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« OLIY MATEMATIKA» KAFEDRASI

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"O l i y   m a t e m a t i k a" fani

A N I Q M A S   I N T E G R A L L A R

bo'limidan mustaqil ish topshiriqlari

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A n i q m a s   i n t e g r a l

Reja

1. Boshlang’ich funksiya va aniqmas integral hossalari.

1.1. Boshlang’ich funksiya va aniqmas integral tushunchasi.

1. 2.Asosiy integrallar jadvali

1.3.Integrallash usullari

1. Bevosita integrallash usuli.

2. Aniqmas integralda o’zgaruvchilarni almashtirish usuli.

3. Bo’laklab integrallash

1.4.Mustaqil ishlash uchun misollar

2.Rasional kasrlar. To’g’ri rasional kasrni ajratish.

3.Aniqmas koeffisientlar metodi.
4.Rasional kasrlarni integrallash.
Mastaqil ishlash uchun misollar

5.Ba’zi irrasional funksiyalarni integrallash.

I. $\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{r_1},\ldots,\left(\frac{ax+b}{cx+d}\right)^{r_s}\right]dx$ ko’rinishdagi integral

II. $J_1=\int\frac{dx}{\sqrt{ax^2+bx+c}}$ ko’rinishdagi integral

6.Trigonometrik funksiyalarni integrallash.
I. $\int R(\sin x, \cos x)dx$ ko’rinishdagi integral

II. $\int \sin^m x\cdot\cos^n xdx$ ko’rinishdagi integral.

III. $\int \sin mx\cos nxdx$, $\int \cos mx\cos nxdx$, $\int \sin mx\sin nxdx$ ko’rinishdagi integrallar.

Mastaqil ishlash uchun misollar

1. Boshlang'ich funksiya va aniqmas integral hossalari. Integrallash usullari.

1.1. Boshlang'ich funksiya va aniqmas integral tushunchasi.

1-Ta'rif. Agar *[a;b]* segmentning hamma nuqtalarida $F'(x)=f(x)$ tenglik bajarilsa, $F(x)$funksiya shu segmentda $f(x)$funksiyaga nisbatan boshlang'ich funksiya deb ataladi.

1-Misol. $f(x)=4x^3$funksiyaning $(-\infty;\ +\infty)$ intervalda boshlang'ich funksiyasi $F(x)=x^4$ bo'ladi,chunki $(-\infty;\ +\infty)$ da $F'(x)=4x^3$.

2-misol. $f(x)=\frac{1}{x}$ funksiyaning $(0;\ +\infty)$ intervalda boshlang'ich funksiyasi $F(x)=\ln x$ bo'ladi,chunki shu intervalda $F'(x)=\frac{1}{x}$.

3-misol. $f(x)=4x^3$funksiyaning $(-\infty;\ +\infty)$ intervalda boshlang'ich funksiyasi $F(x)=x^4+C$ bo'lgani uchun uni aniqmas integrali

$\int 4x^3dx = x^4 + C$ *(C-sonst)* bo'ladi.

4-Misol. $\int\left(x^4 + 4\sin x - 9\right)dx$ ni toping.

Yechilishi. 4-va 5-xossalarga asosan quyidagiga egamiz:

$$\int\left(x^4 + 4\sin x - 9\right)dx = \int x^4dx + 4\int \sin xdx - 9\int dx = \frac{x^5}{5} - 4\cos x - 9x + C.$$

Hosil qilingan natijaning to'g'riligini differensiallash yordamida oson tekshirish mumkin. Haqiqatdan,

$$d\left(\frac{x^5}{5} - 4\cos x - 9x + C\right) = (x^4 + 4\sin x - 9)dx.$$

1. 2.Asosiy integrallar jadvali

1^0 $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C. \qquad (\alpha \neq -1)$

2^0 $\int dx = x + C.$

3^0 $\int \frac{dx}{x} = \ln|x| + C.$

4^0 $\int \sin xdx = -\cos x + C.$

5^0 $\int \cos xdx = \sin x + C.$

6^0 $\int \frac{dx}{\cos^2 x} = tgx + C.$

7^0 $\int \frac{dx}{\sin^2 x} = -ctgx + C.$

8^0 $\int tgxdx = -\ln|\cos x| + C$.

9^0 $\int ctgx dx = \ln|\sin x| + C.$

10^0 $\int a^x dx = \frac{a^x}{\ln a} + C.$

11^0 $\int \ell^x dx = \ell^x + C.$

12^0 $\int \frac{dx}{1+x^2} = arc\,tgx + C.$

13^0 $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arc\,tg \frac{x}{a} + C.$

14^0 $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C.$

15^0 $\int \frac{dx}{\sqrt{1-x^2}} = arc\sin x + C.$

16^0 $\int \frac{dx}{\sqrt{a^2-x^2}} = arc\sin \frac{x}{a} + C.$

17^0 $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

1.3.Integrallash usullari

1^0. Bevosita integrallash usuli.

Aniqmas integralning asosiy xossalaridan va integrallar jadvalidan bevosita foydalanib integrallarni hisoblashga uni bevosita integrallash deyiladi.

5-misol. $\int x^8 dx = \frac{x^9}{9} + C$ (1^0-formula).

6-misol. Integralni toping: $\int \sqrt[3]{x} dx = \int x^{\frac{1}{3}} dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4} x\sqrt[3]{x} + C$ (1^0-formula).

7-misol. $\int \frac{dx}{x^5} = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{x^{-4}}{4} + C = -\frac{1}{4x^4} + C.$

8-misol.
$$\int \frac{dx}{\sin x \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{\sin x} dx =$$
$$= \int \frac{d(\sin x)}{\sin x} - \int \frac{d(\cos x)}{\cos x} = \ln|\sin x| - \ln|\cos x| + C = \ln|tgx| + C.$$
(8^0-9^0 -formulalar).

9-misol.

$$\int (3x+5)^{17} dx = \frac{1}{3} \int (3x+5)^{17} d(3x+5) =$$
$$= \frac{1}{3} \cdot \frac{(3x+5)^{18}}{18} + C = \frac{(3x+5)^{18}}{54} + C.$$
(1^0-formula).

2^0. Aniqmas integralda o'zgaruvchilarni almashtirish usuli. x=φ(t) deb olib, integralostidagi ifodada o'zgaruvchini almashtiramiz; bu erda *φ(t)*-uzluksizfunksiya bo'lib, uzluksiz hosilaga va teskarifunksiyaga ega. U vaqtda *dx=φ′(t)dt*; bu holda ushbu tenglik to'g'ri bo'lishini isbotlaymiz:

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt.$$

10-misol.

$$\int \frac{\sqrt[3]{1+\ln x}}{x}dx = \left| \begin{array}{l} 1+\ln x = t^3 \Leftrightarrow \ln x = t^3 - 1 \Rightarrow \\ \frac{dx}{x} = 3t^2 dt \end{array} \right| = \int t\cdot 3t^2 dt = 3\int t^3 dt = 3\cdot\frac{t^4}{4} + C = \frac{3}{4}\sqrt[3]{(1+\ln x)^4} + C.$$

11-misol. (13^0-formula).

$$\int \frac{dx}{x^2 - 6x + 25} = \left| \begin{array}{r} x^2 - 6x + 25 = (x-3)^2 + 16,\ x - 3 = t \\ dx = dt \end{array} \right| =$$

$$= \int \frac{dt}{t^2 + 16} = \int \frac{dt}{t^2 + 4^2} = \frac{1}{4}arctg\frac{t}{4} + C = \frac{1}{4}arctg\frac{x-3}{4} + C.$$

12-misol. (1^0-formula).

$$\int \cos^5 x\sqrt{\sin x}dx = \int (1-\sin^2 x)^2\sqrt{\sin x}\cos xdx = \left|\sin x = t \Rightarrow \cos xdx = dt\right| =$$

$$= \int (1-t^2)^2\sqrt{t}dt = \int t^{\frac{1}{2}}dt - 2\int t^{\frac{5}{2}}dt + \int t^{\frac{9}{2}}dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2\cdot\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} + C =$$

$$= \frac{2}{3}\sqrt{\sin^3 x} - \frac{4}{7}\sin^2 x\sqrt{\sin^3 x} + \frac{2}{11}\sin^4 x\sqrt{\sin^3 x} + C.$$

3^0. Bo'laklab integrallash

u(x) va *v(x)* funksiyalar x ning differensiallanuvchi funksiyalari bo'lsin. Bufunksiyalar ko'paytmasining differensialini topamiz:

$$d(uv) = vdu + udv, \quad \text{bundan} \quad udv = d(uv) - vdu.$$

Oxirgi tenglikning ikkala qismini integrallab, quyidagini topamiz:

$$\int udv = \int d(uv) - \int vdu \quad \text{yoki} \int udv = uv - \int vdu.$$

Bo'laklab integrallash metodi bilan hisoblanadigan ba'zi integrallarni ko'rsatamiz:

I. $\int P(x)\ell^{kx}dx, \int P(x)\sin kxdx, \int P(x)\cos kxdx$

ko'rinishdagi integrallar, bu erda P*(x)*-ko'pxad, *k*-biror son. Bu tipdagi integrallarni hisoblashda *u=P(x)* deb olinsa bo'ladi.

II. $\int P(x)\ln xdx, \int P(x)arc\sin x\,dx, \int P(x)arc\cos xdx, \int P(x)arc\,tgxdx, \int P(x)arc\,ctgxdx.$

ko'rinishdagi integrallar, bu erda *P(x)*-ko'pxad. Bu xollarning barchasida bo'laklab integrallashda birinchisida *u=lnx,* ikkinchisida *u=arcsinx,* uchinchisida *u=arccosh,* to'rtinchisida *u=arctgx,* beshinchisida *u=arcctgx* deb olish lozim.

III. $\int \ell^{ax}\cos bxdx, \int \ell^{ax}\sin bxdx,$ (bunda *a* va *b* sonlar) ko'rinishdagi integrallar. Bu integrallar ikki marta bo'laklab integrallab topiladi.

13-misol. $\int arctgxdx$ ni toping.

Echilishi. *u=arctgx, dv=dx* deymiz.

Bundan $du=\frac{dx}{1+x^2}$; *v=x*; u holda yuqoridagi formulaga ko'ra:

$$\int arctgxdx = xarctgx - \int \frac{xdx}{1+x^2} = arctgx - \frac{1}{2}\ln(1+x^2)+C.$$

14-misol. $\int (x^3-2x^2+5)\ell^{3x}dx.$

Echilishi. $u=x^3-2x^2+5 \Rightarrow du=(3x^2-4x)dx,$

$dv=\ell^{3x}dx \Rightarrow v=\int \ell^{3x}dx=\frac{1}{3}\ell^{3x}$; bularni yuqoridagi formulaga qo'yib quyidagiga ega bo'lamiz: $\int (x^3-2x^2+5)\ell^{3x}dx=\frac{1}{3}(x^3-2x^2+5)\ell^{3x}-\frac{1}{3}\int \ell^{3x}\cdot(3x^2-4x)dx,$

Bundagi oxirgi integralni yana bo'laklab integrallaymiz:

$u=3x^2-4x \Rightarrow du=(6x-4)dx;$ $dv=\ell^{3x}dx \Rightarrow v=\frac{1}{3}\ell^{3x},$

U holda

$$\int \ell^{3x}(3x^2-4x)dx=\frac{1}{3}(3x^2-4x)\ell^{3x}-\frac{1}{3}\int \ell^{3x}(6x-4)dx=\frac{1}{3}(3x^2-4x)\ell^{3x}-\frac{1}{3}\int \ell^{3x}(6x-4)dx=$$
$$=\frac{1}{3}(3x^2-4x)\ell^{3x}+\frac{4}{3}\int \ell^{3x}dx-2\int x\ell^{3x}dx=\frac{1}{3}(3x^2-4x)\ell^{3x}+\frac{4}{9}\ell^{3x}-2\int x\ell^{3x}dx,$$

Bundagi oxirgi integralni yana bo'laklab integrallaymiz: $u=x \Rightarrow du=dx;$ $dv=\ell^{3x}dx \Rightarrow v=\frac{1}{3}\ell^{3x},$

Unda $\int x\ell^{3x}dx=\frac{1}{3}x\ell^{3x}-\frac{1}{3}\int \ell^{3x}dx=\frac{1}{3}x\ell^{3x}-\frac{1}{9}\ell^{3x},$ buni o'rniga qo'ysak,

$\int \ell^{3x}(3x^2-4x)dx=\frac{1}{3}(3x^2-4x)\ell^{3x}+\frac{4}{9}\ell^{3x}-\frac{2}{3}x\ell^{3x}+\frac{2}{9}\ell^{3x};$ buni o'rniga qo'ysak,

$$\int (x^3-2x^2+5)\ell^{3x}dx=\frac{1}{3}(x^3-2x^2+5)\ell^{3x}-\frac{1}{9}(3x^2-4x)\ell^{3x}-$$
$$-\frac{4}{27}\ell^{3x}+\frac{2}{9}x\ell^{3x}-\frac{2}{27}\ell^{3x}+C=\left(\frac{1}{3}x^3-x^2+\frac{2}{3}x+\frac{13}{9}\right)\ell^{3x}+C.$$

Mustaqil ishlash uchun misollar

Quyidagi integrallarni bevosita integrallash usuli bilan toping:

1^0. $\int x\sqrt{x}dx$; *javob:* $\frac{2}{5}x^2\sqrt{x}+C.$

2^0. $\int \frac{dx}{\sqrt[5]{x}}$; *javob:* $\frac{5}{4}\sqrt[5]{x^4}+C.$

3^0. $\int \frac{2-\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$; *javob: 2 arcsin x-x+C.*

4^0. $\int \frac{2-x^4}{1+x^2}dx$; *javob:* $arctgx+x-\frac{x^3}{3}+C.$

5^0. $\int \ell^{3x}\cdot 3^x dx$; *javob:* $\ell^{3x}\cdot 3^x/(3+\ln 3)+C.$

6^0. $\int tg^2 xdx$; *javob: tgx-x+C.*

7^0. $\int (shx-\sin x)dx$; *javob: chx+cosh+C.*

8^0. $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$; *javob:* $x^2/2-2x+\ln|x|+C.$

9^0. $\int (2tgx+3ctgx)^2 dx$; *javob: 4tgx-9ctgx-x+C.*

10^0. $\int \frac{dx}{x\ln x}$; *javob: ln|lnx|+C.*

11^0. $\int \sqrt{\sin x}\cos xdx$; *javob:* $\frac{2}{3}\sin x\sqrt{\sin x}+C$

12^0. $\int \sin(a+bx)dx$; *javob: -(1/b)cos(a+bx)+C.*

13^0. $\int \frac{dx}{x^2-6x+13}$; *javob:* $\frac{1}{2}arctg\frac{x-3}{2}+C.$.

14^0. $\int \frac{x-1}{\sqrt[3]{x^2}}dx$; *javob:* $\frac{3}{4}(x-4)\sqrt[3]{x}+C$.

Quyidagi integrallarni o'zgaruvchilarni almashtirish usuli bilan hisoblang:

15^0. $\int \ell^{5x}dx.$ *javob:* $\frac{1}{5}\ell^{5x}+C.$

16^0. $\int \cos 5xdx.$ *javob:* $\frac{1}{5}\sin 5x+C.$

17^0. $\int \frac{\ln x}{x}dx.$ *javob:* $\frac{1}{2}\ln^2 x+C.$

18^0. $\int \frac{dx}{\sin^2 3x}.$ *javob:* $-\frac{1}{3}ctg3x+C.$

19^0. $\int \sin^2 x\cos xdx.$ *javob:* $\frac{1}{3}\sin^3 x+C.$

20^0. $\int \cos^3 x\sin dx.$ *javob:* $-\frac{1}{4}\cos^4 x+C.$

$21^0.\ \int \frac{xdx}{\sqrt{2x^2+3}}$; *javob:* $\frac{1}{2}\sqrt{2x^2+3}+C.$

$22^0.\ \int \frac{x^2dx}{\sqrt{x^3+1}}$; *javob:* $\frac{2}{3}\sqrt{x^3+1}+C.$

$23^0.\ \int \frac{\cos xdx}{\sin^2 x}$; *javob:* $-\frac{1}{\sin x}+C.$

$24^0.\ \int \frac{\sin xdx}{\cos^3 x}$; *javob:* $\frac{1}{2\cos^2 x}+C.$

$25^0.\ \int \frac{tgx}{\cos^2 x}dx$; *javob:* $\frac{tg^2 x}{2}+C.$

$26^0.\ \int \frac{dx}{\cos^2 x\sqrt{tgx-1}}$; *javob:* $2\sqrt{tgx-1}+C.$

$27^0.\ \int \frac{\ln(x+1)}{x+1}dx$; *javob:* $\frac{\ln^2(x+1)}{2}+C.$

$28^0.\ \int \frac{\cos xdx}{\sqrt{2\sin x+1}}$; *javob:* $\sqrt{2\sin x+1}+C.$

$29^0.\ \int \frac{\sin 2xdx}{(1+\cos 2x)^2}$; *javob:* $\frac{1}{2(1+\cos 2x)}+C.$

$30^0.\ \int \frac{\sin 2xdx}{\sqrt{1+\sin^2 x}}$; *javob:* $2\sqrt{1+\sin^2 x}+C.$

Quyidagi integrallarni bo’laklab integrallab toping:

$31^0.\ \int \ln xdx$; *javob:* $x(lnx-1)+C.$

$32^0.\ \int x\sin x$; *javob:* $-xcosh+sinx+C.$

$33^0.\ \int x^2\ell^x dx$; *javob:* $\ell^x(x^2-2x+2)+C.$

$34^0.\ \int x\ln xdx$; *javob:* $(x^2/4)(2\ln x-1)+C.$

$35^0.\ \int \arcsin xdx$; *javob:* $x\arcsin x+\sqrt{1-x^2}+C.$

$36^0.\ \int x^2 arctgxdx$; *javob:* $\frac{x^2}{3}arctgx-\frac{1}{6}x^2+\frac{1}{6}\ln(x^2+1)+C.$

$37^0.\ \int (x+1)\ell^x dx$; *javob:* $x\ell^x+C.$

$38^0.\ \int x^2\sin xdx$; *javob:* $-x^2\cos x+2x\sin x+2\cos x+C.$

$39^0.\ \int x^5\cdot\ell^{x^2}dx$; *javob:* $\frac{1}{2}\ell^{x^2}(x^4-2x^2+2)+C.$

$40^0.\ \int (x^2+2x+3)\cos xdx$; *javob:* $(x+1)^2\sin x+2(x+1)\cos x+C.$

$41^0.\ \int \ell^{2x}\cos xdx$; *javob:* $(\ell^{2x}/5)(\sin x+2\cos x)+C.$

$42^0.\ \int \sin\sqrt{x}dx$; *javob:* $-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C.$

$43^0. \int \ln(x+\sqrt{1+x^2})dx;$ *javob:* $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C.$

$44^0. \int \frac{\arcsin x}{\sqrt{1+x}}dx;$ *javob:* $2\sqrt{1+x}\arcsin x+4\sqrt{1-x}+C.$

2.Rasional kasrlar. To'g'ri rasional kasrni ajratish.

Kasr rasional funksiya yoki oddiy rasional funksiya deb ikkita ko'pxadning bo'linmasiga teng bo'lgan funksiyaga aytiladi: $R(x)=\frac{P_m(x)}{Q_n(x)}$, bu erda $P_m(x)-m$-darajali ko'pxad, $Q_n(x)-n$-darajali ko'pxad. Biz quyida ko'ramizki, xar qanday to'g'ri rasional kasrni eng sodda rasional kasrlar deb ataluvchi quyidagi kasrlarning chekli yig'indisi ko'rinishida tasvirlash mumkin:

$$I.\frac{A}{x-a};\quad II.\frac{A}{(x-a)^n}\ (n=2,3,...);\quad III.\frac{M\tilde{o}+N}{x^2+px+q};\quad IV.\frac{M\tilde{o}+N}{(x^2+px+q)^n}\ (n=2,3,...)$$

bu erda *A, a, r, q, M* va *N*-xaqiqiy sonlar x^2+rx+q kvadrat uchxad xaqiqiy ildizlarga ega emas, ya'ni $\frac{p^2}{4}-q<0.$ Shuning uchun biz sodda kasrlarni integrallashni o'rgansak va to'g'ri rasional kasrni soddalarining yig'indisiga ajrata olsak, rasional kasrlarni integrallash masalasi xal qilingan bo'ladi.

$$I.\int\frac{Adx}{x-a}=A\int\frac{d(x-a)}{x-a}=A\ln|x-a|+C.$$

$$II.\int\frac{Adx}{(x-a)^n}=A\int(x-a)^{-n}d(x-a)=\frac{A}{(1-n)(x-a)^{n-1}}+C.$$

$$III.\int\frac{Mx+N}{x^2+px+q}dx.$$

Maxrajda to'la kvadrat ajratib, quyidagiga ega bo'lamiz:

$$x^2+px+q=x^2+2\cdot\frac{p}{2}x+\frac{p^2}{4}+(q-\frac{p^2}{4})=(x+\frac{p}{2})^2+(q-\frac{p^2}{4}).$$

Shartga ko'ra x^2+px+q uchxad xaqiqiy ildizlarga ega bo'lmagani uchun $q-\frac{p^2}{4}>0.$ $q-\frac{p^2}{4}=a^2$ belgilashni kiritamiz. Endi integralga $t=x+\frac{p}{2}$ deb o'zgaruvchini almashtirishni qo'llaymiz. Bundan:

$$x=t-\frac{p}{2},\quad dx=dt,\quad x^2+px+q=(x+\frac{p}{2})^2+(q-\frac{p^2}{4})=t^2+a^2.$$

Demak,

$$\int\frac{Mx+N}{x^2+px+q}dx=\int\frac{M(t-\frac{p}{2})+N}{t^2+a^2}dt=M\int\frac{tdt}{t^2+a^2}+(N-\frac{Mp}{2})\cdot\int\frac{dt}{t^2+a^2}=\frac{M}{2}\ln(t^2+a^2)+\frac{N-\frac{Mp}{2}}{a}arctg\frac{t}{a}+C.$$

$$\int\frac{dt}{t^2+a^2}=\frac{M}{2}\ln(t^2+a^2)+\frac{N-\frac{Mp}{2}}{a}arctg\frac{t}{a}+C.$$

t va *a* ning qiymatlarini o'z o'rniga qo'yib, nixoyat quyidagini hosil qilamiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\frac{M}{2}\ln(x^2+px+q)+\frac{N-\frac{Mp}{2}}{\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C.$$

*1-**Misol.*** $\int \frac{3x+2}{x^2-3x+8}dx$ ni toping.

Echilishi.maxrajni to'la kvadratga ajratamiz: $x^2-3x+8=(x-\frac{3}{2})^2+\frac{23}{4}$;

$$x-\frac{3}{2}=t, \qquad dx=dt, \qquad 3x+2=3(t+\frac{3}{2})+2=3t+\frac{13}{2}.$$

Demak, $\int \frac{3x+2}{x^2-3x+8}dx=\int \frac{3t+\frac{13}{2}}{t^2+\frac{23}{4}}dt=3\int \frac{tdt}{t^2+\frac{23}{4}}+\frac{13}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{23}}{2}\right)^2}=$

$$=\frac{3}{2}\ln(t^2+\frac{23}{4})+\frac{13}{2}\cdot\frac{1}{\frac{\sqrt{23}}{2}}arctg\frac{t}{\frac{\sqrt{23}}{2}}+C=\frac{3}{2}\ln(x^2-3x+8)+\frac{13}{\sqrt{23}}arctg\frac{2x-3}{\sqrt{23}}+C.$$

$IV.\ \int \frac{Mx+N}{(x^2+px+q)^n}dx \qquad (n\geq 2)$ integralni qaraymiz. $t=x+\frac{p}{2}$ deb, avvalgidek yangi t o'zgaruvchini kiritamiz. Bu quyidagini beradi:

$x=t-\frac{p}{2}, \quad dx=dt, \quad x^2+px+q=t^2+a^2,$ bu erda $a^2=q-\frac{p^2}{4}.$

Demak, $\int \frac{Mx+N}{(x^2+px+q)^n}dx=M\int \frac{tdt}{(t^2+a^2)^n}+(N-\frac{Mp}{2})\int \frac{dt}{(t^2+a^2)^n}.$ (1)

(1) tenglikning o'ng tomonidagi integrallardan birinchisi oson hisoblanadi:

$$\int \frac{tdt}{(t^2+a^2)^n}=\frac{1}{2}\int (t^2+a^2)^{-n}d(t^2+a^2)=\frac{1}{2(1-n)(t^2+a^2)^{n-1}}+C.$$

Shunday qilib, $J_n=\int \frac{dt}{(t^2+a^2)^n}$ integralni hisoblash qoladi. Bu integralni quyidagi ko'rinishda yozib olamiz:

$$J_n=\int \frac{dt}{(t^2+a^2)^n}=\frac{1}{a^2}\int \frac{(t^2+a^2)-t^2}{(t^2+a^2)^n}dt=\frac{1}{a^2}[\int \frac{dt}{(t^2+a^2)^{n-1}}-\int \frac{t^2dt}{(t^2+a^2)^n}]$$

$\int \frac{dt}{(t^2+a^2)^{n-1}}=J_{n-1}$ deb quyidagini hosil qilamiz: $J_n=\frac{1}{a^2}\left[J_{n-1}-\int \frac{t^2dt}{(t^2+a^2)^n}\right]$ (2)

$u=t, du=dt,\ dv=\frac{tdt}{(t^2+a^2)^n}, \qquad v=\frac{1}{2(1-n)(t^2+a^2)^{n-1}}$ deb $\int \frac{t^2dt}{(t^2+a^2)^n}$ integralni bo'laklab integrallaymiz:

$$\int \frac{t^2dt}{(t^2+a^2)^n}=\frac{t}{2(1-n)(t^2+a^2)^{n-1}}-\frac{1}{2(1-n)}\int \frac{dt}{(t^2+a^2)^{n-1}}=\frac{1}{2(1-n)(t^2+a^2)^{n-1}}-\frac{1}{2(1-n)}J_{n-1}.$$

Hosil qilingan integralni (2) formulaga qo'yib, quyidagini hosil qilamiz:

$$J_n=\frac{1}{a^2}\left[J_{n-1}-\frac{t}{2(1-n)(t^2+a^2)^{n-1}}+\frac{1}{2(1-n)}\cdot J_{n-1}\right]=\frac{1}{a^2}\left[\frac{3-2n}{2-2n}\cdot J_{n-1}+\frac{t}{2(n-1)(t^2+a^2)^{n-1}}\right].$$

Shunday qilib, $J_n=\frac{1}{a^2}\left[\frac{2n-3}{2n-2}J_{n-1}+\frac{t}{2(n-1)(t^2+a^2)^{n-1}}\right].$ (3)

Hosil qilingan formula keltirish formulasi deyiladi.

3.Aniqmas koeffisientlar metodi.

To'g'ri rasional kasrni koeffisientlarini topishning eng soddametodlaridan biri aniqmas koeffisientlarmetodidir. Bumetodni qo'llanishinimisollarda tushuntiramiz.

3-Misol . $\frac{x^2-2x+6}{(x-2)^2(x^2+3x+4)}$ ni sodda kasrlarga ajrating.

Echilishi: (4) yoyilmadanfoydalanib, yozamiz:

$$\frac{x^2-2x+6}{(x-2)^2(x^2+3x+4)}=\frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{Mx+N}{x^2+3x+4}, \quad (*)$$

bu erda A_1, A_2,*m va N*-lar xozircha noma'lum sonlar. (*) ayniyatning o'ng tomonini umumiymaxrajga keltiramiz:

$$\frac{\tilde{o}^2-2\tilde{o}+6}{(\tilde{o}-2)^2\cdot(\tilde{o}^2+3\tilde{o}+4)}=\frac{\grave{A}_1(\tilde{o}-2)(\tilde{o}^2+3\tilde{o}+4)+\grave{A}_2(\tilde{o}^2+3\tilde{o}+4)+(\grave{I}\ \ \tilde{o}+N)(x-2)^2}{(x-2)^2(x^2+3x+4)}$$

Bu ayniyatda kasrlarning maxraji birxil. Demak, suratlari ham aynan teng:

$x^2-2x+6=A_1x^3+3A_1x^2+4A_1x-2A_1x^2-6A_1x-8A_1+A_2x^2+3A_2x+4A_2+Mx^3-4Mx^2+4Mx+Nx^2-4Nx+4N$

Ikkita ko'pxadx ning birxil darajalarida koeffisientlar birxil bo'lganda vafaqat shunda aynan teng bo'ladi. Bu ko'pxadlarning koeffisientlarinix ning birxil darajalarida tenglab, quyidagi tenglamalar sistemasini hosil qilamiz:

$$\left.\begin{array}{l} \tilde{o}^3\ddot{a}\grave{a}: A_1+M=0 \\ x^2\ddot{a}\grave{a}: A_1+A_2-4M+N=1 \\ x\ \ddot{a}\grave{a}: -2A_1+3A_2+4M-4N=-2 \\ \hat{\imath}\c{c}\hat{\imath}\ddot{a}\quad \tilde{o}\grave{a}\ddot{a}: -8\grave{A}_1+4\grave{A}_2+4N=6 \end{array}\right\}$$

Bu sistemani echib, $A_1=-\frac{1}{14},\ A_2=\frac{3}{7},\ M=\frac{1}{14},\ N=\frac{13}{14}$ larni topamiz. (*)*m*unosabatda A_1, A_2,*m* va *N* lar o'rniga topilgan qiymatlarni qo'yib, uzil-kesil quyidagini hosil qilamiz:

$$\frac{x^2-2x+6}{(x-2)^2(x^2+3x+4)}=-\frac{1}{14(x-2)}+\frac{3}{7(x-2)^2}+\frac{x+13}{14(x^2+3x+4)}.\ (**)$$

4.Rasional kasrlarni integrallash.

4-Misol. $\int\frac{x^3+x^2}{x^2-6x+5}dx$ ni toping.

Echilishi: Integralostidagi noto'g'ri rasional kasrni ko'pxad bilan to'g'ri kasr yig'indisi ko'rinishida ifodalaymiz:

$$\begin{array}{l|l} x^3+x^2 & x^2-6x+5 \\ \hline - & x+7 \\ x^3-6x^2+5x & \\ \hline \quad 7x^2-5x & \\ \quad - & \\ \quad 7x^2-42x+35 & \\ \hline \quad 37x-35 & \end{array}$$

Shunday qilib,

$$\int\frac{x^3+x^2}{x^2-6x+5}dx=\int\left[x+7+\frac{37x-35}{x^2-6x+5}\right]dx=\int(x+7)dx+\int\frac{37x-35}{x^2-6x+5}dx=\frac{x^2}{2}+7x+\int\frac{37x-35}{(x-5)(x-1)}dx.$$

Oxirgi integralni hisoblaymiz:

$$\frac{37x-35}{(x-5)(x-1)}=\frac{A}{x-5}+\frac{B}{x-1};\quad 37x-35=Ax-A+Bx-5B.\quad \begin{cases}A+B=37\\ -A-5B=-35\end{cases}\Rightarrow -4B=2,\quad B=-\frac{1}{2},\quad A=\frac{75}{2}$$

$\frac{37x-35}{(x-5)(x-1)}=\frac{75}{2(x-5)}-\frac{1}{2(x-1)}$. U holda, $\int\frac{37x-35}{(x-5)(x-1)}dx=\frac{75}{2}\int\frac{dx}{x-5}-\frac{1}{2}\int\frac{dx}{x-1}=\frac{75}{2}\ln|x-5|-\frac{1}{2}\ln|x-1|+C,$ buni o'rniga qo'ysak, $\int\frac{x^3+x^2}{x^2-6x+5}dx=\frac{1}{2}x^2+7x+\frac{75}{2}\ln|x-5|-\frac{1}{2}\ln|x-1|+C.$

Mastaqil ishlash uchun misollar

Quyidagi rasionalfunksiyalarning integrallari topilsin.

1^0. $\int \frac{dx}{(x-1)^4}$; *javob:* $-1/[3(x-1)^3]+C.$

2^0. $\int \frac{dx}{(2x+3)^3}$; *javob:* $-1/[4(2x+3)^2]+C.$

3^0. $\int \frac{dx}{x^2-6x+18}$; *javob:* $\frac{1}{3}arctg(x-3)/3+C.$

4^0. $\int \frac{x^2dx}{x^6+2x^3+3}$; *javob:* $\frac{1}{3\sqrt{2}}\cdot arctg\frac{x^3+1}{\sqrt{2}}+C.$

5^0. $\int \frac{x-2}{x^2-4x+7}dx$; *javob:* $\frac{1}{2}\ln(x^2-4x+7)+C.$

6^0. $\int \frac{5x+3}{x^2+10x+29}dx$; *javob:*

$(5/2)\ln(x^2+10x+29)-11arctg(x+5)/2+C.$

7^0. $\int \frac{x+1}{5x^2+2x+1}dx$; *javob:* $(1/10)\ln(5x^2+2x+1)+(2/5)arctg(5x+1)/2+C.$

8^0. $\int \frac{xdx}{(x+1)(2x+1)}$; *javob:* $\ln\frac{|x+1|}{\sqrt{2x+1}}+C$

5.Ba'zi irrasional funksiyalarni integrallash.

I. $\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{r_1},\ldots,\left(\frac{ax+b}{cx+d}\right)^{r_s}\right]dx$ (1)

integralni qaraymiz. Bunda $r_1, r_2, \ldots, r_s$ lar rasional sonlar, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}\neq 0,$ R-o'z argumentlariga nisbatan rasionalfunksiya. $r_1, r_2, \ldots, r_s$ sonlarning umumiymaxraji m bo'lsin: $r_i=\frac{p_i}{m}$, r_i-butun sonlar, i=1,2,...,s.

$t^m=\frac{ax+b}{cx+d}$ (2) almashtirishni bajaramiz. Bundan $x=\frac{dt^m-b}{a-ct^m}=\rho(t);$ (3)

$\rho(t)$ rasionalfunksiyadan iborat, shuning uchun $\rho'(t)$ ham rasionalfunksiya; shundan so'ng (3)dan $dx=\rho'(t)dt,$ (4)

$$\left(\frac{ax+b}{cx+d}\right)^{r_i}=t^{mr_i}=t^{p_i},\ i=1,2,\ldots,s \quad (5)$$

(3), (4) va (5) larni (1) ga qo'yib, qo'yidagiga ega bo'lamiz:

$$\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{r_1},\ldots,\left(\frac{ax+b}{cx+d}\right)^{r_s}\right]dx=\int R\left(\frac{dt^m-b}{a-ct^m},t^{p_1},\ldots,t^{p_s}\right)\rho'(t)dt=\int R^*(t)dt,$$

bunda $R^*(t)=R\left(\frac{dt^m-b}{a-ct^m},t^{p_1},\ldots,t^{p_s}\right)\cdot\rho'(t)$ rasionalfunksiyani integraliga keladi.

1-Misol. $\int\frac{\sqrt[6]{x}}{1+\sqrt[3]{x}}dx$ ni toping.

Echilishi Bunda $\frac{1}{6}$ va $\frac{1}{3}$ kasrlarning umumiymaxraji $m=6$. Quyidagi almashtirishni bajaramiz. $t^6=x,$ u holda $dx=6t^5dt$ va demak,

$$\int\frac{\sqrt[6]{õ}}{1+\sqrt[3]{õ}}dx=\int\frac{t}{1+t^2}\cdot 6t^5dt=6\int\frac{t^6}{1+t^2}dt=6\int\left(t^4-t^2+1-\frac{1}{1+t^2}\right)dt=6\left(\frac{t^5}{5}-\frac{t^3}{3}+t-arctgt\right)+Ñ=$$

$= \frac{6}{5}\sqrt[6]{x^5} - 2\sqrt{x} + 6\sqrt[6]{x} - 6arctg\sqrt[6]{x} + C.$

II. $J_1 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$ integralni qaraymiz. Bunda ildizostida kvadrat uchxadni to'la kvadratga ajratib, $x + \frac{b}{2a} = t,\ dx = dt,\ \frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$ десак, $a > 0$ bo'lganda $\int \frac{dt}{\sqrt{t^2 \pm k^2}}$ yoki $a < 0$ bo'lganda $\int \frac{dt}{\sqrt{k^2 - t^2}}$ integral-larga keladi, bu esa jadvalning *16⁰* - va *17⁰* - ga ko'ra hisoblanadi.

2-Misol. $\int \frac{dx}{\sqrt{x^2 - x - 1}}$ ni toping.

Echilishi: $x^2 - x - 1 = (x - \frac{1}{2})^2 - \frac{5}{4}$. U holda $\int \frac{dx}{\sqrt{x^2-x-1}} = \int \frac{d(x-\frac{1}{2})}{\sqrt{(x-\frac{1}{2})^2 - \frac{5}{4}}} = \ln|x - \frac{1}{2} + \sqrt{x^2-x-1}| + C.$

Ushbu integral $J_2 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ quyidagi almashtirishlar yordami bilan hisoblanadi:

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx = \int \frac{\frac{A}{2a}(2ax+b) + (B - \frac{Ab}{2a})}{\sqrt{ax^2+bx+c}}dx = \frac{A}{2a}\int \frac{d(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} + (B - \frac{Ab}{2a})\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$

$$= \frac{A}{2a}\int (ax^2+bx+c)^{-\frac{1}{2}} d(ax^2+bx+c) + (B - \frac{Ab}{2a}) \cdot J_1 = \frac{A}{2a}(2\sqrt{ax^2+bx+c}) + (B - \frac{Ab}{2a})J_1 + C.$$

3-misol. $\int \frac{3\tilde{o}+2}{\sqrt{\tilde{o}^2+\tilde{o}+2}}dx$ ni toping.

Echilishi: $\int \frac{3\tilde{o}+2}{\sqrt{\tilde{o}^2+\tilde{o}+2}}dx = \int \frac{\frac{3}{2}(2x+1) + (2 - \frac{3}{2})}{\sqrt{\tilde{o}^2+\tilde{o}+2}}dx = \frac{3}{2}\int \frac{2x+1}{\sqrt{\tilde{o}^2+\tilde{o}+2}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{\tilde{o}^2+\tilde{o}+2}} =$

$$= \frac{3}{2}\int (\tilde{o}^2+\tilde{o}+2)^{-\frac{1}{2}} d(\tilde{o}^2+\tilde{o}+2) + \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + \frac{7}{4}}} = 3\sqrt{x^2+x+2} + \frac{1}{2}\ln|x + \frac{1}{2} + \sqrt{x^2+x+2}| + C.$$

6.Trigonometrik funksiyalarni integrallash.

I. $\int R(\sin x, \cos x)dx$ ko'rinishdagi integral

Bunda *R*-rasionalfunksiya. Bu integral $tg\frac{x}{2} = t$ almashtirish bilan rasionalfunksiyaning integraliga keltirilishimumkin.

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

$$tg\frac{x}{2} = t \Rightarrow \frac{x}{2} = arctgt \Rightarrow x = 2arctgt \Rightarrow dx = \frac{2dt}{1+t^2};$$

U holda berilgan integral ushbu ko'rinishdagi rasionalfunksiyaning integraliga keladi, ya'ni $\int R(\sin x,\ \cos x)dx = \int R\left[\frac{2t}{1+t^2}; \frac{1-t^2}{1+t^2}\right]\cdot\frac{2dt}{1+t^2} = \int R^*(t)dt,$ bunda *$R^*(t)$*funksiya *t* ning rasionalfunksiyasi.

4-Misol. $\int\frac{dx}{\sin x}$ integralni hisoblang.

Echilishi Yuqoridagi yozilganformulalarga asosan:

$$\int\frac{dx}{\sin x} = \int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int\frac{dt}{t} = \ln|t| + C = \ln|tg\frac{x}{2}| + C.$$

$tg\frac{x}{2} = t\ \ (-\pi < x < \pi)$ almashtirishga universal almashtirish deyiladi. Bu almashtirish *R(sinx,cosh)* ko'rinishdagi xar qandayfunksiyani integrallash uchun imkon beradi. Lekin praktikada bu almashtirish ko'pincha anchamurakkab rasionalfunksiyagaolib keladi. Shuning uchun universal almashtirish bilan bir qatorda ba'zi xollar uchunmaqsadga tezolib keladigan boshqa almashtirishlarni ham bilishfoydalidir.

1^0. Agar *R(sinx, cosh)*funksiya *sinx* ga nisbatan toq bo'lsa, ya'ni *R(-sinx, cosh)* = - *R(sinx, cosh)* bo'lsa, u holda *cosh=t,* - *sinxdx=dt* almashtirish bilan rasionalfunksiyani integraliga keladi.

2^0.Agar *R(sinx, cosh)* funksiya *cos x* ga nisbatan toq bo'lsa, ya'ni *R(sinx, -cosh)=-R(sinx, cosh)* bo'lsa, u holda *sinx=t* almashtirish bilan rasionalfunksiyani integraliga keltiriladi.

3^0.Agar *R(sinx, cosh)* funksiya *sinx* va *cosh* ga nisba-tan juft bo'lsa, ya'ni *R(-sinx,-cosh)=R(sinx, cosh)* bo'lsa, u holda *tgx=t* almashtirish bilan rasionallashtiriladi.

Bu holda $\sin^2 x = \frac{tg^2x}{1+tg^2x} = \frac{t^2}{1+t^2},\quad \cos^2 x = \frac{1}{1+tg^2x} = \frac{1}{1+t^2},\quad dx = \frac{dt}{1+t^2}.$

5-Misol. $\int\frac{\sin^3 x}{2+\cos x}dx$ integral hisoblansin.

Echilishi: $R(\sin x,\ \cos x) = \frac{\sin^3 x}{2+\cos x}$funksiya sinx ga nisbatan toqfunksiya. Bu integralni $\int R(\cos x)\sin xdx$ ko'rinishga keltirishoson. Xaqiqatdan,

$$\int\frac{\sin^3 x}{2+\cos x}dx = \int\frac{\sin^2 x\sin x}{2+\cos x}dx = \int\frac{1-\cos^2 x}{2+\cos x}\cdot\sin xdx.$$

cosh=t almashtirishni bajaramiz. Bu holda *sinxdx=-dt*. Demak,

$$\int\frac{\sin^3 x}{2+\cos x}dx = -\int\frac{1-t^2}{2+t}dt = \int\frac{t^2-1}{2+t}dt = \int(t-2+\frac{3}{t+2})dt = \frac{t^2}{2} - 2t + 3\ln|t+2| + Ñ = \frac{\cos^2 x}{2} - 2\cos x + 3\ln|\cos x + 2| + Ñ.$$

6-Misol. $\int\frac{dx}{2-\sin^2 x}dx$ integral hisoblansin.

Echilishi: *tgx=t* almashtirishni bajaramiz. Bu holda:

$$\int\frac{dx}{2-\sin^2 x} = \int\frac{dt}{(2-\frac{t^2}{1+t^2})(1+t^2)} = \int\frac{dt}{2+t^2} = \frac{1}{\sqrt{2}}arctg\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}arctg\frac{tgx}{\sqrt{2}} + C.$$

II. $\int\sin^m x\cdot\cos^n xdx$ **ko'rinishdagi integral.**

Bunda uchta xolni qarashga to'g'ri keladi:

a)Berilgan integralda *m* va *n* larning kamida bittasi toq son. Aniqlik uchun *n*-toq son bo'lsin. *n=2p+1* debolib, integralni o'zgartiramiz:

$$\int \sin^m x \cos^{2p+1} x dx = \int \sin^m x \cos^{2p} x \cos x dx = \int \sin^m x (1-\sin^2 x)^p \cos x dx.$$

O'zgaruvchini almashtiramiz: $\sin x = t \Rightarrow \cos x dx = dt.$ U holda

$\int \sin^m x \cos^n x dx = \int t^m (1-t^2)^p dt,$ bu esa *t* ning rasionalfunksiyasining integralidir.

7-Misol. $\int \frac{\cos^5 x}{\sin x} dx$ integral hisoblansin.

Echilishi: $\int \frac{\cos^5 x}{\sin x} dx = \int \frac{\cos^4 x \cdot \cos x}{\sin x} dx = \int \frac{(1-\sin^2 x)^2 \cos x dx}{\sin x}.$ $\sin x = t$ deymiz, $\cos x dx = dt,$ u holda

$$\int \frac{\cos^5 x}{\sin x} dx = \int \frac{(1-t^2)^2}{t} dt = \int \frac{dt}{t} - 2\int t dt + \int t^3 dt = \ln|t| - 2\cdot\frac{t^2}{2} + \frac{t^4}{4} + C = \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C.$$

$$= \ln|t| - 2\cdot\frac{t^2}{2} + \frac{t^4}{4} + C = \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C.$$

b) Berilgan integralda *m* va *n*-manfiy bo'lmagan juft son. *m=2p, n=2q* deymiz. Trigonometriyadanma'lum bo'lganformulalarni yozamiz:

$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x).$ Bularni berilgan integralga qo'yamiz:

$$\int \sin^{2p} x \cdot \cos^{2q} x dx = \int (\frac{1}{2} - \frac{1}{2}\cos 2x)^p \cdot (\frac{1}{2} + \frac{1}{2}\cos 2x)^q dx. \quad (6)$$

Darajaga ko'tarib hamda qavslarniochib, *cos2x* ning juft va toq darajalarini o'z ichigaolgan xadlarni hosil qilamiz.

Toq darajali xadlar a) holda ko'rsatilgandek integrallanadi. Darajaning juft ko'rsatkichlarini (6)formulalarga ko'ra yana pasaytiramiz. Daraja ko'rsatkich-larni pasaytirishnioson integrallanadigan $\int \cos kx dx$ ko'rinishdagi xadlar hosil bo'lguncha shunday davom ettiramiz.

8-misol. $\int cos^4 x dx$ integral hisoblansin.

Echilishi: $\int cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx =$

$$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x dx + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + + \frac{1}{8}\int (1+\cos 4x) dx = \frac{1}{4}x$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}(x + \frac{1}{4}\sin 4x) + \tilde{N} = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \tilde{N}.$$

v) Agar ikkala daraja ko'rsatkich ham juft bo'lib, ulardan kamida bittasimanfiy bo'lsa, yuqorida bayon qilingan usulmaqsadgaolib kelmaydi. Bunda *tgx=t (yoki ctgx=t)* almashtirishni bajarishga to'g'ri keladi.

III. $\int \sin mx \cos nx dx, \int \cos mx \cos nx dx, \int \sin mx \sin nx dx$ **ko'rinishdagi integrallar.**

Bular quyidagi formulalar yordamida hisoblanadi:

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x], \quad (7)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x], \quad (8)$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]. \quad (9)$$

9-Misol. $\int cos\frac{x}{2}\cos\frac{x}{3}dx$ integral hisoblansin.

Echilishi: (8)formulaga ko'ra: $\cos\frac{x}{2}\cdot\cos\frac{x}{3}=\frac{1}{2}\left[\cos\left(\frac{x}{2}+\frac{x}{3}\right)+\cos\left(\frac{x}{2}-\frac{x}{3}\right)\right]=\frac{1}{2}\left(\cos\frac{5}{6}x+\cos\frac{x}{6}\right).$

U holda

$$\int\cos\frac{x}{2}\cos\frac{x}{3}dx=\frac{1}{2}\int\left(\cos\frac{5}{6}x+\cos\frac{1}{6}x\right)dx=\frac{1}{2}\cdot\frac{6}{5}\sin\frac{5}{6}x+\frac{1}{2}\cdot 6\sin\frac{1}{6}x+C=\frac{3}{5}\sin\frac{5}{6}x+3\sin\frac{1}{6}x+C.$$

Mastaqil ishlash uchun misollar

Quyidagi irrasional funksiyalarning integrallari topilsin:

1^0. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}+2\sqrt[4]{x}}$; *javob:* $2\sqrt{x}-3\sqrt[3]{x}-8\sqrt[4]{x}+$

$+6\sqrt[6]{\tilde{o}}+48\sqrt[12]{\tilde{o}}+3\ln(1+\sqrt[12]{x})+\frac{33}{2}\ln(\sqrt[6]{x}-\sqrt[12]{x}+2)-\frac{171}{\sqrt{7}}arctg\frac{2\sqrt[12]{x}-1}{\sqrt{7}}+C.$

2^0. $\int\sqrt{\frac{1-x}{1+x}}\cdot\frac{dx}{x}$, *javob:* $\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+2arctg\sqrt{\frac{1-x}{1+x}}+C.$

3^0. $\int\frac{dx}{\sqrt{1-2x}-\sqrt[4]{1-2x}}$; *javob:* $-\sqrt{1-2x}-2\sqrt[4]{1-2x}-2\ln|\sqrt[4]{1-2x}-1|+C.$

4^0. $\int\frac{dx}{\sqrt{-x^2-2x+8}}$; *javob:* $\arcsin\frac{x+1}{3}+C.$

5^0. $\int\frac{5x+3}{\sqrt{-x^2+4x+5}}dx$; *javob:* $-5\sqrt{-x^2+4x+5}+13\arcsin\frac{x-2}{3}+C.$

Quyidagi trigonometrikfunksiyalarning integrallari topilsin:

1^0. $\int\frac{\sin^3x}{\cos^4x}dx$; *javob:* $\frac{1}{3\cos^3x}-\frac{1}{\cos x}+C.$

2^0. $\int\frac{dx}{\cos x\sin^3x}$; *javob:* $\ln|tgx|-\frac{1}{2\sin^2x}+C$

3^0. $\int\frac{dx}{\cos^3x\sin^3x}$; *javob:* $\frac{1}{2}(tg^2x-ctg^2x)+2\ln|tgx|+C.$

4^0. $\int\frac{dx}{\cos^4x\sin^4x}$; *javob:* $\frac{(tg^2x-1)(tg^4x+10tg^2x+1)}{3tg^3x}+C;$

5^0. $\int\cos^6xdx$; *javob:* $\frac{5}{16}x+\frac{1}{12}\sin 2x\left(\cos^4x+\frac{5}{4}\cos^2x+\frac{15}{8}\right)+C.$

6^0. $\int tg^5xdx$ *javob:* $\frac{1}{4}tg^4x-\frac{1}{2}tg^2x-\ln|\cos x|+C$.

7^0. $\int\frac{cos^4x+\sin^4x}{\cos^2x-\sin^2x}dx$; *javob:* $\frac{1}{4}\ln\left|\frac{1+tgx}{1-tgx}\right|+\frac{1}{2}\sin x\cos x+C;$

8^0. $\int\frac{dx}{(\sin x+\cos x)^2}$; *javob:* $C-\frac{1}{1+tgx}.$

9^0. $\int\frac{dx}{\sin x+\cos x}$; *javobi:* $\frac{\sqrt{2}}{2}\ln\left|tg\left(\frac{\pi}{8}+\frac{x}{2}\right)\right|+C.$

10^0. $\int\frac{dx}{a\cos x+b\sin x}$; *javob:* $\frac{1}{\sqrt{a^2+b^2}}\ln|tg\frac{x+arctg\frac{a}{b}}{2}|+C.$

11^0. $\int\frac{dx}{5-4\sin x+3\cos x}$; *javob:* $\frac{1}{2-tg\frac{x}{2}}+C.$

12^0. $\int\frac{dx}{4-3\cos^2x+5\sin^2x}$; *javob:* $\frac{1}{3}arctg(3tgx)+C.$

Adabiyotlar

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Mundarija