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$\alpha-$**subgarmonik funksiyalar va ularning xossalari**

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0.1. Dissertatsiya ishining umumiy tavsifi.

1.1. Mavzuning dolzarbligi. Magistrlik dissertatsiyasi potensiallar nazariyasida yangi bir sinfni tashkil etuvchi $\alpha-$subgarmonik funksiya va uning xossalarini oʻrganishga bagʻishlanadi.

Ma’lumki, garmonik va subgarmonik funksiyalar sinfi funksiyalar nazariyasida muhim oʻrin tutib, matematik fizika, geometriya masalalarida koʻp qoʻllaniladi. Klassik Dirixle masalasi biror $D \subset R^n$ sohada $\Delta u = 0,\ u\Big|_{\partial D} = \varphi$ tenglamaning yechimlari xususida boʻlib, bunday yechimning mavjudligi, yagonaligi, ularning xossalari garmonik analizning asosini tashkil qiladi. C^n fazoda Laplas operatori $dd^c u$ differensial forma bilan $dd^c u \wedge \beta^{n-1} = (n-1)!\Delta u dV$ kabi bogʻlanishga ega, bu yerda $\beta = dd^c\left|z\right|^2,\ d = \partial + \overline{\partial},\ d^c = \dfrac{\partial - \overline{\partial}}{4i}$. U holda C^n fazoda subgarmonik funksiya umumlashgan ma’noda $dd^c u \wedge \beta^{n-1} \geq 0$ tengsizlik bilan ifodalanadi: ya’ni u yuqoridan yarim uzluksiz, $\overline{\lim\limits_{w \to z}} u(w) \leq u(z)$ va $\forall \omega \geq 0,\ \omega \in F^{(0,0)}$ uchun $dd^c u \wedge \beta^{n-1}(\omega) = \int u dd^c \omega \wedge \beta^{n-1} \geq 0$ tengsizlik bajarilsa, u funksiya $D \subset \mathbb{C}^n$ sohada subgarmonik boʻladi, bu yerda $F^{(p,p)} = \left\{\omega \in \mathscr{F}^{(p,p)}(D) \cap C^{\infty}(D) : \sup p\omega \subset\subset D\right\}$ - silliq va finit differensial formalar fazosi. Keyingi paytlarda standart Keller formasi $\beta = dd^c \left|z\right|^2$ o‘rniga ixtiyoriy musbat α formani olish zaruriyati yuzaga keldi. Bunday holat, jumladan, $m-sh$ funksiyalar sinfini o‘rganish (B. Abdullayev [17], va boshqalar) $p-$plyurisubgarmonik funksiyalar va kalibrlangan geometriya masalalarida (F. Harvey, H. Lawson [18,19,20]), T-subgarmonik funksiyalarda ko‘zga tashlanadi. Shu sabab $\alpha-$subgarmonik funksiyalar nazariyasini qurish muammosi dolzarb hisoblanadi.

1.2. Ishning maqsadi. $\alpha-$subgarmonik funksiyalarning xossalarini o‘rganish.

1.3. Ishning vazifalari. Dissertatsiyada quyidagi asosiy masalalar qaraladi:

- qat’iy musbat differensial forma yordamida yangi $dd^c u \wedge \alpha$ operatorning elliptikligini tadqiq qilish
- $\alpha-$subgarmonik funksiyalar sinfi xossalarini o‘rganish.

1.4. Tadqiqot ob’yekti. Qat’iy musbat differensial formalar, elliptik operatorlar.

1.5. Tadqiqot predmeti. $\alpha-$subgarmonik funksiyalar.

1.6.Tadqiqot usullari. Ko‘p kompleks o‘zgaruvchili funksiyalar nazariyasi va kompleks potensiallar nazariyasi usullari.

1.7. Tadqiqot natijalarining ilmiy jihatdan yangilik darajasi. Mazkur tadqiqot ishida yangi natijalar olingan.

1.8. Tadqiqot natijalarining amaliy ahamiyati va tadbiqi. Olingan natijalar nazariy xarakterga ega. Natijalar va usullarni matematik analiz, funksional analiz va funksiyalar nazariyasining tadbiqlarida foydalanish mumkin.

1.9. Ish tuzilishi va tarkibi. Dissertatsiya ishi kirish, uchta bob hamda adabiyotlar ro‘yhatidan iborat. Har bir bob bo‘limlardan tuziladi.

0.2. Dissertatsiya ishining mazmuni.

Dissertatsiya ishining birinchi bobida tayanch tushunchalar keltiriladi. Bunda potensiallar nazariyasi asosini tashkil etuvchi garmonik funksiyalar, subgarmonik funksiyalar, plyurisubgarmonik funksiyalarning xossalari o‘rganiladi.

Dissertatsiya ishining ikkinchi bobi $dd^c u \wedge \alpha$ operator va $\alpha-$garmonik funksiyalarni o‘rganishga bag‘ishlanadi. Ikkinchi bobning birinchi bo‘limi musbat aniqlangan differensial forma va oqimlarni o‘rganishga bag‘ishlangan bo‘lib, unda differensial forma, musbat differensial forma, oqim tushunchalari batafsil o‘rganiladi.

Ikkinchi bobning ikkinchi bo‘limida $dd^c u \wedge \alpha$ operator kiritilib, uning elliptik operator ekanigi isbotlanadi. Bu operator yordamida $\alpha-$garmonik funksiya tushunchasi kiritiladi.

$\mathbb{C}^n$ kompleks fazoda $(n-1, n-1)$-bidarajali qat’iy musbat α differensial formani tayinlaymiz.

2.2.1-teorema. *$dd^c u \wedge \alpha$ elliptik operator bo‘ladi.*

2.2.1-ta’rif. *$D \subset \mathbb{C}^n$ sohada $dd^c u \wedge \alpha = 0$ tenglamani qanoatlantiruvchi ikki marta silliq $u(z) \in C^2(D)$ funksiyaga $\alpha-$garmonik funksiya deyiladi.*

Xususiy holda $\alpha-$garmonik funksiya tushunchasi kiritilib, uning o‘rta qiymat haqidagi xossasi o‘rganiladi.

$(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}_+^n$ vektor uchun,

$$\varphi(z) = \alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n$$

funksiyani olamiz va ushbu formani qaraymiz:

$$dd^c\varphi = \frac{i}{2}(\alpha_1 dz_1 \wedge d\overline{z}_1 + \alpha_2 dz_2 \wedge d\overline{z}_2 + ... + \alpha_n dz_n \wedge d\overline{z}_n).$$

$\alpha = \left(dd^c\varphi\right)^{n-1}$ deb olib, $dd^c u \wedge \alpha = (n-1)!\sum_j \alpha_1\alpha_2 \overset{\overset{j}{\vee}}{...} \alpha_n \dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV$

operatorni qaraymiz. Bu yerda $\overset{\overset{j}{\vee}}{...}$ belgi shu ko‘paytmada j-had qatnashmasligini bildiradi.

2.2.2-ta'rif. $\sum_j \alpha_1\alpha_2 \overset{\overset{j}{\vee}}{...} \alpha_n \cdot \dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_j} = 0$ *tenglikni qanoatlantiruvchi* $u \in C^2(D)$ *funksiyaga* $\alpha -$*garmonik funksiya deyiladi.*

Ushbu teorema $\alpha -$garmonik funksiyalarning geometrik xususiyatini ifodalaydi:

2.2.2-teorema. *Agar* $u(z)$ *funksiya* D *sohada* $\alpha -$*garmonik bo‘lsa, u holda* $\forall z^0 \in D$ *nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left|z_1 - z_1^0\right|^2 + ... + \alpha_n \left|z_n - z_n^0\right|^2 \le r^2 \right\} \subset\subset D \quad gipershar$$

uchun

$$u(z^0) = \frac{1}{\tilde{V}(r)} \int_{E(z^0, r)} u(\tau) dV(\tau)$$

tenglik o‘rinli bo‘ladi. Bu yerda $\tilde{V}(r)$ *- gipershar hajmi va* $dV(\tau) -$ *hajm elementi.*

Shuningdek quyidagi teoremani o‘rinli ekanligini ko‘rish mumkin.

2.2.3-teorema. *Agar $u(z)$ funksiya D sohada $\alpha-$garmonik bo‘lsa, u holda $\forall z^0 \in D$ nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left| z_1 - z_1^0 \right|^2 + \ldots + \alpha_n \left| z_n - z_n^0 \right|^2 = r^2 \right\} \subset\subset D$$

uchun

$$u(z^0) = \frac{1}{\sigma_n r^{2n-1}} \int\limits_{\partial E(z^0, r)} u(\xi) d\sigma(\xi)$$

tenglik o‘rinli bo‘ladi. Bu yerda $d\sigma(\xi)$ - sirt elementi va $\sigma_n -$ birlik gipersharning sirt yuzasi.

Dissertatsiya ishining uchinchi bobi $\alpha-$subgarmonik funksiyalar va ularning xossalarini o‘rganishga bag‘ishlanadi. Ushbu bobning birinchi bo‘limida umumiy holda $\alpha-$subgarmonik funksiya ta’rifi kiritilib, so‘ngra xususiy holda kiritilgan $\alpha-$subgarmonik funksiyalarning xossalari o‘rganiladi.

$\alpha-$ qat’iy musbat, yopiq, cheksiz silliq koeffisientli bo‘lgan holda biz quyidagi $\alpha-$subgarmonik funksiya ta’rifini beramiz.

3.1.1-ta’rif. $D \subset \mathbb{C}^n$ *sohada berilgan* $u(z) \in L^1_{loc}\left(D\right)$ *funksiya uchun quyidagi shartlar*

1) u yuqoridan yarim uzluksiz, ya’ni $\overline{\lim\limits_{z \to z^0}} u(z) = \lim\limits_{\varepsilon \to 0} \sup\limits_{B(z^0, \varepsilon)} u(z) \le u(z^0)$;

2) umumlashgan funksiya (oqim) ma'nosida

$$\left[dd^c u\wedge\alpha\right]\left(\omega\right)=\int u\wedge\alpha\wedge dd^c\omega, \forall\,\omega\in F\left(D\right), \qquad (1)$$

musbat, ya'ni $\left[dd^c u\wedge\alpha\right]\left(\omega\right)\geq 0, \forall\,\omega\geq 0$ *bajarilsa, bu yerda* $F\left(D\right)-$ *asosiy finit funksiyalar fazosi,* u *ga* $\alpha-$*subgarmonik funksiya deyiladi.*

Endi xususiy holda $\alpha-$subgarmonk funksiyalar sinfi xossalarni o'rganamiz.

$(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbb{R}_+^n$ vektor, $\varphi(z)=\alpha_1 z_1\overline{z}_1+\alpha_2 z_2\overline{z}_2+...+\alpha_n z_n\overline{z}_n$ funksiya va $D(D\subset\mathbb{C}^n)$ sohada berilgan $u\in C^2$ funksiya uchun quyidagi operatorni qaraymiz:

$$dd^c u\wedge\alpha=dd^c u\wedge(dd^c\varphi)^{n-1}=(n-1)!\sum_j\alpha_1\alpha_2\overset{\overset{j}{\vee}}{...}\alpha_n\frac{\partial^2 u}{\partial z_j\partial\overline{z}_j}dV\,,$$

bu yerda $\overset{\overset{j}{\vee}}{...}$ belgi shu ko'paytmada j-had qatnashmasligini bildiradi.

3.1.2-ta'rif. $D\subset\mathbb{C}^n$ soha berilgan bo'lsin. Ushbu

$$\sum_{j=1,n}\alpha_1\alpha_2\overset{\overset{j}{\vee}}{...}\alpha_n\cdot\frac{\partial^2 u}{\partial z_j\partial\overline{z}_j}\geq 0$$

tengsizlikni qanoatlantiruvchi $u\in C^2(D)$ funksiyaga $\alpha-$subgarmonik funksiya deyiladi.

$\alpha-$subgarmonik funksiya quyidagi xossalarga ega.

a) $\alpha-$subgarmonik funksiyaning musbat koeffitsentli chiziqli kombinatsiyasi yana α-subgarmonik funksiya bo'ladi ya'ni

$$u_k(z) \in \alpha - sh(D),\ b_k \in R_+ \Rightarrow b_1u_1(z) + b_2u_2(z) + ... + b_mu_m(z) \in \alpha - sh(D)$$

$$k = \overline{1,m}$$

Bu yerda ham subgarmonik funksiyaning xossasi kabi xossa o'rinli.

3.1.1-teorema. *Agar $u(z)$ funksiya D sohada $\alpha-$subgarmonik bo'lsa, u holda $\forall z^0 \in D$ nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left|z_1 - z_1^0\right|^2 + ... + \alpha_n \left|z_n - z_n^0\right|^2 \le r^2 \right\} \subset\subset D \quad \textit{gipershar}$$

uchun

$$u(z^0) \le \frac{1}{\tilde{V}(r)} \int\limits_{E(z^0,r)} u(\tau) dV(\tau)$$

tenglik o'rinli bo'ladi. Bu yerda $\tilde{V}(r)$ - gipershar hajmi va $dV(\tau)-$hajm elementi.

Bu o'rta qiymat haqidagi teorema yordamida $\alpha-$subgarmonik funksiyalarning quyidagi xossalari kelib chiqadi.

b) Chekli sondagi $\alpha-$subgarmonik funksiyaning maksimumi ham $\alpha-$subgarmonik funksiya bo'ladi, ya'ni

$$u_1(z), u_2(z), ..., u_m(z) \in \alpha - sh(D) \Rightarrow \max\{u_1(z), u_2(z), ..., u_m(z)\} \in \alpha - sh(D).$$

c) Monoton kamayuvchi $\alpha-$subgarmonik funksiyalar ketma-ketligining limiti $\alpha-$subgarmonik funksiya bo'ladi, ya'ni

$$u_j(z) \in \alpha - sh(D),\ u_j(z) \geq u_{j+1}(z) \Rightarrow \lim_{j\to+\infty} u_j(z) \in \alpha - sh(D),\ \ j = 1,2,...,n.$$

d)Tekis yaqinlashuvchi α-subgarmonik funksiyalar ketma-ketligi yana α-subgarmonik funksiyaga yaqinlashadi, ya'ni

$$u_j(z) \in \alpha - sh(D),\ \ u_j(z) \rightrightarrows u(z) \Rightarrow u(z) \in \alpha - sh(D),\ \ (k = 1,2,3...)$$

e) *Maksimum prinsipi.* Agar $u(z) \in \alpha - sh(D)$ funksiya biror $z^0 \in D$ nuqtada o'zining maksimumiga erishsa, ya'ni

$$u(z^0) = \sup_{x\in D} u(z)$$

bo'lsa, u holda $u(z) = const$ bo'ladi.

Uchinchi bobning ikkinchi bo'limida $\alpha-$subgarmonik funksiyalarning Riss tasviri isbotlanadi. Bunda $K_\alpha\left(z,w\right):\ dd^cK_\alpha \wedge \alpha = \delta\left(z-w\right)$ fundamental yechimning mavjudligi izohlanib, u yordamida Riss tasviri o'rinli bo'lishi ko'rsatiladi.

3.2.1-teorema. Agar $u \in \alpha - sh\left(D\right)$ *bo'lsa, u holda istalgan* $G \subset\subset D$ *kompakt qism sohada quyidagi tasvir o'rinli:*

$$u\left(z\right) = \int_G K_\alpha\left(z-w\right)d\mu_w + \Phi\left(z\right),\ \ z \in G, \tag{2}$$

bu yerda $\Phi\left(z\right)-$ *G sohadagi biror* $\alpha-$*garmonik funksiya.*

Xususiy holda kiritilgan $dd^cu \wedge \alpha = (n-1)!\sum_j \alpha_1\alpha_2...\overset{j}{\vee}...\alpha_n \dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV$ operator uchun Riss yadrosining aniq ko'rinishi quyidagicha bo'lishi ko‘rsatiladi.

3.2.2-teorema. Berilgan $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$, $n \geq 2$ vektor uchun ushbu

$$K(z) = \frac{1}{(n-1)\cdot(\alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n)^{n-1}}$$

funksiya quyidagi

$$dd^c u \wedge \alpha = 0$$

elliptik tenglamaning fundamental yechimi bo‘ladi.

I - BOB. FUNKSIYALARNING BA'ZI MUHIM SINFLARI

1.1 Garmonik funksiyalar va ularning xossalari.

$D \subset \mathbf{R}^n$ sohada ushbu

$$\Delta u = 0$$

tenglamani qanoatlantiruvchi $u \in C^2\left(D\right)$ funksiya *garmonik funksiya* deyiladi, bunda

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

Laplas operatori.

Barcha garmonik funksiyalar to'plami $H(D)$ kabi belgilanadi.

1.1.1 – teorema. Agar $u(x)$ funksiya $D \subset R^n$ sohada garmonik bo'lsa, u holda ixtiyoriy $x^o \in D$ nuqta va $S(x^o, r) \subset D$ sfera uchun

$$u(x^o) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^o,r)} u(x)\, d\sigma \qquad (1.1.1)$$

bo'ladi, bunda σ_n - birlik sferaning yuzi.

Aksincha, agar D dan olingan har bir $x^o \in D$ nuqta va yetarlicha kichik r $(0 < r \le r_0(x^o))$ lar uchun (1.1.1) formula o'rinli bo'lsa, u holda $u(x) \in H(D)$ bo'ladi.

Endi garmonik funksiyalarning ba'zi bir xossalarini keltiramiz.

a) Agar $u, v \in H(D)$ bo'lib, $\lambda, \mu \in R$ bo'lsa, u holda

$$\lambda u + \mu v \in H\left(D\right)$$

bo‘ladi;

b) agar umumlashgan funksiya f uchun $\Delta f = 0$, ya’ni ixtiyoriy $\phi \in F\left(D\right)$ da $f(\Delta\phi) = 0$ tenglik bajarilsa, u holda f garmonik funksiya bo‘ladi, aniqroq qilib aytganda, shunday $v(x) \in C^2(D)$, $\Delta v = 0$ bo‘lgan funksiya mavjudki,

$$f(\phi) = v(\phi) = \int v(x)\phi(x)dV$$

bo‘ladi;

v) agar $u(x)$ funksiya $B(x_o, r)$ sharda garmonik, uning yopig‘ida esa uzluksiz, ya’ni

$$u(x) \in H(B(x_o, r)) \cap C(\overline{B(x_o, r)})$$

bo‘lsa, u holda

$$u(x) = \int\limits_{S(x^o, r)} u(y)P(x,y)d\sigma(y)$$

formula o‘rinli bo‘ladi, bunda

$$P(x,y) = \frac{r^2 - \left|x - x^0\right|^2}{\sigma_n r\left|x - y\right|^n}, \qquad n \geq 2.$$

Puasson yadrosi, $\sigma - R^n$ dagi birlik sfera yuzi, $n \geq 2$.Ikkinchi tomondan, agar $\phi(y)$ funksiya $S(x^o, r)$ sferada uzluksiz bo‘lsa, u holda

$$u(x) = \int\limits_{S(x^o,r)} \phi(y)P(x,y)d\sigma(y)$$

funksiya $B(x^0,r)$ sharda Dirixle masalasi $\Delta u = 0$, $u\Big|_{S(x^o,\ r)} = \phi(y)$ ning yechimi bo‘ladi.

Dirixle masalasini chegarasi silliq bo‘lgan ixtiyoriy $D \subset \mathbf{R}^n$ sohada ham qarash mumkin: $\phi(y) \in C(\partial D)$ uchun yagona $u(x) \in C(\overline{D})$ mavjudki, $\Delta u = 0$, $u\Big|_{\partial D} = \phi\ (y)$ bajariladi;

g) kompleks tekislik $\mathbf{C} \approx \mathbf{R}^2$ da garmonik funksiyalar golomorf funksiyalarga bevosita bog‘langandir.

Agar $z = x + iy$ deyilsa, unda

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

bo‘ladi va har bir golomorf $f \in \mathcal{O}\left(D\right)$ funksiya uchun

$$\operatorname{Re} f = \frac{f + \overline{f}}{2}, \quad \operatorname{Im} f = \frac{f - \overline{f}}{2i}$$

tengliklardan

$$\Delta \operatorname{Re} f = 0, \quad \Delta \operatorname{Im} f = 0\ ,$$

ya’ni $\operatorname{Re} f$ va $\operatorname{Im} f$ larning garmonik funksiyalar ekanligi kelib chiqadi.

Shuningdek, agar $u(z) \in H\left(D\right)$ bo‘lsa, u holda har bir $z^o \in D$ nuqta va uning atrofi $B(z^o,r) \subset D$ uchun shunday $f \in \mathcal{O}\left(D\right)$ funksiya topiladiki,

$\operatorname{Re} f(z) = u(z), \quad z \in B(z^o, r)$ bo‘ladi.

Haqiqatan ham, $z^0 = 0$ deb $B = B(0,r)$ doirada Puasson formulasini ushbu

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\xi)\frac{r^2 - |z|^2}{|\xi - z|^2} dt = \frac{1}{2\pi}\int_0^{2\pi} u(\xi) \operatorname{Re}\frac{\xi + z}{\xi - z} dt,$$

bunda $\xi = re^{it}$ ko‘rinishda ifodalaymiz. Bu formuladan $B(o,r)$ da golomorf

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\xi)\frac{\xi + z}{\xi - z} dt \tag{1.1.2}$$

funksiyasi uchun $\operatorname{Re} f(z) = u(z)$ ekanligi kelib chiqadi.

(1.1.2) formula golomorf funksiyani uning haqiqiy qismi orqali ifodalashda ham ishlatiladi: ixtiyoriy $f(z) \in \mathcal{O}(B) \bigcap C(\bar{B})$ uchun ushbu

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\xi)\frac{\xi + z}{\xi - z} dt + i \operatorname{Im} f(0)$$

formula o‘rinlidir.

1.2. Subgarmonik funksiyalar

Faraz qilaylik, $D \subset \mathbf{R}^n$ sohada aniqlangan lokal integrallanuvchi

$$u: \ D \to [-\infty, +\infty)$$

funksiya berilgan bo‘lsin: $u \in L^1_{loc}(D)$.

Agar bu funksiya quyidagi ikki shartni bajarsa:

1) $u(x)$ yuqoridan yarim uzluksiz:

ya’ni $\forall x^o \in D$ uchun $\overline{\lim_{x \to x^o}} u(x) \le u(x^o)$

(bundan $u(x)$ funksiyaning D ga tegishli ixtiyoriy kompaktda yuqoridan chegaralanganligi kelib chiqadi);

2) har bir $x^o \in D$ nuqta uchun shunday $r(x^o) > 0$ topiladiki, $r \leq r(x^o)$ uchun

$$u(x^o) \leq \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(x^o,r)} u(x)d\sigma \qquad (1.2.1)$$

u holda $u(x)$ funksiya D sohada *subgarmonik funksiya* deyiladi.

Subgarmonik funksiyalar sinfi $Sh\left(D\right)$ kabi belgilanadi. Kelgusida qulaylik uchun biz trivial $u(z) \equiv -\infty$ funksiyasi ham D da $Sh(D)$ ga tegishli deb qaraymiz. Endi subgarmonik funksiyalarning xossalarini keltiramiz.

a) subgarmonik funksiyalarning musbat koeffisentli chiziqli kombinatiyasi subgarmonik funksiya bo‘ladi:

$$\begin{gathered} u_k(x) \in Sh(D), \quad a_k \in R_+ \quad (k = 1,2,...,m) \;\Rightarrow \\ \Rightarrow a_1u_1(x) + a_2u_2(x) + ... + a_mu_m(x) \in Sh(D) \; ; \end{gathered}$$

b) chekli sondagi subgarmonik funksiyalarning maksimumi subgarmonik funksiya bo‘ladi:

$$\begin{gathered} u_1(x), u_2(x),..., u_m(x) \in Sh(D) \;\Rightarrow \\ \Rightarrow \;\max\left\{ u_1(x), u_2(x),..., u_m(x) \right\} \in Sh(D) \; ; \end{gathered}$$

v) monoton kamayuvchi subgarmonik funksiyalar ketma – ketligining limiti subgarmonik funksiya bo‘ladi:

$$u_j(x) \in Sh(D)\ , u_j(x) \geq u_{j+1}(x) \quad (j = 1,2,...) \Rightarrow$$
$$\Rightarrow \lim_{j \to \infty} u_j(x) \in Sh(D)\ ;$$

$g)$ tekis yaqinlashuvchi subgarmonik funksiyalar ketma–ketligi subgarmonik funksiyaga yaqinlashadi:

$$u_j(x) \in Sh(D)\ \left(k = 1,2,3,...\right), u_j\left(x\right) \to u\left(x\right) \Rightarrow u\left(x\right) \in Sh\left(D\right).$$

$a)$ – $g)$ xossalarning isbot bevosita yuqorida keltirilgan ta'rifdan kelib chiqadi.

$d)$ *Maksimumlar prinsipi.* Agar $u(x) \in Sh(D)$ funksiya biror $x^o \in D$ nuqtada o'zining maksimumiga erishsa, ya'ni

$$u(x^o) = \sup_{x \in D} u(x) \tag{1.2.2}$$

bo'lsa, u holda $u(x) \equiv const$ bo'ladi.

Ushbu

$$M = \left\{ x \in D :\ u(x) = u(x^o) \right\}$$

to'plamni qaraylik. $u(x)$ funksiyaning yuqoridan yarim uzluksiz bo'lganligidan va (1.2.2) shartdan M to 'plamning D da yopiqligi kelib chiqadi.

Ikkinchi tomondan, ixtiyoriy $p \in M$ uchun (1.2.1) formulaga ko'ra

$$u(x^o) = u(p) \leq \frac{1}{\sigma_n r^{n-1}} \int\limits_{S(p,r)} u(x) d\sigma \leq u(x^o)\ , \qquad r \leq r(p)\ ,$$

bo'ladi. Bu munosabatdan $u\Big|_{S(p,r)} \equiv u(x^o)$, ya'ni p ning $B(p, r(p))$ atrofida

$u(x)=u(x^o)$ bo‘lishini ko ‘ramiz. Bu esa M to‘plamning D da ochiq ekanligini ko‘rsatadi. Demak, $M=D$.

$e)$ Agar $\vartheta(x)\in Sh(D)$, $u(x)\in H(D)$ funksiyalar uchun $S(x^o,r)\subset\subset D$ sferada

$$\vartheta\Big|_S\le u\Big|_S$$

bo‘lsa, u holda $B(x^o,r)$ sharda $\vartheta(x)\le u(x)$ tengsizlik o‘rinli bo‘ladi.

Bu xossaning isboti yuqoridagi $d)$ xossadan kelib chiqadi;

$e)$ – xossadan quyidagi muhim xulosaga kelamiz. Faraz qilaylik,

$$\vartheta(x)\in Sh(D)\bigcap C(D)$$

berilgan bo‘lib, $B(x^o,r)\subset\subset D$ bo‘lsin. Ushbu

$$u(x)=\int\limits_{S(x^o,r)}\vartheta(y)P(x,y)d\sigma(y)\ , \qquad x\in B(x^o,r)\ ,$$

Puasson integralini qaraylik. $u(x)\in h(B)$, $u\Big|_S=\vartheta\Big|_S$ bo‘lib, $e)$ – xossaga ko‘ra $B=B(x^o,r)$ da $\vartheta(x)\le u(x)$ bo‘ladi. Uzluksiz funksiyalar uchun isbotlangan bu xossa ixtiyoriy $\vartheta\in Sh\big(D\big)$ uchun ham o‘rinlidir: $\vartheta(x)\le u(x)$ va deyarli barcha $x\in S$ lar uchun $\vartheta(x)=u(x)$ o‘rinlidir;

j) aytaylik, $u(x)\in Sh(D)$ funksiya va $x^o\in D$ nuqta berilgan bo‘lsin.

Ushbu

$$m(x^o,r)=\frac{1}{\sigma_n r^{n-1}}\int\limits_{S(x^0,r)}u(x)d\sigma$$

va

$$n(x^o,r)=\frac{1}{V_n r^n}\int\limits_{B(x^o,r)} u(x)dV\,,$$

bunda $V_n r^n$ miqdor $B(x^o,r)$ ning hajmi, integrallarni (o‘rta qiymatlarni) qaraylik.

1.2.1 – teorema. Faraz qilaylik, $u(x)\in Sh(D)$ va $u(x)\not\equiv-\infty$ bo‘lsin. U holda ixtiyoriy $x^o\in D$ uchun

$$u(x^0)\le n(x^o,r)\le m(x^o,r)\ ,\ n(x^o,r)>-\infty\ ,\ 0<r<\rho(x^o,\partial D)\ ,$$

tengsizliklar o‘rinli bo‘ladi. Bundan tashqari, agar r monoton kamayib nolga intilsa, unda $n(x^o,r)$ va $m(x^o,r)$ o‘rta qiymatlar monoton kamayib, $u(x^o)$ ga intiladi.

1.2.1 – teoremadan ushbu muhim natijaga ega bo‘lamiz: $D\subset\mathbf{R}^n$ sohada subgarmonik $u(x)\neq-\infty$ funksiya D da lokal integrallanuvchidir, ya’ni ixtiyoriy $B\subset\subset D$ uchun $\int\limits_B u(x)dV$ integral mavjuddir;

z) subgarmonik funksiya D sohaning har bir nuqtasida uzilishga ega bo‘lishi mumkin. Ayni paytda, quyidagi teorema o‘rinli bo‘ladi.

1.2.2 – teorema. Agar $u(z)\in Sh(D)$ bo‘lsa, u holda shunday monoton to‘plamlar

$$D_j \subset D_{j+1} \subset D \ , \quad \bigcup_{j=1}^{\infty} D_j = D$$

ketma – ketligi va shunday monoton kamayuvchi funksiyalar

$$u_j(x) \in Sh(D_j) \cap C^{\infty}(D_j) \ , \qquad j = 1,2,3,...$$

ketma – ketligi mavjudki,

$$u(x) = \lim_{j \to \infty} u_j(x) \qquad (x \in D)$$

bo ‘ladi.

Isboti. Ushbu

$$K\left(x\right) = \begin{cases} ce^{-\frac{1}{1-\left|x\right|^2}}, & agar \ \left|x\right| \leq 1 \ \ bo'lsa, \\ 0, & agar \ \left|x\right| > 1 \ \ bo'lsa. \end{cases}$$

funksiyani qaraylik. Bu funksiya $C^{\infty}(\mathbf{R}^n)$ sinfga tegishli bo ‘lib, uning havzasi $\sup\mathrm{pK(x)} = \mathrm{B}(0{,}1)$ dir. Endi $K(x)$ ning ifodasidagi o ‘zgarmas c ni shunday tanlab olamizki,

$$\int_{\mathbf{R}^n} K(x) dV = 1$$

bo ‘lsin. $u(x) \not\equiv -\infty$ deb, bu yadro yordamida ushbu

$$u_{\delta}(x) = \frac{1}{\delta^n} \int_{\left|y-x\right| \leq \delta} u(y) K\left(\frac{y-x}{\delta}\right) dV \ , \quad \delta > 0 \ , \tag{1.2.3}$$

integralni qaraylik. $u_{\delta}(x)$ funksiyasi

$$D_\delta = \left\{ x \in D: \ \rho(x, \partial D) < \delta \right\}$$

ochiq to'plamda aniqlangan bo'lib, $x \in D_\delta$ da

$$u_\delta(x) = \frac{1}{\delta^n} \int\limits_{\mathbf{R}^n} u(y) K\left(\frac{y-x}{\delta}\right) dV = \int\limits_{\mathbf{R}^n} u(x+\delta y) K(y) dV$$

bo'ladi. Keyingi tenglikdagi birinchi integral $C^\infty(D_\delta)$ sinfga, ikkinchi integral esa $Sh(D_\delta)$ sinfga tegishli bo'ladi. Binobarin,

$$u_\delta(x) \in Sh(D_\delta) \bigcap C^\infty(D_\delta).$$

$u(x)$ ning subgarmonik funksiya ekanligidan, u_δ ning $\delta \downarrow 0$ da, monoton kamayuvchi bo'lishligi va

$$\int\limits_{\mathbf{R}^n} K(x) dV = 1$$

tenglikdan, $u_\delta(x)$ ning $u(x)$ ga intilishini topamiz.

Teorema isbotlandi.

i) Subgarmonik funksiyalar umumiy holda C^2 sinfga tegishli bo'lmasligidan, Δu - Laplas operatorini faqat umumlashgan funksiya sifatida qarash mumkin bo'ladi:

$$\Delta u(\phi) = \int u \Delta \phi, \qquad \phi \in F(D).$$

1.2.3-teorema. Har qanday subgarmonik funksiya $u(x) \in Sh(D)$, $u \not\equiv -\infty$, uchun umumlashgan ma'noda $\Delta u \geq 0$ bo'ladi, ya'ni ixtiyoriy $\phi \in F(D)$, $\phi \geq 0$ uchun $\int u \Delta \phi \geq 0$ munosabat o'rinlidir.

Jumladan, $u(x)$ funksiya C^2 sinfga tegishli bo‘lib, $u(x)$ subgarmonik bo‘lsa, $u \in Sh(D) \bigcap C^2(D)$, u holda $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_n^2} \geq 0$ dir.

Yuqorida keltirilgan teoremaning aksi ham o‘rinli: agar umumlashgan funksiya u uchun

$$\Delta u(\phi) = \int u \Delta \phi \geq 0 \ , \quad \phi \in F(D) \ , \quad \phi \geq 0 \ ,$$

munosabat bajarilsa, u holda u subgarmonik funksiya bo‘ladi.

1.3. Plyurisubgarmonik funksiyalar.

1.3.1.Plyurigarmonik funksiyalar.

$\mathbf{C}^n$ kompleks fazodagi $D \subset \mathbf{C}^n$ sohada $u(z)$ funksiya berilgan bo‘lib, $u(z) \in C^2(D)$ bo‘lsin.

Agar ixtiyoriy $z^0 \in D$ va bu nuqtadan o‘tuvchi ixtiyoriy kompleks to‘g‘ri chiziq $l : z = z^0 + w\xi$, $w \in \mathbf{C}^n$, $\xi \in \mathbf{C}$ uchun ushbu $\varphi(\xi) = u \,/\, l = u(z^0 + w\xi)$ kesim–funksiya $\xi = 0$ nuqtada garmonik, ya’ni $\xi = 0$ da $\frac{\partial^2 \phi}{\partial \xi \partial \overline{\xi}} = 0$ bo‘lsa, $u(z)$ funksiya D da *plyurigarmonik funksiya* deyiladi.

Plyurigarmonik funksiyalar sinfi $Ph(D)$ kabi belgilanadi. Aytaylik, $u(z) \in Ph(D)$ bo‘lsin. Unda

$$\frac{\partial^2 u(z^0 + w\xi)}{\partial \xi \, \partial \overline{\xi}} = 0$$

bo‘lib, murakkab funksiyaning differensiallash qoidasidan $\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k} = 0$, $w \in \mathbf{C}^n$,ekanligini ko‘ramiz. Bundan va $w \in \mathbf{C}^n$ vektorning ixtiyoriyligidan

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = 0 \qquad (k, j = 1,2,...,n) \qquad (1.3.1)$$

bo‘lishi kelib chiqadi.

Demak, plyurigarmonik funksiyalar sinfi (1.3.1) tenglamalar sistemasi bilan aniqlanadi va ba’zan bu sistema plyurigarmonik funksiyaning ta‘rifi sifatida ham qaraladi.

1.3.1–teorema. Agar $f(z) = u(z) + iv(z)$ funksiya $D \subset \mathbf{C}^n$ sohada golomorf bo‘lsa, u holda

$$\operatorname{Re} f(z) = u(z), \quad \operatorname{Im} f(z) = v(z)$$

funksiyalar D sohada plyurigarmonik bo‘ladi va aksincha, agar $u(z)$ funksiya D sohada plyurigarmonik bo‘lsa, u holda ixtiyoriy $B(z^0, r) \subset D$ atrofda $u(z)$ funksiya biror $f(z) \in \mathcal{O}(B(z^0, r))$ golomorf funksiyaning haqiqiy qismi bo‘ladi: $u(z) \equiv \operatorname{Re} f(z)$.

1.3.2. Plyurisubgarmonik funksiyalar

Agar $D \subset \mathbf{C}^n$ sohada aniqlangan $u(z): D \to [-\infty, \infty)$ funksiya quyidagi ikki shartni bajarsa:

1) $u(z)$ yuqoridagi yarim uzluksiz;

2) ixtiyoriy l kompleks to‘g‘ri chiziq uchun $u/_l$ funksiya $l\cap D$ da subgarmonik bo‘lsa, $u(z)$ funksiya D sohada *plyurisubgarmonik* funksiya deyiladi.

Plyurisubgarmonik funksiyalar to‘plami $Psh(D)$ kabi belgilanadi. $D\subset \mathbf{C}^n$ sohada plyurisubgarmonik bo‘lgan funksiya $D\subset \mathbf{R}^{2n}$ da aniqlangan funksiya sifatida subgarmonik funksiya bo‘ladi. Uni subgarmonik funksiyalarning 2 – shartini bajarishini ko‘rish qiyin emas. Binobarin, subgarmonik funksiyalarning xossalari plyurisubgarmonik funksiyalar uchun ham saqlanadi.

Endi plyurisubgarmonik funksiyalarning xossalarini keltiramiz:

a) plyurisubgarmonik funksiyalarning musbat koeffitientli chiziqli kombinatiyasi plyurisubgarmonik funksiya bo‘ladi;

b) agar $u(z)\in Psh(D)$ funksiya $z^0\in D$ nuqtada maksimumga erishsa, ya’ni $u(z^0)\geq u(z)\quad (z\in D)$ bo‘lsa, u holda $u(z)\equiv const$ bo‘ladi;

v) monoton kamayuvchi yoki tekis yaqinlashuvchi plyurisubgarmonik funksiyalar ketma – ketligining limiti plyurisubgarmonik funksiya bo‘ladi;

g) agar $u(z)\in Psh(D)$ bo‘lsa, u holda $u_\delta(z)$ funksiya $Psh(D_\delta)\cap C^\infty(D_\delta)$ sinfga tegishli bo‘lib, $\delta\downarrow 0$ da $u_\delta(z)\downarrow u(z)$ bo‘ladi;

d) D sohada plyurisubgarmonik funksiyalar sinfi $\{u_\alpha\}_{\alpha\in\wedge}$ berilgan bo‘lib, $u=\sup\limits_{\alpha\in\wedge} u_\alpha$ yuqoridan yarim uzluksiz funksiya bo‘lsin. U holda $u(z)$

plyurisubgarmonik funksiya bo'ladi, $u \in Psh(D)$. Jumladan, $u_j(z) \in Psh(D) \quad (j = 1,2,...,k)$ bo'lsa,

$$\sup\{u_1(z),\ u_2(z),...,u_k(z)\} \in Psh(D)$$

bo 'ladi;

e) agar $f(z) \in O(D)$ bo'lsa, $\alpha \cdot \ln\left|f(z)\right| \in Psh(D)$ bo'ladi, bunda $\alpha \geq 0$;

j) agar $u(z) \in Psh(D)$ bo'lsa, u holda $e^{u(z)} \in Psh(D)$ dir, agar $u(z) \in Psh(D)$, $u\big|_D < 0$ bo'lsa, u holda $-\ln[-u(z)] \in Psh(D)$ dir.

z) D sohada aniqlangan $u(z) \in C^2(D)$ funksiyaning D da plyurisubgarmonik bo'lishi uchun ushbu

$$L(u,w) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \qquad (1.3.2)$$

kvadratik formaning (Levi formasining) ixtiyoriy $z \in D$ da musbat aniqlangan bo'lishi zarur va yetarlidir. (1.3.2) munosabatda $w = (w_1, w_2, ..., w_n) \in \mathbf{C}^n$ bo'lib, kvadratik formaning musbat aniqlanganligi $\forall w \in \mathbf{C}^n$ da $L(u,w) \geq 0$ bo'lishini anglatadi.

Agar $\forall w \in \mathbf{C}^n$, $w \neq 0$ da $L(u,w) > 0$ bo'lsa, $u(z)$ funksiya z nuqtada ***qat'iy plyurisubgarmonik*** funksiya deyiladi; agar $u(z)$ funksiya D sohaning har bir nuqtasida qat'iy plyurisubgarmonik bo'lsa, $u(z)$ ***sohada qat'iy plyurisubgarmonik*** funksiya deyiladi.

II BOB. α –garmonik funksiyalar

2.1. Musbat aniqlangan differensial formalar va oqimlar

2.1.1. Differensial formalar. $f \in C^1(G)$ funksiya $G \subset C^n$ sohada berilgan bo`lsin.

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$$

differensial 1-darajali differensial formaga misol bo`ladi.

Umuman aytganda

$$w = \sum_{j=1}^{n} a_j(x) dx_j$$

ko`rinishdagi ifoda 1-darajali differensial forma deyiladi, $\deg w = 1$.

Yuqori darajali differensial formalarni ta'rifini berish uchun $\wedge$-tashqi ko`paytma tushunchasini kiritamiz.Bu ko`paytma quyidagi xossalarni qanoatlantiradi:

a) $a(x) \wedge dx_j = dx_j \wedge a(x) = a(x) dx_j$

b) $dx_k \wedge dx_j = -dx_j \wedge dx_k$

bu yerdan $dx_j \wedge dx_j = 0$ bo`lishi kelib chiqadi.

2.1.1-ta'rif. Ushbu

$$w = \sum_{0 \le j_1 \le j_2 \le ... \le j_p} a_{j_1 j_2 ... j_p} dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_p}$$

Ifodaga p darajali differensial forma deyiladi, bu yerda $a_{j_1 j_2 ... j_p}(x)$ G da aniqlangan funksiya va $\deg w = p$.

p-darajali differensial formalar to`plami $F^p(G)$ orqali belgilanadi. $w = F^p(G)$ differensial formani quyidagi ekvivalentlik formada yozish qulay

$$w = \sum_{1 \le j_1 < j_2 < ... < j_p \le n} a_{j_1 j_2 ... j_p} dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_p} = \sum_J a_J dx_J$$

bu yerda $dx_J = dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_p}$, $J = (j_1, j_2, ..., j_p)$, $1 \le j_1, j_2, ..., j_p \le n$-multindeks 0-darajali differensial forma-bu $a(x)$ skalyar funksiya, n-darajali differensial forma esa $w = a(x)dx_1 \wedge ... \wedge dx_n$ ko`rinishga ega. $p > n$ da $p$-darajali forma nolga teng bo`ladi.

Differensial formalarning tashqi ko`paytmasi yig'indini ko`paytirish qoidasiga bo`ysunadi.

$$\left(\sum_J a_J dx_J\right) \wedge \left(\sum_J b_J dx_I\right) = \sum_{I,J} a_J b_J dx_J \wedge dx_I$$

Shunday qilib 2 ta formaning $w_1 \wedge w_2$ ko`paytmasida, bu yerda $\deg w_1 = n$, $\deg w_2 = p$ bo`lsa, biz 0 ga ega bo`lamiz. Agar $w$ differensial formaning barcha koeffitsientlari $C^k$ sinfdan olingan bo`lsa, u holda w forma C^k sinfga tegishli deyiladi, $w \in C^k$ bilan belgilanadi. Silliq formalar differensiallashga ruhsat beradi.

Agar

$$w = \sum_J a_J dx_J$$

bo`lsa,u holda

$$dw = \sum da_J(x) \wedge dx_J dw = \sum_J \left(\frac{\partial a_J}{\partial x_1} dx_1 + ... + \frac{\partial a_J}{\partial x_k} dx_k\right) \wedge dx_j = \sum_J \sum_{k=1} dx_k \wedge dx_J$$

faqat bu yerda o`xshash differensiallarni tartibga keltirish kerak.
Buning uchun $dx_J \wedge dx_J$ ko`rinishdagi tashqi ko`paytma qatnashgan qo`shiluvchilarni nolga tenglash kerak. Ta'rifdan ko`rinadiki, p-darajali w differensial formaning differensiali $p+1$-darajali differensial formani ifodalaydi:

$$D : F^p(G) \to F^{p+1}(G).$$

Differensiallash operatori quyidagi xossalarni qanoatlantiradi:

a) $d(w_1 + w_2) = dw_1 + dw_2$

b) $d(w_1 \wedge w_2) = dw_1 \wedge dw_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

v) $d(dw) = d^2 w = 0$

$\mathbb{C}^n \approx \mathbb{R}^{2n}$ kompleks fazoda differensial formalarni dz_j va $d\overline{z_j}$ lardan foydalanib yozish mumkin.

$z = (z_1, ..., z_n)$ lar $\mathbb{C}^n \approx \mathbb{R}^{2n}$ dagi kompleks koordinatalar bo`lsin,

$z_j = x_j + ix_{n+j}, \quad j = 1,2,....$

U holda,

$$dx_j = \frac{dz_j + d\bar{z}_j}{2},$$

$$dx_{n+j} = \frac{dz_j - d\bar{z}_j}{2i}$$

va

$w = \sum_J a_J(x) dx_J$ differensial formani dz_j va $d\bar{z}_k$ larning chiziqli kombinatsiyasi ko`rinishida ifodalash mumkin.

Bunda har bir hadda dz_J va $d\bar{z}_k$ lar soni turlicha bo`lishi mumkin.

Masalan, $\mathbb{C}^2 \approx \mathbb{R}^4$ da

$$dx_1 \wedge dx_2 \wedge dx_3 = \frac{i}{4}(-dz_1 \wedge dz_2 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_1 \wedge dz_1)$$

bo`lishi mumkin.

Ba'zan w da barcha qo`shiluvchilarda $dz_1$ larning soni bir xil $p$ ga va $d\bar{z}_k$ larning soni $q$ ga teng bo`ladi. Bunday hollarda biz w differensial forma (p,q) bidarajaga ega deb ataymiz va $w \in F^{p,q}$ deb yozamiz.

Agar

$$\partial a = \frac{\partial a}{\partial z_1} dz_1 + ... + \frac{\partial a}{\partial z_n} dz_n$$

va

$$\bar{\partial} a = \frac{\partial a}{\partial \bar{z}_1} d\bar{z}_1 + ... + \frac{\partial a}{\partial \bar{z}_n} d\bar{z}_n$$

deb belgilasak, u holda differensiallash operatori $d = \partial + \bar{\partial}$ ko`rinishga ega bo`ladi. Bu yerda shuni ta'kidlaymizki

$$d^2w = ddw = (\partial + \bar{\partial})(\partial + \bar{\partial})w = \partial^2 w + \overline{\partial^2 w}$$

Bu yerdan $d^2w = 0$ ligidan $\partial^2 w + \overline{\partial^2 w} = 0$ tenglik istalgan w differensial forma uchun bajarilishi kelib chiqadi. Shu bilan birga $w \in F^{p,q}$ bo`lsa, u holda $d^2w \in F^{p+2,q}$ va $\overline{\partial^2} w \in F^{p,q+2}$ va $\partial^2 w + \overline{\partial^2 w} = 0$ tenglikdan $\partial^2 w = 0, \overline{\partial^2 w} = 0$ ekani kelib chiqadi.

Quyidagi qo`shma differensiallash operatorini kiritamiz:

$$d^c = \frac{\partial - \bar{\partial}}{4i} = \frac{1}{4}\sum_{j=1}^{n}(-\frac{\partial}{\partial y_j}dx_j + \frac{\partial}{\partial x_j}dy_j)$$

U holda

$$dd^c \left|z\right|^2 = \frac{i}{2}(dz \wedge d\bar{z}1 \wedge ... \wedge dz_n \wedge d\overline{z_n})$$

va

$$(dd^c \left|z\right|^2)^n = \underbrace{dd^c \left|z\right|^2 \wedge ... \wedge dd^c \left|z\right|^2}_{nmarta} = (\frac{i}{2})^n (dz_1 \wedge d\bar{z}1 + ... + dz_n \wedge d\overline{z_n}) =$$

$$= dx_1 \wedge dx_2 \wedge ... \wedge dx_n = dV$$

$\mathbb{C}^n \approx \mathbb{R}^{2n}$ dagi hajm elementi hosil bo`ladi.

Yuqorida differensial forma tushunchasi $\mathbb{C}^n$ Evklid fazosida kiritildi. $X = \varphi(y)$ o`zgaruvchi almashtirishda bu yerda $\varphi(y) = (\varphi_1(y),...,\varphi_2(y))$ o`rniga qo`yish va differensiallash qonunlariga muvofiq hisoblanadi.

Agar

$$W(x) = \sum_J a_J dx_J = \sum_{1 \le j_1 < j_2 < ... < j_p \le n} a_J dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_p}$$

bo`lsa,u holda yangi koordinatalarda

$$\bar{w}(x) = \sum_{1 \le j_1 < j_2 < ... < j_p \le n} a_J(\varphi(y))(\sum_{i=1}^{n}\frac{\partial \varphi_{j_1}}{\partial y_i}dy_i) \wedge ... \wedge (\sum_{i=1}^{n}\frac{\partial \varphi_{j_p}}{\partial y_i}dy_i) = dx_{j_2} \wedge ... \wedge dx_{j_p}$$

2.1.2-ta'rif. Ushbu

$$\omega = \mathop{\wedge}_{j=1}^{p} \frac{i}{2} dl_j \wedge d\overline{l_j}$$

ko ‘rinishdagi differensial forma $\left(p,p\right)$ bidarajali bosh kuchli musbat differensial forma deyiladi, bu yerda $l_j = a_{j_1} dz_1 + ... + a_{j_n} dz_n - \left(dz_1,...,dz_n\right)$ bazis bo ‘yicha chiziqli kombinatsiya, $a_{jk} \in \mathbb{C},\ j = 1,2,...,p, k = 1,2,...,n$ Bunday formalarning ixtiyoriy chiziqli kombinatsiyasi kuchli musbat differensial forma deb ataladi, ya’ni kuchli musbat differensial forma bu-

$$\omega = \sum_s \lambda_s\left(z\right) \mathop{\wedge}_{j=1}^{p} \frac{i}{2} dl_{j_s} \wedge d\bar{l}_{j_k}$$

ko‘rinishdagi formadir, bu yerda $\lambda_s\left(z\right)$ - qaralayotgan $G \subset \mathbb{C}^n$ sohadagi manfiy bo‘lmagan funksiya.

Agar $z_j = x_j + ix_{n+j}$ kompleks koordinatalardan $x_j, x_{n+j},\ j = 1,2,...,n$ haqiqiy koordinatalarga o‘tsak, unda ushbu

$$\left(\frac{i}{2}\right)^p dz_1 \wedge d\overline{z_1} \wedge ... \wedge dz_p \wedge d\overline{z_p} = dx_1 \wedge dx_{n+1} \wedge ... \wedge dx_p \wedge dx_{n+p}$$

ifoda $\mathbb{R}^{2n}$ fazoda $2p$ darajali musbat forma bo ‘ladi.

$\left(n,n\right)$ bidarajali differensial forma (kuchli) musbat bo ‘ladi, faqat va faqat shu holdaki, qachonki u

$$\omega = \lambda\left(z\right) \mathop{\wedge}_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z_j} = \lambda(z) dx_1 \wedge dx_{n+1} \wedge ... \wedge dx_n \wedge dx_{2n}$$

ko‘rinishga ega bo‘lsa,bu yerda $\lambda\left(z\right) -$ musbat funksiya.

$\left(n,n\right)$ bidarajali $\Omega = \mathop{\wedge}_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z_j} = dV$ differensial forma $\mathbb{C}^n$ da hajm formasi rolini o‘ynaydi.

$p=1$ da $\omega=\sum\lambda_{jk}dz_j\wedge d\overline{z_k}$ – differensial forma musbat bo‘ladi, faqat va faqat shu holdaki, qachonki $\sum\limits_{jk}\lambda_{jk}\xi_j\xi_k$ – ermit kvadratik formasi musbat bo‘lsa. Shuningdek, ixtiyoriy $z^o\in G$ tayinlangan nuqtada unitar almashtirish yordamida uni

$$\omega=\frac{i}{2}\sum_{j=1}^{n}\mu_j dz_j\wedge d\overline{z_j},\;\;\mu_j\geq 0$$

ko ‘rinishda yozish mumkin.

2.1.3-ta’rif. $\omega\in F^{p,p}$ differensial forma kuchsiz musbat yoki musbat deyiladi, agar ixtiyoriy kuchli musbat $\nu\in F^{n-p,n-p}$ forma uchun $\omega\wedge\nu\in F^{n,n}$ forma musbat bo‘lsa.

$p=0,1,n-1,n$ da kuchli musbatlik kuchsiz musbatlikka ekvivalent. Biroq boshqa p lar uchun bu sinflar ustma-ust tushmaydi. Buni aniq tushunish uchun kuchli musbat formani quyidagi ko‘rinishda yozamiz:

$$\begin{aligned}\omega&=\sum_s\lambda_s(z)\mathop{\wedge}_{j=1}^{p}\frac{i}{2}dl_{j_s}\wedge d\bar{l}_{j_s}=\frac{i^{p^2}}{2^p}\sum_s\lambda_s(z)\mathop{\wedge}_{j=1}^{p}\frac{i}{2}dl_{j_s}\mathop{\wedge}_{j=1}^{p}d\bar{l}_{j_s}=\\&=\frac{i^{p^2}}{2^p}\sum_s\lambda_s(z)dl_{j_s}\wedge d\bar{l}_{j_s}\end{aligned}\qquad(1.3.3)$$

Agar $\theta=\sum\limits_s\mu_s(z)dl_s$ formani qarasak, bunda μ_s – kompleks qiymatli funksiya, unda $\frac{i^{p^2}}{2^p}\theta\wedge\bar{\theta}$ ko ‘rinishdagi forma (kuchsiz) musbat bo‘ladi, biroq $2 \leq p \leq n-2$ da u har doim ham $\left(1.3.3\right)$ ko‘rinishga ega bo‘lavermaydi.

2.1.2. Oqimlar. M ko‘pxillikdagi oqimni koeffitsiyentlari umumlashgan funksiyalardan iborat differensial forma deb qarash mumkin.

Tayinlangan $G \subset \mathbb{R}^n$ soha uchun asosiy diffferensial formalar fazosi quyidagicha aniqlanadi:

$$F^p = F^p\left(G\right) = \left\{\omega \in F^p\left(G\right) \bigcap C^\infty\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$$

F^p da topologiya asosiy funksiyalar fazosidagi topologiya kabi aniqlanadi:

2.1.4-ta’rif. F^p fazodagi uzluksiz chiziqli (kompleks qiymatli) T funksional $n-p=q$ darajali oqim deyiladi.Barcha $q-$ darajali oqimlar fazosini $\Im^q$ orqali belgilaymiz. $\Im^q$ oqimlar fazosi umumlashgan funksiyalarning barcha xossalarini qanoatlantiradi.Xususan,ular uchun kuchsiz yaqinlashish, kuchsiz chegaralanganlik tushunchalarini kiritish mumkin va $T_\delta \in \Im^q \cap C^\infty$

o‘ramani aniqlash mumkinki, $\delta \to 0$ da oqim sifatida T da yaqinlashadi, ya’ni $T_\delta \circ \omega \to T \circ \omega, \quad \forall \omega \in \Im^p$ .Undan tashqari T oqimni differensial forma kabi differensiallashimiz mumkin: $dT \circ \omega = \left(-1\right)^p T \circ d\omega$

2.1.5-ta’rif. Agar $F^p = \left\{\omega \in \Im^p\left(G\right) \bigcap C^\infty\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$ fazodan olingan T oqim (funksional) $\left\{\omega \in \Im^p\left(G\right) \bigcap C^m\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$ fazoga uzluksiz davom qilsa, unda T oqim m dan ko'p bo'lmagan singulyarlikka ega deyiladi bunda $0 \le m \le \infty$. Agar undan tashqari T oqim

$$\left\{w \in F^p\left(G\right) \cap C^{m-1}\left(G\right) : \operatorname{supp} w \subset\subset G\right\}$$

fazoga davom qilmasa, unda T oqim aniq m singulyarlikka ega deyiladi. $T = \sum_{|k|=q} \beta_k dx_k$ koeffitsiyentlari umumlashgan funksiya bo'lgan differensial forma oqimni ifodalaydi.

2.1.1-teorema. Ixtiyoriy $q-$darajali oqim koeffitsiyentlari umumlashgan funksiya bo 'lgan differensial forma bo 'ladi, shu bilan birga singulyarligi nolga teng bo'lgan oqim o'lchov tipidagi oqim bo'ladi va uni barcha koeffitsiyentlari kompleks qiymatli o'lchov bo 'ladi.

O'lchov tipidagi oqimlar fazosini $\Im_0^q$ orqali belgilaymiz. $\Im_0^q$ nafaqat $G \subset \mathbb{R}^n$ soha uchun, balki ixtiyoriy $E \subset \mathbb{R}^n$ borel to'plami uchun ham aniqlangan. Xususan, $K \subset \mathbb{R}^n$ kompakt uchun differensial forma koeffitsiyentlaridan tashkil topgan vektorlarni $F_0^q\left(K\right) = F_0^q\left(K\right) \cap C\left(K\right)$ fazoga qo'shma fazo bo'lgani uchun u Banax fazosi bo 'ladi. $T \in \Im_0^q$ oqim uchun o'lchov tipidagi oqim $T \circ \omega = \mu\left(z\right)\Omega$ bo'ladi,bu yerda $\mu-$ kompleks qiymatli o 'lchov bo'lib, u $\omega \in \Im_0^p$ ga bog'liq va $\Omega = dx_1 \wedge ... \wedge dx_n$,shu bilan birga funksional qiymati $T \circ \omega = \int_G d\mu$ bo'ladi.

Xuddi o 'lchovlar fazosi kabi quyidagi xossalar o'rinli:

a) $\left\{T_j\right\} \subset \Im_0{}^q$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, ya'ni

$\lim\limits_{j\to\infty} T_j \circ \omega = T \circ \omega,\ \forall \omega \in \Im_0{}^p$ faqat va faqat shu holdaki,qachon unga mos

T_j dan olingan koeffitsiyentlardan tuzilgan o'lchovlar ketma-ketligi

yaqinlashuvchi bo'lsa.

b) $\left\{T_j\right\} \subset \Im_0{}^q$ oqimlar ketma-ketligi T ga kuchsiz yaqinlashadi, faqat va

faqat shu holdaki,qachon T_j norma bo 'yicha chegaralangan bo 'lsa

$\left\|T_j\right\|_k \leq c,\ \forall K \subset\subset G,\ j = 1,2,...$ va $j \to \infty$ da $T_j \circ \omega \to T \circ \omega$ limitik

munosabat hamma yerda zich biror $E \subset \Im_0{}^p\left(K\right)$ to 'plamdan olingan

barcha ω lar uchun o 'rinli bo 'lsa, masalan barcha cheksiz

differensiallanuvchi $\omega \in F^p \bigcap C^\infty\left(K\right)$ lar uchun

d)ixtiyoriy kuchsiz chegaralangan $E \subset \Im_0{}^q\left(K\right)$ qiymatlar to'plami ham

kuchsiz kompakt ya'ni uni ixtiyoriy ketma-ketligi kuchsiz yaqinlashuvchi

qismiy ketma-ketligi $\left\{T_j\right\} \subset E$ ni o'zida saqlaydi.

2.1.6-ta'rif. $q-$ darajali differensial forma (oqim) yopiq deyiladi, agar uni differensiali nolga teng bo'lsa;u aniq deyiladi, agar u biror $q-1$ darajali formani(oqimni) differensialini ifodalash.Aniq formalar(oqimlar) yopiq bo'ladi, teskari tasdiq uchun G sohaga qo 'shimcha geometrik shartlarni qo'yish zarur.

Endi $\mathbb{C}^n$ da asosiy differensial formalar fazosini qaraymiz:

$$\Im^{p,p} = \Im^{p,p}\left(G\right)\left\{\omega \in \Im^{p,p}\left(G\right) \bigcap C^{\infty}\left(G\right) : \operatorname{supp}\omega \subset\subset G\right\}$$

2.1.7-ta'rif. $F^{p,p}$ fazoda uzluksiz chiziqli(kompleks qiymatli) T funksionalga $\left(n-p, n-p\right) = \left(q,q\right)$ bidarajali oqim deyiladi.

2.1.2-teorema.Musbat oqimlar o'lchov tipidagi oqimlar bo'ladi.

2.1.1-misol. Koeffitsiyentlari L^{1}_{loc} sinfdan olingan ixtiyoriy $T \in F^{q,q}$ differensial forma $\left(q,q\right)$ bidarajali oqim bo'ladi.

$T \circ \omega$ funksional $T \circ \omega = \int\limits_{G} T \wedge \omega, \quad \forall \omega \in F^{p,p}$ kabi aniqlanadi.Bunday oqimlar regulyar oqim deyiladi.

2.1.2-misol. $A \subset \mathbb{C}^{n} - p$ o 'lchamli analitik to'plam bo'lsin.U holda $\left[A\right] \circ \omega = \int\limits_{A} \omega$ integral $\left(q,q\right)$ bidarajali oqimni aniqlaydi. Ko'rish qiyin emaski,$\left[A\right]$ musbat oqim bo 'ladi.

2.2. $dd^c u \wedge \alpha$ operator va $\alpha-$garmonik funksiyalar

$\mathbb{C}^n$ kompleks fazoda $(n-1, n-1)$-bidarajali qat'iy musbat α differensial formani tayinlaymiz.

2.2.1-teorema. $dd^c u \wedge \alpha$ elliptik operator bo'ladi.

Isbot. α differensial forma quyidagi ko'rinshda tasvirlanadi deb faraz qilamiz:

$$\alpha = \frac{(i)^{(n-1)^2}}{2^{n-1}} \sum_{j,k} \alpha_{jk}(z) dz \overset{j,\bar{k}}{\vee} \wedge d\bar{z}, \tag{2.2.1}$$

Bu yerda $dz \overset{j,\bar{k}}{\vee} \wedge d\bar{z}$ belgilash $dz_1 \wedge dz_2 \wedge ... \wedge dz_n \wedge d\bar{z}1 \wedge d\bar{z}2 \wedge ... \wedge d\bar{z}n$ ko'paytmada dz_j va $\overline{dz_k}$ differensiallar tushib qolishini bildiradi. .

Ma'lmki, istalgan musbat forma ω haqiqiy bo'ladi, ya'ni $\bar{\omega} = \omega$ tenglik o'rinli bo'ladi. Lokal koordinatalarda agar $\omega = i^{p^2} \sum_{|I|=|J|=p} a_{I,J} dz_I \wedge d\bar{z}J$ bo'lsa, u holda koeffisientlar ermit skalyar ko'paytmani qanoatlantiradi: $\overline{a_{I,J}} = a_{J,I}$ (q. [18]). Binobarin, (2.2.1) formada quyidagi tenglik o'rinli: $\overline{\alpha_{j\,k}(z)} = \alpha_{k\,j}(z)$.

Endi ixtiyoriy musbat $(1,1)-$bidarajali formani olamiz: $\omega = \frac{i}{2} \sum_{j,k} a_{jk} dz_j \wedge d\bar{z}k$. Formaning musbatligiga ko'ra va

$idz_1 \wedge d\bar{z}_1 \wedge ... \wedge idz_n \wedge d\bar{z}_n = i^{n^2} dz \wedge d\bar{z}$ o‘rin almashtirish qoidasidan foydalanib biz quyidagiga ega bo‘lamiz:

$$\alpha \wedge \omega = \frac{i^{(n-1)^2}}{2^{n-1}} \sum_{j,k} \alpha_{jk}(z) dz \overset{j,\bar{k}}{\vee}{\wedge} d\bar{z} \wedge \frac{i}{2} \sum_{j,k} a_{jk} dz_j \wedge d\bar{z}_k =,$$

$$= \left(\frac{i}{2}\right)^n (-1)^{\frac{(n-1)(n-2)}{2}} \sum_{j,k} (-1)^{3n-j-k-1} a_{jk} \alpha_{jk}(z) dz \wedge d\bar{z} =$$

$$= \sum_{j,k} (-1)^{2n-j-k} a_{jk} \alpha_{jk}(z) dV > 0, \qquad (2.2.2)$$

bu yerda $dV = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n -$ $\mathbb{C}^n$ dagi hajm elementi.

$\mathbb{R}^{2n}$ fazodagi m tartibli chiziqli differensial operator $P(x, \boldsymbol{\mathcal{D}})u = \sum_{|\alpha| \le m} a^{\alpha}(x) \boldsymbol{\mathcal{D}}^{\alpha} u$ bilan beriladi, bu yerda $x = (x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$,

$\alpha = (\alpha_1, ..., \alpha_{2n}) -$ multiindeks, $|\alpha| = \alpha_1 + ... + \alpha_{2n}$, $\boldsymbol{\mathcal{D}}^{\alpha} = \boldsymbol{\mathcal{D}}_1^{\alpha_1} \boldsymbol{\mathcal{D}}_2^{\alpha_2} ... \boldsymbol{\mathcal{D}}_{2n}^{\alpha_{2n}}$,

$\boldsymbol{\mathcal{D}}_j = \frac{\partial}{\partial x_j}$, $j = \overline{1, 2n}$.

Istalgan $u(z) \in C^2(D)$ uchun $dd^c u \wedge \alpha$ operator quyidagi ko‘rinishda tasvirlanadi:

$$dd^c u \wedge \alpha = P(z, \boldsymbol{\mathcal{D}}) u(z),$$

bu yerda $dd^c u \wedge \alpha = \frac{i}{2} \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \frac{i^{(n-1)^2}}{2^{n-1}} \sum_{j,k} \alpha_{jk}(z) dz \overset{j,\bar{k}}{\vee}{\wedge} d\bar{z} =$

$$=\left(\frac{i}{2}\right)^n(-1)^{\frac{(n-1)(n-2)}{2}}\sum_{j,k}(-1)^{n+k+j-3}\alpha_{jk}(z)\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}dz\wedge d\bar{z}=$$

$$=\left(\frac{i}{2}\right)^n(-1)^{\frac{n(n-1)}{2}}\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}dz\wedge d\bar{z}=\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\frac{\partial^2 u}{\partial z_j\partial \bar{z}_k}dV$$

$$=\frac{1}{4}\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right)u(z)dV=$$

$$=\frac{1}{4}\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\left(\mathcal{D}_j-i\mathcal{D}_{n+j}\right)\left(\mathcal{D}_k+i\mathcal{D}_{n+k}\right)u(z)dV\,.$$

Bu yerda $z_k=x_k+ix_{n+k},\ k=1,...,n$ deb o‘zgaruvchilarni almashtirdik.

Endi $dd^c u\wedge\alpha$ operatorning xarakteristik formasini tuzamiz. U quyidagi ko‘rinishga ega bo‘lib, $\xi\in R^{2n}$ ning ikkinchi darajasi bo‘yicha bir jinsli ko‘phadni ifodalaydi:

$$P(z,\xi)=\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\left(\xi_j-i\xi_{n+j}\right)\left(\xi_k+i\xi_{n+k}\right).$$

Bu yerda $\overline{\alpha_{jk}(z)}=\alpha_{kj}(z)$ shartga ko‘ra $\overline{P(z,\xi)}=P(z,\xi)$ tenglikni olamiz. .

Binobarin, $P(z,\xi)$ ko‘phad haqiqiy va u quyidagi ko‘rinishda bo‘ladi:

$$P(z,\xi)=\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\xi_j\xi_k+\sum_{j,k}(-1)^{k+j-2}\alpha_{jk}(z)\xi_{n+j}\xi_{n+k}$$

Agar biz $\omega_1=\sum_{j,k}\xi_j\xi_k d\,z_j\wedge d\,\bar{z}_k$ va $\omega_2=\sum_{j,k}\xi_{n+j}\xi_{n+k}d\,z_j\wedge d\,\bar{z}_k$ formalarni qarasak, bu formalar bosh musbat formalarni aniqlashini ko‘ramiz. Bu yerdan (2.2.2) munosabatga asosan $P(z,\xi)>0$ tengsizlikka ega bo‘lamiz.

Bundan $P(z,\xi)=0$ tenglik faqat va faqat $0=\xi\in\mathbb{R}^{2n}$ da bajarilishi kelib chiqadi. Bu $dd^c u\wedge\alpha$ operatorning elliptikligini ko‘rsatadi (q. [14]).

2.2.1-ta’rif. *$D\subset\mathbb{C}^n$ sohada $dd^c u\wedge\alpha=0$ tenglamani qanoatlantiruvchi ikki marta silliq $u(z)\in C^2(D)$ funksiyaga $\alpha-$garmonik funksiya deyiladi.*

Endi quyidagi xususiy hol uchun $\alpha-$garmonik funksiyalar sinfini o‘rganamiz. $(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbb{R}^n_+$ vektor uchun,

$$\varphi(z)=\alpha_1 z_1\overline{z}_1+\alpha_2 z_2\overline{z}_2+...+\alpha_n z_n\overline{z}_n$$

funksiyani olamiz va ushbu formani qaraymiz:

$$dd^c\varphi=\frac{i}{2}\partial\overline{\partial}\varphi=\frac{i}{2}\sum_{\substack{j=1,\\k=1}}^{n}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}dz_j\wedge d\overline{z}_k=$$

$$=\frac{i}{2}(\alpha_1 dz_1\wedge d\overline{z}_1+\alpha_2 dz_2\wedge d\overline{z}_2+...+\alpha_n dz_n\wedge d\overline{z}_n).$$

$\alpha=\left(dd^c\varphi\right)^{n-1}$ deb olsak, u (n-1,n-1) bidarajali qat’iy musbat differensial formani beradi. $D(D\subset\mathbb{C}^n)$ sohada berilgan $u\in C^2$ funksiya uchun quyidagi operatorni qaraymiz:

$$dd^c u\wedge\alpha=dd^c u\wedge(dd^c\varphi)^{n-1} \quad .$$

$$\left(dd^c\varphi\right)^{n-1}=(n-1)!\sum_{j=1}^{n}\alpha_1\alpha_2\cdot\overset{\overset{j}{\vee}}{...}\cdot\alpha_n dz_1\wedge d\overline{z}_1\wedge\overset{\overset{j}{\vee}}{...}\wedge dz_n\wedge d\overline{z}_n$$

Bu yerda $\overset{j}{\overset{\vee}{\ldots}}$ belgi shu koʻpaytmada j-had qatnashmasligini bildiradi.

$(dd^c u) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} dz_j \wedge \overline{d} z_k$ tengliklardan

$$dd^c u \wedge \alpha = (n-1)! \sum_j \alpha_1 \alpha_2 \overset{j}{\overset{\vee}{\ldots}} \alpha_n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV$$

ifodaga ega boʻlamiz.

Oxirgi tenglikdan koʻrinadiki u funksiya uchun $dd^c u \wedge \alpha = 0$ tenglikni bajarilishi

$$\sum_j \alpha_1 \alpha_2 \overset{j}{\overset{\vee}{\ldots}} \alpha_n \cdot \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} = 0 \qquad (2.2.3)$$

tenglikni bajarilishi bilan ekvivalent ekan.

2.2.2-ta'rif. (2.2.3) tenglikni qanoatlantiruvchi $u \in C^2(D)$ funksiyaga $\alpha-$garmonik funksiya deyiladi.

$\alpha-$garmonik funksiyalar sinfi $\alpha - h(D)$ kabi belgilanadi. $\alpha = dd^c \left|z\right|^2$ boʻlgan holda bu oʻzimizga ma'lum garmonik funksiyalar sinfiga ega boʻlamiz. $\alpha-$garmonik funksiyalarning koʻpchilik xossalari garmonik funksiyalarning xossalari kabi boʻlib, biz ularni keltirib oʻtirmaymiz.

Ushbu teorema $\alpha-$garmonik funksiyalarning geometrik hususiyatini ifodalaydi:

2.2.2-teorema. *Agar $u(z)$ funksiya D sohada $\alpha-$garmonik boʻlsa, u* *holda* $\forall z^0 \in D$ *nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left| z_1 - z_1^0 \right|^2 + ... + \alpha_n \left| z_n - z_n^0 \right|^2 \le r^2 \right\} \subset\subset D \quad \textit{gipershar}$$

uchun

$$u(z^0) = \frac{1}{\tilde{V}(r)} \int\limits_{E(z^0, r)} u(\tau) dV(\tau)$$

tenglik o'rinli bo'ladi. Bu yerda $\tilde{V}(r)$ *- gipershar hajmi va* $dV(\tau) -$ *hajm elementi.*

Isbot: Umumiylikni buzmagan holda $z^0 = (0, 0, ..., 0) \in D$ bo'lsin deb olishimiz mumkin. Teoremaning isboti quyidagi sodda tasdiqdan kelib chiqadi: Agar $u \in \alpha - h(D)$ bo'lsa, u holda

$$v(z_1, ..., z_n) = u\left(\sqrt{\alpha_2 ... \alpha_n} z_1, ..., \overset{\overset{j}{\vee}}{\sqrt{\alpha_1 ... \alpha_n}}\, z_j, ..., \sqrt{\alpha_1 ... \alpha_{n-1}} z_n \right)$$

funksiyaning D sohaning yuqoridagi chiziqli akslantirish natijasida hosil bo'lgan obrazi D' da garmonik bo'ladi. Demak, v garmonik funksiya uchun o'rta qiymat haqidagi teoremaga asosan ixtiyoriy

$B(0, \rho) = \left\{ z \in \mathbb{C}^n : |z_1|^2 + ... + |z_n|^2 \le \rho^2 \right\} \subset\subset D'$ shar uchun

$$v(0) = \frac{1}{V(\rho)} \int\limits_{B(0, \rho)} v(\xi) dV(\xi)$$

Ushbu integralda o'zgaruvchini quyidagicha almashtiramiz:

$$\sqrt{\alpha_1 \overset{j}{\vee} \alpha_n}\,\xi_j = \tau_j \Rightarrow d\xi_j = (\sqrt{\alpha_1 \overset{j}{\vee} \alpha_n})^{-1} d\tau_j, \qquad j = 1,2,...,n.$$

va $\alpha_1 \left|\tau_1\right|^2 + ... + \alpha_n \left|\tau_n\right|^2 = \rho^2 \alpha_1 ... \alpha_n$ tenglikda $r^2 = \rho^2 \alpha_1 ... \alpha_n$ deb belgilab

olamiz va bundan $\rho = \dfrac{r}{\sqrt{\alpha_1 ... \alpha_n}}$ kelib chiqadi. Ellipsoidning hajmi uchun

$$\tilde{V}(r) = \int_{E(0,r)} dV(\tau) = \int_{B(0,\rho)} (\alpha_1 ... \alpha_n)^{n-1} dV(\xi) = (\alpha_1 ... \alpha_n)^{n-1} V(\rho)$$

munosabat o‘rinli bo‘lishidan, quyidagi tenglik kelib chiqadi

$$\begin{aligned} u(0) &= \frac{1}{V(r)(\alpha_1 ... \alpha_n)^{n-1}} \int_{\alpha_1|\tau_1|^2 + ... + \alpha_n|\tau_n|^2 \le \rho^2} u(\tau_1,...,\tau_n) dV(\tau) = \\ &= \frac{1}{\tilde{V}(\rho)} \int_{\alpha_1|\tau_1|^2 + ... + \alpha_n|\tau_n|^2 \le \rho^2} u(\tau_1,...,\tau_n) dV(\tau). \end{aligned} \tag{2.2.4}$$

Shunday qilib, (2.2.4) integral ellipsoid bilan chegaralangan soha bo‘yicha o‘rta qiymatni bildiradi. Teorema isbot bo‘ldi.

Shuningdek quyidagi teoremani o‘rinli ekanligini ko‘rish mumkin.

2.2.3-teorema. *Agar $u(z)$ funksiya D sohada $\alpha -$garmonik bo‘lsa, u holda $\forall z^0 \in D$ nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left|z_1 - z_1^0\right|^2 + ... + \alpha_n \left|z_n - z_n^0\right|^2 = r^2 \right\} \subset\subset D$$

uchun

$$u(z^0) = \frac{1}{\sigma_n r^{2n-1}} \int_{\partial E(z^0, r)} u(\xi) d\sigma(\xi)$$

tenglik o‘rinli bo‘ladi. Bu yerda $d\sigma(\xi)$ - sirt elementi va $\sigma_n -$ birlik gipersharning sirt yuzasi.

III-BOB. $\alpha-$subgarmonik funksiyalar

3.1. $\alpha-$subgarmonik funksiyalar va ularning sodda xossalari

$D \subset C^n$ sohada qat'iy musbat $(n-1, n-1)$ bidarajali differensial formani va $dd^c u \wedge \alpha$ operatorni qaraymiz. Bu operator elliptik operator bo'lishini yuqorida ko'rib o'tdik.

$\alpha-$ qat'iy musbat, yopiq, cheksiz silliq koeffisientli bo'lgan holda biz quyidagi $\alpha-$subgarmonik funksiya ta'riflarini beramiz.

3.1.1-ta'rif. *$D \subset \mathbb{C}^n$ sohada berilgan $u(z) \in L^1_{loc}(D)$ funksiya uchun quyidagi shartlar*

1) u yuqoridan yarim uzluksiz, ya'ni $\overline{\lim\limits_{z \to z^0}} u(z) = \lim\limits_{\varepsilon \to 0} \sup\limits_{B(z^0, \varepsilon)} u(z) \le u(z^0)$;

2) umumlashgan funksiya (oqim) ma'nosida

$$\left[dd^c u \wedge \alpha\right](\omega) = \int u \wedge \alpha \wedge dd^c \omega, \forall\, \omega \in F(D), \qquad (3.1.1)$$

musbat, ya'ni $\left[dd^c u \wedge \alpha\right](\omega) \ge 0, \forall\, \omega \ge 0$ bajarilsa, bu yerda $F(D)-$ asosiy finit funksiyalar fazosi, u ga $\alpha-$subgarmonik funksiya deyiladi.

Bu yerda agar $\alpha = \left(dd^c |z|^2\right)^{n-1}$ bo'lsa, biz mos ravishda potensiallar nazariyasidagi yaxshi tadqiq qilingan subgarmonik funksiya ta'rifiga ega bo'lamiz.

$\alpha-$subgarmonk funksiyalar sinfini $\alpha - sh(D)$ orqali belgilaymiz. (3.1.1) dan $u \in \alpha - sh\left(D\right)$ uchun $dd^c u \wedge \alpha$ oqim (umumlashgan funksiya) musbat va shuning uchun u musbat borel o‘lchovini ifodalashi kelib chiqadi: $dd^c u \wedge \alpha = \mu$. μ o‘lchov $u\left(z\right)$ funksiyaning assosirlangan o‘lchovi deyiladi.

Endi xususiy holda $\alpha-$subgarmonk funksiyalar sinfi xossalarni o‘rganamiz.

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}_+^n$ vektor, $\varphi(z) = \alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n$ funksiya va $D(D \subset \mathbb{C}^n)$ sohada berilgan $u \in C^2$ funksiya uchun quyidagi operatorni qaraymiz:

$$dd^c u \wedge \alpha = dd^c u \wedge (dd^c \varphi)^{n-1} = (n-1)! \sum_j \alpha_1 \alpha_2 \overset{\overset{j}{\vee}}{\dots} \alpha_n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV \quad .$$

ifodaga ega bo‘lamiz. Bu yerda $\overset{\overset{j}{\vee}}{\dots}$ belgi shu ko‘paytmada j-had qatnashmasligini bildiradi.

3.1.2-ta’rif. $D \subset \mathbb{C}^n$ soha berilgan bo‘lsin. Ushbu

$$\sum_{j=1,n} \alpha_1 \alpha_2 \overset{\overset{j}{\vee}}{\dots} \alpha_n \cdot \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} \geq 0$$

tengsizlikni qanoatlantiruvchi $u \in C^2(D)$ funksiyaga $\alpha-$subgarmonik funksiya deyiladi.

$\alpha-$subgarmonik funksiya quyidagi xossalarga ega.

a) $\alpha-$subgarmonik funksiyaning musbat koeffitsentli chiziqli kombinatsiyasi yana α-subgarmonik funksiya bo'ladi ya'ni

$$u_k(z) \in \alpha - sh(D),\ b_k \in R_+ \Rightarrow b_1u_1(z) + b_2u_2(z) + ... + b_mu_m(z) \in \alpha - sh(D).$$

$$k = \overline{1,m}$$

Bu yerda ham subgarmonik funksiyaning xossasi kabi xossa o'rinli.

3.1.1-teorema. *Agar $u(z)$ funksiya D sohada $\alpha-$subgarmonik bo'lsa, u holda $\forall z^0 \in D$ nuqta va*

$$E(z^0, r) = \left\{ z \in \mathbb{C}^n : \alpha_1 \left|z_1 - z_1^0\right|^2 + ... + \alpha_n \left|z_n - z_n^0\right|^2 \le r^2 \right\} \subset\subset D \quad gipershar$$

uchun

$$u(z^0) \le \frac{1}{\tilde{V}(r)} \int\limits_{E(z^0,r)} u(\tau)dV(\tau)$$

tengsizlik o'rinli bo'ladi. Bu yerda $\tilde{V}(r)$ - gipershar hajmi va $dV(\tau)-$hajm elementi.

Bu o'rta qiymat haqidagi teorema yordamida $\alpha-$subgarmonik funksiyalarning quyidagi xossalari kelib chiqadi.

b) Chekli sondagi $\alpha-$subgarmonik funksiyaning maksumimi ham $\alpha-$subgarmonik funksiya bo'ladi ya'ni

$$u_1(z), u_2(z), ..., u_m(z) \in \alpha - sh(D) \Rightarrow \max\{u_1(z), u_2(z), ..., u_m(z)\} \in \alpha - sh(D).$$

c) Monoton kamayuvchi $\alpha-$subgarmonik funksiyalar ketma-ketligining limiti $\alpha-$subgarmonik funksiya bo‘ladi ya’ni

$$u_j(z)\in\alpha-sh(D),\ u_j(z)\ge u_{j+1}(z)\Rightarrow \lim_{j\to+\infty}u_j(z)\in\alpha-sh(D),\ \ j=1,2,...,n.$$

d) Tekis yaqinlashuvchi α-subgarmonik funksiyalar ketma-ketligi

yana α-subgarmonik funksiyaga yaqinlashadi ya’ni

$$u_j(z)\in\alpha-sh(D),\ \ u_j(z)\rightrightarrows u(z)\Rightarrow u(z)\in\alpha-sh(D),\ \ (k=1,2,3...)$$

e) *Maksimum prinsipi.* Agar $u(z)\in\alpha-sh(D)$ funksiya biror $z^0\in D$

nuqtada o‘zining maksimumiga erishsa, ya’ni

$$u(z^0)=\sup_{x\in D}u(z)$$

bo‘lsa, u holda $u(z)=const$ bo‘ladi..

f) $\alpha-$subgarmonik funksiyalar fazosida metrika quyidagi ko‘rinishga

yega:

$$\rho(z,w)=\sqrt{\sum_{i=1}^{n}\alpha_i\left|z_i-w_i\right|^2}$$

$D\subset\mathbb{C}^n$ sohada aniqlangan

$$u:D\to[-\infty,\infty)$$

funksiya berilgan bo‘lsin.

3.1.3-ta’rif. Quyidagi ikkita shart o‘rinli bo‘lsa:

1) $u(z)$ yuqoridan yarim uzluksiz, ya’ni

$\forall z^0\in D$ uchun $\overline{\lim_{z\to z^0}}u(z)\le u(z^0)$

2) har bir $z^0 \in D$ nuqta uchun shunday $\rho(z^0) > 0$ topiladiki, $\rho \le \rho(z^0)$ uchun

$$u(z^0) \le \frac{1}{\tilde{V}(\rho)} \int\limits_{\alpha_1\left|z_1 - z_1^0\right|^2 + \ldots + \alpha_n\left|z_n - z_n^0\right|^2 \le \rho^2} u(\tau)dV(\tau)$$

shartni qanoatlantirsa, u holda $u(z)$ funksiya D sohada $\alpha-$ subgarmonik funksiya deyiladi.

$\alpha-$subgarmonik funksiya va subgarmonik funksiyalarni farqli jihatlarini koʻrsatuvchi misollarni koʻrib oʻtamiz.

3.1.1-misol: $\mathbb{C}^3$ fazoda bizga $\alpha = (3,1,2) \in \mathbb{R}_+^3$ vektor berilgan boʻlsin. $u = -5\left|z_1\right|^2 + 6\left|z_2\right|^2 - 2\left|z_3\right|^2$ bu funksiya subgarmonik funksiya emas, lekin $\alpha-$subgarmonik funksiya boʻladi.

Haqiqatan ham,

$$\Delta u = 4\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + 4\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} + 4\frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} = 4(-5+6-2) = -4 \le 0$$

bundan koʻrinib turibdi u funksiya subgarmonik funksiya emas ekan, lekin quyidagiga asosan

$$\Delta_\alpha u = 2\cdot 4\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + 6\cdot 4\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} + 3\cdot 4\frac{\partial^2 u}{\partial z_3 \partial \overline{z}_3} = 4\cdot(-10+36-6) = 80 \ge 0$$

ya'ni u $\alpha-$subgarmonik funksiya ekan.

3.2. $\alpha-$subgarmonik funksiyalarning Riss tasviri

Elliptik operatorlar nazariyasidan yaxshi ma'lumki, $dd^c u \wedge \alpha$ operator ikkinchi tartibli elliptik operator bo‘lgani uchun $\mathbb{C}^n$ fazoda tayinlangan $w\in\mathbb{C}^n$ nuqtaga nisbatan $K_\alpha(z,w)$ fundamental yechim mavjud: $dd^cK_\alpha\wedge\alpha=\delta(z-w)$, bu yerda $\delta(z-w)-$ $w\in\mathbb{C}^n$ nuqtadagi Dirak o‘lchovi va shu bilan birga $K_\alpha(z,w)$ $\mathbb{C}^n$ da $z=w$ dan tashqarida cheksiz silliq funksiyani ifodalaydi va shu bilan birga $z=w$ nuqta atrofida quyidagi baholash o‘rinli: (q. [1],[13])

$$\left|D^\beta K_\alpha(z,w)\right|\le const\begin{cases}\left|z-w\right|^{-2n-|\beta|+2}, & 2n+|\beta|>2\\ \ln\left|z-w\right|, & n=1,\ \beta=0\end{cases} \qquad (3.2.1)$$

$K_\alpha(z)$ fundamental yechim $dd^cu\wedge\alpha$ operator subyechimining Riss tasviri yadrosi bo‘lib xizmat qiladi.

3.2.1-teorema. Agar $u\in\alpha-sh(D)$ *bo‘lsa, u holda istalgan* $G\subset\subset D$ *kompakt qism sohada quyidagi tasvir o‘rinli:*

$$u(z)=\int_G K_\alpha(z-w)d\mu_w+\Phi(z),\ z\in G, \qquad (3.2.2)$$

bu yerda $\Phi(z)-$ G *sohadagi biror* $\alpha-$*garmonik funksiya.*

Isbot. Bu teoremani isbotlash uchun quydagi ayirmani qaraymiz:

$$\Phi(z)=u(z)-\int_G K_\alpha(z-w)d\mu_w\,.$$

Yuqorida aytib o‘tganimizdek, agar $u\in\alpha-sh(D)$ bo ‘lsa, u holda Riss-Markov teoremasiga (q. [6]) ko‘ra unga μ assosiirlangan o‘lchov mos keladi: $dd^c u \wedge \alpha = \mu$. Endi $\upsilon_\mu(z) = \int\limits_G K_\alpha\left(z-w\right)d\mu_w$ deb belgilaymiz va $dd^c\upsilon_\mu \wedge \alpha = \mu$ ekanini ko'rsatamiz. Haqiqatan ham, $K_\alpha(z)$ fundamental yechim uchun tayinlangan $w \in \mathbb{C}^n$ nuqtaga nisbatan $dd^c K_\alpha(z-w)\wedge\alpha = \delta\left(z-w\right)$ tenglikka ega bo'lamiz. Istalgan $\omega \in F^{(0,0)}\left(D\right)$, $\omega \ge 0$ uchun α formaning yopiqligidan foydalanib,

$$\left[dd^c\upsilon_\mu \wedge \alpha\right]\left(\omega\right) = \int\left(\int\limits_G K_\alpha(z-w)d\mu_w\right)\wedge\alpha\wedge dd^c\omega = \int\limits_G d\mu_w \int K_\alpha(z-w)\wedge\alpha\wedge dd^c\omega$$
$$= \int\limits_G \left[dd^c K_\alpha(z-w)\wedge\alpha\right](\omega)d\mu_w = \int\limits_G \delta(z-w)d\mu_w = \mu$$

tenglikni olamiz. Shuning uchun G sohada

$$dd^c\Phi\left(z\right)\wedge\alpha = dd^c u\left(z\right)\wedge\alpha - dd^c\upsilon_\mu\left(z\right)\wedge\alpha = \mu - \mu = 0.$$

tenglikka egamiz. Binobarin $\Phi\left(z\right)$ funksiya G sohada $\alpha-$garmonik funksiyani ifodalaydi.

Endi bu tasvirning yagona ekanini ko'rsatamiz. Haqiqatan G sohada $u(z) = \upsilon_\nu(z) + \Phi_G(z)$ tenglik o'rinli bo'lsin, bu yerda $\nu -$ G sohadagi biror chekli o'lchov, $\Phi_G(z) \in \alpha - h(G)$, u holda

$$dd^c u \wedge \alpha = dd^c\upsilon_\nu \wedge \alpha + dd^c\Phi_G(z)\wedge\alpha = \nu\,,$$

ya'ni $\nu(G) = \mu(G)$ ni olamiz. Bu yerdan $\Phi(z) = \Phi_G(z)$ ekani kelib chiqadi. Teorema isbotlandi.

Endi 3.1 da kiritilgan hususiy hol uchun $\alpha-$subgarmonik funksiyalar sinfida yuqoridagi masalani qarab chiqamiz.

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$ vektor uchun,

$$\varphi(z) = \alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n$$

funksiyani olamiz va $\alpha = \left(dd^c \varphi\right)^{n-1}$ bo 'lgan holda quyidagi operatorni qaraymiz:

$$dd^c u \wedge \alpha = (n-1)! \sum_j \alpha_1 \alpha_2 \overset{\overset{j}{\vee}}{...} \alpha_n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} dV,$$

bu yerda $\overset{\overset{j}{\vee}}{...}$ belgi shu ko'paytmada j-had qatnashmasligini bildiradi,

$dV = \left(\frac{i}{2}\right)^n dz_1 \wedge d\overline{z}_1 \wedge ... \wedge dz_n \wedge d\overline{z}_n$ - $\mathbb{R}^{2n}$ dagi hajm formasi.

Bu operator ham elliptik operator bo'lib, quyidagi teorema uning fundamental yechimi ko'rinishini aniq tasvirlash imkonini beradi.

3.2.2-teorema. Berilgan $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$, $n \geq 2$ vektor uchun ushbu

$$K(z) = \frac{1}{(n-1) \cdot (\alpha_1 z_1 \overline{z}_1 + \alpha_2 z_2 \overline{z}_2 + ... + \alpha_n z_n \overline{z}_n)^{n-1}} \tag{3.2.3}$$

funksiya quyidagi

$$dd^c u \wedge \alpha = 0 \tag{3.2.4}$$

elliptik tenglamaning fundamental yechimi bo'ladi.

Isbot: Bu yadrodan xususiy hosilalarni olamiz:

$$\frac{\partial K(z)}{\partial z_1}=\frac{1}{(n-1)}\cdot\frac{\alpha_1\overline{z}_1(n-1)}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}=\frac{\alpha_1\overline{z}_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n},$$

$$\frac{\partial K(z)}{\partial z_2}=\frac{1}{(n-1)}\cdot\frac{\alpha_2\overline{z}_2(n-1)}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}=\frac{\alpha_2\overline{z}_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n},$$

$$... \quad ... \quad ... \quad ... \quad ... \quad ...$$

$$\frac{\partial K(z)}{\partial z_n}=\frac{1}{(n-1)}\cdot\frac{\alpha_n\overline{z}_n(n-1)}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}=\frac{\alpha_n\overline{z}_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}$$

endi bu xususiy hosiladan $\overline{z}_j,\ (j=1,..,n)$ bo‘yicha hosila olamiz:

$$\frac{\partial^2 K(z)}{\partial z_1\partial\overline{z}_1}=\frac{\alpha_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_1}^2 z_1\overline{z}_1}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}},$$

$$\frac{\partial^2 K(z)}{\partial z_2\partial\overline{z}_2}=\frac{\alpha_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_2}^2 z_2\overline{z}_2}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}},$$

$$... \quad ... \quad ... \quad ... \quad ...$$

$$\frac{\partial^2 K(z)}{\partial z_n\partial\overline{z}_n}=\frac{\alpha_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^n}-\frac{n{\alpha_n}^2 z_n\overline{z}_n}{(\alpha_1 z_1\overline{z}_1+...+\alpha_n z_n\overline{z}_n)^{n+1}}$$

Topilgan bu qiymatlarni (3.2.4) tenglikga qo‘yamiz:

$$\sum_{j=1}^{n}\alpha_1\alpha_2\overset{\overset{j}{\vee}}{\ldots}\alpha_n\cdot\frac{\partial^2 u}{\partial z_j\partial\overline{z}_j}=\alpha_2\alpha_3\ldots\alpha_n\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}+\ldots+\alpha_1\alpha_3\ldots\alpha_{n-1}\frac{\partial^2 u}{\partial z_n\partial\overline{z}_n}=$$

$$=\alpha_2\alpha_3\ldots\alpha_n\left(\frac{\alpha_1}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n{\alpha_1}^2z_1\overline{z}_1}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)+$$

$$+\alpha_1\alpha_3\ldots\alpha_n\left(\frac{\alpha_2}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n{\alpha_2}^2z_2\overline{z}_2}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)+\ldots+$$

$$+\alpha_1\alpha_2\ldots\alpha_{n-1}\left(\frac{\alpha_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n{\alpha_n}^2z_n\overline{z}_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)=$$

$$=\left(\frac{\alpha_1\alpha_2\alpha_3\ldots\alpha_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n{\alpha_1}^2\alpha_2\alpha_3\ldots\alpha_nz_1\overline{z}_1}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)+$$

$$+\left(\frac{\alpha_1\alpha_2\alpha_3\ldots\alpha_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n\alpha_1{\alpha_2}^2\alpha_3\ldots\alpha_nz_2\overline{z}_2}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)+\ldots+$$

$$+\left(\frac{\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{n\alpha_1\alpha_2\ldots\alpha_{n-1}{\alpha_n}^2z_n\overline{z}_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)=$$

$$=n\alpha_1\alpha_2\ldots\alpha_{n-1}\alpha_n\left(\frac{1}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^n}-\frac{\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n}{(\alpha_1z_1\overline{z}_1+\ldots+\alpha_nz_n\overline{z}_n)^{n+1}}\right)=0$$

bundan esa (3.2.3) ning (3.2.4) ni fundamental yechimi ekanligi kelib chiqadi.

Xulosa

Ushbu magistrlik dissertatsiyasida subgarmonik funksiyalarning kengaytmasi bo'lgan $\alpha-$subgarmonik funksiyalar sinfining ba'zi xossalari o'rganilgan. $\alpha-$qat'iy musbat $(n-1,n-1)$ bidarajali differensial forma bo'lgan holda $dd^c u\wedge\alpha$ operatorning elliptikligi isbotlanib, bu operatorning subyechimi sifatida $\alpha-$subgarmonik funksiyaning Riss tasviri o'rganildi. Xususiy holda $\alpha-$subgarmonik funksiyalarning o'rta qiymat haqidagi xossasi isbotlandi va Riss yadrosining aniq tasviri topildi.

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