MINISTRY OF PUBLIC EDUCATION OF THE REPUBLIC OF UZBEKISTAN

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named after AJINIYAZ

Physics and Mathematics Faculty

Department of methods of teaching mathematics

FINAL QUALIFYING WORK

of Asalxon Sobirova Baxtiyor qizi, the student of 4 “a” group of methods of teaching mathematics
on the topic of

Classification of second order partial differential equation depended on two variables

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Content

Introduction

“***Education is rescuer***

power of the world”

I. Karimov

Mathematics is playing an ever more important role in the physical an biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as classical and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

Solving many problems of natural science and technology that described considering actions processes belong to Mathematics and Physical actions. For example: Unknown functions and their derivatives when we know dependent on connections of each functions then we may find solution of these functions. As like connections we may call differential equations.

As a conclusion I say a lot of things about my topic of Qualification Paper. During preparing it I have learned a lot of information depend on my topic. For example: How to use first order equation in our real life. The role of it is large place in natural sciences, physics and others. We may take examples belong to the role differential equation in nature.

Example: Speed of linear motion actions, cooling down of thing, action of bullet, reactive action and others.

Solving equation of science and technologies’ problems as well as general actions in nature depend on a function which belonged to many variables.

My qualification paper is about on topic “Classification of second order partial differential equation depended on two variables” consists of two chapter:

First chapter is about general information of differential equation. It is named “General information about differential equations”. First chapter consist of 6 subtopics. They are general information about differential equations, types of equation, first order differential equation general information, geometrical interpretation of differential equation, the linear equation, high order differential equation.

Second chapter is on topic “Second order partial differential equation depend on two variables. It consists of 4 subtopics. They are transformation of variables, characteristic lines and the classification, canonical form, Initial and boundary conditions.

Chapter I

1.1 General information about differential equations

Ordinary differential equations (*ODEs*) arise naturally whenever a rate of change of some entity is known. This may be the rate of increase of a population, the rate of change of velocity, or maybe even the rate at which soldiers die on a battlefield. ODEs describe such changes of *discrete entities*. Respectively, this may be the capita of a population, the velocity of a particle, or the size of a military force.

Partial differential equations (*PDEs*) are analogous to ODEs in that they involve rates of change; however, they differ in that they treat *continuous media*. For example, the cloth could just as well be considered to be some kind of continuous sheet. This approach would most likely lead to only 3 (maybe 4) partial differential equations, which would represent the entire continuous sheet, instead of a set of ODEs for each particle.

Many of the concepts of the previous section may be summarized in this example. We won't deal with the PDE just yet. Consider heat flow along a laterally insulated rod. Let's call the temperature of the rod u, and let $u = u(x,t)$, where t is time and x represents the position along the rod. To reemphasize, the temperature depends both on time and position along the rod, which is exactly what $u = u(x,t)$ says.

Let's say that the rod has unitless length 1, and that its **initial** temperature (again unitless) is known to be $u(x,0) = \sin(\pi x)$. This states the **initial condition**, which depends on x.

Let's also say that the temperature is somehow fixed to 0 0 at both ends of the rod, at $x = 0$ and at $x = 1$. This would result in $u(0,t) = u(1,t) = 0$, which specifies **boundary conditions**. The BCs state that *for all* t, u at $x = 0$ and $x = 1$

A PDE can be written to describe the situation. This and the IC/BCs form an initial boundary value problem (IBVP). The solution to this IBVP is (with a physical constant taken to be 1):

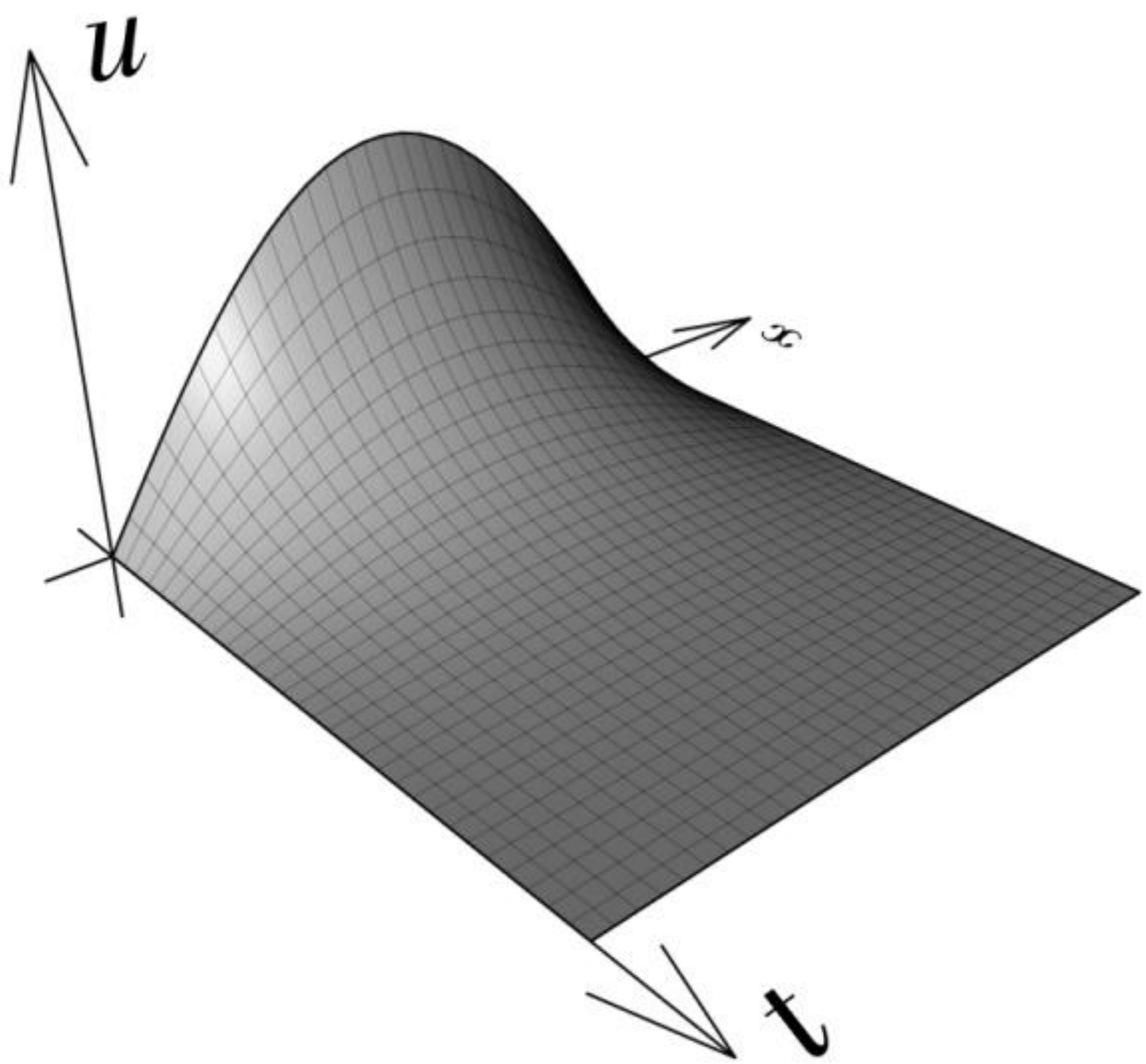
$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

Note that;

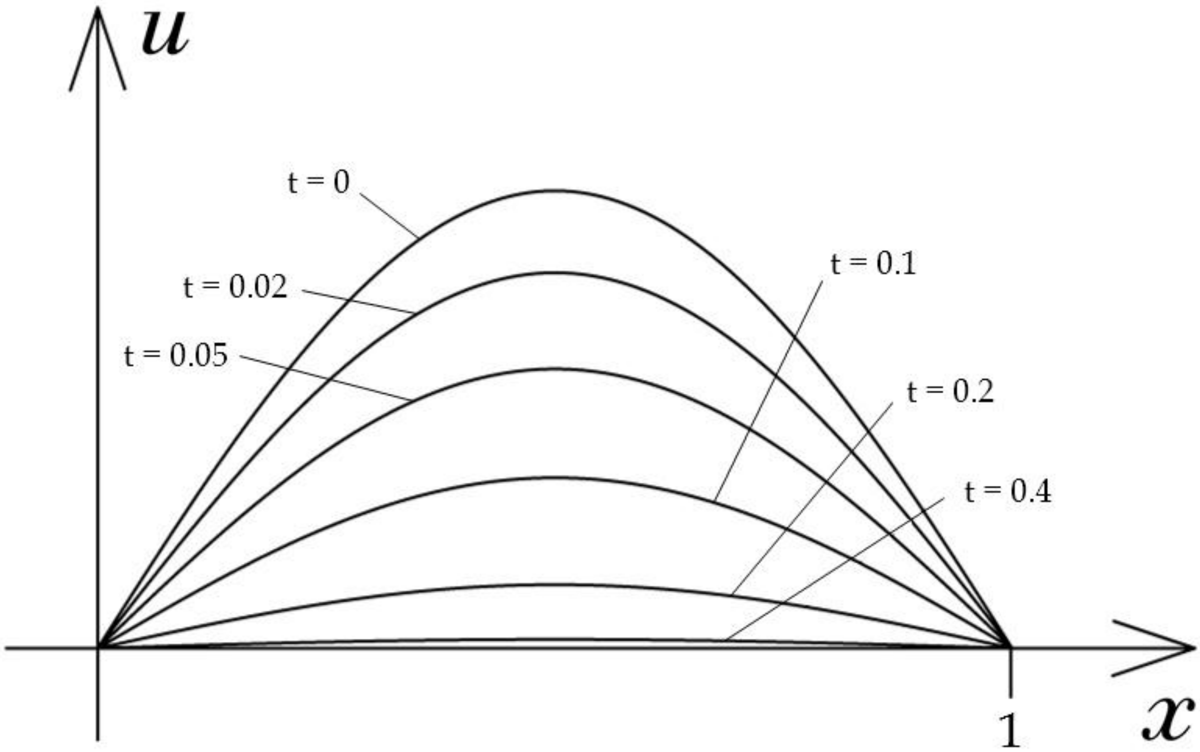
$$\begin{aligned} u(x,0) &= e^{-\pi^2 0} \sin(\pi x) = \sin(\pi x) \\ u(0,t) &= e^{-\pi^2 t} \sin(\pi \cdot 0) \\ u(1,t) &= e^{-\pi^2 t} \sin(\pi \cdot 1) \end{aligned}$$

It also satisfies the PDE, but (again) that'll come later.

This solution may be interpreted as a surface, it's shown in the figure below with *x* going from 0 to 1, and *t* going from 0 to $0{,}5$.



$u(x,t)$ from $t=0$ to $t=0{,}5$ and $x=0$ to $x=1$. Surfaces may or may not be the best way to convey information, and in this case a possibly better way to draw the picture would be to graph $u(x,t)$as a curve at several different choices of t, this is portrayed below.



$u(x,t)$ in the domain of interest for various interesting values of t.

PDEs are extremely diverse, and their ICs and BCs can radically affect their solution method. As a result, the best (read: easiest) way to learn is by looking at many different problems and how they're solved.

1.2Types of equation

We consider functions $u(x,t)$, defined by suitable partial differential equations; in the case of first-order equations, these are represented by the relation

$$f\left(u,x,y,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0.$$

We should note, at this stage, that we shall limit our discussion to functions of two variables, although some of the ideas go over to higher dimensions.

We start with linear, homogeneous equations that contain only derivative terms:

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=0,$$

and then extend the ideas to *quasi-linear* equations:

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c\;\; x,y,u$$

linear in the two derivatives, but otherwise nonlinear and inhomogeneous. The second-order equations that we discuss are linear in the highest derivatives, in the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+2b(x,y)\frac{\partial^2 u}{\partial y\partial x}+c\;\; x,y\;\; \frac{\partial^2 u}{\partial y^2}=d\left(x,y,\mathrm{u},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right);$$

this is the *semi-linear* equation. (The equation in which a,b and c also depend on u and its two first partial derivatives is the *quasi-linear* equation, which will not be discussed here, although many of the principles that we develop work equally well in this case.) Examples of the three types mentioned above are

$$x\frac{\partial u}{\partial x}+(x+y)\frac{\partial u}{\partial y}=0;$$
$$x\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=u;$$
$$y\frac{\partial^2 u}{\partial x^2}+(x+y)\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=xu+\frac{\partial u}{\partial x}+y\left(\frac{\partial u}{\partial y}\right)^2,$$

respectively.

1.3. First order differential equation general information

The right side of first order equations is might have been depend on x, y and y' so that the general form of the first order differential equation is as following:

$$F(x,y,y')=0 \qquad (1.3.1)$$

Usually (1.3.1) equation is solved according to derivative form

$$y' = f(x,y) \quad (1.3.2)$$

or the participation of differentials are tried to expressed as

$$M(x,y)dx + N(x,y)dy = 0 \quad (1.3.3)$$

It is easy to change the (1.3.2) form to (1.3.3) form or oppositely. Actually, if we exchange y' with $\frac{dy}{dx}$ in (1.3.2) equation and both sides of it multiply to dx and if we move all bounds to one side, we create as following:

$$f(x,y)dx - dy = 0 \ .$$

This is (1.3.3) itself, here $M(x,y) = f(x,y), N(x,y) = -1$ oppositely, if we exchange the first bound of equation to right side and suppose it as $N(x,y) \neq 0$ and if we divide second sides of equation into $N(x,y)dx$, we create $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ the (1.3.2) form , here

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}.$$

so, the (1.3.2) and (1.3.3) form are completely equivalent; next time we will use very position which is comfortable among them.

For distinguishing the functions, we need terms such as $x = x_0$ or $y = y_0$. This is called initial term. It is expressed as following:

$$y(x_0) = y_0 \quad (1.3.4)$$

The (1.3.4) equality is private solution of (1.3.2) equation.

Example:

$x^2y' + 3y = 0$ is an equation as (1.3.1) form,

$y' = -\frac{3y}{x^2}$ is like (1.3.2) equation,

$x^2dy + 3ydx = 0$ equation is as (1.3.3).

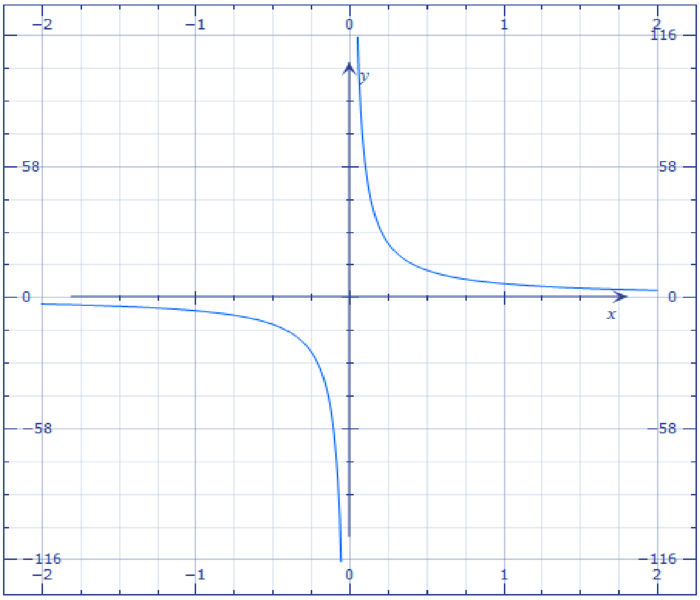
Example: $xy' + y = 0$

$$x\frac{dy}{dx} = -y \Rightarrow xdy = -ydx \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$
$$\int \frac{dy}{y} = \int \left(-\frac{dx}{x}\right) \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} + \ln c;$$
$$\ln|y| = -\ln|x| + \ln c \Rightarrow \ln|y| = \ln\frac{c}{|x|}.$$

Here $\ln|y| = \ln\frac{c}{|x|}$ is general solution.

One of the private solution of equation is $xy = 6 \Rightarrow y = \frac{6}{x}$.

$$y = \frac{6}{x}.$$



Here the initial term is as following:

$$y(x_0) = y_0 : y(3) = 2 \ (x_0 = 3, y_0 = 2)$$

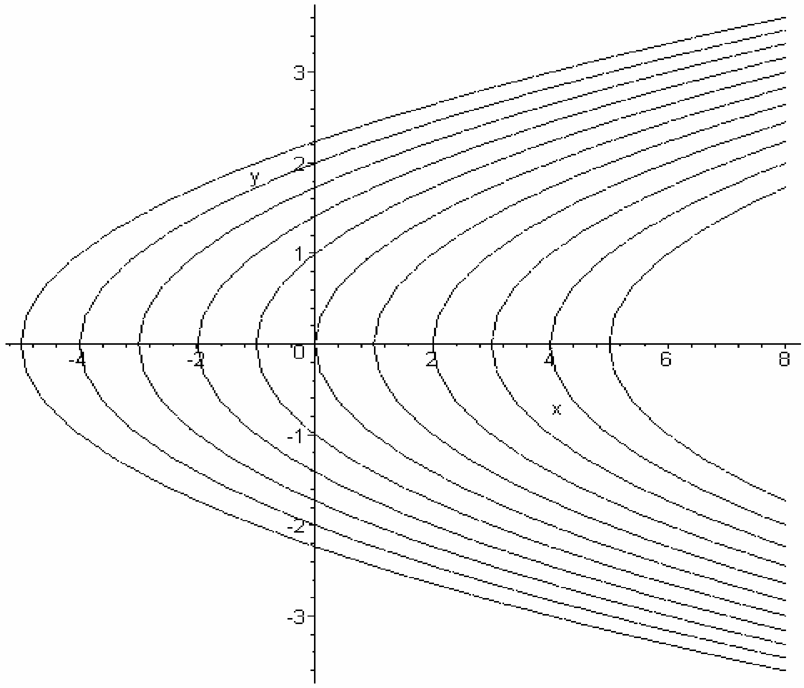
Description: *If C constant is changed depend on $y(x_0) = y_0$ and the function that dependent on C constant $y = \varphi(x, C)$ is called general solution.*

So we say about first-order equations depend on two variables.

The fundamental idea that we exploit is best introduced *via* a couple of elementary functions. First, let us suppose that we are given any function, $f(x)$, then it is obvious (and apparently of no significance) that $f(x)$ is constant whenever x is constant; this is usually described by stating that f is constant on lines $x=$ constant. Expressed like this, we are simply re-interpreting the description in terms of the conventional rectangular Cartesian coordinate system: $y = f(x)$, and then $y = f(x) =$ constant when $x =$ constant. To take this further, let us now suppose that we have a function of two variables, $f(x,y)$, but one that depends on a specific combination of x and y.

$$f(x,y) = \sin(x - y^2).$$

This function changes as x and y vary (independently), but it has the property that $f =$ constant on lines $x - y^2 =$ constant; these lines are shown below.



The function takes (in general different) constant values on each line. It is quite apparent that this interpretation of the function provides more information than simply to record that it is some function of the two variables. But we may take this still further.

The conventional Cartesian coordinate axes, and the lines parallel to them, are defined by lines $x =$ constant and $y =$ constant, which is usually regarded as sufficient and appropriate for describing functions $f(x,y)$. However, functions such as our example above, $\sin(x - y^2)$, are better described by lines $x - y^2 =$ constant (perhaps together with either $x =$ constant or $y =$ constant). These should then be the appropriate lines to use (in place of $x =$ constant, $y =$ constant); these 'coordinate' lines are those reproduced in the figure above. The essence of our approach to solving partial differential equations is to find these special coordinate lines, usually called *characteristic lines*.

1.4 Geometrical Interpretation of differential equation

To investigate the geometrical content of a first-order, partial differential equation, we begin with a general, quasi-linear equation. To investigate the geometrical content of a first-order, partial differential equation, we begin with a general, quasi-linear equation

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \quad (1.5.1)$$

We assume that the possible solution of (1) in the form $u = u(x,y)$ or in an implicit form

$$f(x,y,u) = u(x,y) - u = 0 \qquad (1.5.2)$$

represents a possible solution surface in (x,y,u) space. This is often called an integral surface of the equation (1.5.1). At any point (x,y,u) on the solution surface, the gradient vector $\nabla f = (f_x, f_y, f_u) = (u_x, u_y, -1)$ is normal to the solution surface. Clearly, equation (1.5.1) can be written as the dot product of two vectors

$$au_x + bu_y - c = (a,b,c)(u_x u_y - 1) = 0. \quad (1.5.3)$$

This clearly shows that the vector $(a,\mathrm{b},\mathrm{c})$ must be a tangent vector of the integral surface (1.5.2) at the point (x,y,u) and hence, it determines a direction field called the the *characteristic direction or Monge axis*. This direction is of fundamental importance in determining a solution of equation (1.5.1). To summarize, we have shown that $f(x,y,u) = u(x,y) - u = 0$, as a surface in the (x,y,u)-space, is a solution of (1.5.1) if and only if the direction vector field

(a, b, c) lies in the tangent plane of the integral surface $f(x, y, u) = 0$ at each point (x, y, u), where $\nabla f \neq 0$ as shown in Figure1.

A curve in (x, y, u)-space, whose tangent at every point coincides with the characteristic direction field (a, b, c), is called a *characteristic curve*. If the parametric equations of this characteristic curve are

$$x = x(t), \quad y = y(t), \quad u = u(t), \quad (1.5.4)$$

then the tangent vector to this curve is $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$ which must be equal to (a, b, c). Therefore, the system of ordinary differential equations of the characteristic curve is given by

$$\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = c(x, y, c) \qquad (1.5.5)$$

These are called the characteristic equations of the quasi-linear equation (1.5.1).

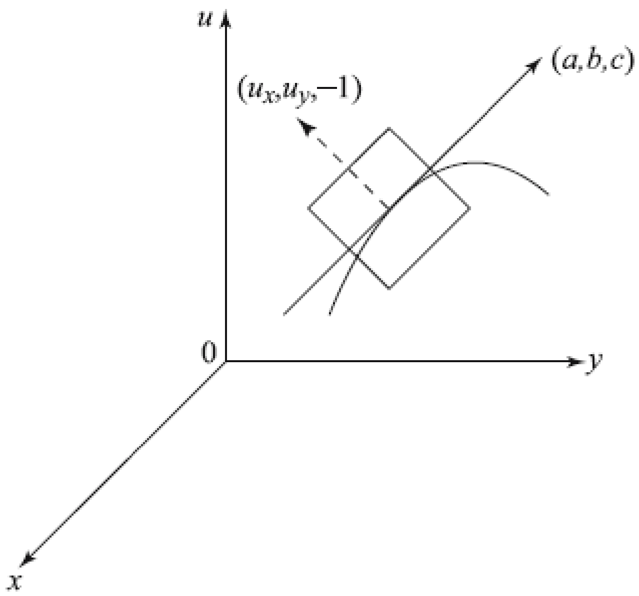


Figure1. Tangent and normal vector fields of solution surface at a point (x, y, u).

In fact, there are only two independent ordinary differential equations in the system (1.5.5); therefore, its solutions consist of a two-parameter family of curves in (x, y, u)-space.

The projection on $u=0$ of a characteristic curve on the (x,t) -plane is called a characteristic base curve or simply characteristic.

Equivalently, the characteristic equations (1.5.5) in the nonparametric form are

$$\frac{dx}{a}=\frac{dy}{b}=\frac{du}{c}. \qquad (1.5.6)$$

The typical problem of solving equation (1.5.1) with a prescribed u on a given plane curve C is equivalent to finding an integral surface in (x,y,u) space, satisfying the equation (1.5.1) and containing the three-dimensional space curve G defined by the values of u on C, which is the projection on $u=0$ of G.

Remark1. The above geometrical interpretation can be generalized for higher-order partial differential equations. However, it is not easy to visualize geometrical arguments that have been described for the case of three space dimensions.

Remark2. The geometrical interpretation is more complicated for the case of nonlinear partial differential equations, because the normal to possible solution surfaces through a point do not lie in a plane.

We conclude this section by adding an important observation regarding the nature of the characteristics in the (x,t)-plane. For a quasi-linear equation, characteristics are determined by the first two equations in (1.5.5) with their slopes

$$\frac{dy}{dx}=\frac{b(x,y,u)}{a(x,y,u)}. \qquad (1.5.7)$$

If (1) is a linear equation, then a and b are independent of u, and the characteristics of (1) are plane curves with slopes

$$\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)} \qquad (1.5.8)$$

By integrating this equation, we can determine the characteristics which represent a one-parameter family of curves in the (x,t)-plane. However, if a and b are constant, the characteristics of equation (1.5.1) are straight lines.

We can use the geometrical interpretation of first-order, partial differential equations and the properties of characteristic curves to develop a method for finding the general solution of quasi-linear equations. This is usually referred to as the method of characteristics due to Lagrange. This method of solution of quasi-linear equations can be described by the following result.

Theorem 1. The general solution of a first-order, quasi-linear partial differential equation

$$a(x,y,u)u_x + b(x,y,u) = c(x,y,u) \quad (1.5.9)$$

is

$$f(\phi,\psi) = 0 \quad (1.5.10)$$

where f is an arbitrary function of $\varphi(x,y,u)$ and $\psi(x,y,u)$, and φ =constant = c_1 and ψ = constant = c_2 are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \quad (1.5.6)$$

The solution curves defined by $\varphi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ are called the families of characteristic curves of equation (1.5.9).

Proof. Since $\phi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ satisfy equations (1.5.6), these equations must be compatible with the equation

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0 \quad (1.5.11)$$

This is equivalent to the equation

$$a\phi_x + b\phi_y + c\phi_u = 0 \quad (1.5.12)$$

Similarly, equation (6) is also compatible with

$$a\psi_x + b\psi_y + c\psi_u = 0 \quad (1.5.13)$$

We now solve (1.5.12), (1.5.13) for a, b and c to obtain

$$\frac{dx}{\dfrac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{dy}{\dfrac{\partial(\phi,\psi)}{\partial(u.x)}} = \frac{du}{\dfrac{\partial(\phi,\psi)}{\partial(x,y)}}. \quad (1.5.14)$$

It has been shown earlier that $f(\varphi,\psi) = 0$ satisfies an equation similar to

$$p\frac{\partial(\phi,\psi)}{\partial(y,z)}+q\frac{\partial(\phi,\psi)}{\partial(z,x)}=\frac{\partial(\phi,\psi)}{\partial(x,y)}. \qquad (1.5.15)$$

Substituting, (1.5.14) in (1.5.15), we find that $f(\varphi,\psi)=0$ is a solution of (1.5.9). This completes the proof.

Note that an analytical method has been used to prove *Theorem 1*. Alternatively, a geometrical argument can be used to prove this theorem. The geometrical method of proof is left to the reader as an exercise. Many problems in applied mathematics, science, and engineering involve partial differential equations. We seldom try to find or discuss the properties of a solution to these equations in its most general form. In most cases of interest, we deal with those solutions of partial differential equations which satisfy certain supplementary conditions. In the case of a first-order partial differential equation, we determine the specific solution by formulating an *initial-value problem* or a *Cauchy problem.*

Theorem 2. (*The Cauchy problem for a first-order partial differential equation*). Suppose that C is a given curve in the (x,y)-plane with its parametric equations

$$x=x_0(t), \qquad y=y_0(\text{t}), \qquad (1.5.16)$$

where t belongs to an interval $I\subset R$, and the derivatives $x_0'(t)$and $y_0'(t)$ are piecewise continuous functions, such that $(x_0')^2+(y_0')^2\neq 0.$ Also, suppose that $u=u_0(t)$ is a given function on the curve C. Then, there exists a solution $u=u(x,y)$ of the equation

$$F(x,y,u,u_x,u_y)=0 \qquad (1.5.17)$$

in a domain D of R^2 containing the curve C for all $t\in I$, and the solution $u(x,y)$ satisfies the given initial data, that is,

$$u(x_0(t),y_0(t))=u_0(t) \qquad (1.5.18)$$

for all values of $t\in I$.

In short, the *Cauchy problem* is to determine a solution of equation (1.5.17) in a neighborhood of C, such that the solution $u = u(x,y)$ takes a prescribed value $u_0(t)$ on C. The curve C is called the *initial curve* of the problem, and $u_0(t)$ is called the initial data. Equation (1.5.18) is called the *initial condition* of the problem.

The solution of the Cauchy problem also deals with such questions as the conditions on the functions $F, x_0(t), y_0(t)$ and $u_0(t)$ under which a solution exists and is unique.

We next discuss a method for solving a Cauchy problem for the first order, quasi-linear equation (1.5.9). We first observe that geometrically $x = x_0(t)$, $y = y_0(t)$, and $u = u_0(t)$ represent an initial curve G in (x,y,u)-space. The curve C, on which the Cauchy data is prescribed, is the projection of G on the (x,y)-plane. We now present a precise formulation of the Cauchy problem for the first-order, quasi-linear equation (1.5.9).

Theorem 3 (*The Cauchy Problem for a Quasi-linear Equation*). Suppose that $x_0(t), y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in a closed interval, $0 \le t \le 1$, and that a, b and c are functions of x, y, and u with continuous first-order partial derivatives with respect to their arguments in some domain D of (x,y,u) -space containing the initial curve

$$\text{G}: x = x_0(t), \quad y = y_0(t), \quad u = u_0(t), \qquad (1.5.19)$$

where $0 \le t \le 1$ and satisfying the condition

$$y'_0(t)a(x_0(t), y_0(t), u_0(t)) - x'_0(t)b(x_0(t), y_0(t), u_0(t)) \neq 0 \qquad (1.5.20)$$

Then there exists a unique solution $u = u(x,y)$ of the quasi-linear equation (1.5.9) in the neighborhood of $C: x = x_0(t), y = y_0(t)$, and the solution satisfies the initial condition

$$u_0(t) = u(x_0(t), y_0(t)), \text{ for } 0 \le t \le 1 \qquad (1.5.21)$$

Note: The condition (1.5.20) excludes the possibility that C could be a characteristic.

Example: Find the general solution of the first-order linear partial differential equation.

$$xu_x + yu_y = u \qquad (E_1^1)$$

The integral surfaces are the solutions of the characteristic equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}. \qquad (E_2^1)$$

This system of equations gives the integral surfaces

$$\varphi = \frac{y}{x} = C_1 \text{ and} \psi = \frac{u}{x} = C_2,$$

where C_1 and C_2 are arbitrary constants. This, the general solution of

(1) is $f\left(\frac{y}{x}, \frac{u}{x}\right) = 0, \qquad (E_3^1)$

$$u(x,y) = xg\left(\frac{y}{x}\right), \qquad (E_4^1)$$

where g is an arbitrary function.

Example: Obtain the general solution of the linear Euler equation

$$xu_x + yu_y = nu. \qquad (E_1^2)$$

The integral surfaces are the solutions of the characteristic equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}. \qquad (E_2^2)$$

From these equations, we get

$$\frac{y}{x} = C_1, \quad \frac{u}{x^n} = C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, the general solution of (1) is

$$f\left(\frac{y}{x}, \frac{u}{x^n}\right) = 0. \qquad (E_3^2)$$

This can also be written as

$$\frac{u}{x^n} = g\left(\frac{y}{x}\right)$$

or

$$u(x,y) = x^n g\left(\frac{y}{x}\right). \qquad (\mathrm{E}_4^2)$$

This shows that the solution *u (x, y)* is a homogeneous function of x and y of degree n.

Example: Find the general solution of the linear equation

$$x^2 u_x + y^2 u_y = (x+y)u. \qquad (\mathrm{E}_1^3)$$

The characteristic equations associated with (E_1^3) are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}. \qquad (\mathrm{E}_2^3)$$

From the first two of these equations, we find

$$\frac{1}{x} - \frac{1}{y} = C_1, \qquad (\mathrm{E}_3^3)$$

where C_1 is an arbitrary constant.

It follows from (E_3^3) that

$$\frac{dx - dy}{x^2 - y^2} = \frac{du}{(x+y)u}$$

or

$$\frac{d(x-y)}{x-y} = \frac{du}{u}.$$

This gives

$$\frac{x-y}{u} = C_2, \qquad (\mathrm{E}_4^3)$$

where C_2 is a constant. Furthermore, (E_3^3) and (E_4^3) also give

$$\frac{xy}{u} = C_3, \qquad (\mathrm{E}_5^3)$$

where C_3 is a constant.

Thus, the general solution (E_1^3) is given by

$$f\left(\frac{xy}{u}, \frac{x-y}{u}\right) = 0, \qquad (\mathrm{E}_6^3)$$

where g is an arbitrary function, or, equivalently,

$$u(x,y) = xy\, h\left(\frac{x-y}{xy}\right), \qquad (\mathrm{E}_7^3)$$

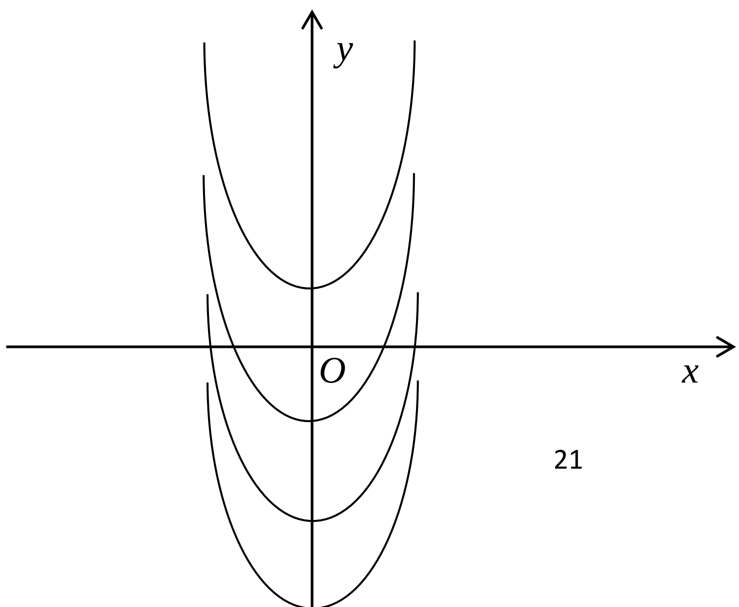
where h is an arbitrary function.

Example: $y' - 2x = 0$

$$y' - 2x \Rightarrow y' = 2x \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dy = 2xdx;$$

$$\int dy = \int 2xdx \Rightarrow \int dy = 2\int xdx \Rightarrow y = 2\left(\frac{x^2}{2} + \frac{C}{2}\right) \quad \Rightarrow y = x^2 + C$$

General solution is expressed in system of cordinate as following:

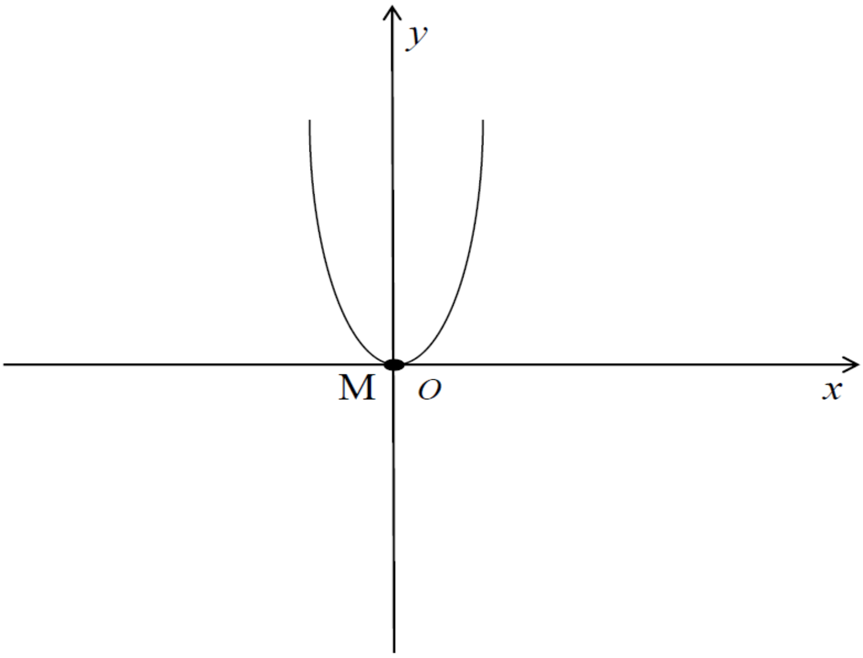


So, general solution of the first order differential equation depended on one variable usually is expressed curve in the plain. We may say that solution of the equation consists of family of initial function (or family of curve).

We find $C = 0$ by the following initial term:

$$x_0 = 0, y_0 = 0$$

At this position $y = x^2$. So, we try to paint graphic of $y = x^2$ in system of coordinate in plain. $y = x^2$ is expressed parabola which passed through $M(x_0, y_0)$ point.



So, in geometry general solution of $y' - 2x = 0$ means that the curves which dependent on ***C*** parameter. The private solution is consisted of curve which passed through the point $M_0(x_0 y_0)$.

1.5 The linear equation

Here we consider the equation

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=0.$$

Where the coefficients *a* and *b* will be assumed continuous throughout the domain (D) where the solution, $u(x,y)$ is to exist. To proceed, we seek a solution that depends on x (say) and $\xi=\xi(x,y)$; the solution is defined in 2-space so, in general, we must transform into some corresponding 2-space (that is, using two independent variables). The aim is to determine the function $\xi(x,y)$ so that the equation for u becomes sufficiently simple, allowing it to be integrated; indeed, we hope that this results in a solution that depends – essentially – on only one variable (namely, ξ). We note, in passing, that if we choose $\xi=y$, then we simply recover the original problem (which does confirm that a transformation exists).

Let us write, for clarity,

$$u(x,y)=U\left[x,\xi(x,y)\right]$$

then

$$\frac{\partial u}{\partial x}=\frac{\partial U}{\partial x}+\frac{\partial \xi}{\partial x}\frac{\partial U}{\partial \xi} \text{ and } \frac{\partial u}{\partial y}=\frac{\partial \xi}{\partial y}\frac{\partial U}{\partial \xi};$$

this the equation for u (now U) becomes

$$a(x,y)(U_x+\xi_x U_\xi)+b(x,y)\xi_y U_\xi=0,$$

where we have used subscripts to denote the partial derivatives. Now we choose $\xi(x,y)$ such that $a\xi_x+b\xi_y=0$ (and we note that is no more than the original partial differential equation!) which leaves the equation for U as simply $aU_x=0$, and so provided $a\neq 0$ throughout D, then

$$U_x=0 \text{ or } U=F(\xi) \text{ so on } u(x,y)=F\left[\xi(x,y)\right],$$

where F is an arbitrary function; this constitutes the general solution and confirms that we may, indeed, seek a solution that depends on one (specially chosen) variable. (It should be clear that arbitrary constants in the solution of ordinary differential equations go over to arbitrary functions in the solution of partial differential equations.) The function $\xi(x,y)$ is determined completely when we consider lines ξ = constant for then

$$\xi_x + \frac{dy}{dx}\xi_y = 0 \text{ or } \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{b(x,y)}{a(x,y)}.$$

Note that, since we now know that we may describe the solution as u = constant on certain lines, we may equally write directly that u = constant on lines $y' = \frac{b}{a}$, without the need to introduce ξ at all. Nevertheless, as we shall see, the introduction of characteristic lines is fundamental to any generalization of this technique. So the integration of the *ordinary* differential equation $y' = \frac{b(x,y)}{a(x,y)}$ yields the characteristic lines $\xi(x,y)$ = constant (this being the arbitrary constant of integration), and then $u = F(\xi)$ is the required general solution (which is equivalent, of course, to u = constant on lines $\xi(x,y)$ = constant).

Example: Find the general solution of the partial differential equation $yu_x + x^2u_y = 0$.

The characteristic lines are given by the solution of the ordinary differential equation

$$y' = \frac{x^2}{y}(y \neq 0) \text{ and so } \frac{1}{2}y^2 = \frac{1}{3}x^3 + \text{constant},$$

or $\xi(x,y) = 3y^2 - 2x^3$ = constant , which describes the characteristic lines. This the general solution is

$$u(x,y) = F(3y^2 - 2x^3),$$

where F is an arbitrary function.

Comment: We can check this solution directly (at least, if F is a differentiable function), for then $u_x = -6x^2F'(\xi)$ and $u_y = 6yF'(\xi)$, so that $yu_x + x^2u_y = -6yx^2F' + 6yx^2F' = 0$, and observe that this does not require the condition $y \neq 0$.

The next issue that we must address is how the arbitrary function, F is determined in order to produce – we hope a unique solution of a particular problem.

1.6 High order differential equation

$F(y^{(n)}, y^{(n-1)}, y^{(n-2)}, ..., y', y, x) = 0$ is called high order differential equation.

It is may be expressed as following:

$$y^{(n)} = f(x, y, y', ..., y^{(n-2)}, y^{(n-1)}).$$

Solving of high order differential equation is difficult than first order differential equation. But high order differential equation often solves as first order differential equation.

We may decrease its order as following:

$y'' = (y')', y''' = (y'')', ..., y^{(n)} = (y^{(n-1)})'$

Example:

$$y^{(n)} = f(x) \Rightarrow y^{(n-1)} = \int f(x)dx + C_1, ...$$

Example: Find the general solution of $y'' - xy' - y = 0$ (1).

Solving: We know $y'' - y'x - y = (y' - xy)' = 0$. So solving of $y' - xy = C$ (2) is enough for us. $y' - xy = C$ is linear equation. At this position we change *y*. We use uv instead of y.

$$y = uv. \ (3)$$

So we need y'.

$$y' = u'v + v'u. \ (4)$$

We put (3) and (4) on (2)

$$u'v + u(v' - xv) = C.$$

At this position we take $v' - vx = 0$ and find v

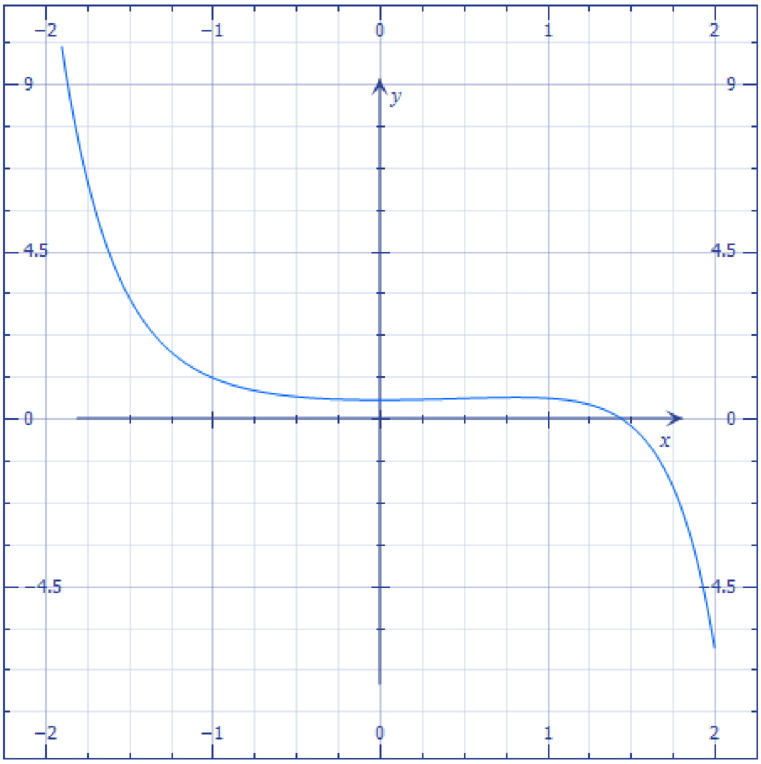
$$v' - vx = 0 \Rightarrow \frac{dv}{dx} = vx \Rightarrow \frac{dv}{v} = xdx, \ln v = \frac{x^2}{2} \Rightarrow v = e^{\frac{x^2}{2}}$$

$$u'v = C \Rightarrow u'e^{\frac{x^2}{2}} = C \Rightarrow u' = Ce^{-\frac{x^2}{2}}, u = C\int\left(-\frac{x^2}{2}\right)dx + C_1,$$

$$u = -\frac{1}{2}C\left(\frac{x^3}{3} + C_2\right) + C_1$$

$$y = uv = e^{\frac{x^2}{2}} \cdot \left(-\frac{1}{2}C\left(\frac{x^3}{3} + C_2\right) + C_1\right). \qquad (5)$$

(5) expression is general solution of (1).$y = e^{\frac{x^2}{2}}\left(-\frac{1}{2}C\left(\frac{x^3}{3} + C_2\right) + C_1\right)$



This figure is expressed one of solution of the equation.

Chapter II

2.1. Transformation of variables

The general equation that we consider here is

$$a(x,y)\,u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$$

where *a*, *b*, *c* and *d* are given; we have written partial derivatives using the subscript notation. The basic procedure follows that which was so successful for the wave equation, namely, to find a suitable transformation of variables. This will necessitate the consideration of three cases, which leads to the essential classification of these equations and then to the standard (*canonical*) versions of the equation.

Although we eventually require the solution $u(x,y)$, we represent this in the form

$$u(x,y) = U\left[\,\xi(x,y), \eta(x,y)\right]$$

for suitable choices of the new coordinates

$$\xi(x,y)=\text{constant},\ \eta(x,y)=\text{constant},$$

which replace the conventional Cartesian set: $x =$ constant , $y =$ constant. This we have, for example, which replace the conventional Cartesian set: $x =$ constant, y =constant. This we have, for example,

$$u_x = \xi_x U_\xi + \eta_x U_\eta \,; u_y = \xi_y U_\xi + \eta_y U_\eta,$$

and then U_ξ and U_η exist provided that the *Jacobian* $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$ (and note that the choice $\xi = x, \eta = y$ – which is no transformation at all, of course – generates *J* = 1, so some ξ, η certainly do exist). [K.G.J. Jacobi, 1804-1851, German mathematician, who did much to further the theory of elliptic functions.] However, we also require second partial derivatives; for example, expressed as differential operators, we have

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right) = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)$$

and we may choose to use either the first version, or the second, or a mixture of the two. In particular, we elect to use the former when we differentiate ξ_x and η_x, but the latter when we operate on $\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial \eta}$; the result is

$$\frac{\partial^2}{\partial x^2} = \xi_{xx} \frac{\partial}{\partial \xi} + \eta_{xx} \frac{\partial}{\partial \eta} + \xi_x^2 \frac{\partial^2}{\partial x^2} + 2\xi_x \eta_x \frac{\partial^2}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2}{\partial \eta^2};$$

there are corresponding results for $\frac{\partial^2}{\partial x \partial y}$ and $\frac{\partial^2}{\partial y^2}$. Our original equation now takes the form

$$AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} = D(\xi, \eta, U, U_\xi, U_\eta),$$

where the coefficients on the left-hand side are given by

$$A = a\xi_x^2 + 2b\xi_x \xi_y + c\xi_x \,; B = a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + c\xi_y \eta_y \,;$$
$$C = a\eta_x^2 + 2b\eta_x \eta_y + c\eta_y^2,$$

and D is a combination of d evaluated according to the transformation and the first derivative terms that arise from the transformation used on the left-hand side. The first observation that we make concerns the coefficients A, B and C; in particular, we form $B^2 - AC$ (which, as we shall see shortly, naturally arises – or a version of it – in what we do later). This gives

$$
\begin{aligned}
&B^2 - AC = \left[a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\right]^2 - \\
&-(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2) = \\
&= a^2(\xi_x^2\eta_x^2 - \xi_x^2\eta_x^2) + b^2\left[\ \xi_x\eta_y + \xi_y\eta_x\ {}^{2} - 4\xi_x\xi_y\eta_x\eta_y\right] + \\
&+c^2(\xi_y^2\eta_y^2 - \xi_y^2\eta_y^2) + ac(2\xi_x\xi_y\eta_x\eta_y - \xi_x^2\eta_y^2 - \xi_y^2\eta_x^2) + \\
&+ab\left[2\xi_x\eta_x(\xi_x\eta_y + \xi_y\eta_x) - 2\xi_x^2\eta_x\eta_y - 2\eta_x^2\xi_x\xi_y\right] + \\
&+bc\left[2\xi_y\eta_y(\xi_x\eta_y + \xi_y\eta_x - 2\xi_y^2\eta_x\eta_y - 2\eta_y^2\xi_x\xi_y\right] = \\
&= b^2(\xi_x\eta_y - \xi_y\eta_x)^2 - ac(\xi_x\eta_y - \xi_y\eta_x)^2 = \ b^2 - ac\ \ J^2
\end{aligned}
$$

where $J = \xi_x\eta_y - \xi_y\eta_x$ is the Jacobian introduced above. For the transformation from (x, y) to (ξ, η) to exist, we must have $J \neq 0$, and then the sign of $B^2 - AC$ is identical to the sign of $b^2 - ac$ (which uses the coefficients given in the original equation). Thus, no matter what (valid) transformation we choose to use, the sign of $B^2 - AC$ is controlled by that of $b^2 - ac$, and this suggests that this property of $b^2 - ac$ is fundamental to the construction of a solution; the intimate connection with the method of solution will be demonstrated in the next section.

2.2 Characteristic lines and the classification

Let us address the question of how to choose the new coordinates, ξ and η; lines $\xi(x, y)$ = constant imply that on them

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

(and correspondingly $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$ on lines $\eta(x,y) =$ constant). With these functions, we may write

$$A = \xi_y^2 \left[a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c \right] \text{ and } C = \eta_y^2 \left[a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c \right],$$

and then both A and C are zero if we elect to use as the definition of the characteristic lines

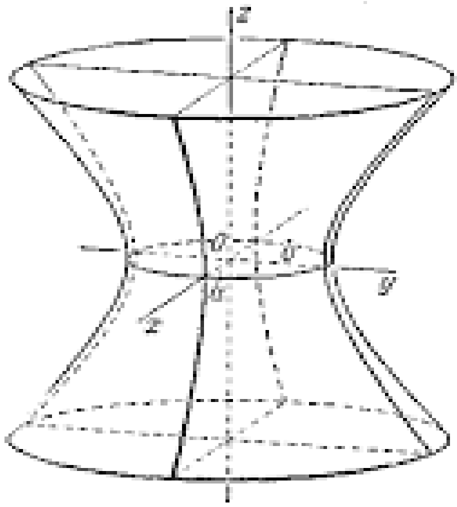
$$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{a}\ b \pm \sqrt{b^2 - ac}\ .$$

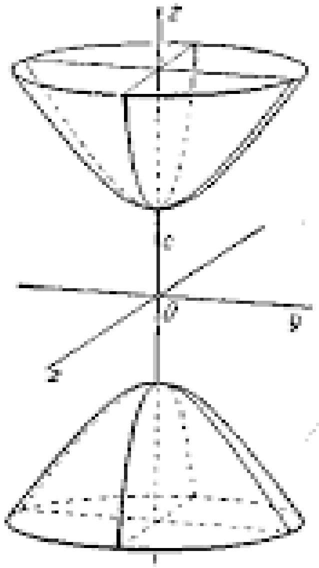
This if $b^2 > ac$ we have two real families of curves (defined by the solution of the ordinary differential equation) and we may identify one family as $\xi =$ constant and the other as $\eta =$ constant: we have determined a choice of ξ and η that simplifies the original equation – it now becomes simply

$$2BU_{\xi\eta} = D.$$

Further, it is clear that we have three cases: $b^2 > ac$, $b^2 = ac$ and $b^2 < ac$, and we should note that $b^2 - ac$ will, in general, vary over the (x,y)-plane, so there should be no expectation that it will remain single-signed. These three cases provide the classification.

I. $\Delta = b^2 - ac > 0$ **(hyperbolic)** usually solution of hyperbolic typical equation is expressed as the following figures:





This is the most straightforward case, as we have just seen. The characteristic lines, $\xi(x,y)$= constant and $\eta(x,y)$= constant, are defined by the two (real) solutions of the first-order equation

$$\frac{dy}{dx} = \frac{1}{a}\ \ b \pm \sqrt{b^2 - ac}\ \ ;$$

this is referred to as the *hyperbolic* case, and the partial differential equation is then said to be of hyperbolic type (a terminology that will be explained below).

Example1.

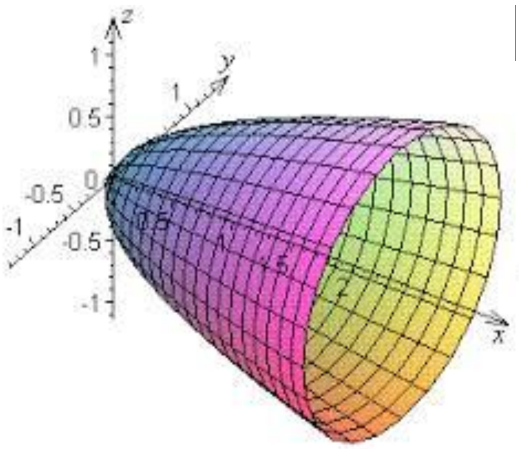
Find the characteristic lines of the wave equation

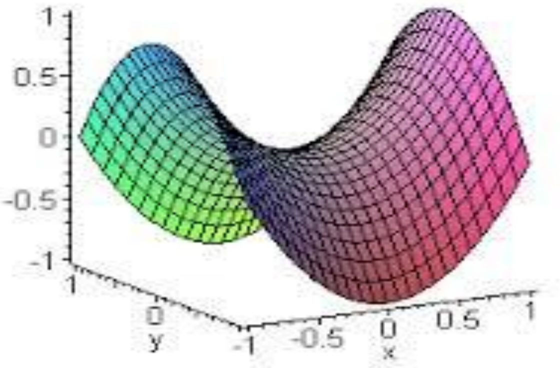
$$u_{xx} - k^2 u_{yy} = 0 \ \ (k > 0 \text{ , constant}).$$

Here we have $a = 1$, $b = 0$and $c = -k^2$, so that $b^2 - ac = k^2 > 0$ (and so the equation is hyperbolic everywhere); thus the characteristic lines are given by

$$\frac{dy}{dx} = \pm\sqrt{k^2} = \pm k \ \Rightarrow y \mp kx = \text{constant}$$

II. $b^2 - ac = 0$ **(parabolic)** usually solution of parabolic equation is expressed as the following figures:





We now have only one solution of the ordinary differential equation, because we have repeated roots; we call this the *parabolic* case. To proceed, we choose one characteristic, ξ say, which is defined by the solution of $y' = \dfrac{b}{a}$; the other is defined in any suitable way, provided that it is independent of the family $\xi(x, y) =$ constant i.e. it results in $J \neq 0$. Typically, the choice $\eta = x$ is made, although other choices may be convenient for particular equations.

Example 2.

Find the characteristic lines for the heat conduction (diffusion) equation

$$u_y = ku_{xx}$$

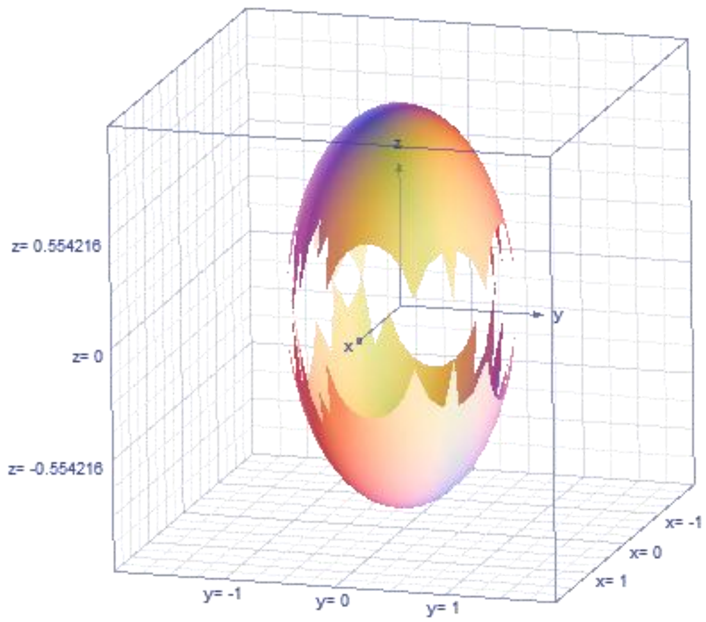
($k > 0$, constant).

First write the equation as $ku_{xx} = u_y$, then we identify $a = k, b = c = 0$ which gives $b^2 - ac = 0$, so parabolic everywhere. This $y' = 0$; we may use $\xi = y =$ constant with $\eta = x =$ constant, which is no transformation at all. This original equation is already, as one of parabolic type, written in its simplest form.

Example: Find the characteristic equation $xyu_{xx} + 4x^2yu_{xy} + 4x^3yu_{yy} = u$

Solving: We know $a = xy, b = 2x^2y, c = 4x^3y$ and $\dfrac{dy}{dx} = \dfrac{1}{a}$

III. $b^2 - ac < 0$ **(elliptic)** usually solution of elliptical typical equation is expressed as the following figure:



This case presents us – or so it would appear – with a much more difficult situation: the equation defining the characteristic lines is no longer real, so we might hazard that no transformation exists in this case. It is clear that, because we have the identity $B^2 - AC = \; b^2 - ac \;\; J^2 < 0$, then A and C must have the same sign and cannot be zero; this we choose to define the transformation to produce $A = C$ and $B = 0$.

$$A - C = a \;\; \xi_x^2 - \eta_x^2 \;\; + 2b(\xi_x \xi_y - \eta_x \eta y) + c(\xi_y^2 - \eta_y^2) = 0 \;\; ,$$
$$B = a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + \mathrm{c}\,\xi_y \eta_y$$

Let us define the complex quantity $\lambda = \xi + i\eta$, then we have $\lambda_x = \xi_x + i\eta_x$ and $\lambda_y = \xi_y + i\eta_y$ and so

$$a\lambda_x^2 + 2b\lambda_x\lambda_y + c\lambda_y^2 =$$
$$= a(\xi_x^2 - \eta_x^2) + 2b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) +$$
$$+ 2i(a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y) = 0$$

this lines $\lambda(x,y)$= constant are exactly as before: solutions of

$$\frac{dy}{dx} = \frac{1}{a}\ \ b \pm i\sqrt{ac - b^2}\ \ .$$

However, the solution of this differential equation is necessarily complex-valued (called the *elliptic* case), so we write this as $\lambda(x,y) = \xi(x,y) + i\eta(x,y) = \alpha + i\beta$ = constant where $\alpha + i\beta$ is a complex constant. This choice of the new coordinates is given by $\xi(x,y) = \alpha$= constant and $\eta(x,y) = \beta$= constant (both real!) i.e. we follow the procedure used in the hyperbolic case, but here we apply the principle to the real and imaginary parts separately. So there is a transformation, even though the characteristic lines, defined by the ordinary differential equation, are certainly not real.

Example 3.

Find the characteristic lines for Laplace's equation: $u_{xx} + u_{yy} = 0$. Here we have $a = c = 1$ and $b = 0$, so $b^2 - ac = -1 < 0$ elliptic everywhere, and then $y' = \pm i$ or $y \mp ix$= constant. This we may choose the transformation $\xi = y$ and $\eta = x$ (or *vice versa*); as in the previous example, this is no transformation – the Laplace equation is already in its simplest form.

The simple results obtained in the last two examples lead naturally to the notion of the canonical form.

2.3 Canonical form

The general equation, following a general transformation, is

$$AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} = D$$

and then the three cases give

I. Hyperbolic $A = C = 0$: $2BU_{\xi\eta} = D$;

II. Parabolic (e.g. $A = 0$, then $B^2 - ac = 0 \Rightarrow B = 0$): $CU_{\eta\eta} = D$;

III. Elliptic ($A = C, B = 0$): $A(U_{\xi\xi} + U_{\eta\eta}) = D$.

These constitute the *canonical* forms (and so we confirm that both $ku_{yy} = u_x$ and $u_{xx} + u_{yy} = 0$, Examples 2 and 3, are already in this form). Here, we use the word 'canonical' in the sense of 'standard' or 'accepted'. The terminology (hyperbolic, parabolic, elliptic) as applied to the classification of partial differential equations, was introduced in 1889 by Paul du Bois-Reymond (1831-1889, French mathematician) because he interpreted the underlying differential equation

$$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$$

as being associated with the algebraic form

$$ay^2 - 2bxy + cx^2 = \text{terms linear in } x \text{ and } y.$$

Then $a = c = 0$ gives e.g. $xy =$ constant, the rectangular hyperbola; $a = b = 0$ gives e.g. $x^2 = y$, a parabola; $b = 0$ gives e.g. $x^2 + k^2y^2 =$ constant, an ellipse (and a circle if $a = c$).The construction of the canonical form, via the appropriate characteristic variables, will be explored in three further examples (and then we will briefly examine a few specific and relevant applications of these ideas).

Example 4.

Show that the equation $y^2u_{xx} - 4x^2u_{yy} = 0$ is of hyperbolic type (for $x \neq 0, y \neq 0$), find the characteristic variables and hence write the equation in canonical form.

We have $b^2 - ac = 4x^2y^2 > 0$ for $x \neq 0, y \neq 0$, so everywhere else the equation is of hyperbolic type. The characteristic lines, where the equation is hyperbolic, are given by the solution of the equation $y' = \pm\frac{\sqrt{4x^2y^2}}{y^2} = \pm\frac{2x}{y}$

(and note that $y = 0$ must be avoided here, anyway) so that $y^2 \mp 2x^2 =$ constant; we set $\xi = y^2 - 2x^2$ and $\eta = y^2 + 2x^2$, to give

$$\frac{\partial}{\partial x} = 4x\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \text{ and } \frac{\partial}{\partial y} = 2y\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right).$$

Then we obtain

$$\frac{\partial^2}{\partial x^2} = 4x\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) + 16x^2\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right),$$

$$\frac{\partial^2}{\partial x^2} = 2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) + 4y^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right);$$

the original equation becomes, with $u(x, y) = \mathrm{U}(\xi, \eta)$,

$$\begin{aligned} &4y^2(U_\eta - U_\xi) + 16x^2y^2(U_{\eta\eta} - 2U_{\xi\eta} + U_{\xi\xi}) - \\ &-8x^2(U_\xi + U_\eta) - 16x^2y^2(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}) = 0 \end{aligned}$$

this

$$64x^2y^2U_{\xi\eta} = 4\ \ y^2 - 2x^2\ \ U_\eta - 4\ \ y^2 + 2x^2\ \ U_\xi$$

where we now write $y^2 = \frac{1}{2}\ \ \xi + \eta$ and $x^2 = \frac{1}{4}\ \ \eta - \xi$ giving

$$2\ \ \eta^2 - \xi^2\ \ U_{\xi\eta} = \xi U_\eta - \eta U_\xi,$$

which is the canonical form of the equation (because the only second-order derivative is $U_{\xi\eta}$).

Example 5.

Show that the equation $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$ is of parabolic type, choose appropriate characteristic variables, write the equation in canonical form and hence find the general solution.

Here we have $b^2-ac=(xy)^2-x^2y^2=0$, so the equation is parabolic (everywhere).

One characteristic line is given by the solution of $y'=\dfrac{xy}{x^2}=\dfrac{y}{x}$ (so, technically, we must avoid $x=0$) i.e. $xy=$ constant ; this we introduce $\xi=xy$ (slightly more convenient than xy) and choose $\eta=x$, to give

$$\frac{\partial}{\partial x}=\frac{1}{y}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\ \text{ and }\ \frac{\partial}{\partial y}=-\frac{x}{y^2}\frac{\partial}{\partial\xi}.$$

Then we obtain

$$\frac{\partial^2}{\partial x^2}=\left(\frac{1}{y}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\left(\frac{1}{y}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right);\ \frac{\partial^2}{\partial x\partial y}=-\frac{1}{y^2}\frac{\partial}{\partial\xi}-\frac{x}{y^2}\left(\frac{1}{y}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\frac{\partial}{\partial\xi};$$
$$\frac{\partial^2}{\partial y^2}=\frac{2x}{y^3}\frac{\partial}{\partial\xi}+\frac{x^2}{y^4}\frac{\partial^2}{\partial\xi^2},$$

and so the equation becomes, with $u(x,y)=U(\xi,\eta)$,

$$x^2\left(\frac{1}{y^2}U_{\xi\xi}+\frac{2}{y}U_{\xi\eta}+U_{\eta\eta}\right)+2xy\left(-\frac{1}{y^2}U_{\xi}-\frac{x}{y^3}U_{\xi\xi}-\frac{x}{y^2}U_{\xi\eta}\right)$$
$$+y^2\left(\frac{2x}{y^3}U_{\xi}+\frac{x^2}{y^4}U_{\xi\xi}\right)=0.$$

This simplifies to

$$U_{\eta\eta}=0\ \text{and so}\ U=F(\xi)+\eta G(\xi),$$

where F and G are arbitrary functions; thus the general solution is

$$u(x,y)=F\left(\frac{x}{y}\right)+xG\left(\frac{x}{y}\right).$$

Example 6.

Show that the equation

$$y^2 u_{xx} + 2xyu_{xy} + \left(x^2 + 4y^4\right) u_{yy} = \frac{2y^2}{x} u_x + \frac{1}{y}\left(y^2 + x^2 + 4x^4\right) u_y$$

(for $x \neq 0, y \neq 0$) is of elliptic type, find suitable characteristic variables and hence write the equation in canonical form.

We have $b^2 - ac = \left(xy\right)^2 - y^2(x^2 + 4x^4) = -4x^4y^2 < 0$ for $x \neq 0, y \neq 0$, so elliptic and the characteristic lines are given by the solution of the equation

$$y' = \frac{xy \pm \sqrt{-4x^4y^2}}{y^2} = \frac{x}{y} \pm 2i\frac{x^2}{y}.$$

This $\frac{1}{2}y^2 = \frac{1}{2}x^2 \pm i\frac{2}{3}x^3 +$ constant or $y^2 - x^2 \mp i\frac{4}{3}x^3$=constant;

We choose $\xi = y^2 - x^2$ and $\eta = x^3$, although we could use just $\eta = x$; the current choice will produce the simplest version of the canonical form – indeed, we could even include the factor $\frac{4}{3}$ (and we comment on this later). This

$$\frac{\partial}{\partial x} = -2x\frac{\partial}{\partial \xi} + 3x^2\frac{\partial}{\partial \eta} \text{ and } \frac{\partial}{\partial y} = 2y\frac{\partial}{\partial \xi},$$

and then

$$\frac{\partial^2}{\partial x^2} = -2\frac{\partial}{\partial \xi} + 6x\frac{\partial}{\partial \eta} - 2x\left(-2x\frac{\partial}{\partial \xi} + 3x^2\frac{\partial}{\partial \eta}\right)\frac{\partial}{\partial \xi} + 3x^2\left(-2x\frac{\partial}{\partial \xi} + 3x^2\frac{\partial}{\partial \eta}\right)\frac{\partial}{\partial \eta}$$

with $\frac{\partial^2}{\partial x \partial y} = 2y\left(-2x\frac{\partial}{\partial \xi} + 3x^2\frac{\partial}{\partial \eta}\right)\frac{\partial}{\partial \xi}$ and $\frac{\partial^2}{\partial y^2} = 2\frac{\partial}{\partial \xi} + 4y^2\frac{\partial^2}{\partial \xi^2}$.

This the original equation, with $u(x, y) = U(\xi, \eta)$, becomes

$$y^2 \ \ -2U_\xi + 6xU_\eta + 4x^3U_{\xi\xi} - 12x^3U_{\xi\eta} + 9x^4U_{\eta\eta}$$
$$+2xy \ \ -4xyU_{\xi\xi} + 6x^2yU_{\xi\eta} \ \ + \ \ x^2 + 4x^4 \ \ \ 2U_\xi + 4y^2U_{\xi\xi}$$
$$= \frac{2y^2}{x} \ \ -2xU_\xi + 3x^2U_\eta \ \ + \frac{1}{y} \ \ y^2 + x^2 + 4x^4 \ \ \ 2yU_\xi \ \ ,$$

which simplifies to give

$$x^4y^2 \ \ 9U_{\eta\eta} + 16U_{\xi\xi} \ \ = 0 \text{ or } 9U_{\eta\eta} + 16U_{\xi\xi} = 0.$$

This equation is essentially the classical Laplace equation (and therefore the required canonical form); it can be written in precisely the conventional form if, for example, we replace η by $\frac{3}{4}\eta$ the new$\eta = \frac{4}{3}x^3$, which is exactly the transformation suggested by the solution of the ordinary differential equation.

2.4 Initial and boundary conditions

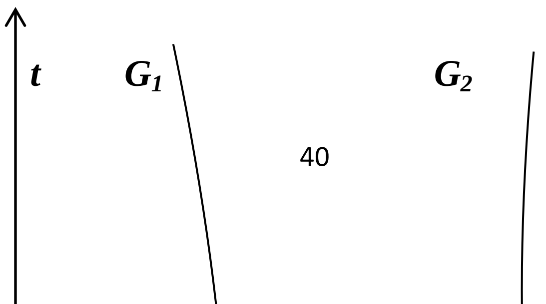
Any differential equation will normally be provided with additional constraints on the solution: the given boundary and/or initial data (as appropriate). Indeed, any physical problem or practical application will almost always have such auxiliary conditions. However, what forms these should take in order to produce a well-posed problem for partial differential equations is not a trivial investigation.
We have already touched on this aspect for first order equations and for the wave equation; we will now discuss these ideas a little further (although it is beyond the scope of this text to produce any formal proofs of the various assertions of uniqueness and existence)

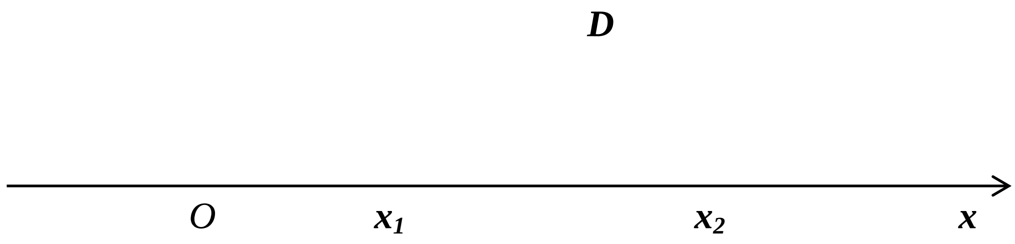
I. Hyperbolic equations

The standard type of data – Cauchy data – is to be given both *u* and $\frac{\partial u}{\partial n}$ on some curve, G, which intersects the characteristic lines i.e. at no point is G parallel to a characteristic line (so a characteristic line and G do not have a common tangent at any point). Here, $\frac{\partial u}{\partial n}$ is the normal derivative on G (and this situation is exactly what we encountered for the wave equation: $u(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0)$ were prescribed). Further, it is quite usual to seek solutions that move away (along characteristic lines) from the curve G on one side only.

II. Parabolic equations

It will be helpful, in this brief overview, to consider the canonical form of the parabolic equation, written with x (distance) replacing h and t (time) replacing x; the simplest such equation is $u_{xx} = u_t$. The characteristic lines, as we have seen, appear as a repeated pair defined by $\dfrac{dt}{dx} = 0$ (see Example 2); interpreting this in the form $\dfrac{dt}{dx}$, we see that propagation on the characteristic lines $t =$constant is at infinite speed, implying that the whole domain is affected instantaneously (although often to an exponentially small degree well away from the initial disturbance). Then we may have data on $t = 0$ (initial data) and, if the solution is defined in the domain $t > 0,\ -\infty < x < \infty,$ no further information is required (although a boundedness condition may need to be invoked the solution decays as $\left|x\right| \to \infty$). However, more often than not, the region is bounded, usually by one or two lines x=constant, although any pair of curves in (x,t)-space will suffice to describe the region where the solution is to exist; see the figure below.





The solution is in D, bounded by the curves G_1, G_2 and $x_1 < x < x_2$ (on $t = 0$).

G_1 and G_2, must be parallel to the characteristic lines i.e. no point of these curves must have a slope parallel to the x - axis.

The data given on the curves, G_1 and G_2, will be either u (the *Dirichlet* problem) or $\frac{\partial u}{\partial n}$ (the *Neumann* problem) or a mixture of the two, each on different sections of G_1 and G_2 (the *mixed* problem).

III. Elliptic equations

This class of problems is the easiest to describe in terms of boundary conditions. First, initial data has no meaning here, for the two variables appear symmetrically and there are no real characteristics, so there is no exceptional variable such as 'time'. Indeed, elliptic equations in two variables arise exclusively in two spatial dimensions. Then we simply need to prescribe u (Dirichlet) or $\frac{\partial u}{\partial n}$ (Neumann), or a mix of these two, or a linear combination of them (Robin) on the boundary of a region, D, in order to define a unique solution throughout D. (Note that, by the very nature of Neumann data, the solution in this case will be known only up to an arbitrary constant.)

Example: Explain type of the equation and write canonical form. $u_{yy} - u_{xx} = 0.$

Solving: we know '$a = -x, b = 0, c = 1,$' $b^2 - ac = x > 0$,

So original equation is hyperbolic.

$$\frac{dy}{dx} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right) = \frac{1}{-x}\left(\pm\sqrt{x}\right) \Rightarrow -\frac{dy}{dx} = \pm\frac{1}{\sqrt{x}} \Rightarrow$$

$$-dy = \pm\frac{1}{\sqrt{x}}dx \Rightarrow -\int dy = \pm\int\frac{1}{\sqrt{x}} \Rightarrow -y = \pm 2\sqrt{x} \Rightarrow y \pm 2\sqrt{x} = 0$$

$$\xi = y + 2\sqrt{x}, \eta = y - 2\sqrt{x};$$

$$u_x = U_\xi \xi_x + U_\eta \eta_x = -\frac{1}{\sqrt{x}}U_\xi + \frac{1}{\sqrt{x}}U_\eta; \; u_y = U_\xi + U_\eta;$$

$$u_{xx} = \frac{1}{x}\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) + \frac{1}{\sqrt{x^3}}\left(U_\xi - U_\eta\right); \; u_{yy} = U_{\xi\xi} + 2U_{\eta\xi} + U_{\eta\eta};$$

$$u_{yy} - u_{xx} = U_{\xi\xi} + 2U_{\eta\xi} + U_{\eta\eta} -$$

$$-x\left(\frac{1}{x}\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) + \frac{1}{\sqrt{x^3}}\left(U_\xi - U_\eta\right)\right) = 4U_{\eta\xi} = 0$$

$$u_{yy} - u_{xx} = 4U_{\eta\xi} + \frac{1}{\sqrt{x}}\left(U_\xi - U_\eta\right) = 0 \Rightarrow 2U_{\xi\eta} = \frac{U_\eta - U_\xi}{\xi - \eta}$$

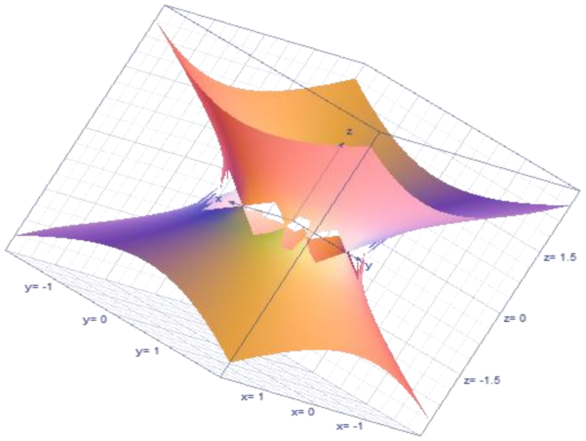
$2U_{\xi\eta} = \dfrac{U_\eta - U_\xi}{\xi - \eta}$ is canonical form of the equation.

Example: Explain type of the equation the equation: $4yu_{xx} + 2(y-1)u_{xy} - u_{yy} = 0$

Solving: We know '$a = 4y, \; b = y - 1, \; c - 1, \; d = 0$' so we can find Δ .

$$\Delta = b^2 - ac = \left(y - 1\right)^2 - 4y\left(-1\right) =$$
$$= y^2 + 2y + 1 = \left(y + 1\right)^2 > 0$$

If $\Delta > 0$, $4yu_{xx} + 2(y-1)u_{xy} - u_{yy} = 0$ is hyperbolic type.



the figure is expressed approximate graphic of the solution.

Conclusion

By my qualification paper I tried to give general information about differential equations. When I was writing the qualification paper, I had many difficulties. Although, I have learnt a lot of things. During the process I used internet information and references depend on my topic.

I wrote my qualification paper on topic "Classification of second order partial differential equation depend on two variable" consists of two chapters:

First chapter is about general information of differential equation. It is named "General information about ordinary differential equations". First chapter consist of 6 subtopics. They are general information about differential equations, types of equation, first order differential equation general information, geometrical interpretation of differential equation, the linear equation, high order differential equation.

Second chapter is on topic "Second order partial differential equation depend on two variables. It consists of 4 subtopics. They are Transformation of variables, Characteristic lines and the classification, Canonical form, Initial and boundary conditions.

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