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On One Boundary-Value Problem for an Equation of Higher Even Order

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Abstract—We study boundary-value problem for an equation of even order in a rectangular domain. By the spectral method we obtain necessary and sufficient conditions of uniqueness of a solution. The solution is constructed in the form of infinite series in eigenfunctions. We obtain sufficient conditions under which this series is a regular solution.

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Introduction. Let us consider partial differential equation

$$Lu = D_t^{2n} u\,(x,t) - D_x^{2n} u\,(x,t) = 0 \tag{1}$$

in a rectangular domain $\Omega = \{(x,t) : 0 < x < l, 0 < t < T\}$, where l and T are given positive numbers, $n \in \mathbb{N}$, and formulate the following problem.

Problem A. *In a domain Ω, find a function $u\,(x,t)$, satisfying the conditions*

$$u \in C^{2n-1}(\overline{\Omega}) \cap C^{2n}\,(\Omega)\,, \tag{2}$$

$$Lu\,(x,t) \equiv 0, \;\; (x,t) \in \Omega, \tag{3}$$

$$D_x^k u\,(0,t) = D_x^k u\,(l,t) = 0, \;\; 0 \le t \le T, \tag{4}$$

$$D_t^k u\,(x,0) = \varphi_k\,(x)\,, \;\; 0 \le x \le l, \tag{5}$$

$$D_t^k u\,(x,T) = \psi_k\,(x)\,, \;\; 0 \le x \le l, \tag{6}$$

where $k = 0, \ldots, n-1$, $\varphi_k\,(x)$, $\psi_k\,(x)$ are given sufficiently smooth functions.

With $n = 1$ Eq. (1) is the well-known string equation. Equation (1), where edge conditions are given for derivatives of even orders (the Dirichlet problem) was investigated in [1], where one shows that the irrationality of relation T/l is a necessary and sufficient condition of uniqueness of the solution to the Dirichlet problem for Eq. (1) with any $n \in \mathbb{N}$. Estimates were found, which allow to establish the existence of a solution to the Dirichlet problem. The Dirichlet problem for a more general equation of order $2n$, when $l = 1$, $T \le 1$, was investigated in [2], where one shows the resolvability of the problem for almost all T. The spectral problem for an equation of even order with the potential $q\,(x,y)$ and edge Dirichlet conditions was investigated in [3]. The singularity of the Dirichlet problem is the fact that solving this problem by the Fourier method, the proper functions are composed of sines, and the eigenvalues are explicitly calculated. It is not true for problem A.

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Making certain transformations, in what follows we will investigate the equation

$$Lu = a^{2n} D_t^{2n} u\,(x,t) - D_x^{2n} u\,(x,t) = 0, \tag{7}$$

where $a = T/l$, in the square $\Omega = \{(x,t) : 0 < x < 1, 0 < t < 1\}$ with boundary conditions (4)–(6) with $T = l = 1$.

1. Uniqueness of solution to problem A. The following theorem holds true.

Theorem 1. *If problem* A (*for Eq.* (7)) *has a solution, then it is unique for almost all values of* a.

Proof. Solving the problem by the Fourier method $u\,(x,t) = X\,(x)\,Y\,(t)$, with respect to the function $X(x)$ we obtain the following one-dimensional spectral problem:

$$\begin{gathered} X^{(2n)}\,(x) = (-1)^n \lambda^{2n} X\,(x)\,, \;\; \lambda > 0; \\ X^{(k)}\,(0) = X^{(k)}\,(1) = 0, \;\; k = \overline{0, n-1}. \end{gathered} \tag{8}$$

Preliminarily we prove the following lemma.

Lemma . *For proper values of problem* (8) *we have*

$$\lambda_k = \frac{\pi}{2} + \pi k + \varepsilon_{1k}, \;\; k = 1, 2, 3, \ldots, \;\; \textit{if n is even},$$

$$\lambda_k = \pi k + \varepsilon_{2k}, \;\; k = 1, 2, 3, \ldots, \;\; \textit{if n is odd},$$

here $\lim\limits_{k\to+\infty} \varepsilon_{1k} = \lim\limits_{k\to+\infty} \varepsilon_{2k} = 0.$

Proof. Let us investigate the case, when n is even, i.e., $n = 2m$. We consider the problem

$$\begin{gathered} X^{(4m)}\,(x) = \lambda^{4m} X\,(x)\,, \;\; \lambda > 0; \\ X^{(k)}\,(0) = X^{(k)}\,(1) = 0, \;\; k = \overline{0, 2m-1}. \end{gathered} \tag{9}$$

In order to understand the general case, we study the case when $m = 2$ (the case $m = 1$ was investigated in [4]):

$$\begin{gathered} X^{(8)}\,(x) = \lambda^8 X\,(x)\,, \;\; \lambda > 0; \\ X^{(k)}\,(0) = X^{(k)}\,(1) = 0, \;\; k = \overline{0, 3}. \end{gathered} \tag{10}$$

The general solution to Eq. (10) has the form $X(x) = X_1\,(x) + X_2\,(x)$, where

$$X_1\,(x) = c_1 e^{\lambda x} + e^{\lambda \alpha_1 x}\,(c_2 \cos \lambda\beta_1 x + c_3 \sin \lambda\beta_1 x) + c_4 \cos \lambda x,$$

$$X_2\,(x) = c_5 e^{-\lambda x} + e^{\lambda \alpha_2 x}\,(c_6 \cos \lambda\beta_2 x + c_7 \sin \lambda\beta_2 x) + c_8 \sin \lambda x,$$

$$\alpha_1 = \cos\theta_1 > 0, \;\; \alpha_2 = \cos\theta_2 < 0, \;\; \beta_1 = \sin\theta_1, \;\; \beta_2 = \sin\theta_2, \;\; \theta_1 = \frac{\pi}{4}, \;\; \theta_2 = \frac{3\pi}{4},$$

c_i are arbitrary constants. For derivatives of order $p = \overline{0, 3}$ we have

$$X_1^{(p)}\,(x) = \lambda^p \left\{ c_1 e^{\lambda x} + e^{\lambda\alpha_1 x} \left[c_2 \cos\left(\lambda\beta_1 x + p\theta_1\right) + c_3 \sin\left(\lambda\beta_1 x + p\theta_1\right)\right] + c_4 \cos\left(\lambda x + \tfrac{\pi}{2} p\right) \right\},$$

$$X_2^{(p)}\,(x) = \lambda^p \left\{ (-1)^p\, c_5 e^{-\lambda x} + e^{\lambda\alpha_k x} \left[c_6 \cos\left(\lambda\beta_k x + p\theta_k\right) + c_7 \sin\left(\lambda\beta_k x + p\theta_k\right)\right] + c_8 \sin\left(\lambda x + \tfrac{\pi}{2} p\right) \right\}.$$

Satisfying boundary conditions, we obtain the system of equations for finding c_i^k; the determinant of system has the form

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

where

$$A=\begin{pmatrix}1 & 1 & 0 & \cos\left(0\cdot\frac{\pi}{2}\right)\\ 1 & \cos\theta_1 & \sin\theta_1 & \cos\left(1\cdot\frac{\pi}{2}\right)\\ 1 & \cos 2\theta_1 & \sin 2\theta_1 & \cos\left(2\cdot\frac{\pi}{2}\right)\\ 1 & \cos 3\theta_1 & \sin 3\theta_1 & \cos\left(3\cdot\frac{\pi}{2}\right)\end{pmatrix},\quad B=\begin{pmatrix}1 & 1 & 0 & \sin\left(0\cdot\frac{\pi}{2}\right)\\ -1 & \cos\theta_2 & \sin\theta_2 & \sin\left(1\cdot\frac{\pi}{2}\right)\\ 1 & \cos 2\theta_2 & \sin 2\theta_2 & \sin\left(2\cdot\frac{\pi}{2}\right)\\ -1 & \cos 3\theta_2 & \sin 3\theta_2 & \sin\left(3\cdot\frac{\pi}{2}\right)\end{pmatrix},$$

$$C=\begin{pmatrix}e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,0} & e^{\lambda\alpha_1}\sin\sigma_{1,0} & \cos\left(\lambda+0\cdot\frac{\pi}{2}\right)\\ e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,1} & e^{\lambda\alpha_1}\sin\sigma_{1,1} & \cos\left(\lambda+1\cdot\frac{\pi}{2}\right)\\ e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,2} & e^{\lambda\alpha_1}\sin\sigma_{1,2} & \cos\left(\lambda+2\cdot\frac{\pi}{2}\right)\\ e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,3} & e^{\lambda\alpha_1}\sin\sigma_{1,3} & \cos\left(\lambda+3\cdot\frac{\pi}{2}\right)\end{pmatrix},$$

$$D=\begin{pmatrix}e^{-\lambda} & e^{\lambda\alpha_2}\cos\sigma_{2,0} & e^{\lambda\alpha_2}\sin\sigma_{2,0} & \sin\left(\lambda+0\cdot\frac{\pi}{2}\right)\\ -e^{-\lambda} & e^{\lambda\alpha_2}\cos\sigma_{2,1} & e^{\lambda\alpha_2}\sin\sigma_{2,1} & \sin\left(\lambda+1\cdot\frac{\pi}{2}\right)\\ e^{-\lambda} & e^{\lambda\alpha_2}\cos\sigma_{2,2} & e^{\lambda\alpha_2}\sin\sigma_{2,2} & \sin\left(\lambda+2\cdot\frac{\pi}{2}\right)\\ -e^{-\lambda} & e^{\lambda\alpha_2}\cos\sigma_{2,3} & e^{\lambda\alpha_2}\sin\sigma_{2,3} & \sin\left(\lambda+3\cdot\frac{\pi}{2}\right)\end{pmatrix},$$

$$\sigma_{k,p}=\lambda\beta_k+p\theta_k.$$

The proper values of problem (10) are zeros of this determinant. Preliminarily we make certain transformations in the determinant. We present an example of a part of determinant composed of two columns:

$$\begin{vmatrix}1 & 0\\ e^{\alpha_1}\cos\beta_1 & e^{\alpha_1}\sin\beta_1\\ e^{\alpha_2}\cos\beta_2 & e^{\alpha_2}\sin\beta_2\end{vmatrix}=\frac{1}{i}\begin{vmatrix}1 & 0\\ e^{\alpha_1}\cos\beta_1 & ie^{\alpha_1}\sin\beta_1\\ e^{\alpha_2}\cos\beta_2 & ie^{\alpha_2}\sin\beta_2\end{vmatrix}=\frac{1}{i}\begin{vmatrix}1 & 0\\ e^{\alpha_1}\cos\beta_1+ie^{\alpha_1}\sin\beta_1 & ie^{\alpha_1}\sin\beta_1\\ e^{\alpha_2}\cos\beta_2+ie^{\alpha_2}\sin\beta_2 & ie^{\alpha_2}\sin\beta_2\end{vmatrix}$$

$$=\frac{1}{i}\begin{vmatrix}1 & 0\\ e^{\alpha_1}e^{i\beta_1} & e^{\alpha_1}\frac{e^{i\beta_1}-e^{-i\beta_1}}{2}\\ e^{\alpha_2}e^{i\beta_2} & e^{\alpha_2}\frac{e^{i\beta_2}-e^{-i\beta_2}}{2}\end{vmatrix}=-\frac{i}{2}\begin{vmatrix}1 & 0\\ e^{\alpha_1}e^{i\beta_1} & e^{\alpha_1}\left(e^{i\beta_1}-e^{-i\beta_1}\right)\\ e^{\alpha_2}e^{i\beta_2} & e^{\alpha_2}\left(e^{i\beta_2}-e^{-i\beta_2}\right)\end{vmatrix}$$

$$=-\frac{i}{2}\begin{vmatrix}1 & -1\\ e^{\alpha_1}e^{i\beta_1} & e^{\alpha_1}\left(e^{i\beta_1}-e^{-i\beta_1}\right)-e^{\alpha_1}e^{i\beta_1}\\ e^{\alpha_2}e^{i\beta_2} & e^{\alpha_2}\left(e^{i\beta_2}-e^{-i\beta_2}\right)-e^{\alpha_2}e^{i\beta_2}\end{vmatrix}$$

$$=-\frac{i}{2}\begin{vmatrix}1 & -1\\ e^{\alpha_1}e^{i\beta_1} & -e^{\alpha_1}e^{-i\beta_1}\\ e^{\alpha_2}e^{i\beta_2} & -e^{\alpha_2}e^{-i\beta_2}\end{vmatrix}=(-1)^2\,\frac{i}{2}\begin{vmatrix}1 & 1\\ e^{\alpha_1}e^{i\beta_1} & e^{\alpha_1}e^{-i\beta_1}\\ e^{\alpha_2}e^{i\beta_2} & e^{\alpha_2}e^{-i\beta_2}\end{vmatrix}.$$

Now we make the above-stated transformations in the determinant Δ combining the following columns:

the second and third ones, the fourth and eighth ones, the sixth and seventh ones. Then we have

$$\Delta = \left(\frac{i}{2}\right)^3 \begin{vmatrix} A^* & B^* \\ C^* & D^* \end{vmatrix},$$

where

$$A^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\theta_1} & e^{-i\theta_1} & i \\ 1 & e^{2i\theta_1} & e^{-2i\theta_1} & i^2 \\ 1 & e^{3i\theta_1} & e^{-3i\theta_1} & i^3 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & e^{i\theta_2} & e^{-i\theta_2} & -i \\ 1 & e^{2i\theta_2} & e^{-2i\theta_2} & (-i)^2 \\ -1 & e^{3i\theta_2} & e^{-3i\theta_2} & (-i)^3 \end{pmatrix},$$

$$C^* = \begin{pmatrix} e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,0}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,0}} & e^{i\left(\lambda+0\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,1}} & e^{\lambda\alpha_1}e^{-i\sigma_{01}} & e^{i\left(\lambda+1\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,2}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,2}} & e^{i\left(\lambda+2\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,3}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,3}} & e^{i\left(\lambda+3\cdot\frac{\pi}{2}\right)} \end{pmatrix},$$

$$D^* = \begin{pmatrix} e^{-\lambda} & e^{\lambda\alpha_2}e^{i\sigma_{2,0}} & e^{\lambda\alpha_2}e^{-i\sigma_{2,0}} & e^{-i\left(\lambda+0\cdot\frac{\pi}{2}\right)} \\ -e^{-\lambda} & e^{\lambda\alpha_2}e^{i\sigma_{2,1}} & e^{\lambda\alpha_2}e^{-i\sigma_{2,1}} & e^{-i\left(\lambda+1\cdot\frac{\pi}{2}\right)} \\ e^{-\lambda} & e^{\lambda\alpha_2}e^{i\sigma_{2,2}} & e^{\lambda\alpha_2}e^{-i\sigma_{2,2}} & e^{-i\left(\lambda+2\cdot\frac{\pi}{2}\right)} \\ -e^{-\lambda} & e^{\lambda\alpha_2}e^{i\sigma_{2,3}} & e^{\lambda\alpha_2}e^{-i\sigma_{2,3}} & e^{-i\left(\lambda+3\cdot\frac{\pi}{2}\right)} \end{pmatrix}.$$

In order to show the existence of zero of the determinant and obtain the asymptotics of the determinant root, it suffices to calculate the expression, which contains the maximal positive degree of exponent, because the rate of increasing or decreasing of determinant value depends on it with large λ. This expression with accuracy up to the sign is calculated as follows:

$$\left(\frac{i}{2}\right)^3 \left(\det C^* \det B^* - \det C_1^* \det B_1^*\right),$$

where

$$C_1^* = \begin{pmatrix} e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,0}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,0}} & e^{-i\left(\lambda+0\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,1}} & e^{\lambda\alpha_1}e^{-i\sigma_{01}} & e^{-i\left(\lambda+1\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,2}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,2}} & e^{-i\left(\lambda+2\cdot\frac{\pi}{2}\right)} \\ e^{\lambda} & e^{\lambda\alpha_1}e^{i\sigma_{1,3}} & e^{\lambda\alpha_1}e^{-i\sigma_{1,3}} & e^{-i\left(\lambda+3\cdot\frac{\pi}{2}\right)} \end{pmatrix}, \quad B_1^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & e^{i\theta_2} & e^{-i\theta_2} & i \\ 1 & e^{2i\theta_2} & e^{-2i\theta_2} & i^2 \\ -1 & e^{3i\theta_2} & e^{-3i\theta_2} & i^3 \end{pmatrix}.$$

Let us separately calculate each determinant:

$$\det C^* = e^{\lambda(1+2\alpha_1)}e^{i\lambda} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\theta_1} & e^{-i\theta_1} & i \\ 1 & e^{2i\theta_1} & e^{-2i\theta_1} & i^2 \\ 1 & e^{3i\theta_1} & e^{-3i\theta_1} & i^3 \end{vmatrix},$$

further we use the method of calculation of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \prod_{j>i} (x_j - x_i),$$

hence we have

$$\det C^* = e^{\lambda(1+2\alpha_1)} e^{i\lambda} \left(i - e^{-i\theta_1}\right)\left(i - e^{i\theta_1}\right)(i-1)\left(e^{-i\theta_1} - e^{i\theta_1}\right)\left(e^{-i\theta_1} - 1\right)\left(e^{i\theta_1} - 1\right),$$

$$\det B^* = \left(-i - e^{-i\theta_2}\right)\left(-i - e^{i\theta_2}\right)(-i+1)\left(e^{-i\theta_2} - e^{i\theta_2}\right)\left(e^{-i\theta_2} + 1\right)\left(e^{i\theta_2} + 1\right),$$

$$\det C_1^* = e^{\lambda(1+2\alpha_1)} e^{-i\lambda} \left(-i - e^{-i\theta_1}\right)\left(-i - e^{i\theta_1}\right)(-i-1)\left(e^{-i\theta_1} - e^{i\theta_1}\right)\left(e^{-i\theta_1} - 1\right)\left(e^{i\theta_1} - 1\right),$$

$$\det B_1^* = \left(i - e^{-i\theta_2}\right)\left(i - e^{i\theta_2}\right)(i+1)\left(e^{-i\theta_2} - e^{i\theta_2}\right)\left(e^{-i\theta_2} + 1\right)\left(e^{i\theta_2} + 1\right),$$

further

$$\det C^* = -8e^{\lambda(1+2\alpha_1)} e^{i\lambda}(i-1)\cos\theta_1 \sin\theta_1 (1-\cos\theta_1), \quad \det B^* = 8(1-i)\cos\theta_2 \sin\theta_2 (1+\cos\theta_2),$$

$$\det C^* \det B^* = 2^6 e^{\lambda(1+2\alpha_1)} e^{i\lambda} (1-i)^2 (1-\cos\theta_1)(1+\cos\theta_2) \prod_{j=1}^{2} (\cos\theta_j \sin\theta_j),$$

$$\det C_1^* = -8e^{\lambda(1+2\alpha_1)} e^{-i\lambda}(i+1)\cos\theta_1 \sin\theta_1 (1-\cos\theta_1),$$

$$\det B_1^* = -8(1+i)\cos\theta_2 \sin\theta_2 (1+\cos\theta_2),$$

$$\det C_1^* \det B_1^* = 2^6 e^{\lambda(1+2\alpha_1)} e^{-i\lambda} (1+i)^2 (1-\cos\theta_1)(1+\cos\theta_2) \prod_{j=1}^{2} (\cos\theta_j \sin\theta_j).$$

So,

$$\left(\frac{i}{2}\right)^3 (\det C^* \det B^* - \det C_1^* \det B_1^*) = -i\left(\frac{i}{2}\right)^3 2^7 e^{\lambda(1+2\alpha_1)} (1-\cos\theta_1)(1+\cos\theta_2)$$
$$\times \prod_{j=1}^{2} \cos\theta_j \sin\theta_j \left(e^{i\lambda} + e^{-i\lambda}\right) = -2^5 e^{\lambda(1+2\alpha_1)} (1-\cos\theta_1)(1+\cos\theta_2) \prod_{j=1}^{2} (\cos\theta_j \sin\theta_j)\cos\lambda$$
$$= N_1 e^{\lambda(1+2\alpha_1)} \cos\lambda,$$

where $N_1 = -2^5(1-\cos\theta_1)(1+\cos\theta_2)\prod_{j=1}^{2}(\cos\theta_j \sin\theta_j) \neq 0$. Then the general form of determinant is

$$\Delta = N_1 e^{\lambda(1+2\alpha_1)} \cos\lambda + O\left(e^{\lambda(1+\alpha_1)}\right) \ \text{ or } \ \Delta = N_1 e^{\lambda(1+2\alpha_1)}\left(\cos\lambda + O\left(e^{-\lambda\alpha_1}\right)\right).$$

Now we consider the general case, i.e., problem (9). Roots of the characteristic equation have the form

$$\mu_k = \lambda(\alpha_k + i\beta_k), \ \alpha_k = \cos\theta_k, \ \beta_k = \sin\theta_k, \ \theta_k = \frac{\pi k}{2m}, \ k = \overline{0, 4m-1},$$

$$\mu_0 = \lambda,\ \ \mu_m = i\lambda,\ \ \alpha_k > 0,\ \ k = \overline{0, m-1};\ \ \alpha_s < 0,\ \ s = \overline{m+1, 2m-1}.$$

The general solution to Eq. (9) is $X(x) = X_1(x) + X_2(x)$, where

$$X_1(x) = c_1^0 e^{\lambda x} + \sum_{k=1}^{m-1} e^{\lambda \alpha_k x}\left(c_1^k \cos \lambda\beta_k x + c_2^k \sin \lambda\beta_k x\right) + c_3 \cos \lambda x,$$

$$X_2(x) = c_4^0 e^{-\lambda x} + \sum_{k=m+1}^{2m-1} e^{\lambda \alpha_k x}\left(c_4^k \cos \lambda\beta_k x + c_5^k \sin \lambda\beta_k x\right) + c_6 \sin \lambda x,$$

c_i^k are arbitrary constants. For derivatives of order $p = \overline{0, 2m-1}$ we have the formula

$$X_1^{(p)}(x) = \lambda^p \left\{ c_1^0 e^{\lambda x} + \sum_{k=0}^{m} e^{\lambda \alpha_k x}\left[c_1^k \cos(\lambda\beta_k x + p\theta_k) + c_2^k \sin(\lambda\beta_k x + p\theta_k)\right]\right\} + \lambda^p c_3 \cos\left(\lambda x + \frac{\pi}{2} p\right),$$

$$X_2^{(p)}(x) = \lambda^p \left\{ (-1)^p c_4^0 e^{-\lambda x} + \sum_{k=m+1}^{2m-1} e^{\lambda \alpha_k x}\left[c_4^k \cos(\lambda\beta_k x + p\theta_k) + c_5^k \sin(\lambda\beta_k x + p\theta_k)\right]\right\} + \lambda^p c_6 \sin\left(\lambda x + \frac{\pi}{2} p\right).$$

Satisfying boundary conditions, we obtain the system of equations for finding c_i^k; the main determinant of system has the form

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

where

$$A = \begin{vmatrix} 1 & 1 & . & 1 & 0 & . \ . & 1 \\ 1 & \cos\theta_1 & . & \cos\theta_{m-1} & \sin\theta_1 & . \ . & \cos\theta_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos(2m-1)\theta_1 & . & \cos(2m-1)\theta_{m-1} & \sin(2m-1)\theta_1 & . \ . & \cos(2m-1)\theta_m \end{vmatrix},$$

$$B = \begin{vmatrix} 1 & 1 & . & 1 & 0 & . & 0 \\ -1 & \cos\theta_{m+1} & . & \cos\theta_{2m-1} & \sin\theta_{m+1} & . & \sin\theta_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{2m-1} & \cos(2m-1)\theta_{m+1} & . & \cos(2m-1)\theta_{2m-1} & \sin(2m-1)\theta_{m+1} & . & \sin(2m-1)\theta_m \end{vmatrix},$$

$$C = \left(C^1 \ \ C^2\right),$$

$$C^1 = \begin{pmatrix} e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,0} & . & e^{\lambda\alpha_{m-1}}\cos\sigma_{m-1,0} \\ e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,1} & . & e^{\lambda\alpha_{m-1}}\cos\sigma_{m-1,1} \\ \cdots & \cdots & \cdots & \cdots \\ e^{\lambda} & e^{\lambda\alpha_1}\cos\sigma_{1,2m-1} & . & e^{\lambda\alpha_{m-1}}\cos\sigma_{m-1,2m-1} \end{pmatrix}, \ \ \sigma_{k,p} = \lambda\beta_k + p\theta_k,$$

$$C^2=\begin{pmatrix} e^{\lambda\alpha_1}\sin\sigma_{1,0} & . & e^{\lambda\alpha_{m-1}}\sin\sigma_{m-1,0} & \cos\sigma_{m,0}\\ e^{\lambda\alpha_1}\sin\sigma_{1,1} & . & e^{\lambda\alpha_{m-1}}\sin\sigma_{m-1,1} & \cos\sigma_{m,1}\\ \dots & \dots & \dots & \dots\\ e^{\lambda\alpha_1}\sin\sigma_{1,2m-1} & . & e^{\lambda\alpha_{m-1}}\sin\sigma_{m-1,2m-1} & \cos\sigma_{m,2m-1}\end{pmatrix},\quad D=\begin{pmatrix}D_1 & D_2\end{pmatrix},$$

$$D_1=\begin{pmatrix} e^{-\lambda} & e^{\lambda\alpha_{m+1}}\cos\sigma_{m+1,0} & . & e^{\lambda\alpha_{2m-1}}\cos\sigma_{2m-1,0}\\ -e^{-\lambda} & e^{\lambda\alpha_{m+1}}\cos\sigma_{m+1,1} & . & e^{\lambda\alpha_{2m-1}}\cos\sigma_{2m-1,1}\\ \dots & \dots & \dots & \dots\\ (-1)^{2m-1}e^{-\lambda} & e^{\lambda\alpha_{m+1}}\cos\sigma_{m+1,2m-1} & . & e^{\lambda\alpha_{2m-1}}\cos\sigma_{2m-1,2m-1}\end{pmatrix},$$

$$D_2=\begin{pmatrix} e^{\lambda\alpha_{m+1}}\sin\sigma_{m+1,0} & . & e^{\lambda\alpha_{2m-1}}\sin\sigma_{2m-1,0} & \sin\sigma_{m,0}\\ e^{\lambda\alpha_{m+1}}\sin\sigma_{m+1,1} & . & e^{\lambda\alpha_{2m-1}}\sin\sigma_{2m-1,1} & \sin\sigma_{m,1}\\ \dots & \dots & \dots & \dots\\ e^{\lambda\alpha_{m+1}}\sin\sigma_{m+1,2m-1} & . & e^{\lambda\alpha_{2m-1}}\sin\sigma_{2m-1,2m-1} & \sin\sigma_{m,2m-1}\end{pmatrix}.$$

Let us find the expression, which contains the maximal positive degree of exponent, used with calculation of determinant Δ. For that at first we calculate the product of determinants $|C|\cdot|B|$. We separately calculate each determinant, making the same actions as with the determinant Δ in the case $m=2$:

$$\det C=e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)}\cdot\left(\frac{i}{2}\right)^{m-1}$$

$$\times\begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1 & \cos\sigma_{m,0}\\ 1 & e^{\theta_1 i} & . & e^{\theta_{m-1}i} & e^{-\theta_1 i} & . & e^{-\theta_{m-1}i} & \cos\sigma_{m,1}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 1 & e^{(2m-1)\theta_1 i} & . & e^{(2m-1)\theta_{n-1}i} & e^{-(2m-1)\theta_1 i} & e^{-(2m-1)\theta_1 i} & e^{-(2m-1)\theta_{m-1}i} & \cos\sigma_{m,2m-1}\end{vmatrix}.$$

We introduce the denotation

$$\Delta_1=e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)}\left(\frac{i}{2}\right)^{m-1}\begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1\\ 1 & e^{\theta_1 i} & . & e^{\theta_{m-1}i} & e^{-\theta_1 i} & . & e^{-\theta_{m-1}i}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 1 & e^{(2m-2)\theta_1 i} & . & e^{(2m-2)\theta_{m-1}i} & e^{-(2m-2)\theta_1 i} & . & e^{-(2m-2)\theta_{m-1}i}\end{vmatrix}$$

$$=e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)}\overline{\Delta}_1,$$

which is the Vandermonde determinant and can be directly calculated:

$$\Delta_1=e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)}\left(\frac{i}{2}\right)^{m-1}\prod_{j=1}^{m-1}\left(e^{i\theta_j}-1\right)\left(e^{-i\theta_j}-1\right)\prod_{j>t}\left(e^{i\theta_j}-e^{i\theta_t}\right)\left(e^{-i\theta_j}-e^{-i\theta_t}\right)$$

$$\times \prod_{j=1}^{m-1} \left(e^{-i\theta_j} - e^{i\theta_j}\right) \times \prod_{j\neq t} \left(e^{-i\theta_j} - e^{i\theta_t}\right)\left(e^{-i\theta_t} - e^{i\theta_j}\right).$$

Making certain transformations, we obtain

$$\Delta_1 = 2^{2(m-1)} e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)} \prod_{j=1}^{m-1} \sin\theta_j \prod_{j=1}^{m-1} \sin^2\frac{\theta_j}{2} \prod_{j>t} 4\sin^2\frac{\theta_j-\theta_t}{2} \prod_{j\neq t} (-4)\sin^2\frac{\theta_j+\theta_t}{2}, \qquad (11)$$

Since in expression (11) each summand differs from zero, we have $\Delta_1 \neq 0$. Taking into account that

$$\cos\sigma_{m,p} = \frac{e^{(\lambda\beta_m+p\theta_m)i} + e^{-(\lambda\beta_m+p\theta_m)i}}{2},$$

we obtain

$$\det C = \frac{\Delta_1}{2}\left(e^{\lambda i}(i-1)\prod_{j=1}^{m-1}\left(i-e^{i\theta_j}\right)\left(i-e^{-i\theta_j}\right) + e^{-\lambda i}(-i-1)\prod_{j=1}^{m-1}\left(-i-e^{i\theta_j}\right)\left(-i-e^{-i\theta_j}\right)\right)$$

or

$$\det C = \frac{\Delta_1(i-1)}{2}(2i)^{m-1}\prod_{j=1}^{m-1}\cos\theta_j\left[(-1)^{m-1}e^{\lambda i} + ie^{-\lambda i}\right].$$

Now we calculate the determinant of the matrix B. Using again the Euler formula, we have

$$\det B = \left(\frac{i}{2}\right)^{m-1}$$

$$\times \begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1 & 0 \\ -1 & e^{\theta_{m+1}i} & . & e^{\theta_{2m-1}i} & e^{-\theta_{m+1}i} & . & e^{-\theta_{2m-1}i} & \sin\theta_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{2m-1} & e^{(2m-1)\theta_{m+1}i} & . & e^{(2m-1)\theta_{2m-1}i} & e^{-(2m-1)\theta_{m+1}i} & . & e^{-(2m-1)\theta_{2m-1}i} & \sin(2m-1)\theta_m \end{vmatrix}.$$

We denote

$$\Delta_2 = \left(\frac{i}{2}\right)^{m-1} \begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1 \\ -1 & e^{\theta_{m+1}i} & . & e^{\theta_{2m-1}i} & e^{-\theta_{m+1}i} & . & e^{-\theta_{2m-1}i} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{2m-2} & e^{(2m-2)\theta_{m+1}i} & . & e^{(2m-2)\theta_{2m-1}i} & e^{-(2m-2)\theta_{m+1}i} & . & e^{-(2m-2)\theta_{2m-1}i} \end{vmatrix},$$

taking into account the formula $\sin p\theta_n = \frac{e^{ip\theta_n} - e^{-ip\theta_n}}{2i}$, we obtain

$$\det B = \frac{\Delta_2}{2i}\left[(i+1)\prod_{j=1}^{m-1}\left(i-e^{i\theta_{m+j}}\right)\left(i-e^{-i\theta_{m+j}}\right) + (i-1)\prod_{j=1}^{m-1}\left(i+e^{i\theta_{m+j}}\right)\left(i+e^{-i\theta_{m+j}}\right)\right],$$

after certain transformations we have

$$\det B = \frac{(1+i)\Delta_2}{2i}(2i)^{m-1}\prod_{j=1}^{m-1}\cos\theta_{m+j}\left((-1)^{m-1} + i\right),$$

hence

$$\det C\ \det B = (-1)^m\, 4^{m-1}\Delta_1\Delta_2,\ \left(\sin\lambda + (-1)^{m-1}\cos\lambda\right)\prod_{j=1,j\neq m}^{2m-1}\cos\theta_j.$$

Now we calculate the product of determinants, where

$$\det C_1 = e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)}\left(\frac{i}{2}\right)^{m-1}$$

$$\times\begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1 & \sin\sigma_{m,0} \\ 1 & e^{\theta_1 i} & . & e^{\theta_{m-1}i} & e^{-\theta_1 i} & . & e^{-\theta_{m-1}i} & \sin\sigma_{m,1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & e^{(2m-1)\theta_1 i} & . & e^{(2m-1)\theta_{m-1}i} & e^{-(2m-1)\theta_1 i} & . & e^{-(2m-1)\theta_{m-1}i} & \sin\sigma_{m,2m-1} \end{vmatrix},$$

$$\det B_1 = \left(\frac{i}{2}\right)^{m-1}$$

$$\times\begin{vmatrix} 1 & 1 & . & 1 & 1 & . & 1 & 1 \\ -1 & e^{\theta_{m+1}i} & . & e^{\theta_{2m-1}i} & e^{-\theta_{m+1}i} & . & e^{-\theta_{2m-1}i} & \cos\theta_m \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (-1)^{2m-1} & e^{(2m-1)\theta_{m+1}i} & . & e^{(2m-1)\theta_{2m-1}i} & e^{-(2m-1)\theta_{m+1}i} & . & e^{-(2m-1)\theta_{2m-1}i} & \cos(2m-1)\theta_m \end{vmatrix},$$

using the formulas

$$\sin\sigma_{m,p} = \frac{e^{(\lambda\beta_m+p\theta_m)i} - e^{-(\lambda\beta_m+p\theta_m)i}}{2i},\ \ \cos p\theta_m = \frac{e^{ip\theta_m}+e^{-ip\theta_m}}{2},$$

we obtain

$$\det C_1 = \frac{\Delta_1}{2i}\left(e^{\lambda i}(i-1)\prod_{j=1}^{m-1}\left(i-e^{i\theta_j}\right)\left(i-e^{-i\theta_j}\right) - e^{-\lambda i}(-i-1)\prod_{j=1}^{m-1}\left(-i-e^{i\theta_j}\right)\left(-i-e^{-i\theta_j}\right)\right),$$

after transformations we have

$$\det C_1 = \frac{\Delta_1(i-1)}{2i}(2i)^{m-1}\prod_{j=1}^{m-1}\cos\theta_j\left[(-1)^{m-1}e^{\lambda i} - ie^{-\lambda i}\right],$$

further,

$$\det B_1 = \frac{\Delta_2}{2}\left[(i+1)\prod_{j=1}^{m-1}\left(i-e^{i\theta_{m+j}}\right)\left(i-e^{-i\theta_{m+j}}\right) - (i-1)\prod_{j=1}^{m-1}\left(i+e^{i\theta_{m+j}}\right)\left(i+e^{-i\theta_{m+j}}\right)\right]$$

or

$$\det B_1 = \frac{(1+i)\,\Delta_2}{2}(2i)^{m-1}\prod_{j=1}^{m-1}\cos\theta_{m+j}\left((-1)^{n-1} - i\right),$$

$$\det C_1\ \det B_1 = (-1)^m\, 4^{m-1}\Delta_1\Delta_2\prod_{\substack{j=1\\ j\neq m}}^{2m-1}\cos\theta_j\left(\sin\lambda - (-1)^{m-1}\cos\lambda\right),$$

hence the expression, which contains the maximal positive degree of exponent and belongs to the addend with the calculation of determinant with accuracy up to the sign, has the form

$$\det C\ \det B - \det C_1\ \det B_1 = \left(-2^{2m-1}\overline{\Delta}_1\,\Delta_2 \prod_{\substack{j=1\\ j\neq m}}^{2m-1} \cos\frac{\pi}{2m}j\right) e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)} \cos\lambda,$$

$$\Delta(\lambda) = N_2 e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)} \left(\cos\lambda + O\left(e^{-\lambda\alpha_{m-1}}\right)\right), \qquad (12)$$

where $N_2 = -2^{2m-1}\overline{\Delta}_1\,\Delta_2 \prod\limits_{\substack{j=1\\ j\neq m}}^{2m-1} \cos\frac{\pi}{2m}j \neq 0$. Since proper values of problem (9) are zeros of the determinant $\Delta(\lambda)$, taking into account (12), we have $\lambda_k = \frac{\pi}{2} + \pi k + \varepsilon_{1k}$, $k = 1, 2, 3, \ldots$, where $\lim\limits_{k\to+\infty} \varepsilon_{1k} = 0$. With odd n, making analogous calculations, we obtain

$$\Delta(\lambda) = N_3 e^{\lambda\left(1+2\sum\limits_{p=1}^{m-1}\alpha_p\right)} \left(\sin\lambda + O\left(e^{-\lambda\alpha_{m-1}}\right)\right), \qquad (13)$$

where N_3 is a certain nonzero constant independent of λ_k; $\alpha_k = \cos\theta_k$, $\theta_k = \frac{2k+1}{2(2m+1)}\pi$, hence

$$\lambda_k = \pi k + \varepsilon_{2k},\ \ k = 1, 2, 3, \ldots,$$

where $\lim\limits_{k\to+\infty} \varepsilon_{2k} = 0$. □

From Eqs. (12) and (13) we can define proper values, these equations have the countable number of solutions (because of alternating-sign property of sine and cosine). Proper values of problem (8) can be arranged in ascending order:

$$\lambda_1^{2n} < \lambda_2^{2n} < \lambda_3^{2n} < \lambda_4^{2n} < \cdots. \qquad (14)$$

We denote proper functions by $X_k(x)$, $k = 1, 2, 3, \ldots$, and assume that they are orthonormal. Let now $u(x, y)$ be a solution to Eq. (7) with homogeneous conditions (4)–(6). We consider the Fourier coefficients of this function by the system of proper functions $\{X_k\}_{k=1}^{k=\infty}$ of problem (8)

$$u_m(t) = \int_0^1 u(x,t)\, X_m(x)\, dx,\ \ 0 \le t \le 1,\ \ m = 1, 2, \ldots \qquad (15)$$

We note that problem (8) is self-conjugate. It is known in the theory of differential operators ([4]; [5], P. 91) that the system of proper functions $X_m(x)$ of a self-conjugate operator is complete in the space $L_2[0,1]$. Based on (15), we introduce auxiliary functions

$$u_m(t) = \int_\varepsilon^{1-\varepsilon} u(x,t)\, X_m(x)\, dx, \qquad (16)$$

where $\varepsilon > 0$ is a sufficiently small number. Differentiating (16) $2n$ times with respect to $t \in (0,1)$, taking into account (7), we obtain

$$u_{m,\varepsilon}(t) = \frac{1}{a^{2n}} \int_\varepsilon^{1-\varepsilon} D_x^{2n} u(x,t)\, X_m(x)\, dx.$$

Integrating the latter expression $2n$ times by part and taking into account homogeneous boundary conditions (4)–(6) as $\varepsilon \to 0$ we obtain the following boundary-value problem for an ordinary differential equation:

$$u_m^{(2n)}(t) = (-1)^n \left(\frac{\lambda_m}{a}\right)^{2n} u_m(t),$$
$$u_m^{(k)}(0) = u_m^{(k)}(1) = 0, \quad k = \overline{0, n-1}. \tag{17}$$

We again have problem (8). If $\frac{\lambda_m}{a}$ is not a root of expression (12) or (13) for the cases of even or odd n, respectively, then problem (17) has a trivial solution, only, i.e., the Fourier coefficients of the function $u(x,t)$ equal zero, hence due to the completeness of the system of functions $\{X_k\}_{k=1}^{k=\infty}$ in $L_2[0,1]$ the function $u(x,t) = 0$ almost everywhere, but it is continuous, therefore $u(x,t) \equiv 0$. Let now $\frac{\lambda_m}{a}$ be a root of expressions (12) or (13), then $\left(\frac{\lambda_m}{a}\right)^{2n}$ equals one of numbers of series (14), i.e., there exist numbers k and m such that

$$a = \frac{\lambda_m}{\lambda_k}, \tag{18}$$

but the set of such numbers is no more than countable. Theorem 1 has been proved.

We note that if condition (18) is fulfilled, then the uniqueness of solution to problem is violated, because we can take products of proper functions of problems (8) and (17) as a solution to the problem with zero boundary conditions. Therefore, the following theorem is proved.

Theorem 2. *If there exists a solution to Problem* A, *then it is unique only if condition* (18) *is not fulfilled.*

We note that the above reasoning is also true for more general edge conditions (if only edge conditions were identical with respect to the variables x and t). For example, in the Dirichlet problem, which is considered in paper [1], we have

$$a = \frac{\lambda_m}{\lambda_k} = \frac{\pi m}{\pi k} = \frac{m}{k},$$

i.e., for the uniqueness of solution to the Dirichlet problem it is necessary and sufficient that the number a (the ratio of lengths of rectangle sides) be not rational, that has been showed in the above mentioned paper.

2. Existence of a solution. Let numbers $\frac{\lambda_k}{a}$, $k = 1, 2, 3, \ldots$, be not roots of expressions (12) or (13). We will formally seek the solution to the stated problem in the form

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x) Y_k(t), \tag{19}$$

where $X_k(x)$ are proper functions of problem (8), and $Y_k(t)$ is a solution to the problem

$$Y_k^{(2n)}(t) = (-1)^n \left(\frac{\lambda_k}{a}\right)^{2n} Y_k(t),$$
$$Y_k^{(s)}(0) = \varphi_{sk},$$
$$Y_k^{(s)}(1) = \psi_{sk}, \quad s = \overline{0, n-1},$$

here

$$\varphi_{sk} = \int_0^1 \varphi_s(x) X_k(x)\, dx, \quad s = \overline{0, n-1}, \quad \psi_{sk} = \int_0^1 \psi_s(x) X_k(x)\, dx, \quad s = \overline{0, n-1}.$$

For the justification of convergence of series (19) and its differentiability till necessary orders, we need to obtain necessary estimates for the functions $X_k(x), Y_k(t)$. As distinct from paper [1], we have no explicit forms of these functions, therefore we deal as follows: We reduce problem (8) to an equivalent integral equation, constructing the Green function $G(x,\xi)$, and then we obtain the necessary estimates. The Green function satisfies the following conditions

1) with $x \neq \xi$ it is a solution to the following problem

$$\frac{\partial^{2n}G}{\partial x^{2n}} = 0,$$
$$\frac{\partial^{s}G}{\partial x^{s}}(0,\xi) = \frac{\partial^{s}G}{\partial x^{s}}(1,\xi) = 0, \quad s = \overline{0,n-1};$$

2) it is continuous till the derivative of the $(2n-2)$nd order;

3) $\frac{\partial^{2n-1}G}{\partial x^{2n-1}}(+\xi,\xi) - \frac{\partial^{2n-1}G}{\partial x^{2n-1}}(-\xi,\xi) = 1$;

4) $G(x,\xi) = G(\xi,x)$.

Theorem 3. *The Green function of problem* (8) *has the form*

$$G(x,\xi) = -\frac{1}{(2n-1)!}\begin{cases} G_1(x,\xi), & 0 \le x \le \xi; \\ G_2(x,\xi), & \xi \le x \le 1, \end{cases}$$

where

$$G_1(x,\xi) = (1-\xi)^n x^n \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}(-1)^k C_{2n-1}^{k} C_{n-1+j}^{j} x^{n-k-1}\xi^{j+k},$$

$$G_2(x,\xi) = (1-x)^n \xi^n \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}(-1)^k C_{2n-1}^{k} C_{n-1+j}^{j} \xi^{n-k-1}x^{j+k}, \;\; C_n^m = \frac{n!}{m!\,(n-m)!}.$$

Proof. We will seek the functions

$$G_1(x,\xi) = (1-\xi)^n x^n \left(x^{n-1}P_0(\xi) + x^{n-2}P_1(\xi) + \cdots + xP_{n-2}(\xi) + P_{n-1}(\xi)\right),$$

$$G_2(x,\xi) = (1-x)^n \xi^n \left(\xi^{n-1}P_0(x) + \xi^{n-2}P_1(x) + \cdots + \xi P_{n-2}(x) + P_{n-1}(x)\right),$$

where $P_k(x) = \sum\limits_{j=0}^{n-k-1} a_{k+j}^{k} x^{k+j}$.

The fulfillment of conditions 1) and 4) is evident, for the fulfillment of conditions 2) and 3) we choose $P_i(x)$ so that the identity

$$G_1(x,\xi) - G_2(x,\xi) = (x-\xi)^{2n-1} \tag{20}$$

holds true. In (20) we consider derivatives with respect to ξ till order $(n-1)$ inclusively, at point $\xi = 0$ we equate coefficients with identical degrees of x:

$$\frac{\partial^{p}G_2}{\partial \xi^{p}}(x,0) = 0, \;\; p = \overline{0,n-1},$$

$$\frac{\partial^{p}(x-\xi)^{2n-1}}{\partial \xi^{p}}(x,0) = (-1)^p \frac{(2n-1)!}{(2n-1-p)!}x^{2n-1-p}. \tag{21}$$

Now we consider the function

$$F_0(x,\xi) = x^{2n-1}(1-\xi)^n P_0(\xi), \;\; \frac{\partial^{p}F_0}{\partial \xi^{p}} = x^{2n-1}\sum_{j=0}^{p}\left((1-\xi)^n\right)^{(j)} P_0^{(p-j)}(\xi)\, C_p^j,$$

$$\left((1-\xi)^n\right)^{(j)} = (-1)^j \frac{n!}{(n-j)!}(1-\xi)^{n-j}, \;\; \left((1-\xi)^n\right)^{(j)}_{\xi=0} = (-1)^j \frac{n!}{(n-j)!},$$

$$P_0^{(p-j)}(0) = (p-j)!a_{p-j}^0,$$

$$\frac{\partial^p F_0}{\partial \xi^p}(x,0) = y^{2n-1}\sum_{j=0}^{p}(-1)^j\frac{p!}{j!\,(p-j)!}\frac{n!}{(n-k)!}(p-j)!a_{p-k}^0$$
$$= x^{2n-1}\sum_{j=0}^{p}(-1)^j\frac{p!n!}{j!\,(n-j)!}a_{p-j}^0 = x^{2n-1}p!\sum_{j=0}^{p}(-1)^j C_n^j a_{p-j}^0. \quad (22)$$

Equating coefficients with the identical degrees of x in correlations (21) and (22), we obtain the system of equations for finding coefficients a_j^0, $j=\overline{0,n-1}$:

$$\begin{aligned}
&a_0^0 C_n^0 = 1,\\
&(-1)^1 a_0^0 C_n^1 + (-1)^0 a_1^0 C_n^0 = 0,\\
&(-1)^2 a_0^0 C_n^2 + (-1)^1 a_1^0 C_n^1 + (-1)^0 a_2^0 C_n^0 = 0,\\
&\qquad\qquad\qquad \cdots\\
&(-1)^{n-1} a_0^0 C_n^{n-1} + (-1)^{n-2} a_1^0 C_n^{n-2} + \cdots + (-1)^0 a_{n-1}^0 C_n^0 = 0.
\end{aligned}$$

Using the formula from [6] $C_n^0 C_{n+m-1}^m - C_n^1 C_{n+m-2}^{m-1} + \cdots + (-1)^m C_{n-1}^0 C_n^m = 0$, we obtain the solution to system in the form $a_j^0 = C_{n+j-1}^j$. So, $P_0(\xi) = \sum\limits_{j=0}^{n-1} C_{n+j-1}^j \xi^j$.

Analogously we can find other unknown polynomials $P_k(\xi)$.

Now problem (8) can be reduced to the following integral equation:

$$X_k(x) = (-1)^n \lambda_k^{2n}\int_0^1 G(x,\xi)X_k(\xi)\,d\xi,$$

$$\frac{X_k(x)}{\lambda_k^{2n}} = \int_0^1 G_1(x,\xi)X_k(\xi)\,d\xi,$$

where $G_1(x,\xi) = (-1)^n G(x,\xi)$, further, following [7] (P. 212), we consider the expression

$$0 \le \int_0^1\left(G_{1\xi}^{(n)}(x,\xi) - \sum_{k=1}^{N}\frac{X_k(x)\,X_k^{(n)}(\xi)}{\lambda_k^{2n}}\right)^2 d\xi = \int_0^1\left(G_{1\xi}^{(n)}(x,\xi)\right)^2 d\xi$$
$$-2\sum_{k=1}^{N}\frac{X_k(x)}{\lambda_k^{2n}}\int_0^1 G_{1\xi}^{(n)}(x,\xi)X_k^{(n)}(\xi)\,d\xi + \sum_{s,k=1}^{N}\frac{X_k(x)\,X_s(x)}{\lambda_k^{4n}}\int_0^1 X_s^{(n)}(\xi)\,X_k^{(n)}(\xi)\,d\xi,$$

where N is an arbitrary finite natural number.

We have

$$\int_0^1 G_{1\xi}^{(n)}(x,\xi)X_k^{(n)}(\xi)\,d\xi = G_{1\xi}^{(n-1)}(x,\xi)\,X_k^{(n)}(\xi)\Big|_{\xi=0}^{\xi=1} - \int_0^1 G_{1\xi}^{(n-1)}(x,\xi)X_k^{(n+1)}(\xi)\,d\xi$$
$$= -\int_0^1 G_{1\xi}^{(n-1)}(x,\xi)X_k^{(n+1)}(\xi)\,d\xi = (-1)^n\int_0^1 G_1(x,\xi)X_k^{(2n)}(\xi)\,d\xi$$
$$= \lambda_k^{2n}\int_0^1 G_1(x,\xi)X_k(\xi)\,d\xi = X_k(x),$$

$$\int_0^1 X_s^{(n)}(\xi)\,X_k^{(n)}(\xi)\,d\xi = (-1)^n\int_0^1 X_s(\xi)\,X_k^{(2n)}(\xi)\,d\xi = \lambda_k\int_0^1 X_s(\xi)\,X_k(\xi)\,d\xi = \begin{cases}0, & k\ne s;\\ \lambda_k, & k=s,\end{cases}$$

hence

$$0 \le \int_0^1 \left(G_{1\xi}^{(n)}(x,\xi)\right)^2 d\xi - 2\sum_{k=1}^{N} \frac{X_k(x)}{\lambda_k^{2n}} X_k(x) + \sum_{k=1}^{N} \frac{X_k(x) X_k(x)}{\lambda_k^{4n}} \lambda_k^{2n}$$

or

$$\sum_{k=1}^{N} \frac{X_k^2(x)}{\lambda_k^{2n}} \le \int_0^1 \left(G_{1\xi}^{(n)}(x,\xi)\right)^2 d\xi,$$

due to the arbitrariness of N and in view of the fact that the function $G_{1\xi}^{(n)}(x,\xi)$ is continuous on the compact $[0,1] \times [0,1]$ and, hence, it is uniformly bounded:

$$\sum_{k=1}^{\infty} \frac{X_k^2(x)}{\lambda_k^{2n}} \le \int_0^1 \left(G_{1\xi}^{(n)}(x,\xi)\right)^2 d\xi \le A < \infty, \tag{23}$$

where A is a certain positive number. Now we go to the investigation of solution to problem (19). It is not difficult to obtain the estimates

$$\left|Y_k^{(m)}(t)\right| \le N_6 \frac{\lambda_k^m \sum\limits_{s=0}^{n-1} \{|\varphi_{sk}| + |\psi_{sk}|\}}{\left|\cos \frac{1}{a}\lambda_k + O\left(e^{-\frac{1}{a}\lambda_k \alpha_{n-1}}\right)\right|}, \quad k = 1,2,3,\ldots, \tag{24}$$

with even n,

$$\left|Y_k^{(m)}(t)\right| \le N_7 \frac{\lambda_k^m \sum\limits_{s=0}^{n-1} \{|\varphi_{sk}| + |\psi_{sk}|\}}{\left|\sin \frac{1}{a}\lambda_k + O\left(e^{-\frac{1}{a}\lambda_k \alpha_{n-1}}\right)\right|}, \quad k = 1,2,3,\ldots,$$

with odd n. Here N_6 and N_7 are some positive constants independent of λ_k, $m = 0,1,\ldots,2n$.

Indeed, in the case of even n ($n = 2m$), the solution to problem (20) is

$$Y_k(t) = Y_{1k}(t) + Y_{2k}(t), \quad k = 1,2,3,\ldots,$$

where

$$Y_{1k}(t) = c_1^0 e^{\mu_k t} + \sum_{j=1}^{m-1} e^{\mu_k \alpha_j t}\left(c_1^j \cos \mu_k \beta_j t + c_2^j \sin \mu_k \beta_j t\right) + c_3 \cos \mu_k t,$$

$$Y_{2k}(t) = c_4^0 e^{-\mu_k t} + \sum_{j=m+1}^{2m-1} e^{\mu_k \alpha_j t}\left(c_4^j \cos \mu_k \beta_j t + c_5^j \sin \mu_k \beta_j t\right) + c_6 \sin \mu_k t,$$

$$\alpha_k = \cos\theta_k, \quad \beta_k = \sin\theta_k, \quad \theta_k = \frac{\pi k}{2m}, \quad k = \overline{0, 4m-1}, \quad \mu_k = \frac{\lambda_k}{a}, \quad k = 1,2,\ldots\ .$$

For definition of unknowns c_i^j we obtain a heterogeneous system of algebraic equations, whose left-hand side is identical to the system, which was studied in problem (9), and the right-hand side is composed of the Fourier coefficients φ_{sk}, ψ_{sk}. We estimate

$$\left|c_1^0 e^{\mu_k t}\right| = \left|\frac{\Delta^1}{\Delta}\right| e^{\mu_k t} \le \frac{e^{\mu_k t} \sum\limits_{s=0}^{n-1} \{|\varphi_{sk}| + |\psi_{sk}|\} \left|O\left(e^{2\mu_k \sum\limits_{p=1}^{n-1} \alpha_p}\right)\right|}{\left|N_2\left(\cos\mu_k + O\left(e^{-\mu_k \alpha_{n-1}}\right)\right)\right| e^{\mu_k\left(1 + 2\sum\limits_{p=1}^{n-1} \alpha_p\right)}}$$

$$= \frac{e^{\mu_k t} \sum\limits_{s=0}^{n-1} \{|\varphi_{sk}| + |\psi_{sk}|\} \; |O(1)|}{|N_2(\cos\mu_k + O(e^{-\mu_k \alpha_{n-1}}))| \, e^{\mu_k}} \leq N_6 \frac{\sum\limits_{s=0}^{n-1} \{|\varphi_{sk}| + |\psi_{sk}|\}}{|(\cos\mu_k + O(e^{-\mu_k \alpha_{n-1}}))|},$$

here Δ^1 is the determinant obtained from Δ by the replacement of the first column with the right-hand side of system. In the above transformations we decomposed the determinant Δ^1 by the first column, and calculated algebraic complements by the methodic of calculation of determinant Δ in problem (9). Analogous estimates are true for other addends and derivatives of the function $Y_k(t)$.

Now it is necessary to find conditions, with which the denominator of expression (26) is separated from zero. We have (for even n)

$$\cos\frac{\lambda_k}{a} = \cos\frac{\varepsilon_{1k}}{a} \cos\frac{1}{a}\left(\frac{\pi}{2} + \pi k\right) - \sin\frac{\varepsilon_{1k}}{a} \sin\frac{1}{a}\left(\frac{\pi}{2} + \pi k\right),$$

taking into account (12), we obtain $\cos\left(\frac{\pi}{2} + \pi k + \varepsilon_{1k}\right) = O\left(e^{-\alpha_{n-1}\lambda_k}\right)$, hence

$$\sin\varepsilon_{1k} = O(e^{-\alpha_{n-1}k\pi} \times e^{-\alpha_{n-1}\varepsilon_{1k}}),$$

or for large numbers k we obtain the asymptotics $\varepsilon_{1k} \approx O\left(e^{-\alpha_{n-1}k\pi}\right)$, i.e., ε_{1k} tends to zero more rapidly than any power function. Hence, it necessary to show separability of the expression $\cos\frac{\pi}{2a}(1+2k)$ from zero. Further we deal as in paper [1]. From the theory of numbers it is known [7] that there exist irrational numbers $\frac{1}{a}$, such that for any $\varepsilon > 0$ there exists a sequence $\frac{1+2s_k}{1+2q_k}$, where $1+2s_k$ and $1+2q_k$ are coprime natural numbers, such that

$$\left|\frac{1}{a} - \frac{1+2s_k}{1+2q_k}\right| < \frac{\varepsilon}{(1+2q_k)^2}.$$

For such numbers we have

$$\left|\cos\frac{\pi}{2a}(1+2q_k)\right| = \left|\sin\left(\frac{\pi}{2a}(1+2q_k) - \frac{\pi}{2}(1+2s_k)\right)\right| = \left|\sin\left(\frac{\pi}{2}(1+2q_k)\left(\frac{1}{a} - \frac{1+2s_k}{1+2q_k}\right)\right)\right|$$
$$\leq \left|\frac{\pi}{2}(1+2q_k)\left(\frac{1}{a} - \frac{1+2s_k}{1+2q_k}\right)\right| < \frac{\pi}{2}(1+2q_k)\frac{\varepsilon}{(1+2q_k)^2} = \frac{\pi\varepsilon}{2(1+2q_k)}.$$

Hence it follows that for $\frac{1}{a}$ one can make the denominator of expression (24) arbitrarily small, i.e., series (19) can diverge.

Let now $\frac{1}{a}$ be an algebraic number of degree $n \geq 2$, then from the corollary to the Roth theorem ([8], P. 268) it follows that for an arbitrary positive number $0 < \varepsilon < 1$ there exists a positive number $\delta > 0$ such that with any integer s, k, where $k > 0$,

$$\left|\frac{1}{a} - \frac{1+2s}{1+2k}\right| \geq \frac{\delta}{(1+2k)^{2+\varepsilon}}.$$

Now for any $k \in \mathbb{N}$ we choose $s \in \mathbb{N}$ so that the inequality takes place

$$\left|\frac{1}{a} - \frac{1+2s}{1+2k}\right| < \frac{1}{(1+2k)},$$

in the capacity of $1+2s$ we can choose $\left[\frac{1}{a}(1+2k)\right]$ (the integer part of number), if $\left[\frac{1}{a}(1+2k)\right]$ is odd, and $\left[\frac{1}{a}(1+2k)\right]+1$, if $\left[\frac{1}{a}(1+2k)\right]$ is even.

Now, applying the inequality $\sin x > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, we obtain

$$\left|\cos\frac{\pi}{2a}(1+2k)\right| = \left|\sin\left(\frac{\pi}{2}(1+2k)\left(\frac{1}{a} - \frac{1+2s}{1+2k}\right)\right)\right| > (1+2k)\left|\frac{1}{a} - \frac{1+2s}{1+2k}\right| \geq \frac{C_0}{(1+2k)^{1+\varepsilon}}.$$

We consider the denominator of expression (24):

$$\left|\cos\frac{1}{a}\lambda_k + O\left(e^{-\frac{1}{a}\lambda_k\alpha_{n-1}}\right)\right|$$
$$= \left|\cos\frac{1}{a}\varepsilon_{1k}\cos\frac{\pi}{2a}(1+2k) - \sin\frac{1}{a}\varepsilon_{1k}\sin\frac{\pi}{2a}(1+2k) + O\left(e^{-\frac{1}{a}\lambda_k\alpha_{n-1}}\right)\right|$$
$$= \frac{1}{(1+2k)^{1+\varepsilon}}\left|(1+2k)^{1+\varepsilon}\cos\frac{1}{a}\varepsilon_{1k}\cos\frac{\pi}{2a}(1+2k)\right.$$
$$\left.-(1+2k)^{1+\varepsilon}\left(\sin\frac{1}{a}\varepsilon_{1k}\sin\frac{\pi}{2a}(1+2k) + O\left(e^{-\frac{1}{a}\lambda_k\alpha_{n-1}}\right)\right)\right|.$$

We investigate each addend in this module. Starting from some number $k = k_1$ we have $0 < \delta_1 < \cos\frac{1}{a}\varepsilon_{1k} < 1$, hence, $\left|(1+2k)^{1+\varepsilon}\cos\frac{1}{a}\varepsilon_{1k}\cos\frac{\pi}{2a}(1+2k)\right| \geq C_0\delta_1$. Further, taking into account the asymptotics of ε_{1k}, we obtain

$$\lim_{k\to\infty}\left((1+2k)^{1+\varepsilon}\left(\sin\frac{1}{a}\varepsilon_{1k}\sin\frac{\pi}{2a}(1+2k) + O\left(e^{-\frac{1}{a}\lambda_k\alpha_{n-1}}\right)\right)\right) = 0.$$

Taking into account the above reasoning, we obtain the lower estimate for the denominator of expression (24):

$$\left|\cos\frac{1}{a}\lambda_k + O\left(e^{-\frac{1}{a}\lambda_k\alpha_{n-1}}\right)\right| \geq \frac{N_8}{(1+2k)^{1+\varepsilon}},$$

where N_8 is a real positive number, independent of number k. Hence for numbers, starting from some $k = k_1$, the inequality is fulfilled

$$\left|Y_k^{(m)}(t)\right| \leq N_9\lambda_k^m k^{1+\varepsilon}\sum_{s=0}^{n-1}\{|\varphi_{sk}| + |\psi_{sk}|\} \leq N_{10}\lambda_k^{m+1+\varepsilon}\sum_{s=0}^{n-1}\{|\varphi_{sk}| + |\psi_{sk}|\},$$

where N_8, N_9, and N_{10} are some positive constants. We obtain the same estimate for the case of odd n.

We note that even if the first $k_1 - 1$ addends become too large finite numbers (they cannot equal infinity, because numbers $\frac{\lambda_k}{a}$ are not roots of expression (12) or (13)), there is a finite number of them, therefore they have no influence to series (19).

We considered the case of irrational number $\frac{1}{a}$ only, because in the case of rational one, for example, $\frac{1}{a} = \frac{1+2s}{1+2k}$, $n, s \in \mathbb{N}$, one can make the arbitrarily small denominator.

Now we obtain conditions, with which series (19) is a regular solution to the stated problem. At once we consider the componentwise differentiating of series (19) with respect to the variable t of order $2n$ (the uniform convergence of the series itself, series composed of derivatives of smaller order and derivatives with respect to the variable x can be proved analogously):

$$\frac{\partial^{2n}u}{\partial t^{2n}} = \sum_{k=1}^{k_1-1}X_k(x)\,Y_k^{(2n)}(t) + \sum_{k=k_1}^{\infty}X_k(x)\,Y_k^{(2n)}(t),$$

further we deal with the second series, only:

$$\left|\frac{\partial^{2n}u}{\partial t^{2n}}\right| \leq \left(\frac{1}{a}\right)^{2n}\sum_{k=k_1}^{\infty}|X_k(x)|\left|\lambda_k^{2n}Y_k(t)\right| \leq M\sum_{k=k_1}^{\infty}\left(|X_k(x)|\,\lambda_k^{2n+1+\varepsilon}\sum_{s=0}^{n-1}\{|\varphi_{sk}| + |\psi_{sk}|\}\right), \quad (25)$$

where M is some positive constant. Let us show the convergence of each addend in series (25):

$$\sum_{k=k_1}^{\infty}\left(|X_k(x)|\,\lambda_k^{2n+1+\varepsilon}|\varphi_{0k}|\right) \leq \sqrt{\sum_{k=k_1}^{\infty}\left(\frac{X_k(x)}{\lambda_k^n}\right)^2}\sqrt{\sum_{k=k_1}^{\infty}\left(\lambda_k^{3n+1+\varepsilon}\varphi_{0k}\right)^2}, \quad (26)$$

$$\sum_{k=k_1}^{\infty}\left(\lambda_k^{3n+1+\varepsilon}\varphi_{0k}\right)^2 \leq \sum_{k=k_1}^{\infty}\left(\lambda_k^{4n}\varphi_{0k}\right)^2,$$

$$\lambda_k^{4n}\varphi_{0k} = \lambda_k^{4n}\int_0^1 \varphi_0(x)\,X_k(x)\,dx = (-1)^n\,\lambda_k^{2n}\int_0^1 \varphi_0(x)\,X_k^{(2n)}(x)\,dx,$$

let $\varphi_0^{(s)}(0) = \varphi_0^{(s)}(1) = 0$, $s = \overline{0, n-1}$, then

$$\lambda_k^{4n}\varphi_{0k} = (-1)^n\,\lambda_k^{2n}\int_0^1 \varphi_0^{(2n)}(x)\,X_k(x)\,dx = \int_0^1 \varphi_0^{(2n)}(x)\,X_k^{(2n)}(x)\,dx,$$

let $\varphi_0^{(s)}(0) = \varphi_0^{(s)}(1) = 0$, $s = \overline{2n, 3n-1}$, then

$$\lambda_k^{4n}\varphi_{0k} = \int_0^1 \varphi_0^{(4n)}(x)\,X_k(x)\,dx,$$

we apply the Bessel inequality

$$\sum_{k=k_1}^{\infty}\left(\lambda_k^{4n}\varphi_{0k}\right)^2 \leq \sum_{k=1}^{\infty}\left(\lambda_k^{4n}\varphi_{0k}\right)^2 \leq \int_0^1\left(\varphi_0^{(4n)}(x)\right)^2 dx.$$

If now $\varphi_0^{(4n)}(x) \in L_2(0,1)$, then, taking into account (23) we obtain the absolute and uniform convergence of series (26). Analogously one can prove the convergence of other addends in series (29). Hence, the following theorem is proved.

Theorem 4 (*of existence*)**.** *If* $\varphi_i(x), \psi_i(x) \in C^{4n-1}[0;1]$, $\varphi_i^{(4n)}(x), \psi_i^{(4n)} \in L_2(0,1)$, $\varphi_i^{(s)}(0) = \varphi_i^{(s)}(1) = \psi_i^{(s)}(0) = \psi_i^{(s)}(1) = 0$, *for* $i = \overline{0, n-1}$, $s = \overline{0, n-1}$, $s = \overline{2n, 3n-1}$, *then series* (18) *is a regular solution to Problem* A.

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