OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

TOSHKENT TO'QIMACHILIK VA YENGIL SANOAT INSTITUTI

"Matematika" fanining "Aniqmas va aniq integrallar" bo'limi bo'yicha o'quv metodik qo'llanma (Barcha texnika yo'nalishidagi bakalaviyat talabalari uchun)

TOSHKENT 2017

Annottatsiya

Mazkur metodik qo'llanmada "Aniqmas va aniq integrallar" bo'limi va unga bog'liq bo'lgan mavzular batafsil bayon qilingan. Talabalarning bilimini tekshirish uchun etarli miqdorda misol va masalalar javoblari bilan berilgan. Metodik qo'llanma matematikaning shu bo'limini mustaqil o'rganuvchilar uchun mo'ljallangan.

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kengashida tasdiqlangan

"____"___________2017 yil.

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TTYeSI bosmaxonasida

"____" nusxada ko'paytirilgan

SO'Z BOSHI

Ushbu metodik qo'llanma matematikaning "Aniqmas va aniq integrallar" bo'limiga oid bo'lib, u texnika Oliy ta'lim muassasalarining barcha bakalavriyat ta'lim yo'nalishlarining talabalari uchun tayyorlangan.

Ushbu metodik qo'llanma TTYeSI "Oliy matematika" kafedrasi o'quv rejasining "Aniqmas va aniq integrallar" bo'limiga mos yozilgan bo'lib unda ta'rif va teoremalar, integrallar haqida tushuncha va ularni hisoblash usullari, aniq integralning geometriyaga tadbiqi va xosmas intagrallar haqida ma'lumotlar berilgan.

Talabalarning olgan bilimlarini mustahkamlash uchun misollar javoblari bilan berilgan.

ANIQMAS INTEGRAL

Harakat qonuniga ko`ra harakatdagi jismning tezligini topish masalasi hosila tushunchasiga olib keladi.

Harakatdagi jismning tezligiga ko`ra harakat qonunini topish masalasi esa yuqorida aytilgan masalaga nisbatan teskari masala bo`lib, u funksiyaning aniqmas integrali tushunchasiga olib keladi.

1-§. Boshlang`ich funksiya va aniqmas integral

$f(x)$ funksiya (a,b) da berilgan bo`lib, $F(x)$ esa shu $(a,b)$ berilgan va differensiallanuvchi (ya'ni $F(x)$ hosilaga ega), $x \in (a,b)$ bo`lsin.

Ta'rif. *Agar $F(x)$ funksiyaning hosilasi $F'(x)$ berilgan $f(x)$ funksiyaga teng $F'(x) = f(x)$ yoki $dF(x) = f'(x)dx$ bo`lsa, $F(x)$ funksiya $f(x)$ funksiyaning boshlang`ich funksiyasi deyiladi.*

Masalan,

$$f(x) = x^2, \ (x \in (-\infty; +\infty))$$

funksiyaning boshlang`ich funksiyasi

$$F(x) = \frac{1}{3}x^3$$

bo`ladi, chunki

$$F'(x) = \left(\frac{1}{3}x^3\right)' = \frac{1}{3} \cdot 3x^2 = x^2 = f(x)$$

bo`ladi.

Aytaylik, $f(x)$ funksiya (a,b) da berilgan bo`lib, u shu oraliqda ikkita $F(x)$ va $\varphi(x)$ boshlang`ich funksiyalarga ega bo`lsin. Ta`rifga binoan

$$F'(x) = f(x), \Phi'(x) = f(x) \qquad (x \in (a,b))$$

Keyingi tengliklardan

$$F'(x) = \Phi'(x)$$

bo`lish kelib chiqadi. Unda $F(x)$ va $\varphi(x)$ funksiyalar bir-biridan o`zgarmas songa farq qiladi;

$$\Phi(x) = F(x) + C, \qquad C = const$$

Demak, berilgan $f(x)$ funksiyaning boshlang`ich funksiyalari cheksiz ko`p bo`lib, ular bir-biridan o`zgarmas songa farq qiladi.

Agar $F(x)$ funksiya $f(x)$ funksiyaning boshlang`ich funksiyasi bo`lsa, unda $f(x)$ funksiyaning istalgan boshlang`ich funksiyasi

$$F(x) + C, \qquad C = const$$

ko`rinishida bo`ladi.

Ta`rif. *Ushbu*

$$F(x) + C$$

ifoda $f(x)$ *funksiyaning aniqmas integral deyiladi* va $\int f(x)dx$ kabi belgilanadi:

$$\int f(x)dx = F(x) + C$$

bunda $\int$ integral belgisi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ integral ostidagi ifoda deyiladi.

Masalan,

$$\int x^2 dx = \frac{1}{3}x^3 + C,$$

chunki

$$\left(\frac{1}{3}x^3 + C\right)' = \frac{1}{3} \cdot 3x^2 = x^2$$

bo`ladi.

Odatda funksiyaning aniqmas integralini topish amaliga shu funksiyani integrallash amali deyiladi.

Teorema. (a, b) *da uzluksiz bo`lgan har qanday funksiya boshlang`ich funksiyaga, demak aniqmas integralga ega bo`ladi.*

2-§. Aniqmas integralning xossalari

Aniqmas integral bir nechta xossalarga ega. Biz ularni keltiramiz:

1^0. $f(x)$ funksiya aniqmas integral $\int f(x)dx$ ning differensiali $f(x)dx$ ga teng:

$$d\left[\int f(x)dx\right] = f(x)dx,$$

2^0. Funksiya differensialining aniqmas integarali shu funksiya bilan o`zgarmas son yig`ndisiga teng:

$$\int d\,F(x) = F(x) + C.$$

3^0. Ushbu munosabat o`rinli:

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx \;(k = const, k \neq 0)$$

4^0. Ushbu formula o`rinli:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$$

Bu xossalarning isboti bevosita aniqmas integral ta'rifidan kelib chiqadi. Biz ulardan birini, masalan, 2°-xossaning isbotini keltiramiz.

$F(x)$ funksiya $f(x)$ funksiyaning biror boshlang`ich funksiyasi bo`lsin. Ta'rifga ko`ra

$$F'(x) = f(x)$$

bo`ladi,

$$\int f(x)dx = F(x) + C$$

bo`ladi. Unda

$$\int f(x)dx = \int F'(x)dx = \int d\,F(x)$$

bo`ladi. Keying tengliklardan

$$\int d\,F(x) = F(x) + C$$

bo`lishi kelib chiqadi. Bu esa 2°-xossani isbotlaydi.

3-§. Asosiy integrallar jadvali.

1^0. Sodda funksiyalarning aniqmas integrallari.

Avval sodda funksiyalarning aniqmas integrallarini topamiz. Bunda boshlang`ich funksiya ta`rifidan hamda hosilalar jadvalidan foydalanamiz.

1) $f(x) = x^\alpha$ bo`lsin. $(x > 0,\ \ \alpha -$ haqiqiy son, $\alpha \neq -1)$
Unda

$$\int f(x)dx = \int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

bo`ladi, chunki

$$\left(\frac{x^{\alpha+1}}{\alpha+1} + C\right)' = (\alpha+1)\cdot\frac{x^\alpha}{\alpha+1} = x^\alpha.$$

1) $f(x) = \frac{1}{x}$ bo`lsin $(x \neq 0)$, unda

$$\int \frac{1}{x}dx = ln|x| + C$$

bo`ladi.

2) $f(x) = a^x$ bo`lsin $(a > 0,\ a \neq 1)$. U holda

$$\int f(x)dx = \int a^x dx = \frac{a^x}{lna} + C$$

bo`ladi, chunki

$$\left(\frac{a^x}{lna}+C\right)'=\frac{1}{lna}\cdot a^x\cdot lna=a^x$$

Xususiy, $f(x)=e^x$ bo`lsa,

$$\int f(x)dx=\int e^x\cdot dx=e^x+C$$

bo`ladi.

3) $f(x)=\sin x$ bo`ladi, unda

$$\int f(x)dx=\int \sin x\,dx=-\cos x+C$$

bo`ladi, chunki.

$$(-\cos x+C)'=-(-\sin x)=\sin x.$$

4) $f(x)=\cos x$ bo`lsin, u holda

$$\int f(x)dx=\int \cos x\,dx=\sin x+C$$

bo`ladi, chunki.
$(\sin x+C)'=\cos x.$

5) $f(x)=\frac{1}{cos^2\cdot x}$ bo`lsin, unda

$$\int f(x)dx=\int \frac{1}{cos^2x}dx=tgx+C$$

bo`ladi, chunki

$$(tgx+C)'=\frac{1}{cos^2x}.$$

6) $f(x)=\frac{1}{sin^2x}$ bo`lsin, u holda

$$\int f(x)dx=\int \frac{1}{sin^2x}dx=-ctgx+C$$

bo`ladi, chunki

$$(-ctgx+C)'=-\left(-\frac{1}{sin^2x}\right)=\frac{1}{sin^2x}$$

7) $f(x)=\frac{1}{1+x^2}$ bo`lsin, u holda

$$\int f(x)dx=\int \frac{1}{1+x^2}dx=arctgx+C$$

bo`ladi, chunki

$$(arc\ tgx+C)'=\frac{1}{1+x^2}$$

8) $f(x)=\frac{1}{\sqrt{1-x^2}}$ bo`lsin, unda

$$\int f(x)dx=\int\frac{1}{\sqrt{1-x^2}}dx=arc\sin x+C$$

bo`ladi, chunki

$$(arc\sin x+C')'=\frac{1}{\sqrt{1-x^2}}$$

2^0. Integrallar jadvali. Yuqorida keltirilgan formulalarni jamlab ushbu integrallar jadvalini hosil qilamiz:

1) $\int x^{\alpha}dx=\frac{x^{\alpha-1}}{\alpha+1}+C \quad (\alpha\neq-1)$

2) $\int\frac{1}{x}dx=ln|x|+C \quad (x\neq0)$

3) $\int a^{x}dx=\frac{a^x}{lna}+C \quad (a>0, a\neq1)$

4) $\int e^{x}dx=e^{x}+C,$

5) $\int\sin x\,dx=-\cos x+C,$

6) $\int\cos x\,dx=\sin x+C,$

7) $\int\frac{1}{sin^2x}dx=-ctgx+C,$

8) $\int\frac{1}{coc^2x}dx=tgx+C,$

9) $\int\frac{1}{\sqrt{1-x^2}}dx=arc\sin x+C,$

10) $\int\frac{1}{1+x^2}dx=arctgx+C.$

Bevosita integrallashga doir misollar

1. $\int(3-x^2)^3dx$	1. $27x-9x^3+\frac{9}{5}x^5-\frac{1}{7}x^7+C$
2. $\int\frac{x+1}{\sqrt{x}}dx$	2. $\frac{2}{3}x\sqrt{x}+2\sqrt{x}+C$
3. $\int\frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx$	3. $\frac{4}{5}x\sqrt[4]{x}-\frac{24}{17}x\sqrt[12]{x^5}+\frac{4}{3}\sqrt[4]{x^3}+C$
4. $\int\frac{(1-x)^3}{x\sqrt[3]{x}}dx$	4. $-\frac{3}{\sqrt[3]{x}}\left(1+\frac{3}{2}x-\frac{3}{5}x^2+\frac{1}{8}x^3\right)+C$

5. $\int \frac{\left(\sqrt{2x}-\sqrt[3]{3x}\right)^2}{x}dx$	5. $2x-\frac{12}{5}\sqrt[6]{72x^5}+\frac{3}{2}\sqrt[3]{9x^2}+C$
6. $\int \frac{x^2dx}{1+x^2}$	6. $x-arctgx+C$
7. $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx$	7. $\ln\left\|\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2+1}}\right\|+C$
8. $\int (1+\sin x+\cos x)dx$	8. $x-\cos x+\sin x+C$
9. $\int ctg^2xdx$	9. $-x-ctgx+C$
10. $\int tg^2xdx$	10. $-x+tgx+C$
11. $\int \frac{2+3\sqrt[3]{x^2}+5\sqrt{x}}{\sqrt{x^3}}dx$	11. $\frac{4}{\sqrt{x}}+18\sqrt[6]{x}+5\ln\|x\|+C$
12. $\int \frac{dx}{x^4+x^2}$	12. $-\frac{1}{x}-arktgx+C$
13. $\int (2x^8-5x^5-1)dx$	13. $\frac{2}{9}x^9-\frac{5}{6}x^6-x+C$
14. $\int \frac{(1-x)^2}{x}dx$	14. $x-\frac{1}{x}-2\ln\|x\|+C$
15. $\int \frac{x^2 5x+6}{x+3}dx$	15. $\frac{x^2}{2}+2x+C$
16. $\int \left(\frac{1}{x^2}-\frac{1}{x}\right)dx$	16. $-\frac{1}{x}-\ln\|x\|+C$
17. $\int \frac{2-x+x^2}{3x}dx$	17. $\frac{2}{3}\ln\|x\|-\frac{1}{3}x+\frac{x^2}{6}+C$
18. $\int \frac{x^2+2}{1+x^2}dx$	18. $x+arctgx+C$
19. $\int \frac{3-\sqrt{5+x^2}}{5+x^2}dx$	19. $\frac{5}{\sqrt{5}}arctg\frac{x}{\sqrt{5}}-\ln\left\|x+\sqrt{5+x^2}\right\|+C$
20. $\int \left(2\ell^x+\frac{1}{1+x^2}\right)dx$	20. $2\ell^x+arctgx+C$

21. $\int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)dx$ 21. $tgx - ctgx + C$

22. $\int\frac{\sin 2x}{\cos x}dx$ 22. $-2\cos x + C$

23. $\int\frac{2x\sin^2 x+1}{\sin^2 x}dx$ 23. $x^2 - ctgx + C$

24. $\int\frac{\cos^2 x-\sin x\cos^2 x+1}{\cos^2 x}dx$ 24. $\sin x+\cos x+tgx+C$

25. $\int\frac{x\ell^2-x}{x}dx$ 25. $\ell^x - x + C$

26. $\int\frac{1+\cos^2 x}{1+\cos 2x}dx$ 26. $\frac{1}{2}(tgx+x)+C$

27. $\int\left(\ell^{-x}+\ell^{-2x}\right)dx$ 27. $-\left(\ell^{-x}+\frac{1}{2}\ell^{-2x}\right)$

28. $\int\frac{x^3+3x^2+4x}{x}dx$ 28. $\frac{1}{3}x^3+\frac{3}{2}x^2+4x+C$

29. $\int 2(3x-1)^2\,dx$ 29. $6x^3-6x^2+2x+C$

30. $\int x^3(1+5x)dx$ 30. $\frac{1}{4}x^4+x^5+C$

4-§.Integrallash usullari.

1^0. Bevosita integrallash usuli. Bu usulda integral ostidagi funksiyani (yoki ifodani) ayniy almashtirishlar yo`li bilan, yoki bevosita integralning xossalarini tatbiq etish yo`li bilan jadval integraliga keltirib hisoblanadi.

Ko`p hollarda quyidagi almashtirishlardan foydalaniladi:

$du = d(u+a)$, a-o`zgarmas son,

$$du=\frac{1}{a}d(au),\quad a\neq 0$$

$$udu=\frac{1}{2}d(u^2),$$

$$\cos u\,du=d(\sin u)$$

$$\sin u\,du=-d(\cos u),$$

$$\frac{1}{u}du=d\,(lnu),$$

$$\frac{1}{cos^2u}du = d(tgu),$$

umuman

$$f'(u)du = d(f(u)).$$

Masalan,

$$\int \frac{dx}{x+2} = \int \frac{d(x+2)}{x+2} = ln|x+2| + C,$$

$$\int tgx\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x} = -ln|\cos x| + C$$

2^0. O\`zgaruvchilarni almashtirib integrallash usuli

Ushbu

$$\int f(x)dx$$

integralni hisoblash talab etilsin. Ba\`zan $x$ o\`zgaruvchini boshqa o\`zgaruvchiga almashtirilish natijasida berilgan integral soddaroq, hisoblash uchun qulayiroq integralga keladi.

Aytaylik.

$$\int f(x)dx = F(x) + C \qquad (1)$$

bo\`lsin. Bu integralda

$$x = \varphi(t)$$

almashtirish bajaramiz. ($f(x)$, $\varphi(t)$ va $\varphi'(t)$-uzluksiz funksiyalar).

Unda

$$\int f(\varphi(t)) \cdot \varphi'(t)dt = F(\varphi(t)) + C \qquad (2)$$

bo\`ladi. Shuni isbotlaymiz.

Ma’lumki,

$$F'(x) = f(x).$$

Unda

$$[F(\varphi(t)) + C]' = (F(\varphi(t)))' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t)$$

bo\`ladi. Bu esa (2) munosabatning to\`g\`riligini bildiradi. (1) va (2) tengliklardan topamiz:

$$\int f(x)dx = \int f(\varphi(t)) \cdot \varphi'(t)dt.$$

Masalan

$$\int (2+3x)^5 dx$$

integralni hisoblashda

$$2+3x=t$$

Almashtirish bajarilsa, unda

$$x=\frac{t-2}{3}, \quad dx=\frac{1}{3}dt$$

bo`lib,

$$\int (2+3x)^5 dx=\int t^5\cdot\frac{1}{3}dt=\frac{1}{3}\cdot\frac{t^6}{6}+C=\frac{1}{18}(2+3x)^6+C$$

bo`ladi.

O`rniga qo`yish usuli bilan integrallash.

Misollar: Javoblar:

1. $\int \frac{xdx}{\sqrt{1-x^2}}$ 1. $-\sqrt{1-x^2}+C$

2. $\int x^2\sqrt[3]{1+x^3}dx$ 2. $\frac{1}{4}\left(1+x^3\right)^{\frac{4}{3}}+C$

3. $\int \frac{xdx}{3-2x^2}$ 3. $-\frac{1}{4}\ln\left|3-2x^2\right|+C$

4. $\int \frac{xdx}{\left(1+x^2\right)^2}$ 4. $-\frac{1}{2\left(1+x^2\right)}+C$

5. $\int \frac{xdx}{4+x^4}$ 5. $\frac{1}{4}arctg\frac{x^2}{2}+C$

6. $\int \frac{x^3dx}{x^8-2}$ 6. $\frac{1}{8\sqrt{2}}\ln\left\|\frac{x^4-\sqrt{2}}{x^4-\sqrt{2}}\right\|+C$

7. $\int x\ell^{-x^2}dx$ 7. $-\frac{1}{2}\ell^{-x^2}+C$

8. $\int \frac{\ell^x dx}{2+\ell^x}$ 8. $\ln\left(2+\ell^x\right)+C$

9. $\int \frac{dx}{(1+x)\sqrt{x}}$ 9. $2arctg\sqrt{x}+C$

10. $\int (2x-3)^{10}\,dx$ 10. $\frac{1}{22}(2x-3)^{11}+C$

11. $\int \sqrt[3]{1-3x}\,dx$ 11. $-\frac{1}{4}(1-3x)^{\frac{4}{3}}+C$

12. $\int \frac{dx}{\sqrt{2-5x}}$ 12. $-\frac{2}{5}\sqrt{2-5x}+C$

13. $\int \frac{dx}{\sqrt{(5x-2)^5}}$ 13. $-\frac{2}{15\sqrt{(5x-2)^3}}+C$

14. $\int \frac{dx}{x\sqrt{x^2+1}}$ 14. $-\ln\left|\frac{1+\sqrt{x^2+1}}{x}\right|+C$

15. $\int \frac{\ln^2 x}{x}\,dx$ 15. $\frac{1}{3}\ln^3 x+C$

16. $\int \sin^5 x\cos x\,dx$ 16. $\frac{1}{6}\sin^6 x+C$

17. $\int \frac{\sin x}{\sqrt{\cos^3 x}}\,dx$ 17. $\frac{2}{\sqrt{\cos x}}+C$

18. $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\,dx$ 18. $\frac{3}{2}\sqrt[3]{1-\sin 2x}+C$

19. $\int \frac{dx}{\sin^2 x\sqrt[4]{ctgx}}$ 19. $-\frac{4}{3}\sqrt[4]{ctg^3 x}+C$

20. $\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}}$ 20. $\frac{2}{3}(-2+\ln x)\sqrt{1+\ln x}+C$

21. $\int \frac{3\sqrt{arctgx}}{1+x^2}\,dx$ 21. $\frac{3}{4}(arctgx)^{\frac{4}{3}}+C$

22. $\int \frac{\ell^{tgx}dx}{\cos^2 x}$ 22. $\ell^{tgx}+C$

23. $\int \frac{\ell^{\arcsin x}}{\sqrt{1-x^2}}\,dx$ 23. $\ell^{\arcsin x}+C$

24. $\int \frac{dx}{\ell^x + 1}$ 24. $x - \ln(\ell^x + 1) + C$

25. $\int (x^2+1)^3 2xdx$ 25. $\frac{(x^2+1)^4}{4} + C$

26. $\int \frac{\cos xdx}{\sqrt{3+\sin x}}$ 26. $2\sqrt{3+\sin x} + C$

27. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ 27. $-2\cos\sqrt{x} + C$

28. $\int \frac{x^2dx}{\sqrt{x^6-2}}$ 28. $\frac{1}{3}\ln\left|x^3 + \sqrt{x^6-2}\right| + C$

29. $\int x^3\sqrt{2x^4-3}dx$ 29. $\frac{1}{12}\sqrt{(2x^4-3)^3} + C$

30. $\int \frac{\sin xdx}{\cos^2 x}$ 30. $\frac{1}{\cos x} + C$

3^0. Bo`laklab integrallash usuli.

Aytaylik, $u = u(x)$ va $v = v(x)$ funksiyalar biror oraliqda aniqlangan uzluksiz hamda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. Ma'lumki.

$$d(u \cdot v) = udv + vdu.$$

Bu tenglikni integrallab topamiz:

$$\int d(u \cdot v) = \int udv + \int vdu.$$

Ravshanki,

$$\int d(u \cdot v) = u \cdot v$$

Unda

$$u \cdot v = \int u\, dv + \int v\, du$$

bo`lib, quyidagi formula

$$\int u\, dv = u \cdot v - \int v\, du \qquad (3)$$

hosil bo`ladi.

Odatda (3) bo'laklab integrallash formulasi deyiladi.

(3) formula $\int udv$ integralni $\int vdu$ ni hisoblashga keltiradi.

Masalan, ushbu

$$\int xe^x\,dx$$

integralda

$$u = x, \quad dv = e^x dx$$

deyiladi, unda

$$du = dx,$$

$$\int dv = \int e^x dx, \quad v = \int e^x dx = e^x$$

bo`lib, (3) formulaga ko`ra

$$\int xe^x dx = xe^x - \int e^x dx$$

bo`ladi va berilgan integral

$$\int xe^x\,dx = xe^x - e^x + C = e^x(x-1) + C$$

ga tengligi topiladi.

Endi bo`laklab integrallash usuli yordamida ushbu

$$J_n = \int \frac{dx}{(x^2+a^2)^n} \qquad (n = 1,2,3, \ldots\ldots)\ a\text{-o`zgarmas)}$$

integralni (bu integraldan keyinchalik ko`p foydalaniladi) hisoblaymiz.

Berilgan integralda

$$u = \frac{1}{(x^2+a^2)^n}, \quad dv = dx$$

deb olamiz. Unda

$$du = \left(\frac{1}{(x^2+a^2)^n}\right)' \cdot dx = [(x^2+a^2)^{-n}]' \cdot dx =$$

$$= -n(x^2+a^2)^{-n-1} \cdot 2xdx = -\frac{2n\,x}{(x^2+a^2)^n} \cdot dx,$$

$$dv = dx$$

bo`lishidan esa,

$$v = x$$

bo`lishini topamiz.

Bo`laklab integrallash formulasiga ko`ra

$$J_n=\int\frac{dx}{(x^2+a^2)^n}=x\cdot\frac{1}{(x^2+a^2)^n}-\int x\cdot\left(-\frac{2nx}{(x^2+a^2)^{n+1}}\right)dx=$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$$

bo`ladi. Bu tenglikning o`ng tomonidagi

$$\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$$

integralni quyidagicha yozib olamiz:

$$\int\frac{x^2}{(x^2+a^2)^{n+1}}dx=\int\frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx=\int\frac{dx}{(x^2+a^2)^n}-$$

$$-a^2\int\frac{1}{(x^2+a^2)^{n+1}}dx=J_n-a^2\cdot J_{n+1}$$

demak,

$$J_n=\frac{x}{(x^2+a^2)^n}+2n\cdot J_n-2na^2\cdot J_{n+1}$$

Keyingi tenglikdan J_{n+1} ni topamiz:

$$2na^2\cdot J_{n+1}=\frac{x}{(x^2+a^2)^n}+2n\cdot J_n-J_n=\frac{x}{(x^2+a^2)^n}+(2n-1)\cdot J_n$$

$$J_{n+1}=\frac{x}{2na^2(x^2+a^2)^n}+\frac{2n-1}{2na^2}\cdot J_n \qquad (4)$$

Bu rekurrent formuladan foydalanib, J_1 ni bilgan holda birin-ketin $J_2,\ J_3,\dots$ larni toppish mumkin.

Ravshanki, $n=1$ bo`lganda

$$J_1=\int\frac{dx}{x^2+a^2}=\frac{1}{a}\int\frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2}=\frac{1}{a}arctg\frac{x}{a}+C$$

bo`ladi.

Masalan, (4) formuladan foydalanib,

$$J_2=\frac{dx}{(x^2+a^2)^2}$$

integral quyidagicha hisoblanadi:

$$J_2=\int\frac{x}{2a^2\cdot(x^2+a^2)}+\frac{1}{2a^2}J_1=\frac{x}{2a^2\cdot(x^2+a^2)}+\frac{1}{2a^2}arctg\,\frac{x}{a}+C.$$

Misollar: **Javoblar:**

1. $\int x\cos xdx$ — 1. $x\sin x+\cos x+C$

2. $\int x^2\sin 2xdx$ — 2. $-\frac{2x^2-1}{4}\cos 2x+\frac{x}{2}\sin 2x+C$

3. $\int x\ell^{-x}dx$ — 3. $-(x+1)\ell^{-x}+C$

4. $\int \ln xzdx$ — 4. $x(\ln x-1)+C$

5. $\int arctgxdx$ — 5. $xarctgx-\frac{1}{2}\ln(1+x^2)+C$

6. $\int \arcsin xdx$ — 6. $x\arcsin x+\sqrt{1-x^2}+C$

7. $\int xarctgxdx$ — 7. $-\frac{x}{2}+\frac{1+x^2}{2}arctgx+C$

8. $\int\frac{\arcsin x}{x^2}dx$ — 8. $-\frac{\arcsin x}{x}-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C$

9. $\int\frac{xdxz}{\cos^2 x}$ — 9. $xtgx+\ln|\cos x|+C$

10. $\int x^2\ln xdx$ — 10. $\frac{x^3}{3}\ln x-\frac{x^3}{9}+C$

11. $\int\frac{\ln x}{x^2}dx$ — 11. $-\frac{\ln|x|+1}{x}+C$

12. $\int\sqrt{x}\ln xdx$ — 12. $\frac{2}{3}\sqrt{x^3}\left(\ln|x|-\frac{2}{3}\right)+C$

13. $\int x3^x dx$ — 13. $\frac{3^x}{\ln^2 3}(x\ln 3-1)+C$

14. $\int\frac{\arcsin x}{\sqrt{x+1}}dx$ — 14. $2\sqrt{x+1}\arcsin x+4\sqrt{1-x}+C$

15. $\int\ln(x^2+1)dx$ — 15. $x\ln(x^2+1)-2x+2arctgx+C$

16. $\int x^2 \ell^{-x} dx$ 16. $C - \ell^{-x}\left(2 + 2x + x^2\right)$

17. $\int x\cos^2 x dx$ 17. $\frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

18. $\int xtg^2 x dx$ 18. $xtgx - \frac{x^2}{2} + \ln\left|\cos x\right| + C$

19. $\int arctg\sqrt{x}dx$ 19. $xarctg\sqrt{x} - \sqrt{x} + arctg\sqrt{x} + C$

20. $\int \frac{xarctgx}{\sqrt{1+x^2}}dx$ 20. $\sqrt{1+x^2}arctgx - \ln\left(x + \sqrt{1+x^2}\right) + C$

21. $\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}dx$ 21. $\left(\sqrt{x} - \sqrt{1-x}\arcsin\sqrt{x}\right) + C$

22. $\int \ell^x \sin x dx$ 22. $\frac{\ell^x(\sin x - \cos x)}{2} + C$

23. $\int x\cos 3x dx$ 23. $\frac{x\sin 3x}{3} + \frac{\cos 3x}{9} + C$

24. $\int x2^{-x} dx$ 24. $-\frac{x\ln 2 + 1}{2^x \ln^2 2} + C$

25. $\int \left(x^2 - 2x + 5\right)\ell^{-x} dx$ 25. $-\ell^{-x}\left(x^2 + 5\right) + C$

26. $\int x\arcsin x dx$ 26. $\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{x}{4}\sqrt{1-x^2} + C$

27. $\int \frac{\ln x}{\sqrt{x}dx}$ 27. $2\sqrt{x}\ln x - 4\sqrt{x} + C$

28. $\int 3^x \cos x dx$ 28. $\frac{3^x(\sin x + \cos x \ln 3)}{1 + (\ln 3)^2} + C$

29. $\int \left(x^2 + 5x + 6\right)\cos 2x dx$ 29. $\frac{2x^2 + 10x + 11}{4}\sin 2x + \frac{2x+5}{4}\cos 2x + C$

30. $\int x^3 \ell^{-\frac{x}{3}} dx$ 30. $-3\ell^{-\frac{x}{3}}\left(x^3 + 9x^2 + 54x + 162\right) + C$

5-§. Sodda kasrlar va ularning integrallari.

Ushbu

$$\frac{A}{x-a},\qquad \frac{A}{(x-a)^m},\quad \frac{Bx+C}{x^2+px+q},\quad \frac{Bx+C}{(x^2+px+q)^m}\qquad (m>1)$$

Ko`rinishdagi kasrlar sodda kasrlar deyiladi. Bunda $A, B, C,\ a, p, q$-o`zgarmas haqiqiy sonlar, m-natural son, x^2+px+q -kvadrati uch had haqiqiy ildizlarga ega emas.

1⁰. $\frac{A}{x-a}$ sodda kasrning integrali quyidagicha hisoblanadi.

$$\int \frac{A}{x-a}dx = A\int \frac{dx}{x-a} = A\int \frac{d(x-a)}{x-a} = A\cdot ln|x-a| + C$$

2⁰.$\int \frac{A}{(x-a)^m}dx\ (m>1)$ sodda kasrning aniqmas integrali.

Bu integral quyidagicha hisoblanadi:

$$\int \frac{A}{(x-a)^m}dx = A\int \frac{dx}{(x-a)^m} = A\int (x-a)^{-m}d(x-a) =$$

$$= A\cdot \frac{(x-a)^{-m+1}}{-m+1} + C = \frac{A}{(x-a)^m\cdot(1-m)} + C.$$

3⁰. $\int \frac{Bx+C}{x^2+px+q}\ dx$ sodda kasrning aniqmas integrali.

Avvalo sodda kasr maxrajidagi x^2+px+q kvadrat uchhadni quyidagicha yozib olamiz:

$$x^2+px+q = x^2+2\cdot\frac{p}{2}x + (\frac{p}{2})^2 + q - (\frac{p}{2})^2 = (x+\frac{p}{2})^2 +$$

$$+q-\frac{p^2}{4} = (x+\frac{p}{2})^2 + a^2,\quad a^2 = q-\frac{p^2}{4}$$

U holda

$$\int \frac{Bx+C}{x^2+px+q}\ dx = \int \frac{Bx+C}{(x+\frac{p}{2})^2+a^2}\ dx$$

bo`ladi. Keying integralda o`zgaruvchini almashtiramiz:

$$x+\frac{p}{2} = t.$$

Pavshanki,

$$dx = dt,\qquad x = t-\frac{p}{2}.$$

Natijada

$$\int \frac{Bx+C}{x^2+px+q}\,dx = \int \frac{Bx+C}{(x+\frac{p}{2})^2+a^2}\,dx = \int \frac{B\left(t-\frac{p}{2}\right)+C}{t^2+a^2}\,dx =$$

$$= \int \frac{Bt+C-\frac{p}{2}B}{t^2+a^2}\,dt = B\int \frac{tdt}{t^2+a^2} + (C-\frac{p}{2}B)\int \frac{dt}{t^2+a^2} =$$

$$= \frac{B}{2}\int \frac{d(t^2+a^2)}{t^2+a^2} + \left(C-\frac{Bp}{2}\right)\cdot\frac{1}{a}\int \frac{d\left(\frac{t}{a}\right)}{1+\left(\frac{t}{a}\right)^2} =$$

$$= \frac{B}{2}\ln(t^2+a^2) + \left(C-\frac{Bp}{2}\right)\cdot\frac{1}{a}arctg\frac{t}{a} + C =$$

$$= \frac{B}{2}\ln(x^2+px+q) + \left(C-\frac{Bp}{2}\right)\cdot\frac{1}{\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C.$$

bo`ladi.

4^0. $\int \frac{Bx+C}{(x^2px+q)^m}dx$ $(m > 1)$ sodda kasrning integrali.

Bu integral quyidagicha hisoblanadi:

$$\int \frac{Bx+C}{(x^2px+q)^m}dx = \int \frac{Bx+C}{[(x+\frac{p}{2})^2+q-\frac{p^2}{4}]^m}dx =$$

$$\int \frac{B\cdot\left(t-\frac{p}{2}\right)+C}{(t^2+a^2)^m}dt = B\cdot\int \frac{tdt}{(t^2+a^2)^m} + (C-\frac{Bp}{2})\int \frac{dt}{(t^2+a^2)^m} =$$

$$= \frac{B}{2}\int \frac{d(t^2+a^2)}{(t^2+a^2)^m} + (C-\frac{Bp}{2})\int \frac{dt}{(t^2+a^2)^m} =$$

$$\frac{B}{2}\cdot\frac{1}{(1-m)(t^2+a^2)^{m-1}} + \left(C-\frac{Bp}{2}\right)\int \frac{dt}{(t^2+a^2)^m}$$

Bu integraldagi $\int \frac{dt}{(t^2+a^2)^m}$ aniqmas integral.

yuqorida keltirilgan rekurrent formula yordamida hisoblanadi.

6-§. Ratsional funksiyalarni integrallash.

Ushbu

$$P_n(x) = a_0 + a_1x + a_2x^2 + \ \ldots\ldots + a_nx^n$$

-butun ratsional funksiyaning integrali oson hisoblanadi:

$$\int P_n(x)dx = \int [a_0 + a_1x + a_2x^2 + \cdots \ldots\ldots + a_nx^n]dx =$$

$$= a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1} + C.$$

Kasr ratsional funksiya

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots \ldots\ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots \ldots\ldots + b_mx^m}$$

ni integrallash birmuncha murakkab bo`ladi.

Agar $\frac{P_n(x)}{Q_m(x)}$ notog`ri kasr $(n > m)$ bo`lsa, uning butun qismi ajratilib, butun ratsional funksiya va to`g`ri kasr yig`indisi ko`rinishida yoziladi:

$$\frac{P_n(x)}{Q_m(x)}dx = R_n(x) + \frac{\overline{R_n}(x)}{Q_m(x)}.$$

U holda

$$\int \frac{P_n(x)}{Q_m(x)}dx = \int R_n(x)dx + \int \frac{\overline{R_n}(x)}{Q_m(x)}$$

bo`ladi.

To`g`ri kasrni integrallash uchun, avvalo bu kasrni sodda kasrlar yig`indisi sifatida yozib olinadi so`ng ularning integrallari topiladi.

Quyida ikkita integralning hisoblanishini ko`rsatamiz.

Ushbu

$$\int \frac{xdx}{(x-1)(x-2)^3}$$

integral hisoblansin.

◅ Bu integralni topish uchun integral ostidagi to`g`ri kasrni sodda kasrlar yig`indisi shakilida yozib olamiz:

$$\frac{x}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Bu tenglikni qanoatlantiruvchi A, B, C, D larni toppish uchun tenglikning ikki tomonini

$$(x-1)\cdot(x-2)^3$$

ga ko`paytiramiz:

$$x = A(x-2)^3 + B\cdot(x-1)(x-2)^2 + C(x-1)\,(x-2) + D(x-1)$$

Undan ushbu

$$x = A(x^3-6x^2-12x-8) + B(x^3-5x^2+8x-4) +$$

$$+C(x^2-3x+2) + D(x-1)$$

tenglamaga ega bo`lamiz.

Endi x ning bir xil darajalari oldindagi koeffisientlarini tenglashtirsak, unda ushbu sistema hosil bo`ladi:

$$A + B = 0$$

$$C - 6A - 5B = 0,$$

$$12A + 8B - 3C + D = 1,$$

$$2C - D - 8A - 4B = 0.$$

Bu sistemani yechib

$$A = -1, \qquad B = 1, \;\; C = -1\,, \;\; D = 2$$

bo`lishini topamiz. Demak,

$$\frac{x}{(x-1)(x-2)^3} = \frac{-1}{x-1} + \frac{1}{x-2} - \frac{1}{(x-2)^2} + \frac{2}{(x-2)^3}$$

Natijada

$$\int \frac{xdx}{(x-1)(x-2)^3} = -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} - \int \frac{dx}{(x-2)^2} + 2\int \frac{dx}{(x-2)^3} =$$

$$= -ln|x-1| + ln|x-2| + \frac{1}{x-2} - \frac{1}{(x-2)^2} + C. \;\; \rhd$$

bo`ladi.

Ushbu

$$\int \frac{dx}{(x^2+1)(x^2+4)}$$

integral hisoblansin.

$\lhd$ Integral ostidagi ratsional funksiya (kasr) sodda kasrlarga quyidagicha yoyiladi.

$$\frac{dx}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Dx+E}{x^2+4}.$$

Bundan

$$(Ax+B)(x^2+4)+(Dx+E)(x^2+1)=1$$

bo`lishi kelib chiqadi. Demak,

$$(A+B)x^3+(B+E)x^2+(4A+D)x+(4B+E)=1$$

Bu tenglikdagi x ning bir xil darajalari oldidagi koeffisientlarni tenglashtirib

$$\begin{cases} A+D=0, \\ 4A+D=0, \\ B+E=0, \\ 4B+E=1 \end{cases}$$

sistemani hosil qilamiz. Bu sistemani yechib topamiz:

$$A=0, \qquad D=0, \qquad B=\frac{1}{3}, \qquad E=-\frac{1}{3}.$$

Shunday qilib,

$$\frac{1}{(x^2+1)(x^2+4)}=\frac{\frac{1}{3}}{x^2+1}+\frac{-\frac{1}{3}}{x^2+4}$$

bo'lib,

$$\int\frac{dx}{(x^2+1)(x^2+4)}=\frac{1}{3}\int\frac{dx}{x^2+1}-\frac{1}{3}\int\frac{dx}{x^2+4}=$$

$$=\frac{1}{3}\,arc\,tg\,x-\frac{1}{6}arc\,tg\,\frac{x}{2}+C$$

bo`ladi.

Misol	Javob
1. $\int\frac{5x^3+9x^2-22x-8}{x^3-4x}dx$	1. $5x+2ln\lvert x\rvert+3ln\lvert x-2\rvert+4ln\lvert x+2\rvert+C$
2. $\int\frac{x^2+x-1}{x^3+x^2-6x}dx$	2. $\frac{1}{6}\ln\lvert x\rvert+\frac{1}{2}\ln\lvert x-2\rvert+\frac{1}{3}\ln\lvert x+3\rvert+c$
3. $\int\frac{x^5+x^4-8}{x^3-4x}dx$	3. $\frac{x^3}{3}+\frac{x^2}{2}+4x+2\ln\lvert x\rvert+5\ln\lvert x-2\rvert-3\ln\lvert x+2\rvert+c$
4. $\int\frac{3x+2}{x(x+1)^3}dx$	4. $\frac{4x+3}{2(x+1)^2}+2\ln\left\lvert\frac{x}{x+1}\right\rvert+c$
5. $\int\frac{x^2dx}{(x+2)^2(x+4)^2}$	5. $-\frac{1}{x+2}-\frac{4}{x+4}+2\ln\left\lvert\frac{x+4}{x+2}\right\rvert+c$

6. $\int \frac{dx}{x^3(x-1)^2}$ 6. $-\frac{1}{2x^2}-\frac{2}{x}-\frac{1}{x-1}+3\ln\left|\frac{x}{x-1}\right|+c$

7. $\int \frac{x^3-2x+2}{(x-1)^2(x^2+1)}dx$ 7. $\frac{1}{2(x-1)}+\frac{1}{2}\ln(x^2+1)+\frac{3}{2}\text{arctg}x+c$

8. $\int \frac{2x^2-3x+1}{x^3+1}dx$ 8. $2\ln|x+1|-\frac{2}{\sqrt{3}}\text{arctg}\frac{2x+1}{\sqrt{3}}+c$

9. $\int \frac{dx}{(x^2+2)(x-1)^2}$ 9. $\frac{1}{9}\ln(x^2+2)-\frac{1}{9\sqrt{2}}\text{arctg}\frac{x}{\sqrt{2}}-\frac{1}{3(x-1)}-\frac{2}{9}\ln|x-1|+c$

10. $\int \frac{x^3+3}{(x+1)(x^2+1)^2}dx$ 10. $\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln|x^2+1|+2\text{arctg}x+\frac{x+2}{2(x^2+1)}+c$

11. $\int \frac{2x+3}{(x-2)(x+5)}dx$ 11. $\ln|x-2|+\ln|x+5|+c$

12. $\int \frac{xdx}{(x+1)(x+2)(x+3)}$ 12. $\frac{1}{2}\ln\left|\frac{(x+2)^4}{(x+1)(x+3)^3}\right|+c$

13. $\int \frac{xdx}{(x-1)^2(x^2+2x+2)}$ 13. $-\frac{1}{5(x-1)}+\frac{1}{50}\ln\frac{(x-1)^2}{x^2+2x+2}-\frac{8}{25}\text{arctg}(x+1)+c$

14. $\int \frac{xdx}{(x-1)(x+1)^2}$ 14. $-\frac{1}{2(x+1)}+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+c$

15. $\int \frac{dx}{x^3-2x^2+x}$ 15. $\ln|x|-\ln|x-1|-\frac{1}{x-1}+c$

16. $\int \frac{2x^3+x^2-4x+7}{x^2+x-2}dx$; 16*. $x^2-x+2\ln|x-1|-\ln|x+2|+c$;

17. $\int \frac{x^2-3x+3}{x^3-2x^2+x}dx$; 17*. $3\ln|x|-2\ln|x-1|-\frac{1}{x-1}+c$;

18. $\int \frac{3x^2-3x+3}{x^3+5x^2+9x+5}dx$; 18*. $-\ln|x+1|+2\ln|x^2+4x+5|-$

$-7arctg(x+2)+c$;

19. $\int \frac{2x}{x^2-4x+3}dx$; 19*. $3\ln|x-3|-\ln|x-1|+c$;

20. $\int \frac{x}{x^3-1}dx$; 20*. $\frac{1}{3}(\ln|x-1|-\frac{1}{2}\ln|x^2+x+1|+\sqrt{3}arctg\frac{2}{\sqrt{3}}(x+\frac{1}{2}))+c$;

21. $\int \frac{x^4+1}{x^3-x^2}dx$; 21*. $\frac{1}{2}x^2+x+2\ln|x-1|-\ln|x|+\frac{1}{x}+c$;

22. $\int \frac{17x^2-16x+60}{x^4-16}dx$;

22*. $\ln|x^2+4| + 3\ln|x-2| - 5\ln|x+2| + \frac{1}{2}arctg\frac{x}{2} + c$;

23. * $\int \frac{2x^3-2x^2+7x+3}{(x^2+4)(x-1)^2}dx$; 23*.

$\ln|x-1| - \frac{2}{x-1} + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}arctg\frac{x}{2} + c$;

7-§. Ba'zi irratsional funksiyalarni integrallash.

Biz yuqorida ratsional funksiyalarning integrallari har doim hisoblanishini ko`rdik. Irratsional funksiyalarning integrallarini hisoblashda vaziyat boshqacha, ya'ni irratsional funksiyalarning integrallari o`zgaruvchilarini almashtirish yordamida ratsional funksiyaga keltirilib hisoblanadi.

Quyida ba'zi irratsional funksiyalarning integrallanishini keltirish bilan kifoyalanamiz.

1-misol. Ushbu

$$J = \int \frac{\sqrt{x}}{1+\sqrt{x}}dx$$

integral hisoblansin

⊲ Bu integralda

$$\sqrt{x} = t, \text{ ya'ni } x = t^2$$

almashtirish bajaramiz. Unda

$$dx = 2tdt$$

bo`lib,

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{t}{1+t}2tdt = 2\int \frac{t^2}{1+t}dt$$

bo`ladi. Natijada irratsional funksiyani integrallash ratsional funksiyani integrallanishiga keladi.

Ravshanki,

$$\frac{t^2}{1+t} = t - 1 + \frac{1}{t+1}$$

bo`ladi.

Unda

$$\int \frac{t^2}{1+t}dt = \int \left(t - 1 + \frac{1}{t+1}\right)dx = \frac{t^2}{2} - t + \ln(t+1) + C$$

bo`ladi,

$$J=2\left[\frac{t^2}{2}-t+\ln(t+1)\right]+C=t^2-2t+2\ln(t+1)+C=$$

$$=x-2\sqrt{x}+2\ln\left(\sqrt{x}+1\right)+C$$

bo`ladi. ⊳

2-misol. Ushbu

$$J=\int\frac{dx}{\sqrt[3]{2x-3}+1}$$

integral hisoblansin.

⊲ Bu integralda

$$\sqrt[3]{2x-3}=t, \text{ ya'ni } 2x-3=t^3$$

almashtirish bajaramiz. U holda

$$2x-t^3=3\,,\quad x=\tfrac{1}{2}(t^3+3),$$

$$dx=\frac{3}{2}t^2dt$$

bo`ladi. Demak,

$$\int\frac{dx}{\sqrt[3]{2x-3}+1}=\int\frac{\frac{3}{2}t^2dt}{t+1}=\frac{3}{2}\int\frac{t^2dt}{t+1}=\frac{3}{2}\int\left(t-1+\frac{1}{t+1}\right)dt=$$

$$=\frac{3}{2}\int tdt-\frac{3}{2}\int dt+\int\frac{dt}{t+1}=\frac{3}{4}t^2-\frac{3}{2}t+ln|t+1|+C=$$

$$=\frac{3}{4}(2x-3)^{\frac{2}{3}}-\frac{3}{2}(2x-3)^{\frac{1}{3}}+ln\left|\sqrt[3]{2x-3}+1\right|+C$$

Misol	Javob
1. $\int\sqrt{a^2-x^2}dx$	1. $\frac{a^2}{2}arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$
2. $\int\sqrt{4-x^2}dx$	2. $2\arcsin\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2}+c$
3. $\sqrt{1-x^2}dx$	3. $\frac{1}{2}arcsinx+\frac{x}{2}\sqrt{1-x^2}+c$
4. $\int\frac{dx}{\sqrt{a^2+x^2}}$	4. $ln\frac{\sqrt{a^2+x^2}+x}{a}+c$
5. $\int\frac{dx}{\sqrt{1+x^2}}$	5. $\ln\left(x+\sqrt{1+x^2}\right)+c$
6. $\int\frac{dx}{\sqrt{x^2-a^2}}$	6. $\ln\left(x+\sqrt{x^2-a^2}\right)+c$

7. $\int \frac{dx}{\sqrt{x}+1}$ 7. $2\sqrt{x}-2\ln(\sqrt{x}+1)+c$

8. $\int \frac{dx}{\sqrt{2x}+5}$ 8. $\sqrt{2x}-5\ln(\sqrt{2x}+5)+c$

9. $\int \frac{dx}{x\sqrt{1+x^2}}$ 9. $\left(x=\frac{1}{t}\right); ln\frac{\sqrt{1+x^2}-1}{x}+c$

10. $\int \frac{\sqrt{x}}{x-\sqrt[3]{x^2}}dx$ 10. $2\sqrt{x}+6\sqrt[6]{x}+3ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|+c$

11. $\int \frac{\sqrt{x}}{1+\sqrt{x}}dx$ 11. $x-2\sqrt{x}+2\ln(\sqrt{x}+1)+c$

12. $\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$ 12. $6\sqrt[6]{x}-6arctg\sqrt[6]{x}+c$

13. $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}dx$ 13. $\frac{4}{5}\sqrt[4]{x^5}-x+\frac{4}{3}\sqrt[4]{x^3}-2\sqrt{x}+4\sqrt[4]{x}--4\ln(\sqrt[4]{x}+1)+c$

14. $\int \frac{1}{(1-x)^2}*\sqrt{\frac{1-x}{1+x}}dx$ 14. $\frac{1-x}{1+x}=t^2$ almashtrish bajariladi

J: $\sqrt{\frac{1+x}{1-x}}+c$

15. $\int \frac{\sqrt{1+x}}{x}dx$ 15. $2\sqrt{1+x}+ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|$+c

16. $\int \frac{dx}{x\sqrt{2x^2-5x+3}}$ 16. $x=\frac{1}{t}$ almashtrish bajariladi

J: $\frac{1}{\sqrt{2}}arcsin\frac{4x-3}{5}+c$

17. $\int \frac{dx}{x\sqrt{7x^2+2x+5}}$ 17. $x=\frac{1}{t}$ almashtirish bajariladi

J:$-\frac{1}{\sqrt{3}}arcsin\frac{6+5x}{7x}+c$

18. $\frac{dx}{(x^2+16)\sqrt{9-x^2}}$ 18. $(x=asint)$ almashtrish bajariladi

J:$\frac{1}{20}arctg\frac{5x}{4\sqrt{9-x^2}}+c$

19. $\int \frac{x+1}{\sqrt[3]{3x+1}}dx$19. $\frac{x+2}{5}\sqrt[3]{(3x+1)^2}+c$

8-§. Tirgonometrik funksiyalarni integrallash

Ma'lumki, integrallar jadvalida

$$y = \sin x, \quad y = \cos x,$$

funksiyalarning integrallari ham oson hisoblanadi:

Masalan,

$$\int ctg\, x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = ln|\sin x| + C.$$

Shuningdek

$$y = \sin ax, \quad y = \cos a x, \quad y = tgax, \quad y = ctgax$$

hamda

$$y = \sin(x + a), \quad y = \cos(x + a), \quad y = tg(x + a), \quad y = ctg\,(x + a)$$

funksiyalarning integrallari ham oson hisoblanadi.

Masalan,

$$\int \sin(2x + 1)dx = \frac{1}{2}\int \sin(2x + 1)d(2x + 1) =$$

$$= -\frac{1}{2}\cos(2x + 1) + C.$$

Ko`p hollarda trigonometrik funksiyalarni integrallashda

$$tg\,\frac{x}{2} = t$$

(ba'zan $\sin x = t,$ $\cos x = t,$ $tgx = t$) almashtirish natijasida qaralayotgan aniqmas integral ratsional funksiyalarni integrallashga keladi. Bunda

$$x = 2arctg\, t, \quad dx = \frac{2}{1 + t^2}dt,$$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{2t}{1 + t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

bo`lishini e'tiborga olish kerak bo`ladi.

Masalan,

$$J = \int \frac{dx}{1 + \sin x + \cos x}$$

integralni hisoblashda

$$tg\frac{x}{2} = t$$

almashtirish bajarib hisoblanadi:

Ravshanki,

$$dx = \frac{2}{1+t^2}dt,$$

$$sin\, x = \frac{2t}{1+t^2},$$

$$cos\, x = \frac{1-t^2}{1+t^2}.$$

Unda

$$\int \frac{dx}{1+sin\, x+cos\, x} = \int \frac{\frac{2}{1+t^2}dx}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} =$$

$$= \int \frac{2}{1+t^2}\cdot\frac{1+t^2}{1+t^2+2t+1-t^2}dt = \int \frac{dt}{1+t} = ln|1+t| + c =$$

$$= ln\left|1+tg\frac{x}{2}\right| + C$$

bo`ladi. ▻

Ayrim trigonometrik funksiyalarni integrallashda trigonometriyada ma’lum bo`lgan ushbu

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)],$$

$$\sin\alpha\cdot\cos\alpha = \frac{1}{2}\sin 2\alpha,$$

$$sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

formulalardan foydalanish maqsadga muofiq bo`ladi.

Masalan,

$$\int \sin 3x \cdot \sin 4x \, dx$$

Integral quyidagicha hisoblanadi:

$$\int \sin 3x \cdot \sin 4x \, dx = \frac{1}{2}\int [\cos(3-4)x - \cos(3+4)x]dx =$$

$$= \frac{1}{2}\int \cos x \, dx - \frac{1}{2}\int \cos 7\,x \, dx = \frac{1}{2}\sin x - \frac{1}{14}\sin 7\; x + C.$$

Trigonometrik funktsiyalarni integrallash.

Misollar:	Javoblar:
1. $\int \cos^4 x \sin^3 xdx$	1. $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$
2. $\int \cos^5 x \sin^2 xdx$	2. $\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$
3. $\int \frac{\sin^3 x}{1+\cos^2 x}dx$	3. $\cos x - 2arctg(\cos x) + C$
4. $\int \frac{\cos^3 x}{4\sin^2 x - 1}dx$	4. $-\frac{1}{4}\sin x + \frac{3}{16}\ln\left\|\frac{2\sin x - 1}{2\sin x + 1}\right\| + C$
5. $\int \frac{dx}{3\cos^2 x 4\sin^2 x}$	5. $\frac{1}{2\sqrt{3}} arctg \frac{2tgx}{\sqrt{3}} + C$
6. $\int \cos 4x \cos 7xdx$	6. $\frac{1}{6}\sin 3x + \frac{1}{22}\sin 11x + C$
7. $\int \sin 3x \cos 10xdx$	7. $\frac{1}{14}\cos 7x - \frac{1}{26}\cos 13x + C$
8. $\int \sin\frac{x}{3}\cos\frac{x}{4}dx$	8. $-6\cos\frac{x}{12} - \frac{6}{7}\cos\frac{7x}{12} + C$
9. $\int \sin\frac{x}{3}\sin\frac{x}{2}dx$	9. $3\sin\frac{x}{6} - \frac{3}{5}\sin\frac{5x}{6} + C$
10. $\int \sin^2 x \cos^4 xdx$	10. $\frac{x}{16} + \frac{\sin 2x}{64} - \frac{\sin 4x}{64} - \frac{\sin 6x}{192} + C$
11. $\int \cos^2 xdx$	11. $\frac{x}{2} + \frac{\sin 2x}{4} + C$

12. $\int \sin^4 x dx$ | 12. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

13. $\int \cos^4 x dx$ | 13. $\frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

14. $\int \sin^4 x \cos^2 dx$ | 14. $\frac{x}{10} - \frac{\sin 2x}{64} - \frac{\sin 4x}{64} + \frac{\sin 6x}{192} + C$

15. $\int \cos^6 x dx$ | 15. $\frac{5}{16}x + \frac{1}{4}\sin 2x - \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x + C$

16. $\int \frac{\sin^3 x}{\cos x - 3} dx$ | 16. $\frac{\cos^2 x}{2} + 3\cos x + 8\ln|\cos x - 3| + C$

17. $\int \frac{dx}{5 + 4\sin x}$ | 17. $\frac{2}{3} arctg \frac{5tg\frac{x}{2} + 4}{3} + C$

18. $\int \frac{dx}{3\sin 4x} - 4\cos x$ | 18. $\frac{1}{5}\ln\left|\frac{tg\frac{x}{2} - \frac{1}{2}}{tg\frac{x}{2} + \frac{1}{2}} + C\right|$

19. $\int \frac{dx}{\sin x}$ | 19. $\ln\left|tg\frac{x}{2}\right| + C$

20. $\int \frac{dx}{\cos x}$ | 20. $\ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

21. $\int \cos^5 x dx$ | 21. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

22. $\int \sin^6 x dx$ | 22. $\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

23. $\int \frac{\sin^3 x}{\cos^4 x} dx$ | 23. $\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C$

24. $\int \frac{\cos^4 x}{\sin^3 x} dx$ | 24. $-\frac{3}{2}\cos x - \frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\ln\left|tg\frac{x}{2}\right| + C$

25. $\int \frac{dx}{\sin^3 x}$ | 25. $-\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|tg\frac{x}{2}\right| + C$

26. $\int \frac{dx}{\cos^3 x}$ | 26. $\frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

27. $\int tg^3 x dx$

27. $\frac{tg^2 x}{2} + \ln|\cos x| + C$

28. $\int ctg^3 x dx$

28. $-\frac{ctg^2 x}{2} - \ln|\sin x| + C$

29. $\int \sin^5 x \cos^5 x dx$

29. $-\frac{\cos 2x}{64} + \frac{\cos^3 2x}{96} - \frac{\cos^5 2x}{320} + C$

ANIQ INTEGRAL

1-§. Aniq integral tushunchasi

Aniq integral matematikaning muhim tushunchalaridan hisoblanadi. Bu tushunchani bayon etishdan avval, unga olib keladigan masalalardan birini keltiramiz.

1°. O`tilgan yo`l haqidagi masala

Moddiy nuqta to`g`ri chiziq bo`ylab, $V = V(t)$ tezlik bilan harakat qilsin. Uning $t_0$ momentdan $T$ momentgacha ketgan vaqtda bosib o`tgan yo`lni topish talab etilsin.

Ma'lumki, tezlik V o`zgarmas bo`lganda, o`tilgan yo`

$$S = V \cdot (T - t_0)$$

bo`ladi.

Tezlik o`zgaruvchan bo`lganda, ya'ni u vaqtning funksiyasi $(V = V(t))$ bo`lganda, ravshanki, bu formula bilan moddiy nuqtaning $[t_0, T]$ vaqt oralig`ida bosib o`tgan yo`lini aniq hisoblab bo`lmaydi. O`tilgan yo`lni aniq hisoblash maqsadida $[t_0, T]$ vaqt oralig`ini.

$$t_0, t_1, t_2, \dots, t_n \qquad (t_0 < t_1 < t_2 < \cdots \dots\dots < t_n = T)$$

nuqtalar yordamida n ta bo`lakka bo`lamiz. Natijada $[t_0, T]$ ushbu

$$[t_0, t_1], \qquad [t_1, t_2], \dots, [t_{n-1}, t_n] = [t_{n-1}, T]$$

bo`laklarda ajraladi. Har bir

$$[t_k, t_{k+1}] \qquad (k = 0,1,2, \dots\dots, n-1)$$

da ξ_k nuqta olib, so`ng shu $[t_k, t_{k+1}]$ da nuqtaning tezligi o`zgarmas $V(\xi_k)$ bo`lsin deb, o`tilgan yo`lni

$$V(\xi_k) \cdot (t_{k+1} - t_k) = V(t_k) \cdot \Delta t_k$$

$(\Delta t_k = t_{k+1} - t_k)$ bo`lishini topamiz. Bu ifoda albatta $[t_k, t_{k+1}]$ vaqt oralig`ida o`tilgan yo`lni taqriban ifodalaydi. Unda $[t_0, T]$ vaqt oralig`ida o`tilgan yo`l

$$S \approx V(\xi_0) \cdot \Delta t_0 + V(\xi_1) \cdot \Delta t_1 + \cdots + V(\xi_k) \cdot \Delta t_k +$$

$$+\cdots+V(\xi_{n-1})\cdot\Delta t_{n-1}=\sum_{k=0}^{n-1}V(\xi_k)\cdot\Delta t_k$$

bo`ladi.

Endi $[t_k,\ t_{k+1}]$ $\quad(k=0,1,2,\dots,n=1)$ segmentlar uzunliklari

$$\Delta t_k=t_{k+1}-t_k$$

ning eng kattasini

$$\lambda \qquad (\lambda=\max_k\{\Delta t_0,\Delta t_1,\dots,\Delta t_{n-1}\})$$

deylik.

Unda λ nolga intilganda $(\lambda\to 0)$

$$\sum_{k=0}^{n-1}V(\xi_k)\Delta t_k$$

yig`indining limiti moddiy nuqtaning $[t_0,\ T]$ vaqt oralig`ida bosib o`tilgan yo`lni ifodalaydi.

Demak, o`tilgan $S$ yo`l $\lambda\to 0$ da

$$\sum_{k=0}^{n-1}V(\xi_k)\cdot\Delta t_k$$

yig`indining limiti bo`ladi.

Umuman, ko`p masalalarning yechimi yuqoridagi kabi yig`indining limitini topish bilan hal etiladi. Bu aniq integral tushunchasiga olib keladi.

2^0. Funksiyaning integral yig`indisi.

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo`lsin. $[a,b]$ segmentni

$$x_0,\ x_1,x_2,\dots,x_n \quad (a=x_0<x_1<x_2<\cdots<x_n=b)$$

nuqtalar yordamida n ta bo`lakka bo`lamiz:

$$[x_0,x_1],\ [x_1,x_2],\dots,[x_{n-1},x_n] \qquad (x_0=a,\ x_n=b)$$

Har bir $[x_k,\ x_{k+1}]$ $\quad (k = 0, 1, 2, \dots, n = 1)$

bo`lakchada ixtiyoriy

$$\xi_k \quad (x_k \le \xi_k \le x_{k+1})$$

nuqtani olamiz. So`ng funksiyaning shu nuqtada qiymati $f(\xi_k)$ ni $\Delta x_k = x_{k+1} - x_k$ ga ko`paytirib quyidagi yig`indini tuzamiz:

$$\sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = f(\xi_0) \cdot \Delta x_0 + f(\xi_1) \cdot \Delta x_1 + \dots + f(\xi_n) \cdot \Delta x_k + \dots +$$

$$+ f(\xi_{n-1}) \cdot \Delta x_{n-1}$$

Bu yig`indi $f(x)$ funksiyaning $[a, b]$ segment bo`yicha integral yig`indisi deyiladi va u σ orqali belgilanadi:

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

Masalan, $f(x) = x^2$ funksiyaning $[a, b]$ segmentdagi integral yig`indisi (yuqoridagi bo`linishga nisbatan)

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \sum_{k=0}^{n-1} \xi_k^2 \cdot \Delta x_k$$

bo`ladi.

3^0. Aniq integral ta’rifi.

$f(x)$ funksiya $[a, b]$ segmentda berilgan bo`lsin.

Bu funksiyaning integral yig`indisi

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

ni qaraymiz.

Ta’rif. *Agar $\lambda \to 0$ da (yoki $n \to \infty$ da) $f(x)$ funksiyaning integral yig`indisi*

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

chekli limitga ega bo`lsa, bu limit $f(x)$ *funksiyaning aniq integrali deyiladi* va

$$\int_a^b f(x)dx$$

kabi belgilanadi. Demak,

$$\int_a^b f(x)dx = \lim_{\substack{\lambda \to 0 \\ (n \to \infty)}} \sigma = \lim_{\substack{\lambda \to 0 \\ (n \to \infty)}} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

Bu holda $f(x)$ funksiya $[a, b]$ da integrallanuvchi deyiladi, $a -$ integralning quyi chegarasi, b integralning yuqori chegarasi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ integral ostidagi ifoda deyiladi.

Endi aniq integralning mavjudligini ifodalaydigan teoremani isbotsiz keltiramiz.

Teorema. *Agar* $y = f(x)$ *funksiya* $[a, b]$ *segmentida uzluksiz bo`lsa, u holda.*

$$\int_a^b f(x)dx$$

aniq integral mavjud bo`ladi.

Eslatma. *Funksiyaning uzluksiz bo`lishi sharti uning integrallanuvchi bo`lishining etarli sharti bo`ladi. Agar funksiya* $[a, b]$ *segmentda chegaralangan bo`lib, u shu segmentning chekli sondagi nuqtalarida uzilishga ega bo`lib, qolgan barcha nuqtalarida uzluksiz bo`lsa, u integrallanuvchi, ya'ni uning aniq integrali mavjud bo`ladi.*

Eslatma. *Agar* $f(x)$ *funksiya* $[a, b]$ *segmentda integrallanuvchi bo`lsa,*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \quad \int_a^a f(x)dx = 0$$

deb qaraladi.

Misol. Ushbu

$$J = \int_0^1 x\,dx$$

aniq integral ta’rifga ko`ra topilsin.

◅ Integral ostidagi $f(x) = x$ funksiya $[0,1]$ segmentda uzluksiz. Demak, qaralayotgan integral mavjud. Ta’rifga ko`ra

$$\int_0^1 x\,dx = \lim_{\lambda \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k, \quad (f(x) = x)$$

bo`ladi. Integral yig`indini quyidagicha tuzamiz

1) $[0, \ 1]$ segmentni

$$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}$$

nuqtalar yordamida n ta teng bo`lakka bo`lamiz

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{k}{n}, \frac{k+1}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

Bu holda

$$\Delta x_k = \frac{k+1}{n} - \frac{k}{n} = \frac{1}{n}$$

bo`ladi.

2) Har bir $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ $(k = 0,1,2,\dots,n-1)$ kesmadagi ξ_k nuqta sifatida $\frac{k+1}{n}$ ni olamiz:

$$\xi_k = \frac{k+1}{n}, \quad (k = 0,1,2,\dots,n-1)$$

Berilgan $f(x) = x$ funksiyaning bu nuqtadagi qiymati

$$f(\xi_k) = \frac{k+1}{n}, \quad (k = 0,1,2,\dots,n-1)$$

bo`ladi. Unda $f(x) = x$ funksiyaning integral yig`indisi

$$\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k=\sum_{k=0}^{n-1}\frac{k+1}{n}\cdot\frac{1}{n}=\frac{1}{n^2}\sum_{k=0}^{n-1}(k+1)=$$

$$=\frac{1}{n^2}(1+2+3+\cdots+n)=\frac{1}{n^2}\cdot\frac{n(n+1)}{2}=\frac{1}{2}\left(1+\frac{1}{n}\right)$$

bo`lib,

$$\int_0^1 x\,dx=\lim_{\substack{\lambda\to 0\\(n\to\infty)}}\left[\frac{1}{2}\left(1+\frac{1}{n}\right)\right]=\frac{1}{2}$$

bo`ladi. Demak,

$$\int_0^1 x\,dx=\frac{1}{2}. \quad \rhd$$

2-§. Aniq integralning xossalari.

Aytaylik, $f(x)$ funksiya $[a,\ b]$ segmentda berilgan bo`lib, uning aniq integrali

$$\int_a^b f(x)\,dx$$

mavjud bo`lsin.

Funksiyaning aniq integrali qator xossalarga ega. Ularni keltiramiz.

1^0. Ushbu

$$\int_a^b c\cdot f(x)dx=c\cdot\int_a^b f(x)dx \qquad (c-const)$$

munosabat o'rinli bo'ladi

$\lhd$ $f(x)$ funksiya $[a,\ b]$ segmentda integrallanuvchi. Unda ta'rifga binoan

$$\int_a^b f(x)\,dx=\lim_{\lambda\to 0}\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k$$

bo`ladi. Ravshanki, $c\cdot f(x)$ funksiyaning integral yig`indisi.

$$\sum_{k=0}^{n-1} c\cdot f(\xi_k)\cdot \Delta x_k$$

bo`lib,

$$\sum_{k=0}^{n-1} c\cdot f(\xi_k)\cdot \Delta x_k = c\cdot \sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k$$

bo`ladi. Bu tenglikda $\lambda \to 0$ da limitga o`tib topamiz:

$$\int_a^b c\cdot f(x)dx = c\cdot \int_a^b f(x)dx. \quad \rhd$$

2^0. Agar $f(x)$ va $g(x)$ funksiyalar $[a,\ b]$ da integrallanuvchi bo'lsa u holda $f(x) \pm g(x)$ funksiya ham $[a, b]$ da integrallanuvchi va

$$\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

bo`ladi.

Ma'lumki,

$$\int_a^b f(x)\ dx = \lim_{\lambda\to 0} \sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k,$$

$$\int_a^b g(x)\ dx = \lim_{\lambda\to 0} \sum_{k=0}^{n-1} g(\xi_k)\cdot \Delta x_k.$$

Ravshanki, $f(x) \pm g(x)$ funksiyaning integral yig`indisi

$$\sum_{k=0}^{n-1} [f(\xi_k) \pm g(\xi_k)]\ \Delta x_k$$

bo`lib,

$$\sum_{k=0}^{n-1} [f(x) \pm g(x)]\, \Delta x_k = \sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k \pm \sum_{k=0}^{n-1} g(\xi_k)\cdot \Delta x_k$$

bo`ladi. Bu tenglikda $\lambda \to 0$ da limitga o`tib topamiz:

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx. \quad \rhd$$

Aniq integralning keyingi xossalarini isbotsiz keltiramiz.

3^0. Agar $f(x)$ funksiya $[a,\ b]$ da integrallanuvchi bo`lib, ixtiyoriy $a < c < b$ bo`lsa, u holda

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

bo`ladi.

4^0 Agar $f(x)$ funksiya $[a,\ b]$ da integrallanuvchi bo`lib, ixtiyoriy $x \in [a, b]$ da $f(x) \geq 0$ bo`lsa, u holda

$$\int_a^b f(x)\, dx \geq 0$$

bo`ladi.

5^0. Agar $f(x)$ va $g(x)$ funksiyalar $[a,\ b]$ da integrallanuvchi bo`lib, ixtiyoriy $x \in [a, b]$ da

$$f(x) \leq g(x)$$

bo`lsa, u holda

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)dx$$

bo`ladi.

3-§. Aniq integrallarni hisoblash usullari.

1^0. Nyuton-Leybnis formulasi va uning yordamida aniq integrallarni hisoblash.

Aytaylik, $y = f(x)$ funksiya $[a,\ b]$ segmentda uzliksiz bo`lib, $F(x)$ esa uning boshlang`ich funksiyasi $(F'(x) = f(x))$ bo`lsin. U holda ushbu

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

formula o`rinli bo`ladi. Shuni isbotlaymiz.

◅ $[a, b]$ segmentni

$$a = x_0, x_1, x_2, \dots, x_n = b \quad (x_0 < x_1 < \dots < x_n)$$

nuqtalar yordamida n ta

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

bo`laklarga ajratamiz.

So`ng quyidagi tenglikni qaraymiz:

$$F(b) - F(a) = F(x_n) - F(x_0) =$$

$$=[F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots +$$

$$+[F(x_2) - F(x_1)] + [F(x_1) - F(x_0)].$$

Mazkur kitobda Lagranj formulasi deb ataluvchi ushbu

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

formulani keltirgan edik. Shu formuladan foydalanib yuqoridagi tenglikning o`ng tomonidagi ayirmalarni quyidagicha yozamiz:

$$F(b) - F(a) = F'(\xi_k) \cdot (x_n - x_{n-1}) + F'(\xi_{n-1}) \cdot (x_{n-1} - x_{n-2}) +$$

$$+ \dots + F'(\xi_2) \cdot (x_2 - x_1) + F'(\xi_1) \cdot (x_1 - x_0) =$$

$$= \sum_{k=1}^{n} F'(\xi_k) \cdot \Delta x_k = \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k$$

Demak,

$$F(b) - F(a) = \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k$$

$f(x)$ funksiya $[a, b]$ segmentda uzluksiz bo`lgani uchun u $[a, b]$ da integrallanuvchi. Bunobarin $\lambda \to 0$ da

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k = \int_a^b f(x)dx$$

bo`ladi. Keyingi tenglamadan

$$F(b) - F(a) = \int_a^b f(x)dx \qquad (1)$$

bo`lishi kelib chiqadi.

Odatda (1) Nyuton-Leybnis formulasi deyiladi.

Aniq integrallarni sodda, ayni paytda qulay bo`lgan hisoblash yo`llaridan biri ularni Nyuton-Leybnis formulasi yordamida hisoblashdir.

Agar quyidagi

$$F(b) - F(a) = F(x)|_a^b$$

belgilash kiritilsa, unda Nyuton-Leybnis formulasi ushbu

$$\int_a^b f(x)dx = F(x)|_a^b$$

ko`rinishga keladi.

Nyuton-Leybnis formulasi yordamida

$$\int_a^b f(x)dx$$

aniq integral quyidagicha hisoblanadi:

Avvalo $f(x)$ funksiyaning aniqmas integrali

$$\int f(x)dx$$

topiladi. Aytaylik, bu integral topilib, u $\Phi\ (x)$ ga teng bo`lsin:

$$\int f(x)dx = \Phi(x) + C$$

So`ng bu funksiyaning a va b nuqtalardagi qiymatlari hisoblanib.

$$\Phi(b) - \Phi(a)$$

ayirma topiladi. Bu qiymat Nyuton-Leybnis formulasiga ko`ra

$$\int_a^b f(x)dx$$

integralning qiymati bo`ladi.

Masalan,

$$\int_0^\pi \sin x\ \ dx = -\cos x \Big|_0^\pi = -(\cos\pi - \cos 0) = 2$$

bo`ladi.

2^0. O`zgaruvchilarni amashtirish usuli.

Aytaylik, $f(x)$ funksiya $[a,b]$ da berilgan va uzliksiz bo`lib, uning aniq integrali

$$\int_a^b f(x)dx$$

ni hisoblash talab etilsin. Bu integralda

$$x = \varphi(t)$$

almashtirish bajaramiz. Bunda $\varphi(t)$ funksiya quyidagi shartlarni qanoatlantirsin:

1) $\varphi(t)$ funksiya $[\alpha, \beta]$ segmentda uzluksiz;
2) $\varphi(\alpha) = a, \quad \varphi(\beta) = b$;
3) $\varphi(t)$ funksiya $[\alpha, \beta]$ segmentda uzluksiz $\varphi'(t)$ hosilaga ega. U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt \qquad (2\ )$$

bo`ladi.

◅ Aytaylik, $F(x)$ funksiya $f(x)$ ning boshlang`ich funksiyasi bo`lsin. Unda Nyuton-Leybnis formulasiga ko`ra

$$\int_a^b f(x)dx = F(b) - F(a)$$

bo`ladi. Ma'lumki,

$$\Big(F\big(\varphi(t)\big)\Big)' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t).$$

Demak, $F\big(\varphi(t)\big)$ funksiya $f(\varphi(t)) \cdot \varphi'(t)$ funksiyaning boshlang`ich funksiyasi bo`ladi. Unda Nyuton-Leybnis formulasiga ko`ra

$$\int_a^b f(\varphi(t)) \cdot \varphi'(t)\, dx = (F\big(\varphi(t)\big)\,|_\alpha^\beta = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) =$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

bo`ladi. Bu (2) munosabatni isbotlaydi. ▻

Misol. Ushbu

$$\int_0^2 x^2\sqrt{4-x^2}\,dx$$

Integral hisoblansin.

◅ Bu integralda x o`zgaruvchini quyidagicha almashtiramiz:

$$x = 2\sin t.$$

Bunda

$$dx = (2\sin t)'\cdot dt = 2\cos t\,dt$$

bo`lish, $x = 0$ bo`lganda $t = 0, x = 2$ bo`lganda, $t = \frac{\pi}{2}$ bo`ladi. Unda (2) formuladan foydalanib topamiz.

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} 4\,sin^2\,t\cdot\sqrt{4-4\,sin^2\,t}\cdot 2\cos t\ \,dt.$$

Keyingi integralni hisoblaymiz:

$$\int_0^{\frac{\pi}{2}} 4\,sin^2\,t\cdot\sqrt{4-4\,sin^2\,t}\cdot 2\cos t\ \,dt = 16\int_0^{\frac{\pi}{2}} sin^2\,t\,cos^2\,t\,dt =$$

$$= 16\int_0^{\frac{\pi}{2}} \frac{1}{4}\,sin^2 2t\ \,dt\ = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\,(1-\cos 4\,t)\,dt =$$

$$= 2\left[t\ \Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin 2\,t\ \Big|_0^{\frac{\pi}{2}}\right] = 2\left(\frac{\pi}{2}-0\right) = \pi.$$

Demak,

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \pi.\ \ ▻$$

3^0. Bo`laklab integrallash usuli.

Aytaylik, $u = u(x)$ va $v = v(x)$ funksiya $[a,\ b]$ segmentda aniqlangan, uzuksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo`lsin.

Ravshanki,

$$[u(x)\cdot v(x)]' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Demak, $u(x) +\ v(x)$ funksiya

$$u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

funksiyaning boshlang`ich funksiyasi bo`ladi.

Nyuton-Leybhis fopmulasiga ko`ra

$$\int_a^b [u'(x)\cdot v(x) + u(x)\cdot v'(x)]\ dx = [u(x)\cdot v(x)]\,|_a^b$$

bo`ladi. Agar

$$\int_a^b [u'(x)\cdot v(x) + u(x)\cdot v'(x)]\ dx = \int_a^b v(x)\cdot u'(x)dx +$$

$$+\int_a^b u(x)\cdot v'(x)dx = \int_a^b v(x)\cdot du(x) + \int_a^b u(x)\cdot dv(x)$$

bo`lishini e'tiborga olsak, u holda

$$\int_a^b v(x)\cdot du(x) + \int_a^b u(x)\cdot dv(x) = [u(x)\cdot v(x)]_a^b$$

bo`lib, bundan esa

$$\int_a^b u(x)\cdot dv(x) = [u(x)\cdot v(x)]_a^b - \int_a^b v(x)\cdot du(x) \qquad (3)$$

bo`lishi kelib chiqadi. Bu tenglik aniq integralning bo`laklab integrallash formulasi deyiladi.

U $u(x)dv(x)$ ni integrallashni $v(x)du(x)$ ni integrallashga olib keladi.

Bo`laklab integrallash formulasidan foydalanish uchun integral ostidagi ifodani $u(x)$ va $dv(x)$ lar ko`paytmasi ko`rinishida yozib olinadi, bunda, albatta $v(x)du$ ifodalarning integralini oson hisoblana olinishi lozimligini e'tiborda tutish kerak.

Misol. Ushbu

$$\int_e^{e^2} x \ln x \, dx$$

Integral hisoblaymiz.

◅ Bu integralda

$u = \ln x, \quad dv = x\, dx$

deyiladi. Unda

$$du = \frac{1}{x}dx, \quad v = \int x\, dx = \frac{x^2}{2}$$

bo`ladi. Bo`laklab integrallash formulasi (3) dan foydalanib topamiz:

$$\int_e^{e^2} x \ \ln x \, dx = \frac{x^2}{2}\ln x \ \Big|_e^{e^2} - \int_e^{e^2} \frac{x^2}{2}\cdot\frac{1}{x}\ dx =$$

$$= \frac{x^2}{2}\ln x \ \Big|_e^{e^2} - \frac{1}{2}\int_e^{e^2} x\ dx = \frac{1}{2}\left[e^4\cdot\ln e^2 - e^2\ln e - \frac{x^2}{2}\Big|_e^{e^2}\right] =$$

$$= \frac{1}{2}\left(2e^4 - e^2 - \frac{e^4}{2} + \frac{e^2}{2}\right) = \frac{1}{4}(3e^2 - 1)e^2. \ \triangleright$$

4-§. Aniq integrallarni taqribiy hisoblash.

Biz yuqorida integral ostidagi funksiyaning boshlang`ich funksiyasi ma'lum bo`lsa integralni Nyuton-Leybhis fopmulasi yordamida hisoblash mymkinligini ko`rdik. Ammo boshlang`ich funksiyani topish masalasi doim osngina hal bo`lavermaydi. Agar integral ostidagi funksiya murakkab bo`lsa, tegishli aniq integralni hisoblashni taqribiy usullarini qo`llash lozim bo`ladi.

1^0. To`g`ri to`rtburchaklar formulasi

$f(x)$ funksiya $[a,\ b]$ segmentda berilgan va uzluksiz bo`lsin. Bu funksiyaning aniq integrali.

$$\int_a^b f(x)dx$$

ni taqribiy ifodalovchi formulani keltiramiz, $[a,\ b]$ segmentni

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

nuqtalar yordamida n ta teng bo`lakka bo`lamiz.

Bu holda

$$\Delta x_k = x_{k+1} - x_k = \frac{b-a}{n}, \qquad x_k = a + k\frac{b-a}{n}, \quad (k = 0,1, \dots, n)$$

bo`ladi.

Berilgan $f(x)$ funksiyaning x_k huqtadagi qiymati $f(x_k)$ni hisoblab, $f(x)$ funksiyaning $[x_k,\ x_{k+1}]$ segment bo`yicha aniq integralini quyidagicha

$$\int_{x_k}^{x_{k+1}} f(x)dx \approx f(x_k) \cdot \Delta x_k = f(x_k) \cdot \frac{b-a}{n}$$

taqribiy ifodalaymiz. Bunday taqribiy formulani har bir $[x_k,\ x_{k+1}]$ $(k = 0, 1, 2, \dots \dots, n-1)$ segmentga nisbatan yozib, so`ng ularni hadlab qo`shib topamiz:

$$\int_a^{x_1} f(x)dx \approx f(x_0) \cdot \frac{b-a}{n},$$

$$\int_{x_1}^{x_2} f(x)dx \approx f(x_1) \cdot \frac{b-a}{n},$$

$$\int_{x_2}^{x_3} f(x)dx \approx f(x_2) \cdot \frac{b-a}{n}$$

. . . .

$$\int_{x_{n-1}}^{b} f(x)dx \approx f(x_{n-1}) \cdot \frac{b-a}{n},$$

$$\int_{a}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \dots + \int_{x_{n-1}}^{b} f(x)dx \approx$$

$$\approx \frac{b-a}{n}[f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})]$$

Demak,

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{n}[f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})],$$

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k) \qquad (4)$$

Bu aniq integralni taqribiy hisoblovchi formula to`g`ri to`rtburchaklar formulasi deyiladi.

2°. Trapetsiyalar formulasi.

$f(x)$ funksiya $[a, b]$ segmentda berilgan va uzluksiz bo`lsin $[a, b]$ segmentni yuqoridagidek $n$ ta teng bo`lakka bo`lib, $f(x)$ funksiyaning $[x_k, x_{k+1}]$ segment bo`yicha olingan aniq integralini quyidagicha

$$\int_{x_k}^{x_{k+1}} f(x)dx \approx \frac{f(x_k) + f(x_{k+1})}{2} \cdot \Delta x_k = \frac{f(x_k) + f(x_{k+1})}{2} \cdot \frac{b-a}{n}$$

taqribiy ifodalaymiz. Bunday taqribiy formulani har bir $[x_k, x_{k+1}]$ $(k = 0, 1, 2, \dots\dots, n-1)$ segmentga nisbatan yozib, so`ng ularni hadlab qo`shib topamiz:

$$\int_{a}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \cdots + \int_{x_{n-1}}^{b} f(x)dx \approx$$

$$\approx \frac{b-a}{2n}[(f(x_0) + f(x_1) + (f(x_1) + f(x_2)) + (f(x_2) + f(x_3)) +$$

$$+ \cdots + (f(x_{n-1}) + f(x_n))] =$$

$$= \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Demak,

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{n}\left[\frac{f(x_0) + f(x_n)}{2} + f(x_1 + f(x_2) + \cdots + f(x_{n-1})\right] \qquad (5)$$

Bu aniq integralni taqribiy hisoblovchi formulaga trapetsiyalar formulasi deyiladi.

3^0. Parabolalar (Simnson) Formulasi

$f(x)$ funksiya $[a,\ b]$ segmentda berilgan va uzluksiz bo`lsin. $[a,\ b]$ segmentni

$$a = x_0 < x_1 < x_2 < \cdots < x_{2n-2} < x_{2n-1} < x_{2n} = b$$

nuqtalar yordamida $2n$ ta teng bo`lakka bo`lamiz.

$f(x)$ funksiyaning $[x_{2k},\ x_{2k+2}]$ segment bo`yicha aniq integralini quyidagicha

$$\int_{x_{2k}}^{x_{2k+2}} f(x)dx \approx \frac{b-a}{6}[f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})]$$

taqribiy ifodalaymiz. Bu taqribiy formulani har bir

$$[x_{2k}, x_{2k+2}] \qquad (k = 0, 1, 2, \ldots \ldots, n-1)$$

segmentga nisbatan yozib, so`ng ularni hadlab qo`shib topamiz:

$$\int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + \cdots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx \approx$$

$$\approx \frac{b-a}{6n}[(f(x_0)+4f(x_1)+f(x_2))+(f(x_2)+4f(x_3)+f(x_4))+$$

$$+(f(x_4)+4f(x_5)+f(x_6))+\cdots(f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n}))]=$$

$$=\frac{b-a}{6n}[(f(x_0)+f(x_{2n}))+4(f(x_1)+f(x_3)+f(x_5)+\cdots+f(x_{2n-1}))+$$

$$+2(f(x_2)+f(x_4)+f(x_6))+\cdots+f(x_{2n-2}))]$$

Demak,

$$\int_a^b f(x)dx \approx$$

$$\approx \frac{b-a}{6n}[(f(x_0)+f(x_{2n})+4(f(x_1)+f(x_3)+\cdots+f(x_{2n-1}))+$$

$$+2(f(x_2)+f(x_4)+\cdots+f(x_{2n-2}))] \qquad (6)$$

Bu (6) formula parabolalar (Simnson) formulasi deyiladi.

Misol. Ushbu

$$\int_0^1 e^{-x^2}dx$$

aniq integral to`g`ri to`rtburchaklar, trapetsiyalar va Simpson formulalari yordamida taqribiy hisoblansin.

◅ $[0,1]$ segmentni 5 ta teng bo`lakka bo`lamiz:

$$0=x_0, \quad x_1=0{,}2, \quad x_2=0{,}4, \quad x_3=0{,}6, \quad x_4=0{,}8, \quad x_5=1$$

Bu nuqtalarda $f(x)=e^{-x^2}$ funksiyaning qiymatlari quyidagicha bo`ladi:

$$f(x_0)=1{,}00000,$$

$$f(x_1)=0{,}96079,$$

$$f(x_2)=0{,}85214,$$

$$f(x_3) = 0{,}69768,$$

$$f(x_4) = 0{,}52729,$$

$$f(x_5) = 0{,}36788.$$

Har bir bo`lakning o`rtasini ifodalovchi nuqtalar quyidagicha

$$x_{\frac{1}{2}} = 0{,}1, \qquad x_{\frac{3}{2}} = 0{,}3, \qquad x_{\frac{5}{2}} = 0{,}5, \qquad x_{\frac{7}{2}} = 0{,}7, \qquad x_{\frac{9}{2}} = 0{,}9.$$

bo`lib, bu nuqtalardagi qiymatlari esa quyidagicha bo`ladi:

$$f\left(x_{\frac{1}{2}}\right) = 0{,}99005,$$

$$f\left(x_{\frac{3}{2}}\right) = 0{,}91393,$$

$$f\left(x_{\frac{5}{2}}\right) = 0{,}77680,$$

$$f\left(x_{\frac{7}{2}}\right) = 0{,}61263,$$

$$f\left(x_{\frac{9}{2}}\right) = 0{,}44486.$$

a) To`g`ri tortburchaklar formulasi (4) bo`yicha.

$$\int_0^1 e^{-x^2}dx \approx$$

$$\approx \frac{1}{5}(0{,}99005 + 0{,}91393 + 0{,}77680 + 0{,}61263 + 0{,}44486) =$$

$$= \frac{1}{5} \cdot 3{,}74027 \approx 0{,}74805$$

b). Trapetsiyalar formulasi (5) bo`yicha

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{5}\left(\frac{1{,}00000 + 0{,}36788}{2} + 0{,}96079 + 0{,}85214 +\right.$$

$$\left.+0{,}69768 + 0{,}52729\right) = \frac{1}{5}(0{,}68394 + 3{,}03790) =$$

$$= \frac{1}{5} \cdot 3{,}72184 \approx 0{,}74437$$

v) Simpson formulasi (6) bo`yicha

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{30}[(1{,}00000 + 0{,}36788) +$$

$$+4(0{,}99005 + 0{,}91393 + 0{,}77680 + 0{,}61263 + 0{,}44486) +$$

$$+2(0{,}96079 + 0{,}85214 + 0{,}69768 + 0{,}52729)] =$$

$$\frac{1}{30}(1{,}36788 + 4 \cdot 3{,}74027 + 2 \cdot 3{,}03790) =$$

$$\frac{1}{30}(1{,}36788 + 6{,}07580 + 14{,}96108) \approx 0{,}74682$$

Taqribiy formulalar yordamida hisoblab topilgan

$$\int_0^1 e^{-x^2}dx$$

integralning qiymatini, uning

$$\int_0^1 e^{-x^2}dx = 0{,}74685\ \ldots$$

qiymati bilan taqqoslab, Simpson formulasi yordamida topilgan integralning taqribiy qiymati aniqroq ekanini ko`ramiz.

ANIQ INTEGRALNING BA'ZI BIR TADBIQLARI

1-§. Tekis shaklning yuzini hisoblash

$f(x)$ funksiya $[a,\ b]$ segmentda aniqlangan, uzliksiz hamda ixtiyoriy $x \in [a,\ b]$ da $f(x) \geq 0$ bo`lsin. Yuqoridan $f(x)$ funksiya grafigi, yon tomonlardan $x = a,\ x = b$ vertikal chiziqlar hamda pastdan $OX$ –abssissa o`qi bilan chegaralangan shaklni qaraylik.

Odatda, bunday tekis shakl egri chiziqli trapetsiya deyiladi. Uni $aABb$ deylik.

Agar $f(x)$ funksiya $[a,\ b]$ segmentda o`zgarmas, ya'ni

$$f(x) = C = const$$

bo`lsa, u holda $aABb$ shakl to`g`ri to`rtburchak bo`lib, uning yuzi

$$S = C \cdot (b - a)$$

bo`ladi.

Agar $f(x)$ funksiya ixtiyoriy uzluksiz funksiya bo`lsa, unda $aABb$ shaklning yuzi quyidagicha topiladi:

$[a,\ b]$ segmentni

$$a = x_0, \qquad x_1, \qquad x_2, \dots \dots, x_{n-1}, \qquad x_n = b$$

$$(x_0 < x_1 < x_2 < \cdots < x_n)$$

nuqtalar yordamida n ta bo`lakka bo`lamiz va har bir

$[x_k, x_{k+1}] \qquad (k = 0, 1, 2, \dots \dots, n - 1)$

segmentda ixtiyoriy

$$\xi_k \qquad (\xi_k \in [x_k, x_{k+1}])$$

nuqta olamiz.

So`ng har bir $[x_k,\ x_{k+1}]$ segmentda $f(x)$ funksiyani o`zgarmas va uni $f(\xi_k)$ ga teng qilib olsak, u holda $x_k\, A_k\, B_k\, x_{k+1}$ egri chiziqli trapetsiyaning yuzini

$$f(\xi_k) \cdot (x_{k+1} - x_k)$$

deb olish mumkin bo`lib, $aABb$ shaklning yuzini esa

$$S \approx f(\xi_0) \cdot (x_1 - x_0) + f(\xi_1) \cdot (x_2 - x_1) + \cdots +$$

$$+ f(\xi_k) \cdot (x_{k+1} - x_k) + \cdots + f(\xi_{n-1}) \cdot (x_n - x_{n-1})$$

deyish mumkin. Demak,

$$S \approx \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k \qquad (1)$$

bunda $\Delta x_k = x_{k+1} - x_k$.

$aABb$ egri chiziqli trapetsiyaning yuzini ifodalovchi formula taqribiy formuladir.

Endi $[a,\ b]$ segmentning bo`laklari sonini shunday orttira boraylikki, bunda har bir $[x_k, x_{k+1}.]$ segmentning uzunligi Δx_k nolga intila borsin. U holda

$$\sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

yig`indining miqdori ham o`zgara boradi.

Ma'lumki, bu holda

$$\lim_{\lambda \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \int_a^b f(x) dx.$$

bo`ladi.

Demak, egri chiziqli trapetsiyaning yuzi

$$S = \int_a^b f(x) dx$$

aniq integral orqali topiladi.

Eslatma. Agar egri chiziqli trapetsiya OX o`qidan pastda joylashgan bo`lsa, $(f(x) < 0,\ \ x \in [a,\ b])$ unda bu shaklning yuzi ushbu

$$S = -\int_a^b f(x) dx$$

formula yordamida topiladi.

Masalan, yuqoridan $f(x) = x^2$ parabola, yon tomonlardan $x = 1$, $x = 3$ vertikal to`g`ri chiziqlar va pastdan OX o`qi bilan chegaralangan shaklning yuzi yuqorida keltirilgan formulaga ko`ra

$$S = \int_1^3 x^2\, dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1}{3} = 9 - \frac{1}{3} = \frac{26}{3} = 8\frac{2}{3}$$

bo`ladi.

Agar tekislikdagi shakil quyidagi

$$y = f_1(x), \qquad y = f_2(x), \qquad x = a, \qquad x = b$$

$(f_1(x),\ f_2(x)$ funksiyalar $[a,\ b]$ da uzluksiz va $f_1(x) > 0,\ f_2(x) > 0,\quad f_1(x) > f_2(x)$ chiziqlar bilan chegaralangan shaklning yuzi

$$Q = \int_a^b [f_1(x) - f_2(x)]\, dx$$

formula bilan topiladi.

Demak, manfiy bo`lmagan uzluksiz funksiyaning aniq integral egri chiziqli trapetsiyaning yuziga teng. Bu aniq integralning geometrik ma'nosini ifodalaydi.

2-§. Yoy uzunligi.

$f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzliksiz bo`lsin. Uning grafigi tekislikning

$$\big(a,\ f(a)\big) \text{ va } (b, f(b))$$

nuqtalari orasidagi egri chiziq-yoyni ifodalasin.

Agar $f(x)$ funksiya $[a,b]$ oraliqda o`zgarmas, $f(x) = c,\ c = const$ bo`lsa, bu funksiyaning grafigi tekislikda $(a,b),\ (b,c)$ nuqtalarni birlashtiruvchi to`g`ri chiziq kesmasi bo`lib, uning uzunligi

$$l_1 = b - a$$

bo`ladi.

Agar $f(x)$ funksiya $[a,b]$ oraliqda chiziqli funksiya, ya'ni $f(x) = kx + b$ $(k, b -$o'zgarmas sonlar) bo`lsa, bu funksiyaning grafigi $(a, f(a)),\ (b, f(b))$ nuqtalarni birlashtiruvchi to`g`ri chiziq kesmasi bo`lib, uning uzunligi

$$l_2 = \sqrt{(b-a)^2 + (f(b) - f(a))^2} = (b-a) \cdot \sqrt{1 + k^2}$$

bo`ladi.

Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan va uluksiz bo`lsin. Bu funksiyaning grafigi –chizmada tasvirlangan egri chziq yoyini bo`lsin.

Uni $\breve{AB}$ deb belgilaymiz. $[a,b]$ segmentda ixtiyoriy

$$a = x_0, \qquad x_1, x_2, \dots, x_n = b \quad (x_0 < x_1 < x_2 < \cdots < x_4)$$

nuqtalar olib, uni bu nuqtalar yordamida n ta bo`lakchalarga ajratamiz.

$$[x_0, x_1], [x_1, x_2], \dots, [x_k, x_{k+1}], \dots, [x_{n-1}, x_n]$$

$[a,b]$ segmentni bo`luvchi nuqtalar

$$x_0, x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{n-1}, x_n$$

orqali OY o`qiga parallel to`g`ri chiziqlar o`tkazamiz. Bu to`g`ri chiziqlarning $\overset{\frown}{AB}$ yoy bilan kesishgan nuqtalari

$$A_k(x_k, f(x_k)) \quad (k = 0,1,2, \dots n)$$

$A_0 = A,\ A_n = B$ bo`ladi. $\overset{\frown}{AB}$ yoyidagi bu nuqtalarni bir-biri bilan to`g`ri chiziq kesmalari yordamida birlashtirib, L_n siniq chiziqni hosil qilamiz. L_n siniq chiziq, $\overset{\frown}{AB}$ yoyga chizilgan siniq chiziq deyiladi. Bu siniq chiziq perimetri

$$l_n = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo`ladi.

Ravshanki, l_n perimetr $[a, b]$ oraliqning bo`linishiga, ya'ni $\Delta x_k = x_{k+1} - x_k$ ga bog`liq bo`ladi.

Avvaldagidek, $\Delta x_k = x_{k+1} - x_k$ larning eng kattasini λ bilan belgilaymiz.

Ta'rif. *Agar* $\lambda \to 0$ *da*

$$l_n = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]}$$

perimetr chekli limitga ega bo`lsa, $\overset{\frown}{AB}$ *yoy uzunlikka ega deyiladi* va

$$\lim_{\lambda \to 0} l_n = l$$

limit $\overset{\frown}{AB}$ yoyining uzunligi deyiladi.

$f(x)$ funksiya $[a, b]$ segmentda aniqlangan, uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo`lsin. Bu funksiyaning $[a, b]$ oraliqdagi grafigi $\overset{\frown}{AB}$ yoyni tasvirlasin.

Ma'lumki, $\overset{\frown}{AB}$ yoyga chizilgan siniq chiziqning perimetri

$$l_n = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo`ladi.

Har bir $[x_k, x_{k+1}]$ da $f(x)$ funksiyaga Lagranj teoremasini qo`llaymiz. U holda shunday

$$\xi_k \quad (\xi_k \in [x_k, x_{k+1}])$$

nuqta topiladiki,

$$f(x_{k+1}) - f(x_k) = f'(\xi_k) \cdot (x_{k+1} - x_k)$$

bo`ladi.

Demak,

$$l_n = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + f'^2(\xi_k) \cdot (x_{k+1} - x_k)^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_n$$

Bu tenglikning o`ng tomonidagi yig`indi

$$\sqrt{1 + f'^2(x)}$$

funksiyaning integral yig`indisini eslatadi.

Uning integral yig`indidan farqi shuki, integral yig`indidagi ξ_k nuqta ixtiyoriy bo`lgan holda, yuqoridagi yig`indida esa ξ_k nuqta $[x_k, x_{k+1}]$ oraliqdagi tayin nuqtadir. Ammo

$$\sqrt{1 + f'^2(x)}$$

funksiya integrallanuvchi bo`lganligi (chunki shartga ko'ra, $f'(x)$ uzluksiz) sababli buning ahamiyati yo`q. Demak,

$$\lim_{\lambda \to 0} ln = \lim_{\lambda \to 0} \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k = \int_a^b \sqrt{1 + f'^2(x)}\, dx$$

Bu esa $\overset{\frown}{AB}$ yoyga chizilgan perimetri $\lambda \to 0$ da chekli limitga ega bo`lishini va u limit

$$\int_a^b \sqrt{1 + f^2(x)}\, dx$$

integralga teng ekanini bildiradi.

Demak $\overset{\frown}{AB}$ yoy uzunlikka ega va bu yoy uzunligi

$$l = \int_a^b \sqrt{1 + f'^2(x)}\, dx$$

formula yordamida (ya'ni aniq integral yordamida) hisoblanadi.

Misol. Ushbu

$$f(x) = x^{\frac{3}{2}} \qquad (0 \le x \le 4)$$

funksiya tasvirlagan egri chiziqning uzunligi topilsin.

◅ Avvalo berilgan funksiyaning hosilasini hisoblaymiz:

$$f'(x) = (x^{\frac{3}{2}})' = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}}$$

Undan

$$1 + f'^2(x) = 1 + \frac{9}{4}x, \quad \sqrt{1 + f'^2(x)} = \sqrt{1 + \frac{9}{4}x}$$

bo`lib,

$$l = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

bo`ladi. Bu integralda

$$1 + \frac{9}{4}x = t, \qquad dx = \frac{4}{9}dt, \qquad 1 \le t \le 10$$

almashtirish bajaramiz. Natijada

$$\int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9}\int_1^{10} t^{\frac{1}{2}}\, dt = \frac{8}{27} \cdot t^{\frac{1}{2}}\, |_1^{10} =$$

$$\frac{8}{27}\left(\sqrt{1000} - 1\right) = \frac{8}{27}\left(10\sqrt{10} - 1\right)$$

bo`ladi. Demak, yoy uzunli

$$l = \frac{8}{27}\left(10\sqrt{10} - 1\right)$$

ga teng. ⊳

3-§. O`zgaruvchi kuchning bajargan ishi

Aytaylik, biror jism OX o`qi bo`ylab F kich ta’sirida harakat qilayotgan bo`lsin. Bunda $F$ kich jismning $OX$ o`qidagi xolatiga bog`liq, ya’ni

$F = F(x)$ va uning yo`nalishi harakat yo`nalishi bilan ustma-ust tushsin. Bu kuch ta’sirida jismni a nuqtadan b nuqtaga o`tkazish uchun bajarilgan ishni topish masalasi yozaga keladi.

Ma’lumki, $F = F(x)$ kuch $[a, b]$ oraliqda

$$F(x) = c, \quad c - const$$

bo`lsa, jismni $a$ nuqtadan $b$ nuqtaga o`tkazish uchun bajarilgan ish

$$A = c \cdot (b - a)$$

formula bilan ifodalanadi.

$F = F(x)$ kuch $[a, b]$ da x o`zgaruvchining ixtiyoriy uzluksiz funksiyasi bo`lsin. U holda $[a, b]$ oraliqni ushbu nuqtalar yordamida

$$a = x_0, x_1, x_2, \dots \dots, x_n = b \quad (x_0 < x_1 < x_2 < \dots < x_n)$$

n ta

$[x_0, x_1], [x_1, x_2], \dots, [x_k, x_{k+1}], \dots, [x_{n-1}, x_n]$

bo`laklarga ajratib, har bir bo`lakchada ixtiyoriy $\xi_k \quad (k = 0, 1, 2, \dots \dots, n-1)$

$$\xi_k \in [x_k, x_{k+1}]$$

nuqta olamiz.

Agar har bir $[x_k, x_{k+1}] \quad (k = 0, 1, 2, \dots \dots, n-1)$ oraliqda jismga ta'sir etayotgan $F(x)$ kuchni o`zgarmas va $F(\xi_k)$ ga teng deb olinsa, u holda $[x_k, x_{k+1}]$ oraliqda bajarilgan ish taxminan

$$F(\xi_k) \cdot (x_{k+1} - x_k)$$

formula bilan, $[a, b]$ oraliqda bajarilgan ish esa taxminan

$$A \approx \sum_{k=0}^{n-1} F(\xi_k) \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} F(\xi_k) \cdot \Delta x$$

formula bilan ifodalanadi.

Agar $\lambda \to 0$ da

$$\sum_{k=0}^{n-1} F(\xi_k) \cdot \Delta x_k$$

yig`indi chekli songa intilsa bu sonni $F(x)$ kuchning $[a, b]$ oraliqdagi bajargan ishi deyilishi mumkin. Demak

$$A = \lim_{\lambda \to 0} \sum_{k=0}^{n-1} F(\xi_k) \cdot \Delta x_k$$

Ravshanki, qaralayotgan yig`indi $F = F(x)$ funksiyaning $[a, b]$ oraliq bo`yicha integral yig`indisi bo`ladi. $F = F(x)$ funksiya esa shartga ko`ra $[a, b]$ da uzluksiz. Demak, yig`indining limiti mavjud va u

$$\int_a^b F(x)\, dx$$

ga teng bo`ladi:

$$\lim_{\lambda\to 0}\sum_{k=0}^{n-1} F(\xi_k)\cdot\Delta x_k = \int_a^b f(x)\,dx$$

Shunday qilib, o`zgaruvchi $F(x)$ kuchning $[a,b]$ oraliqdagi bajargan ishi

$$A = \int_a^b F(x)\,dx$$

bo`ladi.

Misol. Vintsimon prujinaning bir uchi mustahkamlangan, ikkinchi uchiga esa $F = F(x)$ kuch ta’sir etib, prujina qisilgan. Agar prujinaning qisilishi unga ta’sir etayotgan $F(x)$ kuchga proportsional bo`lsa, prujinani a birlik qisish uchun $F(x)$ kuchning bajargan ishi topilsin.

⊲ Agar $F(x)$ kuch ta’sirida prujinaning qisilishi miqdorini x orqali belgilasak, u holda

$$F(x) = k\cdot x$$

bo`ladi, bunda k – proportsionallik koeffisienti (qisilish koeffisienti). Yuqoridagi formuladan foydalanib bajarilgan ishni topamiz:

$$A = \int_0^a kx\,dx = k\cdot\frac{x^2}{2}\Big|_0^a = \frac{ka^2}{2}. \quad \triangleright \qquad \blacksquare$$

I. Tekis egri chiziq yoyining uzunligi.

1. $y^2 = x^3$ egri chziqning $x = \frac{4}{3}$ to’g’ri chziq bilan Kesilgan qismining uzunligi. $\frac{112}{27}$;
2. $x^{2/3} + y^{2/3} = a^{2/3}$ egri chiziqning butun uzunligi. 6a;
3. $y^2 = (x+1)^3$ egri chiziqning x=4 to’g’ri chziq bilan kesilgan qismining uzunligi. $\frac{670}{27}$;
4. $x = a(t - sint)$, $y = a(1 - cost)$ Sikloidaning bir davrining uzunligi. 8a;
5. $x = \frac{t^6}{6}$, $y = 2 - \frac{t^4}{4}$ egri chiziqning koordinata o’qlari bilan kesishgan nuqtalari orasidagi qisming $t_1 = 0\ va$ $t_2 = \sqrt[4]{8}$;

uzunligi. $S = \int_0^{\sqrt[4]{8}} \sqrt{t^4+1}\, t^3 dt = 4\frac{1}{3}$;

6. $y = \ln x$ ning $x = \frac{3}{4}$ dan $x = \frac{12}{5}$ gacha bo'lgan qismining uzunligi. ; $S = \int_{\frac{3}{4}}^{\frac{12}{5}} \frac{\sqrt{1+x^2}}{x} dx$;

7. $y^2 = \frac{4}{9}(2-x)^3$ ning $x = -1$ to'g'ri chiziq bilan kesilgan qismining uzunligi. $\frac{28}{3}$;

8. $y = \ln(1 - x^2)$ ning $x = -\frac{1}{2}$ dan $x = \frac{1}{2}$ gacha qismining uzunligi. *2ln3-1;*

9. $y^2 = 2px$ ning $x = \frac{p}{2}$ to'g'ri chiziq bilan kesilgan qismining uzunligi. $P[\sqrt{2} + \ln(1+\sqrt{2})] \approx 2{,}29p$;

Chiziqlar bilan chegaralangan sohaning yuzini hisoblash.

Misollar Javoblar

1. $y = x^2$, $x + y = 2$; ;

2. $y = 2x - x^2$, $x + y = 0$; ;

$= 2^x$, $y = 2$, $x =$

3. ;

4. $y = |\lg x|$, $y = 0$, $x = 0.1$, $x = 10$; $9.9 - 8.1\lg e \approx 6.38$;

5. $y = 2 - x^2$, $y^3 = x^2$;

6. $y = 4x - x^2, \qquad y = 0$ $\qquad \frac{32}{3}$;

7. $y = lnx, \qquad y = 0, \qquad x = e$ $\qquad$ 1;

8. $y^3 = x, \qquad y = 1, \qquad x = 8;$

9. $y = sinx, \qquad y = 0$; $\qquad$ 2;

10. $y = x^3, \qquad y = 8, \qquad x = 0$; $\qquad$ 12;

11. $y = 2x - x^2, \qquad y = -x$; $\qquad$;

12. $y = 3 - 2x, \qquad y = x^2$; $\qquad$;

13. $y = x^2, \qquad y = \frac{x^2}{2}, \qquad y = 2x$; $\qquad$ 4;

14. $y = \frac{x^2}{3}, \qquad y = 4 - \frac{2}{3}x^2$; $\qquad \frac{32}{3}$;

15. $y^2 = 2x + 1, \qquad x - y - 1 = 0$; $\qquad \frac{16}{3}$;

XOSMAS INTEGRALLAR.

1-§. Chegaralari cheksiz (cheksiz oraliq bo`yicha) xosmas integrallar.

$f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan va uzluksiz bo`lsin. Bu oraliqning istalgan chekli $[a,t]$ $(a<t<+\infty)$ qismi bo`yicha

$$\int_a^t f(x)dx$$

integralni qaraylik. Ravshanki, bu integral t ga bog`liq bo`ladi:

$$F(t)=\int_a^t f(x)dx$$

Shunday qilib, berilgan $f(x)$ funksiya yordamida $F(t)$ funksiya yuzaga keldi.

Ta'rif. *Agar* $t\to+\infty$ *da* $F(t)$ *funksiyaning limiti mavjud bo`lsa, bu limit* $f(x)$ *funksiyaning* $[a,+\infty)$ *oraliq bo`yicha xosmas integrali deyiladi* va

$$\int_a^{+\infty} f(x)dx$$

kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx=\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}\int_a^t f(x)dx$$

Odatda

$$\int_a^{+\infty} f(x)dx$$

integral chegarasi cheksiz (yoki cheksiz oraliq bo`yicha) xosmas integral deb ham yuritiladi.

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo`lsa,

$$\int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi deyiladi.

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti cheksiz (yoki limit mavjud bo`lmasa) bo`lsa,

$$\int_a^{+\infty} f(x)$$

xosmas integral uzoqlashuvchi deyiladi.

Aytaylik, $f(x)$ funksiya $(-\infty, a]$ yoki $(-\infty, +\infty)$ oraliqda berilgan va uzluksiz bo`lsin.

Bu funksiyaning $(-\infty, a]$ va $(-\infty, +\infty)$ oraliqlar bo`yicha xosmas integrallari ham yuqoridagi kabi ta'riflanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty} \int_{t}^{a} f(x)dx \qquad (-\infty < t < a),$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{t\to-\infty \\ v\to+\infty}} \int_{t}^{v} f(x)dx \qquad (-\infty < t < v < +\infty)$$

Masalan,

$$\int_{0}^{+\infty} e^{-2x}\, dx$$

integral uchun

$$\int_{0}^{+\infty} e^{-2x}\, dx = \lim_{t\to+\infty} \int_{0}^{t} e^{-2x}\, dx = \lim_{t\to+\infty} \frac{-1}{2} \int_{0}^{t} e^{-2x}\, d\,(-2x) =$$

$$= -\frac{1}{2} \lim_{t\to-\infty} e^{-2x} \,|_0^t = -\frac{1}{2} \lim_{t\to-\infty} (e^{-2t} - e^{0}) = -\frac{1}{2}(0-1) = \frac{1}{2}$$

bo`ladi.

Demak, qaralayotgan xosmas integral yaqinlashuvchi va

$$\int_{0}^{+\infty} e^{-2x}\, dx = \frac{1}{2}$$

Misol. Ushbu

$$J = \int_{a}^{+\infty} \frac{dx}{x^{\alpha}} \qquad (\alpha > 0, a > 0)$$

integral yaqinlashuvchilikka tekshirilsin.

◅ Aytayilk, $\alpha > 1$ bo`lsin. Bu holda

$$\int_{a}^{+\infty} \frac{dx}{x^{\alpha}} = \lim_{t\to+\infty} \int_{a}^{t} \frac{dx}{x^{\alpha}} = \lim_{t\to+\infty} \left(\frac{x^{-\alpha+1}}{-\alpha+1} \right)_{a}^{t} =$$

$$= \lim_{t\to+\infty} \frac{1}{1-\alpha} (t^{1-\alpha} - a^{1-\alpha}) = \frac{-1}{1-\alpha} a^{1-\alpha}$$

bo`ladi. Demak, $\alpha > 1$ bo`lganda

$$\int_{a}^{+\infty} \frac{dx}{x^{\alpha}}$$

xosmas integral yaqinlashuvchi.

Aytaylik, $\alpha < 1$ bo`lsin. Bu holda

$$\int_{a}^{+\infty} \frac{dx}{x^{\alpha}} = \lim_{t\to+\infty} \int_{a}^{t} x^{-\alpha}dx = \lim_{t\to+\infty} (t^{1-\alpha} - a^{1-\alpha})\frac{1}{1-\alpha} = +\infty.$$

Demak, $\alpha < 1$ bo`lganda

$$\int_{a}^{+\infty} \frac{dx}{x}$$

uzoqlashuvchi.

Aytaylik, $\alpha = 1$ bo`lsin. Bu holda

$$\int_{a}^{+\infty} \frac{dx}{x} = \lim_{t\to+\infty} \int_{a}^{t} \frac{dx}{x} = \lim_{t\to+\infty} (\ln t - \ln a) = +\infty$$

bo`ladi. Demak, $\alpha = 1$ bo`lganda

$$\int_{a}^{+\infty} \frac{dx}{x}$$

integral uzoqlashuvchi.

Shunday qilish

$$\int_{a}^{+\infty} \frac{dx}{x^{\alpha}} \qquad (\alpha > 0, a > 0)$$

xosmas integral $\alpha > 1$ bo`lganda yaqinlashuvchi $\alpha \leq 1$ bo`lganda uzoqlashuvchi bo`ladi.

2°. Yaqinlashuvchi xosmas integralning xossalari.

Xosmas integrallar ham aniq integrallarga o`xshash xossalarga ega. Biz ularni isbotsiz keltiramiz.

1) Agar $f(x)$ funksiyaning

$$\int_{a}^{+\infty} f(x)dx$$

xosmas integrali yaqinlashuvi bo`lsa, bu funksiyaning

$$\int_{b}^{+\infty} f(x)dx \quad (a < b < +\infty)$$

xosmas integrali ham yaqinlashuvchi bo`ladi va aksincha. Bunda

$$\int_{a}^{+\infty} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx$$

bo`ladi.

2). Agar

$$\int_{a}^{+\infty} f(x)dx$$

integral yaqinlashuvchi va k o`zgarmas son bo`lsa, u holda

$$\int_{a}^{+\infty} kf(x)dx$$

integral ham yaqinlashuvchi bo`ladi va

$$\int_{a}^{+\infty} kf(x)dx = k\int_{a}^{+\infty} f(x)dx.$$

3). Agar

$$\int_{a}^{+\infty} f(x)dx \ \text{va} \int_{a}^{+\infty} g(x)dx$$

integrallar yaqinlashuvch bo`lsa, u holda

$$\int_{a}^{+\infty} [f(x) \pm g(x)]dx$$

integral ham yaqinlashuvchi bo`ladi va

$$\int_{a}^{+\infty} [f(x) \pm g(x)]dx = \int_{a}^{+\infty} f(x)dx \pm \int_{a}^{+\infty} g(x)dx$$

4). Agar

$$\int_{a}^{+\infty} f(x)dx$$

integral yaqinlashuvchi bo`lib, ixtiyoriy $x \in [a, +\infty)$ da

$$f(x) \geq 0$$

bo`lsa, u holda

$$\int_{a}^{+\infty} f(x)dx \geq 0$$

bo`ladi.

5) Agar ixtiyoriy $x \in [a, +\infty)$ da

$$f(x) \leq g(x)$$

bo`lib,

$$\int_{a}^{+\infty} f(x)dx \text{ va } \int_{a}^{+\infty} g(x)dx$$

integrallar yaqinlashuvchi bo`lsa, u holda

$$\int_{a}^{+\infty} f(x)dx \leq \int_{a}^{+\infty} g(x)dx$$

bo`ladi.

Tasdiq. *$f(x)$ va $g(x)$ funksiyalar $[a, +\infty)$ oraliqda berilgan va uzluksiz bo`lib, ixtiyoriy $x \in [a, +\infty)$ da*

$$f(x) \geq 0, \qquad g(x) \geq 0, \qquad f(x) \leq g(x)$$

bo`lsin. U holda

$$\int_{a}^{+\infty} g(x)dx$$

integral yaqinlashuvchi bo`lsa,

$$\int_{a}^{+\infty} f(x)dx$$

integral ham yaqinlashuvchi bo`ladi.

Misol. Ushbu

$$\int_0^{+\infty} e^{-x^2}\, dx$$

xosmas integral yaqinlashuvchilikka tekshirilsin.

◅ Ma'lumki $x \geq 1$ da

$$e^{-x^2} \leq \frac{1}{x^2}$$

bo`ladi. Ravshanki,

$$\int_1^{+\infty} \frac{dx}{x^2}$$

xosmas integral yaqinlashuvchi. Unda yuqorida keltirilgan tasdiqqa ko`ra

$$\int_1^{+\infty} e^{-x^2} dx$$

xosmas integral ham yaqinlashuvchi bo`ladi.

Avvaldan bilamizki,

$$\int_0^1 e^{-x^2} dx$$

mavjud. Unda 1)-xossadan foydalanib

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{+\infty} e^{-x^2} dx$$

quyidagi

$$J = \int_0^{+\infty} e^{-x^2} dx$$

xosmas integralning yaqinlashuvchi bo`lishini topamiz.

2-§. Chegaralanmagan funksiyaning xosmas integrali.

$f(x)$ funksiya $[a,b)$ yarim intervalda berilgan va uzluksiz bo`lib, $(t,b)$ intervalda $(a<t<b)$ chegaralanmagan bo`lsin. Bu funksiyaning $[a,b)$ ning istalgan $[a,t]$ qismidagi $(a<t<b)$

$$\int_a^t f(x)dx$$

t ga bog`liq bo`ladi:

$$\Phi(t)=\int_a^t f(x)dx$$

Ta'rif. *Agar* $t\to b-0$ *(ya'ni* $t\to b;\;\; t<b$*) da* $\Phi(t)$ *funksiyaning limiti mavjud bo`lsa, bu limit chegaralanmagan* $f(x)$ *funksiyaning* $[a,\ b)$ *oraliq bo`yicha xosmas integrali deyiladi* va

$$\int_a^b f(x)dx$$

kabi belgilanadi.

$$\int_a^b f(x)dx=\lim_{t\to b-0}\Phi(t)=\lim_{\substack{t\to b\\ t<b}}\Phi(t)=\lim_{t\to b-0}\int_a^t f(x)dx,$$

Agar $t\to b-0$ da

$$\Phi(t)=\int_a^t f(x)dx,$$

ning limiti mavjud va chekli bo`lsa,

$$\int_a^b f(x)dx,$$

xosmas integral yaqinlashuvchi deyiladi.

Agar $t\to b-0$ da $\Phi(t)$ funksiyaning limiti cheksiz yoki mavjud bo`lmasa,

$$\int_a^b f(x)dx,$$

xosmas integral uzoqlashuvchi deyiladi.

Chegaralanmagan $f(x)$ funksiyaning $(a,b]$ (yoki (a,b)) oraliq bo`yicha xosmas integrali ham yuqoridagidek ta’riflanadi:

$$\int_a^b f(x)dx = \lim_{t\to a+0}\int_t^b f(x)dx, \quad (a<t<b),$$

$$\left(\int_a^b f(x)dx \lim_{\substack{n\to b-0\\ v\to a+0}}\int_v^n f(x)dx \qquad (a<v<n<b)\right)$$

Misol. Ushbu

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

integral yaqinlashuvchilikka tekishirilsin.

◅ $x = 1$ nuqta atrofida

$$f(x) = \frac{1}{\sqrt{1-x}}$$

funksiya chegaralanmagan.

Demak, berilgan integral chegaralanmagan funksiyaning xosmas integrali. Ta’rifga ko`ra

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = \lim_{t\to 1-0}\int_0^t \frac{1}{\sqrt{1-x}}dx$$

bo`ladi. Agar

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = -\int_0^1 (1-x)^{-\frac{1}{2}}d(1-x) = -\frac{(1-x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_0^t =$$

$$= -2[(1-t)^{-\frac{1}{2}} - \left(1-0\right)^{\frac{1}{2}}\Big] = 2 - 2\sqrt{1-t}$$

bo`lishini e’tiborga olsak, unda

$$\lim_{t\to 1-0}\int_0^1 \frac{1}{\sqrt{1-x}} = \lim_{t\to 1-0}(2-2\sqrt{1-t}) = 2$$

bo`lib,

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

ekanligini topamiz. Demak, berilgan xosmas integral yaqinlashuvi va u 2 ga teng.⊳

Misol. Ushbu

$$\int_0^1 \frac{dx}{x}$$

integral yaqinlashuvchilikka tekshirilsin.

⊲ Ravshanki, $x = 0$ nuqta $f(x) = \frac{1}{x}$ funksiyaning maxsus nuqtasi. Xosmas integral ta’rifiga ko`ra

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0+0} \int_t^1 \frac{dx}{x} = \lim_{t\to 0+0} [\ln x]_t^1 =$$

$$= \lim_{t\to 0+0} [\ln 1 - \ln t] = \infty$$

bo`ladi. Demak berilgan xosmas integral uzoqlashuvchi.⊳

Misol. Ushbu

$$A = \int_a^b \frac{dx}{(x-a)^\alpha}, \quad B = \int_a^b \frac{dx}{(b-x)^\alpha} \qquad (\alpha > 0)$$

integrallar yaqinlashuvchilikka tekshirilsin.

⊲ Aytaylik, $\alpha \neq 1$ bo`lsin. Bu holda

$$\int_a^b \frac{dx}{(x-a)^\alpha} = \lim_{t\to a+0} \int_t^b \frac{dx}{(x-a)^\alpha} = \lim_{t\to a+0} \int_t^b (x-a)^{-\alpha}\, d(x-a) =$$

$$= \lim_{t\to a+0} \left.\frac{(x-a)^{-\alpha+1}}{-\alpha+1}\right|_t^b = \lim_{t\to a+0} \frac{1}{1-\alpha}\left[(b-a)^{1-\alpha} - (t-a)^{1-\alpha}\right]$$

bo`lib, bu limit $\alpha < 1$ bo`lganda chekli, $\alpha > 1$ bo`lganda esa cheksiz bo`ladi.

Aytaylik, $\alpha = 1$ bo`lsin. Bu holda

$$\int_a^b \frac{dx}{x-a} = \lim_{t\to a+0} \int_t^b \frac{dx}{x-a} = \lim_{t\to a+0} \int_t^b \frac{d(x-a)}{x-a} =$$

$$= \lim_{t\to a+0} [\ln(t-a)]_t^b = \infty$$

bo`ladi. Demak,

$$A = \int_a^b \frac{dx}{(x-a)^\alpha} \qquad (\alpha > 0)$$

xosmas integral $\alpha < 1$ bo`lganda yaqinlashuvchi, $\alpha \geq 1$ bo`lganda uzoqlashuvchi bo`ladi.

Xuddi shunga o`xshash

$$B = \int_a^b \frac{dx}{(b-x)^\alpha} \qquad (\alpha > 0)$$

xosmas integral $\alpha < 1$ bo`lganda yaqinlashuvchi, $\alpha \geq 1$ bo`lganda uzoqlashuvchi bo`lishi ko`rsatiladi.

Chegaralanmagan funksiyaning xosmas integrallari ham bayon etilgan chegaralari cheksiz xosmas integralning xossalari kabi xossalarga ega bo`ladi.

Xosmas integrallar.

Agar $y = f(x)$ funktsiya $a \leq x < +\infty$ da uzluksiz bo'lib, $\int_a^b f(x)dx = J(B)$ bo'lsa, bunda $J(b)$ biror uzluksiz funktsiya , ushbu

$$\lim_{b\to+\infty} \int_a^b f(x)dx \quad (1)$$

limit yuqori chegarasi cheksiz bo'lgan xosmas integral deyiladi va $\int_a^{+\infty} f(x)dx$ (2) kabi belgilanadi.

Demak, ta'rifga ko'ra;

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx$$

Agar (1) limit mavjud bo'lsa (2) xosmas integral yaqinlashuvchi, agar (1)limit mavjud bo'lmasa, yoki cheksizlikka intilsa,(2) xosmas integral uzoqlashuvchi deyiladi.

Misollar	Javoblar
1. $\int_0^\infty \frac{dx}{1+x^2}$	$\frac{\pi}{2}$
2. $\int_0^1 \frac{dx}{\sqrt{x}}$	2
3. $\int_0^3 \frac{dx}{((x-1))^2}$	uzoqlashuvshi
4. $\int_1^\infty \frac{dx}{x^2}$	1

5. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 9}$ $\frac{\pi}{\sqrt{5}}$

6. $\int_{-1}^{2} \frac{dx}{x}$ uzoqlashuvshi

7. $\int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}}$ $\frac{\pi}{2}$

8. $\int_{0}^{\infty} sinxdx$ uzoqlashuvshi

9. $\int_{0}^{\frac{1}{2}} \frac{dx}{xlnx}$ uzoqlashuvshi

10. $\int_{0}^{\infty} \frac{arctgx}{x^2 + 1}$ $\frac{\pi^2}{8}$

11. $\int_{1}^{\infty} \frac{dx}{x^4}$

12. $\int_{1}^{\infty} \frac{dx}{\sqrt{x}}$ uzoqlashuvshi

13. $\int_{2}^{\infty} \frac{lnx}{x} dx$ uzoqlashuvshi

14. $\int_{0}^{\infty} xe^{-x^2} dx$

15. $\int_{0}^{\infty} xsinxdx$ uzoqlashuvshi

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