O’zbekiston Respublikasi
Oliy va o’rta maxsus ta’lim vazirligi

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FIZIKA kafedrasi

Matematik fizika metodlari

fanidan

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1-mavzu. Kirish. Kompleks sonlar va ular ustida amallar

Reja:

1. Kompleks sonning shakllari.
2. Kompleks sonlar ustida amallar
3. Kompleks sonning darajasi va ildizi

XVI-XVII asrlarda matematika fani quyidagiga o'xshash masalalarni yechishda boshi berk ko'chaga kirib qoldi.

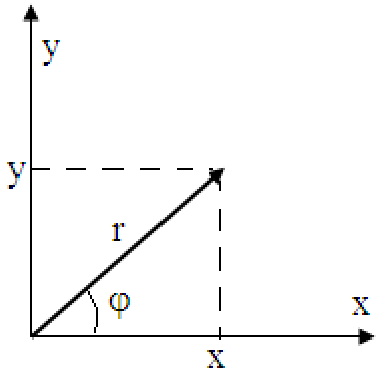
1) $x^2-3x+4=0$
2) $\sin x=2$
3) $\ln(-5)=?$

4) $\int_0^\infty \frac{\cos x}{x}dx$

Bundan qutilish uchun kompleks sonlar tushunchasi kiritildi. Kompleks son deb- ikkita haqiqiy sonlar juftiga aytiladi va u quyidagicha yoziladi:

$z=x+iy$ (1)

yoki quyidagi ko'rinishda tasvirlanadi,



bu yerda x- kompleks sonning haqiqiy, y-mavhum qismi.

$x=Rez$

$y=Imz$

$i=\sqrt{-1}$

x y φ r

Rasmdan ko'rinib turibdiki ,

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad (3)$$

(3) ni (1) ga qo'ysak,

$z=r(\cos\alpha+i\sin\alpha)$ (4),

Agar Eyler formulasiga qo'ysak,

$e^{i\varphi}=\cos\varphi+i\sin\varphi$

$z=re^{i\alpha}$ (5) $\alpha=\varphi$

(1)- formula kompleks sonning algebraik shakli,

(2)- geometrik shakli,

(3)- trigonometrik shakli,

(4)- ko'rsatkichli shakli deyiladi.

Agar kompleks son trigonometrik yoki ko'rsatkichli shaklda berilgan bo'lsa, algebraik shaklga o'tish uchun (3) formuladan foydalaniladi, aksincha o'tish uchun esa

$$\begin{cases} \alpha = arctg\frac{y}{x} \\ r = \sqrt{x^2+y^2} \end{cases} \quad (6)$$

Ifodalardan foydalaniladi.

1-misol: Kopleks son trigonometrik shakldan algebraic shaklga o'tkazilsin

$z=2(\cos\pi/6+i\sin\pi/6)$

yechish:

$x=2\cos\pi/6=2(\sqrt{3}/2)=\sqrt{3}$

$y=2\sin\pi/6=2(1/2)=1$

$z=\sqrt{3}+i$

Agar ikkita kompleks son berilgan bo'lsa

$$z_1 = x_1 + iy$$

$$z_2 = x_2 + iy_2$$

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

Ko' paytirishvabolish

$$z_1 \times z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2 + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2}$$

Ko'rsatkichli ko'rinishda berilgan ikkita kompleks sonning ko'paytmasini topamiz:

$z_1=r_1e^{i\varphi_1}$ $z_2=r_2e^{i\varphi_2}$

$z_1z_2=r_1r_2e^{i(\varphi_1+\varphi_2)}=r_1r_2(\cos(\varphi_1+\varphi_2)+i\sin((\varphi_1+\varphi_2))$ (7)

Demak, kompleks sonlar ko'rsatkichli shaklda berilgan bo'lsa, ularni ko'paytirish uchun modullari ko'paytiriladi, argumentlari qo'shiladi. Xuddi shunungdek n ta kompleks sonni ko'paytiramiz:

$z_1z_2z_3\ldots z_n=r_1r_2r_3\ldots r_ne^{i(\varphi_1+\varphi_2+\varphi_3+\ldots+\varphi_n)}$

$z^n=r^ne^{in\varphi}=r^n(\cos n\varphi+i\sin n\varphi)$ (8)

Kompleks sonni n- darajasiga ko'tarish uchun moduli n- darajaga ko'tariladi, argumenti esa n ga ko'paytiriladi. Agar kompleks son

$z=r(\cos\varphi+i\sin\varphi)$

ni ko'rinishda yozsak va (2) ga qo'ysak

$z^n(\cos\varphi+i\sin\varphi)^n=r^n(\cos n\varphi+i\sin n\varphi)$

Bu yerdan

$(\cos\varphi+i\sin\varphi)^n=\cos n\varphi+i\sin n\varphi)$ (9)

Bu formula Muavr formulasi deyiladi.

Masalan, bo'lganda

$(\cos\varphi+i\sin\varphi)^2=\cos^2\varphi+i2\sin\varphi\cos\varphi-\sin^2\varphi=\cos2\varphi-i\sin2\varphi$

Demak

$\cos^2\varphi-\sin^2\varphi=\cos2\varphi$

$2\sin\varphi\cos\varphi=\sin2\varphi$

Kompleks sonning ildizi.

$$\sqrt[n]{z} = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\varphi k}{n} + i\sin\frac{\varphi k}{n}\right)$$

$$\beta_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right)$$

k=0,1,2,3…n-1 (o'zgaruvchi)

Kompleks sondan n- darajali ildiz chiqarish uchun modulidan n- darajali ildiz chiqariladi, argumenti esa 2πk davr qo'shilib n ga bo'linadi.

Misol: $(\sqrt{3}+I)^{10}=?$

Yechish: $r=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{4}=2$

$\varphi=\mathrm{arctg}(y/x)=1/\sqrt{3}=\pi/6$

$(\sqrt{3}+I)^{10}=2^{10}(\cos10\pi/6+i\sin10\pi/6)=512(1-\sqrt{3}i)$

Nazorat savollari.

1. Kompleks sonning shakllarini yozing.
2. Kompleks sonlar ustida qanday amallar bajarish mumkin.
3. Muavr formulasini yozing

4. Kompleks sonning ildizi qanday hisoblanadi?

2-mavzu. Kompleks o'zgaruvchili funksiyalar

Reja:

1. Kompleks o'zgaruvchili funksiyalar
2. Koshi-Riman shartlari
3. Trigonometrik funksiyalar
4. Logarifmik funksiyalar
5. Teskari trigonometric funksiyalar
6. Teskari giperbolik funksiyalar
7. Kompleks o'zgaruvchili funksiyadan olingan integral

Agar D sohada olingan z ning har bir qiymatiga G sohadan olingan W ning har bir qiymati biror qonuniyat bo'yicha mos keltirilgan bo'lsa D sohada *kompleks o'zgaruvchili funksiya* berilgan deyiladi va quyidagicha yoziladi:

W=f(z)= f(x,y)=u+iv=u(x,y) +iv(x,y)

u-funksiyaning haqiqiy qismi

v- mavhum qismi deyiladi.

Misol: $f(z)=z^2=(x+iy)^2=x^2-y^2-2ixy$ $u=x^2-y^2$ $v=2xy$

Kompleks o'zgaruvchili funksiyaning hosilasi deb-funksiya ortirmasining argument ortirmasiga nisbatining argument ortirmasining ixtiyoriy yo'l bilan 0 ga intigandagi limitiga aytiladi.

$$W' = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta W}{\Delta x + i\Delta y}$$

Hosilaning bu ta'rifidan quyidagi shartlar kelib chiqadi

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Bu shartlar Koshi Riman shartlari deyiladi. Bu shatlarni qanoatlantiruvchi funksiyalar analitik funksiyalar deyiladi

Misol: $W=e^z$ $u=e^x\cos y$ $v=e^x\sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y$$

Trigonometrik funksiyalar

Sinz=$(e^{iz}-e^{-iz})/2i$ $sh\alpha=(e^{\alpha}-e^{-\alpha})/2$

Cosz=$(e^{iz}+e^{-iz})/2$ $ch\alpha=(e^{\alpha}+e^{-\alpha})/2$

***i**fodalar bilan aniqlangan funksiyalar kompleks o'zgaruvchili triginometrik funksiyalar deyiladi.* Bu funksiyalar haqiqiy o'zgaruvchili trigonometrik funksiyalarning barcha xossalariga ega. Bulardan tashqari ikkita xossaga ega.

1.Bu funksiyalarning moduli ixtiyoriy bo'lishi mumkin

2.Analitik

Misol.

1) sinz= sin(x+iy)=sinxcosiy+ cosxsiniy=sinxchy+icosshy

bu yerda $\cos iy=(e^{-y}+e^{y})/2=ch y$ $\sin iy=(e^{-y}-e^{y})/2i=ish y$

Logarifmik funksiyalar

Kompleks o'zgaruvchili logarifmik funksiyalar deb –W=Lnz ga autiladi. Bu yerda z- kompleks o'zgaruvchi bo'lib , haqiqiy , mavhum kompleks bo'lishi mumkin. Bu funksiya ham haqiqiy o'zgaruvchili logarifmik funksiyaning barcha xususiyarlariga ega. Uni hisoblash uchun z ni ko'rsatkichli ko'rinishda yozamiz.

$Lnz=Ln(re^{i\varphi})=Lnr+Lne^{i\varphi}=lnr+i\varphi$

Misol: $Lni=?$

Yechish: $x=0$, $y=1$, $r=1$, $\varphi=\pi/2$

$Lni=ln1+ i(\pi/2)= i(\pi/2)$

Teskari trigonometrik funksiyalar

Teskari trigonometrik funksiya deb,

1. W=arcsinz
2. W=arccosz
3. W=arctgz
4. W=arcctgz larga aytiladi

Ularni hisoblash uchun arcsin ni teskarisini yozamiz.

$sinW=z=e^{iW}-e^{-iW}/2i$ ni $2ie^{iW}$ ga ko'paytiramiz

$2zie^{iW}=e^{iW}-1$

$k=e^{iW}$

$k^2-2zik-1=0$

$$k_{1,2}=\frac{2zi\pm\sqrt{-4z^2+4}}{2}=zi\pm\sqrt{1-z^2}$$

$$e^2=zi\pm\sqrt{1-z^2}$$

$$Ln\left(e^{iW}\right)=Ln\left(zi\pm\sqrt{1-z^2}\right)$$

$$iW=Ln\left(zi\pm\sqrt{1-z^2}\right)$$

$$W=-iLn\left(zi\pm\sqrt{1-z^2}\right)$$

$$\arcsin z=-iLn\left(zi\pm\sqrt{1-z^2}\right)$$

2) $\cos W=z=\dfrac{e^{iW}+e^{-iW}}{2}$ ni $2e^{iW}$ ga ko'paytiramiz

$2ze^{iW}=e^{2iW}+1$ $k=e^{iW}$

$k^2-2zk+1=0$

$$k_{1,2}=\frac{2z\pm\sqrt{4z^2-4}}{2}=z\pm\sqrt{z^2-1}$$

$$e^{iW}=z\pm\sqrt{z^2-1}$$

$$Ln\left(e^{iW}\right)=Ln\left(z\pm\sqrt{z^2-1}\right)$$

$iW=Ln\left(z\pm\sqrt{z^2-1}\right)$ ni (-i) ga kopaytirsak

$$W=-iLn\left(z\pm\sqrt{z^2-1}\right)$$

$$\arccos z=-iLn\left(z\pm\sqrt{z^2-1}\right)$$

$$tgW=z=\frac{e^{iW}-e^{-iW}}{i\left(e^{iW}+e^{-iW}\right)}$$

ni ie^{iW} ga ko'paytiramiz

$$zie^{iW}=\frac{e^{2iW}-1}{e^{iW}+e^{-iW}}$$

$$k=e^{2iW}$$

$$zik+zi=k-1$$

$$k(zi-1)=-1-zi$$

$$k=\frac{1+zi}{1-zi}$$

$$e^{2iW}=\frac{1+zi}{1-zi}$$

$$2iW=Ln\frac{1+zi}{1-zi}$$

ni –(i/2) ga ko’paytirsak

$$W=-\frac{i}{2}Ln\frac{1+zi}{1-zi}$$

$$arctgz=-\frac{i}{2}Ln\frac{1+zi}{1-zi}$$

4) $arcctgW=z=\frac{\cos W}{\sin W}=\frac{i\left(e^{iW}+e^{-iW}\right)}{e^{iW}-e^{-iW}}$ ni e^{iW} ga ko’paytiramiz

$$ze^{iW}=\frac{i\left(e^{2iW}+1\right)}{e^{iW}-e^{-iW}}$$

$$z\left(e^{2iW}-1\right)=i\left(e^{2iW}+1\right)$$

$$k=e^{2iW}$$

$$z(k-1)=i(k+1)$$

$$zk-z-ik-i=0$$

$$k(z-i)=i+z$$

$$k=\frac{z+i}{z-i}$$

$$e^{2iW}=\frac{z+i}{z-i}\Rightarrow 2iw=Ln\frac{z+i}{z-i}$$

$$arcctgz=-\frac{i}{2}Ln\frac{z+i}{z-i}$$

Teskari giperbolik funksiyalar

Teskari giperbolik funksiyalar deb
5)W=arcshz
6)W=arcchz
7)W=arcthz
8)W=arccthz larga aytiladi.

5)shW=z= $\frac{e^{W}-e^{-W}}{2}$ ni $2e^{W}$ ga ko’paytiramiz

$2ze^{W}=e^{2W}-1$ $k=e^{W}$

$k^2-2zk-1=0$

$$k_{1,2}=\frac{2z\pm\sqrt{4z^2+4}}{2}=z\pm\sqrt{z^2+1}$$

$$e^W=z+\sqrt{z^2+1}$$

$$W=Ln\left(z\pm\sqrt{z^2+1}\right)$$

$$arcshz=Ln\left(z\pm\sqrt{z^2+1}\right)$$

6) chW=z=$\frac{e^W+e^{-W}}{2}$ ni $2e^W$ ga ko'paytiramiz

$$2ze^W=e^{2W}+1$$

$$k=e^W$$

$k^2-2zk+1=0$

$$k_{1,2}=\frac{2z\pm\sqrt{4z^2-4}}{2}=z\pm\sqrt{z^2-1}$$

$$e^W=z\pm\sqrt{z^2-1}$$

$$W=Ln\left(z\pm\sqrt{z^2-1}\right)$$

$$arcchz=Ln\left(z\pm\sqrt{z^2-1}\right)$$

7) thW=$\frac{shW}{chW}=z=\frac{e^W-e^{-W}}{e^W+e^{-W}}$ ni e^W ga ko'paytirsak

$$z=\frac{e^{2W}-1}{e^{2W}+1}$$

$$z\left(e^{2W}+1\right)=e^{2W}-1$$

$$k=e^{2W}$$

$$z(k+1)=k-1$$

$$zk-k=-z-1$$

$$k(1-z)=1+z$$

$$k=\frac{1+z}{1-z}$$

$$e^{2W}=\frac{1+z}{1-z}$$

$$2W=Ln\left(\frac{1+z}{1-z}\right)$$

ni ½ ga ko'paytirsak

W=$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$

$$arcthz=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

8) cthW=$\frac{chW}{shW}=z=\frac{e^W+e^{-W}}{e^W-e^{-W}}$ ni e^w ga ko'paytirsak

z=$\frac{e^{2W}+1}{e^{2W}-1}$

$$z\left(e^{2W}-1\right)=e^{2w}+1$$
$$k=e^{2W}$$
$$z(k-1)=k+1$$
$$zk-k=1+z$$
$$k=\frac{z+1}{z-1}$$
$$e^{2W}=\frac{z+1}{z-1}$$
$$2W=Ln\left(\frac{z+1}{z-1}\right)$$
ni ½ ga ko'paytirsak
$$W=\frac{1}{2}Ln\left(\frac{z+1}{z-1}\right)$$
$$arccthz=\frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

Kompleks o'zgaruvchili funksiyadan olingan integral deb
$$I=\int_G f(z)dz$$
ifodaga aytiladi. Bu yerda G integrallash sohasi.

f(z)=u+iv

z=x+iy
$$I=\int_G f(z)dz=\int(u+iv)d(x+iy)=\int udx-vdy+i\int udy+vdx$$
Bu integral haqiqiy o'zgaruvchili funksiyalardan olingan integralning barcha xossalariga ega. Integralning natijasi integrallash sohasiga(yo'liga) bog'liq.

Agar integrallash yo'lining boshi va oxiri biriktirilsa, ya'ni ustma-ust tushsa berk sohaga aylanadi va integrali quyidagicha belgilanadi:

$\int_G \to \oint_C$ bo'lib qoladi

Koshi teoremasi: har qanday analitik funksiyalardan berk soha bo'yicha olingan integrali 0 ga teng ya'ni: $\oint_C f(z)dz=0$.

Quyidagi integralni korib chiqamiz

$\oint_C (z-a)^n dz=?$ n $\neq 1$ bo'lganda bu funksiya barcha sohalarda analitik hisoblanadi. va integralning qiymati teoremaga asosan 0 ga teng bo'ladi, n=-1 bo'lganda 2hol bor;

1) a(.) c konturning tashqarisida
2) a(.) c konturning ichida

1- holda kontur ichida funksiya analitik bo'lganligi uchun bu holda ham integral 0 ga teng

2- hol uchun z-a=re$^{i\varphi}$ o'zgartirish kiritamiz va 2 tomonni hosilasini olsak
$$dz=ire^{i\varphi}d\varphi \Rightarrow \oint_C \frac{dz}{z-a}=\int_0^{2\pi}\frac{ire^{i\varphi}d\varphi}{re^{i\varphi}}=\int_0^{2\pi} id\varphi=2\pi i$$
$$\text{demak}\oint_C (z-a)^n dz\begin{cases} n\neq 1{,}0 \\ n=-1{,}0, a(.)ctashqarida \\ n=-1{,}2\pi i, a(.)cningichida\end{cases}$$

Koshi formulasi

1). $\oint\limits_c (z-a)^n dz$

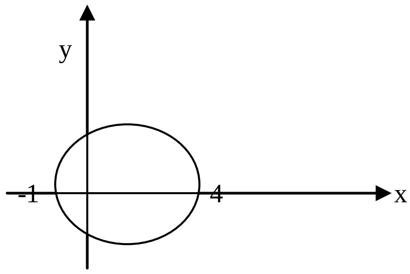
2). $\oint \dfrac{dz}{z-a} = \begin{cases} 0, tashqi \\ 2\pi i, ichki. \end{cases}$

3). $\oint \dfrac{fz}{z-a} dz = \begin{cases} 0, tashqi. \\ 2\pi i, ichki. \end{cases}$

(3) integral ,Koshi – formulasi deyiladi.

Misol: 1). $\oint \dfrac{z^2}{z-2} dz = 2\pi i \times 4 = 8\pi I$

C: $|z-1| = 3$ z=3,



2..) $\oint \dfrac{f(z)}{(z-a)^n} dz = \begin{cases} 0, tashqi \\ 2\pi i \dfrac{f^{(n-1)}}{(n-1)'} \end{cases}$

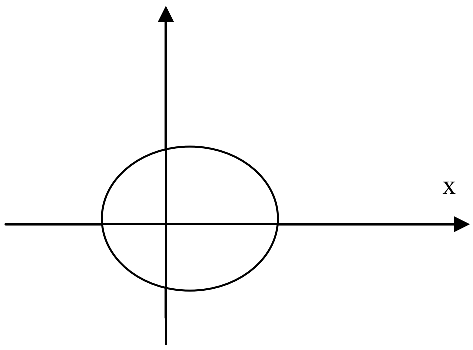
Bu Koshi formulasining natijasi deyiladi.

Misol 2. $\oint \dfrac{z^3}{(z-1)^2} dz = 2\pi i \cdot (z^3)' = 2\pi i \cdot 3z^2 = 6\pi i$ yt

C$\cdot |z-1| = 3$

A=-1

N=2



Nazorat savollari

1. Kompleks o’zgaruvchili funksiya deb nimaga aytiladi?
2. Qanday kompleks o’zgaruvchili funksiyalarni bilasiz?
3. Koshi teoremasini ayting.

3-mavzu. Chegirmalar nazariyasi

Reja:

1. Loran qatori
2. Maxsus nuqtalar
3. Chegirmalar nazariyasi

Loran qatori

$$F(z)=f_1(z)+f_2(z)+f_3(z)+\ldots+=\sum_{n=1}^{\infty} f_n(z).$$

ko’rinishdagi qator kompleks hadli qator deyiladi.

$$f(z)=C_0+C_1(z-a)+C_2(z-a)^2+C_3(z-a)^3+\ldots\ldots=\sum_{n=0}^{\infty} C_n (z-a)^n$$

ko ‘rinishdagi qator darajali qator deyiladi.

Darajali qatorning hususiy hollaridan biri

$$f(z)=f(0)+f^{``}(0)+f^{``}\frac{(0)z^2}{2}+\ldots\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

ko ‘rinishdagi qator Teylor qatori deyiladi.
Ba’zi elementar funksiyalarning Teylor qatorlari quyidagicha.

$$e^x=1+x+\frac{x^2}{2}+\ldots\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$$

$$\sin x=x-\frac{x^3}{6}+\ldots\ldots=\sum_{n=0}^{\infty}(-1)^{2n+1}\frac{x^{2n-1}}{(2n+1)!}$$

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\ldots\ldots=\sum_{n=0}^{\infty}(-1)\frac{x^{2n}}{2n\;!}$$

Har qanqay analitik funksiyaning Teylor qatorga yozish mumkin va aksincha.

Misol: $\arcsin\frac{1}{2}$ z-iln(i $\frac{1}{2}\pm\sqrt{1-\frac{1}{4}}$)=-iln($\frac{1}{2}\pm\frac{\sqrt{3}}{2}$)=-i(ln1±i $\frac{\pi}{6}$)=± $\frac{\pi}{6}$

Loran qatori deb

$$F(z)=\ldots+\frac{c_{-2}}{(z-a)^2}+\frac{c_{-1}}{z-a}+c_0+c_1(z-a)+c_2(z+a)^2+\ldots=\sum_{n=-\infty}^{\infty}c_n(z-a)^n$$

ko’rinishdagi qatorga aytiladi.

Loran qatori ikki qisimdan iborat:

1). $f_1(z)=\sum_{n=0}^{\infty}c_n(z-a)^n$ - to’g’ri qismi.

2). $f_2(z)=\sum_{n=1}^{\infty}\frac{c_n}{(z-a)^n}$ - bosh qismi .

Maxsus nuqtalar

Funksiyaning analitik bo’lgan nuqtalari oddiy nuqtalari analitik bo’lmagan nuqtalari maxsus nuqtalari deyiladi .

Maxsus nuqtalar uch xil bo’ladi.

1). Qutilib bo’ladigan maxsus nuqtalar .

Bunday nuqtada funksiyaning Loran qatoriga yoyilmasida manfiy darajalar qatnashmaydi .

Funksiyaning bu nuqtadagi limiti chekli qiymatga erishadi .

Masalan:

$F(z)=\frac{\sin z}{z}$ funksiya uchun z=0 qutilib bo’ladigan maxsus nuqta bo’lib uning Loran qatori

$$F(z)=\frac{1}{2}(z-\frac{z^3}{6}+\frac{z^5}{5'}+\ldots)=1-\frac{z^2}{6}+\frac{z^4}{5'}\ldots \quad \lim_{n\to 0}f(z)=\lim_{z\to 0}\frac{\sin z}{z}=1$$

2). 2-Maxsus nuqta qutb deyiladi . Qutubda funksiyaning Loran qatoriga yoyilmasida manfiy darajalar soni chekli bo’ladi . Eng katta manfiy daraja qutbning tartibi deyiladi. Bu nuqtada funsiyaning limiti cheksiz bo’ladi .

Masalan : $f(z)=\frac{\cos z}{(z-2)^3}$ funksiya uchun z=2 3-tartibli qutubli hisoblanadi.

$$\lim_{z\to 2}f(z)=\lim_{z\to 2}\frac{\cos z}{(z-2)^2}=\infty$$

3). Muhum maxsus nuqta . Bu nuqtada funksiyaning Loran qatori yoyilmasida manfiy darajalar soni cheksiz bo’ladi . Bu nuqtada funksiyaning limiti mavjud bo’lmaydi .

Misol: f(z) =e $^{\frac{1}{z}}$ funksiya uchun , z=0 muhum maxsus nuqta bo'lib uning Loran qatori

F(z)=1+ $\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\ldots$ to'g'ri keladi.

Chegirmalar nazariyasi

Chegirma – qoldiq – vычet- reside – res – r -c $_{-1}$ ko'rinishda yozilishi mumkin .

Funksiyaning maxsus nuqtadagi chegirmasi deb , funksiyaning shu nuqta atrofida Loran qatoriga yoyilmasidagi manfiy koefisiyentiga aytiladi .

Masalan : f(z)= $\frac{5}{(z-2)^2}+\frac{3}{z-2}+8+9(z-2)+\ldots$

Bu yerda funksiyaning chegirmasi 3 ga

Res(f(z0,z=2)=3 z=2 – maxsus nuqtasi

Funksiyaning maxsus nuqtadagi chegirmasi quyidagicha hisoblanadi .

1).Qutilib bo'ladigan maxsus nuqtadagi chegirma chegirmasi 0 ga teng . res = 0

2). Qutbdagi chegirma

res (f(z),z=a)= $\frac{1}{(n-1)!}\lim\limits_{z\to a}(f(z)(z-a)^n)$

formula orqali hisoblanadi .

3). Muxum maxsus nuqtadagi chegirma funksiyani Loran qatoriga yoyish orqali hisoblanadi .

Misol: 1). res(f(z),z=-1)= $\frac{1}{1!}\lim\limits_{z\to 1}\frac{d^{2-1}}{dz^{2-1}}(f(z)(z-1)^2=\lim\limits_{z\to -1}\frac{d}{dz}(\frac{z^3}{(z+1)^2}(z+1)^2)=\lim\limits_{z\to -1}3z^2=3$

2). F(z)= $\frac{\sin z}{(z-\frac{\pi}{4})^3}=?$ n=3 , a= $\frac{\pi}{4}$

res(f(z)z= $\frac{\pi}{4})=\frac{1}{2!}\lim\limits_{z\to\frac{\pi}{4}}\frac{d^2}{dz^2}(\frac{\sin z}{(z-\frac{\pi}{4})^3}(z-\frac{\pi}{4})^3)=\frac{1}{2}\lim\limits_{z\to\frac{\pi}{4}}\frac{d^2}{dz^2}(\frac{\sin z}{(z-\frac{\pi}{4})^3}(z-\frac{\pi}{4})^3)=$

= $\frac{1}{2}\lim\limits_{z\to\frac{\pi}{4}}(-\sin z)=-\frac{1}{2}\lim\limits_{z\to\frac{\pi}{4}}\sin z=-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{\sqrt{2}}{4}$

3). Res(f(z) z=-2)= $\lim\limits_{z\to -2}\frac{d}{dz}(\frac{z^4+1}{z^2(z+2)}(z+2))=\lim\limits_{z\to 2}(\frac{z^4+1)}{z^2}=\frac{(-2)^{\ 4}+1}{(-2)^2}=\frac{16+1}{4}=\frac{17}{4}=$

=4 $\frac{1}{4}=4{,}25$.

4). F(z)=z $^2\cdot\sin\frac{1}{z}=?$ Loran qatoriga yoyish

f(z)=z $^2(\frac{1}{2}+\frac{1}{6z^3}+\frac{1}{5!z^5}+\ldots)=$ z+ $\frac{1}{6z}+\frac{1}{5!z^3}+\ldots$

res(f(z) z=0)=- $\frac{1}{6}$;

Nazorat savollari

1. Loran qatori deb nimaga aytiladi?
2. Maxsus nuqtalarni ta'riflang?
3. Chegirma deb nimaga aytiladi?

4-mavzu. Chegirmalar yordamida integrallarni hisoblash

Reja:

1. Umumiy hol.

2. Trinomotrek funksiya qatnashgan integrallar.
3. Xosmas integrallar.

Koshi teoremasi , formulalari va chegirmalar nazariyasidan ko'rinadiki , kompleks o'zgaruvchili funksiyadan berk soha boýicha olingan integral

(1) $$\oint_c f(z)dz = \begin{cases} 2\pi i \sum res, ichida(\max susnuqta) \\ 0, \quad tashida(\max susnuqta) \end{cases}$$

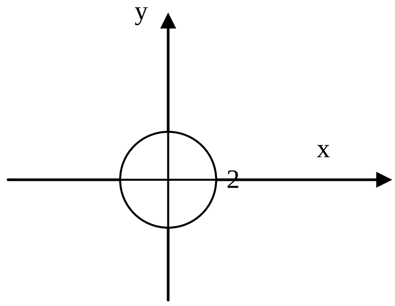
ifoda yordamida aniqlanadi .

Misol: $\oint_c z^4 \sin\frac{1}{2}dz = ?$ c(z)=2 .

$\oint_c z^4 \sin\frac{1}{z}dz = 2\pi i \cdot \frac{1}{5!} = \frac{\pi i}{60}$ ya'ni $\Rightarrow$ a=0 , 0(0;0) r=2

f(z)=$z^4 \sin \frac{1}{2!}\lim_{z\to\frac{z}{4}}\frac{d^2}{dz^2}(z^4(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - ...) = z^3 - \frac{3}{3!} + \frac{1}{5!z} - ...$

res (f(z),z=0)=$\frac{1}{5!} = \frac{1}{120}$



2).Trigonometrik funksuyalar qatnashgan integrallar.

$$\int_0^{2\pi} f(\sin x, \cos x)dx = ?$$

sinx = $\frac{e^{ix} + e^{-ix}}{2i} = \frac{z - \frac{1}{z}}{ei} = \frac{z^2 - 1}{2zi}$

cosx= $\frac{e^{ix} + e^{-ix}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$

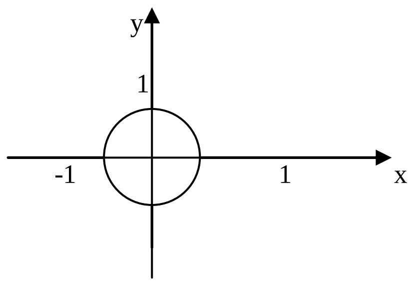
dz=e $^{ix}dx = zdx,$ dx= $\frac{dz}{z}$.

(2) $$\int_0^{2\pi} f(\sin x, \cos x)dx = \oint_{c:(z)=1} f(\frac{z^2 -}{2zi}, \frac{z^2 + 1}{2z})\frac{dz}{z} \Rightarrow \begin{cases} 2\pi i, \quad \sum resichida \\ 0, \quad tashida \end{cases}$$

a). Misol: $$\int_0^{2\pi} \frac{dx}{2 + \cos x} = \oint_{c;(z)=1} \frac{\frac{dz}{z}}{2 + \frac{z^2 + 1}{2z}} = \oint_c \frac{zdz}{z^2 + 4z + 1} = \int \frac{2dz}{(z + 2 - \sqrt{3})(z + 2 + \sqrt{3})} =$$

$=z^2 + 4z + 1 = (z + 2)^2 - 3 = (z + 2 - \sqrt{3})(z + 2 + \sqrt{3}),$

res(f(z) ,z=a)= $\lim_{z\to a_1}(\frac{2}{(z + 2 - \sqrt{3})(z + 2 - \sqrt{3})}) =$



$= \frac{2}{-2 + \sqrt{3} + 2 + \sqrt{3}} = \frac{1}{\sqrt{3}} =$

$=2\pi i \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \pi i$

$a_1=2+\sqrt{3}$ -1

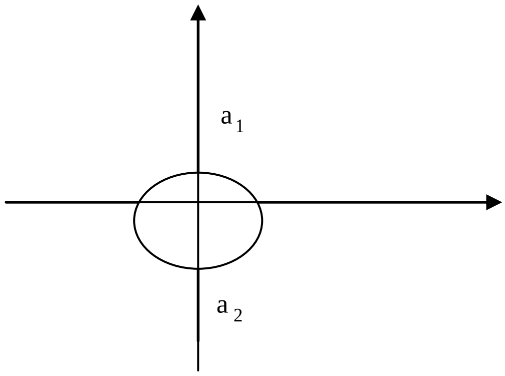
$a_2=2-\sqrt{3}$

$$(3)\quad \int_{-\infty}^{\infty} f(x)dx=\int f(z)dz=\begin{cases}2\pi i, & \sum res, yuqoriyarimqismida\\ 0, & pastda\end{cases}$$

$$a).\quad \int_{-\infty}^{\infty}\frac{x^2}{x^2+1}dx=\int_{a_{1,2}=\pm i}\frac{z^2}{z^2+1}dz=2\pi i\cdot\left(\frac{-1}{2i}\right)=-\pi$$

$$\text{res(f(z) z=a}_1\text{)}=\lim_{z\to a}\frac{z^2}{(z+i)(z-i)}(z-i)=$$

$$=\lim_{z\to i}\frac{z^2}{z+1}=-\frac{1}{2z}$$



Nazorat savollari

1. Chegirmalar yordamida integrallar qanday hisoblanadi?
2. Trigonometrik funksiyalar qatnashgan integrallar qsnday hisoblanadi?
3. Xosmas integrallar qanday hisoblanadi?

5-mavzu. Operatsion hisob

Reja:

1. Laplas almashtirishi va uning asosiy xossalari
2. Elementar funksiyalarning tasvirlari
3. Tasvirlar yordamida differensial tenglamalarni yechish

Xaqiqiy o'zgaruvchili t>0 da aniqlangan istalgan tartibigacha uzluksiz xosilalarga ega bo'lgan f(t) funksiya uchun shunday almashtirish bajarish mumkinki bunda xosila va integral amallari oddiy ko'paytuvchiga aylanadi .Natijada differensiyal va integral tenglamalar oddiy algebraic tenglamalarga aylanadi . Yuqoridagi shartlarni bajaruvchi f(t) funksuya uchun Laplas almashtirishlari quyidagicha aniqlanadi .

$$L\{f(t)\}=f(p)=\int_0^{\infty}e^{-pt}f(t)dt \quad (1)$$

P-kompleks o'zgaruvchi. $F(0)\to f(t)$ yoki $f(t)\overset{\bullet}{\longleftarrow} F(p)$.

Bu yerda f(t) –orginal, F(p) –tasvir deyiladi .

F'(t) ni tasvirini topish uchun f'(t) ni (1) –formulaga qoýamiz .

$$\int_0^{\infty}e^{-pt}f'(t)dt=\begin{Bmatrix}u=e^{-pt}, dv=f'(t)dt\\ du=-Pe^{-pt}, v=f(t)\end{Bmatrix}=e^{-pt}f(t)\Big|_0^{\infty}+p\int_0^{\infty}e^{-pt}f(t)dt=p\int_0^{\infty}e^{-pt}f(t)dt-f(0)$$

Demak , f′(t) tasviri

L{f′(t)}=p·F(p)-f(0) ga teng ekan .

Elementar funksiyalarning tasvirlarini hisoblaymiz.

1). $c \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$

$$F(p)=\int_0^{\infty}e^{-pt}c\,dt=c\cdot\left(-\frac{1}{p}e^{-pt}\right)\Big|_0^{\infty}=\frac{c}{p}$$; demak , $c \overset{\bullet}{\underset{\bullet}{\leftarrow}} \frac{c}{p}$ bo'ladi .

2. a) $t \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$ $\quad F(p)=\int\limits_0^{\infty} e^{-pt} f(t)dt$;

$$F(p)=\int\limits_0^{\infty} e^{-pt} t dt = \left\{ \begin{matrix} u=t, & dv=e^{-pt}dt \\ du=dt, & v=-\dfrac{1}{p}e^{-pt} \end{matrix} \right\} = -\frac{t}{p}e^{-pt}\Big|_0^{\infty} + \frac{1}{p}\int\limits_0^{\infty} e^{-pt}dt = \frac{1}{p}\bullet\frac{1}{p} = \frac{1}{p^2}$$

Demak , $t \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{1}{p^2}$ ga teng bo'ladi .

b) $t^2 \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$ $\quad F(p)=\int\limits_0^{\infty} e^{-pt} t^2 dt = \left\{ \begin{matrix} u=t^2, & dv=e^{-pt} \\ du=2tdt, & v=-\dfrac{1}{p}e^{-pt} \end{matrix} \right\} = t^2 \bullet (-\frac{1}{p}e^{-pt})\Big|_0^{\infty} + \frac{2}{p}\int\limits_0^{\infty} e^{-pt} t dt =$

$= \dfrac{2}{p}\bullet\dfrac{1}{p^2} = \dfrac{2}{p^3}$; demak , $t \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{2}{p^3}$

Demak , $t^n \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{n!}{p^{n+1}}$ ifoda bilan aniqlanar ekan .

c) $t^3 \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$

$$F(p)=\int\limits_0^{\infty} e^{-pt} t^3 dt = \left\{ \begin{matrix} u=t^3, & du=e^{-pt} \\ du=3t^2, & v=-\dfrac{1}{p}e^{pt} \end{matrix} \right\} = t^3(-\frac{1}{p}e^{-pt})\Big|_0^{\infty} + \frac{3}{p}\int\limits_0^{\infty} e^{-p}t^2 dt =$$

$= \dfrac{3}{p}\bullet\dfrac{2}{p^3} = \dfrac{6}{p^4};$ demak, $t^2 \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{6}{p^4}$;

3. $e^{\alpha t} \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$ $\quad F(p)=\int\limits_0^{\infty} e^{-pt} e^{\alpha t} dt = \int\limits_0^{\infty} e^{-(p-\alpha)t} dt = -\dfrac{1}{p-\alpha} e^{-(p-\alpha)t}\Big|_0^{\infty} = \dfrac{1}{p-\alpha};$

demak, $e^{\alpha t} \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{1}{p-\alpha}$ $\qquad \alpha = const$.

Demak , $\alpha^{\pm \alpha t} \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{1}{p \pm \alpha}$ ifoda bilan aniqlanadi .

4. $Sin\omega t \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$ $\quad Sin\omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

$$F(p)=\int\limits_0^{\infty} e^{-pt}\sin\omega t dt = \frac{1}{2}\int\limits_0^{\infty}(e^{-(p\omega)t} - e^{-(p+i\omega)t})dt = \frac{1}{2}(\frac{1}{p-i\omega}) = \frac{\omega}{p^2+\omega^2}$$

$Sin\omega t \overset{\bullet}{\underset{\bullet}{\leftarrow}} \dfrac{\omega}{p^2+\omega^2}$.

5. $Cos\omega t \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$ $\quad Cos\omega t = \dfrac{e^{i\omega t} - e^{-\omega t}}{2}$

$$F(P) = \int\limits_0^{\infty} e^{-pt}\cos\omega t dt = \frac{1}{2}\int\limits_0^{\infty}(e^{(p-i\beta\omega)t} - e^{-(p+i\omega)t})dt = \frac{1}{2}(\frac{1}{p-i\omega} - \frac{1}{p+i\omega}) = \frac{p}{p^2+\omega}$$

$Cos\omega t = \dfrac{p}{p^2+\omega^2}$.

6. $t^n e^{\pm\alpha t} \overset{\bullet}{\underset{\bullet}{\leftarrow}} ?$

$$F(P)=\int_0^\infty e^{-Pt}t^n e^{\pm\alpha}+dt=\int_0^\infty e^{-(p\pm\alpha)t}t^n dt=t^n e^{\pm\alpha t}\doteqdot\frac{n!}{(p\pm\alpha)^{n+1}}$$

1-jadval

№	Original f(t)	Tasvir F'(P)
1	C	$\frac{c}{P}$
2	t	$\frac{1}{p^2}$
3	t^n	$\frac{n!}{p^{n+1}}$
4	$e^{\pm\alpha t}$	$\frac{1}{p^{n\pm\alpha}}$
5	$Sin\,\omega t$	$\frac{\omega}{p^2+\omega^2}$
6	$Cos\,\omega t$	$\frac{p}{p^2+\omega^2}$
7	$t^n e^{\pm\alpha t}$	$\frac{n!}{(p\pm\alpha)^{n+1}}$
8	$f^1(t), y'$	$PF(P)-f(0)$
9	$f''(t), y''$	$P^2F(P)-Pf(0)-f'(0)$

7. $f''(t)\doteqdot ?$

$$F(p)=\int_0^\infty e^{-pt}f''(t)dt=\begin{Bmatrix} u=e^{-pt}\ , & dv=f''(t)dt \\ du=pe^{-pt}, & v=f(t)\end{Bmatrix}=f'(t)e^{-pt}\Big|_0^\infty+P\int_0^\infty f(t)e^{-pt}dt=$$

$$=-f'(0)+p(pF(p)-f(0)=-f'(0)+p^2F(0)-Pf(0)=P^2F(p)-pf(0)-f'(0)$$

8. $f'''(t)\doteqdot ?$

$$F(p)=\int_0^\infty e^{-pt}F'''(t)dt=\begin{Bmatrix} u=e^{-pz}, dv=f''(z)dt \\ du=-pe^{-pz}\ ,\ v=f''(t)\end{Bmatrix}=$$

$$=e^{-pt}f''(t)\Big|_0^\infty+P\int_0^\infty e^{-pt}f''(t)dt=-f''(0)+p(-f'(0)+p^2F(p)-pf(0))=$$

$$=p^3F(p)-P^2f(0)-pf'(0)-f''(0)\qquad .$$

Tasvirlar yordamida differensial tenglamalarni yechish

Tasvirlar yordamida differensial tenglamalarni yechish uchun jadvaldan foydalanib hosilalarning tasvirlari quyidagi teglamaning tasviri topiladi va

$$f(t)=\int e^{+Pt}F(p)dp$$

formula yordamida integral hisoblanadi.

Misol: 1) $y' + y = 2, \qquad y(0) = 1$

$$f(t) = \oint e^{Pt} F(p)dp$$

pF(P)+F(p)–1=2,

F(p) (p+1)=3 ,

$$F(p) = \frac{3}{p+1},$$

$$f(t) = \oint e^{pt} \frac{3}{p+1} dp = 2\pi i \cdot 3e^{-t} = 6\pi i e^{-t}.$$

$$res(f(p), p = -1) = \lim_{p \to -1} \left(\frac{e^{pt}}{p+1}(p+1) \right) = e^{-t}.$$

Nazorat savollari

1. Operatsion hisob deb nimaga aytiladi?
2. Elementar funksiyalr tasvirlarini yozing?
3. Tasvirlar yordamida differensial tenglamalar qanday yechiladi?

6-mavzu. Maxsus funksiyalar

Reja:

1. Lejandr polinomi
2. Chebishev-Ermit polinomi
3. Chebishev-Lagerr polinomi
4. Gamma funksiya
5. Silindrik funksiyalar

Kvant mexanikasi, atom fizikasi va nazariy fizikada

$$aW'' + bW' + cW = 0$$

ko'rinishdagi tenglamalarni yechishga to'g'ri keladi va ularning yechimlari maxsus funksiyalar deyiladi. Bunday tenglamaning birinchisi.

$$(1 - x^2)W'' - 2xW' + 2W = 0 \qquad (1)$$

ko'rinishda bo'lib Lejandr tenglamasi deyiladi. Bu tenglamani yechimini

$$W = \sum_{n=0}^{\infty} a_n \cdot x^n \quad (2)$$

ko'rinishda qidiramiz. Noma'lum a larni topish uchun hosilalar olib tenglamaga oborib qo'yamiz.

$$W' = \sum a_n \cdot nx^{n-1}.$$

$$W'' = \sum a_n \cdot n \cdot (n-1)x^{n-2}.$$

$$\sum a_n \cdot n \cdot (n-1)x^{n-2} - a_n \cdot n \cdot (n-1)x^n - 2a_n \cdot nx^n + \lambda a_n x^n = 0$$

$$\sum a_{n+2}(n+1) \cdot (n+2)x^n - a_n n \cdot (n-1)x^n - 2a_n n \cdot x^n + \lambda a_n x^n = 0.$$

$$\sum \{a_{n+2}(n+1)(n+2) - a_n(n(n+1) - \lambda)\} x^n = 0.$$

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} \cdot a_n.$$

Bu formula noma'lum koeffisentlarni topish uchun rekurent formula deyiladi.

a_0 nolga teng bo'lganda juft hadlar,

a_1 nolga teng bo'lganda toq hadlar nolga aylanadi.

Bu hadlarni topib (2) formulaga qo'ysak (1) tenglamaning yechimi

$$W(x) = P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n \qquad (3).$$

ko’rinishda bo’ladi. Bu Lejandr polinolari deyiladi. Poli –ko’phad ma’nosini bildiradi.
Dastlabki nomerli Lejandr polinomlarini hisoblaymiz.

Misol. $P_0(x) = \frac{1}{2^0 a!} \frac{d^0}{dx^0}(x^2 - 1)^0 = 1.$

Tenglamalarning 2-chisi

$$W'' - 2xW' + \lambda W = 0 \qquad (4)$$

ko’rinishda bo’lib Chebishev- Ermit tenglamasi deyiladi, bu tenglama yechimini ham (2) ko’rinishda qidiramiz. Hosilalarni olib, tenglamaga qo’yib noma’lumlarni topsak tenglamaning yechimi

$$W(x) = H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \cdot e^{-x^2} \qquad (5)$$

ko’rinishda bo’ladi. Bu formula Chebishev – Ermit polinomlari deyiladi.

1) $H_0(x) = 1$

Tenglamalarning 3-chi ko’rinishi quyidagicha ,

$$xW'' + (1-x)W' + \lambda W = 0 \qquad (6).$$

(6) Chebishev – Lagerr tenglamasi deyiladi.

Yuqoridagi amallarni bajargandan keyin tenglamaning yechimi

$$W(x) = L_n(x) = e^x \cdot \frac{d^n}{dx^n} \cdot (x^n \cdot e^n) \qquad (7).$$

ko’rinishda bo’ladi. Bu Chebishev – Lagerr polinomlari deyiladi.

$L_0(x) = 1.$

Maxsus funksiyalarning 4-chisini

$$F(p) = \int_0^\infty x^{p-1} e^{-x} dx \qquad (8).$$

orqali aniqlanib gamma funksiya deyiladi. Bu yerda p≥1

$$\Gamma(1) = \int_0^\infty e^x dx = -e^{-x} \Big|_0^\infty = 1$$

$$\Gamma(n) = (n-1)!$$

$$\int U dV = U \cdot V - \int V dU$$

Demak, Gamma funksiya ixtiyoriy 1 dan katta xaqiqiy sonning faktoriali ekan.

Silindrik funksiyalar

Maxsus funksiyalarning(5) silindrik funksiyalar deyiladi, uning tenglamasi quyidagicha

$$y'' + \frac{1}{x} y' + \left(1 - \frac{P^2}{x^2}\right) y = 0 \qquad (9)$$

Ko’rinishda bo’ladi. Bu funksiyalarning silindrik deyilishiga sabab Laplas tenglamasining silindrik koordinatalar sistemasidagi ko’rinishi shu kabi bo’ladi. Silindrik va Dekart koordinatalar sistemasi orasidagi bog’lanish:

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

(1) tenglamaning yechimini

$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$ ko’rinishda qidiramiz

$$y' = \sum a_n (n+\alpha) x^{n+\alpha-1}$$
$$y'' = \sum a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

Y va uning hosilalarini (1) tenglamaga yuboramiz. X ning bir xildagi darajalar oldidagi koeffisientlarini tenglaymiz va noma'lum a_n koeffisientlarini topamiz. Natijada tenglamaning yechimi

$$y(x) = J_\alpha(x) = \sum_{V=0}^{\infty} (-1)^V - \frac{\left(\frac{x}{2}\right)^{2V+\alpha}}{V \div G(V+\alpha+1)} (2)$$ ko'rinishda bo'ladi. Bu funksiyalar silindrik funksiyalar yoki Bissel funksiyasi deyiladi. α koeffisientlariga qarab silindrik funksiyalar 4 xil bo'ladi

$$J_0(x) = \sum_{v=0}^{\infty} (-1)^v \frac{\left(\frac{x}{2}\right)^{2v}}{v!} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \dots$$

$$J_1(x) = \sum_{v=0}^{\infty} (-1)^v \frac{\left(\frac{x}{2}\right)^{2v+1}}{v!(v+1)!} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} \dots$$

Nazorat savollari

1. Maxsus funksiyalar deb nimaga aytiladi?
2. Maxsus funksiyalar turlarini ayting?
3. Rekurent formula deganda nimani tusunasiz?

7-mavzu. Xususiy hosilali differensial tenglamalar

Reja:

1. Xususiy hosilali differensial tenglamalar va ularning yechimi.
2. Xarakteristik forma tushunchasi.
3. Ikkinchi tartibli differensial tenglamalarning klassifikatsiyasi
4. Erkli o'zgaruvchilarni almashtirish.
5. Elliptik tipdagi tenglamalarning kanonik shakli.
6. Giperbolik tipdagi tenglamalarning kanonik shakli.
7. Parabolik tipdagi tenglamalarning kanonik shakli.

1. Xususiy hosilali differensial tenglamalar va ularning yechimi

$x = (x_1, x_2, ..., x_n) \in D \subset E^n$ bo'lib D-ochiq bog'limli soha bo'lsin. E^n-Evklid fazosi $x_1, x_2, \dots, x_n$ -ortogonal dekart koordinatalar sistemasidagi x nuqta- ning koordinatalari.

Tartiblangan manfiy bo'lmagan n ta butun sonning $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ ketma-ketligi n-tartibli mu'lteindeks deyiladi, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ son mu'lteindeksning ug'unligi deyiladi. $u(x) = u(x_1, x_2, ..., x_n)$ funksiyaning $x \in D$ nuqtadagi $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ tartibli hosilasini

$$D^\alpha u = D^{\alpha_1}{}_1, D^{\alpha_2}{}_2, \dots, D^{\alpha_n}{}_n u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1, \partial^{\alpha_2} x_2, \dots, \partial^{\alpha_n} x_n}, \quad D^\circ u = u(x)$$

Xususiy hoda ko'rinishda belgilashimiz $\alpha = \alpha_i$ bo'lganda

$$D^{\alpha}u=\frac{\partial^{\alpha_i}u}{\partial x_i^{\alpha_i}},\quad D_iu=\frac{\partial u}{\partial x_i},\qquad D_i^2u=\frac{\partial^2u}{\partial x_i^2}=u_{x_ix_i}$$

$F=F(x,...p_{\alpha},...)$ funksiya D sohada x nuqtaning va $p_{\alpha}=p_{\alpha_1},p_{\alpha_2},\dots,p_{\alpha_n}=D^{\alpha}u$, $\alpha_i=0,1,\dots$ haqiqiy o’zgaruvchining berilgan funksiyasi bo’lib, kamida bitta $\frac{\partial F}{\partial p_m}$ $m=\max|\alpha|$ hosila noldan farqli bo’lsin.

Ushbu

$$F=(x,\dots,D^{\alpha}u,\dots)=0 \qquad (1)$$

tenglik noma’lum $u(x)=u(x_1,x_2,...,x_n)$ funksiyaga nisbatan $m-$tartibli xususiy hosilali differensial tenglama deyiladi.

(1) tenglamaning o‘ng tomoni esa xususiy hosilali differensial operator deyiladi.

Agar F barcha p_{α} $(|\alpha|=0,1,\dots,m)$ o’zgaruvchilarga nisbatan chiziqli funksiya bo’lsa, (1) tenglama chiziqli differensial tenglama deyiladi.

Agarda F, $|\alpha|=m$ bo’lganda barcha p_{α} o‘zgaruvchilarga nisbatan chiziqli funksiya bo’lsa, (1) tenglama kvazichiziqli differensial tenglama deyiladi.

Misollar:

1) $$x_1^2\frac{\partial^3u}{\partial x_1^2\partial x_2}+x_1\cdot x_2\frac{\partial^2u}{\partial x_2^2}+\frac{\partial u}{\partial x_1}+u=0\quad u=u(x_1,x_2)$$

- bu uchinchi tartibli ikki o’zgaruvchili chiziqli tenglama.

2) $$u\frac{\partial^2u}{\partial x_1^2}+\left(\frac{\partial u}{\partial x_2}\right)^2+x_3u=0\quad u=u(x_1,x_2,x_3)$$

- bu ikkinchi tartibli uch o’zgaruvchili kvazichiziqli tenglama.

3) $$\left(\frac{\partial^3u}{\partial x_1^2\partial x_2}\right)^3+\frac{\partial^2u}{\partial x_2^2}+u=0\qquad u=u(x_1,x_2)$$

- bu uchinchi tartibli ikki o’zgaruvchili chiziqli bo’lmagan tenglama.

D sohada aniqlangan $u(x)$ funksiya (1) tenglamada ishtirok etuvchi barcha hosilalarri bilan uzluksiz bo’lib, uni ayniyatga aylantirsa, $u(x)$ ga (1) tenglamaning regulyar (klassik) echimi deyiladi.

Xususiy hosilali $m-$tartibli chiziqli differensial tenglamani ushbu

$$Lu=\sum_{|\alpha|\le m}a_{\alpha}(x)D^{\alpha}u=f(x) \qquad (2)$$

ko’rinishda yozib olish mumkun.

Barch $x\in D$ lar uchun (2) tenglamaning o’ng tomoni $f(x)$ nolga teng bolsa, (2) tenglama bir jinsli, $f(x)$ funksiya nolga teng bo’lmasa, bir jinsli bo’lmagan tenglama deyiladi. Agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo’lmagan (2) tenglamaning echimlari bo’lsa, ravshanki $w(x)=u(x)-v(x)$ ayirma bir jinsli $(f=0)$ tenglamaning yechimi bo’ladi.

Agarda $u_i(x), i=1,2,\dots,k$ funksiyalar bir jinsli $(f=0)$ tenglamaning yechimlari bo’lsa, $u(x)=\sum_{i=1}^{k}c_iu_i(x)$ funksiya ham, bu erda c_i-haqiqiy o’zgarmaslar, shu tenglamaning yechimi bo’ladi.

Xususiy hosilali ikkinchi tartibli chiziqli differensial tenglama

$$\sum_{i,j=1}^{m}A_{i,j}(x)\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}B_i(x)\frac{\partial u}{\partial x_i}+C(x)u=F(x) \qquad (3)$$

ko’rinishda yoziladi, bu erda $A_{i,j}$, B_i, C, $F-D$ sohada berilgan haqiqiy funksiyalardir. (3) tenglamaning barcha $A_{i,j}$, $i,j=1,\dots,n$ koeffiisientlari nolga teng bo’lgan $x\in D$ nuqtalarda tenglama ikkinchi tartibli bo‘lmay qoladi, ya’ni bu nuqtalarda tenglamaning tartibi buziladi. Bundan keyin barcha $x\in D$ da

$$\sum_{i,j=1}^{n} A_{i,j}^2(x) \neq 0$$

deb hisoblaymiz. (3) tenglamada $i \neq j$ bo'lganda alohida-alohida $A_{i,j}u_{x_i x_j}$, $A_{j,i}u_{x_j x_i}$ qo'shiluvchilar ishtirok etmay, balki ularning yig'indisi $(A_{i,j} + A_{j,i})u_{x_i x_j}$ ishtirok etadi. Shu sababli ham umumiyatlika ziyon etkazmay hamma vaqt $A_{i,j} = A_{j,i}$ deb hisoblaymiz.

Eslatib o'tamiz D sohada aniqlangan va $K -$ tartibgacha xususiy hosilalri bilan uzluksiz bo'lgan haqiqiy $u(x)$ funksiyalarning to'plamini $C^K(D)$ orqali belgilaymiz.

2. Xarakteristik forma tushunchasi.

Faraz qilaylik (1) tenglamada ishtirok etayotgan $F = F(x,...p_\alpha,...)$ funksiya,

$p_\alpha = p_{\alpha_1,\alpha_2,...,\alpha_n}$ $|\alpha| = m$ o'zgaruvchilar bo'yicha uzluksiz hosilaga ega bo'lsin. (1) tenglamalar nazariy asida $\lambda_1, \lambda_2, \ldots, \lambda_n$ haqiqiy o'zgaruvchilarga nisbatan ushbu

$$K(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{|\alpha|=m} \frac{\partial F}{\partial p_\alpha} \lambda^\alpha \ , \lambda^\alpha = \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, ..., \lambda_n^{\alpha_n} \qquad (4)$$

$m-$tartibli forma m darajali bir jinsli ko'phad muhim ro'l o'ynaydi. Bu forma (1) tenglamaga mos bo'lgan xarakteristik forma deyiladi.

3. Ikkinchi tartibli differensial tenglamalarning klassifikatsiyasi va kanonik ko'rinishi.

Ikkinchi tartibli kvazichiziqli

$$\sum_{i,j=1}^{n} A_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \Phi(x, u, u_{x_1}, u_{x_2}, ..., u_{x_n}) = 0 \qquad (5)$$

differensial tenglama uchun (4) forma

$$Q(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i,j=1}^{n} A_{i,j}(x) \lambda_i \cdot \lambda_j \qquad (6)$$

kvadratik formadan iborat. (5) tenglamani erkli o'zgaruvchilarni almashtirib uni soddaroq ko'rinishga keltirishga harakat qilamiz.

$x = (x_1, x_2, ..., x_n)$ o'rniga $y = y(x)$ ya'ni

$$y_k = y_k(x_1, x_2, ..., x_n) \qquad k = \overline{1,n}$$

ya'ni

$$y_k = \beta_{k_1} x_1 + \beta_{k_2} x_2 + ... + \beta_{k_n} x_n \quad k = \overline{1,n}$$

$y_k(x) \in C^2(D)$ va ushbu yakobian

$$\frac{D(y_1, y_2, ..., y_n)}{D(x_1, x_2, ..., x_n)} \neq 0$$

deb hisoblaymiz. U holda

$$\frac{\partial u}{\partial x_1} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_j \partial x_i} = \sum_{k,e=1}^{n} \frac{\partial^2 u}{\partial y_e \partial y_k} \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i}$$

Buni (5) tenglamaga qo'yib ushbu tenglamaga kelamiz

$$\sum_{i,j=1}^{n} A_{i,j}(x) \left\{ \sum_{k,e=1}^{n} \frac{\partial^2 u}{\partial y_e \partial y_k} \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i} \right\} + \overline{\Phi_0}(y, u, u_{y_1}, \ldots, u_{y_n}) = 0$$

Yoki

$$\sum_{k,e=1}^{n} \overline{A}_{k,e}(y) \frac{\partial^2 u}{\partial y_e \partial y_k} + \Phi_1(y, u, u_{y_1}, \ldots, u_{y_n}) = 0 \qquad (7)$$

Bu erda

$$\overline{A}_{k,e}(y) = \sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial y_e}{\partial x_j} \cdot \frac{\partial y_k}{\partial x_i} \qquad \mathbf{(8)}$$

$$\Phi_1(y, u, u_{y_1}, \ldots, u_{y_n}) = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} \sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial^2 y_k}{\partial x_j \partial x_i} + \overline{\Phi_0}(y, u, u_{y_1}, \ldots, u_{y_n})$$

(5) tenglama tekshirilayotgan D sohada x_0 nuqtani olamiz, va ushbu belgilashlarni kiritamiz.

$$y_0 = y(x_0), \qquad \beta_{ki} = \frac{\partial y_k(x_0)}{\partial x_i}$$

U holda (8) forma x_0 nuqtada quyidagicha yoziladi

$$\overline{A}_{k,e}(y_0) = \sum_{i,j=1}^{n} A_{i,j}(x) \beta_{e,j} \beta_{k,i} \qquad (9)$$

(6) kvadratik formani x_0 nuqtada yozib olamiz

$$Q = \sum_{i,j=1}^{n} A_{i,j}(x_0) \lambda_i \cdot \lambda_j \qquad (10)$$

Maxsus bo'lmagan ushbu

$$\lambda_i = \sum_{k=1}^{n} \beta_{k,i} \zeta_k, \ \det(\beta_{k,i}) \neq 0 \qquad (11)$$

affin almashtirish yordamida (10) kvadratik forma

$$Q = \sum_{i,j=1}^{n} A_{i,j}(x_0) \left(\sum_{k=1}^{n} \beta_{k,i} \zeta_k \right) \cdot \left(\sum_{e=1}^{n} \beta_{e,i} \zeta_e \right) = \sum_{k,e=1}^{n} \zeta_k \zeta_e \left(\sum_{i,j=1}^{n} A_{i,j}(x_0) \cdot \beta_{ki} \cdot \beta_{ej} \right) =$$

$$= \sum_{k,e=1}^{n} \overline{A}_{ke}(y_0) \zeta_k \zeta_e \qquad (12)$$

ga keladi.
Bu kvadratik formaning koeffitsientlari ham (9) formula bilan aniqlanadi.

Shunday qilib, (5) tenglamani x_0 nuqtada x o'zgaruvchilar o'rniga yangi $y = y(x)$ o'zgaruvchilar kiritib soddalashtirish uchun shu nuqtada (10) kvadratik formani maxsus bo'lmagan (11) chiziqli almashtirish yordami bilan soddalashtirihs yetarlidir.

Algebra kursida isbot qilinadiki, hamma vaqt shunday maxsus bo’lmagan (11) almashtirish mavjud bo’lib, uning yordami bilan (10) kvadratik forma quyidagi ko’rinishga olib kelinadi.

$$Q = \sum_{k=1}^{n} \mu_k \zeta_k^{\ 2} \tag{13}$$

bu erda $\mu_k \quad k = 1,\dots,n$ koeffitsientlar 1,-1,0 qiymatlarni qabul qiladi. Shu bilan birga masbat (manfiy) koeffitsientlar soni (inertsiya indeksi) va nolga bo’lgan koeffitsiyentlar soni (forma defekti) affin almashtirishga nisbatan invuriant, ya’ni bu sonlar faqat (10) forma bilan aniqlanib, (11) almashtirishning tanlab olinishiga bog’liq bo’lmaydi.

Bu narsa (5) differensial tenglama $A_{i,j}(x)$ koeffitsiyentlarning x_0 nuqtada qabul qiladigan qiymatlariga qarab, klassifikatsiya qilish imkonini beradi.

Yuqorida aytilganlarga asosan (7) tenglama

$$\sum_{k=1}^{n} \mu_k u_{y_k y_k} + \overline{\Phi}(y, u, u_{y_1}, ..., u_{y_n}) = 0 \tag{14}$$

ko’rinishda yoziladi.

Ikkinchi tartibli differensial tenglamaning aralash hosilalar qatnashmagan bunday ko’rinishi, odatda uning kanonik korinishi deyiladi.

(5) tenglamani bitta nuqtada emas, xech bo’lmaganda $x_0 \in D$ nuqtaning biror kichik atrofida kanonik ko’rinishga olib keluvchi mumkinmi degan savol tug’uladi.

Bu savolga ijobiy javob faqat $n = 2$ bo’lgandagina ma’lum. Bu xolni biz alohida ko’ramiz. Agar barcha $\mu_k = 1$ yoki barcha $\mu_k = -1 \quad k = 1,\dots,n$ bolsa yani Q forma mos ravishda musbat yoki manfiy aniqlangan (gefinit) bo’lsa, (5) tenglama $x \in D$ nuqtada elliptik tipdagi yoki elliptik tenglama deyiladi.

Agar μ_k koyffisientlardan bittasi manfiy, qolganlari musbat (yoki aksincha) bo’lsa, (5) tenglama $x \in D$ nuqtada giperbolik tenglama deb ataladi. μ_k koyffisientlardan ikkitasi, $1 < 2 < n-1$, musbat, qolgan $n-2$ tasi manfiy bo’lsa, (5) tenglamaga ultragiperbolik tipdagi tenglama deyiladi.

Agar koyffisientlardan kamida bittasi nolga teng bo’lsa, (5) tenglama keng manoda $x \in D$ nuqtada parabolik tenglama deb ataladi. Agar (5) tenglama D sohaning xar bir nuqtasida elliptik, giperbolik yoki parabolik bo’lsa, u xolda D sohada mos ravishda elliptik, giperbolik yoki parabolik tipdagi tenglama deb ataladi.

Eslatib o’tamiz, A matritsaning xarakteristik sonlar ushbu

$$\det(A - \lambda E) = 0$$

Algebraik tenglamaning ildizlaridan iborat, bu erda E - birllik matritsa

(5) tenglama berilgan D sohaning ixtiyoriy x nuqtaning A matritsa xarakteristik sonlarning ishorasini aniqlab, (5) tenglamani qaysi tipga tegishli ekanligini aniqlab olish mumkin.

Ushbu

$$\sum_{i,j=1}^{n} A_{i,j}(x) \cdot \frac{\partial \varpi}{\partial x_j} \cdot \frac{\partial \varpi}{\partial x_i} = 0$$

tenglama (5) differensial tanglama xarakteristik tenglamasi deyiladi.

Agar $\varpi(x_1, x_2, ..., x_n)$ funksiya xarakteristikalar tenglamasini qanoatlantirsa

$$\varpi(x_1, x_2, ..., x_n) = c \quad c = const$$

tenglama bilan aniqlangan sirt berilgan (5) differensial tenglamani xarakteristik sirti yoki xarakteristikasi deyiladi.

O’zgaruvchlar soni ikkita bo’lganda xarakteristik egri chiziq haqida so’z boradi.

4. Erkli o‘zgaruvchilarni almashtirish.

Ikkinchi tartibli ikki o‘zgaruvchili kvazichiziqli xususiy hosilali differrensial tenglamalarning shunday xususiyati borki ularni bir nuqtada emas balki butun bir sohada ham kanonik korinishga keltirish mumkin va sohaning nuqtalarida tenglama tipi o‘zgarmaydi.

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \Phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{15}$$

$$A^2(x,y) + B^2(x,y) + C^2(x,y) \neq 0$$

Tenglamani matritsasini yozib olamiz

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

va matritsaning xarakteristik sonlari tenglamasi ushbu ko‘rinishda bo‘ladi.

$$\det(A-\lambda\cdot E)=\begin{vmatrix} A-\lambda & B \\ B & C-\lambda \end{vmatrix}=0$$

yoki

$$\lambda^2-(A+C)\lambda+AC-B^2=0$$

va tenglama haqiqiy echimga ega

$$\lambda_{1,2}=\frac{A+C\pm\sqrt{(A+C)^2-4(AC-B^2)}}{2}=\frac{A+C\pm\sqrt{(A-C)^2+4B^2}}{2}$$

va ildizlar ya‘ni λ_1 va λ_2 bir xil ishorli agar $AC-B^2>0$ bo‘lsa; har xil ishorali agar $AC-B^2<0$ bo‘lsa; ildizlardan biri nolga teng agar $AC-B^2=0$ bo‘lsa. Bu yerdan (15) tenglamani

1) $AC-B^2>0$ bo’lsa tenglama elliptik tipda

2) $AC-B^2<0$ bolsa tenglama giperbolik tipda

3) $AC-B^2=0$ bo’lsa tenglamani parabolik tipda ekanligi kelib chiqadi.

$$A\left(\frac{\partial\omega}{\partial x}\right)^2+2B\frac{\partial\omega}{\partial x}\cdot\frac{\partial\omega}{\partial y}+C\left(\frac{\partial\omega}{\partial y}\right)^2=0 \qquad (16)$$

xarakteristik tenglamani bu holda oddiy differensial tenglamaga olib kelish mumkin. $\omega(x,y)$ -(16) tenglamaning echimi bo‘lsin. Ushbu xarakteristikani qaraymiz

$$\omega(x,y)=const$$

shu xarakteristika yo‘nalishi bo’ylab ushbu munosabat o‘rinlidir.

$$\frac{\partial\omega}{\partial x}dx+\frac{\partial\omega}{\partial y}dy=0$$

yoki

$$\frac{\partial\omega}{\partial x}:\frac{\partial\omega}{\partial y}=-dy:dx \qquad (17)$$

(17) ga asosan (16) tenglama birinchi tartibli oddiy differensial tenglama ko‘rinishini oladi.

$$A\cdot dy^2-2B\cdot dy\cdot dx+C\cdot dx^2=0 \qquad (18)$$

Teskarisi agar $\omega(x,y)=C$ (18) tenglamaning umumi echimi bo‘lsa, u holda $\omega(x,y)$ funksiya (16) tenglamaning echimi bo‘ladi (18) tenglama ushbu ikkita tenglamaga ajraladi

$$\begin{aligned} \frac{dy}{dx}&=\frac{B+\sqrt{B^2-AC}}{A} \\ \frac{dy}{dx}&=\frac{B-\sqrt{B^2-AC}}{A} \end{aligned} \qquad (19)$$

(1) tenglamada erkli o‘zgaruvchilarni quyidagicha almashtiraylik.

$\xi=\xi(x,y)$, $\eta=\eta(x,y)$ u holda (1) tenglama ushbu ko‘rinishni oladi

$$\overline{A}\frac{\partial^2 u}{\partial \xi^2}+2\overline{B}\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}+\overline{C}\frac{\partial^2 u}{\partial \eta^2}+\overline{\Phi}\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right)=0 \qquad \textbf{(20)}$$

$$\overline{A}=A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2$$

$$\overline{C}=A\left(\frac{\partial \eta}{\partial x}\right)^2+2B\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+C\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$\overline{B}=A\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right)+C\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}$$

Bevosita o‘rniga qo‘yish bilan isbotlash mumkini

$$\overline{B}^2-\overline{A}\,\overline{C}=(B^2-AC)\cdot(\Im)^2$$

yani xosmas almashtirish tenglama tipini o‘zgartirmaydi. Hozircha $\xi=\xi(x,y)\,,\eta=\eta(x,y)$ almashtirishlar ixtiyoriy almashtirishlar edi.

5. Elliptik tipdagi tenglamalarning kanonik shakli.

Faraz qilaylik (1) tenglama elliptik tipga tegishli bo’lsin. U holda (5) tenglama ikkita qo‘shma echimga ega bo‘lib

$$\omega(x,y)=\xi(x,y)+i\eta(x,y)=c \qquad (21)$$

(17) tenglamani birinchisini umumiy integral bo‘lsin. (21) ni (15) xarakteristik tenglamaga qo‘yib ushbu tenglikka ega bo‘lamiz

$$A\left(\frac{\partial \omega}{\partial x}\right)^2+2B\frac{\partial \omega}{\partial x}\cdot\frac{\partial \omega}{\partial y}+C\left(\frac{\partial \omega}{\partial y}\right)^2=0$$

$$A(\xi_x+i\eta_x)^2+2B(\xi_x+i\eta_x)\cdot(\xi_y+i\eta_y)+C(\xi_y+i\eta_y)^2=0$$

yoki

$$A\xi_x^2+2B\xi_x\xi_y+C\xi_y^2-(A\eta_x^2+2B\eta_x\eta_y+\eta_y)^2+i\cdot 2(A\xi_x\eta_x+B(\xi_x\eta_y+\eta_x\xi_y)+C\xi_y\eta_y)=0$$

yoki kompleks sonlarning nolga tengligi xossasidan

$$A\xi_x^2+2B\xi_x\xi_y+C\xi_y^2=A\eta_x^2+2B\eta_x\eta_y+C\eta_y{}^2$$

$$A\xi_x\eta_x+2B(\xi_x\eta_y+\eta_x\xi_y)+C\xi_y\eta_y=0$$

Bundan $\overline{A}=\overline{C}\,,\overline{B}=0$ ya’ni tenglama quyidagi ko’rinishda bo’ladi

$$u_{\xi\xi}+u_{\eta\eta}+\frac{1}{\overline{A}}\overline{\Phi}(\xi,\eta,u,u_{\xi},u_{\eta})=0 \qquad (22)$$

Shunday qilib $AC-B^2>0$ bo’lganda (16) tenglamani yechib $\xi(x,y)+i\eta(x,y)=c$ umumiy echimga ko‘ra $\xi=\xi(x,y)\,,\eta=\eta(x,y)$ almashtirish bajarib tenglamani (22) ko‘rinishga olib kelish mumkin

6. Giperbolik tipdagi tenglamalarning kanonik shakli.

(1) tenglama uchun $AC-B^2<0$ bolsin. U holda (5) tenglama ikkita echimga ega va ularning umumiy integrallari ushbu ko‘rinishda bo’lsin

$$\begin{aligned}\omega_1(x,y)&=\xi(x,y)=c\\ \omega_2(x,y)&=\eta(x,y)=c\end{aligned} \qquad (23)$$

u holda (23) ga asosan

$$A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2=0$$

va

$$A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\cdot\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2=0$$

tengliklar o‘rinlidir. Demak

$$\overline{A}=\overline{C}=0$$

Shunday qilib (6) tenglama ushbu ko‘rinishni oladi

$$\frac{\partial^2 u}{\partial\xi\cdot\partial\eta}+\frac{1}{2\overline{B}}\overline{\Phi}(\xi,\eta,u,u_\xi,u_\eta)=0$$

endi $\alpha=\xi+\eta$, $\beta=\xi-\eta$ almashtirish bajarib ushbu tenglamaga kelamiz

$$\frac{\partial^2 u}{\partial\alpha^2}-\frac{\partial^2 u}{\partial\beta^2}+\overline{\overline{\Phi}}\left(\alpha,\beta,u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\beta}\right)=0 \qquad (24)$$

Bu giperbolik tipdagi tenglamaning kanonik ko‘rinishidir.

7. Parabolik tipdagi tenglamalarning kanonik shakli.

Faraz qilaylik (1) tenglama uchun $AC-B^2=0$ bo’lsin. U holda (5) tenglamalar bitta $\xi(x,y)=const$ umumiy integralga ega bo‘ladi. (15) tenglamada $\xi=\xi(x,y)$ deb $\eta=\eta(x,y)$ -ixtiyoriy (lekin $\xi=\xi(x,y)$ bilan $\frac{D(\xi,\eta)}{D(x,y)}\neq 0$ munosabatda bo‘lgan funksiya) almashtirish bajaramiz u holda $\overline{A}=0$ va $\overline{B}^2-\overline{A}\overline{C}=(B^2-AC)\cdot\Im^2$ tenglikdan $\overline{B}^2-\overline{A}\overline{C}=0$ kelib chiqadi, bundan esa $\overline{B}=0$, lekin $\overline{C}\neq 0$, demak (22) tenglama

$$\frac{\partial^2 u}{\partial\eta^2}+\overline{\overline{\Phi}}\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0$$

kanonik shakilga ega bo’lamiz.

Nazorat savollari

1. Xususiy hosilali differensiyal tenglamaga ta’rif bering, ularning echimi deb qanday funksiyasiga aytamiz.
2. Kvazi chiziqli tenglama deb qanday tenglamaga aytiladi.
3. Ikkinchi tartibli chiziqli xususiy hosilali differensial tenglamani klassifikatsiyalang.
4. $\det(A-\lambda E)=0$ xarakteristik sonlarining ishorasi nimani aniqlaydi.
5. Ikkinchi tartibli, ikki o‘zgaruvchili kvazi chiziqli tenglamaning xarakteristik tenglamasini yozing.
6. Elliptik tipdagi tenglama uchun uni kanonik ko‘rinishga keltirish uchun qanday almashtirish bajarish kerak.
7. Giperbolik tipdagi tenglamani kanonik ko‘rinishga keltirish uchun qanday almashtirish bajarish kerak.
8. Parabolik tipdagi tenglamani kanonik ko‘rinishga keltirish uchun qanday almashtirish bajarish kerak.

8-mavzu. Matematik fizika tenglamalari

Reja:

1. Tor haqida tushuncha.
2. Tor tebranish tenglamasini keltirib chiqarish.
3. Tebranish, issiqlik tarqalishi, statsionar tenglamalar.

1. Tor haqida tushuncha.

Tor deganda erkin egiladigan ingichka ip tushuniladi, boshqacha aytganda, tor shunday qattiq jisimki, uning uzunligi boshqa o'lchamlaridan ancha ortiq bo'ladi.

Torning chekkli nuqtalari mahkamlangan, o'zi esa qattiq tortilgan bo'lsin. Agar tor muvozanat holatidan chetlashtirilsa tor tebrana boshlaydi. Biz tor tebranishini bir tekislikda ro'y beradi deb faraz qilamiz.

Bu tekislikda to'g'ri burchakli dekart koordinatalar sistemasini olamiz $x\,o\,u$. ox o'qini torning boshlang'ich tinch holati bo'yicha yo'naltiramiz. U holda u torning muvozanat holatidan siljishini beradi. Tor tebranish jarayonida u-chetlanish x va t ga bog'liq bo'ladi ya'ni $u = u(x,t)$. Har bir fiksirlangan t vaqitda $u(x,t)-$ funksiya grafigi tor tebranishi grafigini beradi, $\frac{\partial u}{\partial x} = u_x(x,t)$ esa bu grafikning x nuqtasiga o'tkazilgan urinma burchak koeffisiyentini beradi. $u_t(x,t)-$ harakat tezligi $u_{tt}(x,t)$ harakat tezlanishi.

Bizning maqsadimiz tor harakatini beruvchi $u(x,t)-$ funksiya qanoatlantradigan tenglama tuzish. Buning uchun ba'zi bir cheklanishlar qilamiz

1. Tor absayut egiluvchan. Torga ta'sir qilinib turgan taranglik kuchi etarli katta deb faraz qilamiz. Shu sababli torning egilganda qarshiligini taranglikka nisbatan hisobga olamsa ham bo'ladi.

Agar torning biror nuqtadan bir tomonga yotuvchi qismi olib tashlansa, u holda olib tashlangan qismining ta'sirini almashtiruvchi taranglik kuchi. Shu nuqtada torning urunmasi bo'yicha yo'nalgan bo'ladi.

Torni cho'ziluvchan emas deb faraz qilamiz va u Guk qonuniga bo'ysinadi, ya'ni taranglik kuchini o'zgarish miqdori torning uzunligini o'zgarishiga proporsionaldir. Torni bir jinsli deb faraz qilamiz va uning chiziqli zichligini ρ orqali belgilaymiz (birlik uzunlikka to'g'ri keluvchi massa).

Torga $o\,u$ o'qiga parallel kuchlar ta'sir etadi deb faraz qilamiz ular tor bo'ylab harakat qiladi va x, t ga bog'liq, ularning zichligini $g(x,t)$ deb belgillaymiz. Muhitning qarshilik kuchi e'tiborga olinmaydi. Biz faqat torning kichik tebranishlarini o'rganamiz.

2. Tor tebranish tenglamasini keltirib chiqarish.

Agar $\alpha(x,t)$ orqali torning x nuqtasida t vaqtda o'tkazilgan urinmasini ox o'qining musbat yo'nalishi bilan tashkil etgan burchagini belgilasak u holda torning kichik tebranishi

$$\alpha^2(x,t) \approx 0 \tag{1}$$

ekanligini ko'zda tutadi.

$\sin\alpha$ ning Makloren qatoriga yoyilmasiga asosan

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \dots$$

demak (1) shartga asosan

$$\sin\alpha \approx \alpha \tag{2}$$

Bu erdan

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2} \approx 2\frac{\alpha^2}{4} = \frac{\alpha^2}{2}$$

demak $\cos\alpha \approx 1$.

$$tg\alpha - \sin\alpha = tg\alpha(1-\cos\alpha) \approx 0$$

bu erdan

$$tg\alpha \approx \sin\alpha$$

Lekin

$$\frac{\partial u}{\partial x} = tg\alpha \approx \sin\alpha \approx \alpha$$

$$\left(\frac{\partial u}{\partial x}\right)^2 \approx \alpha^2$$

Bu erda

$$\overline{M_1M_2} = \int_{x_1}^{x_2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\,dx = \int_{x_1}^{x_2} dx = x_2 - x_1$$

ya'ni kichik tebranishlar tor qisimlari cho'zilmaydi va qisqarmaydi.

Endi taranglik kuchi T ni o'zgarmas ekanligini ko'rsatamiz, ya'ni T ni x va t ga bog'liq emasligini isbotlaymiz.

Buning uchun torning M_1M_2 bo‘lagini olamiz

t momentda tashlab yuborilgan bo‘laklar ta‘sirini T_1 va T_2 taranglik kuchlari bilan almashtiramiz shartga ko‘ra tor nuqtalari bir tekislikda harakat qilmoqda demak tashqi kuchlar ham $o\,u$ o‘qiga parallel yo‘nalishda ta’sir ko‘rsatadi. Bu kuchlarning ox o‘qdagi proeksiyasi nolga teng

$$-T_1\cos\alpha_1 + T_2\cos\alpha_2 = 0$$

$$\cos\alpha_1 \approx 1, \cos\alpha_2 \approx 1$$

demak $T_1 = T_2$ M_1 va M_2 nuqtalar uchun M_1 va M_2 nuqtalar ixtiyoriy bo‘lgani uchun ixtiyoriy x uchun t momentga $T(x,t) = T(t)$.

Lekin Guk qonuniga ko‘ra

$$T(t) = K(x_2 - x_1) = const$$

demak

$$T(x,t) = T_0 = const$$

Endi tor tebranishi tenglamasini keltirib chiqaramiz. Buning uchun torning cheksiz kichik M_1M_2 bo‘lagini olamiz va uning ox o‘qdagi proeksiyasi $[x, x+dx]$ bo‘lsin. Unda taranglik kuchlari T_1 va T_2 ta‘sir etadi. Bu kuchlarning $o\,u$ o‘qidagi proeksiyasi

$$-T_1\sin\alpha_1 + T_2\sin\alpha_2 = T_0(\sin\alpha_2 - \sin\alpha_1)$$

Lekin

$$\sin\alpha_1 \approx tg\alpha_1 = u'_x(x,t)$$

$$\sin\alpha_2 \approx tg\alpha_2 = u'_x(x+dx,t)$$

Demak

$$T_0(\sin\alpha_2 - \sin\alpha_1) = T_0[u'_x(x+dx,t) - u'_x(x,t)] = T_0\frac{\partial^2 u}{\partial x^2}dx$$

t momentda torning $\overline{M_1M_2}$ qismiga ta‘sir etuvchi barcha tashqi kuchlarning teng t’sir etuvchisini F orqali belgilaymiz. Demak

$$F \approx g(x,t)\overline{M_1M_2} = g(x,t)dx$$

$g(x,t)$ - birlik uzunlikka ta’sir etuvchi kuchlar zichligi.

Endi M_1M_2 qisim uchun Nyu’tonning ikkinchi qonunini qo‘llaymiz. Bu qonunga ko‘ra M_1M_2 uchastka massasini tezlanish ko‘paytmasi. burcha ta’sir etuvshi kuchlar yig‘indisiga teng. Agar ρ -tor zichligi bo‘lsa, u holda

$$\rho\cdot dx\cdot\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}dx + g(x,t)dx$$

yoki dx ga qisqartirib

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho}g(x,t) \qquad (4)$$

bu erda $a^2 = \dfrac{T_0}{\rho}$

(4) tenglamaga tor tebranish tenglamasi deyiladi yoki bir o‘lchovli to‘lqin tenglamasi deyiladi.

3. Tebranish, issiqlik tarqalishi, statsionar tenglamalar.

a) Mexanikaning (tor, sterjen, membrana, uch o‘lchovli hajimlarning tebranishlari), fizikaning (elektr tebranishlar) ko‘p masalalari

$$\rho\frac{\partial^2 u}{\partial t^2} = div(p\cdot gradu) - qu + F(x,t) \qquad (5)$$

ko‘rinishdagi tebranish tenglamalariga olib kelinadi. Bundagi $u(x,t)$ noma’lum funksiya n ta fazoviy koordinatalarga hamda t vaqitga bog‘liqdir. ρ, p, q - muxitning xossalari bilan aniqlanadi. $F(x,t)$ esa tashqi ta’sirning intensivligini aniqlaydi (5) tenglamada

$$gradu = \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\overrightarrow{e_k}$$

va agar $\vec{p} = p(p_1, p_2, \ldots, p_n)$ -bo‘lsa

$$div\,\vec{p} = \sum_{i=1}^{n} \frac{\partial p_i}{\partial x_i}$$

demak

$$div(pgradu) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i} \right)$$

b) Issiqlik tarqalish yoki muhitda zarrachalarning diffuziya jarayonlari ushbu umumiy differensial tenglama bilan ifodalanadi

$$\rho\frac{\partial u}{\partial t} = div(pgradu) - qu + F(x,t) \qquad (6)$$

e) Statsinar tenglamalar. Statsionan ya‘ni vaqitga bog‘liq bo‘lmagan jarayonlar uchun $F(x,t) = F(x)$, $u(x,t) = u(x)$ (5) tebranish hamda (6) diffurziya tenglamalar ushbu

$$-div(pgradu) + qu = F(x,t)$$

ko‘rinishda bo‘ladi.

Nazorat savollari.

1. Tebranish va diffuziya tenglamalarning umumiy ko‘rinishini yozing.
2. Issiqlik tarqalish tenglamasini umumiy ko‘rinishini yozing.
3. Statsionar tenglamalarning umumiy ko‘rinishini yozing.

9--mavzu. Matematik fizika tenglamalari uchun masalalarning qo‘yilishi

Reja:

1. Asosiy masalalarning qo‘yiliushi.
2. Koshi masalasi va uning qo‘yilishida harakteristikalarning ro‘li.
3. Koshi Kovalevskaya teoremasi.
4. Elliptik tipdagi tenglamalar uchun chegaraviy masalalar.

1. Asosiy masalalarning qo‘yiliushi.

Bizga yahshi ma’lumki, tebranish tenglamalari

$$q(x)\frac{\partial^2 u}{\partial t^2} = div(p(x)gradu) - q(x)u + F(x,t) \qquad (1)$$

issiqlik tarqalish tenglamasi

$$q(x)\frac{\partial u}{\partial t} = div(p(x)gradu) - q(x)u + F(x,t) \qquad (2)$$

stasionar jarayonlar tenglamasi

$$-div(p(x)gradu) + q(x)u + F(x,t) \qquad (3)$$

ko‘rinishda bo‘ladi.

Biror fizik jarayonni to‘la o‘rganish uchun bu jarayonni tasvirlayotgan tenglamalardan tashqari, uning boshlang‘ holatini va jarayon sodir bo‘layotgan sohaning chegarasidagi holatini berish zarurdir.

Matematik nuqtai nazaridan bu narsa differensial tenglamalar echimining yagona emasligi bilan bog‘liq.

Hususiy hosilalai differensial tenglamalar uchun umumiy echimi ihtiyoriy funksiyalarga bog‘liq bo‘lib, bu funksilarning soni tenglama tartibiga teng bo‘ladi. Ihtiyoriy funksiyalar argumentlarning soni echimi argumentlarning sonidan bitta kam bo‘ladi.

Misollar:

1) $$\frac{\partial u(x,y)}{\partial x} = 0$$

tenglamaning yechimi

$$u(x,y) = f(y)$$

2) $$u_x = u_y$$

tenglamaning yechimini topish uchun

$$\xi = x + y \quad , \quad \eta = x - y$$

almashtirish bajaramiz

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi - u_\eta$$

$$2u_\eta = 0$$

$$u = f(\xi) = f(x+y)$$

Xuddi shunga o‘xshash, agar α va β o‘zgarmas sonlar bo‘lsa

$$\alpha \cdot u_x + \beta \cdot u_y = 0$$

tenglamaning umumiy echimi

$$u(x,y) = f(\beta x - \alpha y)$$

bo‘ladi

3) Ushbu

$$u_{xy} = 0$$

tenglamaning umumiy echimi

$$u(x,y) = f(x) + \varphi(y)$$

bo‘ladi.

4) $$u_{xy} = f(x,y)$$

bir jinsli bo‘lmagan, tenglamaning echimi

$$u(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} f(\xi,\eta)d\xi\, d\eta + \vartheta(x) + w(y)$$

ko‘rinishda bo‘ladi

5) Uchinchi tartibli

$$u_{xyy} = 0$$

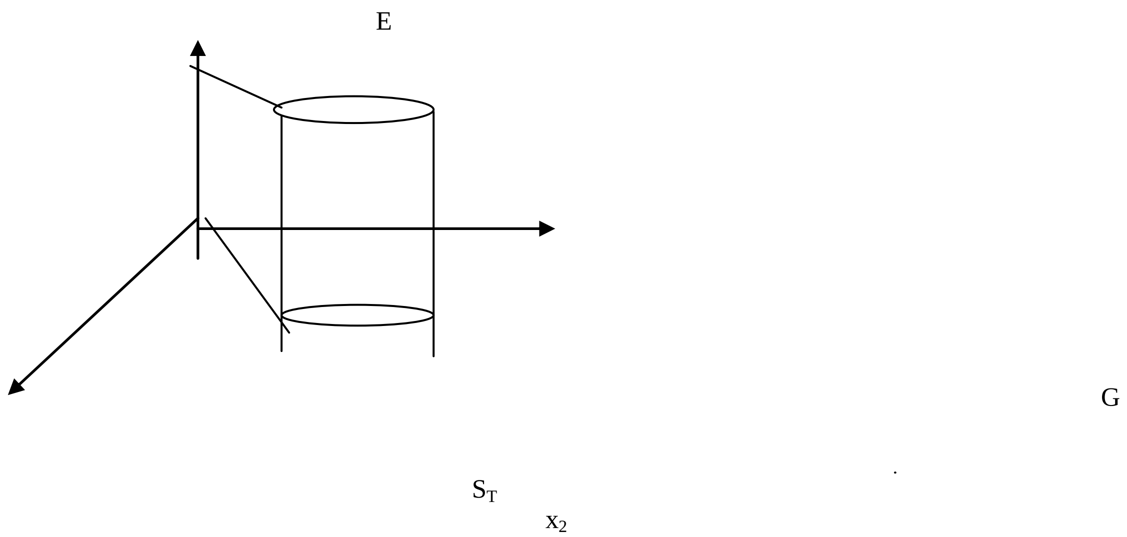
tenglamaning umumiy echimi

$$u(x,y) = \varphi(y) + y\psi(x) + \psi_1(x)$$

dan iborat bo‘ladi.

Shunday qilib, aniq fizik jarayonni ifodalovchi achimni ajratib olish qo‘shimcha shrtlarni berish zarurdir. Bunday qo‘shimcha shartlar boshlang‘ich va chegaraviy shartlardan iboratdir.

Jarayon sodir bo‘layotgan soha $G \in R^n$ bo‘lib, S uning chegarasi bo‘lsin. S ni bo‘laklari silliq sirt deb hisoblaymiz. $\ddot{O}_{\grave{O}} \in G\,x(0,T) -$ bu asosi G soha balandligi T bo‘lgan silindir bo‘lsin, uning chegarasi yon sirti $S\,x[0,T]$, quyi $\overline{G}\,x\{0\}$ va yuqori $\overline{G}\,x\{T\}$ asoslardan iborat.



C

x_1 S

Differensial tenglamnalar uchun, asosan uch tipdagi masalalar bir-biridan farq qiladi.

a) Koshi masalasi. Bu masala, asosan, giperbolik va parabolik tipdagi tenglamalar uchun qo'yiladi. G soha butun $\boldsymbol{R}^n$ fazo bilan ustma-ust tushadi, bu holda chegaraviy shrtlar bo'lmaydi.

v) Chegaraviy masala elliptik tipdagi tenglamalar uchun qo'yiladi, S da chegaraviy shatrlar beriladi, boshlang'ich shartlar tabiiy bo'lmaydi.

g) Aralash masala elliptik tipdagi tenglamalar uchun qo'yiladi: $G \neq R^n$ bo'lib boshlang'ich va chegaraviy shratlar beriladi.

2. Koshi masalasi va uning qo'yilishida xarakteristikalarning roli.

(1) tenglama uchun Koshi masalasi bunday qo'yiladi:

Koshi masalasi: $c^2(t>0)\cap c(t\geq 0)$ sinfga tegishli, $t>0$ yarim fazoda (1) tenglamani va $t=+0$ da

$$u\Big|_{t=+0}=u_0(x)\ ,\ \frac{\partial u}{\partial t}\Big|_{t=+0}=u_1(x) \tag{4}$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,t)$ funksiya topilsin.

(2) tenglama uchun Koshi masalasi quyidagicha qo'yiladi.

Koshi masalasi: $c^2(t>0)\cap c(t\geq 0)$ sinfga tegishli, $t>0$ yarim fazoda (2) tenglamani va

$$u\big|_{t=+0}=u_0(x) \tag{5}$$

boshlang'ich shartlarni qanoatlantiruvchi funksiya topilsin.

Keltirilgan Koshi masalasini umumlashtirish mumkin. Shu maqsadda $x_1, x_2, \ldots, x_n$ o'zgaruvchili ikkinchi tartibli ushbu kvazichiziqli differensial tenglamani tekshiramiz:

$$\sum_{i,j=1}^{n} A_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\hat{O}\left(x,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n}\right)=0 \tag{6}$$

Yetarli siliq S : $\omega(x_1, x_2, \ldots, x_n)$ sirt va bu sirtga urunma bo'lmagan, uning har bir nuqtasida biror ℓ yo'nalish berilgan bo'lsin.

Koshi masalasi: S sirtning biror atrofida (6) tenglamani va

$$u\big|_s=u_0(x)\ ,\ \frac{\partial u}{\partial e}\Big|_s=u_1(x) \tag{7}$$

Koshi shartlarini qanoatlantiruvchi $u(x)$ funksiya topilsin. Bu umumlashtirilgan Koshi masalasidir.

Koshi masalasi qo'yilishida S sirtni xarakteristik sirt bo'lmasligi muhimdir. Agar S sirt xarakteristik sirt bo'lsa, boshlang'ich shartlarda verilgan $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalar o'zaro bog'langan bo'lib qoladi. Demak xarakteristik sirtda boshlang'ich shartlarni ixtiyoriy berilishi mumkin emas. Bu holda Koshi masalasi umuman echimga ega bo'lsa ham u yagona bo'lmaydi.

Misol: Ushbu

$$\frac{\partial^2 u}{\partial x \partial y}=0 \tag{8}$$

tenglamaning

$$u(x)\big|_{y=0}=\varphi_0(x)\ ,\ \frac{\partial u}{\partial y}\Big|_{y=0}=\varphi_1(x) \tag{9}$$

boshlang'ich shrtlarni qanoatlantruvchi echimi topilsin.

Ravshanki, $x=const, y=const$ to'g'ri chiziqlar oilasi, jumladan $y=0$ ham berilgan tenglamaning xarakteristikalardan iborat. Demak boshlang'ich shartlar xarakteristikada berilyapti tekshirilayotgan tenglamaning umumiy echimi

$$u(x,y)=f_1(x)+f_2(x) \tag{10}$$

dan iborat. Umumiylikka ziton etkazmay $f_2(0)=0$ deb hisoblashimiz mumkin.

Boshlang‘ich shartlarga asosan

$$u(x)\Big|_{y=+0}=f_1(x)=\varphi_0(x);\quad \frac{\partial u}{\partial y}\Big|_{y=+0}=f_2'(y)\Big|_{y=0}=\varphi_1(x)$$

Agar $\varphi_1(x)\neq const$ bo‘lsa oxirgi tenglikning bajarilishi mumkin amas, bu holda Koshi masalasi echimga ega bo‘lmaydi.

Shunday qilib, $\varphi_1(x)=const=a$ bo’l gandagina Koshi masalasi echimga ega bo‘ladi. Bu holda

$$f_2(y)=ay+c(y)$$

bu erda $c(y)\in c^2(y\geq 0)$ sinifga tegishli va $c(0)=c^2(0)=0$ shartlarni qanoat-lantiruvchi funksiya.

Agar $\varphi_0(x)\in c^2$ bo’lsa, Koshi masalasining echimi mavjud bo‘lib, u echim

$$u(x,y)=\varphi_0(y)=ay+c(y) \tag{11}$$

formula bilan aniqlanadi lekin echim yagona emas.

3. Koshi Kovalevskaya teoremasi.

N ta noma’lumli $u_1,u_2,\ldots,u_N$ funksiyasi

$$\frac{\partial^{k_i}u_i}{\partial t^{k_i}}=\hat{O}_i(x,t,u_1,u_2,\ldots,u_N,D_t^{\alpha_0}D_x^{\alpha}u_i\ldots) \tag{12}$$

differensial tenglamalar sistemasini qaraymiz $i=\overline{1,N},\ \alpha_0\leq k_i-1\ \ \alpha=\alpha_1+\alpha_2+\ldots+\alpha_n\leq k_i$. Bu holad (12) tenglamalar sistemasi t o’zgaruvchiga nisbatan normal sistema deyiladi.

Agar $f(x),\ x=(x_1,x_2,\ldots,x_n)$ funksiya, α_0 nuqtaning biror atrofida tekis yaqinlashuvchi

$$f(x)=\sum_{\alpha\geq 0}c_\alpha(x-x_0)^\alpha=\sum_{\alpha\geq 0}\frac{D^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha$$

$c_\alpha=c_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, $(x-x_0)^\alpha=(x_1-x_{01})^{\alpha_1}\ldots(x_n-x_{0n})^{\alpha_n}$ darajali qator bilan ifodalansa y x_0 nuqtada analitik funksiya deyiladi.

Agar $f(x)$ funksiya G sohaning har bir nuqtasida analitik bo‘lsa G sohada analitik deyiladi.

t ga nisbatan normal sistema uchun Koshi masalasi bunday qo‘yiladi: (12) sistemaning $t=t_0$ da ushbu.

$$\frac{\partial^k u_i}{\partial t^k}=\varphi_{ik}(x),\quad k=0,1,..k-1,\quad i=1,....,N \tag{13}$$

boshlang‘ich shartlarni qanoatlantiruvchi $u_1,u_2,\ldots,u_N$ echim topilsin. Bu erda $\varphi_{ik}(x)$ -biror $G\in R^n$ sohada berilgan funksiyalar.

Berilgan (13) boshlang‘ich shartlarga asosan φ_j funksiyalarida ishtirok etayotgan barcha hosilalarni hisoblash mumkin.

$$D_x^\alpha u_j(x,t)\Big|_{\substack{t=t_0\\x=x_0}}=D^\alpha\varphi_{j0}(x_0),\quad D_t^{\alpha_0}D_x^\alpha u_j(x,t)\Big|_{\substack{t=t_0\\x=x_0}}=D^\alpha\varphi_{j\alpha_0}(x_0)$$

Koshi-Kovalevskaya teoremasi:

Agar barcha $\varphi_{ik}(x)$ funksiyalan x_0 nuqtaning biror atrofida analitik, $\hat{O}_i(x,t,...,u_{j\alpha_0,\alpha_1,...,})$ funksiya esa $(x_0,t_0,...D^\alpha\varphi_{i\alpha_0}(x_0)...)$ nuqtaning biror atrofida analitik bo‘lsa, u holda (12), (13) Koshi masalasi (x_0,t_0) nuqtaning biror atrofida analitik achimga ega bo‘ladi, shu bilan bu echim analitik funksiyalar sinifida yagona bo‘ladi. Bu teorema analitik funksiyalar sinfida Koshi masalasining echimi etarli kichik sohada mavjud va yagona ekanligini tasdiqlaydi.

4. Elliptik tipdagi tenglamalar uchun chegaraviy masalalar.

$$-div(p\,grad u)+qu=F(x) \qquad (14)$$

(14) tenglama (elliptik tip) uchun chegaraviy masala bunday qo'yiladi; G sohada (14) tenglamani va S chegarada quyidagi shartlardan bittasini qanoatlantiruvchi $u(x)$ funksiya topilsin:

I. $u\big|_S=\varphi_0$.

II. $\dfrac{\partial u}{\partial n}\Big|_S=\varphi_1$.

III. $\alpha u+\beta\dfrac{\partial u}{\partial n}\Big|_S=\varphi_2$

Bu yerda $n-S$ sirtga o'tkazilgan tashqi normal , φ_1, φ_2, α va $\beta-S$ da berilgan uzluksiz funksiyalar.

Masalalarni qo'yishdan darhol shu narsa ma'lumki I holda $u(x)$ funksiya $C^2(G)\cap C(\overline{G})$ sinfga, II, III hollarda esa $C^2(G)\cap C^1(\overline{G})$ sinfga tegishli bo'lishi kerak.

Bu masalardan I ni birinchi chegaraviy masalasi yoki Dirixle masalasi, II ni ikkinchi chegaraviy masala yoki Neyman masalasi, III ni esa uchinchi chegaraviy masala deyiladi.

Yuqorida keltirilgan masalalardan no'malum $u(x)$ funksiya G sohada izlangani uchun ularni mos ravishda ichki masalalar deb yuritiladi. Xuddi shunga o'xshash, chegaralangan G sohaning tashqarisida (tashqi masalalar) chegaraviy masalalar qo'yiladi. Bularning farqi shundaki, S dagi chegaraviy shartlardan tashqari, soha cheksiz bo'lgani uchun, cheksiz uzoqlashgan nuqtada ham shart beriladi.

Masalan, bunday shartlar Laplas tenglamasi uchun $|x|\to\infty$ da

$$u(x)=o(1) \text{ yoki } u(x)=o(1)$$

ko'rinishda bo'lishi mumkin.

Yuqoridagilarga o'xshash G sohada berilgan umumiy ikkinchi tartibli chiziqli

$$\sum_{i,j=1}^{n}A_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}B_1\frac{\partial u}{\partial x_i}+C(x)u=f(x) \qquad (15)$$

tenglama uchun chegaraviy masalalar qo'yiladi.

G sohada (15) tenglamaning

$$\sum_{i,j=1}^{n}\alpha_i(x)\frac{\partial u(x)}{\partial x_i}+\beta(x)u(x)=\gamma(x),\ x\in S \qquad (16)$$

shartni qanoatlantiruvchi regulyar yechimi topilsin.

Bu yerda $a_i(x),\ i=1,.....,n,\ \beta(x)$ va $\gamma(x)-S$ da berilgan funksiyalar; $\dfrac{\partial u(x)}{\partial x_i}$ va $u(x)$ deganda x nuqta G sohaning ichidan S nuqtasiga intilgandagi bu funksiyalarning limit qiymatlari tushuniladi. (15), (16) masala Puankere masalasi deyiladi.

Barcha S da $\alpha_i(x)=0$, $i=1$,....., n, $\beta(x)\neq 0$ bo'lgan holda (15) chegaraviy shartni

$$u(x)\Big|_S=\gamma_1(x),\ \gamma_1(x)=\frac{\gamma(x)}{\beta(x)} \qquad (17)$$

ko'rinishda yozib olish mumkin. Bu masala birinchi chegaraviy masala yoki Dirixle masalasi deyiladi.

S da $\beta(x)=0$ bo'lganda, Puankare masalasining hususiy holi

$$\sum_{i=1}^{n}\alpha_i(x)\frac{\partial u}{\partial x_i}=\gamma(x),x\in S \qquad (18)$$

masala hosil bo'adi. (15), (18) masala qiya hosilali masala deyiladi.

S sirtning ξ nuqtasidagi yo'naltiruvchi kosinuslari

$$\cos(N,\xi_i) = \frac{1}{a}\sum_{j=1}^{n} A_{ij}\cos(n,\xi_i),\ \ i = 1,\dots,n$$

bo'lgan birlik vektorni – konormalni N orqali belgilaymiz. Bu yerda $N - S$ sirtga ξ nuqtada o'tkazilgan tashqi normal,

$$a^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{ij}\cos(n,\xi_i)\right)^2$$

Agar (18) chegaraviy shartda barcha S da

$$a_i(x) = \cos(n, x_i)$$

bo'lsa qiya xosilali masala ikkinchi chegaraviy masala yoki Neyman masalasi deyiladi.

Aralash masala. Tebranishlar tenglamasi (giperbolik tip), yani

$$\rho\frac{\partial^2 u}{\partial t^2} = div(p\, gradu) - qu + F(x,t) \tag{19}$$

(19) tenglama uchun aralash masala bunday qo'yiladi:

$C^2(\ddot{O}_T)\cap\overline{C}\,^1(\ddot{O}_T)$ sinfga tegishli, $\ddot{O}_T$ silindrda (19) tenglamani, $t = 0,\ x \in \overline{G}$ ($\ddot{O}_T$ silidrning quyi asosi) da

$$u\big|_{t=+0} = u_0(x)\ \ ,\ \ \ \frac{\partial u}{\partial t}\bigg|_{t=+0} = u_1(x)$$

boshlang'ich shartlarni va $0 \le t \le T$ ($\ddot{O}_T$ ning yon sirti)da I, II yoki III chegaraviy shartlardan bittasini qanoatlantiruvchi $u(x,t)$ yechimi topilsin.

Boshqa masalalar. Ikkinchi tartibli ikki o'zgaruvchili kanonik ko'rinishga keltirilgan ushbu

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y) \tag{20}$$

umumiy chiziqli tenglama berilgan bo'lsin . Bu tenglamaning xarakteristikalari tenglamasi

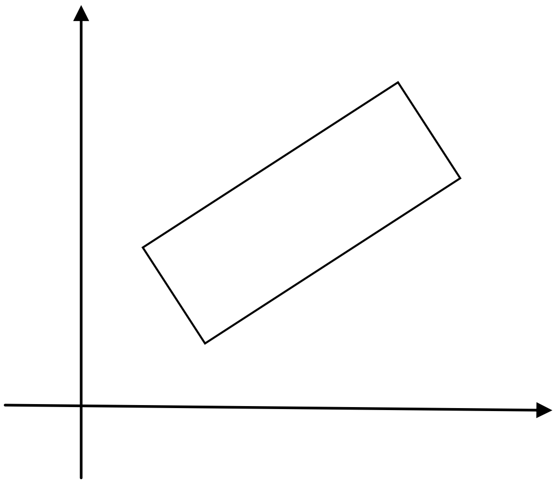
$$dy^2 - dx^2 = 0$$

dan iborat. Bundan darxol $x - y = const,\ x + y = const$ to'gri chiziqlar oilasi kelib chiqadi .

Uchlari A, B, C va D nuqtalarda, tomonlari (20) tenglamaning xarakteritikalaridan iborat bo'lgan to'rtburchakni G orgali belgilab olamiz. Odatda bu to'rtburchak xarakteristik to'rtburchak deyiladi (1-chizma).

Gursa masalasi. G to'rtburchakda regulyar, $\overline{G}$ da uzluksiz va

$$u\big|_{AB} = \varphi\ \ ,\ \ u\big|_{AD} = \psi$$



shartlani qanoatlantiruvchi (20) tenglamaning $u(x,y)$ yechimi topilsin.

Masalaning qo'yilishiga asosan, φ va ψ funksiyalar berilgan sohasida uzluksiz va $\varphi(A) = \psi(A)$ shart bajalishi zarur. Demak, Gursa masalasidas (20) tenglamaning ikkita kesishadigan xarakteristikalarida bitta chegaraviy shart beriladi.

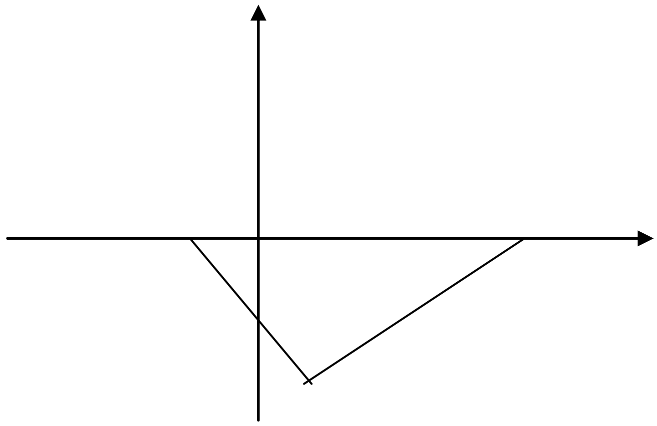
Gursa masalasida shartlar xarakteristikalarda berilgani uchun bu masala xarkteristik masala deb ham yuritiladi.

Endi G orqali $y=0$ o'qining ixtiyoriy $A(x_1,0)$ $B(x_2,0)$ kesmasi va (20) tenglamaning $AC: x+y=-x_1,\ BC: x-y=x_2$ xarakteristikalri bilan chegaralangan uchburchakni belgilaymiz.

Bu uchburchak xarakteristik uchburchak deyiladi (2-chizma). Darbu (Koshi-Gursa) ning birinchi masalasi. G da regulyar, $\overline{G}$ da uzluksiz va

$$u(x,0)=\tau(x)\ ,\quad x_1\le x\le x_2\ ,$$

$$u\big|_{AC}=\psi(x)\ ,\quad x_2\le x\le \frac{x_1+x_2}{2}$$



shartlarni qanoatlantiruvchi (20) tenglamaning yechimii topilsin, bunda $\tau(x)$ va $\psi(x)$ berilgan funksiyalar, shu bilan birga $\tau(x_1)=\psi(x_1)$.

Darbu (Koshi-Gursa) ning ikkinchi maslasi. G da regulyar, $\overline{G}$ da uzluksiz, AB kesmagacha birinchi tartibli hosilalarga ega bo'lsin va

$$\lim_{y\to 0} u_y(x,y)=\nu(x)\quad x_1<x<x_2$$

$$u\big|_{AC}=\psi(x)\ ,\quad x_1\le x\le \frac{x_1+x_2}{2}$$

shartlarni qanoatlantiruvchi (20) tenglamaning yechimi topilsin, bunda $\nu(x)\in C^2(x_1,x_2),\ \psi(x)\in C^2\left[x_1,\frac{x_1+x_2}{2}\right]$

Aralash tipga tegishli

$$\sin g\, y|y|^m u_{xx}+u_{yy}=0\ ,\ m=const>0 \qquad (21)$$

Tenglamani tekshiramiz, $m=1$ bo'lganda bu tenglama Trikomi tenglamasi bilan ustma-ust tushadi, $m=0$ bo'lganda esa (21) tenglama Lavrent'ev-Bidsadze tenglamasi deyiladi.

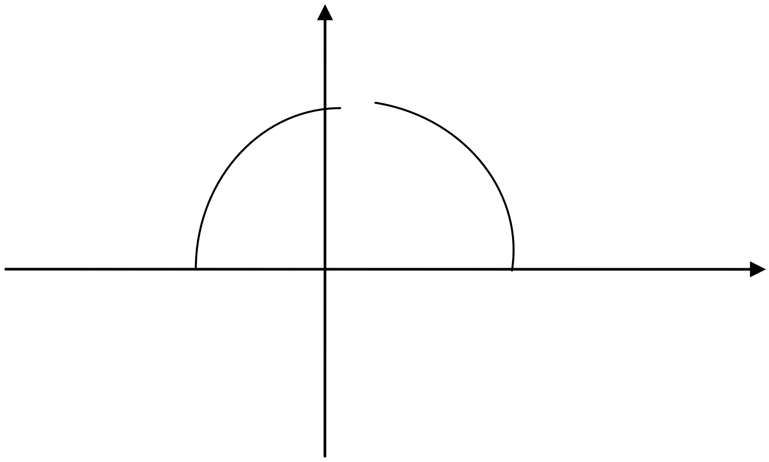
Aralash tipdagi tenglama berilgan soha aralash soha deb yuritiladi.

$G-x,y$ o'zgaruvchilar tekisligida $y>0$ bo'lganda uchlari $A(x_1,0)\ va\ B(x_2,0),\ x_1<x_2$, nuqtalarda bo'lgan Jordan egri chizigi σ bilan $y<0$ da esa (8) tenglamaning

$$AC{:}\,x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=x_1\ ,\quad BC{:}\,x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=x_2$$

xarakteristikalari bilan chegaralangan bir bog'lamli aralash soha bo'lsin (3-chizma)

Trikomi masalasi. G sohada regulyar, $C(\overline{G})\cap G^1(G)$ sinfga tegishli, σ egri chizqda va AC yoki BC xarakteristikalardan bittasida, masalan AC da berilgan qiymatlarni qabul qiluvchi yani



$$u\big|_{\sigma} = \varphi \quad , \quad u\big|_{AC} = \psi$$

shartlarni qanoatlantiruvchi (8) tenglamaning echimi topilsin.

Shu bilan birga $\varphi \in C(\sigma)$, $\psi \in C(AC)$ va $\varphi(x_1) = \psi(x_1)$ bo'lishi zarur. Trikomi masalasi – masala deb yuritiladi.

Korrekt qo'yilgan masala tushunchasi. Biz yuqorida ko'rdikki, matematik fizika masalarining qo'yilishida ayrim funksiyalar (boshlang'ich, chegraraviy shartlar) ishtirok etadi: qo'yilgan masalaning yechimi tabiiy, shu funksiyalarga bog'liq bo'ladi. Bu funksiyalar, odatda, tajriba asosida aniqlanadi, shuning uchun ham ularni absolyut aniq topish mumkin emas.

Demak boshlangich va chegaraviy shartlarda hamma vaqt biror xatolikning bo'lishi muqarrardir.Bu xatolik o'z navbatida yechimga ham ta'sir qiladi. Boshlang'ich va chegaraviy masalalarni tekshirishda, yechimning mavjudligi va yagonaligidan tashqari boshlang'ich va chegaraviy shartlarda qo'yilgan xatolikning yechimga qanday ta'sir qilishini aniqlash ham muhim ahamyatga egadir.

Bu fikrni aniqroq bayon qilish uchun tekshiralayotgan masalani – M orqalai belgilab olamiz. Har qanday M masalaning mohiyati berilgan $\varphi \in E_{\varphi}$ funksiyalarga asosan $u \in E_u$ yechimni topishdan iboratdir, bu yerda, E_u va E_{φ}- metrikalari ρ_u va ρ_{φ} bo'lgan qandaydir metrik fazolar. Bu fazolar masalaning qo'yilishi bilan aniqlanadi. M masalaning yechimi tushunchasi aniqlangan bo'lib, har bir $\varphi \in E_{\varphi}$ elemenga yagona $u = R(\varphi) \in E_u$ yechim mos kelsin.

Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta(\varepsilon) > 0$ sonni ko'rsatish mumkin bo'lib, $\rho_{\varphi}(\varphi_1, \varphi_2) \le \delta(\varepsilon)$ tengsizlikdan $\rho_u(u_1, u_2) \le \varepsilon$ tengsizlik kelib chiqsa, M masala (E_u, E_{φ}) fazolar juftida turg'un masala deyiladi.

Bunda $u_i = R(\varphi_i)$, $u_i \in E_u$, $\varphi_i \in E_{\varphi}$, $i = 1,2,....$ masalaning yechimi berilgan shartlar (boshlang'ich va chegaraviy shartlar, tenglamaning koeffitsentlari, ozod had va h.k) ga uzluksiz bog'liq bo'ladi.

Agar tekshirilayotgan M masala uchun ushbu

1) ixtiyoriy $\varphi \in E_{\varphi}$ uchun $u \in E_u$ yechim mavjud;

2) u yechim yagona;

3) masala (E_u, E_{φ}) fazolar juftligida turg'un shartlar bajarilsa, M masala (E_u, E_{φ}) fazolar juftligida korrekt (to'g'ri) qo'yilgan yoki to'g'ridan-to'g'ri korrekt masala deyiladi.

Aks holda masala korrekt qo'yilmagan masala deyiladi, yani bu holda yuqoridagi talablardan kamida bittasi bajarilmaydi.

Shu narsani ta'kidlab o'tamizki, korrekt qo'yilagan masala ta'rifi berilgan (E_u, E_{φ}) juftlikka taaluqlidir, chunki boshqa metrikalarda shu masalning o'zi korrekt qo'yilgan bo'lishi ham mumkin.

Korrekt qo'yilmagan masalalarga misollar. Koshi-Kovalevskaya teoremasi, uning umumiylik tavsifiga ega ekanligiga qaramasdan differensial tenglamaning normal sistemasi uchun Koshi masalasi korrekt qo'yilganligini to'la hal qilmaydi.

Haqiqatdan ham, bu torema masala yechimining yetarli kichik sohada mavjud va yagonaligini ta'min etadi. Odatda, bu dalillarni avalldan berilgan (umuman kichik bo'lmagan) sohalarda to'g'ri ekanligini ko'rsatish talab qilinadi. Bundan tashqari, tenglamaning ozod hadi va boshlang'ich shartlari, umuman olganda, analitik bo'lmagan funksiyalar bo'ladi, Nihoyat, yechim boshlang'ich shartlarga uzluksiz bog'liq bo'lmasligi ham mumkin. Bu dalilni ko'rsatuvchi misol birinchi marta Adamar tomonidan tuzilgan.

1) **Adamar misoli.** Ushbu

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Laplas tenglamasining $y>0$ yarim tekislikda

$$u(x,0)=0\ ,\ \ u_y(x,0)=e^{-\sqrt{k}}\cos kx$$

boshlang'ich shartlarni qanoatlantiruvchi regulyar yechimi topilsin. Tekshirib ko'rish qiyin emaski, bu masalaning birdan-bir yechimi

$$u(x,y)=\frac{1}{k}e^{-\sqrt{k}}\cos kx shky\quad ,\ k=1,2,....$$

ko'rinishda bo'ladi.

Ko'rinib turibdiki, agar $k\to+\infty$, $e^{-\sqrt{k}}\cos kx$ funksiya nolga tekis intiladi, yani $e^{-\sqrt{k}}\cos kx=0$, lekin

$u(x,y)$ yechim noldan farqli ixtiyoriy y da

$$u(x,y)=\frac{1}{k}e^{-\sqrt{k}}\cos kx\cdot shky-\not\to 0\quad ,\ k\to+\infty$$

bo'ladi.

Shunday qilib, Laplas tenglamasi uchun Koshi masalasi korrekt qo'yilmasgan masala ekan.

2) Ushbu

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

issiqlik o'tkazuvchanlik tenglamasining $t<0$, $0\le x\le\pi$ sohada

$$u(x,0)=\frac{1}{k}e^{-\sqrt{k}}\sin kx,\quad u(0,t)=u(\pi,t)=0$$

shartlarni qanoatlantiruvchi yechimi topilsin.

Bu masalaning yechimi

$$u(x,t)=e^{-\sqrt{k}}e^{-k^2t}\sin kx$$

fuksiyadan iboratdir.

Tabiiy har bir k uchun o'zining boshlang'ich sharti va unga mos bo'lmagan yechimi bor. Shu sababli ham yozilgan yechimni yechimlar ketma-ketligi deb qarash kerak.

$e^{-\sqrt{k}}\sin kx$ funksiya $k\to\infty$ da nolga intiladi, lekin yechim esa huddi Adamar misolidek hech qanday limitga intilmaydi.

3) Tor tebranish tenglamasi

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$$

uchun Direxli masalasini tekshiramaiz

$0<x<\pi$, $0<y<\alpha\pi$ to'rtburchakda,

$$u(0,y)=0,\ u(\pi,y)=0,\ \ 0\le y\le\alpha\pi$$

$$u(x,0)=0,\ u(x,\alpha\pi)=\frac{\sin kx}{\sqrt{k}},\quad 0\le x\le\pi$$

shartlarni qanoatlantiruchi tor tebranish tenglamasining yechimi topilsin. Bu yerda α - musbat irratsional son.

Bu masalaning yechimi

$$u(x,y)=\frac{1}{\sqrt{k}}\frac{\sin kx\sin ky}{\sin k\alpha\pi},\quad k=1,2,...$$

formula bilan aniqlanadi.

Ravshanki $k\to\infty$ da chegaraviy shartdagi $\dfrac{\sin kx}{\sqrt{k}}$ funksiya nolga intiladi. Sonlar nazaryasidan ma'lumki, shunday p_k va q_k butun sonlar ketma – ketligi mavjud bo'lib, har qanday α irratsional son uchun

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q^2{}_k}$$

tengsizliklardan o'rinli bo'ladi.

Bunga asosan

$$\left|\sin q_k \alpha \pi\right| = \left|\sin(q_k \alpha - p_k)\pi\right| < \frac{\pi}{q_k}$$

tengsizlik hosil qilamaiz.

Bu holda, tekshirilayotgan masalaning yechimi $u(x, y)$ funksiya uchun

$$|u(x, y)| = \left|\frac{1}{\sqrt{q_k}} \frac{\sin q_k x \sin q_k y}{\sin q_k \alpha \pi}\right| > \frac{\sqrt{q_k}}{\pi}\left|\sin q_k x \sin q_k y\right|$$

tengsizlikga ega bo'lamiz.

Bundan tor tebranish tenglamasi uchun Direxle masalasi korrekt qo'yilmagan masala emas ekanligi kelib chiqadi.

Nazorat savollari.

1. Xususiy hosilali differensial tenglamaning umumiy echimini tuzilishi qanday bo'ladi.
2. Koshi masalasi qanday tenglamalar uchun qo'yiladi.
3. Chegaraviy masala qanday tenglama uchun qo'yiladi.
4. Aralash masala qanday tenglama uchun qo'yiladi
5. Boshlang'ich shartlarni xarakteristikalarda berish mumkinmi.
6. Normal sistema deb qanday differensial tengalmalar sistemasiga aytiladi.
7. Koshi-Kovalevskaya teoremasini ta'riflang.
8. Elliptik tipdagi tenglamalar uchun asosiy chegaraviy masalalarni ta'riflang.
9. Korrekt qo'yilgan masala ta'rifini bering.

1. Tor tebranish tenglamasi uchun Dirixle maslasi korrekt emasligini ko'rsating.

10-mavzu. Cheksiz tor uchun Dalamber formulasi

Reja:

1. Bir jinsli tor tebranishi tenglamasi uchun Koshi masalasi.
2. Koshi masalasining yechimini Dalamber usulida topish.
3. Xulosa.

Ikki uchi mahkamlangan yoki bir uchi mahkamlangan tor tebranish haqidagi masalani yechishdan oldin osonroq bo'lgan masalani ya'ni cheksiz tor tebranishi haqidagi masalani ko'raylik.

Quyidagi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

birjinsli tor tebranish tenglamasining

$$\left.\begin{aligned} u(x,t)\big|_{t=0} &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= F(x) \end{aligned}\right\} \qquad (2)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topish kerak. Bundagi *f(x), F(x)* lar (-∞,∞) oraliqda berilgan funksiyalardir. Noma'lum *u(x,t)* funksiyaga hech qanday chegaraviy shart qo'yilmagan. (1) tenglamaning (2) boshlang'ich shartlarni qanoatlantiruvchi yechimini topish masalasi Koshi masalasi deyiladi. Bu masalani yechish usuli Dalamber usuli deyiladi. (1) ning umumiy yechimi

$$u(x,t) = \varphi(x - at) + \psi(x + at) \qquad (3)$$

bo'lishligini ko'rsatamiz. Bunda φ, ψ funksiyalar ikki marta differensiallanuvchi funksiyalardir.

Haqiqatan ham

$$u'_x = \varphi'(x-at)+\psi'(x+at), u''_{xx} = \varphi'''(x-at)+\psi'''(x+at)$$
$$u'_t = -a\varphi'(x-at)+a\psi'(x+at), u''_{tt} = a^2\varphi'''(x-at)+a^2\psi'''(x+at)$$
$$u''_{tt} = a^2\varphi''' + a^2\psi''' = a^2(\varphi''' + \psi''') = a^2 u''_{xx}$$
$$u''_{tt} = a^2 u''_{xx}$$
tenglik o'rinli bo'ladi.

Demak, (3) (1)ning umumiy yechimi bo'ladi.

Endi (2) boshlang'ich shartlardan foydalanib, noma'lum φ, ψ funksiyalarni topamiz. (2), (3) dan:
$t=0$ bo'lganda $\varphi(x)+\psi(x)=f(x)$ (4)
hosil bo'ladi.

$$u'_t = -a\varphi'(x-at)+a\psi'(x+at)$$
$$u'_t\big|_{t=0} = -a\varphi'(x-at)+a\psi'(x+at) = F(x)$$
(5)

(5) ni 0 dan x gacha integrallasak,

$$-a\int_0^x \varphi'(x)dx + a\int_0^x \psi'(x)dx = \int_0^x F(x)dx$$
$$-a(\varphi(x)-\varphi(0)) + a(\psi(x)-\psi(0)) = \int_0^x F(x)dx$$
$$-\varphi(x)+\psi(x) = \frac{1}{a}\int_0^x F(x)dx + c \qquad (6)$$
$$c = -\varphi(0)+\psi(0)$$

(4) va (6) dan:

$$\left.\begin{aligned} \varphi(x)+\psi(x) &= f(x) \\ -\varphi(x)+\psi(x) &= \frac{1}{a}\int_0^x F(x)dx \end{aligned}\right\} \to \left.\begin{aligned} \varphi(x) &= \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x F(x)dx \\ \psi(x) &= \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x F(x)dx \end{aligned}\right\} \qquad (7)$$

(7) dagi x o'rniga x-at va x+at qo'ysak,

$$\varphi(x-at) = \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} F(x)dx$$
$$\psi(x+at) = \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} F(x)dx$$

(3) ga asosan

$$u(x,t) = \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} F(x)dx + \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} F(x)dx$$

Ma'lumki, $-\int_0^{x-at} F(x)dx + \int_0^{x+at} F(x)dx = \int_{x-at}^{0} F(x)dx + \int_0^{x+at} F(x)dx = \int_{x-at}^{x+at} F(x)dx$

u holda $u(x,t) = \dfrac{f(x-at)+f(x+at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} F(x)dx.$

(8)

(8) formula tor tebranishi tenglamasi uchun Koshi masalasining Dalamber yechimi deyiladi.

Misollar: 1) $u''_{tt} = u''_{xx}$ tenglamaning

$u\big|_{t=0} = x^2, u'_t\big|_{t=0} = 0$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

$$a = 1, f(x) = x^2, F(x) = 0.$$

$$u(x,t) = \frac{(x-t)^2 + (x+t)^2}{2} = x^2 + t^2, u(x,t) = x^2 + t^2$$

2) $u''_{tt} = 9u''_{xx}$

$$u\big|_{t=0} = 0, u'_t\big|_{t=0} = e^x$$

$$a = 3, f(x) = x^2, F(x) = e^x$$

$$u(x,t) = \frac{1}{6}\int_{x-3t}^{x+3t} e^x dz = \frac{1}{6}e^x\Big|_{x-3t}^{x+3t} = \frac{e^x}{6}\left(e^{3t} - e^{-3t}\right)$$

3) $u''_{tt} = a^2 u''_{xx}$ tenglamaning

$u\big|_{t=0} = \sin x, u'_t\big|_{t=0} = \cos x$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

$$u(x,t) = \sin x \cos at + \frac{1}{a}\cos x \sin at.$$

Endi (3) yechimning fizik ma'nosini aniqlashga to'xtalamiz.

$$u(x,t) = \varphi(x-at) + \psi(x+at).$$

umumiy ifodadagi funksiyalarni alohida qaraymiz.

Avval $\varphi(x-at)$ funksiyani olib, uni t ning o'suvchi $t=t_0$, $t=t_1$, $t=t_2$,... qiymatlariuchun grafiklarni chizamiz. Ikkinchi grafik birinchi grafikka nisbatan at_1 miqdorga o'ng tomonga qarab siljigan, uchinchisi esa at_2 miqdorga o'ng tomonga qarab siljigan va h.k. Agar bu chizmalarni ketma-ket qo'zg'almas ekranga proyektsiyalasak, birinchi grafik o'ng tomonga qarab yugurayotganini ko'ramiz. Endi (3) dagi ikkinchi qo'shiluvchi $\varphi(x+at)$ ni qaraylik. Xuddi yuqoridagiday mulohaza yurgizsak, bu funksiya ham tezlik bilan chap tomonga qarab tarqaladi. $\varphi(x-at)$, $\psi(x+at)$ funksiyalar bilan tavsiflanuvchi to'lqinlar yuguruvchi to'lqinlar deyiladi. Shunday qilib (1) tenglamaning umumiy yechimi $\varphi(x-at)$ va $\psi(x+at)$ to'lqinlarning yig'indisidan (superpozitsiyasidan) iborat ekan.

(8) yechimini tekshiraylik.

F(x)=0 bo'lsin. Bu holda tor nuqtalarning boshlang'ich tezligi nolga teng bo'lib, tor faqatgina boshlang'ich siljitish natijasida tebranma harakat qiladi. Bu holda

$u(x,t) = \frac{1}{2a}f(x-at) + \frac{1}{2a}f(x+at)$ ga ega bo'lamiz.

f(x) funksiya aniq bo'lgani uchun *u(x,t)* ning qiymatini har qanday *x* va *t* lar uchun hisoblash mumkin.

Xuddi yuqorida ko'rganimiz kabi bu yerda ham *u(x,t)* funksiya ikkita $\frac{1}{2}f(x-at)$ va $\frac{1}{2}f(x+at)$ to'lqinlarning superpozitsiyasidan iborat. Birinchi to'lqin a tezlik bilan o'ng tomonga qarab tarqaladi va u to'g'ri to'lqin deyiladi. $t=0$ da esa ikkala to'lqinlarning profili ustma-ust tushadi. Agar $f(x)=0$ bo'lib, $F(x)\neq 0$ bo'lsa

$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} F(z)dz$ bo'ladi.

$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} F(z)dz$ deb faraz qilsak, u holda

$u(x,t) = \frac{1}{2a}f(x-at) + \frac{1}{2a}f(x+at)$ bo'ladi. Bu holda ham tor bo'ylab to'g'ri $u(x,t) = \phi(x+at) + \phi(x-at)$ va teskari $u_1 = -\phi(x-at)$ to'lqinlar tarqaladi.

Tor harakatini tasavvur qilish uchun

$$F(x)=0,\ x\notin(-l,l)$$
$$F(x)=V_0,\ x\in(-l,l)$$

deb olamiz.

$$\phi(x)=\frac{1}{2a}\int_0^x V_o dx=\frac{V_o x}{2a},\ x\in[-l,l]$$

Bu holda

$$\phi(x)=\frac{1}{2a}\int_0^l V_o dx=\frac{V_o l}{2a}=\frac{h}{2},\ x\in(l,\infty)$$

$$\phi(x)=\frac{1}{2a}\int_0^{-l} V_0 dx=-\frac{V_0 l}{2a}=-\frac{h}{2},\ x\in(-\infty,-l),\ h=\frac{V_0}{a}$$

Bundan ko'rinadiki, $\phi(x)$ toq funksiyadir.

Nazorat savollari.
1. Cheksiz tor uchun Koshi masalasi qanday qo'yiladi?
2. Dalamber usuli nimadan iborat?
3. Dalamber formulasi yordamida misollar yeching.

11-mavzu. Chekli tor uchun Furye metodi

Reja

1. O'zgaruvchilarni ajratish metodi yoki Fure metodi haqida.
2. Fure metodi yordamida ikki uchi mahkamlangan torning erkin tebranish tenglamasini yechish.
3. Xulosa.

Matematik fizika tenglamalarini yechishda ko'p qo'llaniladigan metodlardan biri o'zgaruvchilarni ajratish metodi yoki Fere metodidir. Bu metod yordamida ikki uchi biriktirilgan torning erkin tebranish tenglamasini echishni ko'raylik.

Bizga

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

tenglamaning

$$\left.\begin{aligned} u(x,t)\big|_{t=0}&=f(x)\\ \frac{\partial u}{\partial t}\bigg|_{t=0}&=F(x)\end{aligned}\right\} \qquad (2)$$

boshlang'ich shartlarni va

$$u(0,t)=u(l,t)=0$$

chegaraviy shartlarni qanoatlantiruvchi yechimini topish kerak. (1)ning yechimini biri faqat *t* ning funksiyasi 2-si faqat *x* ning funksiyasi bo'lgan ikkita funksiya ko'paytmasi shaklida izlaymiz, ya'ni

$$u(x,t)=X(x)T(t)$$

(4) ni (1) ga qo'ysak,

$\dfrac{T''(t)}{a^2T(t)}=\dfrac{X''(x)}{X(x)}$ ga ega bo'lamiz

Bu tenglamaning chap tomonini faqat *t* ga, o'ng tomoni faqat *x* ga bog'liq bo'lgani uchun, bu nisbatlar o'zgarmas songa teng bo'lishi kerak, ya'ni

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=\mu$$

Bundan

$$T''(t) - \mu a^2 T(t) = 0 \qquad (5)$$
$$X''(x) - \mu X(x) = 0 \qquad (6)$$
bir jinsli oddiy 2-tartibli differensial tenglamaga ega bo'lamiz. (1) tenglamaning (3) chegaraviy shartlarni qanoatlantiruvchi xususiy yechimi *t* ning ixtiyoriy qiymatida

$$\left.\begin{aligned} u(x,t)\big|_{x=0} &= T(t)X(0) = 0 \\ u(x,t)\big|_{x=l} &= T(t)X(l) = 0 \end{aligned}\right\}$$

bo'ladi. Bundan *T(t)=0* desak, *u(x,t)=0* ya'ni (1) tenglama trivial yechimga *(u=0)* ega bo'lamiz. Bizga travial bo'lmagan yechim kerak, bu holda

$$\left.\begin{aligned} X(0) &= 0 \\ X(l) &= 0 \end{aligned}\right\}$$

deb olamiz.

Demak, *X(x)* funksiyani topish uchun quyidagi masalani yechishga kelib qoldik.

$X''(x) - \mu X(x) = 0$ tenglamaning $X(0) = X(l) = 0$ shartni qanoatlantiruvchi yechimini topish kerak.

Echimni $X(x) = e^{rx}$ ko'rinishda izlaymiz va $r^2 - \mu = 0$ xarakteristik tenglamaga ega bo'lamiz.

$$r = \pm\sqrt{\mu} = \pm\lambda, \text{ (} \sqrt{\mu} = \lambda \text{ deb belgiladik)}$$

Bu yerda bir necha hollar bor.

1) $\mu > 0$ bo'lsin, bu holda yechim $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ bo'ladi.

$X(0) = X(l) = 0$ shartga asosan

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\lambda x} + c_2 e^{-\lambda x} &= 0 \end{aligned}\right\} \begin{aligned} c_2 &= -c_1 \\ c_1 e^{\lambda x} + c_2 e^{-\lambda x} &= 0, c_1\left(e^{\lambda x} + e^{-\lambda x}\right) = 0 \end{aligned}$$

$e^{\lambda x} + e^{-\lambda x} \neq 0$ shuning uchun $c_1 = 0, c_2 = 0$

Bu holda $X(x) = 0, u(x,t) = 0$ nol yechimga ega bo'ldik.

2) $\mu = 0$ bo'lsin. Bu holda yechim

$$X(x) = c_1 + c_2 x$$

bo'ladi.

$$c_1 = 0, c_2 l = 0, c_2 = 0 (l \neq 0)$$

Bu holda ham $X(x) = 0, u(x,t) = 0$ travial yechimga ega bo'ldik.

3) $\mu = -\lambda 2 < 0$ bo'lsin. Bu holda yechim

$$X(x) = c_1 \cos\lambda x + c_2 \sin\lambda x \text{ bo'ladi.}$$

Chegaraviy shartlarga asosan:

$$X(0) = c_1 = 0$$

$X(l) = c_2 \sin\lambda l = 0, c_2 \neq 0$ bo'lishi kerak, aks holda yana travial yechimga ega bo'lamiz. Shuning uchun $\sin\lambda l = 0$ bo'lishi kerak.

$$\lambda l = k\pi, \lambda = \frac{k\pi}{l}, k = \pm 1, \pm 2, \ldots$$

k=0 da yana travial yechim hosil bo'ladi.

$X_k(x) = c_k \sin\frac{k\pi}{l} x \;\; c_2 = c_k$ deb belgilanadi.

λ_k - xarakteristik sonlar, $X_k(x)$ xarakteristik funksiyalar deyiladi. $\lambda_k = \frac{k\pi}{l}$ xarakteristik songa $T_k(t)$ funksiya mos keladi. Bu holda (5) tenglama quyidagi

$$T_k''(t)+\left(\frac{ak\pi}{l}\right)^2 T_k(t)=0$$

ko'rinishga keladi. O'tgan darslarga asosan uning umumiy yechimi

$$T_k(t)=B_k\cos\frac{ak\pi}{l}t+D_k\sin\frac{ak\pi}{l}t \qquad (8)$$

B_k, D_k lar ixtiyoriy o'zgarmas sonlar.

(7),(8) ni (4) ga qo'ysak

$$u_k(x,t)=\left(a_k\cos\frac{ak\pi}{l}t+b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x \qquad (9)$$

hosil bo'ladi, bunda $a_k=B_kc_k, b_k=D_kc_k$.

(1) tenglama chiziqli, bir jinsli bo'lgani uchun (9)ning yig'indisi

$$u_k(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t+b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x \qquad (10)$$

ham (1)ning yechimi bo'ladi. Bunda (10) qator yaqinlashuvchi hamda qatorning hadlari ikki marta differensiallanuvchi deb faraz qilamiz. $u_k(x,t)$xususiy yechimlar (3) chegaraviy shartlarni qanoatlantirgani uchun ularning yig'indisi $u(x,t)$ham (3) yegaraviy shartlarni qanoatlantiradi. Endi ixtiyoriy a_k va b_k ixtiyoriy o'zgarmaslarni shunday topish kerakki, (10),(2) boshlang'ich shartlarni qanoatlantirsin.

$$u(x,t)\Big|_{t=0}=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi}{l}x=f(x) \qquad (11)$$

$$\frac{\partial u(x,t)}{\partial t}\Bigg|_{t=0}=\sum_{k=1}^{\infty}\frac{ak\pi}{l}\left(-a_k\sin\frac{ak\pi}{l}t+b_k\cos\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x\Bigg|_{t=0}=$$
$$=\sum_{k=1}^{\infty}\frac{ak\pi}{l}b_k\sin\frac{k\pi}{l}x=F(x) \qquad (12)$$

(11),(12) lardan ko'rinadiki,

$$a_k=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx \qquad (13)$$

$$\frac{ak\pi}{l}b_k=\frac{2}{l}\int_0^l F(x)\sin\frac{k\pi}{l}xdx,\ b_k=\frac{2}{ak\pi}\int_0^l F(x)\sin\frac{k\pi}{l}xdx \qquad (14)$$

(13),(14) larni (10) ga qo'yib (1) tenglamaning (2), (3) shartlarni qanoatlantiruvchi yechimini topamiz:

$$u(x,t)=2\sum_{k=1}^{\infty}\sin\frac{k\pi}{l}x\left[\frac{1}{l}\cos\frac{ak\pi}{l}t\int_0^l f(\xi)\sin\frac{k\pi}{l}\xi d\xi+\frac{1}{ak\pi}\sin\frac{ak\pi}{l}t\int_0^l F(\xi)\sin\frac{k\pi}{l}\xi d\xi\right] \qquad (15)$$

Endi (10) ya'ni

$$u(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t+b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x$$

$$a_k=A_k\sin\varphi_k \qquad A_k=\sqrt{a_k^2+b_k^2}$$
$$b_k=A_k\cos\varphi_k \qquad \varphi_k=arctg\frac{a_k}{b_k}$$

U holda

$$u(x,t)=\sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x \sin\left(\frac{ak\pi}{l}t+\varphi_k\right) \qquad (16)$$

(16) dagi har bir qo'shiluvchi φ_k fazali, $A_k \sin\frac{k\pi}{l}x$ amplitudali, $\omega_k=\frac{ak\pi}{l}$ chastotali garmonik tebranma harakatni ifodalaydi. Shuning uchun (16) dagi har bir qo'shiluvchi garmonika deyiladi. Harbir garmonika tik to'lqin bilan tasvirlanadi.

Umumiy holda $u_k(x,t)$ tik to'lqin $\sin\frac{k\pi}{l}x=0$ tenglama $[0,l]$ da nechta yechimga ega bo'lsa shuncha qo'zg'almas nuqtalarga ega bo'ladi.

Nazorat savollari

1. Fure metodi nima?
2.Ikkinchi o'zgarmas koefisiyentli bir jinsli differensial tenglamaning umumiy yechimi qanday ko'rinishda bo'ladi?

12-mavzu. Fredgol'mning integral tenglamari

Reja:

1. Umumiy tushunchalar.
2. Fredgol'm ikkinchi tur integral tenglamasini yechish.
3. Fredgol'm teoremalari.

1. Umumiy tushunchalar.

Integral tenglamalar deb, noma'lum funksiya integral ishorasi ostida bo'lgan tenglamalarga aytiladi.

Mexanika, matematika fizika va texnikaning juda ko'plab masalalar ushbu

$$\varphi(x)-\lambda\int_a^b K(x,y)\varphi(y)dy = f(x) \qquad (1)$$

ko'rinishdagi integral tenglamalarning tekshirishga olib kelinadi, bu erda $\varphi(x)-$ noma'lum funksiya, $K(x,y)$ va $f(x)$ funksiyalar mos ravishda $a\le x\le b,\ a\le y\le b$ va $a\le x\le b$ (a,b -o'zgarmas sonlar) yopiq sohalarda berilgan uzluksiz haqiqiy funksiyalar. $f(x)$ funksiya (1) integral tenglamaning ozod hadi, $K(x,y)$ tenglmaning yadrosi, sonli λ ko'paytuvchi tenglamaning parametiri deyiladi.

Fredgol'mning birinchi tur integlamasi deb,

$$\int_a^b K(x,y)\varphi(y)dy = f(x) \qquad (2)$$

ko'rinishdagi integral tenglamaga aytiladi.

Agar (1) tenglamada $f(x)\equiv 0$ bo'lsa, ya'ni

$$\varphi(x)-\lambda\int_a^b K(x,y)\varphi(y)dy = 0 \qquad (3)$$

tenglama (1) mos bo'lgan bir jinsli integral tenglama deyiladi.

Bir jinsli

$$\psi(x)-\lambda\int_a^b K(x,y)\psi(y)dy = 0 \qquad (4)$$

tenglama (3) bir jinsli tenglamaga qo'shma integral tenglama deyiladi.

$$\varphi(x) - \lambda \int_a^x K(x,y)\varphi(y)dy = f(x), \quad x > a \tag{5}$$

ko'rinishda bo'lsa, u Volterrning ikkinchi tur integral tenglamasi,

$$\int_a^x K(x,y)\varphi(y)dy = f(x) \tag{6}$$

Tekshirib ko'rish qiyin emaski, agar ikkinchi tur Fredgolintegral tenglamasining umumiy echimi $\Phi(x)$ mavjud bo'lsa

$$\Phi(x) = \varphi^0(x) + \varphi(x) \tag{7}$$

ko'rinishga ega bo'ladi, bunda $\varphi^0(x)$ (3) tenglamaning umumiy echimi,

esa (1) tenglamaning umumiy va xususiy echimlari bo'lsa, bularning ayirmasi $\varphi^0(x) = \Phi(x) - \varphi(x)$ (3) tenglamaning echimidan iborat bo'ladi. Bundan darhol (7) tenglik kelib chiqadi.

2. Fredgol'm ikkinchi tur integral tenglamasini parametr kichik bo'lganda ketma-ket yaqinlashish usuli bilan yechish.

(1) tenglamani tekshiramiz, $K(x,y)$ va $f(x)$ funksiyalar o'zlari aniqlangan sohalrda uzliuksiz bo'lgani uchun

$$\lambda \int_a^x |K(x,y)| \le M, \quad a \le x \le b, \ \max_{a \le x \le b} |f(x)| = m \tag{8}$$

bo'ladi.

Agar (1) tenglama λ parametr

$$|\lambda| < \frac{1}{M} \tag{9}$$

shartni qanoatlantirsa, u holda bu tenglamalarning yagona $\varphi(x)$ echimi mavjud bo'lib, uni ketma-ket yaqinlashish usuli bilan topish mumkin.

Nolinchi yaqinlashish sifatida (1) tenglamaning ozod hadini qabul qilamiz

$$\varphi_0(x) = f(x)$$

Binchi yaqinlashishni

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x,y) f(y)dy$$

munosabat bilan aniqlaymiz. Bu jarayonni davom ettirib, n-yaqinlashishni

$$\varphi_n(x) = f(x) + \lambda \int_a^b K(x,y)\varphi_{n-1}(y)dy, \quad n = 1,2,.....,n \tag{10}$$

munosabat bilan aniqlaymiz.

Shunday qilib, (10) rekurrent munosabatlarni qanoatlantiruvchi

$\varphi_0(x), \varphi_1(x),........, \varphi_n(x),......$

funksional ketma-ketligiga ega bo'lamiz.

Matematik analizdan ma'lumki, (11) ketma-ketlikning yaqinlashishi

$$\varphi_0(x) + \sum_{n=1}^{\infty} [\varphi_n(x) - \varphi_{n-1}(x)] \tag{12}$$

qatorning yaqinlashishiga teng kuchludir. (10) formulani

$$\varphi_n(x) = f(x) + \lambda\int_a^b K(x,y)\left[\varphi_{n-1}(y) - \varphi_{n-2}(y) + \varphi_{n-1}(y)\right]dy =$$

$$= f(x) + \lambda\int_a^b K(x,y)\varphi_{n-2}(y)dy + \lambda\int_a^b K(x,y)\left[\varphi_{n-1}(y) - \varphi_{n-2}(y)\right]dy = \qquad (13)$$

$$= \varphi_{n-1}(x) + \lambda\int_a^b K(x,y)\left[\varphi_{n-1}(y) - \varphi_{n-2}(y)\right]dy \quad n = 2,3,4,.....$$

ko‘rinishda yozib olamiz.
(8) ga asosan (13) dan quyidagi tengsizliklar kelib chiqadi:

$$|\varphi_0(x)| \le m$$

$$|\varphi_1(x) - \varphi_0(x)| \le m|\lambda| M$$

$$|\varphi_2(x) - \varphi_1(x)| \le m|\lambda|^2 M^2$$

....

$$|\varphi_n(x) - \varphi_{n-1}(x)| \le m|\lambda|^n M^n$$

Shunday qilib, (12) qatorning har bir hadi musbat sonli

$$\sum_{n=0}^{\infty} m|\lambda|^n M^n \qquad (14)$$

Fredgol‘mning birinchi teoremasi.

Fredgol‘mning ikkinchi tur

$$\varphi(x) - \lambda\int_a^b K(x,y)\varphi(y)dy = f(x) \qquad (15)$$

integral tenglamasining $K(x,y)$ yadrosi

$$K(x,y) = \sum_{i=1}^{n} p_i(x)q_i(y) \qquad (16)$$

ko‘rinishda bo‘lsa, u aynigan (buzilgan) yadro deb yuritiladi.

Bundagi $p_i(x)$ va $q_i(y)$, $a \le x \le b,\ a \le y \le b$ -berilgan haqiqiy uzluksiz funksiyalardir. Ayrim adabiyotlarda (16) ko‘rinishdagi yadro Pinkerle-Gursa yadrosi, yoki qisqacha PG-yadro deb ham ataladi. Umumiylikka ziyon etkazmay, barcha $p_i(x)$ funksiyalarni ham $q_i(y)$ funksiyalarni ham o‘zaro bog’liq emas deb hisoblaymiz. Aks holda (16) yig’indida qo‘shiluvchilar sonini kamaytirish mumkin. (16) ifodani (15) integral tenglamaga qo‘yib,

$$\varphi(x) - \lambda\sum_{i=1}^{n}\int_a^b p_i(x)q_i(y)\varphi(y)dy = f(x) \qquad (17)$$

tenglamani hosil qilamiz. (17) integral tenglamani

$$\varphi(x) = f(x) + \lambda\sum_{i=1}^{n} c_i p_i(x) \qquad (18)$$

ko‘rinishda yozish mumkin, bunda

$$c_i = \int_a^b q_i(y)\varphi(y)dy, \quad i = 1,2,.....,n$$

lar $\varphi(y)$ noma’lum funksiyaga bog’liq bo‘lgani uchun noma’lum o‘zgarmaslardir.

Endi c_i, $i = 1,2,.....,n$ o‘zgarmas sonlarni shunday tanlashga harakat qilamizki, natijada (18) fomula bilan aniqlangan $\varphi(x)$ funksiya (3) integral tenglamaning echimidan iborat bo‘lsin. Shu maqsadda (18) ifodani (17) tenglamaning chap tomoniga olib borib qo‘yamiz

$$f(x)+\lambda\sum_{i=1}^{n}c_i p_i(x)-\lambda\sum_{i=1}^{n}\int_a^b p_i(x)q_i(y)\left[f(y)-\lambda\sum_{i=1}^{n}c_j p_j(y)\right]dy = f(x)$$

yoki

$$\sum_{i=1}^{n}p_i(x)\left[c_i-\int_a^b q_i(y)f(y)dy-\lambda\sum_{i=1}^{n}\int_a^b c_j p_j(y)q_i(y)dy\right]=0$$

Bundan $p_i(x)$ funksiyalar chiziqli bog’liq bo‘lmagani uchun

$$c_i-\int_a^b q_i(y)f(y)dy-\lambda\sum_{i=1}^{n}\int_a^b c_j p_j(y)q_i(y)dy=0$$

tenglik kelib chiqadi. Ushbu

$$\gamma_i=\int_a^b q_i(y)f(y)dy \qquad \alpha_{ij}=\int_a^b p_j(y)q_i(y)dy=0$$

belgilarni kiritib, avvalgi tenglikni

$$c_i-\lambda\sum_{i=1}^{n}\alpha_{ij}c_j=\gamma_i,\quad i=1,2,.....,n \tag{19}$$

ko‘rinishda yozib olamiz.

Shunday qilib, (17) integral tenglamaning echimini topish masalasini (19) chiziqli algebraik tenglamalar sistemasini echishga olib keldik.

(19) sistema nazariyasida

$$M(\lambda)=\begin{Vmatrix} 1-\lambda a_{11} & -\lambda a_{12}\dots & -\lambda a_{1n}\\ -\lambda a_{21} & 1-\lambda a_{22}\dots & -\lambda a_{2n}\\ -\lambda a_{n1} & -\lambda a_{n2}\dots & 1-\lambda a_{nn}\end{Vmatrix}$$

matrisa muhim rol o‘ynaydi. Bu matrisaning determinantini $D(\lambda)$ orqali belgilab olamiz, ya’ni $\det M(\lambda)=D(\lambda)$. Chiziqli algebra kursida ma’lumki, agar

$$D(\lambda)\neq 0 \tag{20}$$

bo‘lsa, (19) sistema ixtiyoriy γ_i o‘ng tomohlar uchun yagona echimga ega bo‘ladi va bu echim Kramer formulalari bilan aniqlanadi. Ammo, $D(\lambda)$ diterminant λ ga nisbatan n-darajali ko‘phaddan iboratdir.

Demak, $D(\lambda)$ ko‘phadning ildizlari bo‘lgan λ ning shekli sondagi: $\lambda_1,.........,\lambda_m,\ m\leq n$ qiymatlar uchun (6) shart buziladi. λ ning bu qiymatlari $K(x,y)$ yadrosning yoki bunga mos (3) integral tenglamaning xos (xarakteristik) sonlari deyiladi.

Shunday qilib, λ ning λ_k lardan $k=1,2,......,m$ farqli bo‘lgan har bir chekli qiymat uchun (19) sistema yagona $c_1,.........,c_n$ echimga ega bo‘ladi. Bu echimni (18) tenglikning o‘ng tomoniga qo‘yib, (17) integral tenglamaning $\varphi(x)$ echimiga ega bo‘lamiz.

Natijada quyidagi teoremani isbotladik.

Fredgol’mning birinchi teoremasi.

Agar λ $K(x,y)$ yadroning xos soni bo‘lmasa, ixtiyoriy uzluksiz $f(x)$ uchun (17) integral tenglama echimiga ega, shu bilan birga bu echimga yagona bo‘ladi.

Fredgol‘mning ikkinchi teoremasi.

(17) integral tenglamaga mos bir jinsli

$$\varphi(x)-\lambda\int_a^b\sum_{i=1}^{n}p_i(x)q_i(y)\varphi(y)dy=0 \tag{21}$$

tenglama (19) ga mos bo‘lgan ushbu

$$c_i-\lambda\sum_{j=1}^{n}\alpha_{ij}c_j=0 \tag{22}$$

bir jinsli chiziqli algebik sistemsgs keladi. (21) tenglamaga qo‘shma bo‘lgan bir jinsli tenglama,

$$\psi(x) - \lambda\int_a^b \sum_{i=1}^n p_i(y)q_i(y)\psi(y)dy = 0 \qquad (23)$$

ko‘rinishga ega bo‘ladi. (23) tenglamaga ega (22) ga qo‘shma bo‘lgan

$$d_i - \lambda\sum_{j=1}^n \alpha_{ij}d_j = 0$$

bir jinsli sistemaga teng kuchlidir, bunda

$$d_i = \lambda\int_a^b p_i(y)\psi(y)dy, \qquad i = 1,2,.....,n$$

Agar $\lambda = \lambda_k$, $k = 1,2,.....,m$ bo‘lib, $M(\lambda)$ matrisaning rangi r ga teng bo‘lsa, chiziqli algebradan ma’lumki, bir jinsli (8) sistema ham va unga qo‘shma bo‘lgan (10) sistema ham $n-r$ ta

$$c_1^j,......., c_n^j,\ d_1^j,........, d_n^j,\ j = 1,2,..., n-r$$

chiziqli bog’liq bo‘lmagan echimlarga ega bo‘ladi. Bu echimlarni (21) va (34) dan nosil bo‘lgan ushbu

$$\varphi_j(x) = \lambda\sum_{j=1}^n c_i^j p_i(x),$$

$$\psi_j(x) = \lambda\sum_{j=1}^n d_i^j q_i(x), \qquad j = 1,2,....., n-r \qquad (24)$$

formulalarning o‘ng tomonlariga qo‘yib, (21) va (22) bir g’insli integral tenglamalarning $n-r$ tadan chiziqli bog’liq bo‘lmagan echimlarini hosil qilamiz.

Fredgolmning ikkinchi teoremasi.

Bir jinsli (19) tenglama va unga qo‘shma bo‘lgan bir jinsli (9) tenglama bir xil $n-r$ tadan chiziqli bog’liq bo‘lmagan echimlar ega bo‘ladi.

Bir jinsli (19) tenglamaning nolga teng bo‘lmagan $\varphi_j(x)$, $j = 1,2,....., n-r$ echimlari (17) tenglamaning yoki $K(x, y)$ yadroning λ_k xos songa mos bo‘lgan zos funksiyalari deyiladi.

Fredgol‘mning uchinchi teoremasi.

Yana chiziqli algebra kursiga murog’at qilamiz. Ma’lumki, $\lambda = \lambda_k$, $k = 1,2,.....,m$ bo‘lganda (17) sistema ixtiyoriy o‘ng tomonlar uchun echimga ega bo‘lavermaydi. Bu sistemaning echimiga ega bo‘lishi uchun γ_i, $i = 1,2,.....,n$ sonlar

$$\sum_{i=1}^n \gamma_i d_i^j = 0,\ \ j = 1,2,....., n-r \qquad (25)$$

shartlarni qanoatlantirishi zarur va etarlidir. (25) shartlarda γ_i o‘rniga uning qiymatini qo‘yib, (25) ga tang kuchli bo‘lgan quyidagi tengliklar sistemasiga ega bo‘lamiz:

$$\lambda\sum_{i=1}^n \gamma_i d_i^j = \lambda\sum_{j=1}^n d_i^j \int_a^b q_i(x) f(x)dx = \int_a^b f(x)\sum_{i=1}^n d_i^j q_i(x)dx = 0$$

Bundan (24) ga asosan

$$\int_a^b f(x)\psi_j(x)dx = 0, \qquad j = 1,2,....., n-r$$

tengliklar hosil bo‘ladi.

Biror D sohada, xususan (a,b) oraliqda, ikkita funksiyaning ko‘paytmasidan olingan integral nolga teng bo‘lsa, bu funksiyalar ortogonal deb ataladi.

Shunday qilib biz quyidagi teoremani isbot qildik.

Fredgolmning uchunshi teoremasi.

$\lambda = \lambda_k$, $k = 1,2,.....,m$ bo‘lganda (17) integral tenglamaning echimiga ega bo‘lishi uchun uning ozod hadi $f(x)$ ning qo‘shma bir jinsli (22) tenglamaning barcha $\psi_j(x)$, $j = 1,2,....., n-r$ echimlariga ortogonal bo‘lishi zarur va etarlidir.

Uzluksiz yadroli Fredgol'mning ikkinchi tur tenglamasi.

Ushbu

$$\begin{cases} \varphi(x)-\lambda\int\limits_a^b K(x,y)\varphi(y)dy = fx & (A) \\ \varphi(x)-\lambda\int\limits_a^b K(x,y)\varphi(y) = 0 & (B) \\ \psi(x)-\lambda\int\limits_a^b K(y,x)\psi(y)dy = 0 & (C) \end{cases}$$

uzluksiz yadroli Fredgol'mning ikkinchi tur tenglamasini o'rganamiz. Endi (A) tenglamaning yadrosi aynigan bo'lmagan holni tekshiramiz. Agar $K(x,y)$ funksiya $a \le x \le b$ $a \le y \le b$ kvadratda uzluksiz bo'lsa, matematik analiz kursida isbotlanadigan Veyershtrass teoremasiga asosan, har bir $\in > 0$ uchun x va y ga nisbatan yetarli yuqori darajali shunday $K_0(x,y)$ ko'phad mavjud bo'ladiki, barcha R da

$$|K(x,y)-K_0(x,y)| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Ravshanki $K_0(x,y)$ ko'phadning har bir hadi bittasi x ga, ikkinchi y ga bog'liq bo'lgan ikkita ko'paytuvchining ko'paytmasi sifatida ifodalash mumkin. Shu sababli $K(x,y)$ yadroni

$$K(x,y)=\sum_{i=1}^{n} p_i(x)q_1(y)+K_1(x,y) \qquad (26)$$

ko'rinishda yozib olishimiz mumkin. Bundagi $K_1(x,y)$ ifoda R da

$$(b-a)|K_1(x,y)| < \in \qquad (27)$$

sharni qanoatlantiruvchi uzluksiz funksiya.

(26) tenglik $K(x,y)$ yadroni kichik va aynigan yadrolar yig'indisiga ajrtatilganligini ifodalovchi formuladir.

(26) ga sosan Fredgol'm tenglamani

$$\varphi(x)-\lambda\int\limits_a^b K_1(x,y)\varphi(y)dy = F(x) \qquad (28)$$

ko'rinishda yozib olamiz, bunda

$$F(x) = f(x)+\lambda\sum_{i=1}^{n}\int\limits_a^b p_1(x)q_i(y)\varphi(y)dy \qquad (29)$$

λ parametrning ixtiyoriy chekli tayin qiymati uchun ε sonni shunday kichik tanlab olamizki,

$$|\lambda| < \frac{1}{\varepsilon} \qquad (30)$$

tengsizlik o'rinli bo'lsin.

(26) tengsizlikga asosan

$$\int\limits_a^b |K_1(x,y)|dy < \in$$

Demak (3) tenglama (5) ga ko'ra birdan-bir yechimga ega bo'ladi. $K_1(x,y)$ yadroning rezol'ventasini

$R_1(x,y,\lambda)$ oraqali belgilab, (30) tenglama yechimini

$$\varphi(x) = F(x) + \lambda\int_a^b R_1(x,y,\lambda)F(y)dy \qquad (31)$$

ko'rinishda yozib olamiz. (29) tenglikdagi $F(x)$ o'rniga uning qiymati (30) ni olib borib qo'yamiz:

$$\varphi(x) = f(x) + \lambda\sum_{i=1}^{n}\int_a^b p_i(x)q_i(y)\varphi(y)dy +$$

$$+\lambda\int_a^b R_1(x,t,\lambda)\left[f(t) + \lambda\sum_{i=1}^{n}\int_a^b p_i(t)q_i(y)\varphi(y)dy\right]dt.$$

ushbu

$$g(x) = f(x) + \lambda\int_a^b R_1(x,t,\lambda)f(t)dt.$$

$$r_1(x) = p_i(x) + \lambda\int_a^b R_1(x,t,\lambda)p_i(t)dt.$$

belgilashlarni kiritib,

$$\varphi(x) - \lambda\int_a^b\sum_{i=1}^{n} r_1(x)q_i(y)\varphi(y)dy = g(x) \qquad (6)$$

tenglamani hosil qilamiz.

Shunday qilib, λ ning ixtiyoriy chekli tayin qiymati uchun uzluksiz yadroli tenglama aynigan yadroli Fredgol'mning ikkinchi tur (6) tenglamasiga ekvalent ekan.

(6) tenglama uchun avvalgi bandda bayon qilingan Fredgol'm teoremalari o'rinli bo'lgani uchun uzluksiz yadroli tenglama uchun ham o'rinli bo'ladi. Bu teoremalarga asosan quyidagi al'ternativa hosil bo'ladi.

Fredgol'm alternativasi. Agar (A) tenglama mos bir jinsli (B) tenglama λ ning har bir tayin qiymati uchun noldan farqli yechimga ega bo'lmasa, (A) tenglama ixtiyoriy ozod had $f(x)$ uchun yagona yechimga ega bo'ladi; agarda (B) bir jinsli tenglama noldan farqli yechimlarga ega bo'lsa, (B) tenglama ham va unga qo'shma (-) bir jinsli tenglam ham bir xil chekli sondagi chiziqli bog'liq bo'lmagan yechimlarga ega bo'ladi; u holda (A) tenglama ixtiyoriy $f(x)$ uchun yechimga ega bo'lavermaydi, (A) tenglamaning yechimga ega bo'lishi uchun uning ozod hadi $f(x)$ qo'shma (C) bir jinsli tenglamaning barcha yechimlariga ortogonal bo'lishi zarur va yetarlidir, yani

$$\int_a^b f(x)\psi_i(x)dx = 0 \qquad i = 1,2...,k,$$

bunda $\psi_i(x),\ i = 1,...,k,$ - (C) tenglamaning barcha chiziqli bog'liq bo'lmagan yechimlari.

Aynan nolga teng bo'lmagan funksiya, ravshanki bir jinsli (B) tenglamaning va unga qo'shma bo'lgan (C) tenglamaning yechimi bo'ladi. Bundan keyin, bir jinsli (yoki unga qo'shma) tenglamaning yechimi deganda aynan noldan farqli yechimini tushunamiz.

Bir jinsli (B) tenglama $\varphi(x),\ i = 1,...,k,$ yechimlarga ega bo'lgan λ ning qiymatini, $K(x,y)$ yadroning yoki (B) tenglamaning xos soni, $\varphi(x)$ funksiyalarni esa shu yadro yoki tenglamaning λ xos songa mos xos funksiyalari deyiladi.

Yuqorida bayon qilinganlarga asosan, yana bir bor ta'kidlab o'tamizki, berilgan xos songa mos chiziqli bog'liq bo'lmagan xos funksiyalarning soni cheklidir.

$K(x,y)$ yadroning barcha xos sonlarni to'palami bu yadroning spektri deb ataladi.

Volterr tenglamasi yadrosining spektri bo'sh to'plamdan iborat, aynigan yadroli Fredgol'm ikkinchi tur tenglamasining spektri esa chekli sondagi elementlardan iborat.

Endi (A) tenglamani

$$\varphi(y) - \lambda\int_a^b K(y,t)\varphi(t)dt = f(y)$$

ko'rinishda yozib olib, buning har ikki qismini $\lambda K(x,y)$ ga ko'paytiramiz va y bo'yicha a dan b gacha integrallaymiz. U holda

$$\lambda\int_a^b K(x,y)\varphi(y)dy = \varphi(x) - f(x)$$

tenglikni e'tiborga olib,

$$\varphi(y) - \lambda^2\int_a^b K_2(x,t)\varphi(t)dt = f_2(x)$$

tenglamani hosil qilamiz, bunda

$$f_2(y) = f(x) + \lambda^2\int_a^b K(x,y)f(y)dy$$

Bu jarayonni davom ettirib, quyidagi tenglamaga ega bo'lamiz:

$$\varphi(x) - \lambda^2\int_a^b K_n(x,y)\varphi(y)dy = f_n(x) \qquad (32)$$

bunda

$$f_n(x) = f_{n-1}(x) + \lambda^n\int_a^b K(x,y)f_{n-1}(y)dy, \quad f_1(x) = f(x).$$

Shunday qilb, biz ushbu natijaga keldik: agar λ $K(x,y)$ yadroning xos soni, $\varphi(x)$ esa bu xos songa mos xos funksiyasi bo'lsa, u holda λ^n va $\varphi(x)$ iteratsiyalangan $K_n(x,y)$ yadroning xos soni va xos funksiyasidan iborat bo'ladi.

Nazorat savollari

1. Fredgol'mning birinchi teoremasini ta'riflang.
2. Fredgol'mning ikkinchi teoremasini ta'riflang.
3. Fredgol'mning uchinchi teoremasini ta'riflang.
4. Uzluksiz yadroli integral tenglamani aynigan yadroli integral tenglamasiga olib kelish g'oyasini keltiring.
5. Fredgol'm alternativasini ta'riflang.
6. Fredgol'm ikkinchi tur integral tenglamasini ketma-ket yaqinlashish usuli bilan yechishning rekurrent formulasini yozing.

13-mavzu. Vol'terrning integral tenglamalari. Rezol'venta.

Reja:

1. Volterrning ikkinchi tur integral tenglamasi.
2. Iterasiyalangan yadro.
3. Rezol'venta.
4. Volterrning birinchi tur integral tenglamasi.

1. Volterrning ikkinchi tur integral tenglamasi.

$$\varphi(x) = \lambda\int_a^x K(x,y)\varphi(y)dy + f(x) \qquad (1)$$

Integral tenglamani ketma-ket yaqinlashish usuli bilan yechamiz.
7-ma'ruzadagi mulohazalarni qayratib

$$\varphi_0(x),\ \ \varphi_1(x),\ \ \varphi_2(x),\ \ ......\ \ ,\varphi_n(x),\ ....$$

funksiyalar ketma – ketligini hosil qilamiz, bunda

$$\varphi_0(x),\quad \varphi_n = f(x) + \lambda\int_a^x K(x,y)\varphi_{n-1}(y)dy \qquad (2)$$

$$m = \max|f(x)|,\quad N = \max|K(x,y)|,$$

belgilashlarni kiritamiz. Bu holda

$$|\varphi_0(x)| \le m,$$

$$|\varphi_1(x) - \varphi_0(x)| = \lambda\int_a^x K(x,y)\varphi_0(y)d\,y| \le |\lambda| mN(x-a),.....,$$

$$|\varphi_n(x) - \varphi_{n-1}(x)| \le m\frac{|\lambda|^n N^n (x-a)^n}{n!},\quad n = 1,\ 2,....,\qquad (3)$$

tengsizlikga ega bo'lamiz.

Musbat hadli

$$m\sum_{n=o}^{\infty}\frac{|\lambda|^n N^n (x-a)^n}{n} = me^{|\lambda| N(x-a)}$$

funksional qator λ parametrning ixtiyoriy chekli qiymatida tekis yaqinlashuvchi bo'lgani uchun (3) tengsizliklarga asosan (1) funksiyalar ketma – ketligi absolyut va tekis yaqinlashuvchi bo'lib, uning limiti bo'lgan

$$\varphi(x) = \lim_{n\to\infty}\varphi_n(x)$$

funksiya (1) tenglamaning yechimidan iborat bo'ladi.

Endi (1) tenglam yechimining yagona ekanligini ko'rsatamiz.

Faraz qilaylik (1) tenglama ikkita $\varphi(x)$ va $\psi(x)$ uzluksiz yechimlarga ega bo'lsin. Bularning ayirmasi $\omega(x) = \varphi(x) - \psi(x)$ bir jinsli

$$\omega(x) = \lambda\int_a^x K(x,y)\omega(y)dy \qquad (4)$$

tenglamani qanoatlantiradi.

$m^* = \max|\omega(x)|$ deb belgilab olsak, (4) dan darhol

$$|\omega(x)| \le |\lambda|\int_a^x |(K(x,y)||(\omega(y)|dy \le |\lambda| Nm^*(x-a)$$

tengsizlik kelib chiqadi. Bundan foydalanib (4) tenglikdan

$$|\omega(x)| \le |\lambda|\int_a^x |K(x,y)||\omega(y)|dy \le |\lambda|^2 Nm^*\frac{(x-a)^2}{2}$$

tengsizlikni hosil qilamiz. Bu jarayonni davom ettirib, ixtiyoriy natural n uchun

$$|\omega(x)| \le m^* |\lambda| N^n \frac{(x-a)^n}{n!}$$

tengsizlikni hosil qilamiz. Bu tengsizlikdan $n \to \infty$ da $\omega(x) = 0$ yoki $\varphi(x) = \psi(x)$ ekanligi kelib chiqadi.

Shunday qilib quyidagi xulosaga keldik.

Volterrning ikkinchi tur (1) integral tenglamasi, uning yadrosi $K(x, y)$ va ozod hadi $f(x)$ uzluksiz fuksiyalar bo'lgabda λ parametrning har bir chekli qiymatida yagona yechimga ega bo'ladi.

Shu dalil bilan Volterrning ikkinchi tur integral tenglamasi har bir λ uchun yechimga ega bo'lavermaydigan Fredgolmning ikkinchi tur integral tenglamasidan tubdan farq qiladi.

2. Iterasiyalangan yadro. $|\lambda| < \frac{1}{m}$ tengsizlik bajarilganda

(3) funksiyalar ketm – ketligi (1) tenglamaning $\varphi(x)$ yechimga yaqinlashishi isbotlangan edi. Endi shu ketma – ket yaqinlashish strukturasini batafsil o'rganamiz. Ma'lumki,

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x, y) f(y) dy,$$

So'ngra

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t) \varphi_1(t) dt =$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b K(x,t) dt \int_a^b K(t, y) f(y) dy$$

Ikkilangan integralda integrallsh tartibini o'zgartirib,

$$K_2(x, y) = \int_a^b K(x,t) K(t, y) dt$$

deb belgilab olib,

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x, y) f(y) dy + \lambda^2 \int_a^b K(x, y) f(y) dy$$

tenglikni hosil qilamiz.

Bu jarayonni davom ettirib,

$$\varphi_n(x) = f(x) + \lambda \int_a^b \sum_{i=1}^n \lambda^{i-1} K_i(x, y) f(y) dy \tag{5}$$

Tenglikga ega bo'lamiz, bunda $K(x, y)$ lar

$$K_1(x, y) = K(x, y),$$

$$K_i(x, y) = \int_a^b K(x,t) K_{i-1}(t, y) dt, \quad i = 2, 3, \dots$$

rekurrent munosabatlar bilan aniqlanadi. $K_i(x, y)$ funksiyalar iteratsiyalangan (takrorlangan) yadrolar deb ataladi.

3. Rezol’venta. (3) ketma – ketlikning yaqinlashishi isbotlangandagi mulohazalarni qaytarib, $|\lambda| < \frac{1}{m}$ shart bajarilganda $a \le x \le b,\ a \le y \le b$ kvadratda

$$\sum_{i=1}^{\infty} \lambda^{i-1} K_1(x, y)$$

qatorning tekis yaqinlashishiga ishonch hosil qilish mumkin.

Bu qatorning yig’indisi $R(x, y, \lambda)$ ni $K(x, y)$ yadroning yoki (1) integral tenglamaning rezol’ventasi yoki hal qiluvchi yadrosi deyiladi.

(5) da $n \longrightarrow \infty$ da limitga o’tib, (1) tenglamaning yechimini rezol’venta yordamida

$$\varphi(x) = f(x) + \lambda \int_a^b R(x, y, \lambda) f(y) dy$$

ko’rinishda yozishimiz mumkin.

4. Volterrning birinchi tur integral tenglamasi.

Volterrning birinchi tur

$$\int_a^x K(x, y)\varphi(y)dy = f(x) \qquad (6)$$

integral tenglamasini tekshiramiz

Faraz qilaylik, (6) tenglamaning yadrosi va ozod hadi quyidagi shartlarni qanoatlantirsin:

1) $K_x(x, y),\ f'(x)$ lar mavjud uzluksiz funksiyalar,

2) $K(x, x)$ hech qaerda nolga aylanmaydi.

Bu holda (6) tenglamani x bo‘yicha differensiyallab, Volterrning ikkinchi tur integral tenglamasiga olib kelamiz:

$$K(x, x)\varphi(x) + \int_a^x K_x(x, y)\varphi(y)dy = f'(x) \qquad (7)$$

yoki

$$\varphi(x) + \int_a^x K^*(x, y)\varphi(y)dy = f^*(x)$$

bunda

$$K^*(x, y) = \frac{K_x(x, y)}{K(x, x)}, \quad f^*(x) = \frac{f'(x)}{K(x, x)}$$

Yuqorida keltirilgan shartlardan eng muhimi $K(x, x)$ ning nolga aylanmaslik shartidir, chunki x ning biror qiymati $K(x, x)$ nolga teng bo‘lib qolsa, Volterrning birinchi tur integral tenglamasini tekshirishda katta qiyinchiliklarga duch kelinadi.

Nazorat savollari.

1. Iterasiyalangan integral tenglama bilan dastlabki integral tenglama o‘rtasida bog‘lanish.
2. Kuchsiz yadroli integral tenglamalar.
3. Volterrning birinchi tur integral tenglamasi.
4. $K(x, y)$ yadroning rezol’ventasini yozing
5. Tenglama yechimini rezol’venta orqali ifodalang.