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Chegaraviy sharti spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun teskari masala

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M U N D A R I J A

Kirish

Mavzuning dolzarbligi. Matematik fizikaning bir qator masalalari Shturm-Liuvill operatorining xos qiymatlarini va ortonormallangan xos funksiyalarini topishga keltiriladi. Jumladan, klassik matematik fizikaning asosiy tenglamalari hisoblangan tor tebranishi va issiqlik o‘tkazuvchanlik tenglamalarini Furye usuli bilan yechishda Shturm-Liuvill chegaraviy masalasining xos qiymatlarini, ortonormallangan xos funksiyalarini aniqlashga va ixtiyoriy funksiyani ular yordamida Furye qatoriga yoyishga to‘g‘ri keladi. Bu yo‘nalishdagi ilk natijalar D.Bernulli, J.Dalamber, L.Eyler, Liuvill va Shturmlar tomonidan olingan. Shturm-Liuvill operatori spektral nazariyasining asosiy g‘oyalari XX-asrda G.D.Birkgoff, G.Veyl, D.Gilbert, V.A.Steklov, E.Ch.Titchmarsh, N.Levinson, B.M.Levitan va boshqa olimlar tomonidan rivojlantirildi.

Shturm-Liuvill chegaraviy masalasining $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o‘zgarmaslariga yoki spektral funksiyasiga spektral xarakteristikalar deyiladi. Spektral xarakteristikalarni topish va ularning xossalarini o‘rganish masalasiga spektral analizning to‘g‘ri masalasi deyiladi.

Shturm-Liuvill chegaraviy masalasi xos funksiyalarining $L_2[0,\pi]$ fazoda to‘laligi haqidagi teoremani ilk bor XIX asrning oxirlarida V.A.Steklov isbotlashga muvaffaq bo`lgan.

Spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasi koeffitsientini va chegaraviy shartlarni aniqlash masalasiga spektral analizning teskari masalasi deyiladi.

Teskari masalalar nazariyasining rivojiga muhim turtki bo‘lgan ilk natija 1929 yilda taniqli astronom V.A.Ambartsumyan tomonidan olingan: *agar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$y'(0) = 0, \; y'(\pi) = 0$$

Shturm-Liuvill chegaraviy masalasining xos qiymatlari $\lambda_n = n^2, \quad n = 0,1,2,\dots$ *bo‘lsa, u holda* $q(x) \equiv 0$ *bo‘ladi.*

Ambartsumyanning bu natijasi muhim ekanliliga birinchi bo‘lib, 1946 yilda Shved matematigi G.Borg e’tibor bergan, u Ambartsumyanning natijasi umumiy qoidadan istisno ekanligini ko`rsatib bergan, ya’ni faqat bitta spektr orqali Shturm-Liuvill chegaraviy masalasini bir qiymatli tiklab bo`lmasligini ko`rsatdi. G.Borg, faqat bitta chegaraviy sharti bilan farq qiluvchi ikkita Shturm-Liuvill masalasining spektrlari ma’lum bo`lsa, bu spektrlar orqali dastlabki masalalar bir qiymatli tiklanishini isbot qildi. Hozirda bu tasdiq, G.Borgning yagonalik teoremasi deb yuritiladi: *agar* $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$ *sonlar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$\begin{cases} y'(0) - hy(0) = 0, \;\; h \in R, \\ y'(\pi) + Hy(\pi) = 0, \;\; H \in R \end{cases}$$

chegaraviy masalaning xos qiymatlari va $\mu_0 < \mu_1 < \ldots < \mu_n < \ldots$ *sonlar quyidagi*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$\begin{cases} y'(0) - hy(0) = 0, \;\; h \in R, \\ y'(\pi) + H_1 y(\pi) = 0, \;\; H \neq H_1, \; H_1 \in R \end{cases}$$

chegaraviy masalaning xos qiymatlari bo‘lsa, u holda λ_n *va* μ_n *xos qiymatlar ketma-ketligi orqali* $q(x)$ *haqiqiy funksiya va* h, H, H_1 *sonlar bir qiymatli aniqlanadi.*

Hozirgi kunda chegaraviy sharti spektral parametrga bog`liq bo`lgan chegaraviy masalalar uchun teskari masalalarning analoglarini olish masalasi dolzarb masalalardan biri hisoblanadi.

Tadqiqotning maqsadi. Chegaraviy shartlari spektral parametrga bog`liq bo`lgan chegaraviy masala uchun teskari spektral masalaning analogini olish.

Tadqiqotning vazifasi. Chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun xos qiymatlarning xaqiqiyligini, karrasizligini va cheksiz ko‘pligini isbotlash, Krum-Kreyn almashtirishining analogini olish, Krum-Kreyn almashtirishi yordamida xos qiymatlarning asimptotikasini va regulyarlashtirilgan izlar formulasini keltirib chiqarish, V.A.Ambartsumyan teoremasining analogini isbot qilish. Xos funksiyalar uchun ossillyatsiya teoremasi analogini isbotlash. Veyl-Titchmarsh funksiyasining hossalarini o`rganish.

Tadqiqotning obyekti. Chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi.

Tadqiqotning predmeti. Spektral analizning to`g`ri va teskari masalalari.

Tadqiqotning usullari. Teskari spektral masala usuli.

Tadqiqotning natijalarining ilmiy jihatdan yangilik darajasi. Mazkur ishning asosiy natijalari yangi.

Tadqiqotning natijalarining amaliy ahamiyati va tatbiqi. Olingan natijalar va qo‘llanilgan usullar nochiziqli evolyutsion tenglamalarni integrallashga tatbiq qilinadi.

Ish tuzilishi va tarkibi. Dissertatsiya ishi kirish, uchta bob hamda adabiyotlar ro‘yhatidan iborat. Har bir bob bo‘limlardan tuzilgan.

Bajarilgan ishning asosiy natijalari. 1) Chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun xos qiymatlarning asimptotikasi va regulyarlashtirilgan izlar formulasi keltirib chiqarilgan; 2) V.A.Ambartsumyan teoremasining analogi isbot qilingan; 3) Veyl-Titchmarsh funksiyasi spektral berilganlar orqali ifodalangan.

Xulosa va takliflarning qisqacha umumlashtirilgan ifodasi. Mazkur dissertatsiyada chegaraviy shartlari spektral parametrga bog`liq bo`lgan chegaraviy masala uchun teskari spektral masalaning analogi olingan. Qo‘llanilgan usullarni chegaraviy shartlari spektral parametrga murakkab ravishda bog`liq bo`lgan chegaraviy masala uchun qo‘llashni taklif qilamiz.

I-BOB. CHEGARAVIY SHARTLARI SPEKTRAL PARAMETRGA BOG‘LIQ BO‘LGAN SHTURM-LIUVILL MASALASINING XOS QIYMATLARI VA XOS FUNKSIYALARNING XOSSALARI

1-§. Xos qiymatlarning va xos funksiyalarning sodda xossalari

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0, \pi], \tag{1.1}$$

$$\begin{cases} y'(0) = (h_0 + h_1\lambda)y(0), \\ y'(\pi) = -(H_0 + H_1\lambda)y(\pi). \end{cases} \tag{1.2}$$

Bu yerda $q(x) \in C^1[0, \pi]$ haqiqiy qiymatli funksiya, h_0, h_1, H_0, H_1 haqiqiy sonlar, λ esa kompleks parametr. Biz $h_1 < 0$ va $H_1 < 0$ bo‘lgan holni o‘rganamiz.

Odatda $q(x)$ funksiyaga (1.1)+(1.2) Shturm-Liuvill masalasining potentsiali deyiladi.

Ta’rif. Agar λ parametrning biror $\lambda = \lambda_0$ qiymatida (1.1)+(1.2) chegaraviy masala noldan farqli $y(x, \lambda_0) \not\equiv 0$ yechimga ega bo‘lsa, λ_0 songa (1.1)+(1.2) chegaraviy masalaning xos qiymati deyiladi, $y(x, \lambda_0)$ yechimga esa λ_0 xos qiymatga mos keluvchi xos funksiyasi deyiladi.

(1.1)+(1.2) Shturm-Liuvill masalasining barcha xos qiymatlaridan tuzilgan to‘plamga (1.1)+(1.2) masalaning spektri deyiladi.

1-xossa. $y_1(x, \lambda)$ va $y_2(x, \lambda)$ funksiyalar (1.1) tenglamaning ixtiyoriy yechimlari bo‘lsin. U holda ulardan tuzilgan

$$W\{y_1(x, \lambda), y_2(x, \lambda)\} = \begin{vmatrix} y_1(x, \lambda) & y_2(x, \lambda) \\ y_1'(x, \lambda) & y_2'(x, \lambda) \end{vmatrix}$$

Vronskiy determinanti x o‘zgaruvchiga bog‘liq bo‘lmaydi.

Isbot. Buning uchun ushbu

$$\frac{dW}{dx} \equiv 0$$

tenglik bajarilishini ko‘rsatish yetarli:

$$\frac{dW}{dx} \equiv (y_1 y_2' - y_1' y_2)' = y_1 y_2'' - y_1'' y_2 = y_1[q(x)y_2 - \lambda y_2] - y_2[q(x)y_1 - \lambda y_1] = 0 \ . \blacksquare$$

2-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining xos qiymatlari haqiqiy bo‘ladi.

Isbot. $\lambda = u + iv$, $(v \neq 0)$ son (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining xos qiymati bo‘lsin deb faraz qilaylik va unga mos keluvchi xos funksiyani $y(x)$ orqali belgilaylik. U holda $\bar{\lambda} = u - iv$ son ham shu chegaraviy masalaning xos qiymati bo‘ladi va unga $\bar{y}(x)$ xos funksiya mos keladi. Quyidagi

$$(\lambda - \bar{\lambda})\int_0^\pi |y(x)|^2 dx = \int_0^\pi (\lambda - \bar{\lambda}) y(x)\bar{y}(x)dx = \int_0^\pi [(\lambda y)\bar{y} - y(\bar{\lambda}\bar{y})]dx =$$

$$= \int_0^\pi \{\bar{y}[-y'' + q(x)y] - y[-\bar{y}'' + q(x)\bar{y}]\}dx = \int_0^\pi (\bar{y}''y - y''\bar{y})dx = \int_0^\pi (\bar{y}'y - y'\bar{y})'dx =$$

$$= \begin{vmatrix} y(\pi) & \bar{y}(\pi) \\ -(H_0 + H_1\lambda)y(\pi) & -(H_0 + H_1\bar{\lambda})\bar{y}(\pi) \end{vmatrix} - \begin{vmatrix} y(0) & \bar{y}(0) \\ (h_0 + h_1\lambda)y(0) & (h_0 + h_1\bar{\lambda})\bar{y}(0) \end{vmatrix} =$$

$$= (\lambda - \bar{\lambda})|y(\pi)|^2 H_1 + (\lambda - \bar{\lambda})|y(0)|^2 h_1$$

tenglikdan

$$\int_0^\pi |y(x)|^2 dx = |y(\pi)|^2 H_1 + |y(0)|^2 h_1$$

ziddiyat kelib chiqadi, chunki $h_1 < 0$ va $H_1 < 0$. $\blacksquare$

Natija. Xos funksiyani haqiqiy qilib tanlash mumkin. Agar λ xos qiymatga $y(x) = u(x) + iv(x)$ xos funksiya mos keluvchi bo‘lsa, $u(x)$ ham $v(x)$ ham (1.1)+(1.2) masalaning yechimi bo‘ladi. Bu yechimlardan kamida bittasi noldan farqli bo‘ladi, chunki $y(x) \not\equiv 0$.

3-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining $\lambda_1 \neq \lambda_2$ xos qiymatlariga mos keluvchi $y_1(x)$, $y_2(x)$ xos funksiyalari uchun ushbu

$$\int_0^\pi y_1(x)y_2(x)dx - h_1 y_1(0)y_2(0) - H_1 y_1(\pi)y_2(\pi) = 0 \tag{1.3}$$

tenglik o‘rinli bo‘ladi.

Isbot. Ushbu

$$(\lambda_1-\lambda_2)\int_0^{\pi} y_1(x)y_2(x)dx=\int_0^{\pi}[(\lambda_1 y_1)y_2-y_1(\lambda_2 y_2)]dx=$$

$$=\int_0^{\pi}\{y_2[-y_1''+q(x)y_1]-y_1[-y_2''+q(x)y_2]\}dx=$$

$$=\int_0^{\pi}(y_2''y_1-y_1''y_2)dx=\int_0^{\pi}(y_2'y_1-y_1'y_2)'dx=\left.\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right|_0^{\pi}=$$

$$=\begin{vmatrix} y_1(\pi) & y_2(\pi) \\ -(H_0+H_1\lambda_1)y_1(\pi) & -(H_0+H_1\lambda_2)y_2(\pi) \end{vmatrix}-\begin{vmatrix} y_1(0) & y_2(0) \\ (h_0+h_1\lambda_1)y_1(0) & (h_0+h_1\lambda_2)y_2(0) \end{vmatrix}=$$

$$=(\lambda_1-\lambda_2)y_1(\pi)y_2(\pi)H_1+(\lambda_1-\lambda_2)y_1(0)y_2(0)h_1$$

ayniyatda $\lambda_1 \neq \lambda_2$ bo‘lgani uchun (1.3) tenglik o‘rinli bo‘ladi. ■

4-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining turli xos qiymatlariga mos keluvchi $y_1(x)$, $y_2(x)$, … xos funksiyalari chiziqli erkli bo‘ladi.

Isbot. Teskarisini faraz qilaylik, ulardan bittasi qolganlari orqali chiziqli ifodalansin:

$$y_n=C_1y_1+C_2y_2+...+C_{n-1}y_{n-1}+C_{n+1}y_{n+1}+... \ .$$

U holda

$$\int_0^{\pi}y_n^2dx=C_1\int_0^{\pi}y_ny_1dx+C_2\int_0^{\pi}y_ny_2dx+...+C_{n-1}\int_0^{\pi}y_ny_{n-1}dx+C_{n+1}\int_0^{\pi}y_ny_{n+1}dx+...,$$

$$\int_0^{\pi}y_n^2dx=C_1[h_1y_1(0)y_n(0)+H_1y_1(\pi)y_n(\pi)]+C_2[h_1y_2(0)y_n(0)+H_1y_2(\pi)y_n(\pi)]+...+$$

$$+C_{n-1}[h_1y_{n-1}(0)y_n(0)+H_1y_{n-1}(\pi)y_n(\pi)]+C_{n+1}[h_1y_{n+1}(0)y_n(0)+H_1y_{n+1}(\pi)y_n(\pi)]+....$$

Bunga ko‘ra

$$\int_0^{\pi}y_n^2dx=h_1y_n^2(0)+H_1y_n^2(\pi).$$

Ziddiyat, chunki chap tomon musbat, o‘ng tomon esa manfiy. ■

5-xossa. (1.1)+(1.2) Shturm-Liuvill masalasining bitta xos qiymatiga mos keluvchi xos funksiyalari bir-biriga proportsional bo‘ladi.

Isbot. λ xos qiymatga $y_1(x)$, $y_2(x)$ chiziqli erkli xos funksiyalar mos keladi deb faraz qilaylik. U holda

$$W\{y_1, y_2\} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} y_1(0) & y_2(0) \\ (h_0 + h_1\lambda)y_1(0) & (h_0 + h_1\lambda)y_2(0) \end{vmatrix} = 0$$

bo‘lgani uchun, $y_1(x)$, $y_2(x)$ chiziqli bog‘liq bo‘ladi. Bu esa farazimizga zid. ■

6-xossa. $y(x,\lambda)$ funksiya Shturm-Liuvill tenglamasining λ bo‘yicha uzluksiz differensiallanuvchi ixtiyoriy yechimi bo‘lsin. U holda ushbu

$$\int_0^{\pi} y^2(x,\lambda)dx = W_{\pi}\{\dot{y}, y\} - W_0\{\dot{y}, y\} \tag{1.4}$$

tenglik bajariladi. Bu yerda $\dot{y} = \dfrac{\partial y(x,\lambda)}{\partial \lambda}$.

Isbot. Ushbu

$$-y'' + q(x)y = \lambda y \tag{1.5}$$

ayniyatdan λ bo‘yicha hosila olsak,

$$-\dot{y}'' + q(x)\dot{y} = y + \lambda\dot{y} \tag{1.6}$$

kelib chiqadi. Agar (1.5) va (1.6) tengliklarni mos ravishda $\dot{y}$ va y funksiyalarga ko‘paytirib bir-biridan ayirsak, ushbu

$$\dot{y}''y - y''\dot{y} = -y^2 \tag{1.7}$$

ayniyat hosil bo‘ladi. Bu tenglikni $[0,\pi]$ kesmada integrallasak, quyidagi

$$\int_0^{\pi} y^2(x,\lambda)dx = \int_0^{\pi} (y'\dot{y} - \dot{y}'y)'dx = W_{\pi}\{\dot{y}, y\} - W_0\{\dot{y}, y\}$$

formula kelib chiqadi. ■

(1.1) differensial tenglamaning quyidagi

$$\varphi(0,\lambda) = 1, \ \ \varphi'(0,\lambda) = h_0 + h_1\lambda$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini $\varphi(x,\lambda)$ orqali belgilaymiz. $\varphi(x,\lambda)$ yechim (1.2) chegaraviy shartlardan birinchisini qanoatlantirishi ravshan. Bu yechimni chegaraviy shartlardan ikkinchisiga qo‘ysak, ushbu

$$\Delta(\lambda) \equiv \varphi'(\pi,\lambda) + (H_0 + H_1\lambda)\varphi(\pi,\lambda) = 0$$

tenglama hosil bo‘ladi. Bu tenglamaga (1.1)+(1.2) Shturm-Liuvill masalasining xarakteristik tenglamasi deyiladi. Uning ildizlari xos qiymatlardan iborat. $\varphi(x,\lambda_n)$ funksiya (1.1)+(1.2) Shturm-Liuvill masalasining xos funksiyasi bo‘ladi.

7-xossa. (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasi uchun ushbu

$$\int_0^\pi \varphi^2(x,\lambda_n)dx = -\varphi(\pi,\lambda_n)\dot{\Delta}(\lambda_n) + H_1\varphi^2(\pi,\lambda_n) + h_1 \qquad (1.8)$$

tenglik o‘rinli bo‘ladi.

Isbot. 6-xossada $y(x,\lambda)$ yechim o‘rnida $\varphi(x,\lambda)$ yechimni olib, $\lambda=\lambda_n$ desak, ushbu

$$\int_0^\pi \varphi^2(x,\lambda_n)dx = \begin{vmatrix} \dot{\varphi}(\pi,\lambda_n) & \varphi(\pi,\lambda_n) \\ \dot{\varphi}'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n) \end{vmatrix} - \begin{vmatrix} \dot{\varphi}(0,\lambda_n) & \varphi(0,\lambda_n) \\ \dot{\varphi}'(0,\lambda_n) & \varphi'(0,\lambda_n) \end{vmatrix} =$$

$$= \begin{vmatrix} \dot{\varphi}(\pi,\lambda_n) & \varphi(\pi,\lambda_n) \\ \dot{\varphi}'(\pi,\lambda_n) & -(H_0+H_1\lambda_n)\varphi(\pi,\lambda_n) \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ h_1 & h_0+h_1\lambda_n \end{vmatrix} =$$

$$= -\varphi(\pi,\lambda_n)[\dot{\varphi}'(\pi,\lambda_n) + (H_0+H_1\lambda_n)\dot{\varphi}(\pi,\lambda_n)] + h_1 \qquad (1.9)$$

tenglik hosil bo‘ladi. Quyidagi

$$\Delta(\lambda) \equiv \varphi'(\pi,\lambda) + (H_0+H_1\lambda)\varphi(\pi,\lambda),$$

$$\dot{\Delta}(\lambda) \equiv \dot{\varphi}'(\pi,\lambda) + (H_0+H_1\lambda)\dot{\varphi}(\pi,\lambda) + H_1\varphi(\pi,\lambda),$$

ifodalarni hisobga olsak, (1.9) tenglik ushbu

$$\int_0^\pi \varphi^2(x,\lambda_n)dx = -\varphi(\pi,\lambda_n)[\dot{\Delta}(\lambda_n) - H_1\varphi(\pi,\lambda_n)] + h_1$$

ko‘rinishni oladi. ■

8-xossa. (1.1)+(1.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari oddiy (karrasiz) bo‘ladi, ya’ni $\dot{\Delta}(\lambda_n)\neq 0$ bo‘ladi.

Isbot. Teskarisini faraz qilaylik, ya’ni $\dot{\Delta}(\lambda_n)=0$ bo‘lsin. U holda (1.8) tenglikka ko‘ra ushbu

$$\int_0^\pi \varphi^2(x,\lambda_n)dx = H_1\varphi^2(\pi,\lambda_n) + h_1$$

munosabat bajariladi. Bu yerda $h_1 < 0$ va $H_1 < 0$ ekanligini hisobga olsak, ziddiyat hosil bo‘ladi. ■

Misol. Ushbu

$$\begin{cases} -y'' = \lambda y, \quad 0 \le x \le \pi \\ y'(0) = h_1 \lambda y(0), \\ y'(\pi) = -H_1 \lambda y(\pi), \end{cases} \tag{1.10}$$

chegaraviy masalaning xos qiymatlarini izlaymiz. Shu maqsadda (1.10) chegaraviy masaladagi differensial tenglamaning quyidagi

$$\varphi(0,\lambda) = 1, \;\; \varphi'(0,\lambda) = h_1 \lambda$$

boshlang‘ich shartlarni qanoatlantiruvchi $\varphi(x,\lambda)$ yechimini aniqlaymiz:

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + h_1\sqrt{\lambda}\sin\sqrt{\lambda}x .$$

Topilgan bu yechim chegaraviy shartlardan birinchisini qanoatlantirishi ravshan. Bu yechimni chegaraviy shartlardan ikkinchisiga qo‘yib, xarakteristik tenglamani keltirib chiqaramiz:

$$\Delta(\lambda) \equiv \varphi'(\pi,\lambda) + H_1\lambda\varphi(\pi,\lambda) =$$

$$= -\sqrt{\lambda}\sin\sqrt{\lambda}\pi + h_1\lambda\cos\sqrt{\lambda}\pi + H_1\lambda(\cos\sqrt{\lambda}\pi + h_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi) = 0,$$

$$\Delta(\lambda) = \sqrt{\lambda}[(h_1 + H_1)\sqrt{\lambda}\cos\sqrt{\lambda}\pi + (h_1 H_1 \lambda - 1)\sin\sqrt{\lambda}\pi] = 0.$$

Bunga ko‘ra

$$\lambda = 0 \text{ yoki } (h_1 + H_1)ctg\sqrt{\lambda}\pi = \frac{1}{\sqrt{\lambda}} - h_1 H_1\sqrt{\lambda} .$$

Bu holda $\lambda = 0$ eng kichik xos qiymat bo‘lishini isbotlaymiz. Buning uchun manfiy xos qiymat yo‘q ekanligini ko‘rsatish kerak. Teskarisini faraz qilaylik, $\lambda = -a^2$ manfiy xos qiymat bo‘lsin. Bu yerda $a > 0$. U holda

$$\sqrt{\lambda} = ia, \;\; ctg\sqrt{\lambda}\pi = \frac{\cos ia\pi}{\sin ia\pi} = \frac{i(e^{-a\pi} + e^{a\pi})}{e^{-a\pi} - e^{a\pi}} = -icth a\pi$$

bo‘lgani uchun

$$(h_1 + H_1)cth a\pi = \frac{1}{a} + h_1 H_1 a$$

bo‘ladi. Ziddiyat, chunki bu tenglikning chap tomoni manfiy, o‘ng tomoni esa musbat. Musbat xos qiymatlar cheksiz ko‘p ekanligini ko‘rsatish maqsadida $\sqrt{\lambda}=t$, $(t>0)$ belgilash kiritib olamiz. Ushbu

$$(h_1+H_1)ctg\pi t-\frac{1}{t}+h_1H_1t=0 \qquad (1.11)$$

tenglamaning chap tomoni har bir $(n-1,n), n=1,2,...$ oraliqda o‘suvchi bo‘lgani uchun bu oraliqlarning har birida (1.11) tenglama yagona $t_n\in(n-1,n)$ ildizga ega bo‘ladi. Agar $t_n=n-1+\delta_n$ belgilash kiritsak, $\delta_n\in(0,1)$ bo‘ladi.

Demak, (1.10) masalaning xos qiymatlari

$$\lambda_0=0,\ \lambda_n=(n-1+\delta_n)^2,\ n=1,2,...$$

bo‘lib, ularga ushbu

$$y_0(x)=1,\ y_n(x)=\cos(n-1+\delta_n)x+h_1(n-1+\delta_n)\sin(n-1+\delta_n)x,\ n=1,2,...$$

xos funksiyalar mos keladi.

2-§. Shturm teoremasi va undan kelib chiqadigan natijalar

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly\equiv -y''+q(x)y=\lambda y,\quad x\in[0,\pi], \qquad (2.1)$$

$$\begin{cases} y'(0)=(h_0+h_1\lambda)y(0), \\ y'(\pi)=-(H_0+H_1\lambda)y(\pi). \end{cases} \qquad (2.2)$$

Bu yerda $q(x)\in C^1[0,\pi]$ haqiqiy qiymatli funksiya, h_0, h_1, H_0, H_1 haqiqiy sonlar, λ esa kompleks parametr. Biz $h_1<0$ va $H_1<0$ bo‘lgan holni o‘rganamiz.

2.1-teorema. (*Shturm*). Quyidagi ikkita

$$-y''+a(x)y=0,\ (0<x<\pi), \qquad (2.3)$$

$$-y''+\tilde{a}(x)y=0,\ (0<x<\pi) \qquad (2.4)$$

tenglama berilgan bo‘lib, $a(x)>\tilde{a}(x),\ x\in[0,\pi]$ bo‘lsin. $\varphi(x)$ va $\tilde{\varphi}(x)$ mos ravishda (2.3) va (2.4) tenglamalarning noldan farqli ixtiyoriy yechimlari bo‘lsin.

U holda $\varphi(x)$ funksiyaning ixtiyoriy ikkita ildizi orasida $\tilde{\varphi}(x)$ funksiyaning kamida bitta ildizi bo‘ladi.

Isbot. Ushbu

$$-\varphi'' + a(x)\varphi = 0, \tag{2.5}$$

$$-\tilde{\varphi}'' + \tilde{a}(x)\tilde{\varphi} = 0 \tag{2.6}$$

ayniyatlarni mos ravishda $\tilde{\varphi}(x)$ va $\varphi(x)$ funksiyalarga ko‘paytirib, birinchisidan ikkinchisini ayirsak,

$$\tilde{\varphi}''\varphi - \varphi''\tilde{\varphi} + [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi} = 0,$$

$$\varphi''\tilde{\varphi} - \tilde{\varphi}''\varphi = [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi},$$

$$(\varphi'\tilde{\varphi} - \tilde{\varphi}'\varphi)' = [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi} \tag{2.7}$$

kelib chiqadi.

x_1 va x_2 orqali $\varphi(x)$ funksiyaning ketma-ket kelgan ixtiyoriy ikkita ildizini belgilaymiz. (2.7) ayniyatni $[x_1, x_2]$ oraliqda integrallasak, ushbu

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) = \int_{x_1}^{x_2} [a(x) - \tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx \tag{2.8}$$

tenglik hosil bo‘ladi.

$\tilde{\varphi}(x)$ funksiya (x_1, x_2) intervalda ildizga ega emas deb faraz qilaylik. Qulaylik uchun (x_1, x_2) intervalda $\varphi(x) > 0$, $\tilde{\varphi}(x) > 0$ deb hisoblashimiz mumkin. U holda (2.8) tenglikning o‘ng tomoni musbat bo‘ladi. $\varphi(x_1) = 0$, $\varphi(x_2) = 0$, $\varphi(x) > 0$, $x \in (x_1, x_2)$ bo‘lgani uchun $\varphi'(x_1) \geq 0,$ $\varphi'(x_2) \leq 0$ bo‘ladi.

Agar $\varphi'(x_1) = 0$ yoki $\varphi'(x_2) = 0$ bo‘lsa, yagonalik teoremasiga ko‘ra $\varphi(x) \equiv 0$ bo‘ladi. Demak, $\varphi'(x_1) > 0$ va $\varphi'(x_2) < 0$ bo‘lar ekan. Bu tengsizliklardan quyidagi

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) \leq 0$$

baholash kelib chiqadi. Oxirgi tengsizlikdan ziddiyat kelib chiqadi, chunki (2.8) tenglikka asosan ushbu

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) > 0$$

tengsizlik o‘rinli bo‘ladi. ■

2.2-teorema. (*Taqqoslash teoremasi*). Quyidagi ikkita

$$-y'' + a(x)y = 0, \quad (0 < x < \pi) \tag{2.9}$$

$$-y'' + \tilde{a}(x)y = 0, \quad (0 < x < \pi) \tag{2.10}$$

tenglama berilgan bo‘lib, $a(x) > \tilde{a}(x),\ x \in [0,\pi]$ bo‘lsin. $\varphi(x)$ va $\tilde{\varphi}(x)$ mos ravishda (2.9) va (2.10) tenglamalarning noldan farqli bir xil boshlang‘ich shartlarni qanoatlantiruvchi haqiqiy qiymatli yechimlari bo‘lsin. U holda

1) agar $\varphi(x)$ yechimning $(0,\pi]$ yarim intervalda m ta ildizi bo‘lsa, $\tilde{\varphi}(x)$ yechimning $(0,\pi]$ yarim intervaldagi ildizlari soni m tadan kam bo‘lmaydi;

2) $\tilde{\varphi}(x)$ funksiyaning k nomerli ildizi $\varphi(x)$ funksiyaning k nomerli ildizidan kichik bo‘ladi.

Isbot. Mavjudlik va yagonalik teoremasining shartlari bajarilganda, ikkinchi tartibli bir jinsli differensial tenglama noldan farqli ixtiyoriy yechimining ildizlari chekli limitik nuqtaga ega bo‘lmasligini ko‘rsatamiz. Teskarisini faraz qilaylik, ya’ni $y(x_n) = 0$ bo‘lib, $x_n \to a, \quad (n \to \infty)$ bo‘lsin. U holda $y(a) = 0$ va

$$y'(a) = \lim_{x_n \to a} \frac{y(x_n) - y(a)}{x_n - a} = 0$$

bo‘ladi. Mavjudlik va yagonalik teoremasiga ko‘ra $y(x) \equiv 0$ bo‘ladi, bu esa $y(x) \not\equiv 0$ shartga zid.

Demak, $y(x)$ yechimning chekli oraliqdagi ildizlari orasida eng kattasi va eng kichigi hamisha mavjud bo‘ladi.

$\varphi(x)$ yechimning $(0,\pi]$ yarim intervaldagi eng kichik ildizini x_1 orqali belgilaylik. $\tilde{\varphi}(x)$ yechim $(0, x_1]$ intervalda kamida bitta ildizga ega ekanligini ko‘rsatamiz. Teskarisini faraz qilamiz, ya’ni $\tilde{\varphi}(x)$ yechim $(0, x_1]$ intervalda ildizga ega emas deb faraz qilamiz. Umumiylikni buzmasdan, qulaylik uchun $(0, x_1]$ intervalda $\varphi(x) > 0$ va $\tilde{\varphi}(x) > 0$ deb hisoblaymiz. $\varphi(x_1) = 0$ bo‘lgani uchun x_1 nuqtaning yetarlicha kichik atrofida $\varphi(x)$ kamayuvchi bo‘ladi, bundan $\varphi'(x_1) \le 0$ kelib chiqadi. Ushbu

$$(\varphi'\tilde{\varphi}-\tilde{\varphi}'\varphi)'=[a(x)-\tilde{a}(x)]\varphi\tilde{\varphi}$$

ayniyatni $[0,x_1]$ oraliqda integrallaymiz:

$$[\varphi'(x_1)\tilde{\varphi}(x_1)-\tilde{\varphi}'(x_1)\varphi(x_1)]-[\varphi'(0)\tilde{\varphi}(0)-\tilde{\varphi}'(0)\varphi(0)]=\int_0^{x_1}[a(x)-\tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx. \tag{2.11}$$

Boshlang‘ich shartlarni va $\varphi(x_1)=0$ tenglikni e’tiborga olsak,

$$\varphi'(x_1)\tilde{\varphi}(x_1)=\int_0^{x_1}[a(x)-\tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx \tag{2.12}$$

kelib chiqadi. Ziddiyat, chunki tenglikning chap tomoni noldan oshmaydi, o‘ng tomoni esa noldan katta.

Demak, $\tilde{\varphi}(x)$ funksiya $(0,x_1)$ intervalda kamida bitta ildizga ega ekan. Bu ildizlardan bittasini $\tilde{x}_1$ bilan belgilaylik, $\varphi(x)$ funksiyaning $(0,\pi]$ yarim intervaldagi ildizlarini

$$x_1<x_2<\ldots<x_m$$

orqali belgilaymiz. Shturm teoremasiga ko‘ra ushbu

$$(x_1,x_2),(x_2,x_3),\ldots,(x_{m-1},x_m)$$

intervallarning har birida $\tilde{\varphi}(x)$ funksiya kamida bitta ildizga ega. Har bir intervaldan bittadan ildiz olib, ularni

$$\tilde{x}_2<\ldots<\tilde{x}_m$$

orqali belgilaymiz. Bu holda quyidagi tengsizliklar o‘rinli bo‘ladi

$$0<\tilde{x}_1<x_1<\tilde{x}_2<x_2<\ldots<x_{m-1}<\tilde{x}_m<x_m\le\pi .$$

Bu tengsizliklardan $\tilde{\varphi}(x)$ funksiyaning ildizlari soni m tadan kam emasligi va $\tilde{\varphi}(x)$ funksiyaning k nomerli ildizi $\varphi(x)$ funksiyaning k nomerli ildizidan kichik bo‘lishi kelib chiqadi. ■

2.3-teorema. Quyidagi

$$-y''+a(x)y=0,\ \ (0<x<\pi) \tag{2.13}$$

tenglama berilgan bo‘lib, $\varphi(x)$ va $\tilde{\varphi}(x)$ uning ikkita chiziqli erkli yechimlari bo‘lsin. U holda agar $\varphi(x)$ funksiya $(0,\pi)$ intervalda kamida ikkita ildizga ega bo‘lsa, $\varphi(x)$ funksiyaning ketma-ket kelgan ixtiyoriy ikkita ildizi orasida $\tilde{\varphi}(x)$ funksiyaning yagona ildizi bo‘ladi, ya’ni (2.13) tenglama chiziqli erkli yechimlarining ildizlari almashinib keladi.

Isbot. x_1 va x_2 orqali $\varphi(x)$ funksiyaning ketma-ket kelgan ikkita ildizini belgilaymiz va $\tilde{\varphi}(x)$ funksiya $(x_1,\, x_2)$ intervalda ildizga ega emas deb faraz qilamiz. U holda $f(x)=\dfrac{\varphi(x)}{\tilde{\varphi}(x)}$ funksiya $(x_1,\, x_2)$ intervalda uzluksiz bo‘lib, uning chetlarida nolga aylanadi. Chekli orttirmalar haqidagi Lagranj teoremasiga ko‘ra shunday $x_0 \in (x_1,\, x_2)$ topiladiki, bunda $f'(x_0)=0$ bo‘ladi, ya’ni

$$f'(x_0)=\frac{\varphi'(x_0)\tilde{\varphi}(x_0)-\varphi(x_0)\tilde{\varphi}'(x_0)}{\tilde{\varphi}^2(x_0)}=\frac{W\{\tilde{\varphi}(x_0),\,\varphi(x_0)\}}{\tilde{\varphi}^2(x_0)}=0\,.$$

Bu esa ziddiyat, chunki $\varphi(x)$ va $\tilde{\varphi}(x)$ chiziqli erkli yechimlardir. ■

$\varphi(x,\lambda)$ orqali (2.1) tenglamaning noldan farqli yechimini belgilaymiz. Ushbu

$$\varphi(x,\lambda)=0, \quad (0\le x\le \pi)$$

tenglamaning ildizlari λ parametrga nisbatan funksiya bo‘lishi ravshan. Bu funksiyalar λ parametrga nisbatan uzluksiz bo‘lishini isbotlaymiz.

2.4-teorema. Agar $x_0 \in (0,\pi)$ son $\varphi(x,\lambda_0)$ funksiyaning ildizi bo‘lsa, ixtiyoriy musbat $\varepsilon>0$ son uchun shunday $\delta>0$ son topiladiki, bunda $|\lambda-\lambda_0|<\delta$ bo‘lganda $\varphi(x,\lambda)$ funksiya $|x-x_0|<\varepsilon$ intervalda faqat bitta ildizga ega bo‘ladi.

Isbot. Agar x_0 son $\varphi(x,\lambda_0)$ funksiyaning karrali ildizi bo‘lsa, ya’ni $\varphi(x_0,\lambda_0)=0$ va $\varphi'(x_0,\lambda_0)=0$ bo‘lsa, mavjudlik va yagonalik teoremasiga asosan $\varphi(x,\lambda_0)\equiv 0$ bo‘ladi. Teorema shartiga ko‘ra bunday bo‘lishi mumkin emas. Demak, $\varphi'(x_0,\lambda_0)\ne 0$ ekan. Aniqlik uchun $\varphi'(x_0,\lambda_0)>0$ deb hisoblaymiz.

Uzluksiz funksiyaning xossalariga ko‘ra bu tengsizlik x_0 nuqtaning yetarlicha kichik atrofida ham bajariladi, ya’ni $\varepsilon>0$ yetarlicha kichik bo‘lib, $|x-x_0|\le\varepsilon$ bo‘lsa, $\varphi'(x,\lambda_0)>0$ bo‘ladi. Bundan $\varphi(x_0-\varepsilon,\lambda_0)<0$, $\varphi(x_0+\varepsilon,\lambda_0)>0$ bo‘lishi kelib chiqadi.

$\varphi'(x,\lambda)$ funksiya λ ga nisbatan uzluksiz bo‘lgani uchun (u xatto λ ga nisbatan butun funksiya bo‘lishini bilamiz) shunday $\delta > 0$ son topiladiki, $|\lambda - \lambda_0| \le \delta$ bo‘lib, $|x - x_0| \le \varepsilon$ bo‘lsa, $\varphi'(x,\lambda) > 0$ bo‘ladi. Demak, $|\lambda - \lambda_0| \le \delta$ bo‘lsa, $\varphi(x,\lambda)$ funksiya $|x - x_0| \le \varepsilon$ oraliqda monoton o‘suvchi funksiya ekan. Bundan $|\lambda - \lambda_0| \le \delta$ bo‘lganda $\varphi(x,\lambda)$ funksiya $|x - x_0| \le \varepsilon$ oraliqda ko‘pi bilan bitta ildizga ega bo‘lishi mumkinligi kelib chiqadi.

$\varphi(x_0 - \varepsilon, \lambda)$ va $\varphi(x_0 + \varepsilon, \lambda)$ funksiyalar uzluksiz bo‘lib, $\varphi(x_0 - \varepsilon, \lambda_0) < 0$, $\varphi(x_0 + \varepsilon, \lambda_0) > 0$ bo‘lgani uchun λ_0 sonning shunday atrofi topiladiki, bunda $\varphi(x_0 - \varepsilon, \lambda) < 0$ va $\varphi(x_0 + \varepsilon, \lambda) > 0$ bo‘ladi. Bu fikrdan esa, $|\lambda - \lambda_0| < \delta$ bo‘lganda, $\varphi(x,\lambda)$ funksiya $|x - x_0| < \varepsilon$ intervalda kamida bitta ildizga ega bo‘lishi kelib chiqadi. ■

Natija. $\lambda_1 < \lambda_2$ bo‘lsa, $\varphi(x,\lambda_1)$ yechim ildizlarining soni $\varphi(x,\lambda_2)$ yechim ildizlarining sonidan oshmaydi, haqiqatan ham, bu holda $a(x) = q(x) - \lambda_1$, $\tilde{a}(x) = q(x) - \lambda_2$ deb olsak, $\lambda_1 < \lambda_2$ bo‘lgani uchun $a(x) > \tilde{a}(x)$ bo‘ladi. Taqqoslash teoremasiga ko‘ra, $\varphi(x,\lambda_2)$ funksiyaning $(0,\pi]$ yarim intervaldagi ildizlari soni $\varphi(x,\lambda_1)$ funksiyaning $(0,\pi]$ yarim intervaldagi ildizlari sonidan kam emas. Bu fikrdan λ oshgani sari $\varphi(x,\lambda)$ yechimning $(0,\pi]$ oraliqdagi ildizlari soni kamaymasligi (ortib borishi) kelib chiqadi.

λ parametr ortgani sari $\varphi(x,\lambda)$ yechimning $(0,\pi]$ oraliqdagi ildizlari nol nuqtaga tomon harakatlanadilar, ya’ni yangi ildiz π nuqta orqali kirib keladi.

2.5-teorema. (*Ossillyatsiya teoremasi*). (2.1)+(2.2) Shturm-Liuvill chegaraviy masalasi cheksizta $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ xos qiymatlarga ega. Bundan tashqari λ_n xos qiymatga mos keluvchi $y_n(x)$ xos funksiya $(0,\pi)$ intervalda n ta ildizga ega bo‘ladi.

Isbot. $\varphi(x,\lambda)$ orqali (2.1) tenglamaning

$$\varphi(0,\lambda) = 1, \;\; \varphi'(0,\lambda) = h_0 + h_1\lambda \tag{2.14}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaylik. Bu yerda λ haqiqiy parametr. $|q(x)| < m$, $x \in [0,\pi]$ bo‘lsin. Taqqoslash teoremasida

$$a(x) = q(x) - \lambda, \;\; \tilde{a}(x) = -m - \lambda$$

desak, $a(x) > \tilde{a}(x)$ buladi. Ushbu

$$-y'' + (-m - \lambda)y = 0$$

tenglamaning

$$\tilde{\varphi}(0,\lambda) = 1, \;\; \tilde{\varphi}'(0,\lambda) = h_0 + h_1\lambda \tag{2.15}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimi

$$\tilde{\varphi}(x,\lambda) = ch\sqrt{-m-\lambda}\,x + (h_0 + h_1\lambda)\frac{sh\sqrt{-m-\lambda}\,x}{\sqrt{-m-\lambda}}$$

formula bilan beriladi. $\tilde{\varphi}(x,\lambda)$ yechimning ifodasiga asosan shunday $M > 0$ son mavjudki $\lambda < -M$ bo‘lganda $\tilde{\varphi}(x,\lambda)$ yechim $(0,\pi]$ oraliqda ildizga ega bo‘lmaydi. Bundan, taqqoslash teoremasiga muvofiq $\lambda < -M$ bo‘lganda $\varphi(x,\lambda)$ yechimning ham ildizi yo‘q ekanligi kelib chiqadi. CHunki $\tilde{\varphi}(x,\lambda)$ funksiyaning ildizlari soni $\varphi(x,\lambda)$ funksiyaning ildizlari sonidan kam bo‘lmaydi.

Agar taqqoslash teoremasida $\tilde{a}(x) = q(x) - \lambda$, $a(x) = m - \lambda$ deb olsak, $a(x) > \tilde{a}(x)$ buladi. U holda

$$-y'' + (m - \lambda)y = 0$$

tenglamaning (2.14) boshlang‘ich shartlarni qanoatlantiruvchi yechimi ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda - m}\,x + (h_0 + h_1\lambda)\frac{\sin\sqrt{\lambda - m}\,x}{\sqrt{\lambda - m}}$$

formula bilan beriladi. λ parametr cheksiz oshgani sari bu yechimning $(0,\pi]$ oraliqdagi ildizlari soni ortib boraveradi. Taqqoslash teoremasiga asosan λ parametr cheksiz oshgani sari ushbu

$$-y'' + [q(x) - \lambda]y = 0$$

tenglamaning (2.15) boshlang‘ich shartlarni qanoatlantiruvchi $\tilde{\varphi}(x,\lambda)$ yechimining ham $(0,\pi]$ oraliqdagi ildizlar soni ortib boraveradi.

Ushbu $\tilde{\varphi}(x,\lambda)=0$ tenglamani ko‘rib chiqamiz. 4-teoremaga asosan bu tenglamaning ildizlari λ parametrga uzluksiz ravishda bog‘liq bo‘ladi. $\tilde{\varphi}(x,\lambda)=0$ tenglamaning $(0,\infty)$ oraliqdagi ildizlarini $x_n(\lambda)$ orqali belgilaymiz. Ikkinchi tomondan λ parametr ortgani sari $\tilde{\varphi}(x,\lambda)$ yechimning ildizlari nol nuqta tomon harakat qilishadi, ammo nol nuqta orqali chiqib ketmaydilar (chunki ular musbat). $(0,\pi]$ oraliqqa yangi ildizlar π nuqta orqali kirib keladi. Boshqacha qilib aytganda λ parametrning shunday μ_n qiymati borki, bunda $x_n(\mu_n)=\pi$ bo‘ladi.

Demak, $\mu_0, \mu_1, \mu_2, \ldots$ sonlar uchun $\tilde{\varphi}(\pi,\mu_n)=0$ bo‘ladi. $\tilde{\varphi}(x,\mu_n)$ funksiyaning $(0,\pi)$ intervaldagi ildizlari

$$x_0(\mu_n), x_1(\mu_n), \ldots, x_{n-1}(\mu_n)$$

bo‘ladi, ya’ni $\tilde{\varphi}(x,\mu_n)$ funksiya $(0,\pi)$ intervalda n ta ildizga ega.

Endi ushbu $u(x)=\tilde{\varphi}(x,\tilde{\lambda}_1), \quad v(x)=\tilde{\varphi}(x,\tilde{\lambda}_2)$ funksiyalarni kiritib olamiz. Bu yerda $\mu_n<\tilde{\lambda}_1<\tilde{\lambda}_2<\mu_{n+1}$.

Taqqoslash teoremasidan $u(x)$ va $v(x)$ funksiyalarning har biri $(0,\pi]$ yarim intervalda $n+1$ ta ildizga ega bo‘lishi kelib chiqadi.

$\tilde{\varphi}(\pi,\lambda)$ funksiyaning $\mu_0, \mu_1, \mu_2, \ldots$ sonlardan boshqa ildizi bo‘lmagani uchun $u(\pi)\neq 0,\ v(\pi)\neq 0$ bo‘ladi.

$u(x)$ funksiyaning $(0,\pi]$ oraliqdagi eng katta ildizini x_n bilan belgilaymiz. $v(x)$ funksiya $[x_n,\pi]$ kesmada ildizga ega emasligini ko‘rsatamiz. Taqqoslash teoremasiga ko‘ra $v(x)$ funksiyaning $(n+1)$-ildizi x_n dan kichik bo‘ladi, ya’ni $v(x)$ funksiya $(0,x_n)$ oraliqda $n+1$ ta ildizga ega. Agar $v(x)$ funksiyaning $[x_n,\pi]$ kesmada ildizi bor bo‘lsa, uning $(0,\pi]$ yarim intervaldagi ildizlari soni $n+1$ dan oshadi. Ziddiyat, chunki $v(x)$ funksiya $(0,\pi]$ da $n+1$ ta ildizga ega.

Quyidagi tengsizlik bajarilishi ravshan:

$$\frac{d}{dx}\left\{u^2\left(\frac{u'}{u}-\frac{v'}{v}\right)\right\}=2uu'\left(\frac{u'}{u}-\frac{v'}{v}\right)+u^2\left(\frac{u''}{u}-\frac{v''}{v}\right)-u^2\left(\frac{u'^2}{u^2}-\frac{v'^2}{v^2}\right)=$$

$$=\frac{(u'v-uv')^2}{v^2}+u^2(\tilde{\lambda}_2-\tilde{\lambda}_1)\geq 0\,. \qquad (2.16)$$

Bu tengsizlikni $[x_n,\pi]$ kesmada integrallasak, quyidagi

$$u^2(\pi)\left\{\frac{u'(\pi)}{u(\pi)}-\frac{v'(\pi)}{v(\pi)}\right\}-u^2(x_n)\left\{\frac{u'(x_n)}{u(x_n)}-\frac{v'(x_n)}{v(x_n)}\right\}>0,$$

$$u^2(\pi)\left\{\frac{u'(\pi)}{u(\pi)}-\frac{v'(\pi)}{v(\pi)}\right\}>0,$$

$$\frac{u'(\pi)}{u(\pi)}>\frac{v'(\pi)}{v(\pi)} \qquad (2.17)$$

tengsizlik kelib chiqadi. Demak, agar $\mu_n<\tilde{\lambda}_1<\tilde{\lambda}_2<\mu_{n+1}$ bo‘lsa, ushbu

$$\frac{\tilde{\varphi}'(\pi,\tilde{\lambda}_1)}{\tilde{\varphi}(\pi,\tilde{\lambda}_1)}>\frac{\tilde{\varphi}'(\pi,\tilde{\lambda}_2)}{\tilde{\varphi}(\pi,\tilde{\lambda}_2)}$$

tengsizlik o‘rinli ekan, ya’ni $\dfrac{\tilde{\varphi}'(\pi,\lambda)}{\tilde{\varphi}(\pi,\lambda)}$ funksiya (μ_n,μ_{n+1}) oraliqda kamayuvchi funksiya ekan. Bu oraliqda xarakteristik tenglamani ushbu

$$\frac{\tilde{\varphi}'(\pi,\lambda)}{\tilde{\varphi}(\pi,\lambda)}+H_0+H_1\lambda=0$$

tarzda yozib olamiz. Bu tenglamaning chap tomoni (μ_n,μ_{n+1}) oraliqda kamayuvchi funksiya bo‘ladi, chunki $H_0+H_1\lambda$ funksiya kamayuvchi. Bunga ko‘ra $\tilde{\varphi}(\pi,\mu_n)=0$ va $\tilde{\varphi}(\pi,\mu_{n+1})=0$ tengliklarga asosan ushbu

$$\frac{\tilde{\varphi}'(\pi,\lambda)}{\tilde{\varphi}(\pi,\lambda)}+H_0+H_1\lambda\to+\infty,\quad (\lambda\to\mu_n),$$

$$\frac{\tilde{\varphi}'(\pi,\lambda)}{\tilde{\varphi}(\pi,\lambda)}+H_0+H_1\lambda\to-\infty,\quad (\lambda\to\mu_{n+1})$$

munosabatlar o‘rinli bo‘ladi.

Demak, (μ_n,μ_{n+1}) intervalda shunday yagona λ_n son topiladiki, bunda quyidagi

$$\frac{\tilde{\varphi}'(\pi,\lambda_n)}{\tilde{\varphi}(\pi,\lambda_n)}+H_0+H_1\lambda_n=0 \qquad (2.18)$$

tenglik bajariladi. (2.18) tenglik (2.2) chegaraviy shartlarning ikkinchisi ham bajarilishini ko‘rsatadi, ya’ni $\tilde{\varphi}(x,\lambda_n)$ funksiya (2.1)+(2.2) masalaning xos funksiyasi va λ_n son xos qiymati bo‘ladi. $(0,\pi)$ intervalda $\tilde{\varphi}(x,\lambda_n)$ va $\tilde{\varphi}(x,\mu_n)$ funksiyalar ildizlarining soni bir xil bo‘lgani uchun $\tilde{\varphi}(x,\lambda_n)$ xos funksiya $(0,\pi)$ intervalda n ta ildizga ega. ■

3-§. Krum-Kreyn almashtirishi

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0,\pi], \tag{3.1}$$

$$\begin{cases} y'(0) = (h_0 + h_1\lambda)y(0), \\ y'(\pi) = -(H_0 + H_1\lambda)y(\pi). \end{cases} \tag{3.2}$$

Bu yerda $q(x) \in C^1[0,\pi]$ haqiqiy qiymatli funksiya, h_0, h_1, H_0, H_1 haqiqiy sonlar, λ esa kompleks parametr. Biz $h_1 < 0$ va $H_1 < 0$ bo‘lgan holni o‘rganamiz.

(3.1)+(3.2) masalaning xos qiymatlarini $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ orqali, ularga mos keluvchi xos funksiyalarni $y_0(x)$, $y_1(x)$, $y_2(x)$, $\ldots$ orqali belgilaymiz. Ostsillyatsiya teoremasiga ko‘ra $y_0(x)$ xos funksiya $(0,\pi)$ intervalda nolga aylanmaydi. Agar $y_0(0) = 0$ yoki $y_0(\pi) = 0$ bo‘lsa, (3.2) chegaraviy shartlardan $y_0(x) \equiv 0$ ziddiyat kelib chiqadi. Demak, $y_0(x)$ xos funksiya $[0,\pi]$ kesmada nolga aylanmaydi.

3.1-lemma. Ushbu

$$v(x) = \frac{y_0'(x)}{y_0(x)} \tag{3.3}$$

funksiya uchun quyidagi tengliklar bajariladi:

$$v' = q(x) - \lambda_0 - v^2, \quad v(0) = h_0 + h_1\lambda_0, \quad v(\pi) = -H_0 - H_1\lambda_0. \tag{3.4}$$

Isbot. (3.3) funksiyadan hosila olsak, ushbu

$$v' = \left(\frac{y_0'}{y_0}\right)' = \frac{y_0''y_0 - (y_0')^2}{y_0^2} = \frac{(q-\lambda_0)y_0^2 - (y_0')^2}{y_0^2} =$$

$$= q - \lambda_0 - \left(\frac{y_0'}{y_0}\right)^2 = q(x) - \lambda_0 - v^2$$

tenglikka ega bo‘lamiz. (3.2) chegaraviy shartlardan esa $v(0) = h_0 + h_1\lambda_0$ va $v(\pi) = -H_0 - H_1\lambda_0$ kelib chiqadi. ■

3.1-teorema. Ushbu $\lambda_n,\ n = 1, 2, ...$ sonlar quyidagi

$$\begin{cases} -z'' + \tilde{q}(x)z = \lambda z, \quad 0 < x < \pi \\ z'(0) - hz(0) = 0, \\ z'(\pi) + Hz(\pi) = 0 \end{cases} \tag{3.5}$$

Shturm-Liuvill masalasining xos qiymatlari bo‘ladi va ularga ushbu

$$z_n(x) = y_n'(x) - v(x)y_n(x),\ n = 1, 2, ... \tag{3.6}$$

xos funksiyalar mos keladi. Bu yerda

$$\tilde{q}(x) = q(x) - 2v'(x),\ \ h = -\frac{1}{h_1} - h_0 - h_1\lambda_0,\ H = -\frac{1}{H_1} - H_0 - H_1\lambda_0. \tag{3.7}$$

Isbot. (3.6) funksiyaning hosilalarini hisoblaymiz:

$$z_n' = y_n'' - v'y_n - vy_n' = (q - \lambda_n)y_n - (q - \lambda_0 - v^2)y_n - vy_n' = (\lambda_0 - \lambda_n + v^2)y_n - vy_n',$$

$$z_n'' = 2vv'y_n + (-\lambda_n + \lambda_0 + v^2)y_n' - v'y_n' - vy_n'' =$$

$$= 2vv'y_n + (-\lambda_n + \lambda_0 + v^2)y_n' - v'y_n' - v(q - \lambda_n)y_n =$$

$$= vy_n(2v' - q + \lambda_n) + (-\lambda_n + \lambda_0 + v^2 - v')y_n' =$$

$$= vy_n(2v' - q + \lambda_n) + (-2v' + q - \lambda_n)y_n' = (q - 2v' - \lambda_n)(y_n' - vy_n) = (\tilde{q} - \lambda_n)z_n.$$

Endi chegaraviy shartlarning birinchisini keltirib chiqaramiz:

$$\frac{z_n'(0)}{z_n(0)} = \frac{(\lambda_0 - \lambda_n + v^2(0))y_n(0) - v(0)y_n'(0)}{y_n'(0) - v(0)y_n(0)} =$$

$$= \frac{(\lambda_0 - \lambda_n)y_n(0)}{(h_0 + h_1\lambda_n)y_n(0) - v(0)y_n(0)} - v(0) =$$

$$= \frac{\lambda_0 - \lambda_n}{(h_0 + h_1\lambda_n) - (h_0 + h_1\lambda_0)} - (h_0 + h_1\lambda_0) = -\frac{1}{h_1} - h_0 - h_1\lambda_0.$$

Shu tarzda quyidagini amalga oshiramiz:

$$\frac{z_n'(\pi)}{z_n(\pi)}=\frac{(\lambda_0-\lambda_n+v^2(\pi))y_n(\pi)-v(\pi)y_n'(\pi)}{y_n'(\pi)-v(\pi)y_n(\pi)}=$$

$$=\frac{(\lambda_0-\lambda_n)y_n(\pi)}{y_n'(\pi)-v(\pi)y_n(\pi)}-v(\pi)=$$

$$=\frac{\lambda_0-\lambda_n}{-(H_0+H_1\lambda_n)-v(\pi)}-v(\pi)=$$

$$=\frac{\lambda_0-\lambda_n}{-(H_0+H_1\lambda_n)+(H_0+H_1\lambda_0)}+(H_0+H_1\lambda_0)=\frac{1}{H_1}+H_0+H_1\lambda_0.$$

Endi $z_n(x)=y_n'(x)-v(x)y_n(x)\not\equiv 0$ ekanligini ko‘rsatamiz. Agar $z_n(x)\equiv 0$ deb faraz qilsak, ushbu

$$z_n=y_n'-\frac{y_0'y_n}{y_0}=y_0\cdot\frac{y_n'y_0-y_0'y_n}{y_0^2}=y_0\cdot\left(\frac{y_n}{y_0}\right)' \tag{3.8}$$

tenglikdan $y_n=y_0C$ ziddiyat kelib chiqadi. Demak, $z_n(x)$ xos funksiya ekan. ■

Izoh. (3.5) Shturm-Liuvill chegaraviy masalasi λ_n, $n=1,2,...$ sonlardan boshqa xos qiymatga ega emas.

4-§. Xos qiymatlarning asimtotikasi

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly\equiv -y''+q(x)y=\lambda y,\quad x\in[0,\pi], \tag{4.1}$$

$$\begin{cases} y'(0)=(h_0+h_1\lambda)y(0), \\ y'(\pi)=-(H_0+H_1\lambda)y(\pi). \end{cases} \tag{4.2}$$

Bu yerda $q(x)\in C^1[0,\pi]$ haqiqiy qiymatli funksiya, h_0, h_1, H_0, H_1 haqiqiy sonlar, λ esa kompleks parametr. Biz $h_1<0$ va $H_1<0$ bo‘lgan holni o‘rganamiz.

(4.1)+(4.2) masalaning xos qiymatlarini $\lambda_0<\lambda_1<\lambda_2<...$ orqali, ularga mos keluvchi xos funksiyalarni $y_0(x)$, $y_1(x)$, $y_2(x)$, … orqali belgilaymiz.

4.1-teorema. (4.1)+(4.2) Shturm-Liuvill chegaraviy masalasi xos qiymatlari uchun quyidagi

$$\lambda_n = (n-1)^2 + c_0 + \frac{\gamma_n}{n}, \ \{\gamma_n\} \in l_2, \tag{4.3}$$

formula o‘rinli. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^\pi q(x)dx - \frac{2}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right). \tag{4.4}$$

Isbot. (3.5) Shturm-Liuvill masalasining xos qiymatlarini $\mu_0 < \mu_1 < \mu_2 < ...$ orqali belgilasak, ular uchun ushbu

$$\mu_n = n^2 + c_0 + \frac{\beta_n}{n}, \ \{\beta_n\} \in l_2,$$

formula o‘rinli bo‘ladi. Bu yerda

$$c_0 = \frac{2h + 2H}{\pi} + \frac{1}{\pi}\int_0^\pi \tilde{q}(x)dx.$$

Bundan ushbu

$$\lambda_n = (n-1)^2 + c_0 + \frac{\gamma_n}{n}, \ \{\gamma_n\} \in l_2,$$

tenglik kelib chiqadi. Bu yerda

$$c_0 = \frac{2h + 2H}{\pi} + \frac{1}{\pi}\int_0^\pi [q(x) - 2v'(x)]dx,$$

$$c_0 = \frac{2h + 2H}{\pi} + \frac{1}{\pi}\int_0^\pi q(x)dx - \frac{2}{\pi}[v(\pi) - v(0)],$$

$$c_0 = \frac{1}{\pi}\int_0^\pi q(x)dx + \frac{2}{\pi}[h + H + H_0 + H_1\lambda_0 + h_0 + h_1\lambda_0],$$

$$c_0 = \frac{1}{\pi}\int_0^\pi q(x)dx - \frac{2}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right).$$

Demak, (4.3), (4.4) tengliklar bajariladi. ■

5-§. Regulyarlashtirilgan izlar formulasi

Ilk bor 1953 yilda I.M.Gelfand va B.M.Levitan [1] tomonidan regulyar Shturm-Liuvill masalasi uchun regulyarlashtirilgan izlar formulasi keltirib chiqarilgan. Shundan so‘ng, L.A.Dikiy [2] Shturm-Liuvill operatorining barcha darajalari uchun regulyarlashtirilgan izlar formulasini keltirib chiqarishga muvaffaq bo‘lgan. Bu sohada keyingi muhim qadam V.B.Lidskiy va V.A.Sadovnichiy [3] tomonidan amalga oshirilgan. Hozirgi kunda regulyar Shturm-Liuvill masalasi uchun regulyarlashtirilgan izlar formulasini keltirib chiqarishning bir qator usullari mavjud [4-7]. Chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasining regulyarlashtirilgan izi N.J.Guliyevning [8] maqolasida va A.Y.Etkin, G.P.Etkinaning [9] maqolasida B.M.Levitan usuli yordamida hisoblangan.

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0,\pi], \tag{5.1}$$

$$\begin{cases} y'(0) = (h_0 + h_1\lambda)y(0), \\ y'(\pi) = -(H_0 + H_1\lambda)y(\pi). \end{cases} \tag{5.2}$$

Bu yerda $q(x) \in C^1[0,\pi]$ haqiqiy qiymatli funksiya, h_0, h_1, H_0, H_1 haqiqiy sonlar, λ esa kompleks parametr. Biz $h_1 < 0$ va $H_1 < 0$ bo‘lgan holni o‘rganamiz.

(5.1)+(5.2) masalaning xos qiymatlarini $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ orqali belgilaymiz.

5.1-teorema. (5.1)+(5.2) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi uchun quyidagi formula o‘rinli:

$$\lambda_0 + \sum_{n=1}^{\infty}(\lambda_n - (n-1)^2 - c_0) = -\frac{1}{4}\left[q(0) + q(\pi)\right] - \frac{1}{2}c_0 - \\ -\frac{1}{2h_1^2} - \frac{1}{2H_1^2} - \frac{h_0}{h_1} - \frac{H_0}{H_1}. \tag{5.3}$$

Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^{\pi} q(x)dx - \frac{2}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right). \tag{5.4}$$

Isbot. (3.5) Shturm-Liuvill masalasining regulyarlashtirilgan izi uchun quyidagi formula o‘rinli bo‘lishi yaxshi ma’lum:

$$\sum_{n=0}^{\infty}(\mu_n - n^2 - c_0) = \frac{1}{4}\left[\tilde{q}(0)+\tilde{q}(\pi)\right] - \frac{1}{2\pi}\int_0^{\pi}\tilde{q}(t)dt - \frac{h+H}{\pi} - \frac{h^2+H^2}{2}.$$

Agar bu yerda ushbu

$$\mu_n = \lambda_{n+1},\ n = 0,1,2,...,$$

$$v' = q(x) - \lambda_0 - v^2,\ v(0) = h_0 + h_1\lambda_0,\ v(\pi) = -H_0 - H_1\lambda_0,$$

$$\tilde{q}(x) = q(x) - 2v'(x),\ \tilde{q}(x) = 2\lambda_0 + 2v^2(x) - q(x)$$

$$h = -\frac{1}{h_1} - h_0 - h_1\lambda_0,\ H = -\frac{1}{H_1} - H_0 - H_1\lambda_0$$

bog‘lanishlarni ishlatsak, quyidagi tenglik kelib chiqadi

$$\sum_{n=0}^{\infty}(\lambda_{n+1} - n^2 - c_0) = -\frac{1}{4}\left[q(0)+q(\pi)\right] - \frac{1}{2\pi}\int_0^{\pi}q(t)dt +$$

$$+\lambda_0 + \frac{1}{2}(h_0 + h_1\lambda_0)^2 + \frac{1}{2}(H_0 + H_1\lambda_0)^2 - \frac{h^2+H^2}{2} - \frac{h_0 + h_1\lambda_0 + H_0 + H_1\lambda_0}{\pi} - \frac{h+H}{\pi}.$$

Bunga ko‘ra

$$\lambda_0 + \sum_{n=1}^{\infty}(\lambda_n - (n-1)^2 - c_0) = -\frac{1}{4}\left[q(0)+q(\pi)\right] - \frac{1}{2\pi}\int_0^{\pi}q(t)dt -$$

$$-\frac{1}{2}\left(\frac{1}{h_1^2} + \frac{1}{H_1^2}\right) - \left(\frac{h_0}{h_1} + \frac{H_0}{H_1}\right) + \frac{1}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right).$$

Bu yerda (5.4) tenglikni hisobga olsak, (5.3) izlar formulasi hosil bo‘ladi. ■

6-§. V.A.Ambartsumyan teoremasining analogi

Quyidagi Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz:

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0,\pi], \tag{6.1}$$

$$\begin{cases} y'(0) = h_1\lambda y(0), \\ y'(\pi) = -H_1\lambda y(\pi). \end{cases} \tag{6.2}$$

Bu yerda $q(x) \in C^1[0,\pi]$ haqiqiy qiymatli funksiya, λ kompleks parametr va $h_1 < 0$, $H_1 < 0$.

(6.1)+(6.2) masalaning xos qiymatlarini $\lambda_0 < \lambda_1 < \lambda_2 < ...$ orqali belgilaymiz.

Ushbu

$$\begin{cases} -y'' = \lambda y, \quad 0 \le x \le \pi \\ y'(0) = h_1 \lambda y(0), \\ y'(\pi) = -H_1 \lambda y(\pi), \end{cases} \tag{6.3}$$

chegaraviy masalaning xos qiymatlarini $\lambda_0^0 < \lambda_1^0 < \lambda_2^0 < ...$ orqali belgilaymiz. Birinchi paragrafda bu masalaning xos qiymatlarini o‘rgangan edik, xususan $\lambda_0^0 = 0$ bo‘lishini ko‘rgan edik.

6.1-teorema. Agar (6.1)+(6.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari (6.3) chegaraviy masalasining xos qiymatlari bilan ustma-ust tushsa, ya’ni $\lambda_n = \lambda_n^0$, $n = 0, 1, 2, ...$ bo‘lsa, u holda $q(x) \equiv 0$ bo‘ladi.

Isbot. 1) (6.1)+(6.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari uchun olingan asimptotik formulalarga ko‘ra ushbu

$$\lambda_n = (n-1)^2 + c_0 + \frac{\gamma_n}{n}, \ \{\gamma_n\} \in l_2, \tag{6.4}$$

formula o‘rinli. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^\pi q(x)dx - \frac{2}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right). \tag{6.5}$$

(6.3) chegaraviy masala uchun bu formulalar quyidagi k o‘rinishda b o‘ladi:

$$\lambda_n^0 = (n-1)^2 + c_0^0 + \frac{\gamma_n^0}{n}, \ \{\gamma_n^0\} \in l_2, \tag{6.6}$$

$$c_0^0 = -\frac{2}{\pi}\left(\frac{1}{h_1} + \frac{1}{H_1}\right). \tag{6.7}$$

Agar (6.4) tenglikdan (6.6) tenglikni ayirsak va $n \to \infty$ da limitga o‘tsak,

$$\int_0^\pi q(x)dx = 0 \tag{6.8}$$

kelib chikadi.

2) (6.1)+(6.2) Shturm-Liuvill chegaraviy masalasining $\lambda_0 = \lambda_0^0 = 0$ xos qiymatiga mos keluvchi xos funksiyasi $\varphi(x)$ bo‘lsin. U holda quyidagilar o‘rinli:

$$\varphi'' = q(x)\varphi, \qquad (6.9)$$

$$\varphi'(0) = 0, \qquad (6.10)$$

$$\varphi'(\pi) = 0, \qquad (6.11)$$

$$\varphi(x) \not\equiv 0. \qquad (6.12)$$

Ossillyatsiya teoremasiga ko‘ra $\varphi(x)$ xos funksiya $(0,\pi)$ intervalda nolga ega emas. Agar $\varphi(0) = 0$ yoki $\varphi(\pi) = 0$ bo‘lsa, u holda (6.10) yoki (6.11) chegaraviy shartlardan $\varphi(x) \equiv 0$ ziddiyat kelib chiqadi. Demak, $\varphi(x) \neq 0$, $x \in [0,\pi]$.

3) Quyidagi integralni ikki xil usulda hisoblaymiz:

$$J = \int_0^\pi \left(\frac{\varphi'}{\varphi}\right)' dx = \frac{\varphi'(\pi)}{\varphi(\pi)} - \frac{\varphi'(0)}{\varphi(0)} = 0. \qquad (6.13)$$

Ikkinchi tomondan, $\varphi'' = q(x)\varphi$ tenglikka ko‘ra

$$J = \int_0^\pi \left(\frac{\varphi'}{\varphi}\right)' dx = \int_0^\pi \frac{\varphi''\varphi - \varphi'^2}{\varphi^2} dx = \int_0^\pi \frac{\varphi''}{\varphi} dx - \int_0^\pi \left(\frac{\varphi'}{\varphi}\right)^2 dx = \int_0^\pi q(x)dx - \int_0^\pi \left(\frac{\varphi'}{\varphi}\right)^2 dx.$$

Bu yerda (6.8) va (6.13) tengliklarni inobatga olsak, ushbu

$$\int_0^\pi \left(\frac{\varphi'}{\varphi}\right)^2 dx = 0$$

munosabatga ega bo‘lamiz. Bunga ko‘ra $\varphi'(x) \equiv 0$ bo‘ladi. Demak, $\varphi(x) \equiv C \neq 0$ ekan. Buni $\varphi'' = q(x)\varphi$ tenglikka qo‘ysak, $0 = q(x)C$, ya’ni $q(x) \equiv 0$ kelib chiqadi. ■

I bob bo‘yicha xulosalar

Ushbu bobda chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasining xos qiymatlari va xos funksiyalarining xossalari o‘rganilgan. Jumladan, xos qiymatlarning xaqiqiyligi, karrasizligi va cheksiz ko‘pligi isbotlangan, Krum-Kreyn almashtirishi olingan, Krum-Kreyn almashtirishi yordamida xos qiymatlarning asimptotikasi va regulyarlashtirilgan izlar formulasi keltirib chiqarilgan. Bundan tashqari, chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun V.A.Ambartsumyan teoremasining analogi isbot qilingan. Xos funksiyalarning chiziqli erkliligi ko‘rsatilgan va ular uchun ossillyatsiya teoremasi isbotlangan.

II-BOB. CHEGARAVIY SHARTI SPEKTRAL PARAMETRGA RATSIONAL BOG‘LIQ BO‘LGAN SHTURM-LIUVILL MASALASI XOS QIYMATLARINING XOSSALARI

1-§. Masalaning qo‘yilishi va Herlots-Nevanlinna funksiyalarining xossalari

Quyidagi masalani ko‘rib chiqamiz

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1] \tag{2.1}$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha \in [0,\pi) \tag{2.2}$$

$$\frac{y'(1)}{y(1)} = f(\lambda). \tag{2.3}$$

Bu yerda $q(x) \in C[0,1]$ - haqiqiy funksiya bo’lib, quyidagi tasvirga ega:

$$f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k} \tag{2.4}$$

$$a \geq 0, \quad b_k \geq 0, \quad b \in R, \quad c_1 < c_2 < \ldots < c_N, \quad N \geq 0. \tag{2.5}$$

Maskur ishda (2.1)-(2.3) masala xos qiymatlarining xossalari, xususan asimptotikasi, xos funksiyalar uchun Krum-Kreyn tipidagi almashtirish o‘rganiladi.

Berilgan N uchun (2.4) ko‘rinishdagi (2.5) shartlarni qanoatlantiruvchi barcha ratsional funksiyalarni R_N orqali belgilaymiz.

Ta’rif. Agar $f:C\to C$ kompleks o‘zgaruvchili funksiya uchun ushbu

1. $\overline{f(z)}=f(\bar{z})$
2. $\operatorname{Im}z\geq 0\Rightarrow \operatorname{Im}f(z)\geq 0$

shartlar bajarilsa, $f(z)$ funksiyaga Herlots-Nevanlinna funksiyasi deyiladi.

Lemma1. Qutb maxsus nuqtalari haqiqiy va oddiy bo‘lgan ratsional funksiya Herlots-Nevanlinna funksiyasi bo‘lishi uchun biror N topilib, $f(z)\in R_N$ bo‘lishi zarur va yetarli.

Isbot. (Yetarliligi). Agar $f(z)\in R_N$ bo‘lsa, u holda ushbu

$$\overline{f(z)}=\bar{a}\cdot\bar{z}+\bar{b}+\sum_{k=1}^{N}\frac{b_k}{\bar{z}-\bar{c}_k}=a\bar{z}+b-\sum_{k=1}^{N}\frac{b_k}{\bar{z}-c_k}=f(\bar{z})$$

tenglik bajariladi. $\operatorname{Im}z\geq 0$ bo‘lsin, ya’ni $z=u+iv,\ v\geq 0$ bo`lsin. U holda

$$f(z)=au+iav+b-\sum_{k=1}^{N}\frac{b_k}{(u-c_k)-iv}=$$

$$=au+b-\sum_{k=1}^{N}\frac{b_k(u-c_k)}{(u-c_k)^2+(v)^2}+i\left\{av+\sum_{k=1}^{N}\frac{b_kv}{(u-c_k)^2+v^2}\right\},$$

$$\operatorname{Im}f(z)=a+\sum_{k=1}^{N}\frac{b_k}{(u-c_k)^2+v^2},\ \cdot v\geq 0.$$

(Zarurligi). $f(z)$ ratsional funksiya bo‘lgani uchun $f(z)=\dfrac{Q(z)}{P(z)}$ bo‘ladi. Bu yerda $Q(z)$va $P(z)$ ko‘phadlar. Agar qutblarni oddiyligini hisobga olsak va bu funksiyani sodda kasrlar yig‘indisiga yoysak, ushbu

$$f(z)=P(z)+az+b-\sum_{k=1}^{N}\frac{b_k}{z-c_k}$$

tenglikni olamiz. Bu yerda $P(z)$ ko‘phad bo‘lib, undagi hadlar darajasi $k\geq 2$ bo‘ladi. $P(z)\equiv 0$ ekanligini isbotlaymiz. $\operatorname{Im}z\geq 0$ bo‘lgani uchun $z=re^{i\varphi}$,

$0 \le \varphi \le \pi$ bo‘ladi. Bunga ko`ra $P(z)$ ko‘phadning hadlari uchun quyidagilar o`rinli:

$$a_k z^k = r_k e^{i\alpha_k} r^k e^{ik\varphi} = r_k r^k [\cos(k\varphi + \alpha_k) + i \sin(k\varphi + \alpha_k)],$$

$$\operatorname{Im}\{a_k z^k\} = r_k r^k \sin(k\varphi + \alpha_k).$$

Agar $0 \le \alpha_k < \pi$ bo‘lsa, u holda

$$0 < \frac{\pi - \alpha_k}{k} < \varphi < \frac{2\pi - \alpha_k}{k} \le \frac{2\pi - \alpha_k}{2} = \frac{\pi - \alpha_k}{2} < \pi$$

bo‘ladi. Bundan esa $\operatorname{Im}\{a_k z^k\} < 0$ ziddiyat kelib chiqadi. Qolgan holler ham shu tarzda qaraladi. Demak, $P(z) \equiv 0$ ekan.

Agar $a < 0$ bo‘lsa, $z = iv$ deb, $v \to +\infty$ desak, $\operatorname{Im} f(z) \to -\infty$ bo‘ladi. Demak, $a \ge 0$ ekan.

$b_1 > 0$ ekanligini isbotlaymiz. Agar $z = c_1 + iv,\ v \to 0\ (v > 0)$ desak, u holda

$$av + \frac{b_1 v}{v^2} + \frac{b_2 v}{(c_1 + c_2)^2 + v^2} + \cdots + \frac{b_N v}{(c_1 + c_N)^2 + v^2} \to b_1 \cdot \infty .$$

Bu limitning qiymati aslida $+\infty$ bo`lishi lozim. Bunga ko`ra $b_1 > 0$ bo`ladi. $b_2 > 0$, …, $b_N > 0$ tengsizliklar ham shu tarzda ko`rsatiladi. **Lemma isbotlandi.**

Quyidagi belgilashlarni kiritib olamiz:

$$R_N^+ = \{ f(z) \in R_N, a > 0\}, \quad R_N^0 = \{ f(z) \in R_N, a = 0\}.$$

Lemma 2. $f(z) \in R_N$ bo‘lsin. U holda

1) $f(\lambda)$ chekli bo‘lsa, $f(\lambda) > 0$ bo‘ladi;

2) $\lim\limits_{\lambda \to c_k + 0} f(\lambda) = -\infty,\ \lim\limits_{\lambda \to c_k - 0} f(\lambda) = +\infty;$

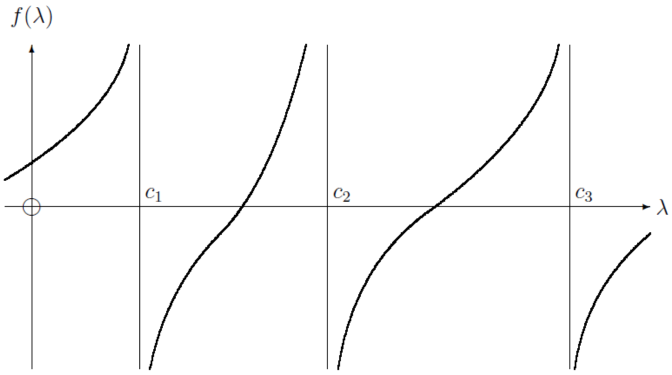
3) $f(z) \in R_N^+$ bo‘lsa, u holda $\lim\limits_{\lambda \to +\infty} f(\lambda) = +\infty,\ \lim\limits_{\lambda \to -\infty} f(\lambda) = -\infty;$

4) $f(z) \in R_N^0$ bo‘lsa, $f(\lambda) \uparrow b,\ (\lambda \to \infty)$ va $f(\lambda) \downarrow b,\ (\lambda \to -\infty)$.

Misol. $f(z) \in R_3^0$ bo‘lsa, u holda

$$f(\lambda) = b - \frac{b_1}{\lambda - c_1} - \frac{b_2}{\lambda - c_2} - \frac{b_3}{\lambda - c_3}$$

bo‘ladi. Bu yerda $b_1>0$, $b_2>0$, $b_3>0$, $c_1<c_2<c_3$. Bu funksiya grafigi quyidagi ko‘rinishda bo‘ladi:



1-rasm

$f(\lambda)\in R_N$ funksiya va $\mu<c_1$ son berilgan bo‘lsin. Ushbu

$$F(\lambda)=-\frac{\lambda-\mu}{f(\lambda)-f(\mu)}-f(\mu) \tag{2.6}$$

funksiyani kiritib olamiz. Bu funksiyani uzluksizligi bo‘yicha davom qildiramiz:

$$F(c_k)=-\frac{c_k-\mu}{f(c_k)-f(\mu)}-f(\mu)=-f(\mu),\quad 1\le k\le N,$$

$$F(\mu)=-\frac{1}{f'(\mu)}-f(\mu).$$

Izoh. Agar $f(\lambda)\in R_N$ bo‘lsa, u holda $F(\mu)\in R_M$, ya’ni

$$F(\lambda)=A\lambda+B-\sum_{k=1}^{M}\frac{B_k}{\lambda-C_k}, \tag{2.7}$$

$A\ge 0$, $B_k\ge 0$, $B\in R$, $C_1<C_2<\ldots<C_M$ ekanligini isbotlaymiz. Bu yerda M soni a ga bog‘liq ravishda N yoki $N-1$ bo`ladi, aniqrog‘i quyidagi teorema o`rinli.

Teorema 1. 1) Agar $f(\lambda)\in R_N^+$ bo‘lsa, u holda $F(\lambda)\in R_N^0$ bo‘lib,

$$\mu<c_1<C_1<c_2<\ldots<c_N<C_N$$

bo‘ladi;

2) Agar $f(\lambda)\in R_N^0$ bo‘lsa, u holda $F(\lambda)\in R_{N-1}^+$ bo‘lib,

$$\mu<c_1<C_1<c_2<\ldots<C_{N-1}<c_N$$

bo‘ladi.

Isbot. Quydagi tengliklarni bir-biridan ayiramiz:

$$-\begin{cases} f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k}, \\ f(\mu) = a\mu + b - \sum_{k=1}^{N} \frac{b_k}{\mu - c_k}, \end{cases}$$

$$f(\lambda) - f(\mu) = a(\lambda - \mu) - \sum_{k=1}^{N} \frac{b_k(\mu - \lambda)}{(\lambda - c_k)(\mu - c_k)}.$$

Bunga ko‘ra

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = a + \sum_{k=1}^{N} \frac{b_k}{(\lambda - c_k)(\mu - c_k)}.$$

Bu yerda

$$d_k = \frac{b_k}{\mu - c_k} < 0 \text{ va } P_N(\lambda) = (\lambda - c_1)\ldots(\lambda - c_N)$$

belgilashlarni kiritamiz. U holda

$$P_N(\lambda) \cdot \sum_{k=1}^{N} \frac{d_k}{\lambda - c_k} = \sum_{k=1}^{N} \frac{d_k \cdot P_N(\lambda)}{\lambda - c_k} = \sum_{k=1}^{N} d_k \cdot \prod_{j=1,\ j\neq k}^{N} (\lambda - c_j) = Q_{N-1}(\lambda),$$

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = a + \frac{Q_{N-1}(\lambda)}{P_N(\lambda)} = \frac{aP_N(\lambda) + Q_{N-1}(\lambda)}{P_N(\lambda)}.$$

Demak,

$$F(\lambda) = -\frac{P_N(\lambda)}{aP_N(\lambda) + Q_{N-1}(\lambda)} - f(\mu) =$$

$$= \frac{P_N(\lambda) - f(\mu) \cdot a \cdot P_N(\lambda) - f(\mu) \cdot Q_{N-1}(\lambda)}{aP_N(\lambda) + Q_{N-1}(\lambda)} = \frac{p(\lambda)}{r(\lambda)}. \tag{2.8}$$

Bu yerda

$$r(\lambda) = a \cdot \prod_{j=1}^{N} (\lambda - c_j) + \sum_{k=1}^{N} d_k \cdot \prod_{j=1, j\neq k}^{N} (\lambda - c_j). \tag{2.9}$$

$r(\lambda)$ ko`phad $a \neq 0$ da $N$ darajali ko‘phad, $a = 0$ da $N-1$ chi darajali ko‘phad bo`ladi. $r(c_n)$ ning ishorasini aniqlaymiz. Ushbu

$$r(c_n) = d_n \prod_{j=1}^{N} (c_n - c_j) = d_n \cdot \prod_{j=1}^{N-1} (c_n - c_j) \cdot \prod_{j=n+1}^{N} (c_n - c_j)$$

tenglikda $d_n < 0$ ekanligini hisobga olsak,

$$sign(r(c_n)) = (-1)^{N-n+1},\ n = 1, 2, \dots, N$$

kelib chiqadi. Demak, $r(\lambda)$ ko‘phad (c_n, c_{n+1}) intervalda μ_n ildizga ega $(n = 1, 2, \dots, N-1)$. Agar $a = 0$ bo‘lsa, bular $r(\lambda)$ ko‘phadning barcha ildizlari, chunki bu holda $\deg(r(\lambda)) = N - 1$.

Agar $a > 0$ bo‘lsa, u holda $sign(r(c_N)) = (-1)^{N-N+1} = -1 < 0$ bo‘lib, $r(\lambda) \to +\infty$, $\lambda \to +\infty$ bo‘lgani uchun $(c_N, +\infty)$ intervalda ham bitta μ_N ildizga ega.

$r(\lambda)$ karrali ildizga ega bo‘lmagani uchun $F(\lambda) = \dfrac{p(\lambda)}{r(\lambda)}$ funksiyaning qutb maxsus nuqtalari oddiy bo‘ladi. (2.8) ga ko‘ra

$$F(\lambda) = \frac{(-1 - af(\mu) \cdot P_N(\lambda)) - f(\mu) \cdot Q_{N-1}(\lambda)}{aP_N(\lambda) + Q_{N-1}(\lambda)}.$$

Agar $a = 0$ bo‘lsa, u holda

$$F(\lambda) = \frac{-P_N(\lambda) - f(\mu) \cdot Q_{N-1}(\lambda)}{Q_{N-1}(\lambda)} = A\lambda + B - \sum_{k=1}^{N-1} \frac{B_k}{\lambda - C_k}$$

bo‘ladi. Bu yerda

$$A = \frac{-1}{d_1 + \cdots + d_N} > 0.$$

Agar $a > 0$ bo‘lsa, $F(\lambda) = B - \sum\limits_{k=1}^{N} \dfrac{B_k}{\lambda - C_k}$ bo‘ladi. Bu yerda $B = -\dfrac{1}{a} - f(\mu)$.

Endi $B_k \geq 0$ ekanligini isbotlash kerak. $F(\lambda)$ ning ta’rifiga ko`ra

$$F(\lambda) = -\frac{\lambda - \mu}{f(\lambda) - f(\mu)} - f(\mu)$$

bo‘lgani uchun $F(\lambda)$ funksiyaning qutb maxsus nuqtalari $f(\lambda) = f(\mu)$ tenglikning μ dan boshqa ildizlari bilan ustma-ust tushadi.

Har-bir (C_k, C_{k+1}) oraliqda ushbu

$$f(\lambda) - f(\mu) \to +0,\ (2.\ \lambda \to C_k + 0),$$

$$f(\lambda)-f(\mu)\to -0,\ (2.\lambda\to C_k-0)$$

munosabatlar o`rinli bo‘ladi. Bunga ko‘ra

$$\lim_{\lambda\to C_k\pm 0} F(\lambda)=\mp\infty .$$

Bundan esa $B_k>0$, $k=1,2,...,M$ bo‘lishi kelib chiqadi. **Teorema isbotlandi.**

Izoh. $f(\lambda)$ dan $F(\lambda)$ ga o‘tish amalini bir qancha marta qo‘llab $R_N\to R_0^0$ akslantirishni ko‘rsatish mumkin. Keyinchalik bu fikirni ishlatamiz.

2-§. Xos qiymatlarning mavjudligi va xos funksiyalarning ossillyatsiya xossalari

Quyidagi

$$-y''+q(x)y=\lambda y,\quad 0\le x\le 1 \tag{2.10}$$

tenglamanining ushbu

$$y(0)\cos\alpha=y'(0)\sin\alpha \tag{2.11}$$

shartni qanoatlantiruvchi biror no‘lmas yechimini $y(x,\lambda)$ orqali belgilaymiz.

Ushbu

$$\begin{cases}\theta'=\cos^2\theta+[\lambda-q(x)]\sin^2\theta,\\ \theta(0,\lambda)=\alpha\end{cases} \tag{2.12}$$

Koshi masalasining yechimini $\theta(x,\lambda)$ orqali belgilaymiz. Bunga ko‘ra

$$ctg\theta(x,\lambda)=\frac{y'(x,\lambda)}{y(x,\lambda)}. \tag{2.13}$$

Bu holda

$$\frac{y'(1,\lambda)}{y(1,\lambda)}=f(\lambda) \tag{2.14}$$

harakteristik tenglama ushbu

$$ctg\theta(1,\lambda)=f(\lambda) \tag{2.15}$$

ko‘rinishini oladi.

Teorema 2. (2.1)-(2.3) masalaning xos qiymatlari haqiqiy.

Isbot. $\lambda = u + iv,\ u, v \in R,\ v \neq 0$ xos qiymat bo‘lsin va bu xos qiymatga $y(x)$ xos funksiya mos kelsin. U holda

$$-y'' + q(x)y = \lambda y,$$

$$\frac{y'(0)}{y(0)} = ctg\alpha\,,$$

$$\frac{y'(1)}{y(1)} = f(\lambda).$$

tengliklarda kompleks qo‘shma olsak, ushbu

$$-\bar{y}'' + q(x)\bar{y} = \bar{\lambda}\bar{y}\,,$$

$$\frac{\bar{y}'(0)}{\bar{y}(0)} = ctg\alpha\,,$$

$$\frac{\bar{y}(1)}{\bar{y}(1)} = f(\bar{\lambda})$$

tengliklar hosil bo‘ladi. Demak, $\bar{\lambda} = u - iv$ ham xos qiymat ekan. Biz $\operatorname{Im}\lambda = v > 0$ deb hisoblaymiz. Endi quydagi tengliklarni ko‘rib chiqamiz:

$$(\lambda - \bar{\lambda}) \cdot \int_0^1 |y(x)|^2 dx = (\lambda - \bar{\lambda}) \cdot \int_0^1 y \cdot \bar{y} dx = \int_0^1 [(\lambda \cdot y)\bar{y} - y(\bar{\lambda} \cdot \bar{y})] dx =$$

$$= \int_0^1 [(-y'' + q(x)y)\bar{y} - y \cdot (\bar{y}'' + q(x)\bar{y})] dx =$$

$$= \int_0^1 (\bar{y}'y - y'\bar{y})' dx = (\bar{y}'y - y'\bar{y})\Big|_0^1 = |y(1)|^2 \cdot \left(f(\bar{\lambda}) - f(\lambda) \right),$$

ya’ni

$$\operatorname{Im}\lambda \cdot \int_0^1 |y(x)|^2 dx = -|y(1)|^2 \cdot \operatorname{Im} f(\lambda)$$

Ziddiyat, chunki $\operatorname{Im}\lambda > 0$ va $\operatorname{Im} f(\lambda) > 0$. **Teorema isbotlandi.**

Teorema 3. (2.1)-(2.3) masalaning xos qiymatlari oddiy (2.karrasiz).

Isbot. $\Delta(\lambda) = \varphi'(1,\lambda) - f(\lambda)\varphi(1,\lambda)$ bo‘lsin. Bu yerda $\varphi(x,\lambda)$ orqali (2.1) tenglamaning $\varphi(0,\lambda) = \sin\alpha$, $\varphi'(0,\lambda) = \cos\alpha$ boshlang`ich shartlarni qanoatlantiruvchi yechimi belgilangan. Ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi$$

ayniyatni λ bo`yichi differensiallasak,

$$-\dot{\varphi}'' + q(x)\dot{\varphi} = \varphi + \lambda\dot{\varphi}$$

kelib chiqadi. Bularga ko`ra ushbu

$$(\dot{\varphi}' \cdot \varphi - \varphi' \cdot \dot{\varphi})' = -\varphi^2$$

ayniyat o`rinli bo`ladi. Bu ayniyatni integrallasak, quyidagi

$$\dot{\varphi}'(1,\lambda) \cdot \varphi(1,\lambda) - \varphi'(1,\lambda) \cdot \dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx \qquad (2.*)$$

formula hosil bo`ladi.

Endi $\Delta(\lambda) = \varphi'(1,\lambda) - f(\lambda)\varphi(1,\lambda)$ funksiyani differensiallaymiz:

$$\dot{\Delta}(\lambda) = \dot{\varphi}'(1,\lambda) - \dot{f}(\lambda) \cdot \varphi(1,\lambda) - f(\lambda) \cdot \dot{\varphi}(1,\lambda).$$

Demak, ushbu

$$\dot{\varphi}'(1,\lambda) = \dot{\Delta}(\lambda) + \dot{f}(\lambda) \cdot \varphi(1,\lambda) + f(\lambda) \cdot \dot{\varphi}(1,\lambda),$$

$$\varphi'(1,\lambda) = \Delta(\lambda) + f(\lambda) \cdot \varphi(1,\lambda)$$

tengliklar o`rinli ekan. Bu ifodalarni (2.*) ayniyatga qo`yamiz:

$$[\dot{\Delta}(\lambda) + \dot{f}(\lambda) \cdot \varphi(1,\lambda) + f(\lambda) \cdot \dot{\varphi}(1,\lambda)] \cdot \varphi(1,\lambda) -$$

$$-[\Delta(\lambda) + f(\lambda) \cdot \varphi(1,\lambda)] \cdot \dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx,$$

$$\dot{\Delta}(\lambda) \cdot \varphi(1,\lambda) + \dot{f}(\lambda)\varphi^2(1,\lambda) + f(\lambda)\varphi(1,\lambda)\dot{\varphi}(1,\lambda) -$$

$$-\Delta(\lambda)\dot{\varphi}(1,\lambda) - f(\lambda)\varphi(1,\lambda) \cdot \dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx,$$

$$\dot{\Delta}(\lambda) \cdot \varphi(1,\lambda) + \dot{f}(\lambda)\varphi^2(1,\lambda) - \Delta(\lambda)\dot{\varphi}(1,\lambda) = -\int_0^1 \varphi^2(x,\lambda)dx. \qquad (2.**)$$

Agar $\lambda = \lambda_0$ karrali xos qiymat bo‘lsa, u holda $\Delta(\lambda_0) = 0$, $\dot{\Delta}(\lambda_0) = 0$ bo‘lgani uchun

$$\dot{f}(\lambda_0)\varphi^2(1,\lambda_0) = -\int_0^1 \varphi^2(x,\lambda_0)dx$$

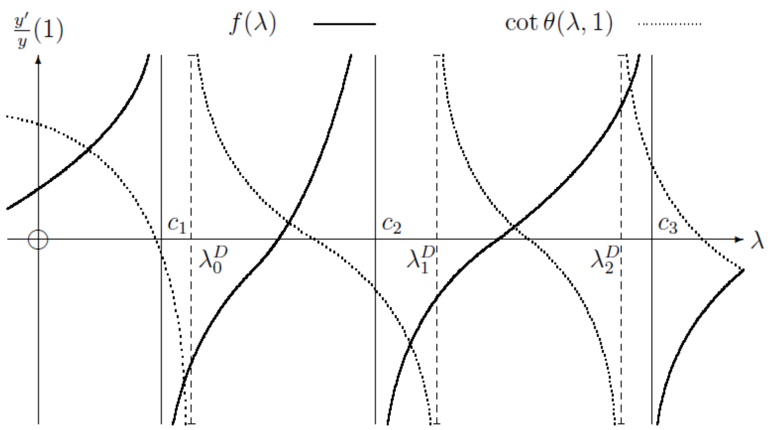
bo‘ladi. $f(\lambda_0) > 0$ bo‘lgani uchun ziddiyat kelib chiqadi. Agar $\lambda = c_k$ xos qiymat bo‘lsa, u holda $y(1, c_k) = 0$, bo‘ladi va bu son oddiy Shturm-Liuvill masalasining xos qiymati bo‘ladi. Demak, u oddiy xos qiymat ekan. **Teorema isbotlandi.**

Teorema 4. (2.1)-(2.3) masala cheksizta xos qiymatga ega. Bu xos qiymatlarni

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

orqali belgilasak, ular ∞ ga intiladi. Bunda $\lambda_0 < c_1$ bo‘ladi.

Isbot. (2.1)-(2.3) masala cheksizta xos qiymatlari ushbu $ctg\theta(\lambda,1) = f(\lambda)$ harakteristik tenglamadan topilishini biz bilamiz. Bu tenglamani grafik usulda yechamiz. Buning uchun $y = ctg\theta(\lambda,1)$ va $y = f(\lambda)$ funksiyalarning grafiklarini bitta chizmada chizamiz:



Bu chizmadan xususan xos qiymatlar soni cheksizta ekanligi va ular ∞ ga ketishi hamda eng kichik xos qiymat c_1 dan kichik ekanligi kelib chiqadi. **Teorema isbotlandi.**

Teorema 5. (2.1)-(2.3) masalada b ning qiymati kamaysa, c_k va $q(x)$ ning qiymatlari ortsa, λ_j larning har biri ortadi. Agar $a > 0$ bo‘lib, a kamaysa va b_k ortsa har bir musbat $\lambda_j > c_k$ ortadi.

Bu teoremaning isboti $f(\lambda)$ ning tuzilishidan kelib chiqadi.

λ_n^D orqali quyidagi masalaning xos qiymatlarini belgilaymiz:

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1],$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha \in [0, \pi),$$
$$y(1) = 0.$$

Endi $\{c_j\}$, $j = 1, 2, ..., N$, $\{\lambda_n^D\}$, $n = 1, ..., \infty$ sonlardan bitta to‘plam tuzamiz va uning elementlarini kamaymaydigan tartibda λ_n^{cD} orqali belgilaymiz. Agar $c_j = \lambda_k^D$ bo‘lsa, λ_n^{cD} ikki karrali deb hisoblaymiz. k_j orqali $c_i \le \lambda_j$ shartni qanoatlantiradigan c_i lar sonini belgilaymiz. $f(\lambda)$ funksiyaning va $ctg\theta(\lambda,1)$ funksiyaning grafiklarini analiz qilsak, quyidagi teorema kelib chiqadi.

Teorema 6. 1) λ_i va λ_j^{cD} ketma-ketliklar quyidagi ma’noda almashinib keladi: $\lambda_0 < \lambda_0^{cD} \le \lambda_1 \le \lambda_1^{cD} \le \cdots$;

2) Ushbu $c_{k_j} \le \lambda_j < c_{k_j+1}$ qo‘sh tengsizliklar bajariladi;

3) Agar $\lambda_{-1}^D = -\infty$ deb olsak, ushbu $\lambda_{j-1-k_j}^D < \lambda_j \le \lambda_{j-k_j}^D$ qo‘sh tengsizliklar o‘rinli bo‘ladi.

Bu teoremaga asoslanib, biz xos funksiyalar uchun ossillyatsiya xossasini keltirib chiqarishimiz mumkin. Agar $\lambda \in (\lambda_{i-1}^D, \lambda_i^D]$ bo‘lsa, u holda (2.1) tenglamaning (2.2) shartni qanoatlantiruvchi noldan farqli yechimi $(0,1)$ oraliqda i ta ildizga ega bo‘ladi.

Natija. λ_j ga mos keluvchi xos funksiyaning $(0,1)$ oraliqdagi ildizlar sonini ω_j orqali belgilaymiz. Bu holda $\omega_j = j - k_j$ bo‘ladi, hususan $\omega_0 = 0$ bo`ladi. Agar $\lambda_j > c_N$ bo‘lsa $\omega_j = j - N$ bo‘ladi.

3-§. Krum-Kreyn almashtirishi

Bu paragrafda (2.1)-(2.3) masalani xos qiymatlaridan ko‘pchiligini saqlagan holda boshqa bir Shturm-Liuvill masalasiga o‘tkazamiz. Natijada hosil bo‘ladigan Shturm-Liuvill masalasida $f(\lambda)$ o‘rnida oldingi paragrafda o‘rganilgan $F(\lambda)$

bo‘ladi. Bu almashtirishni bir necha marta qo‘llab, klassik Shturm-Liuvill masalasini hosil qilamiz.

Hosil bo‘ladigan klassik Shturm-Liuvill masalasining xos qiymatlari eng oldingi Shturm-Liuvill masalasining xos qiymatlaridan cheklitaga farq qiladi halos. (2.1)-(2.3) masalaning $\lambda_0 < \lambda_1 < \cdots$ xos qiymatlariga mos keluvchi xos funksiyalarini $y_0(x), y_1(x), \ldots$ orqali belgilaymiz. μ orqali quyidagi shartni qanoatlantiruvchi biror sonni belgilaymiz: agar $\alpha > 0$ bo‘lsa, $\mu = \lambda_0$, agar $\alpha = 0$ bo‘lsa, $\mu < \lambda_0$.

$\omega(x, \mu)$ orqali (2.1) tenglamaning $\lambda = \mu$ bo‘lgandagi $\omega(1) = 1$, $\omega'(1) = f(\mu)$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Quyidagi belgilashlarni kiritib olamiz

$$z = \frac{\omega'(x, \mu)}{\omega(x, \mu)}, \quad \hat{q}(x) = q(x) - 2z'(x).$$

$F(\lambda)$ oldingi paragrafda aniqlangan funksiya bo‘lsin. $\gamma \in [0, \pi)$ sonni shunday olamizki, bunda agar $\alpha = 0$ bo‘lsa $ctg\gamma = -z(0)$, agar $\alpha > 0$ bo‘lsa, $\gamma = 0$.

Teorema 7. Quyidagi

$$-y'' + \hat{q}(x)y = \lambda y, \tag{2.16}$$

$$\frac{y'(0)}{y(0)} = ctg\gamma, \tag{2.17}$$

$$\frac{y'(1)}{y(1)} = F(\lambda) \tag{2.18}$$

chegaraviy masalaning xos qiymatlari λ_j sonlar bilan, xos funksiyalari esa $U_j(x) = y_j'(x) - z(x)y_j(x)$ funksiyalar bilan ustma-ust tushadi. Bu yerda $\alpha > 0$ bo‘lsa, $j \geq 1$ deb hisoblanadi, $\alpha = 0$ bo‘lsa, $j \geq 0$ deb olinadi.

Isbot. Avvalo $\omega(x)$ funksiya $[0, 1]$ kesmada ildizga ega emasligini ko‘rsatamiz. $\alpha \neq 0$ bo‘lganda bu fikr juda oson, chunki bu holda $\omega(x) = y_0(x)$ bo‘lib, $y_0(x)$ funksiya ossillyatsiya teoremasiga ko‘ra $(0, 1)$ oraliqda nolga aylanmaydi. Bundan tashqari, $\frac{y_0'(0)}{y_0(0)} = ctg\alpha$ va $\frac{y_0'(1)}{y_0(1)} = f(\lambda_0)$ chekli bo‘lishidan $\omega(x)$ funksiya $[0,1]$ kesmada nolga aylanmasligi kelib chiqadi.

Agar $\alpha = 0$ bo‘lsa, u holda $y_0(0) = 0$ bo‘lib, $(0,1)$ oraliqda $y_0(x)$ funksiya nolga aylanmaydi. $\lambda_0 < \lambda_0^D$ bo‘lgani uchun $y_0(1) \neq 0$ bo‘ladi. Endi quyidagi Prufer tenglamasini ko‘rib chiqamiz:

$$\theta' = \cos' \theta + (\mu - q(x))\sin^2 \theta . \tag{2.19}$$

Bu tenglamani ushbu

$$\theta(\mu,1) = arcctgf(\lambda_0) \in (0,\pi) \tag{2.20}$$

shart bilan qaraymiz. $\mu < \lambda_0$ bo‘lgani uchun $\theta(\mu,0) \in (0,\pi)$ bo‘ladi.

Agar biz (2.19) tenglamani $\theta(\mu,1) = arcctgf(\mu)$ shart bilan qarasak, $\theta(\mu,0)$ o‘sadi ammo $(0,\pi)$ intervalda qoladi. Demak, (2.19) tenglamaning bu shart bilan qaralgandagi yechimi $\psi(x)$ faqat $(0,\pi)$ intervaldan qiymatlar qabul qiladi:

$$x \in [0,1], \quad \psi(x) \in (0,\pi).$$

Bu fikr esa $\omega(x)$ funksiyaning $[0,1]$ kesmada ildizga ega emas ekanligiga olib keladi. Demak, $z(x)$ funksiya $[0,1]$ da aniqlangan ekan.

Agar $\alpha > 0$ bo‘lsa, $\psi(x) = \theta(\lambda_0, x)$ deb olamiz. Shunday qilib

$$\psi(0) = \begin{cases} \alpha, & \alpha > 0 \\ \pi - \gamma, & \alpha = 0 \end{cases}$$

To‘g‘ridan-to‘g‘ri hisoblash ishlarini bajarib, $U_j(x)$ funksiyalar (2.16) differensial tenglamani va (2.17) chegaraviy shartni qanoatlantirishini ko‘rsatish mumkin. Bundan tashqari

$$\frac{U_j'(x)}{U_j(x)} = \frac{\mu - \lambda_j}{\frac{y_j'}{y} - z} - z \tag{2.21}$$

bo‘lgani uchun (2.18) chegaraviy shart ham bajariladi. Demak, λ_j sonlar haqiqatdan ham (2.16)-(2.18) chegaraviy masalaning xos qiymatlari ekan.

(2.16)-(2.18) masalaning bulardan boshqa xos qiymati yo‘q ekanligini isbotlash qoldi holos. $\theta(\lambda, x)$ orqali ushbu

$$\theta' = \cos^2\theta + (\lambda - q(x))\sin^2\theta, \quad \theta(\lambda,0) = \alpha \tag{2.22}$$

masalaning yechimini belgilaymiz. $\varphi(\lambda, x)$ orqali esa ushbu

$$\varphi' = \cos^2\varphi + (\lambda - \hat{q}(x))\sin^2\varphi, \quad \varphi(\lambda,0) = \gamma \tag{2.23}$$

masalaning yechimini belgilaylik. U holda quyidagilar o‘rinli:

$$\frac{y_j'(x)}{y_j(x)} = ctg\theta(\lambda_j, x),$$

$$\frac{U_j'(x)}{U_j(x)} = ctg\varphi(\lambda_j, x)$$

$$z(x) = ctg\psi(x).$$

Demak, (2.21) ayniyatga ko‘ra ushbu

$$ctg\varphi(\lambda_j, x) = \frac{\mu - \lambda_j}{ctg\theta(\lambda_j, x) - ctg\psi(x)} - ctg\psi(x) \tag{2.24}$$

tenglik o‘rinli bo‘ladi.

$x \in [0,1]$ da $\psi(x) \in (0,\pi)$ ekanligini ishlatamiz. Agar $\alpha > 0$ bo‘lsa, u holda $\lambda_j > \mu$, $j \geq 1$ bo‘lgani uchun $\theta(\lambda_j, x) > \psi(x)$, $x \in (0,1]$ bo‘ladi. Agar $\alpha = 0$ bo‘lsa, u holda $0 = \theta(\lambda_j,0) < \psi(0)$ o‘rinli bo‘ladi. Bulardan tashqari, agar biror $x \in (0,1)$ uchun $ctg\,\theta(\lambda_j, x) = ctg\,\psi(x)$ bo‘lsa, u holda (2.22) tenglikka ko‘ra

$$\frac{d}{dx}(\theta(\lambda_j, x) - \psi(x)) > 0$$

bo‘ladi, chunki $\sin^2\psi(x) > 0$. Demak, $\theta(\lambda_j, x) - \psi(x)$ o‘suvchi funksiya ekan. (2.21) tenglikka ko‘ra $U_j(x)$ funksiya biror $x \in (0,1)$ da nolga aylanadi. Bu esa faqat $\theta(\lambda_j, x) = \psi(x) + mx$ bo‘lganida bajariladi. Bu yerda m butun son bo‘lib, $\alpha > 0$ bo‘lganida $m > 0$ va $\alpha = 0$ bo‘lganda $m \geq 0$.

Shunday qilib, $U_j(x)$ funksiyaning [0,1] da n ta ildizi bo‘lishi uchun quyidagi tengsizlik bajarilishi zarur va yetarlidir:

$$\alpha > 0, \text{ bo‘lganida } n\pi < \theta(\lambda_j,1) - \psi(1) \le (n+1)\pi,$$

$$\alpha = 0, \text{ bo‘lganida } (n-1)\pi < \theta(\lambda_j,1) - \psi(1) \le n\pi.$$

6-teoremaning 3-bandiga ko‘ra $\lambda_j \in (\lambda_{i-1}^D, \lambda_i^D].$ Bu yerda $i = j - k_j$. Shuning uchun $i\pi < \theta(\lambda_j,1) \le (i+1)\pi$ bo‘ladi. Bundan esa $(i-1)\pi < \theta(\lambda_j,1) - \psi(1) < (i+1)\pi$ kelib chiqadi.

Demak, $U_j(x)$ funksiyaning $[0,1]$ kesmada $\alpha > 0$ bo‘lganda $i-1$ yoki i ta ildizi bor, $\alpha = 0$ bo‘lganida esa i yoki $i+1$ ta ildizi bor.

Bular $\theta(\lambda_j,1) = \psi(1) \le i\pi$ yoki $\theta(\lambda_j,1) - \psi(1) > i\pi$ bo‘lishiga bog‘liq. $f(\lambda)$ va $ctg\theta(\lambda,1)$ tilida aytganda $U_j(x)$ funksiyaning $[0,1]$ kesmada $\alpha > 0$ bo‘lganida $i-1$ yoki i ta ildizi, $\alpha = 0$ bo‘lganda i yoki $i+1$ ta ildizi bo‘lishi $f(\mu) \le ctg\theta(\lambda_j,1)$ yoki $f(\mu) > ctg\theta(\lambda_j,1)$ tengsizliklardan qaysi biri bajarilishiga bog‘liq.

$f(\lambda)$ va $F(\lambda)$ funksiyalar ifodasiga ko‘ra

$$f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k}, \qquad F(\lambda) = A\lambda + B - \sum_{k=1}^{N} \frac{B_k}{\lambda - C_k}.$$

Bu yerda agar $a > 0$ bo‘lsa, $A = 0,$ $M = N$ va $a = 0$ bo‘lsa, $A > 0,$ $M = N - 1$ ga teng bo‘ladi. Bundan tashqari C_k sonlar $f(\lambda) = f(\mu)$ tenglamaning $\lambda = \mu$ ildizidan farq qiluvchi ildizlaridir.

$\lambda = \mu$ bo‘lgan holda a ning katta qiymatlarida $\lambda_j > C_N > c_N$ bo‘ladi. Bundan

$$\frac{y_j'(1)}{y_j(1)} = ctg\theta(\lambda_j,1) = f(\lambda_j) > f(C_N) = f(\mu) \tag{2.25}$$

kelib chiqadi.

Qisqalik uchun $y(x)$ funksiyaning $(0,1)$ intervaldagi ildizlar sonini $osc(y(x))$ orqali belgilaymiz. U holda (2.25) tengsizlikka ko‘ra $osc(U_j) = osc(y_j) - 1.$

Oldingi paragrafning ohirgi natijasiga ko‘ra $\alpha > 0$ bo‘lganida $osc(u_j) = j - N - 1$ bo‘ladi. $\alpha = 0$ bo‘lganida huddi shu tarzda

$osc(u_j) = osc(y_j) = j - N$ ekanligini topamiz. Ikkinchi tomondan λ_j son (2.16)-(2.18) masalaning k - xos qiymati bo‘ladi, $(k \geq 0)$.

Demak, oldingi paragrafning oxirgi natijasiga ko‘ra $osc(u_j) = k - N.$ Shuning uchun $\alpha > 0$ bo‘lsa, $k = j - 1$ va $\alpha = 0$ bo‘lsa $k = j$ bo‘ladi. Bularga asosan $\alpha > 0$ bo‘lganida $\lambda_j,\ j \geq 1$ va $\alpha = 0$ bo‘lganida $\lambda_j,\ j \geq 0$ xos qiymatlar (2.16)-(2.18) masalaning barcha xos qiymatlarini hosil qiladi.

Endi $\alpha = 0$ holni ko‘rib chiqamiz. Bu holda j ning katta qiymatlarida $\lambda_j > C_N$ bo‘lgani uchun

$$\frac{y_j'(1)}{y_j(1)} = ctg\theta(\lambda_j, 1) < b < f(\mu)$$

bo‘ladi. Demak, $\alpha > 0$ bo‘lsa, $osc(u_j) = osc(y_j) = j - N$ va $\alpha = 0$ bo‘lsa

$$osc(u_j) = osc(y_j) + 1 = j - N + 1$$

tenglik o`rinli bo`ladi. **Teorema isbotlandi.**

Bu teoremani bir necha marta qo‘llab quydagi natijaga kelamiz.

Natija. (2.1)-(2.3) Shturm-Liuvill masalasi berilgan bo‘lib, $\lambda_0, \lambda_1, \lambda_2, \ldots$ bu masalaning xos qiymatlari bo‘lsin. Bu yerda $f(\lambda) \in R_N.$ U holda shunday

$$-y'' + q(x)y = \mu y,$$

$$\frac{y'(0)}{y(0)} = ctg\alpha_0,$$

$$\frac{y'(1)}{y(1)} = ctg\alpha_1$$

Shturm-Liuvill masalasi mavjudki, uning $\mu_0 < \mu_2 < \cdots$ xos qiymatlari uchun quyidagilardan biri bajariladi:

1) $\alpha > 0, \quad f \in R_N^+$ bo‘lsa, $\alpha_0 = 0, \quad \mu_j = \lambda_{j+N+1}$;

2) $\alpha = 0, \quad f \in R_N^0$ bo‘lsa, $\alpha_0 = 0, \quad \mu_j = \lambda_{j+N}$

3) boshqa hollarda, $\alpha_0 > 0, \quad \mu_j = \lambda_{j+N}$.

4-§. Xos qiymatlarning asimptotikasi

Bu paragrafda klassik Shturm-Liuvill masalasi xos qiymatlarining asimptotikasi yordamida (2.1)-(2.3) masala xos qiymatlarining asimptotikasini keltirib chiqaramiz.

Bu asimptotikalar albatta $q(x)$ potensialning silliqligiga bog‘liq [20]. $q(x) \in L_1(0,1)$ bo‘lganida [5] ishda xos qiymatlarning asimptotikasi o‘rganilgan, aniqrog‘i quyidagi tearema isbotlangan.

Teorema 8. Agar $q(x) \in L_1(0,1)$ bo‘lsa, ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad 0 \le x \le 1 \\ \dfrac{y'(0)}{y(0)} = \rho_0, \\ \dfrac{y'(1)}{y(1)} = \rho_1 \end{cases} \tag{2.26}$$

masalaning λ'_j xos qiymatlari uchun quyidagi

$$\lambda'_j = j^2\pi^2 + o(j^2), \; j \to \infty \tag{2.27}$$

asimptotik formula o‘rinli bo‘ladi.

Agar (2.26) masalada chegaraviy shartlardan birinchisi $y(0) = 0$ bo‘lsa, u holda

$$\lambda'_j = \left(j + \frac{1}{2} \right)^2 \pi^2 + \int_0^1 q(x)dx - 2\rho_1 + o\left(\frac{1}{j} \right), \; j \to \infty \tag{2.28}$$

asimptotika o‘rinli. Agar (2.26) masalada chegaraviy shartlardan ikkinchisi $y(1) = 0$ bo‘lsa, u holda

$$\lambda'_j = \left(j + \frac{1}{2} \right)^2 \pi^2 + \int_0^1 q(x)dx + 2\rho_0 + o\left(\frac{1}{j} \right), \; j \to \infty \tag{2.29}$$

asimptotika o‘rinli.

Natija. (2.1)-(2.3) masalaning xos qiymatlari uchun ham

$$\lambda_j = j^2\pi^2 + o(j^2) \tag{2.30}$$

asimptotik formula o‘rinli bo‘ladi. Aniqroq asimptotik formulalar quyidagi teoremada olingan.

Teorema 9. 1) Agar (2.1)-(2.3) masalada $f(\lambda) \in R_N^0$ bo‘lsa, u holda quyidagi

$$\lambda_j = \begin{cases} (j-N)^2\pi^2 + \int_0^1 q(x)dx - 2b + 2ctg\alpha + o\left(\frac{1}{j}\right), & \alpha \neq 0 \\ \left(j + \frac{1}{2} - N\right)^2 \pi^2 + \int_0^1 q(x)dx - 2b + o\left(\frac{1}{j}\right), & \alpha = 0 \end{cases} \tag{2.31}$$

asimptotik formula o‘rinli bo‘ladi.

2) Agar (2.1)-(2.3) masalada $f(\lambda) \in R_N^+$ bo‘lsa, u holda quyidagi

$$\lambda_j = \begin{cases} \left(j - \frac{1}{2} - N\right)^2 \pi^2 + \int_0^1 q(x)dx + 2ctg\alpha + \frac{2}{a} + o\left(\frac{1}{j}\right), & \alpha \neq 0 \\ (j-N)^2\pi^2 + \int_0^1 q(x)dx + \frac{2}{a} + o\left(\frac{1}{j}\right), & \alpha = 0 \end{cases} \tag{2.32}$$

asimptotik formula o‘rinli bo‘ladi.

Isbot. $N=0$ bo‘lgan holda bu teorema [5] ishda isbotlangan. Shuning uchun bu teorema $0,1,\ldots,N$ bo‘lgan holda isbotlangan deb olib, $N+1$ uchun isbot qilamiz.

1) $\alpha > 0$ va $f(\lambda) \in R_{N+1}^0$ bo‘lsin. U holda $F(\lambda) \in R_N^+$ bo`ladi. Bu holda Krum-Kreyn almashtirishini qo‘llab, $\hat{q}(x) = q(x) - 2z'$ potensialni va $\hat{\alpha} > 0$ chegaraviy shartdagi o‘zgarmasni olamiz. Induksiya faraziga ko‘ra (2.32) formuladan

$$\lambda_{j+1} = (j-N)^2\pi^2 + \int_0^1 \hat{q}(x)dx + \frac{2}{\hat{a}} + o\left(\frac{1}{j}\right)$$

formula kelib chiqadi. Shuning uchun

$$\lambda_j = (j-(N+1))^2\pi^2 + \int_0^1 q(x)dx - 2[f(\lambda_0) - ctg\alpha + \sum c_k] + o\left(\frac{1}{j}\right) =$$

$$=(j-(N+1))^2\pi^2+\int_0^1 q(x)dx-2b+2ctg\alpha+o\left(\frac{1}{j}\right)$$

o‘rinli bo‘ladi. Bu esa (2.31) formula $N+1$ va $\alpha>0$ holda o‘rinli ekanini ko‘rsatadi.

Endi $\alpha=0$ deb hisoblaymiz. Bu holda $\hat{\alpha}>0$ bo‘lgani uchun (2.32) ga asosan quyidagi

$$\lambda_j=\left(j-\frac{1}{2}-N\right)^2\pi^2+\int_0^1 \hat{q}(x)dx+2ctg\hat{\alpha}+\frac{2}{\hat{a}}+o\left(\frac{1}{j}\right)=$$

$$=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx-2[f(\mu)-z(0)+z(0)+\sum c_k]+o\left(\frac{1}{j}\right)=$$

$$=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx-2b+o\left(\frac{1}{j}\right)$$

o‘rinli bo‘ladi. Bu esa (2.31) formula $N+1$ va $\alpha=0$ holda ham o‘rinli ekanligini ko‘rsatadi.

2) $f(\lambda)\in R^{+}_{N+1}$ holini ko‘rib chiqamiz. Bu holda $F(\lambda)\in R^{0}_{N+1}$ bo‘ladi. Shuning uchun isbotlanayotgan teoremaning birinchi bandidagi natijalardan foydalanamiz. $\alpha>0$ bo‘lganda $z=0$ bo‘ladi. (2.31) formulaga asosan

$$\lambda_{j+1}=\left(j+\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 \hat{q}(x)dx-2\hat{b}+o\left(\frac{1}{j}\right)$$

tenglik o‘rinli bo‘ladi. Bunga ko‘ra

$$\lambda_j=\left(j-\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx-2[f(\lambda_0)-ctg\alpha-(a^{-1}+f(\lambda_0))]+o\left(\frac{1}{j}\right)=$$

$$=\left(j-\frac{1}{2}-(N+1)\right)^2\pi^2+\int_0^1 q(x)dx+2ctg\alpha+\frac{2}{a}+o\left(\frac{1}{j}\right).$$

Bu esa (2.32) tenglik $N+1$ va $\alpha>0$ bo‘lgan holda ham o‘rinli ekanini ko‘rsatadi.

Nihoyat, $\alpha=0$ bajarilganda (2.31) tenglikni ishlatsak

$$\lambda_j=(j-(N+1))^2\pi^2+\int_0^1 \hat{q}(x)dx-a\hat{b}+2ctg\hat{\alpha}+o\left(\frac{1}{j}\right)=$$

$$= (j-(N+1))^2\pi^2 + \int_0^1 q(x)dx - 2[f(\mu)+z(0)-(a^{-1}+ f(\mu)+z(0))] + o\left(\frac{1}{j}\right) =$$

$$= (j-(N+1))^2\pi^2 + \int_0^1 q(x)dx + \frac{2}{a} + o\left(\frac{1}{j}\right).$$

kelib chiqadi. **Teorema isbotlandi.**

II bob bo‘yicha xulosalar

Ushbu bobda bitta chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining xos qiymatlari va xos funksiyalarining xossalari o‘rganilgan. Jumladan, xos qiymatlarning xaqiqiyligi, karrasizligi va cheksiz ko‘pligi isbotlangan, Krum-Kreyn almashtirishi olingan, Krum-Kreyn almashtirishi yordamida xos qiymatlarning asimptotikasi keltirib chiqarilgan. Bundan tashqari, ossillyatsiya teoremasi isbotlangan.

III-BOB. CHEGARAVIY SHARTI SPEKTRAL PARAMETRGA RATSIONAL BOG‘LIQ BO‘LGAN SHTURM-LIUVILL MASALASINING VEYL-TITCHMARSH FUNKSIYASINI SPEKTRAL BERILGANLAR BO‘YICHA TIKLASH

1-§. Zaruriy ma’lumotlar

Quyidagi Shturm-Liuvill masalasini ko‘rib chiqamiz:

$$l(y) \equiv -y'' + q(x)y = \lambda y, \quad 0 < x < 1 \tag{3.1}$$

$$y(0)\cos\alpha = y'(0)\sin\alpha, \tag{3.2}$$

$$\frac{y'(1)}{y(1)} = f(\lambda). \tag{3.3}$$

Bu yerda $q(x)$ haqiqiy differesialanuvchi funksiya bo‘lib,

$$f(\lambda) = \frac{h(\lambda)}{g(\lambda)},$$

$h(\lambda)$ va $g(\lambda)$ haqiqiy koeffitsientli umumiy ildizga ega bo‘lmagan, ko‘phadlardir.

Agar λ ning biror qiymatida $f(\lambda) = \infty$ bo‘lsa, (3.3) chegaraviy shart $y(1) = 0$ bilan almashtiriladi. Bu turdagi chegaraviy masalalar bir qancha ishlarda, masalan: [1, 2, 4, 7, 8, 9, 14, 16, 18, 22, 30, 31, 32, 33] ishlarda o‘rganilgan. Bu turdagi chegaraviy masalaning matematik fizika masalalariga va injeneriyaga tatbiqlari [9] ishda keltirilgan.

Bu ishdagi asosiy maqsad, (3.1)-(3.3) masalaning spektrial berilganlari orqali Veyl-Titchmarsh funksiyasi uchun formula keltirib chiqarishdir.

$\upsilon(x,\lambda)$ orqali (3.1) tenglamaning

$$\upsilon(1,\lambda) = g(\lambda), \quad \upsilon'(1,\lambda) = h(\lambda)$$

shartlarni qanoatlantiruvchi noldan farqli yechimini belgilaymiz.

Ta’rif 1. Ushbu

$$m(\lambda)=\frac{\upsilon'(0,\lambda)\cos\alpha+\upsilon(0,\lambda)\sin\alpha}{\upsilon'(0,\lambda)\sin\alpha-\upsilon(0,\lambda)\cos\alpha} \tag{3.4}$$

funksiyaga Veyl-Titchmarsh funksiyasi deyiladi.

Veyl-Titchmarsh funksiyasining qutb maxsus nuqtalari bilan (3.1)-(3.3) masalaning xos qiymatlari ustma-ust tushadi. Bundan tashqari, qutb nuqtaning tartibi bilan xos qiymatning karrasi ustma-ust tushadi. Bu holda spektrial berilganlar xos qiymatlardan va umumlashgan normallovchi o‘zgarmaslardan iborat bo‘ladi. Chegaraviy shartda λ qatnashmagan holda normallovchi o‘zgarmaslar quyidagicha aniqlanadi

$$\alpha_n^0=\int_0^1\frac{\upsilon^2(x,\lambda_n)}{\upsilon^2(0,\lambda_n)}dx, \quad \alpha\neq 0.$$

Bu holda xos qiymatlar haqiqiy bo‘lmasligi ham mumkin. Bundan tashqari karrali bo‘lishi mumkin. Bu holda umumlashgan normallovchi o‘zgarmaslarni ta’riflash masalasi kelib chiqadi.

Agar $h(\lambda)$ ko‘phadning darajasi $g(\lambda)$ ko‘phadning darajasidan oshmasa, $g(\lambda)$ ko‘phadning darajasini M orqali belgilaymiz va $g(\lambda)$ ko‘phadning ildizlari haqiqiy va karrasiz deb hisoblaymiz.

Agar $h(\lambda)$ ko‘phadning darajasi $g(\lambda)$ ko‘phadning darajasidan katta bo‘lsa, u holda $h(\lambda)$ ko‘phadning darajasini M orqali belgilaymiz va $h(\lambda)$ ko‘phad ildizilari haqiqiy va karrasiz deb hisoblaymiz.

Quyidagi funksiyani kiritib olamiz

$$D(\lambda)=\upsilon'(0,\lambda)\sin\alpha-\upsilon(0,\lambda)\cos\alpha.$$

λ_n xos qiymatlar $D(\lambda)$ funksiyaning nollari bilan ustma-ust tushadi. Bu xos qiymatlarni shunday nomerlaymizki, bunda ularning haqiqiy qismlari kamaymaydigan bo‘lsin:

$$\mathrm{Re}(\lambda_n)\leq\mathrm{Re}(\lambda_{n+1}).$$

Qisqalik uchun (3.1)-(3.3) masalani (α, f, q) orqali belgilaymiz.

$h(\lambda)$ va $g(\lambda)$ ko‘phadlarning xossalaridan $f(\lambda)$ uchun $\deg(h(\lambda))\leq\deg(g(\lambda))$ bo‘lgan holda

$$f(\lambda)=b-\sum_{k=1}^{M}\frac{b_k}{\lambda-c_k} \qquad (3.5)$$

kelib chiqadi. Agar $\deg(h(\lambda))\geq\deg(g(\lambda))$ bo‘lsa, u holda

$$\frac{1}{f(\lambda)}=b-\sum_{k=1}^{M}\frac{b_k}{\lambda-c_k} \qquad (3.6)$$

o‘rinli bo‘ladi. Bu yerda b, b_k, c_k sonlar haqiqiy.

$H=L^2[0,1]\oplus C^M$ fazoda quyidagi bichiziqli formani kiritib olamiz:

$$<Y,Z>=\int_0^1 y\bar{z}dx+\sum_{k=1}^{M}\frac{y_k\bar{z}_k}{b_k}. \qquad (3.7)$$

Bu yerda

$$Y=\begin{pmatrix} y(x) \\ y_1 \\ \ldots \\ y_M \end{pmatrix}, \qquad Z=\begin{pmatrix} z(x) \\ z_1 \\ \ldots \\ z_M \end{pmatrix}.$$

Bu belgilashlardan so‘ng (3.1)-(3.3) masala H Pontryagin fazosida quyidagi

$$LY=\begin{pmatrix} l(y) \\ c_1y_1-b_1y(1) \\ \ldots \\ c_My_M-b_My(1) \end{pmatrix} \qquad (3.8)$$

operatorning xos qiymatlarini topish masalasiga keltiriladi. Bu operatorning aniqlanish sohasi quyidagicha

$$D(L)=\left\{Y\in H: y,y'\in AC, l(y)\in L^2,\ y(0)\cos\alpha=y'(0)\sin\alpha, y'(1)-by(1)=\sum_{k=1}^{M}y_k\right\}.$$

λ son (3.1)-(3.3) masalaning xos qiymati va $y(x)$ unga mos keluvchi xos funksiya bo‘lishi uchun λ son L operatorning xos qiymati bo‘lib, unga

$$Y = \begin{pmatrix} y(x) \\ \dfrac{b_1}{c_1 - \lambda} y(1) \\ \vdots \\ \dfrac{b_M}{c_M - \lambda} y(1) \end{pmatrix}$$

xos funksiya mos kelishi zarur va yetarlidir. Bundan $\lambda \neq c_j$, $j = 1,\ldots,M$ kelib chiqadi.

H Pontryagin fazosida L operator o‘z-o‘ziga qo‘shma, quyidan chegaralangan bo‘lib, uning rezolventasi kompakt operator bo‘ladi. Bunga ko‘ra L operatorning spektri diskret bo‘ladi. Hamda L operator $+\infty$ ga intiladigan xos qiymatlarga ega bo‘lib, ulardan faqat cheklitasi kompleks bo‘lishi mumkin. Bu fikrlar [8] ishda to‘liq isbot qilingan.

$\lambda \neq c_j$, $j = 1,\ldots,M$ bo‘lganida

$$V(x,\lambda) = \begin{pmatrix} \upsilon(x,\lambda) \\ \dfrac{b_1}{c_1 - \lambda} \upsilon(1,\lambda) \\ \vdots \\ \dfrac{b_M}{c_M - \lambda} \upsilon(1,\lambda) \end{pmatrix}$$

vektor-funksiyani kiritib olamiz.

$\lambda = c_j$, bo‘lgan holda esa

$$V(x,c_j) = \begin{pmatrix} \upsilon(x,c_j) \\ 0 \\ \vdots \\ 0 \\ \upsilon'(1,c_j) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \upsilon(x,c_j) \\ 0 \\ \vdots \\ 0 \\ -b_j g'(c_j) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

vektor-funksiya kiritib olamiz.

Bu funksiyalar yordamida esa quyidagi vector-funksiyani tuzib olamiz:

$$U(x,\lambda)=\begin{cases}\sin\alpha\,\dfrac{V(x,\lambda)}{\upsilon(0,\lambda)}, & \alpha\neq 0\\[2mm] \dfrac{V(x,\lambda)}{\upsilon'(0,\lambda)}, & \alpha=0\end{cases}$$

$U(x,\lambda)$ vektor funksiya $\alpha\neq 0$ bo‘lgan holda $\upsilon(0,\lambda)=0$ shartni qanoatlantiradigan λ nuqtada qutb maxsus nuqtaga ega $\alpha=0$ bo‘lgan holda esa $\upsilon'(0,\lambda)=0$ tenglamani qanoatlantiradigan λ nuqtalarda qutb maxsus nuqtaga ega. Shuning uchun bunday nuqtalarda $U(x,\lambda)$ aniqlanmagan.

$\{\mu_n\}_{n=0}^{\infty}$ orqali (α, f, q) chegaraviy masalaning xos qiymatlarini belgilaymiz. m_n orqali μ_n xos qiymatning karrasini begilaymiz.

n ning katta qiymatlarida $\mu_n=\lambda_{n+m}$ bo‘ladi, bu yerda

$$m=\sum_{k=0}^{\infty}(m_k-1). \tag{3.9}$$

Bu yerda $0\leq m<\infty$, ekanligi L operatorning H Pontryagin fazosida o‘z-o‘ziga qo‘shma bo‘lishidan kelib chiqadi (3.[30]).

μ_n xos qiymat uchun quyidagi vektor-funksiyalarni kiritib olamiz (3.[28]):

$$U_n^{(0)}(x)=U(x,\mu_n),$$

$$U_n^{(j)}(x)=\left.\frac{d^jU(x,\mu)}{d\mu^j}\right|_{\mu=\mu_n}, \qquad j=1,\ldots,m_n-1.$$

μ_n xos qiymatning geometrik karrasi 1 ga teng. $U_n^{(j)}(x)$ vektor-funksiyaning birinchi komponentasini $u_n^{(j)}(x)$ orqali belgilaymiz. Xuddi shunday $V^{(j)}(x,\lambda)$ vektor funksiyaning birinchi komponentasini $\upsilon^{(j)}(x,\lambda)$ orqali belgilaymiz. $u(x,\lambda)$ orqali (3.1) tenglamaning $\alpha\neq 0$ bo‘lganda $u(0,\lambda)=\sin\alpha$ shartni qanoatlantiruvchi biror yechimini belgilaymiz. Bu holda

$$U^{(j)}(0,\lambda)=\frac{d^jU(0,\lambda)}{d\lambda^j}=0, \qquad j\geq 1$$

bo‘ladi. $\lambda=\mu_n$ son ushbu

$$u'(0,\lambda)\sin\alpha - u(0,\lambda)\cos\alpha$$

funksiyaning m_n karrali noli bo‘ladi. Shuning uchun

$$U^{(j)}(0,\mu_n)=0, \qquad j=1,\dots,m_n-1,$$

$$U^{(m_n)}(0,\mu_n)\neq 0$$

bo‘ladi.

$\alpha=0$ bo‘lgan holda $U(x,\lambda)$ yechim $U'(x,\lambda)=1$ shartni qanoatlantiruvchi yechim bo‘lib,

$$U'^{(j)}(0,\lambda)=0, \qquad j\geq 1,$$

$$U^{(j)}(0,\mu_n)=0, \qquad j=0,1,\dots,m_n-1,$$

$$U^{(m_n)}(0,\mu_n)\neq 0.$$

Bundan tashqari

$$\tilde{L}V(x,\lambda)=\lambda V(x,\lambda)$$

va

$$\tilde{L}U(x,\lambda)=\lambda U(x,\lambda)$$

bo‘ladi. Bu yerda $\tilde{L}$ orqali L operatorning ushbu

$$D(\tilde{L})=\{Y\in H: y,y'\in AC, l(y)\in L^2[0,1]\}$$

sohadagi kengaytmasi belgilangan.

Endi biz umumlashgan normallovchi o‘zgarmaslarni kiritamiz. Buning uchun ushbu

$$[Y,Z]=<Y,\bar{Z}>=\int_0^1 y(x)z(x)dx+\sum_{k=1}^{M}\frac{1}{b_k}y_k z_k$$

belgilashdan foydalanamiz.

Ta’rif 2. Ushbu

$$\alpha_n^{(j)}=\left[U_n^{(j)},U_n^{(m_n-1)}\right], \qquad j=0,1,\dots,m_n-1$$

sonlarga μ_n xos qiymatlarga mos keluvchi umumlashgan normallovchi o‘zgarmaslar deyiladi.

Izoh. Agar μ_n kompleks xos qiymat bo‘lsa, u holda

$$\overline{\upsilon(x,\mu_n)} = \upsilon(x,\overline{\mu}_n),$$

$$\overline{U^{(j)}(x,\mu_n)} = U^{(j)}(x,\overline{\mu}_n)$$

bo‘ladi.

2-§. Veyl-Titchmarsh funksiyasi uchun Loran yoyilmasi

Lemma1. a nuqta $F(\lambda)$ funksiyaning N karrali noli bo‘lib, $F(\lambda)$ funksiya bu nuqtaning atrofida golomorf funksiya bo‘lsin. U holda a nuqtaning biror atrofida quyidagi tenglik o‘rinli bo‘ladi:

$$\frac{1}{F(\lambda)} = E(\lambda) + \sum_{k=1}^{N} \frac{P_k}{(\lambda - a)^k}.$$

Bu yerda $E(\lambda)$ funksiya $\lambda = a$ nuqtaning biror atrofida golomorf bo‘lib,

$$P_n = \frac{1}{a_N},$$

$$P_{N-j} = -\frac{1}{a_N}\sum_{k=0}^{j-1} P_{N-k}a_{N+j-k}, \qquad j = 1,\dots,N-1,$$

$$a_k = \frac{1}{k!}\frac{d^k F(a)}{d\lambda^k}.$$

Teorema1. Quyidagi tenglik o‘rinli

$$m_n\alpha_n^{(j)}C_{m_n+j}^{m_n} = \begin{cases} U'^{(m_n+j)}(0,\mu_n)\sin\alpha, & \alpha \neq 0 \\ -U^{(m_n+j)}(0,\mu_n), & \alpha = 0. \end{cases}$$

Isbot. Quyidagi hisoblashlarni bajaramiz:

$$(\lambda-\mu)[V(x,\lambda),V(x,\mu)] = [\lambda V(x,\lambda),V(x,\mu)] - [V(x,\lambda),\mu V(x,\mu)] =$$

$$= \left[\tilde{L}V(x,\lambda),V(x,\mu)\right] - \left[V(x,\lambda),\tilde{L}V(x,\mu)\right] =$$

$$= [\upsilon(x,\lambda),\upsilon'(x,\mu) - \upsilon'(x,\lambda),\upsilon(x,\mu)]\Big|_{x=0}^{x=1} +$$

$$+\sum_{k=1}^{M}\frac{\upsilon(1,\lambda)\upsilon(1,\mu)}{b_k}\left[\left(\frac{c_k b_k}{c_k-\lambda}-b_k\right)\frac{b_k}{c_k-\mu} - \left(\frac{c_k b_k}{c_k-\mu}-b_k\right)\frac{b_k}{c_k-\lambda}\right] =$$

$$=[\upsilon(1,\lambda)\upsilon'(1,\mu)-\upsilon'(1,\lambda)\upsilon(1,\mu)]-[\upsilon(0,\lambda)\upsilon'(0,\mu)-\upsilon'(0,\lambda)\upsilon(0,\mu)]+$$

$$+(\lambda-\mu)\upsilon(1,\lambda)\upsilon(1,\mu)\sum_{k=1}^{M}\frac{b_k}{(c_k-\lambda)(c_k-\mu)}=$$

$$=\upsilon(1,\lambda)\upsilon(1,\mu)[f(\mu)-f(\lambda)]-[\upsilon(0,\lambda)\upsilon'(0,\mu)-\upsilon'(0,\lambda)\upsilon(0,\mu)]+$$

$$+\upsilon(1,\lambda)\upsilon(1,\mu)[f(\lambda)-f(\mu)]=\upsilon'(0,\lambda)\upsilon(0,\mu)-\upsilon(0,\lambda)\upsilon'(0,\mu).$$

Bu yerda $[X,Y]$ skalyar ko‘paytmaning ifodasi, $V(x,\lambda)$ vektor-funksiya ta’rifi hamda bo‘laklab integrallash qoidasi ishlatildi. Bunga ko‘ra

$$(\lambda-\mu)[V(x,\lambda),V(x,\mu)]=\upsilon'(0,\lambda)\upsilon(0,\mu)-\upsilon(0,\lambda)\upsilon'(0,\mu) \qquad (3.10)$$

bo‘ladi. Bu tenglik $\lambda\neq c_j$ va $\mu\neq c_j$ bo‘lgan holda keltirib chiqarildi. Ammo (3.10) tenglikdagi funksiyalarning uzluksizligidan (3.10) tenglik $\lambda=c_j$ va $\mu=c_j$ bo‘lgan holda ham o‘rinli bo‘lishi kelib chiqadi.

$\alpha\neq 0$ holini ko‘rib chiqamiz. Bu holda (3.10) tenglikni $\upsilon(0,\mu)\upsilon(0,\lambda)$ ga bo‘lib va $\sin\alpha$ ga ko‘paytirib quyidagi tenglikni olamiz:

$$\frac{(\lambda-\mu)}{\sin\alpha}[U(x,\lambda),U(x,\mu)]=u'(0,\lambda)-u'(0,\mu).$$

Bu tenglikdan λ bo‘yicha j-tartibli hosila olamiz

$$(\lambda-\mu)\left[U^{(j)}(x,\lambda),U(x,\mu)\right]+j\left[U^{(j-1)}(x,\lambda),U(x,\mu)\right]=\sin\alpha u'^{(j)}(0,\lambda). \qquad (3.11)$$

Bu yerda $\lambda=\mu_n$, $\mu=\mu_n$ desak, quyidagi tenglik kelib chiqadi.

$$j\left[U_n^{(j-1)}(x),U_n^{(0)}(x)\right]=\sin\alpha u'^{(j)}(0,\mu_n). \qquad (3.12)$$

Xususan, $j=m_n$ bo‘lganida quyidagi tenglikni olamiz

$$\left[U_n^{(m_n-1)}(x),U_n^{(0)}(x)\right]=\frac{\sin\alpha}{m_n}u'^{(m_n)}(0,\mu_n). \qquad (3.13)$$

(3.11) ayniyatdan μ bo‘yicha k marta hosila olamiz

$$(\mu-\lambda)\left[U^{(j)}(x,\lambda),U^{(k)}(x,\mu)\right]+k\left[U^{(j)}(x,\lambda),U^{(k-1)}(x,\mu)\right]=j\left[U^{(j-1)}(x,\lambda),U^{(k)}(x,\mu)\right].$$

Bu yerda $\lambda=\mu_n$ va $\mu=\mu_n$ desak, quyidagi tenglik kelib chiqadi.

$$k\left[U_n^{(j)}(x),U_n^{(k-1)}(x)\right]=j\left[U_n^{(j-1)}(x),U_n^{(k)}(x)\right].$$

Bundan foydalanib, quyidagi tenglikni keltirib chiqaramiz:

$$C^{k}_{m_n+k-1}\left[U_n^{(m_n-1)},U_n^{(k)}\right]=\left[U_n^{(m_n+k-1)},U_n^{(0)}\right] \tag{3.14}$$

(3.12), (3.13) va (3.14) tengliklarga ko‘ra

$$\sin\alpha u'^{(j+m_n)}(0,\mu)=(m_n+j)\left[U_n^{(j+m_n-1)},U_n^{(0)}\right]=$$

$$=(m_n+j)\frac{(m_n+j-1)!}{(m_n-1)!\,j!}\left[U_n^{(m_n-1)},U_n^{(j)}\right]=m_n\frac{(m_n+j)!}{m_n!\,j!}\alpha_n^{(j)}.$$

bo‘ladi.

Endi $\alpha=0$ holni ko‘rib chiqamiz. (3.10) tenglikni $\upsilon'(0,\mu)\upsilon'(0,\lambda)$ ga bo‘lib quyidagi ayniyatga ega bo‘lamiz:

$$(\lambda-\mu)[U(x,\lambda),U(x,\mu)]=u(0,\mu)-u(0,\lambda).$$

Bu tenglikka yuqoridagi usulni qo‘llasak,

$$-u^{(j+m_n)}(0,\mu_n)=m_n\frac{(m_n+j)!}{m_n!\,j!}\alpha_n^{(j)}$$

formula kelib chiqadi. **Teorema isbotlandi.**

Lemma1 va teorema1 ga asosan $m(\lambda)$ Veyl-Titchmarsh funksiyasi uchun yozilgan Loran yoyilmasining singulyar qismini normallovchi sonlar orqali topishimiz mumkin.

Natija. $m(\lambda)$ Veyl-Titchmarsh funksiyasi μ_n xos qiymat atrofida quyidagi Loran yoyilmasiga ega

$$m(\lambda)=E_n(\lambda)+\sum_{k=1}^{m_n}\frac{p_n^{(k)}}{(\lambda-\mu_n)^k}. \tag{3.15}$$

Bu yerda $E_n(\lambda)$ funksiya μ_n xos qiymatning biror atrofida analitik funksiya bo‘lib,

$$p_n^{(m_n)}=\frac{(m_n-1)!}{\alpha_n^{(0)}}, \tag{3.16}$$

$$p_n^{(m_n-j)}=-\sum_{k=0}^{j-1}p_n^{(m_n-k)}\frac{\alpha_n^{(j-k)}}{(j-k)!\,\alpha_n^{(0)}},\qquad j=1,\dots,m_n-1. \tag{3.17}$$

Isbot. $\alpha\neq 0$ bo‘lsin. Veyl-Titchmarsh funksiyasini quyidagi tarzda yozib olamiz:

$$m(\lambda) = ctg\alpha + \frac{1}{Z(\lambda)}.$$

Bu yerda

$$Z(\lambda) = \sin\alpha\, u'(0,\lambda) - \cos\alpha \sin\alpha.$$

1-lemmani tatbiq qilsak, quyidagilar kelib chiqadi

$$m(\lambda) = E_n(\lambda) + \sum_{k=1}^{m_n} \frac{p_n^{(k)}}{(\lambda - \mu_n)^k}.$$

Bu yerda $E_n(\lambda)$ funksiya μ_n xos qiymatning biror atrofida analitik funksiya.

$$p_n^{(m_n)} = \frac{1}{a_n^{(m_n)}}, \tag{3.18}$$

$$p_n^{(m_n - j)} = -\frac{1}{a_n^{(m_n)}} \sum_{k=0}^{j-1} p_n^{(m_n - k)} a_n^{(m_n + j - k)}, \qquad j = 1,\ldots,M - 1, \tag{3.19}$$

$$a_n^{(k)} = \frac{1}{k!} \frac{d^k Z}{d\lambda^k}(\mu_n).$$

Ammo 1-teoremaga ko‘ra

$$a_n^{(m_n + j)} = \frac{\sin\alpha}{(j + m_n)!} u'^{(m_n + j)}(0,\mu_n) = \frac{\alpha_n^{(j)}}{(m_n - 1)!\, j!}.$$

Buni hisobga olib, (3.18) va (3.19) dan (3.16) va (3.17) tengliklarni keltirib chiqaramiz.

$\alpha = 0$ bo‘lganida quyidagi tenglik o‘rinli:

$$m(\lambda) = -\frac{1}{u(0,\lambda)}.$$

Agar 1-lemmani $Z(\lambda) = -u(0,\lambda)$ ga qo‘llasak quyidagi tenglikni olamiz:

$$a_n^{(m_n + j)} = \frac{-1}{(m_n + j)!} u^{(m_n + j)}(0,\mu_n) = \frac{\alpha_n^{(j)}}{(m_n - 1)!\, j!}.$$

Shundan so‘ng $\alpha \neq 0$ bo‘lgan holdagi usulni qo‘llasak, (3.15) tasvir hosil bo‘ladi.

3-§. Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o‘zgarmaslar orqali tasvirlash

Oldingi paragrafda Veyl-Titchmarsh funksiyasi uchun (3.15) tasvir olingan edi. Bu yerdagi ifodaning singulyar qismi xos qiymatlar va normallovchi o‘zgarmaslar orqali ifodalangan edi. Bu paragrafda (3.15) formuladagi $E_n(\lambda)$ funksiyani xos qiymatlar va normallovchi o‘zgarmaslar orqali ifodalash bilan shug’ullanamiz. Buning uchun quyidagi teoremalardan foydalanamiz.

Teorema 2. Quyidagi asimptotik formulalar o‘rinli

$$m(\lambda) = ctg\alpha + O\left(\frac{1}{\sqrt{|\lambda|}}\right), \ \lambda \to -\infty, \qquad \alpha \neq 0, \tag{3.20}$$

$$m(\lambda) = \sqrt{|\lambda|} + O(1), \ \lambda \to -\infty, \qquad \alpha = 0. \tag{3.21}$$

n ning yetarlicha katta qiymatidan boshlab, barcha xos qiymatlar oddiy, ya’ni karrasiz bo‘ladi. Shuning uchun normallovchi o‘zgarmaslar asimptotikasi faqat oddiy xos qiymatlar bilan bog‘liq.

Teorema 3. $q(x)$ absolyut uzluksiz funksiya bo‘lsin. U holda quyidagi asimptotik formulalar o‘rinli

$$\alpha_n^{(0)} = \frac{1}{2}\sin^2\alpha + O\left(\frac{1}{n^2}\right), \qquad \alpha \neq 0, \tag{3.22}$$

$$\alpha_n^{(0)} = \frac{1}{2\mu_n} + O\left(\frac{1}{n^4}\right), \qquad \alpha = 0. \tag{3.23}$$

Isbot. n ning yetarlicha katta qiymatlarida $m_n = 1$ bo‘ladi.

$$\alpha_n^{(0)} = \left[U_n^{(0)}, U_n^{(0)}\right] = \int_0^1 u_n^{(2)} dx + \sum_{k=1}^{M}\left(\frac{b_k}{c_k - \mu_n}\right)^2 \frac{u_n^{(2)}(1)}{b_k} = \int_0^1 u_n^{(2)} dx + f'(\mu_n) u_n^{(2)}(1). \tag{3.24}$$

[9] ishga ko‘ra

$$u_n^{(2)}(x,\lambda) = \frac{2\cos\sqrt{\mu_n}x}{\sqrt{\mu_n}}\{\sin\alpha\cos\alpha\sin\sqrt{\mu_n}x +$$

$$+\sin^2\alpha\int_0^x q(t)\cos\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dt\Bigg\}+$$

$$+\sin^2\alpha\cos^2\sqrt{\mu_n}x+O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\mu_n}x\right|}}{\mu_n}\right),\quad \alpha\neq 0,$$

$$u_n^{(2)}(x,\lambda)=\frac{\sin^2\sqrt{\mu_n}x}{\sqrt{\mu_n}}+$$

$$+\frac{2\sin\sqrt{\mu_n}x}{\mu_n\sqrt{\mu_n}}\int_0^x q(t)\sin\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dt+O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\mu_n}x\right|}}{\mu_n^2}\right),\qquad \alpha=0.\quad (3.25)$$

Endi [9] ishdagi xos qiymatlarning asimptotikasidan foydalanamiz:

$$\sqrt{\mu_n}=(m+n-M)\pi+O\left(\frac{1}{n}\right),\qquad \alpha\neq 0,\qquad (3.26)$$

$$\sqrt{\mu_n}=\left(m+n+\frac{1}{2}-M\right)\pi+O\left(\frac{1}{n}\right),\qquad \alpha=0.\qquad (3.27)$$

(3.26) va (3.27) ifodalarni (3.25) tengliklarga qo‘ysak,

$$u_n^{(2)}(1)=\sin^2\alpha+O\left(\frac{1}{n}\right),\qquad \alpha\neq 0$$

$$u_n^{(2)}(1)=\frac{1}{\mu_n}+O\left(\frac{1}{n^3}\right),\qquad \alpha=0$$

bo‘ladi. (3.26) va (3.27) asimptotikalardan yanada foydalanamiz:

$$\frac{1}{\mu_n}\int_0^1 2\cos\sqrt{\mu_n}x\sin\sqrt{\mu_n}xdx=\frac{1}{\sqrt{\mu_n}}\int_0^1\sin 2\sqrt{\mu_n}xdx=\frac{1-\cos 2\sqrt{\mu_n}}{2\mu_n}=O\left(\frac{1}{n^2}\right).$$

Yuqoridagilarni hisobga olsak, quyidagi tengliklar kelib chiqadi:

$$\int_0^1\frac{2\cos\sqrt{\mu_n}x}{\sqrt{\mu_n}}\int_0^x q(t)\cos\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dtdx=$$

$$=\int_0^1 2q(t)\cos\sqrt{\mu_n}t\int_t^1\frac{\cos\sqrt{\mu_n}x\sin\sqrt{\mu_n}(x-t)}{\sqrt{\mu_n}}dxdt=$$

$$=\int_0^1\frac{q(t)\cos\sqrt{\mu_n}t}{\sqrt{\mu_n}}\int_t^1\left[\sin\sqrt{\mu_n}(2x-t)-\sin\sqrt{\mu_n}t\right]dxdt=$$

$$= -\frac{1}{2\sqrt{\mu_n}}\int_0^1 (1-t)q(t)\sin 2\sqrt{\mu_n}tdt + O\left(\frac{1}{\mu_n}\right) =$$

$$= -\frac{1}{4\mu_n}\left[q(0) + \int_0^1 \frac{d[(1-t)q(t)]}{dt}\cos 2\sqrt{\mu_n}tdt\right] + O\left(\frac{1}{\mu_n}\right) = O\left(\frac{1}{\mu_n}\right).$$

Xuddi shunday quyidagi tenglikni olamiz:

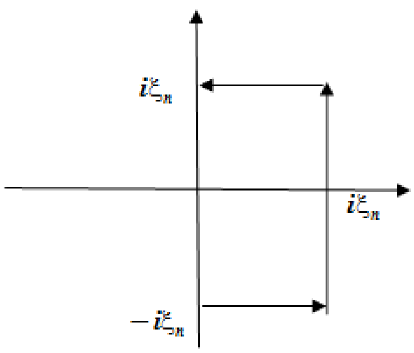
$$\int_0^1 \frac{2\sin\sqrt{\mu_n}x}{\sqrt{\mu_n}}\int_0^x q(t)\sin\sqrt{\mu_n}t\sin\sqrt{\mu_n}(x-t)dtdx = O\left(\frac{1}{\mu_n}\right).$$

Bulardan (3.22) va (3.23) formulalar kelib chiqadi. **Teorema isbotlandi.**

Lemma 2. Agar λ son (3.1)-(3.3) masalaning xos qiymati bo‘lmasa, u holda quyidagi tenglik o‘rinli bo‘ladi.

$$m(\lambda) = m(0) + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} p_n^{(j)}\left[\frac{1}{(\lambda-\mu_n)} - \frac{1}{(-\mu_n)^j}\right]. \tag{3.28}$$

Isbot. $\xi_n = \left(n + \frac{1}{4}\right)\pi$ bo‘lsin. γ_n orqali quyidagi konturni belgilaymiz:



(1-rasm)

$\Gamma_n = \{\xi^2 : \xi \in \gamma_n\}$ bo‘lsin. n ning yetarlicha katta qiymatlarida

$$m(\mu) = O(n), \qquad \mu \in \Gamma$$

bo‘ladi. Agar

$$H(\mu) = \frac{m(\mu)}{\mu(\mu-\lambda)}$$

funksiyani kiritib olsak, u holda

$$H(\mu) = O\left(\frac{1}{n^3}\right), \qquad \mu \in \Gamma_n$$

bo‘ladi va bundan

$$\lim_{n\to\infty}\int_{\Gamma_n} H(\mu)d\mu = 0$$

kelib chiqadi. Shunday qilib, $\lambda \neq 0$ deb va λ xos qiymat emas deb olsak, ushbu

$$0 = \frac{m(\lambda)}{\lambda} - \frac{m(0)}{\lambda} + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(j-1)} \frac{D^{(j-1)}}{D_\mu^{(j-1)}}\left(\frac{1}{\mu} + \frac{1}{\lambda - \mu}\right)\Bigg|_{\mu = \mu_n}$$

tenglik hosil bo‘ladi. Oxirgi tenglikdan esa (3.28) formula kelib chiqadi. **Lemma 2 isbotlandi.**

Teorema 4. Agar $\alpha \neq 0$ va $\lambda \neq \mu_n$, $n = 0,1,2,...$ bo‘lsa, u holda quyidagi tasvir o‘rinli bo‘ladi

$$m(\lambda) = ctg\alpha + \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} \frac{p_k^{(j)}}{(\lambda - \mu_k)^j}. \tag{3.29}$$

Isbot. k ning yetarlicha katta qiymatidan boshlab, $m_k = 1$ bo‘ladi. Shuning uchun (3.15) va (3.22) formulalarga ko‘ra

$$p_k^{(j)} = p_k^{(1)} = \frac{1}{\alpha_k^{(0)}} = O(1)$$

bo‘ladi. (3.28) tenglikda $\lambda \to -\infty$ da limitga o‘tamiz:

$$m(\lambda) \to m(0) - \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j}, \tag{3.30}$$

(3.20) tenglikka ko‘ra

$$m(\lambda) \to ctg\alpha, \quad \lambda \to -\infty. \tag{3.31}$$

bo‘ladi. (3.30) va (3.31) munosabatlarni taqqoslab,

$$m(0) = ctg\alpha + \sum_{n=0}^{\infty}\sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} \tag{3.32}$$

tenglikni olamiz. Bu ifodani (3.28) tenglikka qo‘yib, (3.29) formulani keltirib chiqaramiz. **Teorema isbotlandi.**

4-§. $\alpha = 0$ bo‘lgan holda Veyl-Titchmarsh funksiyasini xos qiymatlar va normallovchi o‘zgarmaslar orqali tasvirlash

Lemma 3. $\alpha = 0$ bo‘lgan holda quyidagi formula o‘rinli bo‘ladi.

$$m\left(-s^2\right) = s + O\left(\frac{1}{s}\right), \qquad s \to \infty. \tag{3.33}$$

Xususan, $m(\lambda) + i\sqrt{\lambda} \to 0, \ \ \lambda \to -\infty.$

Isbot. [9] ishga asosan $\lambda = is$desak,

$$\upsilon\left(0,-s^2\right) = \frac{(-1)^M s^{2M} e^s}{4}\left[2 + O\left(\frac{1}{s}\right)\right],$$

$$\upsilon'\left(0,-s^2\right) = \frac{(-1)^{M+1} s^{2M+1} e^s}{4}\left[2 + O\left(\frac{1}{s}\right)\right]$$

tengliklar o‘rinli bo‘ladi. Bundan quyidagi tenglik kelib chiqadi:

$$m\left(-s^2\right) - s = -\frac{\upsilon'\left(0,-s^2\right) + s\upsilon\left(0,-s^2\right)}{\upsilon\left(0,-s^2\right)} = O\left(\frac{1}{s}\right).$$

Lemma isbotlandi.

Teorema 5. Agar $\alpha = 0$ va $M = \deg(h) > \deg(g)$ bo‘lsa, quyidagi tenglik o‘rinli bo‘ladi

$$m(\lambda) = 1 + 2(m - M) + \sum_{n=0}^{\infty}\left[2\sum_{j=1}^{m_n}\frac{p_n^{(j)}}{(\lambda - \mu_n)}\right]. \tag{3.34}$$

Bu yerdagi $m - M$ o‘zgarmas

$$\sqrt{\mu_n} = (m + n + 1 - M)\pi + O\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{3.35}$$

asimptotikadan aniqlanadi.

Isbot. (3.35) ga ko‘ra

$$\mu_n = (n + 1 + m - M)^2\pi^2 + O(1)$$

o‘rinli bo‘ladi. Quyidagi tenglik bajariladi:

$$\sqrt{\lambda}ctg\sqrt{\lambda} = -i\sqrt{\lambda}\left[1 + \mathrm{O}\left(e^{-\left|\operatorname{Im}\sqrt{\lambda}\right|}\right)\right], \quad \lambda \to -\infty.$$

Bu tenglik va 3-lemmaga ko‘ra

$$\lim_{\lambda \to -\infty}\left[m(\lambda) - \sqrt{\lambda}ctg\sqrt{\lambda}\right] = 0$$

bo‘ladi. (3.28) tenglikni quyidagicha yozib olamiz:

$$m(\lambda) = m(0) + S(\lambda) + \sum_{n=0}^{\infty} p_n^{(1)}\left(\frac{1}{\lambda - \mu_n} + \frac{1}{\mu_n}\right). \tag{3.36}$$

Bu yerda

$$S(\lambda) = \sum_{n=0}^{\infty}\sum_{j=2}^{m_n} p_n^{(j)}\left(\frac{1}{(\lambda - \mu_n)^j} - \frac{1}{(-\mu_n)^j}\right). \tag{3.37}$$

Oxirgi yig’indi aslida chekli yig’indidir, chunki faqat cheklita xos qiymatgina karrali bo‘ladi. Quyidagi belgilashni kiritib olamiz:

$$S_\infty = \lim_{\lambda \to -\infty} S(\lambda) = -\sum_{n=0}^{\infty}\sum_{j=2}^{m_n}\frac{p_n^{(j)}}{(-\mu_n)^j}. \tag{3.38}$$

Mittag-Leffler yoyilmasiga ko‘ra

$$\sqrt{\lambda}ctg\sqrt{\lambda} = -1 - \sum_{n=0}^{\infty}\frac{2\lambda}{n^2\pi^2 - \lambda}$$

bo‘ladi. Bularni hisobga olsak.

$$m(\lambda) - \sqrt{\lambda}ctg\sqrt{\lambda} = m(0) + S(\lambda) + 1 +$$

$$+\sum_{n=0}^{M-m-2} p_n^{(1)}\left(\frac{1}{\lambda - \mu_n} + \frac{1}{\mu_n}\right) + \sum_{n=0}^{\infty}\left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}}\frac{\lambda}{\lambda - \mu_{n+M-m-1}} - 2\frac{\lambda}{\lambda - n^2\pi^2}\right] =$$

$$= m(0) + S(\lambda) + 1 + \sum_{n=0}^{\infty}\left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right]\frac{\lambda}{\lambda - \mu_{n+M-m-1}} +$$

$$+\sum_{n=0}^{M-m-2} p_n^{(1)}\left(\frac{1}{\lambda - \mu_n} + \frac{1}{\mu_n}\right) + \sum_{n=0}^{\infty}\left[\frac{2\lambda}{\lambda - \mu_{n+M-m-1}} - \frac{2\lambda}{\lambda - n^2\pi^2}\right]. \tag{3.39}$$

(3.16) formuladan va ushbu

$$\alpha_n^{(0)} = \frac{1}{2\mu_n} + \mathrm{O}\left(\frac{1}{n^4}\right), \quad n \to \infty$$

tenglikdan foydalansak,

$$\frac{p_n^{(1)}}{\mu_n} = 2 + \mathrm{O}\left(\frac{1}{n^2}\right), \quad n \to \infty \tag{3.40}$$

asimptotik formula kelib chiqadi. Bunga ko‘ra

$$\lim_{\lambda\to-\infty} \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right] \frac{\lambda}{\lambda - \mu_{n+M-m-1}} = \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right]$$

bo‘ladi. Ushbu

$$\mu_{n+M-m-1} - n^2\pi^2 = \mathrm{O}(1)$$

tenglikni inobatga olsak, shunday manfiy, modul jihatdan yetarlicha katta bo‘lgan K son topiladiki, bunda $\lambda \to -\infty$ da quyidagi baholash o‘rinli bo‘ladi

$$\sum_{n=0}^{\infty} \left| \frac{2\lambda}{\lambda - \mu_{n+M-m-1}} - \frac{2\lambda}{\lambda - n^2\pi^2} \right| \le K \sum_{n=0}^{\infty} \left[\frac{|\lambda|}{|\lambda|\left(n^2\pi^2 - \lambda\right)} \right] = -\frac{K\left(\sqrt{\lambda}ctg\sqrt{\lambda} + 1\right)}{2|\lambda|} \to 0.$$

Yuqoridagi munosabatlarni hisobga olsak, ushbu

$$m(\lambda) - \sqrt{\lambda}ctg\sqrt{\lambda} \to m(0) + S_\infty + 1 + \sum_{n=0}^{M-m-2} p_n^{(1)} \frac{1}{\mu_n} + \sum_{n=0}^{\infty} \left[\frac{p_{n+M-m-1}^{(1)}}{\mu_{n+M-m-1}} - 2\right]$$

munosabat kelib chiqadi. Bunga asosan

$$m(0) = \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} - 1 - 2(M - m - 1) - \sum_{n=0}^{\infty} \left[\frac{p_n^{(1)}}{\mu_n} - 2\right]$$

bo‘ladi. Bu ifodani (3.36) tenglikga qo‘ysak, quyidagi formulani olamiz

$$m(\lambda) = 1 - 2M + 2m + \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} \frac{p_n^{(j)}}{(-\mu_n)^j} - \sum_{n=0}^{\infty} \left[\frac{p_n^{(1)}}{\mu_n} - 2\right] + \sum_{n=0}^{\infty} \sum_{j=1}^{m_n} p_n^{(j)} \left[\frac{1}{(\lambda - \mu_n)^j} - \frac{1}{(-\mu_n)^j}\right] =$$

$$= 1 - 2M + 2m + \sum_{n=0}^{\infty} \sum_{j=2}^{m_n} p_n^{(j)} \frac{1}{(\lambda - \mu_n)} + \sum_{n=0}^{\infty} \left[2 + \frac{p_n^{(1)}}{\lambda - \mu_n}\right].$$

Teorema isbotlandi.

Teorema 6. Agar $\alpha = 0$ bo‘lib, $M = \deg(g) \ge \deg(h)$ bo‘lsa, u holda quyidagi tenglik o‘rinli bo‘ladi

$$m(\lambda) = 2(m - M) + \sum_{n=0}^{\infty} \left[2 + \sum_{j=1}^{m_n} \frac{p_n^{(j)}}{(\lambda - \mu_n)^j}\right].$$

Isbot. Bu holda $\mu_n = \left(n+m+\frac{1}{2}-M\right)^2 \pi^2 + \mathrm{O}(1)$ bo‘ladi. Ushbu

$$\sqrt{\lambda}tg\sqrt{\lambda} = i\sqrt{\lambda}\left[1+\mathrm{O}\left(e^{-2\left|\mathrm{Im}\sqrt{\lambda}\right|}\right)\right],\ \lambda\to\infty$$

formulaga asosan quyidagi tenglik (3.[34], 113 bet) kelib chiqadi

$$\frac{tg\sqrt{\lambda}}{\sqrt{\lambda}} = \sum_{k=0}^{\infty}\frac{2}{\left(k+\frac{1}{2}\right)^2\pi^2-\lambda}.$$

Buni hisobga olsak, 3-lemmadan $\lim\limits_{\lambda\to-\infty}\left(m(\lambda)+\sqrt{\lambda}tg\sqrt{\lambda}\right)=0$ tenglik kelib chiqadi.

(3.36), (3.37) va (3.38) tengliklar hamda Mittag-Leffler yoyilmasiga asosan, quyidagi formula o‘rinli:

$$m(\lambda)+\sqrt{\lambda}tg\sqrt{\lambda} = m(0)+S(\lambda)+\sum_{n=0}^{\infty}\left[\frac{p^{(1)}_{n+M-m}}{\mu_{n+M-m}}-2\right]\frac{\lambda}{\lambda-\mu_{n+M-m}}+$$

$$+\sum_{n=0}^{M-m-1}p_n^{(1)}\left(\frac{1}{\lambda-\mu_n}+\frac{1}{\mu_n}\right)+\sum_{n=0}^{\infty}\left[\frac{2\lambda}{\lambda-\mu_{n+M-m}}-\frac{2\lambda}{\lambda-\left(n+\frac{1}{2}\right)^2\pi^2}\right]$$

(3.40) munosabatdan foydalanib, ushbu tenglikni olamiz:

$$\lim_{\lambda\to-\infty}\sum_{n=0}^{\infty}\left[\frac{p^{1}_{n+M-m}}{\mu_{n+M-m}}-2\right]\frac{\lambda}{\lambda-\mu_{n+M-m}} = \sum_{n=0}^{\infty}\left[\frac{p^{1}_{n+M-m}}{\mu_{n+M-m}}-2\right].$$

Ushbu $\mu_{n+M-m}-\left(n+\frac{1}{2}\right)^2\pi^2 = \mathrm{O}(1)$ tenglikni hisobga olsak,

$$\sum_{n=0}^{\infty}\left|\frac{2\lambda}{\lambda-\mu_{n+M-m}}-\frac{2\lambda}{\lambda-\left(n+\frac{1}{2}\right)^2\pi^2}\right| \le K\sum_{n=0}^{\infty}\left[\frac{|\lambda|}{|\lambda|\left(\left(n+\frac{1}{2}\right)^2\pi^2-\lambda\right)}\right]=$$

$$=\frac{Ktg\sqrt{\lambda}}{2\sqrt{\lambda}}\to 0,\qquad \lambda\to-\infty$$

baholash kelib chiqadi. Demak, quyidagi munosabat o‘rinli

$$m(\lambda)+\sqrt{\lambda}tg\sqrt{\lambda}\to m(0)+S_\infty+\sum_{n=0}^{M-m-1}p_n^{(1)}\frac{1}{\mu_n}+\sum_{n=0}^{\infty}\left[\frac{p_{n+M-m}^{(1)}}{\mu_{n+M-n}}-2\right].$$

Bunga ko‘ra

$$0=m(0)+2(M-m)+S_\infty+\sum_{n=0}^{\infty}\left[\frac{p_n^{(1)}}{\mu_n}-2\right]$$

tenglik o‘rinli bo‘ladi. Bu ifodani (3.36) tenglikka qo‘ysak, quyidagi formula kelib chiqadi

$$m(\lambda)=-2(M-m)+\sum_{n=0}^{\infty}\sum_{j=2}^{m_n}\frac{p_n^{(j)}}{(\lambda-\mu_n)^j}+\sum_{n=0}^{\infty}\left[2+\frac{p_n^{(1)}}{\lambda-\mu_n}\right].$$

Teorema isbotlandi.

5-§. Teskari masala yechimining yagonaligi

Teorema 7. (3.1)-(3.3) chegaraviy masala o‘zining $\mu_n,\ n=\overline{0,\infty}$ xos qiymatlari va umumlashgan normallovchi o‘zgarmaslari $\alpha_n^{(k)},\ n=\overline{0,\infty},\ k=\overline{0,m_n}$ orqali bir qiymatli aniqlanadi. Boshqacha qilib aytganda $q(x),\ f(\lambda)$ funksiyalar va α son $\{\mu_n,\alpha_n^{(k)},\ n=\overline{1,\infty},\ k=\overline{0,m_n}\}$ spektral berilganlar orqali bir qiymatli aniqlanadi.

Isbot. Beshinchi va oltinchi teoremalarga ko‘ra $\mu_n,\ \alpha_n^{(k)},\ n=\overline{1,\infty},\ k=\overline{0,m_n}$ spektral berilganlar bo‘yicha $m(\lambda)$ Veyl-Titchmarsh funksiyasi bir qiymatli aniqlanadi. [9] ishda (3.1)-(3.3) masala Veyl-Titchmarsh funksiyasi bo‘yicha bir qiymatli aniqlanishi isbotlangan. **Teorema isbotlandi.**

III bob bo‘yicha xulosalar

Ushbu bobda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun umumlashgan normallovchi o‘zgarmaslar ta’riflangan va Veyl-Titmarsh funksiyasini spektral berilganlar orqali ifodalovchi formulalar olingan. Jumladan, birinchi chegaraviy shartda qatnashadigan α sonning nolga teng bo‘lmagan va nolga teng bo‘lgan hollari alohida qaralgan. Shu bilan birga ikkinchi chegaraviy shartda ishtirok qiluvchi $h(\lambda)$ va $g(\lambda)$ ko‘phadlar darajalarining katta kichikligiga qarab, formulalar o‘zgarishi hisobga olingan. Asosiy formulani keltirib chiqarishda Veyl-Titmarsh funksiyasi uchun Loran yoyilmasidan foydalanilgan.

Veyl-Titmarsh funksiyasini xos qiymatlar va umumlashgan normallovchi o‘zgarmaslar orqali ifodalovchi formulalarga asoslanib, teskari masala uchun yagonalik teoremasi isbotlangan.

Xulosa

Ushbu magistrlik dissertatsiyasining I-bobida chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasining xos qiymatlari va xos funksiyalarining xossalari o‘rganilgan. Jumladan, xos qiymatlarning xaqiqiyligi, karrasizligi va cheksiz ko‘pligi isbotlangan, Krum-Kreyn almashtirishi olingan, Krum-Kreyn almashtirishi yordamida xos qiymatlarning asimptotikasi va regulyarlashtirilgan izlar formulasi keltirib chiqarilgan. Bundan tashqari, chegaraviy shartlari spektral parametrga bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun V.A.Ambartsumyan teoremasining analogi isbot qilingan. Xos funksiyalarning chiziqli erkliligi ko‘rsatilgan va ular uchun ossillyatsiya teoremasi isbotlangan.

II-bobda bitta chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasining xos qiymatlari va xos funksiyalarining xossalari o‘rganilgan. Jumladan, xos qiymatlarning xaqiqiyligi, karrasizligi va cheksiz ko‘pligi isbotlangan, Krum-Kreyn almashtirishi olingan, Krum-Kreyn almashtirishi yordamida xos qiymatlarning asimptotikasi keltirib chiqarilgan. Bundan tashqari, ossillyatsiya teoremasi isbotlangan.

III-bobda chegaraviy sharti spektral parametrga ratsional bog‘liq bo‘lgan Shturm-Liuvill masalasi uchun umumlashgan normallovchi o‘zgarmaslar ta’riflangan va Veyl-Titmarsh funksiyasini spektral berilganlar orqali ifodalovchi formulalar olingan. Jumladan, birinchi chegaraviy shartda qatnashadigan α sonning nolga teng bo‘lmagan va nolga teng bo‘lgan hollari alohida qaralgan. Shu bilan birga ikkinchi chegaraviy shartda ishtirok qiluvchi $h(\lambda)$ va $g(\lambda)$ ko‘phadlar darajalarining katta kichikligiga qarab, formulalar o‘zgarishi hisobga olingan. Asosiy formulani keltirib chiqarishda Veyl-Titmarsh funksiyasi uchun Loran yoyilmasidan foydalanilgan. Veyl-Titmarsh funksiyasini xos qiymatlar va umumlashgan normallovchi o‘zgarmaslar orqali ifodalovchi formulalarga asoslanib, teskari masala uchun yagonalik teoremasi isbotlangan.

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