On cubic stochastic operators and processes

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On cubic stochastic operators and processes [1]

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Abstract. In this paper analogically as quadratic stochastic operators and processes we define cubic stochastic operator (CSO) and cubic stochastic processes (CSP). These are defined on the set of all probability measures of a measurable space. The measurable space can be given on a finite or continual set. The finite case has been investigated before. So here we mainly work on the continual set. We give a construction of a CSO and show that dynamical systems generated by such a CSO can be studied by studying of the behavior of trajectories of a CSO given on a finite dimensional simplex. We define a CSP and drive differential equations for such CSPs with continuous time.

1. Introduction

Dynamical systems generated by quadratic operators have been proved to be a rich source of analysis for the study of dynamical properties and modeling in different domains, such as population dynamics [2, 5], physics [27], economy [3], mathematics [9],[7],[11],[12],[15]. The approach to population genetics, posed within that scheme the problem of an explicit description of evolutionary operators of free populations.

A *quadratic* stochastic operator (QSO) of a free population [15] is a (quadratic) mapping of the simplex

$$S^{m-1} = \{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}$$

into itself, of the form

$$V : x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad (k = 1, ..., m) \tag{1.1}$$

where $p_{ij,k}$ are coefficients of heredity and

$$p_{ij,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad (i, j, k = 1, ..., m) \tag{1.2}$$

Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, ..., m\}$.

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[1] To the memory of our teacher T. A. Sarymsakov on the occasion of his 100th birthday.

Similarly, one can define a *cubic* stochastic operator (CSO) $W : S^{m-1} \to S^{m-1}$ as

$$W : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad (l = 1, ..., m) \tag{1.3}$$

where $P_{ijk,l}$ are coefficients such that

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} P_{ijk,l} = 1, \quad (i, j, k, l = 1, ..., m) \tag{1.4}$$

The population evolves by starting from an arbitrary state (probability distribution on E) $x \in S^{m-1}$ then passing to the state Vx (or Wx) (in the next "generation"), then to the state $V^2 x$ (resp. $W^2 x$), and so on. Thus, states of the population described by the following discrete-time dynamical system

$$x^0, \quad x' = V(x), \quad x'' = V^2(x), \quad x''' = V^3(x), \dots \tag{1.5}$$

where $V^n(x) = \underbrace{V(V(...V}_{n}(x))...)$ denotes the n times iteration of V to x. The main problem for a given dynamical system (1.5) is to describe the limit points of $\{x^{(n)}\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$.

In [10] (see also [19]) a review of the theory of QSOs is given. Note that each quadratic (resp. cubic) stochastic operator can be uniquely defined by a stochastic matrix $\mathbf{P} = \{p_{ij,k}\}_{i,j,k=1}^{m}$ (resp. $\mathbb{P} = \{P_{ijk,l}\}_{i,j,k,l=1}^{m}$). In [6] a constructive description of $\mathbf{P}$ (i.e. a QSO) is given. This construction depends on cardinality of E, which can be finite or continual. Some particular cases of this construction were defined in [7]. In [23] a similar construction for the CSOs on a finite set E is given. We note that for the continual set E one of the key problem is to determine the set of coefficients of heredity which is already infinite dimensional. In this paper we shall give a construction of CSO for a continual set E. In this construction the CSO depends on a probability measure μ given on a measurable space $(E, \mathcal{F})$. We will show that the dynamical systems generated by the CSO depending on μ can be reduced to a dynamical system generated by a Volterra CSO defined on a finite-dimensional simplex.

Next goal of the paper is to define cubic stochastic processes (CSP) and give differential equations for such processes. This investigations will be similar to works [8],[20, 21], [24]-[26], where the authors introduced a continuous-time dynamical system as quadratic stochastic processes (QSP). The reader is referred to very recent book [19] for the theory of QSPs.

2. Definitions and examples

Consider a measurable space $(E, \mathcal{F})$ and let $S(E, \mathcal{F})$ be the set of all probability measures on $(E, \mathcal{F})$.

Definition 2.1. *A mapping* $W : S(E, \mathcal{F}) \to S(E, \mathcal{F})$ *is called a cubic stochastic operator (CSO) if, for an arbitrary measure* $\lambda \in S(E, \mathcal{F})$*, the measure* $\lambda' = W\lambda$ *is defined by*

$$\lambda'(A) = \int_E \int_E \int_E P(x, y, z, A) d\lambda(x) d\lambda(y) d\lambda(z), \quad \forall A \in \mathcal{F},$$

where $P(x, y, z, A)$*, satisfies the following conditions:*

(i) $P(x, y, z, \cdot) \in S(E, \mathcal{F})$ *for any fixed* $x, y, z \in E$*;*

(ii) $P(x, y, z, A)$*, regarded a function of three variables* x*,* y*, and* z *is measurable on* $(E \times E \times E, \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})$ *for any fixed* $A \in \mathcal{F}$*.*

When E is finite, a CSO on $S(E, \mathcal{F}) = S^{m-1}$ is defined as in (1.3) with $P_{ijk,l} = P(i, j, k, l)$.

Let $(E, \mathcal{F}, \mathcal{M})$ be triple, where $\mathcal{F}$ is a σ-algebra of subsets of E and $\mathcal{M} = S(E, \mathcal{F})$, i.e. the set of all probability measures on $(E, \mathcal{F})$. For any three elements $x, y, z \in E$ and a given measure $m_{t_0} \in \mathcal{M}$ at some moment t_0 of the time we assume that we know the law of probability distribution $m_{t_1} \in \mathcal{M}$ of the system E at the moment $t_1 > t_0$ of time.

Denote by $P(t_1, x, y, z, t_2, A)$ the probability of obtaining an element from the set $A \in \mathcal{F}$ at the moment t_2, provided that the elements x, y and z of E interact starting at moment t_1, where $t_2 \geq t_1 + 1$.

Thus if at the moment t_1 we start with a probability measure $m_{t_1} \in \mathcal{M}$, then $m_{t_2} \in \mathcal{M}$ for any $t_2 \geq t_1 + 1$ is defined by

$$m_{t_2}(A) = \int_E \int_E \int_E P(t_1, x, y, z, t_2, A) m_{t_1}(dx) m_{t_1}(dy) m_{t_1}(dz). \tag{2.1}$$

Without loss of generality we assume that the process starts at the moment $t = 0$.

Definition 2.2. *A family $\{P(t_1, x, y, z, t_2, A): x, y, z \in E, A \in \mathcal{F}, t_1, t_2 \in \mathbb{R}^+, t_2 - t_1 \geq 1\}$ is called* cubic stochastic process *(CSP) if it satisfies the following conditions*

(I) $P(t, x, y, z, t+1, A) = P(0, x, y, z, 1, A)$ *for any $t \geq 1$;*

(II) *The value of $P(t_1, x, y, z, t_2, A)$ is independent on any permutations of variables x, y, z for all $x, y, z \in E$, and $A \in \mathcal{F}$;*

(III) $P(t_1, x, y, z, t_2, A)$ *is a probability measure on $(E, \mathcal{F})$ for all $x, y, z \in E$ and $t_1, t_2 \in \mathbb{R}^+$, $t_2 - t_1 \geq 1$;*

(IV) $P(t_1, x, y, z, t_2, A)$ *as function of the three variables x, y and z is measurable with respect to $(E \times E \times E, \quad \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})$ for all $A \in \mathcal{F}$;*

(V) *For any $t_1 < t_2 < t_3$ such that $t_2 - t_1 \geq 1$ and $t_3 - t_2 \geq 1$ the following holds*

$$P(t_1, x, y, z, t_3, A) = \int_E \int_E \int_E P(t_1, x, y, z, t_2, du) P(t_2, u, \vartheta, q, t_3, A) m_{t_2}(d\vartheta) m_{t_2}(dq), \tag{2.2}$$

where m_{t_2} on $(E, \mathcal{F})$ is defined by (2.1).

Remark 2.3. *Now we point out the followings:*

1. *The consideration of CSO and CSP are motivated by their appearance in biology (for example in gene engineering, a triple crossings for different sorts of plants to obtain another sort, and free population with ternary production) [1], in physics (for example in spin systems with ternary interactions) [16]. We note that one may have more general form of stochastic operators (resp. processes) with order $n \geq 1$, and coefficients $P_{i_1 i_2 \ldots i_n, j} \geq 0$ (resp. $P(t_1, x_1, x_2, \ldots, x_n, t_2, A)$) in biology (gene engineering) and physics (n-nary interactions). In this paper, we restrict ourselves to the case $n = 3$, i.e. CSOs and CSPs. Because even for the considered setting associated dynamical systems are very complicated in comparison with $n = 1$ (i.e. Markov chains) and $n = 2$ (i.e. QSOs).*

2. *Note that if one defines a new process by*

$$Q(s, x, y, t, A) = \int_E P(s, x, y, z, t, A) m_s(dz)$$

then one can check that the defined process is QSP (see [20, 24]). We recall that the functions $Q(s, x, y, t, A)$ denote the probability that under the interaction of the elements x and y at time s an event A comes into effect at time t. Since for physical, chemical and biological

phenomena, a certain time is necessary for the realization of an interaction, it is taken the greatest such time to be equal to 1 (see the Boltzmann model [12] or the biological model [15]). Thus the probability $Q(s,x,y,t,A)$ is defined for $t - s \geq 1$. Hence, for CSP, the probabilities $P(s,x,y,z,t,A)$ are also defined for $t - s \geq 1$.

3. *It should be noted that the CSPs are related to CSOs (see [12, 19]) in the same way as Markov processes are related to linear transformations.*

4. *The equation (2.2) in Definition 2.2 is an analogue of Chapman-Kolmogorov equation. In [8], [21],[24]-[26] such equations were extended to QSPs. We note that for QSPs there are two types of the Chapman-Kolmogorov equations: type A and type B. Similarly, one also can define (at least) two types of the Chapman-Kolmogorov equations for a CSP: The equation (2.2) corresponds to the type A of QSP, the following equation is an analogue of the type B for CSP:*

$$P(s,x,y,z,t,A) = \int_E \int_E \int_E \int_E \int_E \int_E P(s,x,y_1,z_1,\tau,du)\, P(s,y,y_2,z_2,\tau,dv)$$

$$\times P(s,z,y_3,z_3,\tau,dw)\, P(\tau,u,v,w,t,A)\, m_s(dy_1)m_s(dz_1)$$

$$m_s(dy_2)m_s(dz_2)m_s(dy_3)m_s(dz_3).$$

In this paper, for the sake of simplicity, we shall only consider CSPs which satisfy (2.2).

Let us provide some examples of CSPs.

Let $E = \{1,2,...,n\}$ and $x^{(0)} = \left\{x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right\}$ be an initial distribution on E.

Denote

$$P_{ijk,l} := P(0,i,j,k,1,\{l\}), \quad P_{ijk,l}^{[s,t]} := P(s,i,j,k,t,\{l\}).$$

By equation (2.1) at the moment $t = 1$ the vector $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, ..., x_n^{(1)}\right)$ is defined as follows

$$x_l^{(1)} = \sum_{i,j,k=1}^n P_{ijk,l} x_i^{(0)} x_j^{(0)} x_k^{(0)}, \quad l = \overline{1,n}.$$

In this case the condition (I) reduces to $P_{ijk,l}^{[t,t+1]} = P_{ijk,l}$.

In general, from (2.1),(2.2) one finds

$$x_l^{(t)} = \sum_{i,j,k} P_{ijk,l}^{[s,t]} x_i^{(s)}, x_j^{(s)} x_k^{(s)} \tag{2.3}$$

$$P_{ijk,l}^{[s,t]} = \sum_{m,\gamma,\delta} P_{ijk,m}^{[s,\tau]} P_{m\gamma\delta,l}^{[\tau,t]} x_\gamma^{(\tau)} x_\delta^{(\tau)}, \tag{2.4}$$

where $\tau - s \geq 1$ and $t - \tau \geq 1$.

Example 2.4. *To define a CSP for $E = \{1,2\}$ we first define a CSO by the matrix $(P_{ijk,l})$, where*

$$P_{111,1} = 1, \quad P_{112,1} = P_{121,1} = P_{211,1} = \frac{2}{3}, \quad P_{122,1} = P_{212,1} = P_{221,1} = \frac{1}{3}, \quad P_{222,1} = 0;$$

$$P_{111,2} = 0, \quad P_{112,2} = P_{121,2} = P_{211,2} = \frac{1}{3}, \quad P_{122,2} = P_{212,2} = P_{221,2} = \frac{2}{3}, \quad P_{222,2} = 1.$$

It is easy to see that this CSO is the identity mapping, i.e. $x_1' = x_1$, $x_2' = x_2$. Take an initial vector $(x, 1-x)$ and using this CSO by formulas (2.3),(2.4) one can define a CSP:

$$P_{111,1}^{[s,t]} = \tfrac{1}{3^{t-s-1}}\left[1 + (3^{t-s-1} - 1)x\right],$$

$$P_{112,1}^{[s,t]} = P_{121,1}^{[s,t]} = P_{211,1}^{[s,t]} = \tfrac{1}{3^{t-s-1}}\left[\tfrac{2}{3} + (3^{t-s-1} - 1)x\right],$$

$$P_{122,1}^{[s,t]} = P_{212,1}^{[s,t]} = P_{221,1}^{[s,t]} = \tfrac{1}{3^{t-s-1}}\left[\tfrac{2}{3} + (3^{t-s-1} - 1)(1-x)\right],$$

$$P_{222,1}^{[s,t]} = \tfrac{3^{t-s-1}-1}{3^{t-s-1}}x,$$

$$P_{ijk,2}^{[s,t]} = 1 - P_{ijk,1}^{[s,t]}, \quad \text{for all} \ \ i,j,k \in E = \{1,2\}.$$

Example 2.5. *Let $E = \{1, 2, \ldots, n\}$. Take a family of stochastic vectors: $a(t) = (a_1(t), a_2(t), \ldots, a_n(t))$, i.e. $a_i(t) \geq 0$, $\sum_i a_i(t) = 1$ for any $t \geq 0$. For each pari s,t we define a stochastic matrix $Q^{[s,t]} = (q_{il}^{[s,t]})_{i,l \in E}$, where $q_{il}^{[s,t]} = a_l(t)$ for all $i \in E$, i.e. it does not depend on s. It is easy to see that this matrix satisfies the Kolmogorov-Chapman equation:*

$$Q^{[s,t]} = Q^{[s,\tau]}Q^{[\tau,t]}, \quad \text{for all} \ \ 0 \leq s < \tau < t.$$

Now define functions

$$P(s, i, j, k, t, \{l\}) = q_{il}^{[s,t]} = a_l(t).$$

One can check that the defined family $\{P(s, i, j, k, t, \{l\})\}$ is a CSP.

Example 2.6. *(cf. Example 4.2.1 of [19]) Let $(E, \mathcal{F})$ be a measurable space and m_0 be an initial measure on this space. Consider the following functions*

$$P(s, x, y, z, t, A) = \frac{1}{3^{t-s-1}}\left(\frac{\delta_x(A) + \delta_y(A) + \delta_z(A)}{3} + (3^{t-s-1} - 1)m_0(A)\right),$$

where $t - s \geq 1$, $x, y, z \in E$ and $A \in \mathcal{F}$,

$$\delta_x(A) = \begin{cases} 1, & \text{if} \ \ x \in A \\ 0, & \text{if} \ \ x \notin A. \end{cases}$$

It is easy to see that the defined family is CSP.

3. A construction of CSOs

Recall that a construction of CSO for finite E was given in [23]. Therefore, in this section we consider the continual case.

Let $G = (\Lambda, L)$ be a countable graph. For a finite set Φ denote by Ω the set of all functions $\sigma : \Lambda \to \Phi$. Let $S(\Omega, \Phi)$ be the set of all probability measures defined on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the σ−algebra generated by the finite-dimensional cylindrical set. Let μ be a measure on $(\Omega, \mathcal{F})$ such that $\mu(B) > 0$ for any finite-dimensional cylindrical set $B \in \mathcal{F}$.

Let $M \subset \Lambda$ be a finite connected subgraph. Two elements $\sigma, \varphi \in \Omega$ are called equivalent if $\sigma(x) = \varphi(x)$ for any $x \in M$, i.e. $\sigma(M) = \varphi(M)$. Let $\xi = \{\Omega_i, i = 1, 2, ..., |\Phi|^{|M|}\}$, be the partition of Ω generated by this equivalent relation, where $|\cdot|$ denotes the cardinality of a set and Ω_i contains all equivalent elements[2].

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[2] We note that ξ depends on M, therefore all quantities which we define using ξ also depend on M. But for simplicity of formulas we will omit M from our formulas.

Denote $\langle ijk, l \rangle = \delta_{il} + \delta_{jl} + \delta_{kl}$, where δ is the Kroneckers symbol, i.e.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Consider

$$P_{\sigma_1 \sigma_2 \sigma_3, \sigma} = \frac{\langle ijk, l \rangle \mu(\Omega_l)}{\mu(\Omega_i) + \mu(\Omega_j) + \mu(\Omega_k)} \quad \text{if } \sigma_1 \in \Omega_i, \sigma_2 \in \Omega_j, \sigma_3 \in \Omega_k, \sigma \in \Omega_l. \tag{3.1}$$

For arbitrary σ it is easy to see that $P_{\sigma_1 \sigma_2 \sigma_3, \sigma}$ is invariant with respect to any permutations of $\sigma_1, \sigma_2, \sigma_3$.

Then the coefficients $P(\sigma_1, \sigma_2, \sigma_3, A) \quad (\sigma_1, \sigma_2, \sigma_3 \in \Omega, \quad A \in \mathcal{F})$ are defined as

$$P(\sigma_1, \sigma_2, \sigma_3, A) = Z(\sigma_1, \sigma_2, \sigma_3) \int_A P_{\sigma_1 \sigma_2 \sigma_3, \sigma} d\mu(\sigma) = Z(\sigma_1, \sigma_2, \sigma_3) \sum_{l=1}^{m} \int_{A \cap \Omega_l} P_{\sigma_1 \sigma_2 \sigma_3, \sigma} d\mu(\sigma),$$

where $m = |\Phi|^{|M|}$ and $Z(\sigma_1, \sigma_2, \sigma_3)$ is the normalizing factor, which is chosen by the condition that $P(\sigma_1, \sigma_2, \sigma_3, \Omega) = 1$.

It is easy to obtain the following

$$P(\sigma_1, \sigma_2, \sigma_3, A) = \begin{cases} \frac{\mu(\Omega_i)\mu(A \cap \Omega_i) + \mu(\Omega_j)\mu(A \cap \Omega_j) + \mu(\Omega_k)\mu(A \cap \Omega_k)}{\mu^2(\Omega_i) + \mu^2(\Omega_j) + \mu^2(\Omega_k)}, \\ \qquad \text{if } \sigma_1 \in \Omega_i, \sigma_2 \in \Omega_j, \sigma_3 \in \Omega_k, i \neq k, j \neq k, i \neq j \\ \frac{2\mu(\Omega_i)\mu(A \cap \Omega_i) + \mu(\Omega_j)\mu(A \cap \Omega_j)}{2\mu^2(\Omega_i) + \mu^2(\Omega_j)}, \quad \text{if } \sigma_1, \sigma_2 \in \Omega_i, \sigma_3 \in \Omega_j, i \neq j \\ \frac{\mu(A \cap \Omega_i)}{\mu(\Omega_i)} \quad \text{if } \sigma_1, \sigma_2, \sigma_3 \in \Omega_i. \end{cases} \tag{3.2}$$

The CSO W acting on the set $S(\Omega, \Phi)$ is determined by coefficients (3.2) is defined as follows: for an arbitrary measure $\lambda \in S(\Omega, \Phi)$, the measure $\lambda' = W\lambda$ is

$$\lambda'(A) = \int_\Omega \int_\Omega \int_\Omega P(\sigma_1, \sigma_2, \sigma_3, A) d\lambda(\sigma_1) d\lambda(\sigma_2) d\lambda(\sigma_3). \tag{3.3}$$

Using (3.2) from (3.3) we obtain

$$\lambda'(A) = \sum_{i=1}^{m} a_i(A) \lambda^3(\Omega_i) +$$

$$3 \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}(A) \lambda^2(\Omega_i) \lambda(\Omega_j) + 6 \sum_{1 \leq i < j < k \leq m} c_{ijk}(A) \lambda(\Omega_i) \lambda(\Omega_j) \lambda(\Omega_k), \tag{3.4}$$

where

$$\begin{aligned} a_i(A) &= \frac{\mu(A \cap \Omega_i)}{\mu(\Omega_i)}, \\ b_{ij}(A) &= \frac{2\mu(\Omega_i)\mu(A \cap \Omega_i) + \mu(\Omega_j)\mu(A \cap \Omega_j)}{2\mu^2(\Omega_i) + \mu^2(\Omega_j)}, \\ c_{ijk}(A) &= \frac{\mu(\Omega_i)\mu(A \cap \Omega_i) + \mu(\Omega_j)\mu(A \cap \Omega_j) + \mu(\Omega_k)\mu(A \cap \Omega_k)}{\mu^2(\Omega_i) + \mu^2(\Omega_j) + \mu^2(\Omega_k)}. \end{aligned} \tag{3.5}$$

It is easy to see that

$$a_i(\Omega_l) = \begin{cases} 1, & \text{if } l = i, \\ 0, & \text{if } l \neq i \end{cases} \qquad b_{ij}(\Omega_l) = \begin{cases} \frac{2\mu^2(\Omega_i)}{2\mu^2(\Omega_i) + \mu^2(\Omega_j)}, & l = i \\ \frac{\mu^2(\Omega_j)}{2\mu^2(\Omega_i) + \mu^2(\Omega_j)}, & l = j \\ 0, & l \neq i, l \neq j. \end{cases} \qquad (3.6)$$

$$c_{ijk}(\Omega_l) = \begin{cases} \frac{\mu^2(\Omega_l)}{\mu^2(\Omega_i) + \mu^2(\Omega_j) + \mu^2(\Omega_k)}, & l \in \{i, j, k\} \\ 0, & l \notin \{i, j, k\}. \end{cases}$$

For a given measure $\lambda \in S(\Lambda, \Phi)$ the trajectory $\{\lambda^{(n)}\}, n = 1, 2, ...$ of the operator (3.3) is defined by $\lambda^{(n+1)}(A) = W(\lambda^{(n)})(A)$, where $n = 0, 1, 2, ...$ and $\lambda^{(0)} = \lambda$, $A \in \mathcal{F}$.

By (3.4) and (3.6) we have

$$\lambda'(\Omega_l) = \sum_{i=1}^{m} a_i(\Omega_l)\lambda^3(\Omega_i) +$$

$$3\sum_{i=1}^{m}\sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}(\Omega_l)\lambda^2(\Omega_i)\lambda(\Omega_j) + 6\sum_{1 \leq i < j < k \leq m} c_{ijk}(\Omega_l)\lambda(\Omega_i)\lambda(\Omega_j)\lambda(\Omega_k). \qquad (3.7)$$

A CSO (1.3) is called Volterra if the coefficients $P_{ijk,l}$ may be nonzero only when $l \in \{i, j, k\}$ and vanish in all the remaining cases (see [13], [14]).

It is easy to see that any Volterra CSO has the following form

$$W : \lambda'_l = \lambda_l \left(\lambda_l^2 + \lambda_l \sum_{\substack{i=1 \\ i \neq l}}^{m} a_{i,l}\lambda_i + \sum_{\substack{i,j=1 \\ i \neq l, j \neq l}}^{m} b_{ij,l}\lambda_i\lambda_j \right), \quad (l = 1, ..., m) \qquad (3.8)$$

where $a_{i,l}$ and $b_{ij,l}$ are some coefficients depending on $P_{ijk,l}$.

Denoting $\lambda_i = \lambda(\Omega_i)$, and $a_{i,l} = 3b_{li}(\Omega_l)$, $b_{ij,l} = 6c_{ijl}(\Omega_l)$ the operator (3.7) can be written as (3.8).

Note that the n-th iteration $\lambda^{(n)} = W^{(n)}\lambda^{(0)}$ of the operator (3.3) (i.e. (3.4)) can be written as

$$\lambda^{(n+1)}(A) = \sum_{i=1}^{m} a_i(A)(\lambda_i^{(n)})^3 +$$

$$3\sum_{i=1}^{m}\sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}(A)(\lambda_i^{(n)})^2\lambda_j^{(n)} + 6\sum_{1 \leq i < j < k \leq m} c_{ijk}(A)\lambda_i^{(n)}\lambda_j^{(n)}\lambda_k^{(n)}, \qquad (3.9)$$

where $\lambda_j^{(n)}$, $j = 1, ..., m$ are coordinates of the trajectory of operator (3.8) for the given λ.

Thus in order to study the trajectory of operator (3.3) it is enough to know the behavior of trajectories of the operator (3.8), i.e. we proved the following

Theorem 3.1. *For any finite $M \subset \Lambda$ the dynamical system generated by the CSO (3.3) is reducible to a dynamical system generated by a Volterra CSO acting on $(m-1)$-dimensional simplex.*

From Theorem 3.1 we get

Corollary 3.2. *Assume for a given measure μ and $\lambda = (\lambda_1, ..., \lambda_m) \in S^{m-1}$ for the trajectory of the Volterra operator (3.8) we have*

$$\lim_{n \to \infty} \lambda^{(n)} = (\lambda_1^*, \lambda_2^*, ..., \lambda_m^*).$$

Then the corresponding trajectory $\{\lambda^{(n)}(A)\}$ of the operator (3.3) has the following limit

$$\lambda(A) = \lim_{n \to \infty} \lambda^{(n)}(A) = \sum_{i=1}^{m} a_i(A)(\lambda_i^*)^3 +$$

$$3 \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}(A)(\lambda_i^*)^2 \lambda_j^* + 6 \sum_{1 \leq i < j < k \leq m} c_{ijk}(A) \lambda_i^* \lambda_j^* \lambda_k^*. \qquad (3.10)$$

Remark 3.3. *As it was mentioned above the theory of Volterra QSOs is well studied (see for example [10]). But Volterra CSOs were not exhaustively studied, because such cubic operators are still complicated. There are just a few articles devoted to Volterra CSOs [13], [14], [22], [23]. Therefore formula (3.10) is already helpful by using the results for the Volterra CSOs studied in these papers.*

4. Integro-differential equations for CSP

The equations which we want to drive here were given in [17] for finite E. So we consider the continual case of E. Let E be a continual set. Consider a CSP on a measurable space $(E, \mathcal{F})$ with initial measure m_0. For $t > s + 2$ from condition (V) of Definition 2.2 we get

$$P(s, x, y, z, t + \Delta, A) - P(s, x, y, z, t, A)$$

$$= \int_E \int_E \int_E P(s, x, y, z, t - 1, du) \{P(t - 1, u, \vartheta, q, t + \Delta, A)$$

$$- P(t - 1, u, \vartheta, q, t, A)\} m_{t-1}(d\vartheta) m_{t-1}(dq).$$

Assume the following limit exists

$$C(t, u, \vartheta, q, A) = \lim_{\Delta \to 0} \frac{P(t - 1, u, \vartheta, q, t + \Delta, A) - P(t - 1, u, \vartheta, q, t, A)}{\Delta}.$$

Then taking limit $\Delta \to 0$ we obtain the *first integro-differential equation*:

$$\frac{\partial P(s, x, y, z, t, A)}{\partial t} = \int_E \int_E \int_E P(s, x, y, z, t - 1, du) \, C(t, u, \vartheta, q, A) m_{t-1}(d\vartheta) m_{t-1}(dq). \qquad (4.1)$$

Similarly, one gets the *second integro-differential equation*:

$$\frac{\partial P(s, x, y, z, t, A)}{\partial s} = - \int_E \int_E \int_E C(s + 1, x, y, z, du) P(s + 1, u, \vartheta, q, t, A) m_{s+1}(d\vartheta) m_{s+1}(dq). \qquad (4.2)$$

Let $E = \mathbb{R}$ and $A_w = (-\infty, w]$, where $w \in \mathbb{R}$. Denote $F(s, x, y, z, t, w) = P(s, x, y, z, t, A_w)$. It is clear that $F(s, x, y, z, t, w)$ as the function of w is monotone, right-continuous and $F(s, x, y, z, t, -\infty) = 0$, $F(s, x, y, z, t, +\infty) = 1$.

The condition (V) of Definition 2.2 for the function $F(s, x, y, z, t, w)$ has the following form

$$F(s, x, y, z, t, w) = \int\limits_E \int\limits_E \int\limits_E dF(s, x, y, z, \tau, u) F(\tau, u, \vartheta, q, t, w) m_\tau(d\vartheta) m_\tau(dq).$$

If the function $F(s, x, y, z, t, w)$ is absolutely continuous with respect to variable w then

$$F(s, x, y, z, t, w) = \int\limits_{-\infty}^{w} f(s, x, y, z, t, u) du,$$

where $f(s, x, y, z, t, w)$ is a non-negative function and measurable with respect to variables x, y, z, w, moreover it satisfies the following conditions

$$\int\limits_{-\infty}^{\infty} f(s, x, y, z, t, w) dw = 1.$$

$$f(s, x, y, z, t, w) = \int\limits_E \int\limits_E \int\limits_E f(s, x, y, z, \tau, u) f(\tau, u, \vartheta, q, t, w) m_\tau(d\vartheta) m_\tau(dq) du. \qquad (4.3)$$

Form (4.1) and (4.2) for $F(s, x, y, z, t, w)$ we get

$$\frac{\partial F(s, x, y, z, t, w)}{\partial t} = \int\limits_E \int\limits_E \int\limits_E \frac{\partial F(s, x, y, z, t-1, u)}{\partial u} C(t, u, \vartheta, q, w) du\, m_{t-1}(d\vartheta) m_{t-1}(dq),$$

$$\frac{\partial F(s, x, y, z, t, w)}{\partial s} = \int\limits_E \int\limits_E \int\limits_E \frac{\partial C(s+1, x, y, z, u)}{\partial u} F(s+1, u, \vartheta, q, t, w) du\, m_{s+1}(d\vartheta) m_{s+1}(dq).$$

Assume the following limit exists:

$$a(t, u, \vartheta, q, w) = \lim_{\Delta \to 0} \frac{f(t-1, u, \vartheta, q, t+\Delta, w) - f(0, u, \vartheta, q, 1, w)}{\Delta}.$$

Then we obtain the following integro-differential equations:

$$\frac{\partial f(s, x, y, z, t, w)}{\partial t} = \int\limits_E \int\limits_E \int\limits_E a(t, u, \vartheta, q, w) f(s, x, y, z, t-1, u) du\, m_{t-1}(d\vartheta) m_{t-1}(dq) \qquad (4.4)$$

$$\frac{\partial f(s, x, y, z, t, w)}{\partial s} = - \int\limits_E \int\limits_E \int\limits_E a(s+1, x, y, z, u) f(s+1, u, \vartheta, q, t, w) du\, m_{s+1}(d\vartheta) m_{s+1}(dq).$$

$$(4.5)$$

5. Reduction of the integro-differential equations to differential equations

Under some conditions the integro-differential equations (4.4) and (4.5) can be reduced to differential equations. In this subsection we illustrate this for equation (4.5).

Let $t > s + 2$, then from (4.3) we get

$$f(s,x,y,z,t,w) - f(s+\Delta,x,y,z,t,w) = \int\limits_E\int\limits_E\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$
$$-f(s+\Delta,x,y,z,s+1+\Delta,u)\}\, f(s+1+\Delta,u,\vartheta,q,t,w)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du \quad (5.1)$$

We consider function f such that the decomposition of $f(s+1+\Delta,u,\vartheta,q,t,w)$ into Taylor's series in a neighborhood of the point (x,y,z) has the form:

$$f(s+1+\Delta,u,\vartheta,q,t,w) = f(s+1+\Delta,x,y,z,t,w) + \frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial u}(u-x)$$

$$+\frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial \vartheta}(\vartheta-y) + \frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial q}(q-z)$$

$$+\frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial u^2}(u-x)^2 + \frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial \vartheta^2}(\vartheta-y)^2$$

$$+\frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial q^2}(q-z)^2 + \frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial u\partial\vartheta}(u-x)(\vartheta-y)$$

$$+\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial u\partial q}(u-x)(q-z) + \frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial\vartheta\partial q}(\vartheta-y)(q-z)$$

$$+\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial u^2\partial\vartheta}(u-x)^2(\vartheta-y) + \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial u^2\partial q}(u-x)^2(q-z)$$

$$+\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial\vartheta^2\partial u}(\vartheta-y)^2(u-x) + \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial\vartheta^2\partial q}(\vartheta-y)^2(q-z)$$

$$+\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial q^2\partial u}(q-z)^2(u-x) + \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial q^2\partial\vartheta}(q-z)^2(\vartheta-y)$$

$$+\frac{1}{6}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial q^3}(u-x)^3 + \frac{1}{6}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial\vartheta^2}(\vartheta-y)^3$$

$$+\frac{1}{6}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial q^2}(q-z)^3 + \frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial u\partial\vartheta\partial q}(u-x)(\vartheta-y)(q-z). \quad (5.2)$$

Substituting (5.2) into (5.1) we consider non-zero summands:

$$\int\limits_E\int\limits_E\int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\}\times$$

$$\frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial x}(u-x)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$=\frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial x}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}\,(u-x)du.$$

Denote

$$a(s,x,y,z,\Delta) = \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)du.$$

Now consider the summands with second order of derivations

$$\int\limits_E \int\limits_E \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\}\times$$

$$\frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,w)}{\partial x^2}(u-x)^2 m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,w)}{\partial x^2}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)^2 du.$$

Denote

$$b^2(s,x,y,z,\Delta) = \int\limits_E \left\{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\right\}(u-x)^2 du$$

Then

$$\int\limits_E \int\limits_E \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\}\times$$

$$\frac{\partial^2 f(s+1+\Delta,x,y,z,w)}{\partial x\partial y}(u-x)(\vartheta-y)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial y}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)du\int\limits_E (\vartheta-y)m_{s+1+\Delta}(d\vartheta).$$

Denote

$$\int\limits_E (\vartheta-y)m_{s+1+\Delta}(d\vartheta) = \alpha(s+1,y,\Delta).$$

Consequently

$$\int\limits_E \int\limits_E \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\}\times$$

$$\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial z}(u-x)(q-z)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial z}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\} \int_E (u-x)du \int_E (q-z)m_{s+1+\Delta}(dq).$$

$$\iiint_{E\ E\ E} \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\} \times$$

$$\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial y}(u-x)^2(\vartheta-y)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial y}\int_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)^2du\int_E (\vartheta-y)m_{s+1+\Delta}(d\vartheta)$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial y}b^2(s,x,y,z,\Delta)\alpha(s+1,y,\Delta).$$

$$\iiint_{E\ E\ E} \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\} \times$$

$$\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial z}(u-x)^2(q-z)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial z}\int_E \{f(s,x,y,z,s+1+\Delta)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)^2du\int_E (q-z)m_{s+1+\Delta}(dq)$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2 \partial z}b^2(s,x,y,z,\Delta)\alpha(s+1,z,\Delta).$$

$$\iiint_{E\ E\ E} \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\} \times$$

$$\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial^2 y \partial x}(\vartheta-y)^2(u-x)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial^2 y \partial x}\int_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)du\int_E (\vartheta-y)^2 m_{s+1+\Delta}(d\vartheta)$$

$$= \frac{1}{2}a(s,x,y,z,\Delta)\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial^2 y \partial x}\cdot \alpha_2(s+1,y,\Delta),$$

where

$$\alpha_2(s+1,y,\Delta) = \int_E (\vartheta-y)^2 m_{s+1+\Delta}(d\vartheta).$$

$$\int\limits_E \int\limits_E \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\} \times$$

$$\frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial z^2 \partial x}(q-z)^2(u-x)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial z^2 \partial x}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)du\int\limits_E (q-z)^2 m_{s+1+\Delta}(dv)$$

$$= \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial z^2 \partial x}a(s,x,y,z,\Delta)\alpha_2(s+1,z,\Delta).$$

$$\int\limits_E \int\limits_E \int\limits_E \{f(s,x,y,z,s+1+\Delta,u) - f(s+\Delta,x,y,z,s+1+\Delta,u)\} \times$$

$$\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial z\partial y\partial z}(u-x)(\vartheta-y)(q-z)m_{s+1+\Delta}(d\vartheta)m_{s+1+\Delta}(dq)du$$

$$= \frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial y\partial z}\int\limits_E \{f(s,x,y,z,s+1+\Delta,u)-$$

$$- f(s+\Delta,x,y,z,s+1+\Delta,u)\}(u-x)du\int\limits_E (\vartheta-y)m_{s+1+\Delta}(d\vartheta)\cdot\int\limits_E (q-z)m_{s+1+\Delta}(dq) =$$

$$= \frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial y\partial z}a(s,x,y,z,\Delta)\alpha(s+1,y,\Delta)\alpha(s+1,z,\Delta).$$

Since other summands are equal to zero, we get

$$f(s,x,y,z,t,w) - f(s+\Delta,x,y,z,t,w) = a(s,x,y,z,\Delta)\frac{\partial f(s+1+\Delta,x,y,z,t,w)}{\partial x}$$

$$+ b^2(s,x,y,z,\Delta)\cdot\frac{1}{2}\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2} + \frac{1}{2}a(s,x,y,z,\Delta)\alpha(s+1,y,\Delta)\times$$

$$\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial y} + a(s,x,y,z,\Delta)\frac{\partial^2 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial z}\alpha(s+1,z,\Delta)$$

$$+ \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2\partial y}b^2(s,x,y,z,\Delta)\alpha(s+1,y,\Delta) + \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x^2\partial z}\times$$

$$b^2(s,x,y,z,\Delta)\alpha(s+1,z,\Delta) + \frac{1}{2}a(s,x,y,z,\Delta)\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial y^2\partial x}\times$$

$$\alpha_2(s+1,y,\Delta) + \frac{1}{2}\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial z^2\partial x}a(s,x,y,z,\Delta)\alpha_2(s+1,z,\Delta) + a(s,x,y,z,\Delta)\times$$

$$\times\alpha(s+1,y,\Delta)\alpha(s+1,z,\Delta)\frac{\partial^3 f(s+1+\Delta,x,y,z,t,w)}{\partial x\partial y\partial z}. \tag{5.3}$$

Dividing both sides of the equality (5.3) by Δ and passing to the limit (assuming the limits exist) as $\Delta \to 0$, and denoting

$$A(s,x,y,z) = \lim_{\Delta \to 0} \frac{a(s,x,y,z,\Delta)}{\Delta}, \quad B^2(s,x,y,z) = \lim_{\Delta \to 0} \frac{b^2(s,x,y,z,\Delta)}{2\Delta},$$

$$D(s+1,y) = \lim_{\Delta \to 0} \alpha(s+1,y,\Delta), \quad D_2(s+1,y) = \lim_{\Delta \to 0} \frac{\alpha_2(s+1,y,\Delta)}{2\Delta}$$

from (5.3) we get

$$\frac{\partial f(s,x,y,z,t,w)}{\partial s} = -A(s,x,y,z)\frac{\partial f(s+1,x,y,z,t,w)}{\partial x} - B^2(s,x,y,z)\frac{\partial^2 f(s+1,x,y,z,t,w)}{\partial x^2}$$

$$-\frac{1}{2}A(s,x,y,z)D(s+1,y)\frac{\partial^2 f(s+1,x,y,z,t,w)}{\partial x \partial y} - A(s,x,y,z)D(s+1,z)\frac{\partial^2 f(s+1,x,y,z,t,w)}{\partial x \partial z}$$

$$-B^2(s,x,y,z)D(s+1,y)\frac{\partial^3 f(s+1,x,y,z,t,w)}{\partial x^2 \partial y}$$

$$-B^2(s,x,y,z)D(s+1,z)\frac{\partial^3 f(s+1,x,y,z,t,w)}{\partial x^2 \partial z} - \frac{1}{2}A(s,x,y,z)D_2(s+1,y)\frac{\partial^3 f(s+1,x,y,z,t,w)}{\partial y^2 \partial x}$$

$$-\frac{1}{2}A(s,x,y,z)D_2(s+1,z)\frac{\partial^3 f(s+1,x,y,z,t,w)}{\partial z^2 \partial x}$$

$$-A(s,x,y,z)D(s+1,y)D(s+1,z)\frac{\partial^3 f(s+1,x,y,z,t,w)}{\partial x \partial y \partial z}. \tag{5.4}$$

Remark 5.1. *We note that the equation (5.4) is known as differential equation with advanced argument. Similarly one can reduce the equation (4.4) to a differential equation which is known as a differential equation with delay argument. For theory of such kind of equations we refer to [4] and [18].*

Example 5.2. *Let $E = \mathbb{R}$. If $m_\tau(du) = r_\tau(u)du$ then from (4.3) we get*

$$f(s,x,y,z,t,w) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(s,x,y,z,\tau,u)f(\tau,u,\vartheta,q,t,w)r_\tau(\vartheta)r_\tau(q)dud\vartheta dq. \tag{5.5}$$

Let $a(s,t)$ and $b(t)$ be strictly positive functions. Consider

$$f(s,x,y,z,t,w) = (\pi a(s,t))^{-1/2}\exp\left[-\frac{(w-x-y-z)^2}{a(s,t)}\right], \tag{5.6}$$

$$r_t(u) = (\pi b(t))^{-1/2}\exp\left[-\frac{u^2}{b(t)}\right].$$

For any $A, B, C \in \mathbb{R}$, with $A > 0$ it is known[3] that

$$\int\limits_{-\infty}^{\infty} \exp(-Ax^2 + Bx + C)dx = \sqrt{\frac{\pi}{A}}\exp\left(\frac{B^2}{4A} + C\right).$$

---

[3] see https://en.wikipedia.org/wiki/Gaussian_integral

Using this formula (three times) one can see that the function (5.6) satisfies (5.5) iff functions $a(s,t) > 0$ and $b(t) > 0$ satisfy the following equation

$$a(s,t) = 2b(\tau) + a(s,\tau) + a(\tau,t), \quad for\ any\ \ s < \tau < t\ \ with\ \ \tau - s \geq 1,\ \ t - \tau \geq 1. \quad (5.7)$$

Thus we obtain a family of CSP generated by functions (5.6) with $a(s,t)$, $b(t)$ satisfying (5.7). For example, take $\epsilon \in (0,1)$, $a(s,t) = t - s - \epsilon$ and $b(t) = \epsilon/2$. These functions satisfy (5.7) and the corresponding f and r are defined by

$$f(s,x,y,z,t,w) = \frac{\exp\left[-\frac{(w-x-y-z)^2}{t-s-\epsilon}\right]}{\sqrt{(t-s-\epsilon)\pi}},$$

$$r_t(u) = \frac{\exp\left[-\frac{2u^2}{\epsilon}\right]}{\sqrt{\epsilon\pi/2}}$$

generate a CSP.

References

[1] Avise JC 2014 *The Hope, Hype, and Reality of Genetic Engineering* (New York: Oxford Univ. Press).
[2] Bernshtein S N 1924 *Uch. Zap. Nauchno-Issled. kaf. Ukr. Otd. Mat.* **1** 83–115.
[3] Dohtani A 1992 *SIAM J.Appl. Math.* **52** 1707-1721.
[4] Elgolz L E and Norkin S B 1971 *Introduction to the theory of differential equations with a deviating agument* (Moscow: Nauka)
[5] Fisher M E and Goh B S 1977 *J. Math. Biol.* **4** 265-274
[6] Ganikhodjaev N N and Rozikov U A 2006 *Regul. Chaotic Dyn.* **11** 467–473
[7] Ganikhodjaev N N 2000 *Doklady Math.* **61** 321–323
[8] Ganikhodjaev N N 1991 *J. Theor. Prob.* **4** 639-653
[9] Ganikhodzhaev R N 1993 *Russian Acad. Sci. Sbornik Math.* **76** 489–506
[10] Ganikhodzhaev R N, Mukhamedov F M and Rozikov U A 2011 *Inf. Dim. Anal. Quant. Prob. Rel. Fields.* **14** 279-335
[11] Erson R C and Stewart J 1971 *Hum. Hered.* **21** 523–542
[12] Jenks R 1969 *J. Differ. Equations.* **5** 497–514
[13] Khamraev A Yu 2009 *Uzbek. Mat. Zh.* **3** 65–71
[14] Khamraev A Yu 2004 *Uzbek. Mat. Zh.* **2** 79–84.
[15] Lyubich Yu I 1992 *Mathematical structures in population genetics* (Berlin: Springer)
[16] Malevanets A and Kapral R 2000 *J. Chem. Phys.* **112** 7260–7269
[17] Mamurov B J 2010 *Uzbek. Mat. Zh.* **4** 113–117
[18] Mishkis A D 1972 *Linear differensial equations with delay of argument* (Moscow: Nauka) (Russian)
[19] Mukhamedov F and Ganikhodjaev N 2015 *Lect. Notes Math.* **2133** (Berlin: Springer)
[20] Mukhamedov F and Supar N A 2015 *Bull. Malay. Math. Sci. Soc.* **38** 1281–1296
[21] Mukhamedov F, Supar N A and Pah Ch H 2013 *J. Phys.: Conf. Ser.* **435** 012013.
[22] Rozikov U A and Khamraev A Yu 2004 *Ukraine Math. Jour.* **56** 1699–1711
[23] Rozikov U A and Khamraev A Yu 2014 *Nonlinear Dyn. Syst. Theory.* **14** 92–100
[24] Sarymsakov T A and Ganikhodzhaev N N 1989 *Soviet Math. Dokl.* **39** 369-373
[25] Sarymsakov T A and Ganikhodzhaev N N 1991 *Soviet Math. Dokl.* **43** 279-283
[26] Sarymsakov T A and Ganikhodzhaev N N 1990 *J. Theor. Prob.* **3** 51-70
[27] Udwadia F E and Raju N 1998 *Physica D* **111** 16-26