(\lambda,\, z)| \le \frac{C}{|\lambda||z_2|^{\frac{1}{2}}} \tag{3.9}$$

Interpolating the inequalities (3.8) and (3.9) with the interpolation parameter

$$\theta := \frac{n^2-n-2}{n^2+n-2} + \frac{2\varepsilon(n+1)}{n-1}, \tag{3.10}$$

where ε is a sufficiently small positive real number, we get:

$$|I_1(\lambda,\, z)| \le \frac{C}{|\lambda|^{\frac{1}{2}+\frac{2}{n+2}-\varepsilon}|z_2|^{\frac{1}{2}(1-\theta)}|z_1|^{\frac{\theta}{n+1}}}$$

It is easy to show that

$$\frac{\chi_{\{|z_1|\le M|z_2|\}}(z)}{|z_2|^{\frac{1}{2}(1-\theta)}|z_1|^{\frac{\theta}{n+1}}} \in L^{n+2}(U),$$

where U is any bounded neighborhood of the origin.

Now, we assume $|z_2| \le \delta|z_1|$, where $\delta = \frac{1}{M}$ is a fixed positive real number. Then from the results of the paper [2] there exists a function $\Omega(z_1,\, \eta)$ satisfying the conditions:

1) for any number $1 \le p < \frac{2n}{n-1}$ and for any z_1 the inequality

$$\int_{|\eta|<\delta} (\Omega(z_1,\, \eta))^p d\eta \le C,$$

holds, with constant C not depending on z_1;

2) the following estimate holds true:

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))|I_1(\lambda,\, z)| \le \frac{\Omega\left(z_1,\, \frac{z_2}{z_1}\right)}{|\lambda||z_1|^{\frac{1}{2}}}$$

Finally, interpolating the last inequality with (3.8) by using interpolating parameter (3.10) we get:

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))|I_1(\lambda,\, z)| \le \frac{\left(\Omega\left(z_1,\, \frac{z_2}{z_1}\right)\right)^{1-\theta}}{|\lambda|^{\frac{1}{2}+\frac{2}{n+2}-\varepsilon}|z_1|^{\frac{2}{n+2}-\varepsilon}}.$$

It is easy to show that the following inclusion holds

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))\frac{\left(\Omega\left(z_1,\frac{z_2}{z_1}\right)\right)^{1-\theta}}{|z_1|^{\frac{2}{n+2}-\varepsilon}}\in L^{n+2}(U).$$

The last inclusion completes a proof of the Theorem 2.

Actually, we prove the following more better result.

Corollary 3. Assume the function ϕ satisfies conditions of the Theorem 2. Then for any positive real number ε there exists a function $\Omega \in L^{n+2}(\underline{\mathrm{R}}^2)$ such that the following inequality

$$|I(\lambda,\, z)| \le \frac{\Omega(z)}{|\lambda|^{\frac{2}{n+2}-\varepsilon}}$$

holds.

3.3. **On classification of M.Sugimoto.** In the paper Sugimoto given definition 1, let $h(y)=h(y_1,y_2)$ be a real analytic function at the origin satisfying $\nabla h(0,0)=0, \quad h''_{11}(0,0)\neq 0$, where $b_1(y_2)$ is solution to the equation

$$\frac{\partial}{\partial y_1}h(b_1(y_2),y_2)=0 \quad b_1(0)=0$$

is type II if

$$b_0(y_2)=h(b_1(y_2),y_2)\equiv 0$$

and δ_1 is the smallest integer $m\ge 2$ such that $b_j^{(m)}(0)\neq 0$ and $\delta_1<\infty$.

Proposition 3.4. *If $h(y)$ is a real analytic function of type II. Then there exists a real analytic function $\omega(y_2)$ such that $\omega(0)\neq 0$ and $b_1(y_2)=y_2^{\delta_1}\omega(y_2)$. Moreover, the function $h(y)$ can be written*

$$h(y_1,y_2)=g(y_1,y_2)(y_1-y_2^{\delta_1}\omega(y_2))^2$$

where $g(0,0)\neq 0$.

Proof. Indeed

$$h(0,0)=0 \quad \nabla h(0,0)=0 \quad \frac{\partial^2 h(0,0)}{\partial y_1^2}\neq 0.$$

By Weierstrass preparation Theorem h can be written as

$$h(y_1, y_2) = g(y_1, y_2)(y_1^2 + g_1(y_2)y_1 + g_2(y_2)^2),$$

where g is a real analytic function $g(0,0) \neq 0$. By the condition $h(b_1(y_2), y_2) \equiv 0$. So we have

$$h(b_1(y_2), y_2) = g(b_1(y_2), y_2)(b_1^2(y_2) + g_1(y_2)b_1(y_2) + g_2(y_2)) \equiv 0.$$

Hence

$$b_1^2(y_2) + g_1(y_2)b_1(y_2) + g_2(y_2) \equiv 0.$$

By we know that $b_1(y_2) + \tilde{b_1}(y_2) = -g_1(y_2)$. So the equation $y_1^2 + g_1(y_2)y_1 + g_2(y_2)^2 = 0$ has two analytic solutions. From this we can write

$$y_1^2 + g_1(y_2)y_1 + g_2(y_2)^2 = (y_1 - b_1(y_2))(y_1 - \tilde{b_1}(y_2)) = 0.$$

Now we produce $b_1(y_2) = \tilde{b_1}(y_2)$. For this we write

$$h(y_1, y_2) = g(y_1, y_2)(y_1 - b_1(y_2))(y_1 - \tilde{b_1}(y_2))$$

$$\frac{\partial h(y_1, y_2)}{\partial y_1} = \frac{\partial g(y_1, y_2)}{\partial y_1}(y_1 - b_1(y_2))(y_1 - \tilde{b_1}(y_2)) + g(y_1, y_2)(y_1 - b_1(y_2)) + g(y_1, y_2)(y_1 - \tilde{b_1}(y_2))$$

There $y_1 = b_1(y_2)$ and so

$$\frac{\partial h(b_1(y_2), y_2)}{\partial y_1} = g(b_1(y_2), y_2)(b_1(y_2) - \tilde{b_1}(y_2)).$$

On the other words $b_1(y_2) = \tilde{b_1}(y_2)$. Therefore,

$$h(y_1, y_2) = g(y_1, y_2)(y_1 - b_1(y_2))^2.$$

Proposition 3.4 is proved.

Proposition 3.5. *The condition of M.Sugimoto given in the Theorem 2.42 is equivalent to the condition*

$$\frac{\partial^\mu}{\partial y_1^\mu}\frac{\partial^\nu g(0,0)}{\partial y_2^\nu} = 0,$$

for $\mu = 1, 2, ..., \delta_1 - 1,\ \nu = 0, 1, 2, ...$

Proof. Suppose that

$$h(y_1, y_2) = g(y_1, y_2)(y_1 - y_2^{\delta_1}\omega(y_2))^2$$

and $g(0,0) \neq 0$, $\omega(0) \neq 0$. There we prove that

$$\frac{\partial^\mu}{\partial y_2^\mu}\{\frac{\partial^\nu}{\partial y_1^\nu} h(y_1, y_2)\}_{|y_2=0} = 0.$$

Furthermore,

$$\frac{\partial h}{\partial y_1} = \frac{\partial g(y_1, y_2)}{\partial y_1}(y_1 - y_2^{\delta_1}\omega(y_2))^2 + g(y_1, y_2)(y_1 - y_2^{\delta_1}\omega(y_2)),$$

$$\frac{\partial^2 h}{\partial y_1^2} = \frac{\partial^2 g(y_1, y_2)}{\partial y_1^2}(y_1 - y_2^{\delta_1}\omega(y_2))^2 + 4\frac{\partial g(y_1, y_2)}{\partial y_1}(y_1 - y_2^{\delta_1}\omega(y_2)) + 2g(y_1, y_2),$$

$$\frac{\partial^3 h}{\partial y_1^3} = \frac{\partial^3 g(y_1, y_2)}{\partial y_1^2}(y_1 - y_2^{\delta_1}\omega(y_2))^2 + 6\frac{\partial^2 g(y_1, y_2)}{\partial y_1^2}(y_1 - y_2^{\delta_1}\omega(y_2)) + 6\frac{\partial g(y_1, y_2)}{\partial y_1},$$

$$\ldots\ldots$$

$$\frac{\partial^\nu h}{\partial y_1^\nu} = \frac{\partial^\nu g(y_1, y_2)}{\partial y_1^\nu}(y_1 - y_2^{\delta_1}\omega(y_2))^2 + 2\nu\frac{\partial^{(\nu-1)} g(y_1, y_2)}{\partial y_1^{(\nu-1)}}(y_1 - y_2^{\delta_1}\omega(y_2)) + \nu(\nu-1)\frac{\partial^{(\nu-2)} g(y_1, y_2)}{\partial y_1^{(\nu-2)}}$$

There we have $y_1 = y_2^{\delta_1}\omega(y_2)$ and we put it in the equation. From this we obtain

$$\frac{d^\mu}{dy_2^\mu}\{\frac{\partial^{(\nu-2)} g(y_2^{\delta_1}\omega(y_2), y_2)}{\partial y_1^{\nu-2}}\}_{|y_2=0} = 0.$$

Thus,

$$\frac{d^\mu}{dy_2^\mu}\{\Phi(y_2^{\delta_1}\omega(y_2), y_2)\}_{|y_2=0} = \frac{\partial^\mu}{\partial y_2^\mu}\Phi(0,0)$$

There $\Phi(y_2^{\delta_1}\omega(y_2), y_2) = y_2^{\delta_1}\omega(y_2)g(y_2) + \Phi(0, y_2)$.

Corollary 4. If the function h has type II then after stationary phase method the phase function $\Phi(y_1^c(y_2, s_1), s_2)$ can be written in the form:

$$\Phi(y_1^c(y_2, s_1), y_2, s_1, s_2) = G(s_1) + s_1 y_2^{\delta_1} A(y_2, s_1) + s_2 y_2,$$

where $A(0,0) \neq 0$.

Proof. We have

$$\Phi(y, s) = g(y_1, y_2)(y_1 - y_2^{\delta_1}\omega(y_2))^2 + s_1 y_1 + s_2 y_2.$$

First, we use change of variables $y_1 - y_2^{\delta_1}\omega(y_2) \mapsto x_1 \quad y_2 \mapsto x_2$ and we obtain

$$\Phi(x, s) = g(x_1 + x_2^{\delta_1}\omega(x_2), x_2)x_1^2 + s_1 x_1 + s_1 x_2^{\delta_1}\omega(x_2) + s_2 x_2.$$

Let's consider equation for the critical point with respect to x_1.

$$\frac{\partial \Phi}{\partial x_1} = 2x_1 g(x_1 + x_2^{\delta_1}\omega(x_2), x_2) + x_1 \frac{\partial g}{\partial x_1}(x_1 + x_2^{\delta_1}\omega(x_2), x_2) + s_1 = 0. \tag{3.11}$$

We prove the following Lemma.

Lemma 3.6. *Solution to the equation 3.11 with respect to x_1 can be written in the form:*

$$x_1^c(s_1, x_2) = s_1(v(s_1) + x_2^{\delta_1} B(x_2, s_1). \tag{3.12}$$

Proof. We can be written as

$$g(y_1, y_2) = g(y_1, 0) + y_2^{\delta_1} g_1(y_1, y_2),$$

where g, is a real analytic function.

Let

$$G(y_1, y_2) = g(y_1, y_2) - g(y_1, 0)$$
$$G(y_1, 0) \equiv 0,$$

and

$$\frac{\partial G(y_1, 0)}{\partial y_2} = \frac{\partial g}{\partial y_2}(y_1, 0) \equiv 0.$$

And so on

$$\frac{\partial^{\delta_1 - 1} G(y_1, 0)}{\partial y_2^{\delta_1 - 1}} \equiv 0.$$

By using integration by parts

$$G(y_1, y_2) = \int_0^{y_2} \frac{\partial G(y_1, z)}{\partial z} dz = \int_0^{y_2} \frac{\partial G(y_1, z)}{\partial z}(z - y_2)' dz =$$
$$\frac{\partial G(y_1, z)}{\partial z}(z - y_2)|_0^{y_2} - \int_0^{y_2} \frac{\partial^2 G(y_1, z)}{\partial z^2}(z - y_2) dz =$$
$$-\int_0^{y_2} \frac{\partial^2 G(y_1, z)}{\partial z^2} \{\frac{(z - y_2)^2}{2!}\}' dz = ... = (-1)^{\delta_1 - 1} \int_0^{y_2} \frac{\partial^{\delta_1} G(y_1, z)}{\partial z^{\delta_1}} \frac{(z - y_2)^{\delta_1 - 1}}{(\delta_1 - 1)!} dz =$$
$$\int_0^{y_2} \frac{\partial^{\delta_1} G(y_1, z)}{\partial z^{\delta_1}} \frac{(z - y_2)^{\delta_1 - 1}}{(\delta_1 - 1)!} dz$$

we use change of variables $z = y_2\xi$, $dz = y_2 d\xi \quad z = 0$, $\xi = 0$ $z - y_2$, $\xi = 1$

$$y_2^{\delta_1} \int_0^1 \frac{\partial^{\delta_1} G(y_1, y_2, \xi)}{\partial z^{\delta_1}} \frac{(1 - \xi)^{\delta_1 - 1}}{(\delta_1 - 1)!} d\xi.$$

We see

$$g_1(y_1, y_2) = \int_0^1 \frac{\partial^{\delta_1} G(y_1, y_2, \xi)}{\partial z^{\delta_1}} \frac{(1-\xi)^{\delta_1 - 1}}{(\delta_1 - 1)!} d\xi.$$

It is a real analytic function

$$g(y_1, y_2) = g(y_1, 0) + y_2^{\delta_1} g_1(y_1, y_1),$$

similarly,

$$\frac{\partial g}{\partial y_1} = \frac{\partial g(y_1, 0)}{\partial y_1} + y_2^{\delta_1} \frac{\partial g_1(y_1, y_2)}{\partial y_1}.$$

Similarly,

$$g(x_1 + x_2^{\delta_1}\omega(x_2), x_2) = g(x_1, 0) + y_2^{\delta_1} \widetilde{g}_1(x_1, x_1),$$

$$\frac{\partial g}{\partial x_1}(x_1 + x_2^{\delta_1}\omega(x_2), x_2) = \frac{\partial g(x_1, 0)}{\partial x_1} + y_2^{\delta_1} \frac{\partial g_1(x_1, x_2)}{\partial x_1},$$

$$2x_1(g(x_1, 0) + x_1 \frac{\partial}{\partial x_1} g(x_1, 0) + x_2^{\delta_1} G_1(x_1, x_2)) = -s_1.$$

Since $g(0,0) \neq 0$ the equation has a unique analytic solution of the form: $x_1 = s_1 H(s_1, x_2)$ $F(x_1, x_2, s_1) = 0$, $x_1 = x_1(x_2, s_1)$.

$$\frac{\partial}{\partial x_2} x_1(0, s_1) = -\frac{\frac{\partial}{\partial x_2} F(x_1, 0, s_1)}{\frac{\partial F}{\partial x_1}}$$

$$H(s_1, x_2) = (\psi(s_1) + x_2^{\delta_1} H_1(s_1, x_2))$$

3.4. **On classification of singularities of phase function and related oscillatory integrals.** Let $\varphi \in C^\omega(R^n \setminus 0)$ be a real analytic on $R^n \setminus 0$ and homogeneous of order 1.

$a_k(\xi) \in C^\infty(R^n)$ is homogeneous of order $-k$ for large $|\xi|$.

Consider the convolution operator given by the kernel.

$$K_k(x) = F^{-1}(e^{i\varphi(\xi)} a_k(\xi). \tag{3.13}$$

Following [26] we can localize the $a_k(\xi)$ in a sufficiently small conic neighborhood of a fixed point $v \in S^2$ where S^2 is a unite sphere centered at the origin of R^3.

Without loss of generality we may assume that $v = (0, 0, 1)$ and

$$\frac{\partial}{\partial \xi_1} \varphi(0, 0) = \frac{\partial}{\partial \xi_2} \varphi(0, 0) = 0$$

$$\varphi(0,0,1) = 0 \quad \frac{\partial}{\partial \xi_1}\varphi(0,0,1) \neq 0.$$

Otherwise, we may use linear change of variables $\xi_1 = \eta_1$, $\xi_2 = \eta_2$, $\xi_3 = \eta_3 - \frac{\partial\varphi(0,0,1)}{\partial\xi_2}\eta_2 - \frac{\partial\varphi(0,0,1)}{\partial\xi_1}\eta_1$. Then for the function

$$\varphi_1(\eta_1,\eta_2,\eta_3) = \varphi(\eta_1,\eta_2,\eta_3 - \frac{\partial\varphi(0,0,1)}{\partial\xi_2}\eta_2 - \frac{\partial\varphi(0,0,1)}{\partial\xi_1}\eta_1).$$

We have $\varphi(0,0,1) = 1$

$$\frac{\partial\varphi_1(0,0,1)}{\partial\eta_1} = \frac{\partial\varphi_1(0,0,1)}{\partial\eta_2} = 0 \quad \frac{\partial\varphi(0,0,1)}{\partial\eta_3} = 1.$$

We remind classification for the function given by M.Sugimoto [26]. We assume

$$h(0,0) \neq 0, \quad \xi_3 = 1 + h(\xi_1,\xi_2), \quad \nabla h(0,0) = 0.$$

Then we define the function $b_0(y_2)$ and $b_1(y_2)$, which are real analytic at the origin, by the equations

$$h_1'(b_1(y_2),y_2) = 0, \qquad b_1(0) = 0 \qquad b_0(y_2) = h(b_1(y_2),y_2). \tag{3.14}$$

They are uniquely determined near the origin by the implicit function theorem. The curve $(b_1(t),t,b_0(t))$ is the ridge of the mountain Σ when we see it parallel to the y_1-axis.

Definition 3.1. Let $h(y) = h(y_1,y_2)$ be a real analytic function at the origin satisfying (2.13), and $b_j(y_2)$ be defined by (2.14) (j-0,1). Then we define δ_j to be the smallest integer $m \geq 2$ such that $b_j^{(m)}(0) \neq 0$, and we say that $h(y)$ is of type I if $\delta_0 < \infty$, type II if $\delta_0 = \infty$, $\delta_1 < \infty$, and type III if $\delta_0 = \delta_1 = \infty$.

The following result holds true.

Proposition 3.7. *The function h is of type I if and only if when h has singularity of type A_{δ_0-1} at the origin. The function h is of type II or III if and only if h has A_∞ type singularity at the origin.*

Proof. By the proposition the phase function $h(y_1,y_2)$ after possible linear change of variables can be written as

$$h(x_1,x_2) = b(x_1,x_2)(x_1 - b_1(x_2))^2 + x_2^n\alpha(x_2),$$

where b_1 is a real analytic function satisfying the condition

$$b_1(0) = b_1'(0) = 0 \quad n \geq 3.$$

Then we have

$$b_0(y_2) = h(b_1(y_2), y_2) = x_2^n \alpha(x_2).$$

If $b_0(y_2)$ has a root of multiplicity at the origin, then we may assume that $n = \delta_0$ and $\alpha(0) \neq 0$. So we can use the diffeomorphic change of variables

$$x_1 = y_1 - b_1(x_2), \quad x_2 = y_2. \tag{3.15}$$

The function h is the x coordinates can be written as

$$h^a(x_1, x_2) = b(x_1 + b_1(x_2), x_2)x_1^2 + x_2^{\delta_0}\alpha(x_2).$$

Then the Newton polygon of the function h^a consists of

$$N(h^a) = R_+^2 \cap \left\{ \frac{t_1}{2} + \frac{t_2}{\delta_0} \geq 1 \right\}.$$

By the Arnol'd theorem on normal forms. h is diffeomorphic equivalent to $\pm x_1^2 \pm x_2^{\delta_0}$. Thus function h has $A_{\delta_0 - 1}$ type singularity at the origin. Similarly if $b_1(y_2)$ has root of multiplicity δ_1 at the origin and $\delta_0 = \infty$ then by using 3.15 change of variables the function $h^a = b(x_1 + b_1(x_2), x_2)x_1^2$. Since $b(0,0) \neq 0$ h can be reduced to the form $\pm x_1^2$. It is the A_∞ type singularities.

Proposition is proved.

Furthermore, we show that the total phase function $\Phi(y_1, y_2, s_1, s_2)$ can have different type of singularities.

Now, we consider the oscillatory integral with $A_{\delta_0 - 1}$ type singularities. Then it is well-known the estimate:

$$|I(\lambda, z)| \leq \frac{C_g}{\lambda^{\frac{1}{2} + \frac{1}{\delta_0}}}$$

$$\Phi(x, s) = b(x_1, x_2)(x_1 - x_2^m \omega(x_2))^2 + x_2^n \beta(x_2) - s_1 x_1 - s_2 x_2.$$

Let $m < \frac{n}{2} + 1$, denote

$$q_m := min\{4, \frac{2(m-1)}{m-2}\}.$$

The following results holds true.

Theorem 3.3. *Assume the phase function h has singularity A_{n-1} and*

$$\delta_0 := m < \frac{n}{2} + 1.$$

Then there exists a neighborhood $U \times V$ of the origin and a function $\psi \in L^{q_m - 0}(V)$ such that for any $a \in C^\infty(U \times V)$ the following estimate holds true:

$$|I(\lambda, s)| \le \frac{\psi(s)}{|\lambda|}.$$

Proof. Without loss of generality we may assume that the phase function of the oscillatory integral has the form:

$$\Phi(x_1, x_2) = b(x_1, x_2)x_1^2 - s_1 x_1 + x_2^n \beta(x_2) - s_1 x_2^m \omega(x_2) - s_2 x_2,$$

where b, β, ω are smooth functions satisfying the conditions $b(0,0) \neq 0$, $\omega(0) \neq 0$, $\beta(0) \neq 0$. First, we consider the following one-dimensional oscillatory integral

$$J(\lambda, s_1, x_2) := \int e^{i\lambda(b(x_1,x_2)x_1^2 - s_1 x_1)} a(x, s) dx_1. \tag{3.16}$$

We may assume that (x_2, s_1) belongs to a sufficiently small neighborhood of the origin. Consider the following equation with respect to the critical point.

$$F(x_1, x_2, s_1) = \frac{\partial b}{\partial x_1}(x_1, x_2)x_1^2 + 2x_1 b(x_1, x_2) - s_1 = 0. \tag{3.17}$$

We claim that the non-degenerate critical point can be written in the form:

$$x_1^c = s_1 B(x_2, s_1),$$

where $B(x_2, s_1)$ is a smooth function with

$$B(0,0) = \frac{1}{2b(0,0)} \neq 0.$$

Indeed,$b(0,0) \neq 0$. By the implicit function Theorem the equation 3.17 has a real analytic solution

$$x_1^c = G(x_2, s_1).$$

In $s_1 = 0$ then for any small x_2 we have $x_1^c(x_2, 0) = G(x_2, 0) = 0$. Hence the function $G(x_2, s_1)$ can be written as

$$G(x_2, s_1) = s_1 B(x_2, s_1),$$

where B is a real analytic function at the origin.

By differentiation of the implicit function we have

$$\frac{\partial x_1^c(0,0)}{\partial s_1} = -\frac{\dfrac{\partial F(0,0,0)}{\partial s_1}}{\dfrac{\partial F(0,0,0)}{\partial x_1}} = -\frac{-1}{2b(0,0)} = \frac{1}{2b(0,0)} \neq 0.$$

On the other hand,

$$\frac{\partial x_1^c}{\partial s_1} = B(0,0).$$

Thus,

$$B(0,0) = \frac{1}{2b(0,0)} \neq 0.$$

Now, we use stationary phase method to the integral 3.16 and have

$$J(\lambda, s_1, x_2) = C\frac{e^{i\lambda s_1^2 \tilde{B}(x_2,s_1)} a(x_1^c(x_2,s_1), x_2, s_1)}{\lambda^{\frac{1}{2}}} + R(\lambda, s_1, x_2),$$

where $R(\lambda, s_1, x_2)$ is a remainder term satisfying the condition

$$|R(\lambda, s_1, x_2)| \leq \frac{C}{\lambda^{\frac{3}{2}}},$$

$$\tilde{B}(x_2, s_1) := b(s_1 B(x_2, s_1), x_2) B^2(x_2, s_1) - B(x_2, s_1),$$

Note that $\tilde{B}(x_2, s_1)$ is a smooth function satisfying the condition

$$\tilde{B}(0,0) = \frac{b(0,0)}{4b^2(0,0)} - \frac{1}{2b(0,0)} = -\frac{1}{4b(0,0)} \neq 0.$$

Hence our two-dimension oscillatory integral can be written in the form:

$$J(\lambda, s) = \frac{1}{\lambda^{\frac{1}{2}}} \int e^{i\lambda\Phi_1(x_2,s)} a_1(x_2, s) dx_2 + O(\frac{1}{\lambda^{\frac{3}{2}}}),$$

where

$$\Phi_1(x_2, s) = x_2^n \beta(x_2) - s_1 x_2^m - s_2 x_2 + s_1^2 \tilde{B}(x_2, s_1),$$

$$a_1(x_2, s) := a(x_1^c(x_2, s_1), x_2, s).$$

Consequently 4 it is enough to consider the following one-dimensional oscillatory integral

$$J_1(\lambda, s) = \int e^{i\lambda\Phi_1(x_2,s)} a_1(x_2, s) dx_2.$$

The following statement holds true.

Lemma 3.8. *If Φ is the phase function satisfying the condition of Theorem 3.3 then there exists a neighborhood $U \times V$ of the origin and a function $\psi \in L^{q_m - 0}(V)$. such that the following estimate holds true:*

$$|J_1(\lambda, s)| \le \frac{\psi(s)\|a(\cdot, s)\|_{C^2}}{\lambda^{\frac{1}{2}}},$$

where $\|a(\cdot, s)\|_{C^2}$ is the norm of the function $a(\cdot, s)$ with respect to a natural norm of the space C^2.

Proof. The function $\tilde{B}(x_2, s_1)$ can be written as

$$\tilde{B}(x_2, s_1) = \tilde{B}(0, s_1) + x_2 \tilde{B}'_{x_2}(0, s_1) + x_2^2 B_2(x_2, s_1),$$

where B_2 is a smooth function. Inserting $\tilde{B}$ to Φ_1 we have

$$\Phi_1(x_2, s_1, s_2) = x_2^n \beta(x_2) - s_1 x_2^m - (s_2 - s_1^2 B'_{x_2}(0, s_1))x_2 + s_1^2 x_2^2 B_2(x_2, s_1).$$

We use change of variables $s_2 - s_1^2 B'_{x_2}(o, s_1) \to s_2$ and have

$$\tilde{\Phi}_1(x_2, s_1, s_2) = x_2^n \beta(x_2) - s_1 x_2^m - s_2 x_2 + s_1^2 x_2^2 \beta_1(x_2, s_1).$$

Now, for the parameters (s_1, s_2) consider two cases.

1-case. $\{(s_1, s_2) : \frac{|s_1|}{|s_2|^{\frac{n-m}{n-1}}} < \varepsilon\}$ where $\varepsilon > 0$ is a fixed sufficiently small positive real number. Then we use change of variables $x_2 = |s_2|^{\frac{1}{n-1}} y_2$ the new variable y_2 again denoting by x_2. We get:

$$J_1(\lambda, s) = |s_2|^{\frac{1}{n-1}} \int a_1(|s_2|^{\frac{1}{n-1}} x_2, s) e^{i\lambda |s_2|^{\frac{1}{n-1}} \Phi_2(x_2, s)} dx_2,$$

where $\Phi_2(x_2, s) = x_2^n \beta(|s_2|^{\frac{1}{n-1}} x_2) - \frac{s_1}{|s_2|^{\frac{n-m}{n-1}}} x_2^m - sign(s_2) x_2 + \frac{s_1^2}{|s_2|^{\frac{n-2}{n-1}}} x_2^2 B(|s_2|^{\frac{1}{n-2}} x_2, s_1)$.
Note that if $\lambda |s_2|^{\frac{n}{n-1}} \lesssim 1$ then we can use Van der Compute Lemma and have

$$|J_1(\lambda, s)| \le \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{n}}} \le \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{n}} |\lambda |s_2|^{\frac{n}{n-1}}|^{\frac{1}{2} - \frac{1}{n}}} = \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{2}} |s_2|^{\frac{n-2}{2(n-1)}}},$$

where $\|a_1(\cdot, s)\|_\nu = |a_1(0, s)| + V[a]$. If $|s_1| \le \varepsilon |s_2|^{\frac{n-m}{n-1}}$ then we have

$$\int \int_{|s_1| \le \varepsilon |s_2|^{\frac{n-m}{n-1}}} \frac{ds_1 ds_2}{|s_2|^{\frac{(n-2)p}{2(n-1)}}} \le \int_0^1 \frac{ds_2}{|s_2|^{\frac{(n-2)p}{2(n-1)} - \frac{n-m}{n-1}}}.$$

The last integral converses whenever $\frac{(n-2)p}{2(n-1)} - \frac{n-m}{n-1} < 1$. Or equivalently $(n-2)p < 2(2n-m-1)$. Thus for $p < \frac{2(2n-m-1)}{n-2}$. Thus for the case when $|\lambda||s_2|^{\frac{n}{n-1}} \lesssim 1$ we have a required bound. Further assume that $|\lambda||s_2|^{\frac{n}{n-1}} \gg 1$. Note that $\frac{|s_1|^2}{|s_2|^{\frac{n-2}{n-1}}} \ll \varepsilon^2$ under condition that $m < \frac{n}{2}+1$. Note that $\frac{2(2n-m-1)}{n-2} \geq \frac{m}{m-1}$.

It is easy to see that there exists a positive real number M such that for any $|x_2| > M$ the phase function Φ_2 has no critical points. We use partition of unity $\{h_1, h_2\}$ corresponding to the covering $(-M-1, M+1)\bigcup(R \setminus [-M, M])$ (so $supp(h_1) \subset (-M-1, M+1)$ and $supp(h_2) \subset R \setminus [-M, M]$).

Let

$$J_1^{\nu}(\lambda, s) = |s_2|^{\frac{1}{n-1}} \int a_1(|s_2|^{\frac{1}{n-1}}x_2, s)e^{i\lambda|s_2|^{\frac{1}{n-1}}\Phi_2(x_2,s)}h_\nu(x_2)dx_2, \quad \nu = 1, 2.$$

Thus $J_1(\lambda, s) = J_1^1(\lambda, s) + J_1^2(\lambda, s)$. Since the phase function of the oscillatory integral $J_1^2(\lambda, s)$ has no critical points by parts arguments we get:

$$|J_1^2(\lambda, s)| \leq \frac{C|s_2|^{\frac{1}{n-1}}\|a(\cdot, s)\|_\nu}{|\lambda|s_2|^{\frac{n}{n-1}}|} \leq \frac{C|s_2|^{\frac{1}{n-1}}\|a_2(\cdot, s)\|_\nu}{|\lambda|s_2|^{\frac{n}{n-1}}|^{\frac{1}{2}}} \leq \frac{C\|a_1(\cdot, s\|_v)}{\lambda^{\frac{1}{2}}|s_2|^{\frac{n-2}{2(n-1)}}}.$$

The last estimate is enough to get a required bound. Now we consider behavior of the integral $J_1^1(\lambda, s)$. By the condition the phase function can be considered as perturbation of the function

$$x_2^n\beta(0) - sign(s_2)x_2.$$

The last function has only non-degenerate critical points (at most two critical points). Therefore, we can use Morse Lemma with parameters and have

$$|J_1^1(\lambda, s)| \leq \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{2}}|s_2|^{\frac{n-2}{2(n-1)}}}.$$

Thus, we are done for the first case. From now on we consider the case.

2-case: $\{(s_1, s_2) : \frac{|s_2|}{|s_1|^{\frac{n-1}{n-m}}} \leq N\}$, where N is a sufficiently large fixed positive real number.

In this case we use change of variables $x_2 = |s_1|^{\frac{1}{n-m}}y_2$ and the new variable y_2 will be denoted by x_2. Thus, we get

$$J_1(\lambda, s) = |s_1|^{\frac{1}{n-m}} \int a_1(|s_1|^{\frac{1}{n-m}}, s)e^{i\lambda|s_1|^{\frac{n}{n-m}}\Phi_3(x_2,s)}d_2,$$

where

$$\Phi_3(x_2, s) = x_2^n\beta(|s_1|^{\frac{1}{n-m}}x_2) - sign(s_1)x_2^m + |s_1|^{\frac{n+2-2m}{n-m}} sign(s_1)x_2^2 B(|s_1|^{\frac{1}{n-m}}x_2) + \frac{s_2}{|s_1|^{\frac{n-1}{n-m}}}x_2.$$

If $\lambda|s_1|^{\frac{n}{n-m}} \lesssim 1$, then we use Van der Corpate type estimate and have

$$|J_1(\lambda, s)| \leq \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{n}}} \leq \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{n}}|\lambda|s_1|^{\frac{n}{n-m}}|^{\frac{1}{2}-\frac{1}{n}}} = \frac{C\|a_1(\cdot, s)\|_V}{|\lambda|^{\frac{1}{2}}|s_1|^{\frac{n-2}{2(n-m)}}}.$$

Note that

$$\int_{|s_2| \leq N|s_1|^{\frac{n-1}{n-m}}} \frac{ds_2 ds_1}{|s_1|^{\frac{(n-2)p}{2(n-m)}}} \leq N\int_0^1 \frac{|s_1|^{\frac{n-1}{n-m}}}{|s_1|^{\frac{(n-2)p}{2(n-m)}}} ds_1.$$

The last integral converges if and only if $\frac{(n-2)p}{2(n-m)} - \frac{n-1}{n-m} < 1$. Consequently,$(n-2)p < 2n-2+2n-2m$. Or equivalent $p < \frac{2(2n-m-1)}{n-2}$. Thus, we have a required bound. There exists a positive real number M such that for any $|x_2| > M$ the phase function has no critical point. As before we consider a partition of unity $\{h_1, h_2\}$ corresponding to the covering $(-M-1, M+1) \cup (\mathbb{R} \setminus ([-M, M]))$. For the integral $J_1^2(\lambda, s)$ we have a required bound. We consider estimate for the integral

$$J_1^1(\lambda, s) = |s_1|^{\frac{1}{n-m}} \int a_1(|s_1^{\frac{1}{n-m}}|x_2, s)e^{i\lambda|s_1|^{\frac{n}{n-m}}\Phi_3(x_2,s)}dx$$

Further, we suppose that $|\lambda||s_1|^{\frac{n}{n-m}} \gg 1$.

The phase function $\Phi_3(x_2, s)$ can be considered as small perfurbaruies of the function

$$x_2^n\beta(0) - sgn(s_1)x_2^m + \xi_2^0 x_2$$

where $\xi_2^0 \in [-N, N]$ is a fixed number. If $\xi_2^0 \neq 0$ then the phase function at worst has A_2 type singularity. Let's consider the case when $\xi_2^0 = 0$. Then the phase function can be considered as perturbation of a_{m-1} type singularities. Thus, the exists a function $\psi \in L^{\frac{2m}{m-1}-0}(V)$ such that for the oscillatory integral $J_1(\lambda, s)$ the following estimate holds true

$$|J_1(\lambda, s)| \leq \frac{\psi(\frac{s_2}{|s_1|^{\frac{n-1}{m-n}}})\|a(\cdot, s)\|_v}{|\lambda|^{\frac{1}{2}}|s_1|^{\frac{n-2}{2(n-m)}}}$$

It is easy to see that

$$\frac{\psi(\frac{s_2}{|s_1|^{\frac{n-1}{m-n}}})}{|s_1|^{\frac{n-2}{2(n-m)}}} \in L^{\frac{2m}{m-1}-0}(U)$$

theorem 3.3 is proved.

The following result improves Theorem proved by M. sugimoto.

Theorem 3.4. *Suppose h has A_{δ_0-1} type singularity and $\delta_1 \geq 3$ then M_k is $L^p \to L^{p'}$ bounded if*

$$k > max\{6(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}, min\{(5-\frac{1}{\delta_1})(\frac{1}{p}-\frac{1}{2}), (5-\frac{2}{\delta_0})(\frac{1}{p}-\frac{1}{2})\}\}.$$

3.5. **On estimates for convolution operators.** In this section, we assume $1 \leq p \leq 2$ and $1/p + 1/p' = 1$, e.g. p, p' are conjugate exponents. We shall consider the problem on $L^p \to L^{p'}$ (where and further $L^q : L^q(R^3)$) boundedness of Fourier multiplier operators of the following type:

$$M_k := F^{-1} e^{i\varphi(\xi)} a_k(\xi) F,$$

where F is the Fourier transform operator, $\varphi \in C^\omega(R^n \setminus \{0\})$ is a non-vanishing real analytic, homogenous function of order 1, $a_k \in C^\infty(R^n)$ is a classical symbol of a pseudo-differential operator of order k. For simplicity we assume that a_k is a smooth homogeneous function of order k for large $|\xi|$. Following, M. Sugimoto we assume $\varphi(\xi) > 0$ for any $\xi \neq 0$ and set:

$$S := \xi \in R^n \setminus 0 : \varphi(\xi) = 1. \tag{3.18}$$

Surely, the case, when $\varphi(\xi) < 0$ for any $\xi \neq 0$, can be treated by using quite similar arguments. By the classical Euler homogeneity relation S is a compact, closed analytic hyper-surface without singularities. The main problem is the following: for which $k > k(p)$ the operator M_k is bounded as a bounded operator from L^p to $L^{p'}$?

We consider estimates for the convolution operators in the case when $n = 3$. Since S is a compact set, we can localize the operator. Thus, we can write the convolution operator as a sum of finite number of operators for which the associated symbol is concentrated in a sufficiently small conic neighborhood of points of the unite sphere. So, following [26], we shall assume that $a_k(\xi)$ is supported in a sufficiently small conic neighborhood Γ of a particular point $v \in S^2$ (where S^2 is the unite sphere centered at the origin of R^3), and $\varphi(\xi) \in C(\Gamma)$. Without loss of generality, we may assume $v = (0,0,1) \in S^2$ and $\varphi(0,0,1) = 1$, $\nabla\varphi(0,01) = (0,0,1)$. Then, in the neighborhood Γ the surface S can be expressed as

$$S \cup \Gamma = \{\xi \in \Gamma : \varphi(\xi) = 1\} = \{(y, 1 + h(y)), y \in U\},$$

where $h \in C^\omega(U)$ (here $U \subset R^2$ is an open neighborhood of the origin) is a real analytic function. After possible linear change of variables we may and shall assume that h is a real analytic function satisfying the conditions: $h(0,0) = 0$, $\nabla h(0,0) = (0,0)$. Lets reduce the classification of the function h given by M. Sugimoto[26]. For the sake of being definite we assume that $det\{\frac{\partial^2 h(0,0)}{\partial x_k \partial x_j}\}_{k,j=1}^2 = 0$. Otherwise, the result can be derived from the main Theorem by Brenner[8]. Then after possible rotation of coordinate axes we may suppose that the function h satisfies the following conditions:

$$\frac{\partial^2 h(0,0)}{\partial x_1^2} \neq 0, \quad \frac{\partial^2 h(0,0)}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 h(0,0)}{\partial x_2^2} = 0. \tag{3.19}$$

Due to the classical implicit function Theorem it is easy to see that the equation $\frac{\partial h(x_1,x_2)}{\partial x_1} = 0$ has a unique solution:

$$x_1 = b_1(x_2) \tag{3.20}$$

with a real analytic function b_1 satisfying the conditions: $b_1(0) = b_1'(0) = 0$. Lets define the new function of one variable given by the function $h :$

$$b_0(x_2) : h(b_1(x_2), x_2). \tag{3.21}$$

The following definition was given in the paper [26]. It gives a classification of singularities of the function h.

Definition 1. Let h be a real analytic function at the origin satisfying 3.19 and b_0, b_1 be the real analytic functions defined by 3.20, 3.21. Then we define δ_j to be the smallest integer $m \geq 2$ such that $b_j^{(m)}(0) \neq 0$, and we say that h is of type I if $\delta_0 < \infty$, type II if $\delta_0 = \infty$, and $\delta_1 < \infty$, type III if $\delta_0 = \delta_1 = \infty$.

First, we give a simple statement, which follows from the classical division Theorem.

Proposition 3.9. *Assume that h is a smooth function satisfying the assumptions 3.19. Then, the function h can be written in the following form on a sufficiently small neighborhood of the origin: $h(x_1, x_2) = u(x_1, x_2)(x_1 - b_1(x_2))^2 + b_0(x_2)$, where u, b_0, b_1 are smooth functions satisfying the conditions: $u(0, 0) \neq 0$, and where $b_1(x_2) = cx_2^{\delta_0} + O(x_2^{\delta_0+1})(c \neq 0)$ unless it is a flat function, $b_0(x_2) = cx_2^{\delta_1} + O(x_2^{\delta_1+1})(c \neq 0)$ unless it is a flat function.*

The following statement characterize the classification given by M. Sugimoto[26].

Proposition 3.10. *Let h be a real analytic function at the origin satisfying 3.18 and b_0, b_1 be the real analytic functions defined by 3.20, 3.21. Then h is of type I if and only if h has A_{δ_0-1} type singularities at the origin; h is of type II or III if and only if h has A_∞ type singularities at the origin. Moreover, if h has A_∞ type singularities at the origin, then it is the type II if and only if the projection of the zero set of the Gaussian curvature to the plane $\mathbb{R}^2$ is a smooth curve, which has no tangent lines with infinite order of contact.*

See [1] for the definition of an A type singularities. In this section we mainly dealt with the A_∞ type singularities. If b_0 is a flat function (we have $b_0(x_2) \equiv 0$ whenever h is a real analytic function) then it has the A_∞ type singularities. It is easy to see that in the latter case, the curve in the plane $\mathbb{R}^2$ is given by the equation $x_1 = b_1(x_2)$ and it coincides with the projection of the zero set of the Gaussian curvature. Hence, the number δ_1 coincides with order of contact with the curve and tangent line at the origin. Note that singularities of the so-called phase function defined by

$$\phi(x_1, x_2, s_1, s_2) = h(x_1, x_2) + s_1x_1 + s_2x_2,$$

depend on classification of the function $h(x)$. In the monograph [1] were given characterizations of singularities of that function and also for the phase function in the case when both in a generic position up to some number of variables and up to some Milnor number. It should be noted that behavior of the convolution operator depends on the Fourier transform of measures supported on the surface S . We introduce a more general signed measures supported on family of hypersurfaces. Let $(S_a), S_a \in \mathbb{R}^(n+1)$ be a family of smooth hyper-surfaces smoothly depending on a parameter $a \in \mathbb{R}^m$, and let $\psi \in C_0^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^m)$ be a smooth function with compact support. The Fourier transform of the signed measure $d\mu_a := \psi(x,a)dS_a$ is determined by the integral

$$\widehat{d\mu_a}(\epsilon) = \int_{S_a} e^{ix\epsilon} d\mu_a(x). \tag{3.22}$$

Theorem 3.5. *Let $S = \{(x, h(x)+1)\}$ be the surface and h be a real analytic function with $h(0,0) = 0$, $\nabla h(0,0) = 0$. If $K(0,0) = 0$ and $\nabla K(0,0) \neq 0$ (where K is the Gaussian curvature of the surface) then M_k is $L^p \to L^{p'}$ bounded if*

$$k > max\{6(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, \frac{14}{3}(\frac{1}{p} - \frac{1}{2})\}$$

provided the cone neighborhood Γ is chosen sufficiently small.

Remark. Actually a solution of the problem posed by M. Sugimoto [26] follows from the more general Theorem 3.5. If $\delta_1 = 2$ then the surface satisfies the conditions of Theorem 3.5. A proof of Theorem 3.5 follows from the following results of the paper [13].

Theorem 3.6. [13] *Let (S_a), $S_a \subset R^{n+1}$, be a family of analytic hyper-surfaces depending on a parameter $a \in \Sigma$ (where $\Sigma \subset R^m$ is a compact set). If for some fixed $a = a_0$ and for any $x \in S_{a_0} \cap Supp(d\mu_a)$ the relation $|K(a_0, x)| + |\nabla K(a_0, x)| \neq 0$, holds, then the maximal function*

$$M_{a_0}(\omega) : sup_{r>0} r^{n/2} |\widehat{d\mu_{a_0}}(r\omega)|$$

belongs to $L^{4-0}(S^n) := \cap_{p<4} L^p(S^n)$. Moreover, there exists a neighborhood V of a_0 such that for any fixed $p < 4$ the integral

$$\int_{S^n} M_a^p d\omega$$

is bounded on $V \cap \Sigma$. The result is sharp in the sense that if the hyper-surface S satisfies the conditions: $K(x_0) = 0$ and $\nabla K(x_0) \neq 0$ at some point x_0, ψ is a smooth function supported in a sufficiently small neighborhood of x_0 and $\psi(x_0) \neq 0$, then the maximal function M corresponding to $\widehat{d\mu(r,\omega)}$ does not belong to $L^4(S^n)$.

Now, we show that if the function h is of type II and $\delta_1 = 2$ then the corresponding surface satisfies the condition of the Theorem 3.5. Actually, the following result holds true.

Proposition 3.11. *Let $S \subset R^3$ be a smooth surface containing the origin of R^3 given as the graph of the function h. If the surface S satisfies the conditions $K(0,0) = 0$ and $\nabla K(0,0) \neq 0$, then either the phase function $\Phi(x_1, x_2, s_1, s_2)$ is the R_+-universal deformation of A_2 type singularities, either the function h is of type II with $\delta_1 = 2$. Conversely, if $\Phi(x_1, x_2, s_1, s_2)$ is the R_+-universal deformation of A_2 type singularities or the function h is of type II with $\delta_1 = 2$ then for the graph of the function h is the surface satisfying the conditions $K(0,0) = 0$ and $\nabla K(0,0) \neq 0$.*

Conclusion.

The master thesis is devoted to solve problem, which $k > k(p)$ the following operator $M_k := F^{-1}e^{i\varphi(\xi)}a_k(\xi)F$ is a bounded operator from L^p to $L^{p'}$.

The following results are obtained:

1. The lower bound for k depends on the surface $\Sigma := \{\xi : \varphi(\xi) = 1\}$.
2. It is proved that behavior of the operator M_k is defined by a small neighborhood of the Gauss map.
3. It is found relation between M. Sugimoto classification of the phase function and singularities Theorem.
4. It is obtained estimate for k in terms of the sharp uniform bound of the measures supported on hyper surfaces Σ.
5. It is given a solution of the M. Sugimoto problem related to A_∞ type singularities.
6. It is given a solution of the M.Sugimoto problem on estimate for M_k for the case when $\delta_1 = 2$.
7. It is obtained result which improves theorem provedby M. Sugimoto for the case when phase function has A_{δ_0-1} type singularities.
8. The results can be applied to investigation of solutions to the Cauchy problem for the hyperbolic equations.

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