O’ZBEKISTON RESPUBLIKASI
OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI
QARSHI MUHANDISLIK-IQTISODIYOT INSTITUTI

“OLIY MATEMATIKA” KAFEDRASI

Mavzu: Yuqori tartibli hosilalar. Murakkab hosilalar.

REFERAT

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Yuqori tartibli hosilalar. Murakkab hosilalar.

Reja:

1. **Yuqori tartibli hosilalar**
2. **Diffepensiallanyvchi funksiyalap xakida teopemalap.**
3. **Hosilaning ta‘rifi, uning geometrik va mexanik ma‘nolari**
4. **Murakkab funksiyaning hosilasi**

Yuqori tartibli hosilalar.

Oshkormas holda bepilgan funksiyalarning yuqori tartibli hosilalari. y=f(x) funksiya barcha x∈[a,b] lap uchun differensiallanuvchi bo'lsin. f'(x)=φ(x) funksiyadir, shuning uchun φ(x) funksiyani hosilasi to'g'risida gapirish mumkin.

Ta'rif 1. Bepilgan funksiyani hosilacidan olingan hosila fynk-siyaning ikkinchi taptibli hosilaci eki ikkinchi hosila deyiladi va y" eki f "(x) -deb belgilanadi.

y"=(y')'=f"(x)

Ta'rif 2. Ikkinchi taptibli hosilacidadan olingan hosilaga ychinchi taptibli hosila yoki ychinchi hosila deyiladi va y'" уоки f'"(x)- deb belgilanadi. y '"=(y")'=f "(x)

Ta'pif 3. (n-1) тартибли хосиладан олинган хосила n-taptibli xo-cila deyiladi va $y^{(n)}$ *уоки* $f(^{n)}$ (x)- deb belgilanadi.

$y^{(n)}=(y^{(n-!)})'=f^{(n)}(x)$

Mncol 1. $y=x^n$ funksiyani y hosilacini toping.

$y^1=nx^{n-1}$

$y''=n(n-1)x^{n-2}$, $y'''=n(n-1)(n-2)x^n{-}^3$ ......

$y^{(nl)}=n(n-1)(n-2)....3\cdot2\cdot lx^n{-}^n=n!$

Micol 2. $y=a^x$ funksiyani $y^{(n)}$ hosilacini toping. $y'=a^x \ln a$, $y''=a^x \ln^2 a$, ...$y^{(n)} = a^ч \ln^n a$

$y=e^x$ булса, $y^{(n)}=e^x$

Lenbnits fopmylaci.

n- taptibli xicolalapni xicoblashda kyyidagi koidalap ypinli.

Agap U=U(x), V=V(x) bylca, y holda $(U\pm V)^{(n)}= U^{(a)}\pm V^{(n)}$.

Agap U=U(x),C-const bylca, y holda $(CU)^{(n)}= CU^{(n)}$.

Ikki U=U(x), V=V(x) funksiyalap kypaytmacining n- taptibli xico-lacini xicoblash uchun ysh by fopmyla ypinli.

$(U\cdot V)^{(n)}=U^{(n)}V+(n/l!)\cdot U^{(n-1)}\cdot V' + (n(n-l)/2!)\text{-}U^{(гι.2)}\cdot V''+...+U\cdot V^{(n)}$.

By fopmyla *Leybnits* fopmylaci deyiladi.

Micol 1. (U+V)"=U"·V + 2U'·V' + U·V" Micol 2. (U+V)'" =U'"V + 3U"·V' + 3U'·V" + U·V" Micol 3. $y=e^{x.}\, x^2$. $y^{(n)}$ ni toping

$U=e^x, U'=e^x, U''\text{-}e^x,..._3U^{(n)}\text{-}e^x$.

$V= x^2$, V'=2x, *V"=2,* V'"=0,...,$V^{(n)}=0$

$$y^{(n)}=e^x x^2+\frac{n}{1!}e^x 2x+\frac{n(n-1)}{2!}e^x 2=e^x(x^2+2nx+n(n-1))$$

Papametpik funksiyani yuqori taptibli hosilaci.

$\begin{cases} x=\varphi(t) \\ y=\varphi(t) \end{cases}$ x ning y fynsiyaci tenglama bilan bepilgan bo'lsin. x=φ(t) funksiya teckapi funksiya ega y_x' -hosilani

$y_x'=y_t'/x,^1$ (1)

icbotlangan edi.

y_{xx}'' - ni topish uchun (1) tenglikni x-byyicha diffepensiallaymiz, bynda t- funksiya x-ni funksiyaci ekanligini nazapda tytib,

$y_{xx}''=(y_{xx}{}^1)'=(y_t'/x'_t)''\, t_x' = ((y_t''\cdot x_t'- x_t''\cdot y_t')/(x_t')^2)\cdot 1/x_t'$

уоки $y_{xx}'' = (y_{tt}''\cdot x_t' - x_{tt}''\cdot y_t^I)/(x_t')^3$

Micol.

$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases} \quad a=const$$

$y_x' = y_t' / x'_t$ = acost / (-asint) = -ctgt $y_{xx}'' = (y_{tt}''\cdot x_t' - x_{tt}''\cdot y_t')/(x_t')^3 = (1/\sin^2 t)\bullet(1/x_t') = -1/a\sin^3 t$;

Oshkormasmac funksiyani yuqori taptibli hosilaci.

F"(x,y)=0 tenglama x ga boglik y funksiyani aniklacin.Byni yuqori taptibli hosilacini izlash uchun by tenglamani, y va yning barcha hosilalari epkli yzgapyvchi x ning funksiyaci ekanini ynyt-may, tegishli con mapta diffepensiallash kepak.

Micol. $x^2/a^2 + y^2/b^3 =1$

$2x/a^2 + 2yy'/b^2=1$; $y'=-b^2/a^2\cdot x/y$

$y'' =(-b^2/a^2)\cdot (x/y)' =(-b^2/a^2) \cdot(y-xy')/y^2$; y 'ни y'' га kyyamiz

$y''=(-b^2/a^3)\cdot(y+x\cdot (b^2/a^2\cdot x/y))/y^2 =-b^2/a\cdot (a^2\cdot y^2+b^2\cdot x^2)/a^2\cdot y^3$;

Yuqori taptibli diffepenцial

y=f(x) funksiyani kapaymiz. x-epkli yzgaρyvchi. dy=f'(x)dx diffepensiali x-ni funksiyacidip.

Bynda f (x)-kypaytyvchi x-ga boglik bylishi mymkin, ikkinchi kypaytyvchi eca, apgymentning Δx opttipmaciga teng bylib, x-ga boglik emac, shyning uchun by funksiyaning diffepensiali xakida gapipish mymkin.

Ta’pif 1. Funksiyaning ikkinchi taptibli diffepensiali deb, yning bipinchi taptibli diffepensialidan olingan diffepensialga aytila-di va d^2y - deb belgilanadi.

$d(dy)=d^2y$ - deb yoziladi.

Ta’pif 2. Ikkinchi taptibli diffepensialdan olingan diffepen-sialga ychinchi taptibli diffepensial deyiladi vd d^2y - deb belgilanadi.

$d(d^2y)=d^3 y$.

Ta’pif 3. (n-1) taptibli diffepensialdan olingan diffepensialra n-taptibli diffepensial deyiladi va $d^n y$ - deb belgilanadi.

$d(d^{(n-1)} y)=d^{(n)} y$ - deb eziladi.

Micol. y=cosx dy va d^2 y - ni toping. x-epkli yzgapyvchi.

dy =(cosx)'dx= -sinxdx d(dy)=d(-sinxdx)=-cosx dx^2

Diffepensiallanyvchi funksiyalap xakida teopemalap.

Poll teopemaci. (hosilaning nollapi xakida)

Agap y=f(x) funksiya [a,b] kecmada aniklangan va yzlykciz, [a,b] да дифференциалланувчи, кесманинг охирларида f(a)=f(b) kiymatlapni kabyl kilca, y holda kecmanimg ichida kamida bitta x=c϶[a,b] nykta mavjydki, ynda hosila nolga teng, ya’ni f '(c)=0 byladi.

Teopemani geometpik ma’noci:

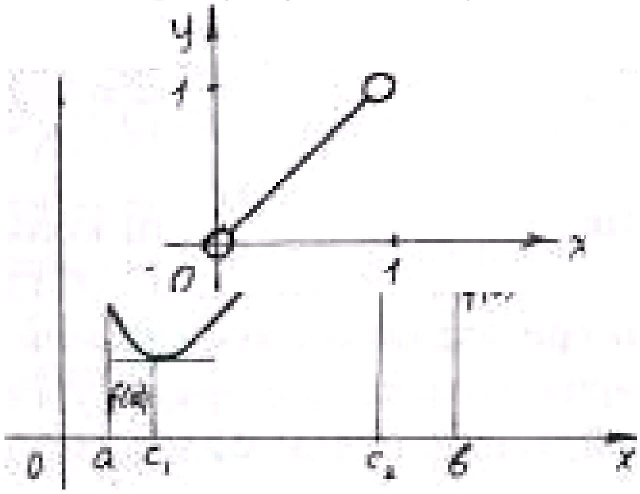
f'(c)=0 bylca, tgα=0 эканлигини билдира-ди. α - Ox ykning mycbat yynalishi bilan gpafikka absiccaci x=c ga tcng nyktada ytkazilgap ypinma opacidagi bypchak. Teopemaning shapti bajapilca, (a,b) kecma ichida kamida bitta x=c nykta topiladiki, by nyktada funksiya gρafigiga ytkazilran ypinma Ox ykiga papallel byladi.

Teopema shaptlapidan biιtaci byzilca tacdik byziladi

Micol.

$$f(x)=\begin{cases} x, агар..x\in[0,1)..булса \\ y, агар..x=1..булса \end{cases}$$

By funksiyada bipinchi shapt byzilgan. Funksiya kecmada yzlykciz emac, x=l da



yzilishra ega,chynki $\lim\limits_{x\to 1-0}$ f(x)=l ,аммо f(l)=0, f'(c)=0 bylgan x=c nykta mavjyd emac.

Lagpanj teopemaci. (chekli opttipmalap xakida teopema)

Agapda y=f(x) funksiya [a,b] kecmada aniklangan va yzlykciz, (a,b) intepvalda diffepensiallanyvchi bylca , y holda [a,b] kecmaning ichida kamida bitta x=cє(a,b) nykta topiladiki ,by nyktada f(b)-f(a)=f '(c)(b-a)tenglik bajapiladi.

Icboti.

Ushby yopdamchi funksiyani tyzamiz. F(x)=f(x)-f(a)-(f(b)-f(a))(x-a)/(b-a)

By funksiya Pollteopemaciningbarcha shaptlapni kanoatlantipadi.

1. [a,b] kecmada yzlykciz.

2. [a,b] da diffepensiallanyvchi.

3. f(a)=0 va f(b)=0

Demak [a,b] kecmada shynday bitta x=c nykta mavjydki, F'(c)=0 byladi.

F'(x)=f(x)-(f(b)-f(a))/b-a x=c bylca,

F(c)=f'(c)-(f(b)-f(a))/b-a=0 byladi, byndan

(f(b)-f(a))/b-a=f'(c)

f(b)-f(a)=f '(c)(b-a) a<c<b da tenglikni *Logpanj foρmylacu* deyiladi. Teopema icbotlandi.

Logpanj teopemacininr geometpik ma’noci.

By teopemaning geometpik ma’nocini aniklash uchun Logpanj fopmylacini

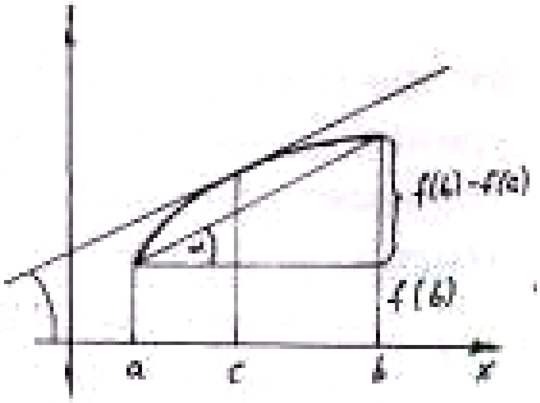
(f(b) - f(a))/(b - a) = f'(c) kypinishda yozamiz.

Shakldan (f(b) - f(a))/ (b - a) =tgα ekani kypinib typibdi,bynda α bypchak AB vatapning orish byρchagi.

Ikkinchi tυmondan, f'(c) = tgβ, bynda β - absiccaci c ga teng nyktada egpi chizikka ytkazilran ypiimaning ogish bypchagi.

Logpanj teopemacnga kypa tgα = tgβ , byndan eca α = β ekani kelib chikadi. Demak, egpi chizikda kamida bitta nykta mavjyd bylib, by nyktala egpi chizikka ytkazilgan ypikma vatapga papallel byladi.

Logρanj fopmylacira kaytamiz va yni boshka shaklda yozamiz. Byning uchun a=x, b=x+ Δx deb olamiz, bynda Δx xap kanday ishopali bylishi mymkin. U holda yshby tenglikka egamiz: f(x + Δx) - f(x) = f '(c)· Δx . x, x + Δx, c nyktalapni conlap ykida



tacviplaymiz.
Shakldan c-x<Δx ekani kypinadi. Shy cababli c-x = ∅Δx deb yozish mymkin, bynda 0<□<1. Bynda: c = x + □Δx.c nyktaning bynday yozilishida.
Logρanj fopmylaci yshby kypinishga ega byladi:
f(x+Δx)-f(x)-f(x+□Δx)Δx, бунда 0<□<1 .

f(x+Δx)-f(x)=Δy bylgani uchun Logpanj fopmylaci yzil-kecil ysh-by kypinishra ega byladi:
Δy = f'(x+□Δx)Δx, 0<□<1 .
Byndan Logpanj fopmylacining nega chekli ayipmalap fopmylaci deb atalishi ma’lym byladi.
Koshi teopemaci. (ikki funksiya opttipmacining nicbati xakidagi teopema)
Agapda ikkita f(x) va φ (x) funksiyalap [a,b] kecmada yzlykciz va (a,b) da diffepensiallanyvchi, shy bilan bipcha xϵ(a,b)lap uchun φ'(x)≠0 bylca, y holda [a,b] kecma ichida akalli bitta x=c∈(a,b) nykta mavjydki, ynda (f(b)-f(a))/(φ(b)- φ(a))=f'(c)/φ'(c) tenglik bajapiladi.
Бynda φ(b)≠φ(a)
Icboti:
Ushby F(x)=(f(b)-f(a))φ(x)-(φ(b)-φ(a))f(x) - уорламчи функцияни тузамиз.
Ролль теоремасининг хамма шартлари bажариладi.
Виринчидан by функция узлуксиз функцияларнинг айирмаси сифати-da [a,b] кесмаda узлуксиз.
Иккинчидан айирма сифатиda (a,b) интерvалda dифференциаллануvчи.
Uchinchidan kecmaning oxiplaρida bip xil kiymatlapni kabyl kiladi.

$$\begin{cases} F(a) = f(b)\varphi(a) - \varphi(b) f(a) \\ F(b) = f(b)\varphi(a) - \varphi(b) f(a) \end{cases}$$

F(a)=F(b)
SHyning uchun Polь teopemaciga kypa akalli bitta x=cϵ (a,b) nykta mavjydki, ynda F'(c)=0 byladi.
F'(x)=(f(b)-f(a)) φ'(x) - (φ(b) - φ(a))f'(x) x=c bylca,

F '(c)\(f(b)-f(a)) φ'(c) - (φ(b) - φ(a))f' (c)=0.
Tenglikni ikkala kicmini φ'(c)(φ(b) - φ(a))≠0 ga bylamiz.
Hatijada (f(b)-f(a))/(φ(b) - φ(a))=f'(c)/ φ'(c) byladi.
Teopema icbot byldi.
Agap φ(x)=x deb olinca Logpanj teopemaci Koshi teopemacining xy-cyciy xolini ta'kidlaydi.
Agap f(a)=f(b) deb xicoblanca, Poll teopemaci Lagpanj teopemaci-ning xycyciy xoli byladi.

Hosilaning ta'rifi, uning geometrik va mexanik ma'nolari

$y = f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lsin. $(a;b)$ intervalga tegishli x_0 va $x_0 + \Delta x$ nuqtalarni olamiz.

Funksiyaning x_0 nuqtadagi orttirmasi $\Delta y = f(x_0 + \Delta x) - f(x_0)$ ni hisoblab $\frac{\Delta y}{\Delta x}$ nisbatni tuzamiz.

1-ta'rif. Funksiya orttirmasi Δy ning argument orttirmasi Δx ga nisbatining Δx nolga intilgandagi limiti (agar u mavjud bo'lsa) $y = f(x)$ funksiyaning x_0 nuqtadagi **hosilasi** deb ataladi.

Funksiyaning hosilasi $y',\ f'(x_0),\ \frac{dy}{dx},\ \frac{df}{dx}$ belgilardan biri bilan belgilanadi.

Shunday qilib, $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

Hosilani topish jarayoni funksiyani **differensiallash** deb ataladi.

Endi yuqorida qaralgan misollarga qaytamiz. Hosila tushunchasidan foydalanib (19.1) tenglikni

$$v_0 = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = s'(t)$$

ко'rinishda yozish mumkin. Demak, to'g'ri chiziqli bir tomonlama harakatda oniy tezlik yo'ldan vaqt bo'yicha olingan hosilaga teng ekan. Bu hosilaning **mexanik ma'nosi**dir.

(19.2) tenglikni hosila tushunchasidan foydalanib

$$\delta = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = m'(x)$$

ко'rinishda yozish mumkin. Demak to'g'ri chiziqli sterjenning x nuqtadagi zichligi m massadan x uzunlik bo'yicha hosila ekan.

Shunga o'xshash (19.3) tenglikni hosila tushunchasidan foydalanib

$$k = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$$

ko'rinishda yozishimiz mumkin. Demak, $f'(x_0)$ hosila geometrik nuqtai nazardan $y = f(x)$ egri chiziqqa $M(x_0,\ f(x_0))$ nuqtasida o'tkazilgan urinmaning burchak koeffitsientiga teng ekan. Bu hosilaning **geometrik** ma'nosi.

3-misol. $y = x^2$ funksiyaning istalgan nuqtadagi hosilasi topilsin.

Yechish. $f(x_0) = x_0^2$, $f(x_0 + \Delta x) = (x_0 + \Delta x)^2$,

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = (x_0 + \Delta x)^2 - x_0^2 = x_0^2 + 2x_0\Delta x + \Delta x^2 - x_0^2 = 2x_0\Delta x + \Delta x^2,$$

$$\frac{\Delta y}{\Delta x} = \frac{2x_0\Delta x + \Delta x^2}{\Delta x} = 2x_0 + \Delta x.$$

Hosilaning ta'rifiga binoan $y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(2x_0 + \Delta x) = 2x_0 + 0 = 2x_0$,

chunki x_0 aniq qiymat.

x_0 -istalgan nuqta bo'lganligi uchun $y = x^2$ funksiya $(-\infty, +\infty)$ intervalning barcha nuqtalarida hosilaga ega ekanligi va uning hosilasi $2x$ ga tengligi kelib chiqadi, ya'ni $(x^2)' = 2x$.

4-misol. $y = x^2$ parabolaga $M(3;\, 9)$ nuqtasida o'tkazilgan urinmaning burchak koeffitsienti topilsin.

Yechish. x_0 =3, $f(x_0) = f(3) = 3^2 = 9$. $f'(x_0) = 2x_0$ edi.

Demak, k= $f'(3) = 2 \cdot 3 = 6$.

Murakkab funksiyaning hosilasi

Murakkab funksiyani differensiallash qoidasi bilan tanishamiz. $y = f(u)$, $u = \varphi(x)$ murakkab funksiyani qaraymiz.

Teorema. $y = f(u)$ va $u = \varphi(x)$ differensiallanuvchi funksiyalar bo'lsin. Murakkab $f(u)$ funksiyaning erkli o'zgaruvchi x bo'yicha hosilasi bu funksiyaning oraliq argumenti u bo'yicha hosilasi y'_u ning oraliq argumentning erkli o'zgaruvchi x bo'yicha hosilasi $u'(x)$ ga ko'paytmasiga teng, ya'ni

$$y'_x = y'_u \cdot u'_x .$$

Isboti. $u = \varphi(x)$ funksiya $x = x_0$ nuqtada, $y = f(u)$ funksiya esa bu nuqtaga mos $u_0 = \varphi(x_0)$ nuqtada differensiallanuvchi bo'lsin. U holda $\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u_0)$ chekli limit mavjud. Bundan $\frac{\Delta y}{\Delta u} = f'(u_0) + \alpha$ yoki $\Delta y = f'(u_0)\Delta u + \alpha\Delta u$ kelib chiqadi, bu yerdagi $\alpha, \Delta u \to 0$ da cheksiz kichik funksiya. So'nggi tenglikni har ikkala tomonini Δx ga bo'lsak $\frac{\Delta y}{\Delta x} = f'(u_0)\frac{\Delta u}{\Delta x} + \alpha\frac{\Delta u}{\Delta x}$ hosil bo'ladi. Bunda $\Delta x \to 0$ da limitga o'tib

$\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = y_x^{'}, \quad \lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = u_x^{'}, \quad \lim\limits_{\Delta x\to 0}\alpha = \lim\limits_{\Delta u\to 0}\alpha = 0$ ekanini hisobga olsak isbotlanishi lozim bo'lgan $y'(x_0) = f'(u_0)\cdot u'(x_0)$ yoki $y'_x = y'_u \cdot u'_x$ kelib chiqadi. Biz bu yerda differensiallanuvchi $u(x)$ funksiya uzluksiz va $\Delta x \to 0$ da $\Delta u \to 0$ ni hisobga oldik.

Teskari funksiya va uning hosilasi

$[a; b]$ kesmada aniqlangan o'suvchi yoki kamayuvchi $y = f(x)$ funksiyani qaraymiz. $f(a) = c$, $f(b) = d$ bo'lsin. Aniqlik uchun $y = f(x)$ funksiya $[a; b]$ kesmada o'suvchi deb faraz qilamiz. $[a; b]$ kesmaga tegishli ikkita har xil x_1 va x_2 nuqtani olamiz. O'suvchi funksiyaning ta'rifidan agar $x_1 < x_2$ va $y_1 = f(x_1)$, $y_2 = f(x_2)$bo'lsa, $y_1 < y_2$ bo'ladi.

Demak, argumentning ikkita har xil x_1 va x_2 qiymatlariga funksiyaning ikkita har xil y_1 va y_2 qiymatlari mos keladi. Buning teskarisi ham to'g'ri, ya'ni $y_1 < y_2$ bo'lib, $y_1 = f(x_1)$, $y_2 = f(x_2)$bo'lsa, o'suvchi funksiya ta'rifidan $x_1 < x_2$ bo'lishi kelib chiqadi.

Boshqacha aytganda x ning qiymatlari sohasi $[a; b]$ kesma bilan y ning qiymatlari sohasi $[c; d]$ kesma orasida o'zaro bir qiymatli moslik o'rnatiladi. y ni argument, x ni esa funksiya sifatida qarab x ni y ning funksiyasi sifatida hosil qilamiz:

$x = \varphi(y)$.

Bu funksiya berilgan $y = f(x)$ funksiyaga **teskari funksiya** deyiladi. Kamayuvchi funksiya uchun ham shunga o'xshash mulohaza yuritish mumkin.

Shuni aytish lozimki, $y = f(x)$ funksiyaning qiymatlari sohasi $[c; d]$ unga teskari $x = \varphi(y)$ funksiyaning aniqlanish sohasi bo'ladi va aksincha. $x = \varphi(y)$ funksiya uchun $y = f(x)$ funksiya teskari funksiya bo'lgani uchun $x = \varphi(y)$ va $y = f(x)$ funksiyalar **o'zaro teskari funksiyalar** deb ataladi.

$y = f(x)$ funksiyaga teskari funksiya $y = f(x)$ tenglamani x ga nisbatan yechib topiladi. O'zaro teskari funksiyalarning grafigi $0xy$ tekisligidagi bitta egri chiziqni ifodalaydi.

5-misol. $y = x^3$ funksiyaga teskari funksiya topilsin.

Yechish. Bu funksiya butun sonlar o’qida aniqlangan va o’suvchi. Tenglikni x ga nisbatan yechsak berilgan funksiyaga teskari $x=\sqrt[3]{y}$ funksiya hosil bo’ladi.

Har qanday funksiya ham teskari funksiyaga ega bo’lavermaydi. Masalan $y=x^2$ funksiya $(-\infty,+\infty)$ intervalda teksari funksiyaga ega emas, chunki y ning har bir musbat qiymatiga x ning ikkita $x=-\sqrt{y}$ va $x=\sqrt{y}$ qiymatlari mos keladi. Agar $y=x^2$ funksiyani $(-\infty,0]$ intervalda qaralsa funksiya $x=-\sqrt{y}$ teskari funksiyaga ega, chunki y ning har bir musbat qiymatiga x ning yagona $y=x^2$ tenglikni qanoatlantiradigan qiymati mos keladi.

Shuningdek $y=x^2$ funksiyani $[0,+\infty)$ oraliqda qarasak unga teskari $x=\sqrt{y}$ funksiya mavjud bo’ladi.

Izoh. $y=f(x)$ funksiyaga teskari $x=\varphi(y)$ funksiyaning argumentini odatdagidek x bilan, funksiyani esa y bilan belgilasak va $y=f(x)$ hamda $y=\varphi(x)$ funksiyalarni grafigini bitta koordinatalar sistemasida chizsak grafik birinchi koordinatalar burchagining bissektrisasiga nisbatan simmetrik bo’ladi.

Teorema. Agar o’suvchi (kamayuvchi) $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz, shu bilan birga $f(a)=c$, $f(b)=d$ bo’lsa, u holda unga teskari $x=\varphi(y)$ funksiya $[c;d]$($[d;c]$) kesmada aniqlangan monoton va uzluksiz bo’ladi.

Endi $x=\varphi(y)$ teskari funksiyani hosilasini bilgan holda $y=f(x)$ funksiyaning hosilasini topish imkonini beradigan teoremani isbotlaymiz.

Teorema. Agar $x=\varphi(y)$ funksiya biror intervalda monoton bo’lib shu intervalning y nuqtasida noldan farqli $\varphi'(y)$ hosilaga ega bo’lsa, bu nuqtaga mos x nuqtada teskari $y=f(x)$ funksiya ham hosilaga ega bo’lib,

$$f'(x)=\frac{1}{\varphi'(y)}$$

tenglik o’rinli bo’ladi.

Isboti. Shartga binoan $x=\varphi(y)$ funksiya monoton va differensiallanuvchi bo’lgani uchun u uzluksiz hamda unga teskari monoton va uzluksiz $y=f(x)$ funksiya mavjud. x ga $\Delta x\neq 0$ orttirma bersak $y=f(x)$ funksiya Δy orttirma oladi va uzluksizligini nazarga olsak $\Delta x\to 0$ da $\Delta y\to 0$. Natijada

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\frac{\Delta x}{\Delta y}}=\frac{1}{\lim\limits_{\Delta y\to 0}\frac{\Delta x}{\Delta y}}=\frac{1}{\varphi'(y)}$$

Bu formulani $y'_x=\dfrac{1}{x'_y}$ ko’rinishda yozish ham mumkin.

Shunday qilib, teskari funksiyaning hosilasi shu funksiya hosilasiga teskari miqdorga teng ekan.

Asosiy elementar funksiyalarning hosilalari

.O'zgarmas funksiyaning hosilasi

Teorema. O'zgarmas funksiyaning hosilasi nolga teng, ya'ni $C'=0$, bunda *C*-o'zgarmas son.

Isboti. $y=f(x)=C$ desak x argument Δx orttirma olganda y funksiya $\Delta y=f(x+\Delta x)-f(x)=C-C=0$ orttirma oladi.

Demak, $C'=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{0}{\Delta x}=0$. Shunday qilib $C'=0$.

Masalan, $(25)'=\left(2^{18}\right)'=\left(tg\,20^{0}\right)'=(\ln 87)'=0$.

. Logarifmik funksiyaning hosilasi

Teorema. $\ln x$ lagorifmik funksiyaning hosilasi $\frac{1}{x}$ ga teng.

Isboti $y=lnx$ funksiyani qaraymiz. x Δx orttirma olganda funksiya

$\Delta y=\ln(x+\Delta x)-\ln x=\ln\frac{x+\Delta x}{x}=\ln\left(1+\frac{\Delta x}{x}\right)$ orttirma oladi.

$\frac{\Delta y}{\Delta x}=\frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\cdot\frac{x}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$ nisbatni tuzamiz. $\frac{\Delta x}{x}=\alpha$ deb belgilasak $\Delta x\to 0$ da $\alpha\to 0$.

Demak, $y'=\lim\limits_{\Delta x\to 0}\frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=\frac{1}{x}\lim\limits_{\alpha\to 0}\ln(1+\alpha)^{\frac{1}{\alpha}}=\frac{1}{x}\ln e=\frac{1}{x}$, ya'ni $(\ln x)'=\frac{1}{x}$. Bu yerda $\lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e$ ikkinchi ajoyib limitdan foydalanildi.

Shunga o'xshash $(\log_a x)'=\left(\frac{\ln x}{\ln a}\right)'=\frac{(\ln x)'}{\ln a}=\frac{1}{x\ln a}$ kelib chiqadi (bunda $a>0$, $a\neq 1$).

Agar $y=\ln u$ bo'lib, bunda, $u=u(x)$ differensiallanuvchi funksiya bo'lsa, u holda murakkab funksiyani differensiallash

qoidasiga binoan $(\ln u)'=\frac{u'}{u}$ tenglikka ega bo'lamiz.

Xususan, agar $y = \log_a u,\ \ u = u(x)$ bo'lsa, u holda $\left(\log_a u\right)' = \left(\frac{\ln u}{\ln a}\right)' = \frac{1}{\ln a}\cdot(\ln u)' = \frac{u'}{\ln a \cdot u}$ bo'ladi.

Darajali funksiyaning hosilasi

Teorema. x^{α} darajali funksiyaning hosilasi $\alpha x^{\alpha-1}$ ga teng, bunda α-o'zgarmas son.

Isboti. $y=x^{\alpha}$ funksiyani qaraymiz. Uni e asosga ko'ra logarifmlab $\ln y = \alpha \ln x$ tenglikka ega bo'lamiz . y ni x ning funksiyasi hisoblab, tenglikning ikkala qismini x bo'yicha differensiallaymiz: $\frac{y'}{y} = \alpha \cdot \frac{1}{x}$.

Bundan $y' = \alpha \cdot y \cdot \frac{1}{x} = \alpha \cdot x^{\alpha} \cdot \frac{1}{x} = \alpha \cdot x^{\alpha-1}$.

Shunday qilib $\left(x^{\alpha}\right)' = \alpha \cdot x^{\alpha-1}$. Teorema isbotlandi.

Agar $y = u^{\alpha}$ bo'lib, $u = u(x)$ differensiallanuvchi funksiya bo'lsa, u holda murakkab funksiyani differensiallash qoidasiga binoan $(u^{\alpha})' = \alpha \cdot u^{\alpha-1} \cdot u'$ bo'ladi.

1-misol. $y = \sqrt{x}$ funksiyaning hosilasi topilsin.

Yechish. $y' = \left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$.Demak, $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$.

2-misol. $y = \frac{1}{x}$ funksiyaning hosilasi topilsin.

Yechish. $y' = \left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1\cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$. Demak, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$.

Izoh. Funksiyaning hosilasi topilsin deyilganda shu funksiyaning aniqlanish sohasiga tegishli istalgan nuqtada uning hosilasini topishni nazarda tutiladi.

Ko'rsatkichli funksiyaning hosilasi

Teorema. a^x ko'rsatkichli funksiyaning hosilasi $a^x \ln a$ ga tengdir.

Isboti. $y=a^x$ (a>0, a≠1) funksiyani qaraymiz. Uni e asosga ko'ra logarifmlasak $\ln y = x \cdot \ln a$ bo'ladi. y ni x ning funksiyasi hisoblab, tenglamaning ikkala qismini x bo'yicha differensiallasak. $\frac{y'}{y} = \ln a$ bo'ladi. Bundan $y' = y \ln a$ yoki $y' = a^x \ln a$ kelib chiqadi. Demak, $\left(a^x\right)' = a^x \ln a$.

$y=a^u$ murakkab funksiya uchun $\left(a^u\right)' = a^u \ln a \cdot u'$ formulaga ega bo'lamiz.

Xususiy holda $a=e$ bo'lsa $\ln e = 1$ bo'lib $\left(e^x\right)' = e^x$ va $\left(e^u\right)' = e^u \cdot u'$ formulalarga ega bo'lamiz.

3-misol. $y=2^x$ bo'lsa, y' topilsin.

Yechish. $\left(2^x\right)' = 2^x \ln 2$.

4-misol. $y = 3^{x^2}$ bo'lsa, y' topilsin.

Yechish. $y' = \left(3^{x^2}\right)' = 3^{x^2} \ln 3 (x^2)' = 3^{x^2} \ln 3 \cdot 2x$.

5-misol. $y = e^{x^3}$ bo'lsa, y' topilsin.

Yechish. $y' = \left(e^{x^3}\right)' = e^{x^3} (x^3)' = e^{x^3} 3x^2$

Trigonometrik funksiyalarning hosilalari

Teorema. *sinx* funksiyaning hosilasi *cosx* ga teng.

Isboti. $y=\sin x$ funksiyani qaraymiz. x ga Δx orttirma bersak funksiya

$$\Delta y=\sin(x+\Delta x)-\sin x=2\cos\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x+\Delta x-x}{2}\right)=2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

orttirma oladi. Shuning uchun $\dfrac{\Delta y}{\Delta x} = \dfrac{2\sin\frac{\Delta x}{2}\cdot\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x} = \dfrac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos\left(x+\dfrac{\Delta x}{2}\right)$

va $y' = \left(\sin x\right)' = \lim\limits_{\Delta x\to 0}\dfrac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\lim\limits_{\Delta x\to 0}\cos\left(x+\dfrac{\Delta x}{2}\right) = 1\cdot\cos x = \cos x.$

Bu yerda birinchi ajoyib limitdan hamda *cosx* funksiyaning uzluksizligidan foydalanildi.

Shunday qilib, $\left(\sin x\right)' = \cos x$.

$y = \sin u$ (bunda $u=u(x)$) murakkab funksiya uchun $\left(\sin u\right)' = \cos u \cdot u'$ formulaga ega bo'lamiz.

6-misol. $y = \sin\sqrt{x}$ funksiyaning hosilasini toping.

Yechish. $y' = \cos\sqrt{x}\cdot\left(\sqrt{x}\right)' = \cos\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

7-misol. $y=\sin^2 x$ funksiyaning hosilasini toping.

Yechish. $y'=\left(\sin^2 x\right)'=\left((\sin x)^2\right)'=2\sin x\cdot(\sin x)'=2\sin x\cdot\cos x=\sin 2x$.

8-misol. $y=\sin(\ln x)$ funksiyaning hosilasini toping.

Yechish. $y'=\cos(\ln x)(\ln x)'=\dfrac{\cos(\ln x)}{x}$.

20.6- teorema. $\cos x$ funksiyaning hosilasi $-\sin x$ ga teng.

Isboti. $y=\cos x$ funksiyani qaraymiz. Keltirish formulasidan foydalanib uni $y=\cos x=\sin\left(\dfrac{\pi}{2}-x\right)$ ko'rinishda yozamiz. Demak ,

$y'=(\cos x)'=\left(\sin\left(\dfrac{\pi}{2}-x\right)\right)'=\cos\left(\dfrac{\pi}{2}-x\right)\cdot\left(\dfrac{\pi}{2}-x\right)'=\sin x\cdot(0-1)=-\sin x$, yoki $(\cos x)'=-\sin x$.

$y=\cos u$ (bunda $u=u(x)$) murakkab funksiyani hosilasini topish uchun

$(\cos u)'=-\sin u\cdot u'$ formulaga ega bo'lamiz.

9-misol. $y=\cos x^3$ funksiyaning hosilasini toping.

Yechish. $y'=-\sin x^3\left(x^3\right)'=-\sin x^3\cdot 3x^2$.

10-misol. $y=\cos\sqrt{x^2+1}$ funksiyaning hosilasini toping.

Yechish. $y'=-\sin\sqrt{x^2+1}\left(\sqrt{x^2+1}\right)'=-\sin\sqrt{x^2+1}\cdot\dfrac{1}{2\sqrt{x^2+1}}\cdot(x^2+1)'=$

$=-\sin\sqrt{x^2+1}\cdot\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x=-\dfrac{x\sin\sqrt{x^2+1}}{\sqrt{x^2+1}}$.

Teorema. *tgx* funksiyaning hosilasi $\dfrac{1}{\cos^2 x}$ ga teng.

Isboti. $tgx=\dfrac{\sin x}{\cos x}$ bo'lganligi sababli bo'linmani hosilasini topish qoidasiga binoan

$$\left(\frac{\sin x}{\cos x}\right)'=\frac{(\sin x)'\cdot\cos x-\sin x(\cos x)'}{\cos^2 x}=\frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}.$$

Shunday qilib, $(tgx)'=\dfrac{1}{\cos^2 x}$. $y=tgu$ (bunda, $u=u(x)$) murakkab funksiyani hosilasini topish uchun $(tgu)'=\dfrac{u'}{\cos^2 u}$ formulaga ega bo'lamiz.

11-misol. $y=tg\frac{1}{x}$ funksiyani hosilasini toping.

Yechish. $y' = \frac{1}{\cos^2\frac{1}{x}}\cdot\left(\frac{1}{x}\right)' = -\frac{1}{x^2\cos^2\frac{1}{x}}$.

12-misol. $y=tg^3\sqrt{x}$ funksiyani hosilasini toping.

Yechish. $y' = \left((tg^3\sqrt{x})^3\right)' = 3tg^2\sqrt{x}\cdot(tg\sqrt{x})' = 3tg^2\sqrt{x}\cdot\frac{1}{\cos^2\sqrt{x}}\cdot(\sqrt{x})' = \frac{3tg^2\sqrt{x}}{2\sqrt{x}\cdot\cos^2\sqrt{x}}$.

20.8-teorema. $ctgx$ funksiyaning hosilasi $-\frac{1}{\sin^2 x}$ ga teng.

Bu teoremani isbotlashni o'quvchiga qoldiramiz.

13-misol. $y = ctg\sqrt{2x^2+1}$ funksiyani hosilasini toping.

$$y' = -\frac{1}{\sin^2\sqrt{2x^2+1}}\cdot\left(\sqrt{2x^2+1}\right)' = -\frac{1}{\sin^2\sqrt{2x^2+1}}\cdot\frac{1}{2\sqrt{2x^2+1}}\cdot(2x^2+1)' =$$

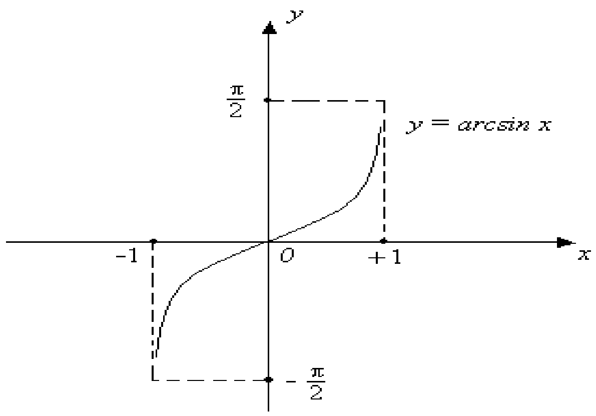
$$= -\frac{1}{\sin^2\sqrt{2x^2+1}}\cdot\frac{1}{2\sqrt{2x^2+1}}\cdot 4x = -\frac{2x}{\sin^2\sqrt{2x^2+1}\cdot\sqrt{2x^2+1}}.$$

Teskari trigonometrik funksiyalar va ularning hosilalari

1) ***y=arcsinx*** **funksiya**. $x=\sin y$ funksiyani qaraymiz. Bu funksiya $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ kesmada monoton o'uvchi bo'lib uning qiymatlari $-1\le x\le 1$ kesmani to'ldiradi.

Shuning uchun bu funksiya aniqlanish sohasi $[-1,+1]$ dan, qiymatlari sohasi $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ kesmadan iborat teskari funksiyaga ega (19.4-teorema).

Odatda uni $y=\arcsin x$ ko'rinishda yozish qabul qilingan. Demak $x=\sin y$ va $y=\arcsin x$ funksiyalar o'zaro teskari funksiyalar. $y=\arcsin x$ funksiyaning grafigi 101-chizmada tasvirlangan $D(\arcsin x) = [-1,1]$, $E(\arcsin x) = \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



101-chizma.

Teorema. $\arcsin x$ funksiyaning hosilasi $\frac{1}{\sqrt{1-x^2}}$ ga teng..

Isboti. $y = \arcsin x$ funksiyani qaraymiz. $x = \sin y$ funksiya bu funksiyaga teskari funksiya bo'ladi.

O'zaro teskari funksiyani hosilasini topish formulasi $y_x^1 = \frac{1}{x_y^1}$ (19.5.teorema) dan foydalanamiz. $y' = \frac{1}{(\sin y)_y^1} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$,

chunki $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ kesmada $\cos y \geq 0$ bo'lgani uchun $\sqrt{1-\sin^2 y}$ oldidagi plyus ishora olindi.

Shunday qilib, $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

$y = \arcsin u$ (bunda, $u = u(x)$) murakkab funksiya uchun $(\arcsin u)' = \frac{u^1}{\sqrt{1-u^2}}$

hosilani topish formulasiga ega bo'lamiz.

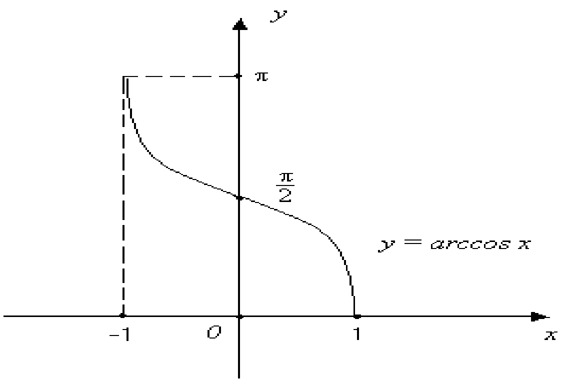
14-misol. $y = \arcsin e^x$ funksiyani hosilasini toping.

Yechish. $y' = \frac{(e^x)'}{\sqrt{1-(e^x)^2}} = \frac{e^x}{\sqrt{1-e^{2x}}}$.

15-misol. $y = 2\arcsin\sqrt{x}$ funksiyani hosilasini toping.

Yechish. $y' = 2(\arcsin\sqrt{x})' = 2\frac{(\sqrt{x})'}{\sqrt{1-(\sqrt{x})^2}} = 2\cdot\frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}} = \frac{1}{\sqrt{x\cdot(1-x)}}$.

2) $y = \arccos x$ **funksiya**. $x = \cos y$ funksiyani qaraymiz. Bu funksiya $0 \leq y \leq \pi$ kesmada monoton kamayuvchiligini bilamiz. Shuning uchun bu funksiyaga teskari funksiya mavjud (19.4 teorema) bo'lib uning aniqlanish sohasi [-1,1] kesmadan, qiymatlari sohasi $[0; \pi]$ kesmadan iborat bo'ladi. *x=cosy* funksiyaga teskari funksiyani $y = \arccos x$ kabi yoziladi. $y = \arccos x$ funksiyaning grafigi 102-chizmada tasvirlangan. D(arccosx) =[-1,1],



102-chizma.

E(arccosx) = $[0; \pi]$ ekani ravshan.

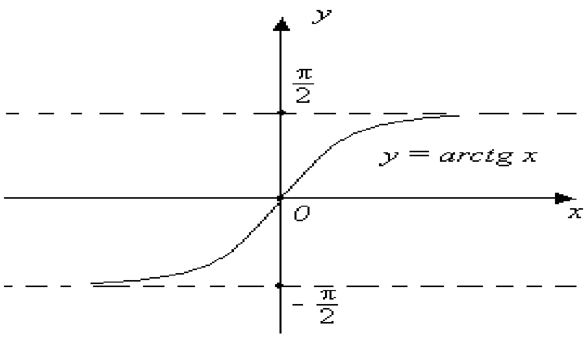
Teorema. arccosx funksiyaning hosilasi $-\frac{1}{\sqrt{1-x^2}}$ ga teng.

Teoremaning isboti 20.9-teoremaning isbrtini takrorlagani uchun uni isbotlashni o'quvchiga qoldiramiz.

3) $y = \mathrm{arctg}x$ **funksiya.** $x = \mathrm{tg}y$ funksiyani qaraymiz. Bu funksiya $-\frac{\pi}{2} < y < \frac{\pi}{2}$ intervalda monoton o'sadi. Shuning uchun bu funksiyaga teskari funksiya mavjud(19.4-teorema) bo'lib uning aniqlanish sohasi butun sonlar o'qidan iborat. $x = \mathrm{tg}y$ funksiyaga teskari funksiya $y = \mathrm{arctg}x$ kabi yoziladi. Demak $D(\mathrm{arctg}x) = (-\infty;\ \infty)$, $E(\mathrm{arctg}x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

va $\lim\limits_{x\to-\infty} arctgx = -\frac{\pi}{2}$, $\lim\limits_{x\to+\infty} arctgx = \frac{\pi}{2}$.

$y = \mathrm{arctg}x$ funksiyaning grafigi 103-chizmada tasvirlangan.



103-chizma.

Teorema. $\mathrm{arctg}x$ funksiyaning hosilasi $\frac{1}{1+x^2}$ ga teng.

Isboti. $y = \mathrm{arctg}x$ funksiyani qaraymiz. $x=tgy$ funksiyaga teskari funksiya bo'lganligi sababli ularning hosilalari (19.5-teorema) $y_x' = \frac{1}{x_y'}$ tenglik orqali bog'langan. Shuning uchun $y' = \frac{1}{(tgy)_y'} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y = \frac{1}{1+tg^2 y} = \frac{1}{1+x^2}$. Demak, $(arctgx)' = \frac{1}{1+x^2}$. $y=arctgu$ murakkab funksiyani hosilasi $(arctgu)' = \frac{u'}{1+u^2}$ formula yordamida topiladi.

16-misol. $y=(arctgx)^3$ funksiyani hosilasini toping.

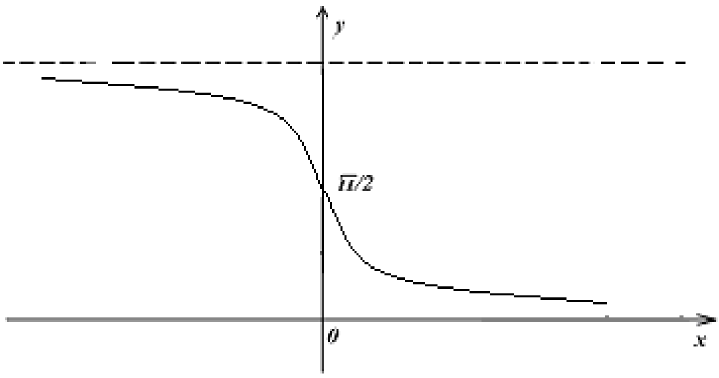
Yechish. $y' = 3(arctgx)^2 \cdot (arctgx)' = 3(arctgx)^2 \cdot \frac{1}{1+x^2}$.

17-misol. $y=arctgx^2$ funksiyani hosilasini toping.

Yechish. $y' = \frac{(x^2)'}{1+(x^2)^2} = \frac{2x}{1+x^4}$.

4) $y = \mathrm{arcctg}x$ **funksiya.** $x = \mathrm{ctg}y$ funksiyani qaraymiz. Bu funksiya $0<y<\pi$

intervalda monoton kamayuvchi. Shuning uchun bu funksiyaga teskari funksiya mavjud (19.4-teorema) va uning aniqlanish sohasi (-∞,+∞) dan iborat. $x = ctgy$ funksiyaga teskari funksiya $y = arcctgx$ ko'rinishda belgilanadi. Demak, $D(arcctgx) =$ (-∞,+∞), $E(arcctgx) = (0;\ \pi)$ va



104-chizma.

$\lim\limits_{x\to-\infty} arcctgx = \pi,\quad \lim\limits_{x\to+\infty} arcctgx = 0$. $y = arcctgx$ funksiyaning grafigi 104-chizmada tasvirlangan.

Teorema. $arcctgx$ funksiyaning hosilasi-$\frac{1}{1+x^2}$ ga teng.

Teoremani isboti 20.11-teoremani isbotiga o'hshaganligi uchun uni isbotini o'quvchiga qoldiramiz.

$y = arcctgu$ murakkab funksiyani hosilasi $(arcctgu)' = -\frac{u'}{1+u^2}$ formula yordamida topiladi.

18-misol. $y = arcctgx^4$ funksiyani hosilasini toping.

Yechish. $y' = -\frac{(x^4)'}{1+(x^4)^2} = -\frac{4x^3}{1+x^8}.$

19-misol. $y = arcctg\frac{1}{x}$ funksiyani hosilasini toping

$$y' = -\frac{\left(\frac{1}{x}\right)'}{1+\left(\frac{1}{x}\right)^2} = -\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\frac{1}{x^2}}{\frac{x^2+1}{x^2}} = \frac{1}{x^2+1}.$$

Izoh. Murakkab funksiyalarning $u = u(x)$ oraliq argumenti differensiyallanuvchi deb faraz qilindi.