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Sobolev funksional fazosida yaqinlashish tartibi bo’yicha optimal kubatur formulalar

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DISSERTATSIYA

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MUNDARIJA

Asosiy belgilashlar ro’yxati.

$R^n - n -$ o’lchovli Evklid fazosi

R - haqiqiy sonlar to’plami

mes Ω - Ω sohani o’lchovi

$L_p^{(m)}(\Omega)$ - S.L. Sobolevning factor fazosi

B – Banax fazosi

C - uzluksiz funksiyalar fazosi

$C^{(m)}(\Omega)$ – Ω sohada m marta differensiallanuvchi uzluksiz funksiyalar fazosi.

$C^{\infty}(\Omega)$ – Ω sohada cheksiz marta differensiallanuvchi uzluksiz funksiyalar fazosi.

$\ell(x)$ – ummumlashgan funksiya, $\varphi(x)$ – asosiy funksiya

$< \ell(x), \varphi(x) >$ - $\ell(x)$ funksionalning $\varphi(x)$ funksiyaga tasiri

$$\gamma^{\alpha} = \prod_{j=1}^{n} \gamma_j^{\alpha_j}$$

$D^{\alpha} = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ - diferensiallash operatori

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha \in Z^n$ va $\alpha_i \geq 0$

$\alpha! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_n!$

$\overset{0}{C}_{\lambda}$ - kubatur (kvadratur) formulaning optimal koeffisiyenlari

$\delta(x)$ - Dirakning delta funksiyasi

$Y_{k,\ell}(\theta)$ – k - tartibli ℓ ko’rinishdagi ortonormallangan sferik garmonikalar

$\sigma(n,k)$ – k - tartibli chiziqli bog’liq bo’lmagan sferik garmonikalar soni

$S - n -$ o’lchovli birlik sfera

$P_k \cos(\theta)$ - Lejandr ko’phadi

$\psi_{\ell}(o,\varphi)$ – ekstremal funksiya

$$|k| = \left(\sum_{j=1}^{n} k_j^{\,2} \right)^{\frac{1}{2}}$$

Kirish.

O’zbekiston Respublikasi mustaqillik odimlarini dadil qo’yayotgan hozirgi davrda axborotlashgan jamiyat qurish masalasi mamlakatimiz uchun naqadar katta ahamiyat kasb etayotgani hech kimga sir emas. Mamlakatimiz rivojlanishining muhim sharti zamonaviy iqtisodiyot, fan, madaniyat, texnika, texnilogiya rivoji asosida kadrlar tayyorlashning takomillashgan tizimining amal qilishiga erishishdir. “Ta`lim to`g`risida”gi Qonun va “Kadrlar tayyorlash milliy dasturi” ning qabul qilinishi bilan uzluksiz ta`lim tizimi orqali zamonaviy kadrlar tayyorlashning asosi yaratildi. Bu esa mamlakatimiz kelajagi uchun qurilgan poydevor hisoblanadi.

Vatanimiz mustaqillikka erishgach, hayotimizning barcha jabhalarida bo’lganidek, xalq ta'limi tizimida ham ulkan islohotlar yo’lga qo’yildiki, bunda ta'limtarbiya jarayoniga zamonaviy axborot texnologiyalarini joriy etish, ta'limni kompyuterlashtirish muammolarini qal qilish muhim ahamiyat kasb etadi.

Bu muammoga hukumatimiz tomonidan alohida e'tibor bilan qaralmoqda. Respublikamiz Prezidenti I.A.Karimovning 2001 yil Oliy Majlisning 5-sessiyasida so’zlagan nutqida axborot texnologiyalari va kompyuterlarni jamiyat hayotiga, kishilarning turmush tarziga, maktab va oliy o’quv yurtlariga jadallik bilan olib kirish g’oyasi ilgari surilgan edi. Prezidentimiz tashabbusi bilan O’zbekiston Respublikasi Vazirlar Mahkamasining 2001 yil 23 maydagi 230-sonli «2001—2005 yillarda kompyuter va axborot texnologiyalarini rivojlantirish», shuningdek, «Internet»ning xalqaro axborot tizimlariga keng kirib borishini ta'minlash dasturini ishlab chiqishni tashkil etish chora-tadbirlari to’qrisida»gi qarorlari qabul qilindi.

Har bir jamiyatning kelajagi uning hayotiy zarurati va ajralmas qismi bo`lgan ta`lim tizimining qay darajada rivojlangani bilan belgilanadi. Mamlakatimiz rivojlanishining muhim sharti zamonaviy iqtisodiyot, fan, madaniyat, texnika, texnalogiya rivoji asosida kadrlar tayyorlashning takomillashgan tizimining amal qilishiga erishishdir.

Ta`lim - bu chuqur, har taraflama asosli ta`lim-tarbiya berish, mutaxassis kadrlar tayyorlashning turli shakl, usul, vosita, uslub va yo`nalishlarining mukammal uyg`unligidan iboratdir.

Ta`limning barcha bosqichlariga oid umumiy pedagogik va didaktik talab o’quvchilarning dasturiy bilim, tasavvur va ko`nikmalari asosida mustaqil ishlash samaradorligini takomillashtirish, ilmiy fikrlashga, o`quv faniga qiziqishini kuchaytirish, kasbiy bilimlarini chuqurlashtirish, nazariy va amaliy mashg`ulot mobaynida ularning faolligini oshirishdan iboratdir.

Ta`limning bugungi vazifasi o`quvchilarni kun sayin oshib borayotgan axborot-ta`lim muhiti sharoitida mustaqil ravishda faoliyat ko`rsata olishga, axborot oqimidan oqilona foydalanishga o`rgatishdan iboratdir. Buning uchun ularga uzluksiz ravishda mustaqil ishlash imkoniyati va sharoitini yaratib berish zarur.

O`zbekiston Respublikasi demakratik, huquqiy va fuqarolik jamiyatini qurish yo`lidan borayotgan bir paytda ta`lim sohasida amalga oshirilayotgan islohotlarning bosh maqsadi va harakatga keltiruvchi kuchi har tomonlama rivojlangan barkamol insonni tarbiyalashdir.

Mavzuning dolzarbligi va o’rganilganlik darajasi. Hisoblash matematikasi sohasida nazariy izlanishlar asosan, tipik matematik masalalarni yechishning sonli metodlar atrofida guruhlanadi. Bu sohaning klassik masalalaridan biri bu integrallarni taqribiy hisoblash formulalarini qurishdan iborat.

Bir karrali integrallarni son qiymatlari geometrik nuqtai nazardan qisqacha kvadratura deb ataladi.

Bunday masalalar bilan ko’pincha buyuk olimlar shug’ullanganlar. Masalan: Gauss, Chebishev, Eyler, Nyuton va boshqalar.

Kvadratur formula deganda quyidagi taqribiy tenglikni tushunamiz:

$$\int_a^b f(x)\,dx \approx \sum_{\lambda=1}^{N} C_\lambda f(x^{(\lambda)}), \tag{0.1}$$

bu yerda C_λ - kvadratur formulaning koeffistientlari, $x^{(\lambda)}$ - tugun nuqtalari, N - tugun nuqtalar soni.

Faraz qilaylik $f(x)$ uzluksiz funkstiyani $[a,b]$ kesmada aniq integralni taqribiy hisoblovchi formulani qurish talab qilinsin.

Integralning eng sodda taqribiy ifodasi asosan $[a,b]$ kesma balandligi $f(x)$ ning $\frac{a+b}{2}$ nuqtadagi $f(\frac{a+b}{2})$ qiymatiga teng bo'lgan to'g'ri turtburchak yuzasining kattaligidan iborat.

Quyidagi kvadratur formulani hosil qilamiz.

$$\int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2}), \qquad (0.2)$$

Katta nazariy va amaliy ahamiyatga ega bo'lgan kvadratur formulalar bilan bog'liq hisoblash algoritmlarini optimizizastiyalash masalalari bilan ko'pgina olimlar shug'ullanib kelgan. Bulardan S.L.Sobolev, A.N.Kolmogorov, S.M.Nikolskiy va boshqalar.

Kvadratur formulani $p(x)$ - vazn funkstiyasi bilan qaraydigan bo'lsak

$$\int_a^b p(x)f(x)\,dx \approx \sum_{\lambda=1}^{N} C_\lambda f(x^{(\lambda)}), \qquad (0.3)$$

ko'rinishda bo'ladi.

Bu holda xatolik funkstionali quyidagi ko'rinishda bo'ladi.

$$\ell(x) = p(x)\varepsilon_{[a,b]}(x) - \sum_{\lambda=1}^{N} C_\lambda \delta(x - x^{(\lambda)}).$$

Integrallarni taqribiy hisoblash uchun formula qurish hisoblash matematikasi va sonlar nazariyasining bir sinf masalasini tashkil qiladi. Bunday masalalarni yechish bilan juda ko'p taniqli matematiklar shug'ullanishgan, shuninig uchun ham juda ko'p formulalar ularning nomlari bilan ataladi, masalan Nyuton, Eyler, Gauss, Chebishev, Markov formulalarini misol keltirish mumkin.

Integrallarni taqribiy hisoblashda integral ostidagi funkstiya bir o’zgaruvchili bo’lsa unda kvadratur formula deyiladi, bunday masalalar bilan birinchi bo’lib Sard, Nikolskiylar shug’ullangan, agar integral ostidagi funkstiya ikki va undan ortiq o’zgaruvchili bo’lsa unda kubatur formula deyiladi, bunday masalalar bilan birinchi bo’lib Sobolev shug’ullangan.

Integrallarni taqribiy hisoblash nazariyasida quyidagi yondashuvlar mavjud.

1. Klassik
2. Ehtimolli, statistik
3. Sonli
4. Funkstional.

Ushbu ilmiy ish funkstional yondashuvga mos keladi.

Quyidagi kubatur formula berilgan bo’lsin.

$$\int_{\Omega} f(x)dx \cong \sum_{\lambda=1}^{N} C_{\lambda} f(x^{(\lambda)}), \qquad (0.4)$$

Ω - Evklid fazosidagi qandaydir soha

C_{λ} - kubatur formulaning koeffistientlari,

$x^{(\lambda)} = (x_1^{(\lambda)}, x_2^{(\lambda)}, \cdots, x_n^{(\lambda)})$ - tugun nuqtalar. N – tugun nuqtalar soni.

(0.4) ko’rinishdagi kubatur formulaning xatoligi deganda biz quyidagi ayirmani tushinamiz.

$$< \ell^N, f >= \int_{\Omega} f(x)dx - \sum_{\lambda=1}^{N} C_{\lambda} f(x^{(\lambda)}) = \int_{R^n} \ell^N(x) f(x)dx, \qquad (0.5)$$

bu yerda

$$\ell^N(x) = \varepsilon_{\Omega}(x) - \sum_{\lambda=1}^{N} C_{\lambda} \delta(x - x^{(\lambda)}) \qquad (0.6)$$

$\ell^N(x)$ - kubatur formulaning xatolik funkstionali deyiladi.

$\varepsilon_{\Omega}(x)$ - xarakteristik funkstiya f(x) funkstiya ***B*** Banax fazosiga tegishli bo'lsin, unda $\ell^N(x)$ funkstional unga qo'shma bo'lgan ***B**** funkstional fazoga tegishli bo'ladi. ***B*** fazo berilgan Ω sohada uzluksiz funkstiyalar fazosida joylashgan bo'lsin. Yani $B \to C(\Omega)$.

Unda $\ell^N(x)$ funkstional ***B**** fazoda chiziqli, uzluksiz bo'ladi va quyidagi tengsizlik o'rinli bo'ladi.

$$\left| < \ell^N, f > \right| \le \left\| \ell^N \middle| B^* \right\| \cdot \left\| f \middle| B \right\| \tag{0.7}$$

Bu shartdan kubatur formula xatolik funkstionalining normasini hisoblash ta'rifi kelib chiqadi.

$$\left\| \ell^N \middle| B^* \right\| = \sup_{\substack{f \in B \\ f \ne 0}} \frac{\left| < \ell^N, f > \right|}{\left\| f \middle| B \right\|} \tag{0.8}$$

Bundan ko'rinadiki ushbu norma tugun nuqtalar va koeffistientlarga bog'lik. Agar biz shunday C_λ yoki $x^{(\lambda)}$ ning qiymatini topsakki ushbu norma eng kichik qiymatga erishsa unda qurilgan formula optimal kubatur formula bo'ladi.

Amaliyotda quyidagi masalalar qaraladi.

1. B fazoda kubatur formula xatolik funkstionalini normasini hisoblash.
2. Optimal kubatur formula qurish.
3. Yaqinlashish tartibi buyicha optimal kubatur formula qurish.
4. Asimptotik optimal kubatur formula qurish.

1975 yilda Salixov tomonidan (0.4) ko'rinishdagi kubatur formula uchun 1- masala yechilgan. Jalolovning ilmiy ishlarida esa umumiy ko'rinishda bajarilgan.

Yani (0.4) – ko'rinishdagi kubatur formula xatolik funkstionalining normasi quyidagiga tengligi isbotlangan.

$$\ell^N(x) = \varepsilon_\Omega(x) - \sum_{\lambda=1}^{N} C_\lambda \delta(x - x^{(\lambda)})$$

$$\left\| \ell_N \middle/ L_2^{m^*}(S) \right\| = \left\{ \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\sum_{\lambda=1}^{N} c_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right) \right]^2}{k^m (k+n-2)^m} \right\}^{1/2} \tag{0.9}$$

Ushbu ishda kubatur formulada vazn qatnashgan hollari uchun ham bu masalani qarab chiqish masalasi qo'yildi va 3- masalani yechish quyildi.

Tadqiqot ob'yekti. $L_2^{(m)}(S)$ Sobolev fazosi, kubatur formulalar, kubatur formulaning xatolik funksionali, sferik garmonikalar, yaqinlashish tartibi bo'yicha optimal kubatur formulalar.

Tadqiqot predmeti. Hisoblash matematikasi, umumlashgan funksiyalar nazariyasi, funksional fazolar uchun sferik garmonikalar nazariyasi.

Tadqiqot maqsadi.

1. Kvadratur va kubatur formulalarni o'rganish.
2. $L_2^{(m)}(S)$ Sobolev fazosi va sferik garmonikalar nazariyasini o'rganish.
3. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali uchun ekstremal funksiyani aniqlash.
4. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasini hisoblash.
5. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasi uchun yuqoridan baho olish.
6. Yaqinlashish tartibi bo'yicha optimal kubatur formula qurish $L_2^{(m)}(S)$ Sobolev fazosida.

Tadqiqot vazifalari. Tadqiqot maqsadidan kelib chiqqan holda quyidagi tadqiqot vazifalari belgilandi. Integrallarni taqribiy hisoblash nazariyalarini o'rganish. $L_2^{(m)}(S)$ Sobolev fazosini o'rganish. $L_2^{(m)}(S)$ Sobolev fazosida funksiyalarni sferik garmonikalar bo'yicha yoyishni o'rganish. Ekstremal funksiyani topishni algoritmini o'rganish. Funksiyani normasini hisoblash algoritmini o'rganish.

Tadqiqotning metodologik asosi. Tadqiqotning metodologik asosi sifatida hisoblash matematikasining asosiy metodlari, funksiyalarni sferik garmonikalar orqali ifodalash usullari, umumlashgan funksiyalar va ularning qo'llanilishi, kvadratur va kubatur formulalar qurish metodlari, xatolik funksionali uchun norma va ekstremal funksiyani topish algoritmlari asos qilib olindi.

Olingan asosiy natijalar.

1. Kvadratur va kubatur formulalar o'rganildi.
2. $L_2^{(m)}(S)$ Sobolev fazosi va sferik garmonikalar nazariyasi o'rganildi.
3. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali uchun ekstremal funksiyani topildi.
4. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasini hisoblandi.
5. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasi uchun yuqoridan baho olindi.
6. Yaqinlashish tartibi bo'yicha optimal kubatur formula qurildi $L_2^{(m)}(S)$ Sobolev fazosida.

Natijalarning ilmiy yangiligi va amaliy ahamiyati. Jalolovning ilmiy ishlarida $L_2^{(m)}(S)$ Sobolev fazosida umumiy ko'rinishdagi kubatur formulalar uchun ushbu masalalar qarab chiqilgan. Ushbu dissertasiya ishida esa $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulalar uchun yuqorida keltirilgan masalalar qarab chiqilgan.

Tadbiq etish darajasi va iqtisodiy samaradorligi. Dissertasiya nazariy xarakterga ega. Funksiyalarni integrallashda, agar funksiya jadval ko'rinishda berilgan bo'lsa, integral tenglamalarni yechishda foydalanish mumkin.

Ishning hajmi va tuzilishi. Ushbu magistrlik dissertasiyasi 70 betdan iborat bo'lib, asosiy belgilashlar, kirish, 3 ta bob, xotima, foydalanilgan adabiyotlar ruyxatidan iborat.

I. BOB. KVADRATUR FORMULALAR

1.1. Interpolyasion kvadratur formulalar.

Eng sodda kvadratur formulalarni oddiy mulohazalar asosida qurish mumkin. Aytaylik, $\int_a^b f(x)dx$ integralni hisoblash talab qilinsin. Agar qaralayotgan oraliqda $\mathrm{f(x)=const}$ bo'lsa u vaqtda

$$\int_a^b f(x)dx \approx (b-a)\, f\left(\frac{a+b}{2}\right), \tag{1.1}$$

deb olishimiz mumkin. Bu formula to'g'ri turtburchaklar formulasi deyiladi.

Faraz qilaylik, $f(x)$ funkstiya chiziqli funkstiyaga yaqin bo'lsin., u holda tabiiy ravishda integralning balandligi $b-a$ va asoslari $f(a)$ va $f(b)$ga teng bo'lgan trapestiya yuzi bilan almashtirish mumkin. U holda

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)), \tag{1.2}$$

deb olishimiz mumkin. Bu formula trapestiya formulasi deyiladi.

Nihoyat $f(x)$ funkstiya $[a,b]$ oraliqda kvadratik formulaga yaqin bo'lsin, u holda $\int_a^b f(x)dx$ ni taqribiy ravishda Ox o'qi va $x=a,\quad x=b$ to'g'ri chiziqlar hamda $y=f(x)$ funkstiya grafigining absstissalari $x=a,\quad x=\frac{a+b}{2},\quad x=b$ bo'lgan nuqtalardan o'tuvchi ikkinchi tartibli parabola orqali chegaralagan yuza bilan almashtirish mumkin. U holda quyidagiga ega bo'lamiz:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left\{\ f(a)-f\left(\frac{b-a}{2}\right)+f(b)\ \right\}, \tag{1.3}$$

Bu formulani ingliz matematigi **Simpson** 1743 yilda taklif etgan edi. Bu formulani hosil qilinish uslubidan ko'rinib turibdiki, u barcha ikkinchi darajali

$$\ell_2(x) = a_a + a_1 x + a_2 x^2$$

Ko'phadlar uchun aniq formuludir. Shunday qilib, biz uchta eng sodda kvadratur formulaga ega bo'ldik.

(1.1) formulani chizishda u o'zgarmas son $f(x) = c$ ni aniq integrallashni talab qilgan edik. Lekin u $f(x) = a_0 + a_1 x$ chiziqli funkstiyasi ham aniq integrallaydi, chunki $(b-a) f(\frac{a+b}{2})$ balandligi va o'rta chizig'i $f(\frac{a+b}{2})$ bo'lgan ixtiyoriy trapestiya yuziga teng.

Shunga o'xshash Simpson formulasi ham biz kutgandan ko'ra ham yaxshiroq formuladir. U uchunchi darajali

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = P_2(x) + a_3 x^3$$

u vaqtda

$$\int_a^b P_3(x)dx = \int_a^b P_2(x)dx + a_3 \int_a^b x^3 dx = \int_a^b P_2(x)dx + \frac{a_3}{4}(b^4 - a^4), \qquad (1.4)$$

Lekin bizga ma'lumki,

$$\int_a^b P_2(x)dx = \frac{b-a}{2}[P_2(a) + 4P_2(\frac{a+b}{2}) + P_2(b)], \qquad (1.5)$$

Ikkinchi tomondan

$$\frac{a_3}{4}(b^4 - a^4) = \frac{b-a}{2}\{a_3 a^3 + 4a_3(\frac{a+b}{2})^3 + a_3 b^3\}, \qquad (1.6)$$

ayniyat o'rinlidir. Endi (1.5)-(1.6) ni (1.4) ga qo'shib,

$$\int_a^b P_3(x)dx = \frac{b-a}{2}\{P_3(a) + 4P_3(\frac{a+b}{2}) + P_3(b)\}$$

ni hosil qilamiz.

Shunday qilib biz uchta kvadratur formulani qurdik. Ulardan ikkitasi to'g'riturtburchak va trapestiya formulalari- birinchi darajali ko'phad uchun aniq formula bo'lib, Simpson formulasi uchinchi darajali ko'phad uchun aniq formuladir.

Buyuk matematik Gauss kvadratura nazariyasiga butunlay yangi va juda muhim g'oyani kiritdiki, u amaliy analizning kop sohalari rivojlanishi uchun asos bo'lib qoldi. Faraz qilaylik, ba'zi bir $y = f(x)$ integrallanuvchi funkstiya x o'zgaruvchining uzluksiz oraliqni har bir nuqtasida emas balkim, shu oraliqda etuvchi maxsus tanlangan $x_1, x_2, x_3, \ldots, x_n$ nuqtalarda berilgan bo'lsin. Biz bu yerda faqat chekli oraliqni qaraymiz. Shuning uchun uni darhol normalab qo'yamiz.

Oraliqni

$$-1 \leq x \leq 1$$

ga keltiramiz va $x_1, x_2, x_3, \ldots, x_n$ nuqtalar ham qaysikim, $y = f(x)$ funkstiya berilgan oraliqda tegishli bo'lsin. Umuman olganda n ning katta bo'lishidan qat'iy nazar,

$$y_1 = f(x_1),\ y_2 = f(x_2),\ \ldots,\ y_n = f(x_n)$$

ordinatalar $f(x)$ funlstiyani aniqlash uchun yetarli emas. Lekin biz $f(x)$ funkstiyani oraliq nuqtalari uchun integrallashga harakat qilamiz. Shu maqsadda x ning darajali funkstiyalaridan foydalanamiz. Biz shunday $n-1$ darajali $P_{n-1}(x)$ ko'phad topishimiz mumkinki, u ham x_n nuqtalarda y_n qiymatga ega bo'ladi. Odatda chekli ayirmalarni hisoblashda berilgan $x = x_n$ nuqtalar teng taksimlangan qilib taksimlanadi.

Gaussning g'oyasi shundan iboratki nuqtalarning holatini oldindan belgilamasdan ushanday sondagi ordinatalar bilan yuqori aniqlikka erishish mumkinligi kabi, bu yerda nuqtalar shunday joylashtiriladiki natijada eng yaxshi natijalar olinadi. Bu yo'lda Gauss kvadratur formulalarning nafaqat eng yuqori aniqlikka erishdi, balkim bu jarayon ko'phadlar bilan teng taqsimli interpolyastiyalashda xavfdan ham holidir. Qaysikim bu xavf u davrda ham ma'lum emasdi. Faraz qilaylik $x = x_k$ interpolyastiyalash nuqtalari tamoman erkin bo'lsin va biz bu nuqtalarda $y_1, y_2, y_3, \ldots, y_n$ qiymatlarni qabul qiladigan

$U = P_{n-1}(x)$ ko'phadni topamiz. Bu masalani hal qiladigan formula Lagranjning interpolyastion formulasi sifatida ma'lum. U

$$F_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n).$$

fundamental ko'phadni qurishga va uni ketma-ket har bir n ta ikki hadliga bo'lishga asoslangandir.

Shunday qilib biz quyidagi xossalarga ega bo'lgan

$$Q_i(x) = \frac{F_n(x)}{F_n'(x_i)(x - x_i)} \qquad \text{(i=1,2,…,n)},$$

ko'phadni oldik. $Q_i(x)$ $x = x_i$ nuqtadan tashqari barcha $x = x_k$ nuqtalarda nolga teng, $x = x_i$ da esa birga teng. Agar f_{ik} - **Kroneker** simvolini kiritsak, ya'ni

$$Q_i(x_k) = f_{ik} = \begin{cases} 1, agar \ \ i = k \\ 0, agar \ \ i \neq k \end{cases},$$

Bu holda kurish mumkinki,

$$P_{n-1}(x) = y_1 Q_1(x) + y_2 Q_2(x) + \ldots + y_n Q_n(x),$$

Ko'phad kuyilgan shartni kanoatlantiradi: yani $x = x_k$ nuqtalarda y $y = y_k$ $(k = 1, 2, \ldots, n)$ qiymatlarni qabul qiladi.

$P_{n-1}(x)$ - ko'phadning yagonaligi shu dalildan kelib chikadiki, $P_{n-1}(x)$ ko'phad bilan ikkinchi gipotetik $\overline{P}_{n-1}(x)$ ko'phad o'rtasidagi ayirma birga $x = x_k$ nuqtalarda nolga aylanadi. Lekin $P_{n-1}(x) - \overline{P}_{n-1}(x)$ ayirma ham yana $n-1$ darajali ko'phad bo'lib, u esa aynan nolga aylanmasdan $n-1$ tadan ko'p ildizga ega bo'lmaydi: bu esa

$$P_{n-1}(x) = \overline{P}_{n-1}(x)$$

ekanligini bildiradi.

Endi agar biz $P_{n-1}(x)$ ni $y = f(x)$ fuknstiyaga yetarlicha yaqinlashgan deb hisoblasak,

$$\overline{A} = \int_{-1}^{1} P_{n-1}(x)dx = \sum_{k=1}^{n} y_k \int_{-1}^{1} Q_k(x)dx, \tag{1.7}$$

hisoblasak, amaliyotda noma'lum $f(x)$ egrilik ostidagi yuzaga ega bo'lamiz. Berilgan ayrim taksimlangan $x = x_k$ nuqtalar uchun $Q_k(x)$ ko'phadlar bir qiymatli aniqlangan va shuning uchun ham

$$\int_{-1}^{1} Q_k(x)dx = \omega_k, \tag{1.8}$$

aniq integrallar ba'zi bir sonli qiymatlarga ega bo'ladiki, qaysikim ular uchun jadvallar tuzish mumkin.

Bizni qiziktiruvchi yuza uchun bu qiymatlar tamoman $y = f(x)$ funkstiyaning tabiatiga bog'lik emas.

Oldingi $x = x_i$ nuqtalarni o'zgartirmasdan yangi $x = x_{n+1}$ qo'shimcha nuqtani qo'shamiz. Qo'shimcha $x - x_{n+1}$ ikki hadni kiritib, $Q_{n+1}(x)$ - qo'shimcha ko'phadni hosil qilamiz. Ta'rifdan $Q_i(x)$ uchun kelib chiqadiki, $Q_{n+1}(x)$ ko'phad $F_n(x)$ ko'phadga proporstionaldir, qaysikim $(x - x_{n+1})$ yangi ko'paytuvchi qiskarib ketadi. Xuddi shunday yangi y_{n+1} ordinata ko'paytiriladigan vaznli ω_{n+1} vaznli ko'paytuvchi

$$\int_{-1}^{1} F_n(x)dx, \tag{1.9}$$

aniq integralga proporstionaldir.

Shunga uxshash, agar yangi

$$x = x_{n+1}, x_{n+2}, \dots, x_{n+m} \tag{1.10}$$

nuqtalarni ularni ordinatalari bilan kiritsak, u holda ularga mos

$\omega_{n+1}, \omega_{n+2}, \dots, \omega_{n+m}$ vaznlar

$$\omega_{n+i} = \int_{-1}^{1} F_n(x)\, \xi_{m-1}^{i}(x)\, dx \tag{1.11}$$

integral bilan aniqlanadi, bu yerda $\xi_{m-1}^{i}(x)$ - ayrim $m-1$ darajali ko'phadlardir. Ixtiyoriy $\xi_{m+1}(x)$ ko'phad, $x^0, x^1, x^2, \ldots, x^{m-1}$ darajali funkstiyalarning chiziqli superpozistiyasidan iborat ekanligidan, agar $F_n(x)$ quyidagi integrallarni qanoatlantirsa, bu hamma vaznlar avtomatik ravishda nolga aylanadi.

$$\int_{-1}^{1} F_n(x)dx = 0, \ldots, \int_{-1}^{1} F_n(x)\, x^{m-1}\, dx = 0,$$

haqiqattan ham bizning talablarimiz $m=n$ gacha borib,

$$\int_{-1}^{1} F_n(x)\, x^{\alpha}\, dx = 0 \qquad (\alpha = 0,1,2,\ldots,n-1), \qquad (1.12)$$

integral shartining bajarilishidir.

Natijada bizning boshida berilgan n ta nuqtani ixtiyoriy ravishda qo'shsak ham baribir hech bir yangi ordinata oldingi natijalarni o'zgartirmaydi.

Oldingi natija shundan iboratki, xuddi biz $2n$ ta ordinata bilan ish ko'rib, haqiqattan esa biz n ta ordinatadan foydalanamiz, yangi qurilgan ordinatalar esa hisoblanayotgan yuzaga hech nima qo'shmaydi.

Bu jarayonda biz

$$\overline{A} = \sum_{k=1}^{2n} y_k\ \omega_k$$

yig'indiga n ta hadni tejayimz. Bu fikrlashlar yuqoridagi mulohazalar uchun yetarlicha emasdir. To'liqroq bo'lishi uchun quyidagi mulohazani tavsiya etamiz.

Haqiqattan ham yangi $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ nuqtalarning berilishi nafaqat $Q_{n+m}(x)$ $(m=1,2,\ldots,n)$ yangi ko'phadlarni qo'shadi, xatto oldingi

$Q_i(x)$ $(i=1,2,\dots,n)$ ko'phadlar ham o'zgaradi: har bir yangi x_{n+m} nuqta $Q_i(x)$ ga qo'shimcha $\dfrac{x-x_{n+m}}{x_i-x_{n+m}}$ ko'paytuvchini kiritadi.

Shunday qilib, yangi m ta $x_{n+1},x_{n+2},\dots,x_{n+m}$ nuqtalarning kiritilishi oldingi $Q_i(x)$ ko'phadni

$$Q_i^*(x)=Q_i(x)\ \frac{x-x_{n+1}}{x_i-x_{n+1}}\cdot\frac{x-x_{n+2}}{x_i-x_{n+2}}\cdot\ldots\cdot\frac{x-x_{n+m}}{x_i-x_{n+m}}, \qquad (1.13)$$

ko'phadga aylantiradi.

Yuqoridagi mulohazalarning haqiqat ekanligi shakli o'zgartirilgan $Q_i^*(x)$ ko'phadlarning quyidagi xossalarga ega ekanligidan kelib chiqadi:

1^0. $Q_i^*(x)=\delta_{ik}\,(k=1,2,\dots,n).$

2^0. $\int_{-1}^{1}Q_i^*(x)\,dx=\int_{-1}^{1}Q_i\,(x)\,dx=\omega_i$

endi bu xossalarni isbotini ko'ramiz.

Birinchi xossa bevosita (1.13) munosabatdan kelib chiqadi. Ikkinchisi uchun esa

$$\frac{x-x_{n+k}}{x_i-x_{n+k}}=1+\frac{x-x_i}{x_i-x_{n+k}}$$

dan foydalanamiz.

Bundan shuni xulosa qilamizki, (1.13) tenglikning o'ng tomonidagi qo'shimcha ko'paytuvchilarni ko'paytirishni $1-\xi_{m-1}^i(x)$ ko'rinishda tasvirlash mumkin ekan, bu yerda $\xi_{m-1}^i(x)-$ $m-1$ darajali ko'phad. (1.12) shartning kuchiga asosan 2^0 - tenglik bajariladi. Isbotlangan 1^0 va 2^0 lar ko'rsatadiki yangi ordinatalar oldingi olingan natijalarni o'zgartirmaydi.

Muhimrog'i shundan iboratki, bizlar qo'shimcha $y_{n+1},y_{n+2},\dots,y_{2n}$ ordinatalarni bilishimiz shart emas.

$$\overline{A} = \sum_{k=1}^{n} y_k\ \omega_k\ , \qquad (1.14)$$

Yig'indi n ordinata yordami bilan shunday aniqlikdagi yuzani beradiki, agar biz $2n$ - ordinata olsak ham o'zgarmaydi.

(1.12) - tipdagi integral shart ortogonallash sharti deyiladi. Biz ko'rsatamizki, $F_n(x)$ ko'phad $1, x^1, \ x^2, ..., x^{n-1}$ darajali funkstiyalarga ortogonaldir. Bunday shartlarni oldin ortogonal funkstiyalar sistemasini ko'rib chiqqanda o'rganganmiz.

Biz Yakobi ko'phadlarini tekshirib chiqdikki, u (1.12) shart ma'nosida ko'phad darajasidan past bo'lgan barcha x ning darajalariga ortogonallik xossalariga egadir. Ammo ortogonallik sharti umumiy holda yana $\rho(x)$ vazn ko'paytuvchini ham integral ostiga oladi. Faqat maxsus hollarda "Lejandr ko'phadlari" da bu vazn ko'paytuvchi 1 ga teng bo'ladi va shunday qilib, ortogonallik oddiy ortogonallikka aylanib qoladi. Shunday qilib, $F_n(x)$ funkstiyani tanlash masalasi hal qilinadi:

Gauss metodi $F_n(x)$ ni n- Lejandr ko'phadlari bilan mos qo'yishni talab qiladi: bu ko'phad ildizlari bizga shunday nuqtalarni beradiki, qaysikim $f(x)$ funkstiya qiymatlari berilgan bo'ladi. ω_i koeffistientlarning sonli qiymatlari bilan birga shu ildizlarning juda aniq jadvallari borki, u (1.8) formula bilan hisoblanadi.

Bizga ma'lumki, $[a,b]$da n nuqtali interpolyastion formulaning

$$\int_a^b \rho(x) f(x) dx \approx \sum_{k=1}^{n} A_k\, f(x_k)\ , \qquad (1.15)$$

tugun nuqtalari $[a,b]$ oraliqda qanday joylashganliklaridan qat'iy nazar, $(n-1)$- darajali ko'phadlar aniq integrallanishi qaraladi. Chekli $[a,b]$ oraliq va $\rho(x) \equiv 1$ uchun Gauss quyidagi masalani qaragan edi. $x_1, x_2, \ldots, x_n$

tugunlar shunday tanlanganki, (1.15) formula mumkin qadar darajasi eng Yuqori bo’lgan ko’phadlarni aniq integrallasin. (1.15) formula n ta parametr - tugunlarni maxsus ravishda tanlash yo’li bilan uning aniqlik darajasini n birlikka ortirishni ko’rish mumkin. Haqiqattan ham $x_1, x_2, \ldots, x_n$ tugunlarni maxsus ravishda tanlash orqali (1.15) formulaning darajasini $2n-1$ dan ortmaydigan barcha $f(x)$ ko’phadlar uchun aniq bo’lishga erishishni Gauss ko’rsatdi.

Qanchalik Gaussning natijasi ixtiyoriy oraliq va vazn funkstiyalar uchun umumlashtirildi. Bunday formulalar Gauss tipidagi kvadratur formulalar deyiladi. Qulaylik uchun x_n tugunlar o’rnida

$$\omega_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$$

ko’phad bilan ish ko’ramiz. Agar x_k lar ma’lum bo’lsa, u holda $\omega_n(x)$ ham ma’lum bo’ladi va aksincha. Lekin x_n larni topishni $\omega_n(x)$ ni topish bilan almashtirsak, u holda biz $\omega_n(x)$ ni ildizlari haqiqiy, har xil va ularning $[a,b]$ oraliqda yotishini ko’rsatishimiz shart.

Teorema 1.1.1 (1.15) kvadratur formula darajasi $2n-1$ dan ortmaydigan barcha ko’phadlarni aniq integrallashi uchun quyidagi shartlarning bajarilishi zarur va etarlidir:

1) U interpolyastion va 2) $\omega_n(x)$ ko’phad $[a,b]$ oraliqda $\rho(x)$ vazn bilan darajasi n dan kichik bo’lgan barcha $Q(x)$ ko’phadlarga ortogonal bo’lishi kerak.

$$\int_a^b \rho(x)\omega_n(x)Q(x)dx = 0, \qquad (1.16)$$

Isbot. Zarurligi. Faraz qilaylik, (1.15) formula darajasi $2n-1$ dan oshmaydigan barcha ko’phadlarni aniq integrallasin. U holda u interpolyastiondir.

Endi darajasi n dan kichik bo'lgan ixtiyoriy $Q(x)$ ko'phadni olib, $f(x)=\omega_n(x)Q(x)$ deb olamiz. Shuning uchun ko'rinib turibdiki, $f(x)$ darajasi $2n-1$ dan ortmaydigan ko'phad. Shuning uchun ham uni (1.15) formula aniq integrallaydi:

$$\int_a^b \rho(x)\omega_n(x)Q(x)dx=\sum_{k=1}^{n}A_k\omega_k(x_k)Q(x_k).$$

Bu yerda, $\omega_n(x_k)=0 \quad (k=\overline{1,n})$ ni hisobga olsak (1.16) tenglik kelib chiqadi, chunki $r(x)$ darajasi n dan kichik ko'phad va (1.15) formula interpolyastiondir.

Demak,

$$\int_a^b \rho(x)f(x)dx=\sum_{k=1}^{n}A_k r(x_k), \qquad (1.17)$$

lekin (1.17) ga ko'ra $r(x)=f(x)$. Shuning uchun

$$\int_a^b \rho(x)f(x)dx=\sum_{k=1}^{n}A_k f(x_k).$$

Shu bilan birga teoremaning yetarli sharti isbot bo'ldi.

$\omega_n(x)$ ko'phad $\rho(x)$ vazn bilan $[a,b]$ oraliqda darajasi n dan kichik bo'lgan barcha ko'phadlar bilan ortogonal va bosh koeffistienti birga teng bo'lishi uchun, bunday $\omega_n(x)$ ko'phad yagona hamda uning ildizlari haqiqiy, har xil va $[a,b]$ oraliqda yotadi. Demak, agar $\rho(x)$ vazn $[a,b]$ oraliqda uz ishorasini saklasa, u holda har bir $n=1,2,\ldots..$ uchun $2n-1$ darajali ko'phadlarni aniq integrallaydigan yagona (1.15) kvadratur formula mavjud.

Faraz qilaylik bizga $f(x)$ furkstiyaning $x_1,x_2,x_3,\ldots,x_n$ nuqtalardagi $f(x_1),f(x_2),f(x_3),\ldots,f(x_n)$ qiymatlari berilgan bo'lib, maqsad shu nuqtalar bo'yicha $\int_a^b f(x)\,dx$ integralning taqribiy qiymatini mumkin qadar Yuqori aniqlikda topishdan iboratdir. Demak, A_k koeffistientlar aniqlanishi kerak. Buning uchun $f(x)$ ni uning berilgan qiymatlaridan foydalanib, $(n-1)$- darajali ko'phad bilan interpolyastiyalaymiz:

$$f(x) = L_{n-1}(x) + r_n(f,x) = \sum_{k=1}^{n} \prod_{i=1,\, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} f(x_k) + r_n(f,x), \qquad (1.18)$$

endi bu tenglikni $\rho(x)$ ga ko'paytirib, a dan b gacha integrallaylik:

$$\int_a^b \rho(x) f(x)\,dx - \sum_{k=1}^{n} A_k f(x_k) = \int_a^b \rho(x) r_n(f,x)\,dx, \qquad (1.19)$$

qoldiq hadni tashlasak,

$$\int_a^b \rho(x) f(x)\,dx \approx \sum_{k=1}^{n} A_k f(x_k),$$

$$A_k = \int_a^b \rho(x) \prod_{i=1,\, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}, \qquad (1.20)$$

kvadratur formulalarga ega bo'lamiz.

Bu formula qurilish usuliga ko'ra interpolyastion formula deyiladi. Bunday formulalar uchun ushbu teorema o'rinlidir.

Teorema 1.1.2 Quyidagi

$$\int_a^b \rho(x) f(x)\,dx \approx \sum_{k=1}^{n} A_k f(x_k), \qquad (1.21)$$

kvadratur formulaning interpolyastion bo'lishi uchun uning barcha $(n-1)$- darajali algebraik ko'phadlarni aniq integrallashi zarur va kifoyadir.

Isbot. Zarurligi. Agar $f(x)$ $(n-1)$- darajali ko'phad bo'lsa, u holla (1.18) tenglikda $r_n(f,x) \equiv 0$ bo'lib,

$$f(x)=\sum_{k=1}^{n}\prod_{i=1,\ i\neq k}^{n}\frac{x-x_i}{x_k-x_i}f(x_k)$$

tenglik urinli bo'ladi va (1.21) hamda interpolyastion bo'lganidan (1.20) ga ko'ra:

$$\int_a^b \rho(x)f(x)=\sum_{k=1}^{n}f(x_k)\int_a^b \rho(x)\prod_{i=1,\ i\neq k}^{n}\frac{x-x_i}{x_k-x_i}dx=\sum_{k=1}^{n}A_k f(x_k).$$

Demak, (1.21) formula $(n-1)$- darajali $f(x)$ ko'phadni aniq integrallaydi.

Yetarligi. (1.21) formula $(n-1)$- darajali ixtiyoriy ko'phad uchun aniq formuladir. Xususiy holda, u $(n-1)$- darajali ushbu

$$\omega_m(x)=\prod_{i=1,\ i\neq m}^{n}\frac{x-x_i}{x_m-x_i}\quad (m=1,2,\ldots,n)$$

ko'phad uchun ham aniq bo'ladi.

Agar $\omega_m(x_k)=0 \quad (k\neq m)$ va $\omega_m(x_m)=1$ ekanligini hisobga olsak,

$$\int_a^b \rho(x)\prod_{i=1,\ i\neq m}^{n}\frac{x-x_i}{x_m-x_i}\ dx=\int_a^b \rho(x)\omega_m(x)\ dx=\sum_{k=1}^{n}A_k\omega_m(x_k)=A_m$$

kelib chiqadi. Demak, (1.21) ham interpolyastiondir, shu bilan teorema isbot bo'ldi.

Bu teoremadan ko'rinadiki, n nuqtali interpolyastion kvadratur formulaning algebraik aniqlik darajasi $n-1$ dan kichik bo'lmasligi kerak. Bularga asosan, ishonch hosil qilish mumkinki, to'g'ri turtburchak, trapestiya va Simpson formulalari interpolyastion kvadratur formulalardir.

1.2. Effektiv kvadratur formulalar.

Amaliy analizning ko'pgina masalalari differenstial tenglamalar orqali aniqlanadi. Agar bunday funkstiyalarni integrallash kerak bo'lsa, u holda faqat oraliqning chetki nuqtalarida funkstiya va uning hosilalarining qiymatlaridan foydalanish maqsadga muvofiq deb hisoblanadiki, agar chetki nuqtalarda chegaraviy nuqtalarni biz bilsak, ketma-ket hosilalarning qiymatlarini funkstiyani aniqlovchi differenstial tenglamalardan osongina hisoblashimiz mumkin. Shuning uchun bizning asosiy maksadimiz shundan iboratki, effektiv kvadratur formulalarni hosil qilish uchun biz ichki ordinatalardan emas, balkim chegaraviy ordinatalardan va bu nuqtalarda hosilalarning qiymatlaridan foydalanamiz. Bunday formulalar aniqlikni oshirish uchun emas, balki integraldarni hisoblash uchun chegaraviy axborotlardan foydalaniladi. Quyidagi teoremani isbotlaymiz.

<u>**Teorema 1.2.1**</u> Agar ikkita $u(x)$ va $v(x)$ funkstiyalar $[a,b]$ oraliqda aniqlangan, uzluksiz va m-tartibli uzluksiz hosilalarga ega bo'lsa, u holda quyidagi tenglik o'rinli bo'ladi;

$$\int_a^b u(x)v^{(m)}(x)dx = \left[\sum_{\alpha=0}^{m-1} u^{(\alpha)}(x)v^{(m-\alpha-1)}(x)(-1)^{\alpha}\right]\Bigg\|_a^b + R_m(x), \qquad (1.22)$$

bu yerda,

$$R_m(x) = \int_a^b (-1)^m v(x)u^m(x)dx \qquad (1.23)$$

va α -hosila tartibi.

Isbot: Har bir $u(x)$ va $v(x)$ funkstiyalar $[a,b]$ oraliqda differenstiallanuvchi va undan tashqari bu oraliqda $u(x)$ $v'(x)$ funkstiya uchun boshlangich funkstiya mavjud bo'lsin. U holda $[a,b]$ oraliqda $u(x)$ $v'(x)$ funkstiya uchun boshlangich funkstiya mavjud bo'lib, bo'laklab integrallash formulalari o'rinlidir. Ya'ni

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx, \qquad (1.24)$$

yoki boshqacha yozsak

$$\int_a^b u(x)v^{'}(x) = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u^{'}(x)dx \,, \qquad (1.25)$$

Shunday qilib (1.22) formulaning ung tomoni uchun (1.25) formulani m marta qo'llasak quyidagiga ega bo'lamiz.

$$\int_a^b u(x)\,v^m(x) - \int_a^b (-1)^m v(x)\,u^{(m)}(x)\,dx = \left[u(x)\,v^{(m-1)}(x) - u^{'}(x)\,v^{(m-2)}(x) + \ldots + \right]\Big|_a^b =$$

$$= \left[\sum_{\alpha=0}^{m-1} u^{(\alpha)}(x)v^{(m-\alpha-1)}(x)(-1)^{\alpha}\right]\Big|_a^b \qquad (1.26)$$

Bundan esa quyidagini olamiz.

$$\int_a^b u(x)v^m(x)dx = \left[\sum_{\alpha=0}^{m-1} u^{(\alpha)}(x)v^{(m-\alpha-1)}(x)(-1)^{\alpha}\right]\Big|_a^b + \int_a^b (-1)^m u^{(m)}(x)v(x)dx \,, \quad (1.27)$$

va teorema shu bilan isbotlanadi.

(1.22) formuladan biz quyidagicha foydalanamiz. Integrallash oraligini $(0,1)$ oraliqgacha normallaymiz. Ya'ni quyidagi belgilashlarni kiritamiz.

$$u(x) = f(x),$$

$$v(x) = \frac{g_m(x)}{\beta_m^m m!} \qquad (1.28)$$

Bu yerda

$$g_m(x) = \beta_m^m x^m + \beta_{m-1}^m x_{m-1} + \ldots + \beta_0^m \,, \qquad (1.29)$$

ko'phadni erkin tanlaymiz

$u(x)$ va $v(x)$ funkstiyalarni bunday tanlashlar natijasida (1.22) formula quyidagi ko'rinishda yozilishi mumkin.

$$\int_0^1 f(x)dx = \frac{1}{\beta_m^m m!}\left[\sum_{\alpha=0}^{m-1} f^{(\alpha)}(x)g_m^{(m-\alpha-1)}(x)(-1)^{\alpha}\right]\Big|_0^1 + R_m \,, \qquad (1.30)$$

Bu yerda

$R_m(x)$ - quyidagi aniq integralni bildiradi:

$$R_m = (-1)^m \int_0^1 \frac{g_m(x)}{\beta_m^m m!} f^{(m)}(x)dx, \qquad (1.31)$$

(1.30) formulani biz quyidagicha tushunamiz va u shundan iboratki oraliqning chetki nuqtalarida egri chiziqning chegaraviy qiymatlari va hosilalarining qiymatlari uchun tegishli yuzani hisoblaydigan kvadratur formulani ifodalaydi.

$$\bar{g}_m \frac{1}{\beta_m^m m!}\left[\sum_{\alpha=0}^{m-1} f^{(\alpha)}(x) g_m^{m-\alpha-1}(x)(-1)^{\alpha}\right]\Big|_0^1, \qquad (1.32)$$

Shu vaqtda (1.31) formuladagi qiymat kvadratur formulaning R_m qoldigini tasvirlaydi.

Shunday qilib biz funkstiya va uning hosilalarining chegaraviy qiymatlarida hisoblanadigan kvadratur formulalar va uning qoldiq hadiga ega bo'ldik.

Odatda xos qiymat deb ataluvchi nomalum doimiy λ parametrni o'z ichiga olgan chiziqli differenstial tenglama berilgan bo'lsin. Yani shunday bir jinsli chegaraviy shartlar berilganki, tenglamaning yechimi faqat λ parametrni tanlash bilan topish mumkin. Masala shundan iboratki, shunday λ eng kichik qiymatni yoki bir nechta λ eng kichik qiymatlarni topish mumkin bo'lsaki, masala qechimga ega bo'lsin.

Bunday xos qiymatlar bilan bog'liq masalalarni yechish uchun effektiv kvadratur formulalarning qo'llanilishi yaqqol yordam berishi mumkin, chunki u faqat oraliqning chetki nuqtalarida funkstiya va uning hosilalarini bilishga asoslangandir. Bu qiymatlar esa berilgan differenstial tenglamalar va chegaraviy shartlar asosida olinadi.

Bu metodni qo'llanishini namoyish qilish uchun biz oddiy bir misol olamiz. Lekin metod esa murakkab shartlar asosida qo'llaniladi. Bizning asosiy maqsadimiz metodning jiddiy qirralarini o'rganishdan iborat bo'ladi. Murakkablashgan texnik qiyinchiliklari bundan mustasnodir. Shuning uchun biz o'zgarmas koeffistentli ikkinchi tartibli differenstial tenglamaga to'xtalamiz.

$$y'' + y' + \lambda y = 0, \qquad (1.33)$$

Chegaraviy shartlari quyidagicha

$$y(0),=y(1)=0, \tag{1.34}$$

Berilgan oraliq $[0,1]$ ga keltirilgan. Bir jinsli chiziqli differenstial tenglama o'zgarmas amplitudali ko'paytuvchi qoldirganligidan $x=0$ nuqtadagi hosila uchun ixtiyoriy $y'(0)=1$ qiymatni yozish mumkin. Bu shartlar bilan (1.33) differenstial tenglama $x=0$ nuqtada barcha hosilalarning qiymatlarini aniqlaydi. Biz ularni ketma–ket differenstiallash bilan yoki uni quyidagi $y=a_0+a_1x+a_2x+\ldots$ darajali qatorga yoyib va o'rniga qo'yib olamiz. Barcha o'xshash hadlarni to'playmiz va x^k oldidagi olingan koeffistentlarni nolga tenglashtiramiz. Bizning oddiy misolimizda $a_0=0$, $a_1=1$, $a_2=-\frac{1}{2}$, $a_3=\frac{1-\lambda}{6}$, $y(0)$, $y'(1)=0$, $y''(1)=-1$, $y'''(1)=1-\lambda$ ni topamiz.

Agar biz koeffistentlarni ketma–ket topishni qancha davom ettirsak, shuncha katta aniqlikni kutishimiz mumkin. Bizning maqsadimiz uchun bu yerda biz a_3 ga to'xtalamiz. Boshqa chetki nuqta uchun xuddi shunday

$$x=1+\xi, \quad y=b_0+b_1\xi+b_2\xi^2+b_3\xi^3$$

deb olamiz.

Oldingiga o'xshagan differenstial tenglamaga etib qo'yish metodidan foydalansak, biz b_0 ni emas balki, barcha keyingi koeffistentlar b_0 koeffistentning chiziqli funkstiyasi bo'lar ekan.

$$b_0=b_1, \qquad b_2=-\frac{\lambda b_0}{2}, \qquad b_3=-\frac{\lambda b_0}{6}$$

$$y(+1)=b_0, \quad y'(1)=0, \quad y''(1)=-\lambda b_0, \quad y'''(1)=\lambda b_0 \tag{1.36}$$

Endi biz $y''(x)$ ni $f(x)$ boshlangich funkstiya sifatida qabul qilamiz va kvadratur formulani qo'llaymiz. Xuddi shunday $f(x)=y''(x)$ va $f'(x)=y'''(x)$ ikkala chetki nuqtalarda berilgan.

$$f(0) = -1 \qquad f(1) = -\lambda b_0$$

$$f'(0) = 1 - \lambda \qquad f'(1) = \lambda b_0$$

$n = 2$ bo'lganda effektiv kvadratur formula

$$\int_0^1 y''(x)dx = y'(1) - y'(0) = \frac{6(-1-\lambda b_0) + 1(1-\gamma-\gamma b_0)}{12};$$

$$-1 = \frac{-5 - 7\lambda b_0 - \lambda}{12}$$

ni hosil qilamiz, bu quyidagi munosabatga olib keladi.

$$b_0 = \frac{7-\lambda}{7\lambda}, \tag{1.37}$$

Endi biz yana bir marta $y'(x)$ ni $f(x)$ sifatida olib, effektiv kvadratur formuladan foydalanamiz;

$$f(0) = 1, \qquad f(1) = 0$$

$$f'(0) = 1, \qquad f'(1) = -\lambda b_0$$

$$f''(0) = 1 - \lambda, \qquad f''(1) = \lambda b_0$$

Hozir biz ikkala chetkilar uchun uch juft berilganlarga egamiz va formulani $n = 3$ bo'lganda qo'llaymiz.

$$\int_0^1 y'(x)dx = y(1) - y(0) = \frac{60(1+0) + 12(-1+\lambda b_0) + (1-\lambda+\lambda b_0)}{120}$$

$$b_0 = \frac{49 - \lambda + 13\lambda b_0}{120}$$

Bu esa yangi munosabatni beradi.

$$b_0 = \frac{49-\lambda}{120-13\lambda} \tag{1.38}$$

(1.37) va (1.38) larning o'ng tomonlarini tenglashtirib λ-xos qiymatni aniqlash uchun quyidagi kvadrat tenglamani olamiz.

$$20\lambda^2 - 554\lambda + 840 = 0$$

Biz ikkita

$$\lambda_1 = 1{,}6095, \qquad \lambda_2 = 26{,}0905, \tag{1.39}$$

ildizlarga ega bo'lamiz.

Faqat kichik ildizning o'ziga xos qiymati bor, kattasini olsak, hisoblashlarda katta o'zgarishlar bo'ladi. Kichik ildiz uchun biz differenstiallash jarayonini davom ettirsak, unda o'zgarish bo'lmaydi. Biz $y'''(x)$ ga kelib qoldiq. Agar biz yana bir qadam bajarsak, $y'''(0)$ va $y'''(1)$ ni kiritsak, effektiv kvadratur formulaning birinchi yaqinlashishi $n = 3$ da ikkinchisi esa $n = 4$ da bo'ladi. Bu uchun biz ikkita munosabatni olamiz.

$$b_0 = \frac{71 - 10\lambda}{\lambda(73 - \lambda)} \quad \text{va} \quad b_0 = \frac{679 - 18\lambda}{168 - 201\lambda + \lambda^2)}$$

Va ular λ ni aniqlash uchun

$$28\lambda^3 - 4074\lambda^2 + 80638\lambda - 119280 = 0$$

kub tenglamani beradi. Bu yerda yechimi

$$\lambda_1 = 1{,}608467,$$

dan iborat bo'ladi. λ_1 qiymatning kichkina o'zgarishi, (1.39) da topilganga nisbattan ko'rsatadiki, birinchi qo'pol yaqinlashish haqqatga juda yaqin ekanki, aniqlik 0,07% gacha bo'ldi. Ko'rgan oddiy misolimizda natijalarimizni tekshirishimiz mumkin. λ_1 - nazariy jihatdan

$$\lambda = \frac{1}{4} + \theta^2, \tag{1.40}$$

$$\theta \ \text{esa} \ tg\theta = 2\theta$$

transtendent tenglamani echimidir. Bu tenglamaning eng kichik ildizi $\theta_1 = 1{,}1655618$ eki $\lambda_1 = 1{,}608534$ ni beradi.

Shunday qilib $n = 2{,}3$ uchun yaqinlashish xatoligi $\eta = -0{,}0010$ ga, $n = 3{,}4$ da esa $\eta = 0{,}000067$ ga teng bo'ldi.

Xuddi shunday silliq $f(x) = y^{''}(x) \;\; n = 3$ funkstiyalar uchun uni qo'llasak yaqinlashish juda tez bo'ladi.

Ma'lumki xos qiymatlarni olish uchun Rem- Ritst metodi ba'zi bir integrallarni minimallashtirishga asoslangandir, shuning uchun ham u metod faqat o'ziga qo'shma differenstial operatorlarga qo'llanishi mumkin.

Tavsiya etilgan metod uchun esa differenstial operator va chegaraviy shartlar o'z-o'ziga qo'shma bo'lishi talab qilinmaydi. Shuning uchun bu metod ancha umumiy hollarda ham qo'llaniladi va xatto bu metodik qo'llash uchun differenstial tenglamaning chiziqli bo'lishi shart emas.

Bu g'oyani yana ham ilgari suradigan bo'lsak shunday xulosaga kelamizki, effektiv kvadratur jarayoni nafaqat xos qiymatlarni topish uchun balkim, differenstial tenglamalarning haqiqiy yechimlarini topishga ham qo'llanilishi mumkin.

1.3. Davriy funksiyalarni integrallash.

Bu paragrafda 2π davrli funkstiyalarni taqribiy integrallash masalasini ko'ramiz. Bu yerda tabiiyki, kvadratur formulaning aniqlik darajasi algebraik ko'phadga emas, balki trigonometrik ko'phadga nisbatan qaraladi.

Agar ushbu kvadratur formula

$$\int_0^{2\pi} f(x)\,dx \approx \sum_{k=1}^{n} A_k f(x_k), \quad \left(x_k \in [0,2\pi]\right), \tag{1.41}$$

Ixtiyoriy $m-1$ tartibli trigonometrik ko'phadlar uchun aniq bo'lib, birorta m- tartibli trigonometrik ko'phad uchun aniq bo'lmasa, u holda bu formulaning trigonometrik aniqlik darajasi (tartibi) $m-1$ ga teng deyiladi.

<u>Teorema 1.3.1</u> n tugunli kvadratur formulalar to'plamida $x_1, x_2, \ldots, x_n$ tugunlari $[0,2\pi]$ oraliqda tekis joylashgan va koefistientlari o'zaro teng bo'lgan kvadratur formula eng yuqori trigonometrik aniqlik tartibiga ega bo'lib, bu tartib $n-1$ ga teng.

<u>Isbot.</u> Avvalo (1) ko'rinishdagi ixtiyoriy kvadratur formulaning aniqlik darajasi $n-1$ dan ortmasligini ko'rsatamiz.

Kvadratur formulaning x_k tugun nuqtalaridan foydalanib,

$$f(x) = \prod_{k=1}^{n} \sin^2 \frac{x - x_k}{2}$$

funkstiyani tuzaylik. Har bir ko'payyuvchi birinchi tartibli trigonometrik ko'phad bo'lgani uchun, $f(x)$ n - tartibli ko'phaddir. Bu ko'phad uchun (1.41) formula aniq emas, chunki

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \prod_{k=1}^{n} \sin^2 \frac{x - x_k}{2}\,dx > 0$$

va

$$\sum_{k=1}^{n} A_k f(x_k) = 0$$

Demak, n tugunli kvadratur formulaning trigonometrik aniqlik tartibi $n-1$ dan ortmaydi. Endi ixtiyoriy $\alpha \in [0, \frac{2\pi}{n}]$ uchun ushbu

$$\int_0^{2\pi} f(x)\,dx \approx \frac{2\pi}{n}\sum_{k=1}^{n} f\,[\alpha + (k-1)\frac{2\pi}{n}], \qquad (1.42)$$

kvadratur formula barcha

$$f(x) = \cos kx, \quad \sin kx \quad (k = 1,2,\ldots,n-1)$$

funkstiyalar uchun aniq ekanini ko’rsatamiz. Buning uchun uning barcha

$$f(x) = e^{ikx} \quad (k = 1,2,\ldots,n-1)$$

funkstiyalar uchun aniq ekanini ko’rsatish kifoyadir. Agar $k = 0$ bo’lsa, $f(x) = 1$ bo’lib, (1.42) formula aniq ekani ravshandir. Endi $0 < k \le n-1$ bo’lsin. U holda

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} e^{ikx}\,dx = \frac{1}{k} e^{ikx}\Big|_0^{2\pi} = 0\,.$$

Shu bilan birga kvadratur yig’indi ham nolga teng :

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} e^{ik(\alpha+(i-1)\frac{2\pi}{n})} = e^{ik\alpha}\sum_{i=1}^{n} e^{ik(i-1)\frac{2\pi}{n}} = e^{ik\alpha}\,\frac{e^{ikn\frac{2\pi}{n}} - 1}{e^{ik\frac{2\pi}{n}} - 1} =$$

$$= e^{ik\alpha}\,\frac{e^{2\pi ik} - 1}{e^{ik\frac{2\pi}{n}} - 1} = 0$$

Shunday qilib, (1.42) formulaning trigonometrik aniqlik tartibi $n-1$ ga teng ekan.

Ixtiyoriy $\alpha \in [0, \frac{T}{n}]$ uchun

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx \approx \frac{T}{n}\sum_{k=1}^{n} f\,[\alpha + (k-1)\frac{T}{n}], \qquad (1.43)$$

kvadratur formulaning $n-1$ - tartibli ixtiyoriy

$$T_{n-1}(x)=a_0+\sum_{k-1}^{n}\left(a_k\cos\frac{2\pi}{T}kx+b_k\sin\frac{2\pi}{T}kx\right)$$

trigonometrik ko'phad uchun aniq tenglikka aylanishini ko'rsatish qiyin emas.

Misol sifatida ushbu

$$K(k)=\int_0^{2\pi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

to'liq elliptik integralning $k=0{,}5$ dagi qiymatini turt xona aniqlikda hisoblaylik. Integral ostidagi funkstiya juft va π davrli bo'lganligi sababli $K(0{,}5)$ ni

$$K(0{,}5)=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-0{,}25\cdot\sin^2\varphi}}$$

kurinishda yozish mumkin. Bu integralni hisoblash uchun (1.43) da $T=\pi$ deb olamiz. Endi $n=6$ deb olib, tugunlarni $\varphi=0$ nuqtaga nisbatan simmetrik ravishda joylashtiramiz:

$$\varphi_{\pm1}=\pm\frac{\pi}{12},\quad \varphi_{\pm2}=\pm\frac{\pi}{4},\quad \varphi_{\pm3}=\pm\frac{5\pi}{12}.$$

U holda

$$K(0{,}5)=\frac{1}{2}\cdot\frac{\pi}{6}\cdot 2\left[f(\frac{\pi}{12})+f(\frac{\pi}{4})+f(\frac{5\pi}{2})\right]=$$

$$=\frac{\pi}{6}\left[\frac{1}{\sqrt{0{,}9983}}+\frac{1}{\sqrt{0{,}8750}}+\frac{1}{\sqrt{0{,}7668}}\right]=$$

$$=\frac{\pi}{6}(1{,}0009+1{,}0691+1{,}419)=1{,}6816.$$

$K(0{,}5)$ ning jadvaldagi qiymati 1,6858. Demak, xato 0,0042 ga teng.

II.BOB. SFERIK GARMONIKALAR VA ULARNI QATORGA YOYISH.

2.1. Shar ko'phadlari va sferik garmonikalar.

Ushbu bobda boshlang'ich ma'lumotlar keltirilgan. Sababi keyingi tadqiqoq ishlarida va S.L. Sobolev funksional fazosida asosiy natijalarni olishda ulardan foydalanilgan. Sferik funksiyalar Laplas tenglamalarni yechishni o'rganishda va potensiallar nazariyasida qo'llaniladi. Sferik koordinatalar n – o'lchovli R^n Evklid fazosida quyidagi ko'rinishga ega.

Belgilaymiz: $r = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$,

$\theta = \dfrac{x}{|x|}$ va

$$\left.\begin{aligned}
&\theta_1 = \cos\varphi_1, \\
&\theta_2 = \sin\varphi_1 \cos\varphi_2, \\
&\theta_3 = \sin\varphi_1 \sin\varphi_2 \cos\varphi_3, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\theta_{n-2} = \sin\varphi_1 ... \sin\varphi_{n-3} \cos\varphi_{n-2}, \\
&\theta_{n-1} = \sin\varphi_1 ... \sin\varphi_{n-3} \sin\varphi_{n-2} \cos\varphi_{n-1}, \\
&\theta_n = \sin\varphi_1 ... \sin\varphi_{n-3} \sin\varphi_{n-2} \sin\varphi_{n-1},
\end{aligned}\right\} \qquad (2.1)$$

Bu yerda $0 \le \varphi_j \le \pi,\ \ j = \overline{1,(n-2)},\ \ 0 \le \varphi_{n-1} < 2\pi$.

Laplas operatorini berilgan sferik koordinatalarda ikkita operatorning yig'indisi ko'rinishida yozish mumkin.

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_1^2} = \Delta = \Delta_r + \frac{1}{r^2}\Delta_\theta, \qquad (2.2)$$

bu yerda

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}, \qquad \Delta_\theta = \sum_{j=1}^{n-1} \frac{1}{g_j [\sin\varphi_j]^{n-j-1}} \frac{\partial}{\partial \varphi_j}\left(\sin^{n-j-1}\varphi_j \frac{\partial}{\partial \varphi_j}\right),$$

$$q_1 = 1, \qquad q_j = \left(\prod_{k=1}^{j-1} \sin\varphi_k\right)^2,\ \ j > 1.$$

Δ_r operator Δ ning qattiy qismi deyiladi, Δ_θ bo'lsa Laplasning sferik operatori deyiladi.

Ta'rif 2.1.1. Bir jinsli k- tartibli n o'zgaruvchili garmonik ko'phadlar shar ko'phadlari deyiladi.

H_k fazoda barcha bir jinsli k- tartibli n o'zgaruvchili shar ko'phadlari chiziqli qism fazoni hosil qiladi. O'lchami esa quyidagiga teng.

$$\sigma(n,k)=\frac{(k+n-3)!}{k!(n-2)!}(n+2k-2). \tag{2.3}$$

Ma'lumki H_k fazoning o'lchami esa quyidagiga teng $\frac{(n+k-1)}{k!(n-1)!}$. Bundan quyidagi kelib chiqadi.

$$\sigma(n,k)=\frac{(n+k-1)!}{k!(n-1)!}-\frac{(n+k-3)!}{(k-2)!(n-1)!}=\frac{(n+k-3)!}{k!(n-2)!}(n+2k-2).$$

Ta'rif 2.1.2. $P_k(x)$ - ixtiyoriy k- tartibli shar ko'phadi bo'lsin. Unda quyidagi munosabat $\frac{P_k(x)}{|x|^k}=Y_k(\theta)$ sferik funksiya yoni k- tartibli garmonikalar deyiladi.

k – tartibli sferik funksiyalar differensial tenglamaning yechimi bo'ladi.

$$\Delta_\theta Y_k(\theta)+k(n+k-2)Y_k(\theta)=0 \tag{2.4}$$

Haqiqatdan ham, agar $Y_k(\theta)$ - k- tartibli sferik garmonikalar bo'lsa , unda $r^k Y_k(\theta)=u_k(x)$ k- tartibli shar ko'phadlari deyiladi.

Quyidagi lemma kelib chiqadi.

Lemma 2.1.1. $Y_k(\theta)$ va $Y_{k'}(\theta)$ - mos ravishda k va k' tartibli sferik garmonikalar bo'lsin, bu yerda $k\neq k'$, unda

$$\int_{S^{n-1}} Y_k(\theta)Y_{k'}(\theta)d\theta=0, \tag{2.5}$$

(2.5) dan ko'rinadiki, $Y_k(\theta)$ sferik garmonikalar k ning turlixil qiymatlarida sferaning sirtida orthogonal bo'ladi.

Isboti. $\theta \neq 0$ uchun R^n da $r=|x|$ va $\theta=\frac{x}{r}$ deb olamiz. Agar $Y_k(\theta)$ va $Y_{k'}(\theta)$- sferik garmonikalar k va k' tartibli bo'lsa, unda $r^k Y_k(\theta)=u_k(\theta)$ va $r^{k'} Y_{k'}(\theta)=\vartheta_{k'}(\theta)$ - mos ravishda k va k' tartibli shar ko'phadlari bo'ladi. Bunda $\theta \neq 0$ va $u_k(0)=\vartheta_{k'}(0)=0$bo'ladi hech bir k, yoki k' nolga teng bo'lmaganda. Agar aytaylik $k=0$bo'lsin, unda $Y_k(\theta)$ - o'zgarmas bo'ladi va $u_k(\theta)$ ni bu tenlik orqali ifodalaymiz. $u_k(\theta)$ va $\vartheta_{k'}(\theta)$ funksiyalarning hosilalari $\theta \in S^{n-1}$ nuqtada quyidagiga teng bo'ladi.

$$\left(dr^k/dr\right)Y_k(\theta)=kY_k(\theta) \text{ и } \left(dr^{k'}/dr\right)Y_{k'}(\theta)=k'Y_{k'}(\theta). \qquad (2.6)$$

Shunday qilib, $Y_k(\theta)$ va $Y_{k'}(\theta)$ - sferik garmonikalar bo'lsa , unda $u_k(\theta)$ va $\vartheta_{k'}(\theta)$ -lar esa fazodagi sferik garmonikalar bo'ladi. Shunday qilib (2.2) va Grin teoremasidan quyidagini olamiz.

$$0=\int\limits_{|\theta|\leq 1}\left[u_k(\theta)\Delta\vartheta_{k'}(\theta)-\vartheta_{k'}(\theta)\Delta u_k(\theta)\right]dx=\int\limits_{S^{n-1}}\left[u_k(\theta)\frac{\partial\vartheta_{k'}(\theta)}{\partial_r}-\vartheta_{k'}\frac{\partial u_k(\theta)}{\partial_r}\right]d\theta=$$

$$=\int\limits_{S^{n-1}}\left[k'Y_k(\theta)Y_{k'}(\theta)-kY_k(\theta)Y_{k'}(\theta)\right]d\theta=(k'-k)\int\limits_{S^{n-1}}Y_k(\theta)Y_{k'}(\theta)d\theta. \qquad (1.7)$$

Olingan natijaning (2.7) har ikkala tomonini $(k'-k)$ ga bo'lsak, $k' \neq k$ bo'lgan holda, teoremaning isboti kelib chiqadi.

2.2. Funkyiyalarni sferik garmonikalar orqali qatorlarga yoyish.

Yuqorida ko'rib o'tdikki turki xil tartibli sferik garmonikalar o'zaro ortogonal bo'ladi. Barcha k- tartibli shar ko'phadlaridan iborat fazoni o'chami $\sigma(n,k)$ teng bo'ladi.

$$Y_{k,\ell}(\theta), \qquad \ell = 1,2,...,\sigma(n,k); \quad k = 0,1,2,.... \tag{2.8}$$

Bu yerda $Y_{k,\ell}(\theta)$ funksiya k tartibli ℓ ko'rinishdagi sferik garmonikalar deyiladi.

Ixtiyoriy funksiyani $f(\theta) \in L_1$ sferik garmonikalar orqali Fure qatoriga yoyish mumkin.

$$f(\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell} Y_{k,\ell}(\theta), \tag{2.9}$$

bu yerda

$$a_{k,\ell} = \int_S f(\theta) Y_{k,\ell}(\theta) d\theta. \tag{2.10}$$

Bu qator L_2 fazoda $f(\theta)$ funksiyaga yaqinlashadi, agar $f(\theta) \in L_2$ tegishli bo'lsa. Ko'rinib turibdiki, $Y_k(\theta)$ (2.4) tenglamaning barcha xos funksiyalaridan iborat sistemani tashkil qiladi.

Har qanday boshqa xos funksiya barcha sferik garmonikalarga ortogonal boladi va nolga aylanishi kelib chiqadi.

Shunday qilib quyidagi teorema kelib chiqadi.

Teorema 2.2.1. k ning yetarlicha katta qiymatida (2.8) funksiyani ixtiyoriy aniqlikda radiusi 1 ga teng bo'lgan sferada uzluksiz funksiyalar orqali tekis yaqinlashuvchi qatorga yoyish mumkin.

Ushbu teoremani isbotini bir qancha usullari bor [82].

Lemma 2.2.1. Radiusi birga teng bo'lgan sferada ixtiyoriy $f(\theta)$ uzluksiz funksiyani kophad ko'rinishda qanchalik katta aniqlikda kerak bo'lsa ifodalash mumkin.

$$Q(x) = \sum_{|\alpha|=k} A_\alpha x^\alpha , \tag{2.11}$$

bu yerda $\alpha=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$, α_i - butun sonlar , $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$ va $x^{\alpha}=x_1^{\alpha_1}\cdot x_2^{\alpha_2}...x_n^{\alpha_n}$.

2.2.1 lemmaning isbotini $n=3$ uchun keltirilgan([82,84]).

θ va θ' - birlik vektorlar bo'lsin. Unda γ burchak quyidagiga teng.

$\gamma=\arccos\left(\theta,\theta'\right)$ yoki $\cos\gamma=\sum_{j=1}^{n}\theta_j\theta_j'$.

Integralning qiymati

$$F\left(k\right)=\int_S\left(\frac{1+\cos\gamma}{2}\right)^k d\theta' \qquad (2.12)$$

θ nuqtaning joylashuviga bog'liq emas, uni x_1 o'qidan olish mumkin.

Unda $\cos\gamma=\cos\varphi_1'$ va ushbu integralni hisoblab, topamiz (φ_i' ni φ_i ga almashtiramiz)

$$F\left(k\right)=\underbrace{\int_0^{\pi}...\int_0^{\pi}}_{n-2\ \text{ðàçà}}\int_0^{2\pi}\left(\frac{1+\cos\varphi_1}{2}\right)^k\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2...\sin\varphi_{n-2}d\varphi_1...d\varphi_{n-1}=$$

$$=\underbrace{\int_0^{\pi}...\int_0^{\pi}}_{n-3\ \text{ðàçà}}\int_0^{2\pi}\sin^{n-3}\varphi_2...\sin\varphi_{n-2}d\varphi_2...d\varphi_{n-1}\int_0^{\pi}\left(\frac{1+\cos\varphi_1}{2}\right)^k\sin^{n-2}\varphi_1 d\varphi_1. \qquad (2.13)$$

(2.13) ning o'ng tomonida ifodaning oldiga $\left(n-2\right)$ - ta aniq integral turibdi, malumki, равно значению поверхности сферы $\left(n-1\right)$ - o'lchovli fazoda sfera sirtini qiymati $\dfrac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$ gat eng.

(2.13) ning ong tomonini oxirgi integralini hisoblab topamiz.

$$\int_0^{\pi}\left(\frac{1+\cos\varphi_1}{2}\right)^k\sin^{n-2}\varphi_1 d\varphi_1=2^{n-2}\frac{\Gamma\left(k+\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(k+n-1\right)}. \qquad (2.14)$$

Shunday qilib, (2.13) va (2.14) ga asosan (2.12) integral uchun oxirgi ifoda ko'rinishda bo'ladi.

$$F(k)=\left(2\sqrt{\pi}\right)^{n-1}\frac{\Gamma\left(k+\frac{n-1}{2}\right)}{\Gamma(k+n-1)}.$$

Strirling formulasini qo'llab $F(k)$ uchun k ni oshirish orqali asimtotik yurish mumkin. Hisoblashlar ko'rsatadiki , $k\to\infty$ ga intilganda

$F(k),\ Ak^{-\frac{n-1}{2}}$,

$$F(k)\approx\frac{A}{k^{\frac{n-1}{2}}}. \tag{2.15}$$

Quyidagi ifodani ko'raylik

$$Q_k=\frac{1}{F(k)}\int_S\left(\frac{1+\cos\gamma}{2}\right)^k f(\theta')d\theta'. \tag{2.16}$$

Ma'lumki, Q_k - k - tartibli ko'phad θ ga nisbatan, yoki $x=(x_1,...,x_n)$. Shunday qilib, agar $Q_k-f=\varepsilon(k)$ ning farqi nolga tekis intilsa $k\to\infty$ intilganda, unda lemma isbotlangan bo'ladi.

θ nuqtaning atrofini sferik soha bilan $\gamma<\delta$ belgilaymiz, barcha θ' lar uchun bu sohada quyidagi tengsizlik bajarilsa.

$$\left|f(\theta)-f(\theta')\right|<\frac{\varepsilon}{2}. \tag{2.17}$$

uzluksiz $f(\theta)$ funksiya uchun sferada shunday M o'zgarmas topiladiki

$$\left|f(\theta)\right|\le M \tag{2.18}$$

Shunday qilib,

$$\left|Q_n-f\right|=\frac{1}{F(k)}\left|\int_S\left(\frac{1+\cos\gamma}{2}\right)^k\left[f(\theta')-f(\theta)\right]d\theta'\right|\le\frac{1}{F(k)}\int_{\gamma\le\delta}\left(\frac{1+\cos\gamma}{2}\right)^k\cdot$$

$$\cdot\left|f(\theta')-f(\theta)\right|d\theta'+\frac{1}{\tilde{A}(k)}\int_{\gamma\ge\delta}\left(\frac{1+\cos\gamma}{2}\right)^k 2Md\theta'<\frac{\varepsilon}{2}+\frac{2M}{F(k)}\left(\frac{1+\cos\delta}{2}\right)^k\int_{\lambda\ge\partial}d\theta'\le$$

$$\le\frac{\varepsilon}{2}+\frac{2M}{\delta(k)}\left(\frac{1+\cos\delta}{2}\right)^k\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{2.19}$$

$F(k)$ funksiya k ni oshirish orqali tez kamaymaydi, $k^{-\frac{n-1}{2}}$ ga nisbatan, $\left(\frac{1+\cos\delta}{2}\right)<1$ da, qanchalik k ni katta tanlasak quyidagi tengsizlik o'rinli bo'ladi.

$$\frac{4\pi^{\frac{\pi}{2}}M}{\Gamma\left(\frac{n}{2}\right)\Gamma(k)}\left(\frac{1+\cos\delta}{2}\right)^{k}<\frac{\varepsilon}{2}.$$

Lemma 2.2.1 isbotlandi.

Endi 2.2.1 teoremani isboti 2.2.1 lemmadan kelib chiqadi.

Shubday qilib L_1 dagi ixtiyoriy funksiya funksiya (2.8) funksiyaga yaqinlashadi. Har qanday funksiyani $f(\theta)\in L_1$ sferik garmonikalar orqali Fure qatoriga yoyish mumkin.

$$f(\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}Q_{k,\ell}Y_{k,\ell}(\theta), \tag{2.20}$$

bu yerda

$$Q_{k,\ell}=\int_S f(\theta)Y_{k,\ell}(\theta)d\theta. \tag{2.21}$$

Agar $f(\theta)\in L_2^m(S)$ fazoga tegishli bo'lsa yoki kamida $f(\theta)\in L_2(S)$ fazoga tegishli bo'lsa bu funkstiyani yaqinlashuvchi sferik garmonikalar qatoriga yoyish mumkin $f(\theta)=\sum_{k=1}^{\infty}Y_k(\theta)$ bu yerda qator yaqinlashuvchi bo'lishi uchun 2m>n shart bajarilishi kerak.

Buni sferik garmonikalar bo'yicha Fur'e qatoriga yoysak $f(\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)$ ga teng bo'ladi.

Bu yerda $Y_{k,\ell}(\theta)$ - k – tartibli ℓ ko'rinishdagi sferik garmonikalar

$a_{k,\ell}=\int_S f(\theta)Y_{k,\ell}(\theta)$

1.3. $L_2^{(m)}(S)$ fazoda funksiyalarni sferik garmonikalar orqali qatorlarga yoyish.

Bu paragraf sferik garmonikalarni muhim xossalanini aniqlashga va $L_2^m(S)$ fazoda funksiyani sferik garmonikalar orqali qatorlarga yoyishga bag'ishlangan.

Funksiyani integrallash uchun sferada qaraladigan masala berilgan funksiyani integralini hisoblash qulay hisoblanadi funksiyani sferada o'rta qiymati bo'yicha integralini hisoblashdan ko'ra.

Bir qancha sferada integrallanuvchi funksiyalarni sferada o'rta qiymatiga teng bo'lgan o'zgarmas funksiyalar yig'indisi ko'rinishida ifodalash mumkin. Funksiyaning o'rta qiymati nolga teng.

$L_2(S)$ funksiyalar fazosi birlik S sferada berilgan bo'lsin va funksiyalar S sferada kvadrati bilan integrallanuvchi bo'lsin. $\tilde{L}_2(S)$ fazo orqali mos ravishda $L_2(S)$ fazoda sferada o'rta qiymati nolga teng bo'lgan funksiyalar fazosini belgilaymiz.

Agar funksiya $f(\theta)\in L_2(S)$ fazoga tegishli bo'lsa, unda uni sferik garmonikalar orqali qatorga yoyish mumkin.

$$f(\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{\sigma(n.k)}Q_{k,\ell}Y_{k,\ell}(\theta)=\sum_{k=0}^{\infty}Y_k(\theta), \tag{2.22}$$

bu yerda

$$Q_{k,\ell}=\int_S f(\theta)Y_{k,\ell}(\theta)dS\,. \tag{2.23}$$

Bu yerda $Y_{k,\ell}(\theta)$ - k – tartibli ℓ ko'rinishdagi sferik garmonikalar

$\sigma(n,k)=\dfrac{(k+n-3)!}{k!(n-2)!}(n+2k-2)$ - chiziqli bog'liq bo'lmagan n o'zgaruvchili sferik garmonikalar soni.

Salixovning ishlarida $L_2^m(S)$ fazoda funksiyani sferik garmonikalar orqali ifodalab funksiyaning normasi quyidagicha kiritilgan.

$$\left\| f / L_2^m(S) \right\|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m (k+n-2)^m . \qquad (2.24)$$

$\tilde{L}_2^m(S)$ fazo orqali mos ravishda $L_2^m(S)$ fazoda sferada o'rta qiymati nolga teng bo'lgan funksiyalar fazosini belgilaymiz.

Yarim norma (2.24) $\tilde{L}_2^m(S)$ da norma bo'ladi, quyida ko'rsatiladiki, $\tilde{L}_2^m(S)$- fazoni to'la ekenligi.

$\overline{f}(x)$ orqali quyidagi belgilashni kiritamiz.

$$D^\alpha \overline{f}(x) = \frac{\partial^{|\alpha|} \overline{f}(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

Bu yerda hosila dekart koordinatalar bo'yicha olinadi $\theta \in S$ bunda $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

$L_2^{(m)}(S)$ - fazo bu shunday fazoki bunda funksiyalar S da aniqlanib, m – tartibli umumlashgan hosilalari kvadrati bilan jamlanuvchi funksiyalar fazosi va unda funksiyani normasi quyidagicha kiritilgan.

$$\left\| f / L_2^{(m)}(S) \right\| = \left\{ \int_S \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(D^\alpha f(\theta) \right)^2 dS \right\}^{1/2}. \qquad (2.25)$$

Agar $f(\theta)$ funksiyalar uchun (1.24) norma chekli bo'lsa, unda (1.25) norma chekli bo'ladi va aksincha.

Ushbu faktni isbotini keltiramiz.

$f(\theta)$ funksiya uchun (2.25) chekli bo'lsin. Unda Mixlinda keltirilgan teoremaga asosan kelib chiqadiki, (2.22) ni koeffisiyetlarini quyidagi ko'rinishda ifodalash mumkin.

$$a_{k,\ell} = k^{-m} \beta_{k,\ell}, \quad k \ge 1, \qquad (2.26)$$

bu yerda

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \left| \beta_{k,\ell} \right|^2 < \infty . \qquad (2.27)$$

Shunday qilib,

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m (k+n-2)^m = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} k^{-2m}\beta_{k,\ell}^2 k^m (k+n-2)^m =$$
$$= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \beta_{k,\ell}^2\left(1-\frac{n}{k}-\frac{2}{k}\right)^m \le (n-1)^m \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \beta_{k,\ell}^2 < \infty. \tag{2.28}$$

(2.28) dan kelib chiqadiki, agar (2.25) chekli bo'sa (2.24) ham chekli bo'ladi.

Quyidagi belgilashni kiritamiz.

$$a_{k,\ell} k^{\frac{m}{2}} (k+n-2)^{\frac{m}{2}} = \gamma_{k,\ell}\,. \tag{2.29}$$

bundan

$$k^{-m}\gamma_{k,\ell} = a_{k,\ell}\left(1+\frac{n}{k}-\frac{2}{k}\right)^{\frac{m}{2}}$$

yoki

$$a_{k,\ell} = k^{-m}\gamma_{k,\ell}\left(1+\frac{n}{k}-\frac{2}{k}\right)^{-\frac{m}{2}} = k^{-m}\beta_{k,\ell}\quad . \tag{2.30}$$

Keyingi hisoblashlarda quyidagiga ega bo'lamiz.

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \left|\beta_{k,\ell}\right|^2 = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \left|\gamma_{k,\ell}\right|^2\left(1+\frac{n}{k}-\frac{2}{k}\right)^{-m} \le \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \left|\gamma_{k,\ell}\right|^2 < \infty\,. \tag{2.31}$$

Mixlin teoremasiga asosan va (2.30), (2.31) lardan (2.25) chekli ekanligi kelib chiqadi.

Shunday qilib Salixov isbotladiki $L_2^m(S)$ va $L_2^{(m)}(S)$ fazolar elementlarining tuzilishiga ko'ra bir xil fazolar ekan. $f(\theta)$ va $g(\theta)$ funksiyalar uchun $\tilde{L}_2^m(S)$ fazoda skalyar ko'paytmani quyidagi tenglik orqali aniqlaymiz.

$$\left[f(\theta), g(\theta)\right] = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell} B_{k,\ell} k^m (k+n-2)^m, \tag{2.32}$$

bu yerda

$$B_{k,\ell} = \int_S g(\theta) Y_{k,\ell}(\theta)\, dS\,. \tag{2.33}$$

Unda $\tilde{L}_2^m(S)$ fazo skalyar ko'paytmaga nisbatan unitar fazoni tashkil qiladi. Uning to'la fazo ekanligi isbotlandi va u gilbert fazosi bo'ladi.

$\{f_N(\theta)\}$ ketma-ketlik - $\tilde{L}_2^m(S)$ fazodagi fundamental funksiyalar ketma-ketligi bo'lsin. Shunday funksiya mavjud bo'lsinki $f(\theta)\in\tilde{L}_2^m(S)$, quyidagi bajarilsin.

$$\lim_{N\to\infty}\left\| f_N(\theta)-f(\theta)/\tilde{L}_2^m(S)\right\|=0. \quad (2.34)$$

$\{f_N(\theta)\}$ ketma-ketlik, ko'rinib turibdiki fundamental va normasi quyidagiga teng.

$$\|f\|^2=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}^2\,. \quad (2.35)$$

Shunday qilib $\tilde{L}_2(S)$ fazoda (2.35) ko'rinishda funksiyani normasi aniqlandi, demak $\tilde{L}_2(S)$ fazoni to'laligidan kelib chiqadiki, shunday element mavjudki $f(\theta)\in\tilde{L}_2(S)$ da

$$f_N(\theta)\xrightarrow[N\to\infty]{}f(\theta) \qquad \tilde{L}_2(S) \text{ fazoda}. \quad (2.36)$$

$f(\theta)$ funksiya uchun norma chekliligi isbotlanadi (2.21) dan va

$$\lim_{N\to\infty}\left\| f_N(\theta)-f(\theta)/\tilde{L}_2^m(S)\right\|=0. \quad (2.37)$$

Fazo $\{a_{k,\ell}\}$ ketma – ketliklarni chekli normasi (2.24) bilan to'la bo'ladi bundan kelib chiqadiki, bu fazoda shunday $\{a_{k,\ell}\}$ ketma-ketlik topiladiki, qaysiki bu fundamental ketma-ketlik $\{a_{k,\ell}^{(k)}\}$ bo'ladi, $f_N(\theta)$ funksiya sferik garmonikalar bo'yicha yoyiladi:

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\left(a_{k,\ell}^{(N)}-a_{k,\ell}\right)k^m(k+n-2)^m\xrightarrow[N\to\infty]{}0. \quad (2.38)$$

Isbotlandiki, $a_{k,\ell}$ Fure koeffisiyentlari $f(\theta)$funksiyaning. Из (1.38) имеем

$$a_{k,\ell}^{(N)}\xrightarrow[N\to\infty]{}a_{k,\ell}\,, \quad (2.39)$$

(2.36) dan

$$a_{k,\ell}^{(N)} = \int_S Y_{k,\ell}(\theta)\, f_N(\theta)\, dS \xrightarrow[N\to\infty]{} \int_S Y_{k,\ell}(\theta)\, f(\theta)\, dS\,. \tag{2.40}$$

(2.39) va (2.40) dan kelib chiqadiki

$$a_{k,\ell} = \int_S Y_{k,\ell}(\theta)\, f(\theta)\, dS\,.$$

Oxirgi tenglikda va (2.38) munosabatdan (2.34) kelib chiqadi. Asosiysi $\tilde{L}_2^m(S)$ fazoni to'laligi isbotlandi.

$\tilde{L}_2^m(S)$ fazoga nisbatan yana bir faktni keltirish mumkin, agan $2m > n$ bo'lsa, unda $\tilde{L}_2^m(S)$ fazo $C(S)$ uzluksiz funksiyalar fazosida yotadi.

Haqiqatdan ham (2.24) dan ixtiyoriy k uchun quyidagi kelib chiqadi

$$k^m (k+n-2)^m \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 \le \left\| f / \tilde{L}_2^m(S) \right\|^2 \tag{2.41}$$

yoki

$$\left\| Y_k / L_2(S) \right\|^2 = \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 \le \frac{A(n)}{k^{2m}} \left\| f / \tilde{L}_2^m(S) \right\|^2. \tag{2.42}$$

Boshqa tomondan ma'lumki [1]

$$\left| Y_k(\theta) \right| \le B(n) \left\| Y_k / \tilde{L}_2(S) \right\| k^{\frac{n}{2}-1}. \tag{2.43}$$

(2.42) va (2.43) baholardan foydalanib quyidagi tengsizlikni yozish mumkin

$$\max_{\theta\in S} \left| Y_k(\theta) \right| \le C(n)\, k^{-m+\frac{n}{2}-1} \left\| f / \tilde{L}_2^m(S) \right\|. \tag{2.44}$$

(2.44) bahodan quyidagi funksional qator kelib chiqadi

$$f(\theta) = \sum_{k=0}^{\infty} Y_k(\theta) \tag{2.45}$$

Bu tekis yaqinlashati S da. Bu yerda $f(\theta)$ funksiya S da uzlusiz (2.45).

$C(S)$ da uni normasini quyidagicha baholaymiz:

$$\|f(\theta)/C(S)\| = \max_{\theta\in S}|f(\theta)| = \max_{\theta\in S}\left|\sum_{k=0}^{\infty}Y_k(\theta)\right| \le \sum_{k=0}^{\infty}\max_{\theta\in S}|Y_k(\theta)| \le$$
$$\le C(n)\|f(\theta)/\tilde{L}_2^m\|\sum_{k=1}^{\infty}k^{\frac{n-2m}{2}-1} = C_1(n_1,m)\|f(\theta)/\tilde{L}_2^m(S)\|, \qquad (2.46)$$

bu yerda $C_1(n_1,m)$ - o'zgarmas.

Yuqoridagi mulohazalardan quyidagi teorema kelib chiqadi, bu teorema Salixov tomonidan isbotlangan [1].

Teorema 2.3.1. Agar $f(\theta)\in L_2^m(S)$ bo'sa, unda (2.45) qator absolyut va tekis yaqinlashuvchi bo'lishi uchun $2m>n$ shart bajarilishi yetarli.

Shunday qilib Salixov n o'lchovli birlik sferada $f(\theta)\in L_2^m(S)$ fazoda quyidagilarni aniqladi. $f(\theta)\in L_2^m(S)$ fazoga tegishli bo'lsin yoki kamida $f(\theta)\in L_2(S)$ fazoga tegishli bo'lsa bu funkstiyani yaqinlashuvchi sferik garmonikalar qatoriga yoyish mumkin $f(\theta)=\sum_{k=1}^{\infty}Y_k(\theta)$ bu yerda qator yaqinlashuvchi bo'lishi uchun 2m>n shart bajarilishi kerak.

Buni sferik garmonikalar bo'yicha fur'e qatoriga yoysak

$f(\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)$ ga teng bo'ladi.

Bu yerda $Y_{k,\ell}(\theta)$ - k – tartibli ℓ ko'rinishdagi sferik garmonikalar

$$a_{k,\ell}=\int_S f(\theta)Y_{k,\ell}(\theta)$$

Fserik garmonikalarning turli xil ko'rinishlari bor. Ulardan ba'zilarini keltirib chiqarishni ko'rib chiqamiz [1].

$$Y_k(\theta)=C_k^{\frac{n-2}{2},k_1}(\cos\varphi_1)C_{k_1}^{\frac{n-3}{2},k_2}(\cos\varphi_2)\ldots C_{k_{n-3}}^{\frac{1}{2},k_{n-2}}(\cos\varphi_{n-2})e^{\pm ik_{n-2}\varphi_{n-1}} \qquad (2.47)$$

bu yerda $k\ge k_1\ge k_2\ge\ldots\ge 0$ - butun sonlar va

$$C_p^{\ell,m}(x)=const(1-x^2)^{\frac{m}{2}}F\left(\frac{P+m+2\ell}{2},-\frac{P-m}{2},\frac{1}{2},x^2\right), \qquad (2.48)$$

yoki

$$C_P^{\ell,m}(x) = const\left(1-x^2\right)^{\frac{m}{2}} xF\left(\frac{P+m+2\ell+1}{2}, -\frac{P-m-1}{2}, \frac{3}{2}, x^2\right) \qquad (2.49)$$

$P-m$ ning juft yoki toqligiga bog'liq.

N o'lchovli fazoda qutb koordinatalarga otamiz.

$$\left.\begin{array}{l}
\theta_1 = \cos\eta_1\cos\varphi_1, \\
\theta_2 = \cos\eta_1\sin\varphi_1, \\
\theta_3 = \sin\eta_1\cos\eta_2\cos\varphi_2, \\
\theta_4 = \sin\eta_1\cos\eta_2\sin\varphi_2, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\theta_{2q-2} = \sin\eta_1\sin\eta_2 ...\cos\eta_{q-1}\sin\varphi_{q-1}, \\
\theta_{2q-1} = \sin\eta_1\sin\eta_2 ...\sin\eta_{q-1}\cos\varphi_q, \\
\theta_{2q} = \sin\eta_1\sin\eta_2 ...\sin\eta_{q-1}\sin\varphi_q,
\end{array}\right\} \qquad (2.50)$$

$$0 \le \eta \le \frac{\pi}{2}, \quad 0 \le \varphi_j,\ \varphi_q < 2\pi, \quad j = 1,2,...,q-1,$$

agar $n = 2q$, va oxirgi ikkita qator (2.50) quyidagi ko'rinishda bo'lsa

$$\left.\begin{array}{l}
\theta_{2q} = \sin\eta_1\sin\eta_2 ...\cos\eta_q\sin\varphi_q, \\
\theta_{2q+1} = \sin\eta_1\sin\eta_2 ...\sin\eta_q,
\end{array}\right\} \qquad (2.51)$$

agar $n = 2q+1$; bunda $-\frac{\pi}{2} \le \eta_q \le \frac{\pi}{2}$.

Bu qutb koordinatalarda k – tartibli sferik garmonikalar quyidagi ko'rinishda bo'ladi [1].

$$Y_k^n = e^{\pm im_1\varphi_1} D_{k_2 m_1}^{q_1 k_1}\left(\cos\eta_1\right) e^{\pm im_2\varphi_2} \times D_{k_1 m_2}^{q-1,k_2}\left(\cos\eta_2\right) ... D_{k_{q-2},m_{q-1}}^{1,k_{q-1}}\left(\cos\eta_{q-1}\right) e^{\pm ik_{q-1}\varphi_q}, \qquad (2.52)$$

agar $n = 2q$, va

$$Y_k^n = e^{\pm im_1\varphi_1} D_{k_1 m_1}^{q-\frac{1}{2},k_1}\left(\cos\eta_1\right) e^{\pm im_2\varphi_2} D_{k_1,m_2}^{q-\frac{3}{2},k_2}\left(\cos\eta_2\right) ... D_{k,m_q}^{\frac{1}{2},0}\left(\cos\eta_q\right), \qquad (2.53)$$

agar $n = 2q+1$.

bunda

$$D_{k,m}^{q,\ell}(x) = x^{\ell}\left(1-x^2\right)^{\frac{m}{2}} \times F\left(\frac{k+\ell+m+2q}{2}, -\frac{n-\ell-m}{2}; q-1; x^2\right), \qquad (2.54)$$

bu yerda $\quad k-m-\ell = 2p, \quad p = 0,1,2,...$.

(2.52) va (2.53) sferik garmonikalar aniq holarlarda quyidagi ko'rinishlarga ega bo'ladi.

$n=3$ uchun uning ko'rinishi quyidagicha bo'ladi:

$$Y_k^3 = e^{\pm im\varphi} P_k^{(m)}(\cos\eta), \tag{2.55}$$

bunda $P_k^{(m)}(\cos\eta)$ - birlashtirilgan Lejandr funksiyalari.

$n=4$ ushun uning ko'rinishi quyidagicha bo'ladi [1]:

$$Y_k^4 = e^{\pm i(m_1\varphi_1 + m_2\varphi_2)} P_k^{(m_1,m_2)}(\cos\eta) \quad , \tag{2.56}$$

bunda

$$\begin{gathered} P_k^{(m_1,m_2)}(\cos\eta) = \cos^{(m_1)}\eta\Bigg[(\sin\eta)^{k-|m_1|} + \sum_{j=0}^{\left[\frac{k-|m_1|-|m_2|}{2}\right]-1} (-1)^{j+1} \times \\ \times \prod_{p=0}^{j} \frac{\left[(k-|m_1|-2p)^2 - m_2^2\right]}{\left[(|m_1|+2p+2)^2 - m_1^2\right]} (\cos\eta)^{2j+2} (\sin\eta)^{k-|m_1|-2_j-2} \end{gathered} \tag{2.57}$$

va

$$\left.\begin{aligned} |m_1| + |m_2| \le k \\ (m_1 + m_2) \equiv k(\operatorname{mod} 2) \end{aligned}\right\}. \tag{2.58}$$

n=5 bo'lgan holda uning ko'rinishi quyidagicha bo'ladi [1]:

$$Y_k^5 = e^{\pm i(m_1\varphi_1 + m_2\varphi 2)} P_k^{(m_1,m_2,\mu)}(\cos\eta_1, \sin\eta_2) \ , \tag{2.59}$$

bunda

$$\begin{gathered} P_k^{(m_1,m_2,\mu)} = \cos_{\eta_1}^{(m)} \sin_{\eta_1}^{\mu} \sin_{\eta_2}^{\mu} \Bigg[(\sin\eta_1 \cos\eta_2)^{k-|m_1|-\mu} + \sum_{j=0}^{\left[\frac{k-|m_1|-|m_2|}{2}\right]-1} (-1)^{j+1} \times \\ \times \prod_{p=0}^{j} \frac{(k-|m_1|-2p-\mu)^2 - m_2^2 + \mu(\mu+1)(\mu+2)(\mu-2|m_2|)}{(|m_1|+2p+2)^2 - m_1^2} \times \\ \times (\cos\eta)^{2j+2} (\sin\eta_1 \cos\eta_2)^{k-|m_1|-2j-\mu-2} \Bigg]. \end{gathered} \tag{2.60}$$

bu yerda
$$\left.\begin{aligned} |m_1| + |m_2| + \mu \le k \\ (m_1 + m_2 + \mu) \equiv k(\operatorname{mod} 2) \end{aligned}\right\}. \tag{2.61}$$

III. BOB. $L_2^{(m)}(S)$ SOBOLEV FAZOSIDA YAQINLASHISH TARTIBI BO'YICHA OPTIMAL KUBATUR FORMULALAR

3.1. $L_2^{(m)}(S)$ Sobolev fazosida oddiy kubatur formulaning xatolik funksionali normasini hisoblash va ekstremal funksiyani aniqlash.

Oxirgi vaqtlarda integrallarni taqribiy hisoblash uchun kubatur formulalar qurishda sfera sirtida, sferuk garmonikalar nazariyasidan foydalaniladi. Bunda integral ostidagi funksiya sferik garmonikalar bo'yicha qatorlarga yoyiladi. Bu bo'yicha ma'lumotlar oldingi bobda to'liq keltirilgan, bu bobda esa asosan oddiy va vaznli kubatur formulalarni qarab chiqamiz. Quyidagi oddiy kubatur formulani qarab chiqamiz

$$\int_S f\ (\theta)d\theta \approx \sum_{\lambda=1}^{N} C_\lambda f(\theta^{(\lambda)}), \tag{3.1}$$

$L_2^{(m)}(S)$ fazoda va uning xatolik funksionalini normasini hisoblaymiz.

$L_2^{(m)}(S)$ fazo berilgan bo'lsin

$$\theta = \frac{x}{|x|} = (\theta_1, \theta_2, ..., \theta_{n-1}).$$

$$D^\alpha \overline{f}(\theta) = \left. \frac{\partial^{|\alpha|} \overline{f}(\theta)}{\partial\theta_1^{\alpha_1} ....\partial\theta_{n-1}^{\alpha_{n-1}}} \right|_{|\theta|=1}$$

$$\theta = (\theta_1, ..., \theta_{n-1}) \in S\ , \alpha = (\alpha_1, \alpha_2, ..., \alpha_{n-1}),$$
$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_{n-1}\ \ è\ \ \alpha! = \alpha_1!.....\alpha_{n-1}!;$$

Ta'rif 3.1.1. $L_2^{(m)}(S)$ fazo quyidagicha aniqlangan funksiyalar fazosi , S birlik sferada berilgan va m tartibli umumlashgan hosilalari kvadrati bilan jamlanuvchi funksiyalar fazosi, funksiya normasi quyidagicha kiritilgan.

$$\| f \mid L_2^m(S) \|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m \left(k+n-2\right)^m \tag{3.2}$$

$L_2^{(m)}(S)$ fazoda aslida Sobolev ta'rifi bo'yicha norma quyidagicha kiritilgan.

$$\left\| f / L_2^{(m)}(S) \right\| = \left\{ \int_S \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(D^\alpha f \right)^2 ds \right\}^{\frac{1}{2}}, \tag{3.3}$$

Shubday qilib $L_2(S)$ - S birlik sferada berilgan va kvadrati bilan jamlanuvchi funksiyalar fazosini bildiradi.

Agar funksiya $L_2(S)$ fazoga tegishli bo'lsa $f(\theta) \in L_2(S)$, unda uni ortonormallangan sferik garmonikalar bo'yicha qatorga yoyish mumkim [1]

$$f(\theta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell} Y_{k,\ell}(\theta) = \sum_{k=1}^{\infty} Y_k(\theta),$$

bu yerda

$a_{k,\ell} = \int_S f(\theta) Y_{k,\ell}(\theta) d\theta$, k – tartibli ℓ ko'rinishdagi ortonormallangan sferik garmonikalar ;

$\sigma(n,k) = (2k+n-2)\dfrac{(k+n-3)!}{(n-2)!k!}$ - k – tartibli chiziqli bog'liq bo'lmagan sferik garmonikalar soni.

$L_2^m(S)$ fazo aniqlaydiki, $f(\theta) \in L_2(S)$, funksiyalar uchun yarim norma bo'lishini [1]:

$$\| f \mid L_2^m(S) \|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m (k+n-2)^m .$$

$L_2^m(S)$ fazo elementlarining tuzilishi bo'yicha $L_2^{(m)}(S)$ S.L.Sobolev fazosi bilan ustma ust tushadi [2].

(3.1) ko'rinishdagi kubatur formulaning xatolik funksionali quyidagicha bo'ladi.

$$\ell_N(\theta) = \varepsilon_S(\theta) - \sum_{\lambda=1}^{N} C_\lambda \, \delta(\theta - \theta^{(\lambda)}), \tag{3.4}$$

bu yerda - Dirakning delta funksiyasi, $\varepsilon_S(\theta)$ - S sohaning xarakteristik funksiyasi

$$\varepsilon_{(S)}(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

$<\delta(x), f(x)> = f(0)$

Kubatur formulaning xatoligi deganda biz quyidagi ayirmani tushunamiz:

$$<\ell_N, f>=\int_S f(\theta)d\theta-\sum_{\lambda=1}^{N} C_\lambda f(\theta^{(\lambda)})=\int_{R^n}\ell_N(\theta)f(\theta)d\theta,$$

$$\ell_N(\theta)=\varepsilon_S(\theta)-\sum_{\lambda=1}^{N}C_\lambda\,\delta(\theta-\theta^{(\lambda)})$$

$$\varepsilon_{(S)}(x)=\begin{cases}1, & x\in S\\ 0, & x\notin S\end{cases}, \qquad <\delta(x), f(x)>=f(0)$$

(3.1) ko'rinishdagi kubatur formulaning xatoligi $L_2^m(S)$ fazoda chiziqli funksionalni tashkil qiladi. Bundan $2m>n$, $L_2^{(m)}(S)\to C(S)$ kelib chiqadi [1].

Quyidagi teoremani qarab chiqamiz. Bu teorema G'.N. Salixov tomonidan kiritilgan va isbotlangan lekin isboti qisqa keltirilgan. Biz uning isbotini to'liq keltiramiz sababi keyingi olinadigan asosiy natijalarda ham biz undan foydalanamiz.

Teorema 3.1.1. (3.1) ko'rinishdagi kubatur formulaning xatolik funksionalini normasi $L_2^m(S)$ fazoda quyidagiga teng

$$\left\|\ell_N/L_2^{m*}(S)\right\|=\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m(k+n-2)^m}\right\}^{1/2}. \tag{3.5}$$

Isboti. Ma'lumki agar funksiyimiz $f(\theta)\in L_2^m(S)$ ga tegishli bo'lsa, unda quyidagi qator absolyut tekis yaqinlashuvchi bo'ladi.

$$f(\theta)=\sum_{k=0}^{\infty}Y_k(\theta),$$

bu yerda $Y_k(\theta)$ - k- tartibli sferik garmonikalar, bunda $2m>n$ shart bajarilishi yetarli.

Shunday qilib funksiyani $f(\theta)\in L_2^m$ sferik garmonikalar bo'yicha absolyut tekis yaqinlashuvchi qatorga yoysak

$$f(\theta)=\sum_{k=1}^{\infty}Y_k(\theta)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta), \tag{3.6}$$

bu yerda $Y_{k,\ell}(\theta)$ - sferik garmonikalar k - tartibli ℓ ko'rinishdagi.

Xatolik funksionali (3.4) va (3.6) dan foytalanib quyidagini topamiz

$$\left|<\ell_N(\theta), f(\theta)>\right|=\left|<\varepsilon_s(\theta)-\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}Y_k(\theta)>\right|=$$

$$=|<\varepsilon_s(\theta),\sum_{k=1}^{\infty}Y_k(\theta)>-<\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}Y_k(\theta)>|=$$

$$=|\int_S\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)d\theta-<\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)>|=$$

$$=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}\int_S Y_{k,\ell}(\theta)d\theta-\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}\sum_{\lambda=1}^{N}C_\lambda<\delta\left(\theta-\theta^{(\lambda)}\right),Y_{k,\ell}(\theta)>\right|=$$

$$=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right|. \tag{3.7}$$

Agar (3.7) ni o'ng tomonidagi $a_{k,\ell}$ ni $k^{\frac{m}{2}}(k+n-2)^{\frac{m}{2}}$ ga ko'paytirib, yig'indini shu ko'paytuvchiga bo'lsak va Koshi tengsizligini qo'llasak, (3.2) ga asosan quyidagini olamiz

$$\left|<\ell_N, f>\right|=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}k^{\frac{m}{2}}(k+n-2)^{\frac{m}{2}}\cdot\left[\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]\Big/k^{\frac{m}{2}}(k+n-2)^{\frac{m}{2}}\right|\le$$

$$\le\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}^2k^m(k+n-2)^m\right\}^{\frac{1}{2}}\cdot\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m(k+n-2)^m}\right\}^{\frac{1}{2}}=$$

$$=\left\| f/L_2^m(S)\right\|\cdot\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m\left(k+n-2\right)^m}\right\}^{\frac{1}{2}}. \tag{3.8}$$

(3.8) dan quyidagi kelib chiqadi

$$\left\|\ell_N/L_2^{m*}(S)\right\|\le\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m\left(k+n-2\right)^m}\right\}^{\frac{1}{2}}. \tag{3.9}$$

Quyidagi funksiyani qarab chiqamiz

$$U(\theta)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}b_{k,\ell}Y_{k,\ell}(\theta), \tag{3.10}$$

bunda

$$b_{k,\ell}=\frac{\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m\left(n+k-2\right)^m}. \tag{3.11}$$

Sferik funksiyalar uchun quyidagi baho o'rinli [1]

$$\max\ \left|Y_k(\theta)\right|\le C(n)k^{-m+\frac{n}{2}-1}\left\| f(\theta)/L_2^m(S)\right\|,$$

(3.11) dan kelib chiqadiki (3.10) ning koeffisiyetlari $U(\theta)\in L_2^m(S)$

Bu funksiya uchun (3.8) kubatur formula xatoligini hisoblasak quyidagi tenklikni olamiz:

$$\left|<\ell_N,U>\right|=\left|<\varepsilon_s(\theta)-\sum_{\lambda=1}^{N}C_\lambda\delta\ \left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m\left(k+n-2\right)^m}Y_{k,\ell}(\theta)>\right|=$$

$$=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m\left(k+n-2\right)^m}\cdot\left[<\varepsilon_s(\theta),Y_{k,\ell}(\theta)>-\sum_{\lambda=1}^{N}C_\lambda<\delta\ \left(\theta-\theta^{(\lambda)}\right),Y_{k,\ell}(\theta)>\right]\right|=$$

$$=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\sum_{\lambda=1}^{N}C_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^{m}\left(k+n-2\right)^{m}}\cdot\left[\int_{S}Y_{k,\ell}\left(\theta\right)d\theta-\sum_{\lambda=1}^{N}C_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]\right|=$$

$$=\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^{2}}{k^{m}\left(k+n-2\right)^{m}}\right|=\left\|U/L_{2}^{m}\left(S\right)\right\|^{2}. \qquad (3.12)$$

(3.9) va (3.11) dan quyidagi kelib chiqadi.

$$\left\|\ell_{N}/L_{2}^{m*}\left(S\right)\right\|=\left\|U/L_{2}^{m}\left(S\right)\right\|,$$

bu yerda $U\left(\theta\right)$ funksiya (3.1) ko'rinishdagi kubatur formula uchun ekstremal funksiya bo'ladi. Bu teoremaga asosan (3.1) ko'rinishdagi kubatur formula xatolik funksionali uchun $L_{2}^{m}\left(S\right)$ fazoda quyidagi baho o'rinli bo'ladi:

$$\left|<\ell_{N}\left(\theta\right),f\left(\theta\right)>\right|\leq\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\sum_{\lambda=1}^{N}C_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^{2}}{k^{m}\left(k+n-2\right)^{m}}\right\}^{\frac{1}{2}}\cdot\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}^{2}k^{m}\left(k+n-2\right)^{m}\right\}^{\frac{1}{2}}.$$

Teoreama 3.1.2. (3.12) tenglik shuni tasdiqlaydiki,

$$U\left(\theta\right)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\sum_{\lambda=1}^{N}\tilde{N}_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^{m}\left(n+k-2\right)^{m}}Y_{k,\ell}\left(\theta\right)$$

$U\left(\theta\right)$ funksiya (3.1) ko'rinishdagi kubatur formula uchun ekstremal funksiya bo'ladi va $U\left(\theta\right)\in L_{2}^{m}\left(S\right)$, bu yerda $Y_{k,\ell}(\theta)$- sferik garmonikalar k- tartibli ℓ ko'rinishdagi va $\sigma(n,k)$- k – tartibli chiziqli bog'liq bo'lmagan sferik garmonikalar soni.

Endi n+1=3 hol uchun quyidagi formulani qarab chiqamiz.

$$\int_0^{2\pi}\int_0^{\pi} f\left(\theta,\varphi\right)\sin\theta\, d\theta d\varphi \approx \sum_{\lambda=1}^{N} C_\lambda f\,(\theta^{(\lambda)},\varphi^{(\lambda)})\,. \tag{3.13}$$

Bubda $Y_{k,\ell}(\theta)=\sqrt{\dfrac{(2k+1)(k-|\,\ell\,|)!}{2\,(k+|\,\ell\,|)!}}e^{\pm i\ell\varphi}P_k^{|\ell|}(\cos\theta)$ ko'rinishda bo'ladi, bu yerda

$P_k^{|\ell|}(\cos\theta)$ - Lejandr funksiyasi. (3.5) ga asosan quyidagicha bo'ladi

$$\left\|\ell_N\,|\,L_2^{m*}(S)\right\|^2=\sum_{k=1}^{\infty}\sum_{\ell=-k}^{k}\frac{\left(\sum\limits_{\lambda=1}^{N} C_\lambda\sqrt{\dfrac{(2k+1)(k-|\,\ell\,|)!}{2\,(k+|\,\ell\,|)!}}e^{\pm i\ell\varphi^{(\lambda)}}P_k^{|\ell|}(\cos\theta^{(\lambda)})\right)^2}{k^m(k+1)^m}\,. \tag{3.14}$$

Endi $L_2^{(m)}(S)$ fazoda xatolik funksionali normasini baholaymiz.

Quyidagi formula berilgan bo'lsin

$$\int_0^{\pi} f\left(\cos\theta\right)\sin\theta\, d\,\theta \approx \sum_{\lambda'=1}^{N'} C_{\lambda'}\; f\left(\cos\theta_{\lambda'}\right) \tag{3.15}$$

va

$$\int_0^{\pi}\cos^{\alpha}\theta d\cos\theta=\sum_{\lambda'=1}^{N'} C_{\lambda'}\cos^{\alpha}\theta_{\lambda'},\quad \left(\alpha=0,1,...,m-1\right), \tag{3.16}$$

yoki

$$\int_{-1}^{1} f\,(t)dt\;\approx \sum_{\lambda'=1}^{N'} C_{\lambda'}\; f\left(t_{\lambda'}\right) \tag{3.17}$$

va

$$\int_{-1}^{1} t^{\alpha}dt\;=\sum_{\lambda'=1}^{N'} C_{\lambda'}\; t_{\lambda'}^{\alpha}\,,\quad (\alpha=0,1,...,m-1)\,.$$

Belgilash kiritamiz:

$$\sum_{\lambda'=1}^{N'} C_{\lambda'} f(\cos\theta_{\lambda'})=L(f;0,\pi) \qquad \text{yoki} \qquad \sum_{\lambda'=1}^{N'} C_{\lambda'} f(t_{\lambda'})=L(f;-1,1)\,.$$

Agar $f\,(t)\in L_2^{(m)}(-1,1)$, unda

$$\int_{-1}^{1} f(t)\,d\,t-L\left(f;-1,1\right)=\frac{1}{(m-1)!}\int_{-1}^{1} F_m\left(u\right)\; f^{(m)}\left(u\right)du, \tag{3.18}$$

bu yerda

$$F_m(u) = \int_u^1 (t-u)^{m-1}\, dt - \sum_{\lambda'=1}^{N'} C_{\lambda'}\, k_m(t-u)$$

va

$$k_m(x) = \begin{cases} x^{m-1}, & x \ge 0 \quad \text{uchun} \\ 0 \quad , & x < 0 \quad \text{uchun} \end{cases}$$

(3.13) kubatur formulada , $0 \le \varphi \le 2\pi$, φ bo'yicha davriy bo'lganligi uchun to'g'ri to'rtburchaklar formulasini qo'llaymiz, unda $x^{(\lambda)} = \frac{2\pi\lambda}{N}$,

$C_1 = C_2 = ... = C_N = \frac{2\pi}{N}$, $\quad 0 \le \theta \le \pi$ davriy bo'lmagan hol uchun yuqori darajali kvadratur formulani qo'llaymiz.

Yuqoridagi fikrlarimizga asosan kubatur formula quyidagi ko'rinishda keladi.

$$\int_0^{2\pi}\int_{-1}^{1} f(t,\varphi)dtd\varphi \approx \frac{2\pi}{N_2} \sum_{\lambda_2=1}^{N_2}\sum_{j=1}^{N_1} L(f(t_{\lambda'}, \frac{2\pi\lambda_2}{N_2}); y_j, y_{j+1}), \tag{3.19}$$

bu yerda $L(f(t_{\lambda'}, \frac{2\pi\lambda_2}{N_2}); y_j, y_{j+1})$ - kvadratur formula:

$$y_{j+1} - y_j = h = \frac{2}{N_1}, \quad j = 1, 2, ..., N_1.$$

Shunday qilib (3.19) ko'rinishdagi kubatur formulani xatolik funksionali normasini baholaymiz: ma'lumki

$$\|\ell_N \quad |L_2^{(m)*}(S)\|^2 = \|\Psi_\ell \,|\, L_2^{(m)}(S)\|^2 = \left|<\ell_N, \Psi_\ell>\right| \tag{3.20}$$

(3.20) dan quyidagi kelib chiqadi

$$|<\ell_N, \Psi_\ell>| = \left| \int_0^{2\pi}\int_{-1}^{1} \Psi_\ell(t,\varphi)d\,t\,d\,\varphi - \frac{2\pi}{N_2}\sum_{\lambda_1=1}^{N_1}\sum_{\lambda_2=1}^{N_2} C_{\lambda_1}\Psi_\ell(t_{\lambda_1}, \frac{2\pi\lambda_2}{N_2}) \right| =$$

$$= |\int_0^{2\pi}\int_{-1}^{1} \Psi_\ell(t,\varphi)d\,t\,d\,\varphi - \int_0^{2\pi}\sum_{\lambda_1=1}^{N_1} C_{\lambda_1}\Psi_\ell(t_{\lambda_1}, \varphi)d\,\varphi +$$

$$+\int_0^{2\pi}\sum_{\lambda_1=1}^{N_1} C_{\lambda_1}\Psi_\ell(t_{\lambda_1},\varphi)d\,\varphi-\frac{2\pi}{N_2}\sum_{\lambda_1=1}^{N_1}\sum_{\lambda_2=1}^{N_2} C_{\lambda_1}\Psi_\ell(t_{\lambda_1},\frac{2\pi\lambda_2}{N_2})\,|\le$$

$$\le\left|\int_0^{2\pi}\left[\int_{-1}^{1}\Psi_\ell(t,\varphi)d\,t-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\Psi_\ell(t_{\lambda_1},\varphi)\right]d\,\varphi\right|+$$

$$+\left|\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\int_0^{2\pi}\Psi_\ell(t_{\lambda_1},\varphi)\,d\,\varphi-\frac{2\pi}{N_2}\sum_{\lambda_2=1}^{N_2}\Psi_\ell(t_{\lambda_2},\frac{2\pi\lambda_2}{N_2})\right|\le$$

$$\le\int_0^{2\pi}\left|\int_{-1}^{1}\Psi_\ell(t,\varphi)\,d\,t-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\Psi_\ell(t_{\lambda_1},\varphi)\right|d\,\varphi+$$

$$+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\left|\int_0^{2\pi}\Psi_\ell(t_{\lambda_1},\varphi)\,d\,\varphi-\frac{2\pi}{N_2}\sum_{\lambda_2=1}^{N_2}\Psi_\ell(t_{\lambda_1},\frac{2\pi\lambda_2}{N_2})\right|\,. \tag{3.21}$$

endi (3.21) formulaning o’ng tomonidagi har bir qo’shiluvchini baholaymiz.

(3.18) ni hisobga olsak quyidagicha bo’ladi,

1)

$$\left|\int_{-1}^{1}\Psi_\ell(t,\varphi)\,d\,t-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\;\Psi_\ell(t_{\lambda_1},\varphi)\right|=$$

$$=\left|\int_{-1}^{1}\Psi_\ell(t,\varphi)\,d\,t-\sum_{j=1}^{N_1}L(\Psi_\ell(t_j,\varphi);y_j,y_{j+1})\right|=$$

$$=\left|\sum_{j=1}^{N_1}\left\{\int_{y_j}^{y_{j+1}}\Psi_\ell(t,\varphi)d\,t-L(\Psi_\ell(t_j,\varphi);y_j,y_{j+1})\right\}\right|=$$

$$=\left|\frac{h}{2}\sum_{j=1}^{N_1}\left\{\int_{-1}^{1}\Psi_\ell(y_j+\frac{h}{2}(\vartheta+1),\varphi)\,d\,\vartheta-\right.\right.$$

$$\left.\left.-L\,[\,\Psi_\ell(y_j+\frac{h}{2}\,(\vartheta+1)\,,\varphi);-1;1\,]\right\}\right|=$$

$$=\left|\frac{h^{m+1}}{2^{m+1}(m-1)!}\quad\sum_{j=1}^{N_1}\int_{-1}^{1}F_m(\vartheta)\;\;\Psi_\ell^{(m)}(y_j+\frac{h}{2}(\vartheta+1),\,\varphi)\;d\;\vartheta\right|. \tag{3.22}$$

Koshi – Bunyakovskiy tengsizligini qo’llasak (3.22) ni o’ng tomoniga quyidagini olamiz.

$$\left|\int_{-1}^{1}\Psi_{\ell}(t,\varphi)\,d\,t-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\Psi_{\ell}\,(t_{\lambda_1},\varphi)\right|\leq\frac{h^{m+1}}{(m-1)!2^{m+1}}\sum_{j=1}^{N_1}\left(\int_{-1}^{1}|F_m(\vartheta)|^2\,d\vartheta\right)^{1/2}\times$$

$$\times\left\{\int_{-1}^{1}|\Psi_{\ell}^{(m)}\left(y_j+\frac{h}{2}(\vartheta+1),\varphi\right|^2\,d\vartheta\right\}^{1/2}=\frac{2^{1/2}h^{m+1-1/2}}{2^{m+1}}M\sum_{j=1}^{N_1}\left\{\int_{y_j}^{y_{j+1}}|\Psi_{\ell}^{(m)}(t,\varphi)|^2\,dt\right\}^{1/2}\leq$$

$$\leq\frac{h^{m+1/2}}{2^{m+1/2}}M\left\{\sum_{j=1}^{N_1}\int_{y_j}^{y_{j+1}}|\Psi_{\ell}^{(m)}(t,\varphi)|^2\,dt\right\}^{1/2}\left\{\sum_{j=1}^{N_1}1^2\right\}^{1/2}=\frac{M}{N_1^m}\left\{\int_{-1}^{1}|\Psi_{\ell}^{(m)}(t,\varphi)|^2\,dt\right\}^{1/2},\qquad(3.23)$$

bu yerda

$$M=\frac{1}{2^{m+1}(m-1)!}\left(\int_{-1}^{1}|F_m(\vartheta)|^2\,d\vartheta\right)^{1/2}.$$

Shunday qilib $\Psi_{\ell}(t,\varphi)\in L_2^{(m)}(S)$, bundan kelib chiqadiki

$$\left\{\int_{-1}^{1}|\Psi_{\ell}^{(m)}(t,\varphi)|^2\,dt\right\}^{1/2}\leq C_1\,\|\,\Psi_{\ell}\,|\,L_2^{(m)}(S)\,\|\,,\qquad C_1-\text{o'zgarmas.}\qquad(3.24)$$

(3.24) va (3.23) dan foydalanib quyidagini olamiz

$$\left|\int_{-1}^{1}\Psi_{\ell}(t,\varphi)dt-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\Psi_{\ell}(t_{\lambda_1},\varphi)\right|\leq$$

$$\leq\frac{M}{N_1^m}C_1\,\|\,\Psi_{\ell}\,|\,L_2^{(m)}(S)\|=\frac{1}{N_1^m}M'\,\|\,\Psi_{\ell}\,|\,L_2^{(m)}(S)\,\|\,,\qquad MC_1=M'.\qquad(3.25)$$

2) (3.21) ning o'ng tomonidagi ikkinchi qo'shiluvchi uchun olingan natijalardan foydalansak quyidagicha bo'ladi.

$$\left|\int_{0}^{2\pi}\Psi_{\ell}(t_{\lambda_1},\varphi)d\varphi-\frac{2\pi}{N_2}\sum_{\lambda_2=1}^{N_2}\Psi_{\ell}\left(t_{\lambda_1},\frac{2\pi\lambda_2}{N_2}\right)\right|\leq\frac{1}{N_2^m}\left\{\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right\}^{1/2}\left\{\int_{0}^{2\pi}|\Psi_{\ell}^{(m)}(t_{\lambda_1},\varphi)|^2\,d\varphi\right\}^{1/2}\leq$$

$$\leq \frac{1}{N_2^m}\left\{\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right\}^{1/2} d\,\|\Psi_\ell\,|\,L_2^{(m)}(S)\,\|, \qquad d \text{ – o'zgarmas.} \tag{3.26}$$

(3.25) va (3.26) ni (3.21) ga qo'ysak quyidagi bahoni olamiz:

$$\left|<\ell_N\,|,\Psi_\ell>\right| \leq \int_0^{2\pi}\left\{\frac{1}{N_1^m}M'\,\|\Psi_\ell\,/\,L_2^{(m)}(S)\,\|\right\}d\varphi +$$

$$+\sum_{\lambda_1=1}^{N_1}|C_{\lambda_1}|\,\frac{1}{N_2^m}\left\{\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right\}^{1/2} d\,\|\Psi_\ell\,|\,L_2^{(m)}(S)\,\|=$$

$$=\|\Psi_\ell\,|\,L_2^{(m)}(S)\,\|\left\{\frac{2\pi}{N_1^m}M' + \frac{d}{N_2^m}\left\{\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right\}^{1/2}\sum_{\lambda_1}^{N_1}|C_{\lambda_1}|\right\}, \tag{3.27}$$

yoki (3.27) dan quyidagi kelib chiqadi

$$\|\ell_N\ \left|L_2^{(m)*}(S)\right\| \leq \frac{2\pi M'}{N_1^m} + \frac{d}{N_2^m}\left(\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right)^{\frac{1}{2}}|\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}|.$$

Ma'lumki kvadratur formulaning koeffisiyentlari musbat bo'lsa (masalan Gaus tipidagi kvadratur formulalar) quyidagicha bo'ladi.

$$\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right| = \sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\,.$$

Shunday qilib quyidagi lemmani isbotladik.

Lemma 3.1.1. $L_2^{(m)}(S)$, $(2m \geq 3)$ fazoda (3.19) ko'rinishdagi kubatur formulaning xatolik funksionali normasi uchun quyidagi baho o'rinli:

$$\|\ell_N\left|L_2^{(m)*}(S)\right\| \leq \frac{2\pi M'}{N_1^m} + \frac{d}{N_2^m}\left(\sum_{k=1}^{\infty}\frac{2}{k^{2m}}\right)^{\frac{1}{2}}|\sum_{\lambda_1}^{N_1}C_{\lambda_1}|, \tag{3.28}$$

bu yerda

$$M' = \frac{d_1}{(m-1)!}\left\{\int_{-1}^{1}|F_m(\vartheta)|^2\,d\,\vartheta\right\}^{\frac{1}{2}}, \quad F_m(\vartheta) = \begin{cases}\vartheta^{m-1}, & \vartheta \geq 0 \quad uchun\\ 0\ \ , & \vartheta < 0 \quad uchun\end{cases}, \quad d \text{ va } d_1$$

o'zgarmaslar.

Quyidagi o'rinli bo'lsin

$$N_1 = N_2 \qquad \text{va} \qquad N = N_1 \times N_2\,. \tag{2.29}$$

(3.29), va (3.28) dan foydalanib quyidagi bahoni olamiz

$$\| \ell_N \quad | L_2^{m*}(S) \| \leq \frac{1}{N^{\frac{m}{2}}} \left\{ 2\pi M' + d \left(\sum_{k=1}^{\infty} \frac{2}{k^{2m}} \right)^{\frac{1}{2}} \sum_{\lambda_1}^{N_1} | C_{\lambda_1} | \right\}. \tag{3.30}$$

Baxvalov teoremasini keltiramiz.

Teorema. (N.S. Baxvalov.) Ω sohaga bog'liq bo'lgan shunday K >0 o'zgarmas topiadiki quyidagi tengsizlik o'rinli bo'ladi,

$$\left\| \ell_\Omega \middle| L_p^{(m)*} \right\| \geq K h^m \ , \tag{3.31}$$

bunda $$h = \left(\frac{|\Omega|}{N} \right)^{\frac{1}{n}}, \qquad |\Omega| = mes\Omega.$$

Yuqoridagi Lemma 3.1.1 va Baxvalov teoremasidan quyidagi teorema kelib chiqadi.

Teorema 3.1.3. (3.19) ko'rinishdagi kubatur formula $N_1=N_2$ va $N=N_1 \times N_2$ uchun $L_2^{(m)}(S)$ fazoda yaqinlashish tartibi bo'yicha optimal formula bo'ladi.

ya'ni $$\left\| \ell_N \mid L_2^{(m)*}(S) \right\| = O\left(N^{-\frac{m}{2}} \right).$$

3.2. $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasini hisoblash va ekstremal funksiyani aniqlash.

Bu bo'limda biz vaznli kubatur formulani qarab chuqamiz. Vaznli kubatur formula quyidagi ko'rinishda bo'ladi:

$$\int_S p(\theta) f(\theta) d\theta \cong \sum_{\lambda=1}^{N} C_\lambda f(\theta^{(\lambda)}) \tag{3.32}$$

$L_2^m(S)$ - fazoda sfera sirtida, by yerda $S-n-$ o'lchovli birlik sfera, $\theta=(\theta_1,\theta_2,...,\theta_n)$, $\|\theta\|=1$, $p(\theta)$ - S sferada integrallanuvchi funksiya, ya'ni $\int\limits_S p(\theta)d\theta<\infty$

va $$\sum_{\lambda=1}^{N} C_\lambda=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\hat{p}_{0,0}$$ bunda $\hat{p}_{k,\ell}=\int\limits_S P(\theta)Y_{k,\ell}(\theta)d\theta$,

bu yerda $Y_{k,\ell}(\theta)$ - sferik garmonikalar k - tartibli ℓ ko'rinishdagi $1\le\ell\le\sigma(n,k)$.

$\sigma(n,k)=(2k+n-2)\dfrac{(k+n-3)!}{(k-2)!k!}$ - k – tartibli chiziqli bog'liq bo'lmagan sferik garmonikalar soni. $Y_{k,\ell}(\theta)$ funksiyani S sferada orthogonal deb hisoblaymiz.

(3.32) ko'rinishdagi vaznli kubatur formulaning xatolik funksionali quyidagicha bo'ladi:

$$\ell_N(\theta)=p(\theta)\varepsilon_S(\theta)-\sum_{\lambda=1}^{N}C_\lambda\delta(\theta-\theta^{(\lambda)}), \tag{3.33}$$

bu yerda $\delta(\theta)$ - Dirakning delta funksiyasi, C_λ va $\theta^{(\lambda)}$ - (3.32) ko'rinishdagi vaznli kubatur formulaning koeffisiyetlari va tugun nuqtalari.

Teorema 3.2.1. (3.32) ko'rinishdagi vaznli kubatur formulaning xatolik funksionalini normasi $L_2^m(S)$ fazoda quyidagiga teng

$$\left\|\ell_N/L_2^{m*}(S)\right\|=\left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\left[\hat{p}_{k.\ell}-\sum\limits_{\lambda=1}^{N}C_\lambda Y_{k,\ell}(\theta)\right]^2}{k^m(k+n-2)^m}\right\}^{1/2},$$

bunda $\hat{p}_{k,\ell}=\int\limits_S p(\theta)Y_{k,\ell}(\theta)d\theta.$

Ushbu teoremani to'liq isbotini keltiramiz.

Isboti. Ma'lumki agar funksiymiz $f(\theta)\in L_2^m(S)$ ga tegishli bo'lsa, unda quyidagi qator absolyut tekis yaqinlashuvchi bo'ladi.

$$f(\theta)=\sum_{k=0}^{\infty}Y_k(\theta),$$

bu yerda $Y_k(\theta)$ - k- tartibli sferik garmonikalar, bunda $2m>n$ shart bajarilishi yetarli.

Shunday qilib funksiyani $f(\theta)\in L_2^m$ sferik garmonikalar bo'yicha absolyut tekis yaqinlashuvchi qatorga yoysak

$$f(\theta)=\sum_{k=0}^{\infty}Y_k(\theta)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta), \tag{3.34}$$

bu yerda $Y_{k,\ell}(\theta)$ - sferik garmonikalar k - tartibli ℓ ko'rinishdagi;

$a_{k,\ell}=\int\limits_S Y_{k,\ell}(\theta)f(\theta)d\theta;$ $\sigma(n,k)$ - k – tartibli chiziqli bog'liq bo'lmagan sferik garmonikalar soni: $\sigma(n,k)=\dfrac{(k+n-3)!}{k!(n-2)!}(n+2k-2).$

(3.32) ni chap qismiga (3.34) keltirib qo'ysak quyidagini topamiz :

$$<\ell_N(\theta),f(\theta)>=<p(\theta)\varepsilon_S(\theta)-\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}Y_k(\theta)>=$$

$$=<p(\theta)\varepsilon_S(\theta),\sum_{k=1}^{\infty}Y_k(\theta)>-<\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}Y_k(\theta)>=$$

$$=\int\limits_S p(\theta)\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)d\theta-<\sum_{\lambda=1}^{N}C_\lambda\delta\left(\theta-\theta^{(\lambda)}\right),\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}Y_{k,\ell}(\theta)>=$$

$$=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}p(\theta)Y_{k,\ell}(\theta)d\theta-\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n.k)}a_{k,\ell}\sum_{\lambda=1}^{N}C_\lambda<\delta\left(\theta-\theta^{(\lambda)}\right),Y_{k,\ell}(\theta)>=$$

$$=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}a_{k,\ell}\left[\hat{p}_{k,\ell}-\sum_{\lambda=1}^{N}C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]. \tag{3.35}$$

Agar (3.35) ni o'ng tomonidagi $a_{k,\ell}$ ni $k^{\frac{m}{2}}\left(k+n-2\right)^{\frac{m}{2}}$ ga ko'paytirib, yig'indini shu ko'paytuvchiga bo'lsak va Koshi tengsizligini qo'llasak, (3.2) ga asosan quyidagini olamiz

$$\left|<\ell_N, f>\right| = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell} k^{\frac{m}{2}}\left(k+n-2\right)^{\frac{m}{2}} \cdot \frac{\hat{p}_{k,\ell} - \sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^{\frac{m}{2}}\left(k+n-2\right)^{\frac{m}{2}}} \le$$

$$\le \left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m \left(k+n-2\right)^m\right\}^{\frac{1}{2}} \cdot \left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\hat{p}_{k,\ell} - \sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m\left(k+n-2\right)^m}\right\}^{\frac{1}{2}} =$$

$$= \left\| f / L_2^m(S) \right\| \cdot \left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\hat{p}_{k,\ell} - \sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m\left(k+n-2\right)^m}\right\}^{\frac{1}{2}}. \tag{3.36}$$

(3.36) dan quyidagi kelib chiqadi

$$\left\| \ell_N / L_2^{m*}(S) \right\| \le \left\{\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\hat{p}_{k,\ell} - \sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)\right]^2}{k^m\left(k+n-2\right)^m}\right\}^{\frac{1}{2}}. \tag{3.37}$$

Quyidagi funksiyani qarab chiqamiz

$$U(\theta) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)} b_{k,\ell} Y_{k,\ell}(\theta), \tag{3.38}$$

bu yerda

$$b_{k,\ell} = \frac{\hat{p}_{k,\ell} - \sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m\left(n+k-2\right)^m}. \tag{3.39}$$

Sferik funksiyalar uchun quyidagi baho o'rinli [1]

$$\max \left|Y_k(\theta)\right| \le C(n) k^{-m+\frac{n}{2}-1} \left\| f(\theta) / L_2^m(S) \right\|,$$

(3.39) dan kelib chiqadiki (3.48) ning koeffisiyetlari $U(\theta) \in L_2^m(S)$ tegishli bo'ladi

Bu funksiya uchun (3.32) kubatur formula xatoligini hisoblasak quyidagi tenklikni olamiz:

$$\left|<\ell_N, U>\right| = \left| < p(\theta)\varepsilon_S(\theta) - \sum_{\lambda=1}^{N} C_\lambda \delta\left(\theta - \theta^{(\lambda)}\right), \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\hat{p}_{k,\ell} - \sum_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m (k+n-2)^m} Y_{k,\ell}(\theta) > \right| =$$

$$= \left| \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\hat{p}_{k,\ell} - \sum_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m (k+n-2)^m} \cdot \left[< p(\theta)\varepsilon_S(\theta), Y_{k,\ell}(\theta) > - \sum_{\lambda=1}^{N} C_\lambda < \delta\left(\theta - \theta^{(\lambda)}\right), Y_{k,\ell}(\theta) > \right] \right| =$$

$$= \left| \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\hat{p}_{k,\ell} - \sum_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^m (k+n-2)^m} \cdot \left[\int_S p(\theta) Y_{k,\ell}(\theta) d\theta - \sum_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right) \right] \right| =$$

$$= \left| \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\hat{p}_{k,\ell} - \sum_{\lambda=1}^{N} C_\lambda Y_{k,\ell}\left(\theta^{(\lambda)}\right) \right]^2}{k^m (k+n-2)^m} \right| = \left\| U / L_2^m(S) \right\|^2. \tag{3.40}$$

(3.37) va (3.40) dan quyidagi kelib chiqadi

$$\left\| \ell_N / L_2^{m*}(S) \right\| = \left\| U / L_2^m(S) \right\|,$$

Bu yerda $U(\theta)$ funksiya (3.32) vaznli kubatur formula uchun ekstremal funksiya bo'ladi, ya'ni $\ell_N(\theta)$ xatolik funksionali uchun $U(\theta)$ - Ris funksiyasi bo'ladi, shunday qilib quyidagi teorema isbotlandi.

Teorema 3.2.2. (3.40) tenglik tasdiqlaydiki,

$$U(\theta)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\sigma(n,k)}\frac{\hat{p}_{k,\ell}-\sum\limits_{\lambda=1}^{N}C_{\lambda}Y_{k,\ell}\left(\theta^{(\lambda)}\right)}{k^{m}(n+k-2)^{m}}Y_{k,\ell}(\theta)$$

Haqiqatdan ham $U(\theta)$ funksiya (3.32) vaznli kubatur formula uchun ekstremal funksiya bo'ladi va $U(\theta)\in L_2^m(S)$, bunda $Y_{k,\ell}(\theta)$- ortonormallangan sferik garmonika k - tartibli ℓ ko'rinishdagi va $\sigma(n,k)$- k – tartibli chiziqli bog'liq bo'lmagan sferik garmonikalar soni:

3.3. $L_2^{(m)}(S)$ Sobolev fazosida yaqinlashish tartibi bo'yicha optimal vaznli kubatur formula qurish.

Bir o'lchovli vaznli kubatur formulaning xatolik funksionali quyidagi ko'rinishda bo'lsin:

$$\ell_{N_i}(\theta_i)=p_i(\theta_i)\varepsilon_{\Omega_i}(\theta_i)-\sum_{\lambda_i=1}^{N_i}C_{\lambda_i}\delta(\theta_i-\theta_i^{(\lambda_i)}),\qquad p(\theta)=\prod_{i=1}^{n}p_i(\theta_i)$$

bunda $$\Omega_i=\begin{cases}[0,2\pi],\ agar\quad i=n\ ,\\ [0,\pi],\ agar\quad i=\overline{1,(n-1)}\,,\end{cases}$$

va (3.32) ko'rinishdagi vaznli kubatur formulaning xatolik funksionali quyidagi ko'rinishda bo'ladi:

$$\ell_N(\theta)=\ell_{N_1}(\theta_1)\cdot\ell_{N_2}(\theta_2)\cdot\ldots\cdot\ell_{N_n}(\theta_n)\,.$$

Quyidagicha belgilash kiritamiz $\ell_N(\theta)=\prod\limits_{i=1}^{n}\ell_{N_i}(\theta_i)$.

Salixovning ishlarida ko'rsatilganki $L_2^m(S)$ fazo elementlarining tuzilishi bo'yicha $L_2^{(m)}(S)$ S.L.Sobolev fazosi bilan ustma ust tushadi [1].

$L_2^{(m)}(S)$ fazoda Sobolev ta'rifi bo'yicha funksiyaning normasi quyidagicha aniqlangan :

$$\left\| f / L_2^{(m)}(S) \right\| = \left\{ \int_S \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(D^\alpha f(\theta) \right)^2 d\theta \right\}^{\frac{1}{2}}.$$

Bir o'lchovli hol uchun ta'rifga asosan norma quyidagicha aniqlanadi $f_i \in L_2^{(m)}(\Omega_i)$:

$$\left\| f_i / L_2^{(m)}(\Omega_i) \right\| = \left\{ \int_{\Omega_i} \left(\frac{d^m}{d\theta_i^m} f(\theta_i) \right)^2 d\theta_i \right\}^{\frac{1}{2}}.$$

Quyidagi lemma o'rinli

Lemma 3.3.1. Agar $f(\theta)$ funksiya uchun $L_2^{(m)}(S)$ fazoda (3.32) ko'rinishdagi vaznli kubatur formulaning xatolik funksionali $\ell_N(\theta)$ uchun quyidagi shartlar bajarilsa

$$\ell_N(\theta) = \ell_{N_1}(\theta_1) \cdot \ell_{N_2}(\theta_2) \cdot \ldots \cdot \ell_{N_n}(\theta_n)$$

va

$$\left\| \ell_{N_i} \mid L_2^{(m)}(\Omega_i) \right\| \le K_i N_i^{-m} \tag{3.41}$$

bu yerda $\Omega_i = \begin{cases} [0, 2\pi], & agar \;\; i = n \\ [0, \pi], & agar \;\; i = \overline{1, n-1} \end{cases}$

unda quyidagi o'rinli bo'ladi

$$\left\| \ell_N \mid L_2^{(m)*}(S) \right\| \le \sum_{i=1}^{n} K_i^1 N_i^{-m}.$$

Isboti. Juda ko'p usullardan ma'lumki n – integralni birlik sferada qaralganda, n marta har bir o'zgaruvchi bo'yicha vaznli kvadratur formula alohida alohida qo'llaniladi. Shunday qilib, agar $Q_1, Q_2, \ldots, Q_n$ - vaznli kvadratur formulalar bo'lsa $\Omega_i \; (i = \overline{1, n})$ sohada integral uchun , unda

$$\int_S p(\theta) f(\theta) d\theta = \int_{\Omega_1} \int_{\Omega_2} \ldots \int_{\Omega_n} p(\theta_1, \ldots, \theta_n) f(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n \approx$$

$$\approx \int_{\Omega_1} \ldots \int_{\Omega_{n-1}} Q_1(f; \theta_1) d\theta_2, d\theta_3 \ldots d\theta_{n-1} \approx$$

$$\approx \int_{\Omega_1} \cdots \int_{\Omega_{n-2}} Q_2(Q_1(f;\theta_1);\theta_2)d\theta_3 \ldots d\theta_{n-1} \approx$$
$$\approx Q_n(Q_{n-1}(\ldots Q_1(f;\theta_1);\theta_2);\ldots;\theta_n). \qquad (3.42)$$

Oxirgi amalda (3.42) f funksiyaning chiziqli kombinatsiyasi bo'ladi.

Haqiqatdan, agar

$$Q_j(g) = \sum_{\lambda_i=1}^{N_i} C_{\lambda_i} g(\theta_j^{\lambda_i}), \quad j = \overline{1,n},$$

unda (3.42) quyidagicha bo'ladi:

$$\sum_{\lambda_1=1}^{N_1}\sum_{\lambda_2=1}^{N_2}\cdots\sum_{\lambda_n=1}^{N_i} C_{\lambda_1}C_{\lambda_2}\ldots C_{\lambda_{n-1}} f(\theta_1^{(\lambda_1)}, \theta_2^{(\lambda_2)}, \ldots, \theta_n^{(\lambda_n)}).$$

Shunday qilib bizga ma'lumli

$$\left| \int_{\Omega_i} f(\theta_i) p_i(\theta_i) d\theta_i - Q_i(f, \theta_i) \right| = E_i \qquad (3.43)$$

Barcha $\theta_1, \theta_2, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n$ qiymatlarda, Ω_i interval bo'yiir o'lchovli xatolikni belgilaymiz, i 1 dan n gacha.

Keyinchalik A_i bilan Q_i formuladagi koeffisiyentlarni absolyut qiymatlanini yig'indisini belgilaymiz. (3.32) ko'rinishdagi vaznli kubatur formulani xatoligini baholash uchun $\ell_N(\theta)$ xatolik funksionalini ekstremal funksiyasi tushunchasidan foydalanamiz [2]. Teorema 3.2.2 ga asosan quyidagicha bo'ladi,

$U(\theta) = \psi_\ell(\theta) \in L_2^{(m)}(S).$

Ma'lumki

$$\left\| \ell_N \mid L_2^{(m)*}(S) \right\|^2 = \left\| \psi_\ell \mid L_2^{(m)}(S) \right\|^2 = |< \ell_N(\theta), \psi_\ell(\theta) >|.$$

Shunday qilib quyidagi kelib chiqadi

$$\left| < \ell_N(\theta), \psi_\ell(\theta) > \right| = \left| \int_S \psi_\ell(\theta) ds - \left(\prod_{i=1}^n Q_i\right)\psi_\ell \right| =$$

$$= \left| \int_{\Omega_1} \cdots \int_{\Omega_n} \psi_\ell(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n - \sum_{\lambda_1=1}^{N_1} C_{\lambda_1} \sum_{\lambda_2=1}^{N_2} C_{\lambda_2} \cdots \sum_{\lambda_n=1}^{N_n} C_{\lambda_n} \psi_\ell(\theta_1^{(\lambda_1)}, \ldots, \theta_n^{(\lambda_n)}) \right| =$$

$$=\left|\int_{\Omega_1}\dots\int_{\Omega_n}\psi_\ell(\theta_1,\dots,\theta_n)d\theta_1\dots d\theta_n-\int_{\Omega_2}\dots\int_{\Omega_n}\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\psi_\ell(\theta_1^{(\lambda_1)},\dots,\theta_n)d\theta_2\dots d\theta_n+\right.$$

$$\left.+\int_{\Omega_2}\dots\int_{\Omega_n}\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\psi_\ell(\theta_1^{(\lambda_1)},\dots,\theta_n)d\theta_2\dots d\theta_n-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\dots\sum_{\lambda_n=1}^{N_n}C_{\lambda_n}\psi_\ell(\theta_1^{(\lambda_1)},\dots,\theta_n^{(\lambda_n)})\right|\le$$

$$\le\int_{\Omega_2}\dots\int_{\Omega_n}d\theta_2\dots d\theta_n\left|\int_{\Omega_1}\psi_\ell d\theta_1-\sum_{\lambda_1=1}^{N_1}C_{\lambda_1}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2,\dots,\theta_n)\right|+$$

$$+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\left|\int_{\Omega_2}\dots\int_{\Omega_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2,\dots,\theta_n)d\theta_2\dots d\theta_n-\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\dots\sum_{\lambda_n=1}^{N_n}C_{\lambda_n}\psi_\ell(\theta_1^{(\lambda_1)},\dots,\theta_n^{(\lambda_n)})\right|=$$

$$=\int_{\Omega_2}\dots\int_{\Omega_n}d\theta_2\dots d\theta_nE_1+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\left|\int_{\Omega_2}\dots\int_{\Omega_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2\dots,\theta_n)d\theta_2\dots d\theta_n-\right.$$

$$-\int_{\Omega_3}\dots\int_{\Omega_n}\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\psi_\ell(\theta_2^{(\lambda_1)},\theta_2^{(\lambda_2)},\theta_3,\dots,\theta_n)d\theta_3\dots d\theta_n+$$

$$\left.+\int_{\Omega_3}\dots\int_{\Omega_n}\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\psi_\ell(\theta_2^{(\lambda_1)},\theta_2^{(\lambda_2)},\theta_3,\dots,\theta_n)d\theta_3\dots d\theta_n-\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\dots\sum_{\lambda_n=1}^{N_n}C_{\lambda_n}\psi_\ell(\theta_1^{(\lambda_1)},\dots,\theta_n^{(\lambda_n)})\right|\le$$

$$\le C_1E_1+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\left\{\int_{\Omega_3}\dots\int_{\Omega_n}d\theta_3\dots d\theta_n\left|\int_{\Omega_2}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2\dots,\theta_n)d\theta_2-\sum_{\lambda_2=1}^{N_2}C_{\lambda_2}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2^{(\lambda_2)},\dots,\theta_n)\right|+\right.$$

$$\left.+\sum_{\lambda_2=1}^{N_2}\left|C_{\lambda_2}\right|\left|\int_{\Omega_3}\dots\int_{\Omega_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2^{(\lambda_2)},\dots,\theta_n)d\theta_3\dots d\theta_n-\sum_{\lambda_3=1}^{N_3}C_{\lambda_3}\dots\sum_{\lambda_n=1}^{N_n}C_{\lambda_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2^{(\lambda_2)},\dots,\theta_n^{(\lambda_n)})\right|\right\}=$$

$$=C_1E_1+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\int_{\Omega_3}\dots\int_{\Omega_n}d\theta_3\dots d\theta_nE_2+\sum_{\lambda_1=1}^{N_1}\left|C_{\lambda_1}\right|\sum_{\lambda_2=1}^{N_2}\left|C_{\lambda_2}\right|\left|\int_{\Omega_3}\dots\int_{\Omega_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2^{(\lambda_2)},\dots,\theta_n)d\theta_3\dots d\theta_n-\right.$$

$$\left.-\sum_{\lambda_3=1}^{N_3}C_{\lambda_3}\dots\sum_{\lambda_n=1}^{N_n}C_{\lambda_n}\psi_\ell(\theta_1^{(\lambda_1)},\theta_2^{(\lambda_2)},\dots,\theta_n^{(\lambda_n)})\right|\le$$

$$\le\dots\le C_1E_1+C_2A_1E_2+C_3A_1A_2E_3+\dots+C_nA_1A_2\dots A_{n-1}E_n. \tag{3.44}$$

Har bir i uchun yozadigan bo'lsak:

$$\left\{\int_{\Omega_i}\frac{d^m}{d\theta_i^m}\left|\psi_\ell(\theta)\right|^2d\theta_i\right\}^{1/2}\le C\left\|\psi_\ell|L_2^{(m)}(S)\right\| \tag{3.45}$$

va (3.41) va (3.45) baholardan, (3.43) dan quyidagi kelib chiqadi

$$E_i(f) \le C \left\| \ell_{N_i} | L_2^{(m)*}(\Omega_i) \right\| \cdot \left\| \psi_\ell | L_2^{(m)}(S) \right\| \le C K_i N_i^{-m} \left\| \psi_\ell \left| L_2^{(m)}(S) \right\|. \right. \tag{3.46}$$

(3.46) ni (3.44) ga keltirib qo'ysak, quyidagi kelib chiqadi

$$\left\| \ell_N | L_2^{(m)*}(S) \right\| \le \sum_{i=1}^{n} K_i^1 N_i^{-m} . \tag{3.47}$$

Lemma isbotlandi.

Teorema 3.3.1. quyidagilar o'rinli bo'lsin

$$\left\| \ell_{N_i} \mid L_2^{(m)*}(\Omega_i) \right\| \le K_i N_i^{-m} ,$$

bunda, $N_1 = N_2 = ... = N_n$ va $\prod_{i=1}^{n} N_i = N$,

unda (3.32) ko'rinishdagi vaznli kubatur formula quyidagi funksional xatoligi bilan

$$\ell_N(\theta) = \ell_{N_1}(\theta_1) \cdot \ell_{N_2}(\theta_2) \cdot ... \cdot \ell_{N_n}(\theta_n)$$

$L_2^{(m)}(S)$ fazoda yaqinlashish tartibi bo'yicha optimal bo'ladi:

ya'ni

$$\left\| \ell_N / L_2^{(m)*}(S) \right\| = O\left(N^{-\frac{m}{n}} \right).$$

Isboti. Lemma 3.3.1 ga asosan, $N_1 = N_2 = ... = N_n$ bo'lsa , undan

$\prod_{i=1}^{n} N_i = N$ bo'ladi, bundan $N_1 = N^{\frac{1}{n}}$ kelib chiqadi. (3.48)

(3.37) ga (3.48) ni quyib hisoblasak, quyidagi kelib chiqadi

$$\left\| \ell_{N_i} / L_2^{(m)*}(S) \right\| \le N^{-\frac{m}{n}} \sum_{i=1}^{n} K_i^1 . \tag{3.49}$$

XOTIMA

Integrallarni taqribiy hisoblash uchun formula qurish hisoblash matematikasi va sonlar nazariyasining bir sinf masalasini tashkil qiladi. Bunday masalalarni yechish bilan juda ko'p taniqli matematiklar shug'ullanishgan, shuninig uchun ham juda ko'p formulalar ularning nomlari bilan ataladi bularga Nyuton, Eyler, Gauss, Chebishev, Markov formulalarini misol keltirish mumkin.

Integrallarni taqribiy hisoblashda integral ostidagi funkstiya bir o'zgaruvchili bo'lsa unda kvadratur formula quriladi, agar integral ostidagi funkstiya ikki va undan ortiq o'zgaruvchili bo'lsa unda kubatur formula quriladi. Kubatur formulalar qurish bilan birinchi bo'lib Sobolev shug'ullangan. Ushbu magistrlik dissertasiyasida ham kubatur formula qurish uchun birinchi kubatur formulani xatolik funksionalini normasini hisoblash berilgan funksional fazoda va xatolik funksionalini normasi uchun yuqoridan baho olish masalalari ko'rib chiqilgan. Dissertasiyani 1-bobida interpolyatsion kvadratur formulalar kurib chiqilgan, ularni xatoliklari tahlil qilinib, effektivligi ko'rib chiqilgan va Gauss tipidagi kvadratur formular uchun algoritm va dastur tuzilib misollarda qo'llanilgan. Dissertasiyani ikkinchi va uchinchi bobida esa asosiy olingan natijalar keltirilgan. Dissertasiyani bajarish davomida quyidagi natijalar olindi.

- Kvadratur va kubatur formulalar o'rganildi.
- $L_2^{(m)}(S)$ Sobolev fazosi va sferik garmonikalar nazariyasi o'rganildi.
- $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali uchun ekstremal funksiyani topildi.
- $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasini hisoblandi.
- $L_2^{(m)}(S)$ Sobolev fazosida vaznli kubatur formulaning xatolik funksionali normasi uchun yuqoridan baho olindi.
- Yaqinlashish tartibi bo'yicha optimal kubatur formula qurildi $L_2^{(m)}(S)$ Sobolev fazosida.

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