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NOZIK ANALITIK FUNKSIYALAR VA GONCHAR SINFI

Annotatsiya. *Ushbu maqolada Gonchar sinfidan olingan funksiyalarni nozik analitik davom qilishi isbotlangan.*

Аннотация. *В этой работе доказывается тонко-аналитическое продолжение функций из класса Гончара.*

Annotation. *In this work is proved fine-analytic extention of functions from Gonchar`s class.*

Kalit so'zlar: *Gonchar sinfi, nozik analitik funksiya, ratsional approksimatsiya, nozik topologiya.*

Ключевые слова: *Класс Гончара, тонко-аналитическая функция, рациональная аппроксимация, тонкая топология.*

Key words: *Gonchar`s class, fine-analytic function, rational approximation, fine topology.*

1972-yilda A.A.Gonchar [1] tomonidan analitik funksiyalarni tez ratsional approksimatsiya qilinishi va ularning bir qiymatli analitik davom etishi o'rtasida uzviy bog'liqlik mavjudligi ko'rsatib berilgan. Keyinchalik, analitik funksiyalarning bir qiymatliligini lokal shartli ko'p kompleks o'zgaruvchili funksiyalar uchun ham o'rinli bo'lishi isbotlangan. Analitik funksiyalarni ratsional funksiyalar bilan tez yaqinlashtirish mumkin bo'lgan sinfi (Gonchar sinfi) alohida ahamiyat kasb etgani bois, keyingi tadqiqotlarda bu sinfning xususiyatlarini o'rganish dolzarb masalalarga aylandi. A.Sadullayev tomonidan[2] R^0 sinfga tegishli bo'lish kriteriyasi va R^0 sinfga tegishli bo'lgan funksiyalarni maxsusliklari va strukturalari o'rganilgan. Shu sababli Gonchar sinfining nozik topologiyadagi

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[1] Гончар А.А. Локальное условие однозначности аналитических функций. Мат.сб., Т. 89(131), (1972), 148-164.

[2] Садуллаев А. Рациональные аппроксимации и плюриполярные множества. Мат. сб., Т.119:1, (1982), 96-118.

analitik funksiyalar bilan bog‘liqligini o‘rganish masalalari yuzaga keldi. Ushbu maqolada Gonchar sinfiga tegishli bo‘lgan ma’lum yaqinlashish tartibiga ega funksiyalarning nozik analitik bo‘lishi ko‘rsatiladi.

1-ta’rif. $0 \in \mathbb{C}^n$ *nuqtada analitik bo‘lgan* f *funksiyani biror* $\overline{B} = \overline{B}(0,r),\ \ r > 0$ *atrofda tez ratsional approksimatsiya qildirish mumkin bo‘lsa, ya’ni* f *funksiyaning* $\{r_m : \deg r_m \le m\}$ *ratsional funksiyalar to‘plamidan og‘ishishi* $\rho_m = \rho_m(f, \overline{B}) = \inf_{r_m} \sup_{\overline{B}} |f(z) - r_m(z)|$ *uchun ushbu*

$$\lim_{m\to\infty} \rho_m^{\frac{1}{m}}(f, \overline{B}) = 0$$

tenglik bajarilsa, u holda bu funksiya R^0 *sinfga tegishli deyiladi.*

R^0 belgilashda "0" yuqori indeks $f \in R^0$ analitik funksiya qaysi nuqtada qaralayotganligini anglatadi.

$\mathbb{C}$ tekislikdagi nozik topologiya deb, subgarmonik funksiyalarning uzluksizligini saqlovchi topologiyalarning eng kuchlisiga aytiladi. Nozik topologiya $\{u \in sh(\mathbb{C}) :\ u(z) > \alpha\}$ va $\{u \in sh(\mathbb{C}) :\ u(z) < \alpha\}$ ko‘rinishdagi to‘plamlar orqali hosil qilinadi. z^0 nuqtaning nozik atrofi deb, z^0 nuqtani o‘z ichiga oluvchi nozik topologiyadagi ochiq $V \subset \mathbb{C}$ to‘plamga aytiladi. Bunda, shunday $U = U(z^0, r)$ doira va U da subgarmonik bo‘lgan $u \in sh(U)$ funksiya mavjudki, $\lim_{z\to z^0, z\notin V} u(z) < u(z^0)$ tengsizlik o‘rinli bo‘ladi. z^0 nuqtaning yopiq nozik atrofi deb, $\bar{U}(z^0, r) \setminus G$ ko‘rinishdagi kompaktga aytiladi, bunda $r > 0, G \subset U(z^0, r)$ orqali z^0 nuqtani o‘z ichiga olgan ochiq “uzilgan” to‘plam belgilangan.

2-ta’rif. *Agar* $f(z)$ *funksiya* D *sohada sig‘im bo‘yicha deyarli hamma yerda aniqlangan bo‘lsa, ya’ni biror* $E \subset D$ *polyar to‘plamning tashqarisida chekli qiymat qabul qilib, har bir* $z^0 \in D \setminus E$ *nuqta uchun* z^0 *nuqtaning yopiq nozik* F *atrofi mavjud bo‘lib, bu atrofda ratsional funksiyalar bilan tekis yaqinlashtirish mumkin bo‘lsa, u holda* $f(z)$ *funksiya* $D \subset \mathbb{C}$ *sohada nozik analitik deyiladi.*

1-teorema. *Agar* $f \in R^0$ *funksiya uchun ushbu*

$$\overline{\lim_{m\to\infty}} \frac{\ln m}{\ln \dfrac{\ln m}{\rho_m^{\frac{1}{m}}}} < +\infty$$

munosabat o'rinli bo'lsa, u holda bu funksiya butun $\mathbb{C}$ *tekislikka nozik analitik davom qiladi, ya'ni butun* $\mathbb{C}$ *tekislikda nozik analitik bo'lgan* $\tilde{f}(z)$ *funksiya topilib,* $\tilde{f}|_U \equiv f$ *bo'ladi.*

Isbot. Isbotni bir necha qismlarga bo'lamiz:

1. Umumiylikni buzmasdan, f funksiya $B(0,r) \subset \mathbb{C}, r > 1$ doirada golomorf bo'lsin deb hisoblaymiz. Teorema shartiga ko'ra $f \in R^0$ funksiya chekli tartibga ega va

$$t = \overline{\lim_{m\to\infty}} \frac{\ln m}{\ln \dfrac{\ln m}{\rho_m^{\frac{1}{m}}}} < +\infty,$$

bunda $\rho_m = \rho_m(f, \overline{B})$ va $\overline{B} = \overline{B}(0,1)$. ρ_m ning ta'rifiga asosan ixtiyoriy $s > 0$ son uchun shunday $r_m(z) = \dfrac{p_m(z)}{q_m(z)}$, $m = 1,2,3,...$ ratsional funksiyalar ketma-ketligi mavjudki, bunda $\| f - r_m \|_{\overline{B}}^{1/m} = \rho_m^{1/m} \le \dfrac{\ln m}{m^{1/s}}, m \ge m'$ va $m = k^l$ $\forall k > 0$. Bu yerda $l > 7s$. $f(z)$ funksiyani

$$f(z) = r_{m_1}(z) + \sum_{k=1}^{\infty} \left[r_{m_{k+1}}(z) - r_{m_k}(z) \right] \tag{1}$$

deb olamiz. (1) qator $\overline{B}(0,1)$ yopiq doirada tez tekis yaqinlashadi.

2. Ushbu $r_{m_{k+1}}(z) - r_{m_k}(z)$ ayirmani $m_k \ge m'$ uchun baholaymiz:

$$\left\| r_{m_{k+1}}(z) - r_{m_k}(z) \right\|_{\overline{B}} \le \left\| r_{m_{k+1}}(z) - r_{m_{k+1}-1}(z) \right\|_{\overline{B}} + \left\| r_{m_{k+1}-1}(z) - r_{m_{k+1}-2}(z) \right\|_{\overline{B}} + ... +$$

$$+ \left\| r_{m_k+1}(z) - r_{m_k}(z) \right\|_{\overline{B}} \le \left\| r_{m_{k+1}}(z) - f(z) \right\|_{\overline{B}} + 2\left\| r_{m_{k+1}-1}(z) - f(z) \right\|_{\overline{B}} + ... +$$

$$+2\left\|r_{m_k+1}(z)-f(z)\right\|_{\overline{B}}+\left\|r_{m_k}(z)-f(z)\right\|_{\overline{B}}\le 2\left\{\frac{\ln m_{k+1}}{\left[m_{k+1}\right]^{m_{k+1}/s}}+\ldots+\frac{\ln m_k}{\left[m_k\right]^{m_k/s}}\right\}.$$

Hisoblashlardan kelib chiqadiki, $m_k=k^l$ uchun

$$\frac{\ln m_{k+1}}{\left[m_{k+1}\right]^{m_{k+1}/s}}+\ldots+\frac{\ln m_k}{\left[m_k\right]^{m_k/s}}\le l\cdot k\left(\frac{1}{(k+l)^{\frac{l\cdot(k+1)^l}{s}}}+\ldots+\frac{1}{k^{\frac{l\cdot k^l}{s}}}\right)\le 2lk\left[\frac{1}{k^l}\right]^{\frac{k^l}{s}}.$$

Bu yerda biz $\ln(k+1)\le k$ ekanidan foydalandik.

Natijada, ushbu

$$\left\|r_{m_{k+1}}(z)-r_{m_k}(z)\right\|\le 4kl\left[\frac{1}{k^l}\right]^{\frac{k^l}{s}},\ m_k\ge m' \tag{2}$$

baholash kelib chiqadi.

3. Endi $r_{m_{k+1}}(z)-r_{m_k}(z)$ ayirmani butun $\mathbb{C}$ tekislikda baholaymiz. Buning uchun $r_m(z)=\dfrac{p_m(z)}{q_m(z)}$ ifodaning surat va maxrajini o‘zgarmas songa ko‘paytirib, $q_m(z)$ ko‘phadning modul bo‘yicha maksimal koeffitsientini 1 ga teng bo‘ladigan qilib olamiz. U holda

$$r_{m_{k+1}}(z)-r_{m_k}(z)=\frac{p_{m_{k+1}}(z)q_{m_k}(z)-p_{m_k}(z)q_{m_{k+1}}(z)}{q_{m_k}(z)q_{m_{k+1}}(z)}$$

bo‘lgani uchun quyidagi tengsizlik o‘rinli

$$\left\|p_{m_{k+1}}(z)q_{m_k}(z)-p_{m_k}(z)q_{m_{k+1}}(z)\right\|_{\overline{B}}\le$$
$$\le\left\|r_{m_{k+1}}(z)-r_{m_k}(z)\right\|_{\overline{B}}\cdot\left\|q_{m_{k+1}}(z)\cdot q_{m_k}(z)\right\|_{\overline{B}}. \tag{3}$$

Ushbu $q_m(z)$ ko‘phadning modul bo‘yicha maksimal bo‘lgan koeffitsiyentlari 1 ga teng ekanligidan $\left\|q_m(z)\right\|_{\overline{B}}\le m+1$ bo‘lishi kelib chiqadi. (2) va (3) munosabatlardan

$$\left\|p_{m_{k+1}}(z)q_{m_k}(z)-p_{m_k}(z)q_{m_{k+1}}(z)\right\|_{\overline{B}}\le A_1\left[\frac{1}{k^l}\right]^{\frac{k^l}{s}},\quad m\ge m' \tag{4}$$

tengsizlikka ega bo‘lamiz, bunda $A_1 = 4kl(k^l+1)((k+1)^l+1)$. Endi, Bernshteyn-Uolsh tengsizligidan foydalanamiz, bunga ko‘ra ixtiyoriy $p(z)$ ko‘phad uchun

$$|p(z)| \le \|p(z)\| \cdot |z|^{\deg p}, \quad z \in \mathbb{C}$$

o‘rinli. Bundan va (4) dan kelib chiqadiki,

$$\left| p_{m_{k+1}}(z) q_{m_k}(z) - p_{m_k}(z) q_{m_{k+1}}(z) \right|_{\overline{B}} \le A_1 \left[\frac{1}{k^l} \right]^{\frac{k^l}{s}} \cdot |z|^{d_k}, \quad z \in \mathbb{C}.$$

Natijada, ushbu

$$\left| r_{m_{k+1}}(z) - r_{m_k}(z) \right| \le A_1 \left[\frac{1}{k^l} \right]^{\frac{k^l}{s}} \cdot |z|^{d_k} \cdot \frac{1}{\left| q_{m_k}(z) q_{m_{k+1}}(z) \right|}, \quad z \in \mathbb{C},\ m_k \ge m' \tag{5}$$

tengsizlik hosil bo‘ladi. Bu yerda $d_k = k^l + (k+l)^l$.

4. Ushbu $Q_k = \left\{ z \in \mathbb{C} : \left| q_{m_k}(z) q_{m_{k+1}}(z) \right|^{\frac{1}{d_k}} < \frac{1}{k^l} \right\}$ to‘plamni kiritib olamiz. Tekislikda kompakt bo‘lgan $D = B(0,R),\ R > 1$ sohani olamiz. Agar $Q_{k,D} = Q_k \cap D$ deb baholasak, u holda bu to‘plam sig‘imi uchun quyidagi baholash o‘rinli bo‘ladi

$$C\left(Q_{k,D}\right) = C\left(Q_{k,D}, D\right) \le \frac{A_2}{k^2}.$$

Bunda A_2 orqali R ga bog‘liq o‘zgarmas belgilangan. C sig‘imning sanoqli subadditiv ekanligidan $N \to 0$ da $C\left(\bigcup_{k \ge N} Q_{k,D} \right) \to 0$ munosabatga ega bo‘lamiz.

Demak, $\bigcap_{N=1}^{\infty} \bigcup_{j \ge N} Q_{k,D}$ va $E = \bigcap_{N=1}^{\infty} \bigcup_{j \ge N} Q_k$ to‘plamlar polyar ekan. Agar $z^0 \notin \bigcup_{j \ge N} Q_k$ bo‘lsa, u holda bu nuqtada $\left| q_{m_k}(z^0) q_{m_{k+1}}(z^0) \right|^{\frac{1}{d_k}} < \frac{1}{k^2},\ k \ge N$ bo‘lib, (5) tengsizlikka asosan quyidagi baholash bajariladi

$$\left| r_{m_{k+1}}(z^0) - r_{m_k}(z^0) \right| \le A_1 \left[\frac{1}{k^l} \right]^{\frac{k^l}{s}} \cdot \left| z^0 \right|^{d_k} \cdot k^{2d_k} = A_1 k^{\left[2d_k - \frac{l}{s} k^l \right]} \left| z^0 \right|^{d_k},\ k \ge N. \tag{6}$$

Agar $d_k = k^l + (k+1)^l$ ekanligidan foydalansak, ushbu

$$2d_k - \frac{l}{s}k^l = \left\{2\left[1+\left(1+\frac{1}{k}\right)^l\right] - \frac{l}{s}\right\}k^l$$

tenglikka ega bo’lamiz. $\frac{l}{s} > 7$ bo‘lishidan, yetarlicha katta $k \geq k_0$ lar uchun

$2d_k - \frac{l}{s}k^l \leq -3k^l$ bo‘lishi kelib chiqadi. Bunga ko‘ra ushbu

$$\left|r_{m_{k+1}}(z^0) - r_{m_k}(z^0)\right| \leq A_1 k^{-3k^l} \left|z^0\right|^{d_k}, \quad k \geq \max\{k_0, N\}, \ \ m_k \geq m' \qquad (7)$$

tengsizlik bajariladi. Qayd qilish kerakki,

$$\left|r_{m_{k+1}}(z^0) - r_{m_k}(z^0)\right|^{\frac{1}{d_k}} \leq A_1^{\frac{1}{d_k}} k^{-3\frac{k^l}{d_k}} \left|z^0\right|, \quad k \geq \max\{k_0, N\}, \ \ m_k \geq m'.$$

Bunda

$$A_1^{\frac{1}{d_k}} = \left[4kl(k^l+1)((k+1)^l+1)\right]^{\frac{1}{k^l+(k+1)^l}} \to 1, \ k \to \infty$$

$$\frac{k^l}{d_k} = \frac{k^l}{k^l+(k+1)^l} \to \frac{1}{2}, \quad k \to \infty$$

bo‘lgani bois, yetarlicha katta $k \geq k(z^0)$ uchun quyidagi tengsizlik o‘rinli:

$$\left|r_{m_{k+1}}(z^0) - r_{m_k}(z^0)\right|^{\frac{1}{d_k}} < \frac{\left|z^0\right|}{k}, \qquad k \geq k(z^0)\,.$$

Bu tengsizlikning isbotidan $z \in \overline{D} \setminus \bigcup_{j \geq N} Q_j$ uchun bu tengsizlik tekis bajarilishi ko‘rinadi, ya`ni shunday butun $k_0 \in N$ son mavjudki, bunda

$$\left|r_{m_{k+1}}(z) - r_{m_k}(z)\right|^{\frac{1}{d_k}} < \frac{R}{k}, \qquad k \geq k_0\,, \ \ z \in \overline{D} \setminus \bigcup_{j \geq N} Q_j \qquad (8)$$

bo‘ladi.

5. $Z = \bigcup_{k=1}^{\infty}\{z \in \mathbb{C} : \ q_k(z) = 0\}$ to‘plam polyar bo‘lgani uchun $v(z) \in sh(\mathbb{C})$ subgarmonik funksiya mavjudki, $v \not\equiv -\infty, v|_Z \equiv -\infty$ bo‘ladi. Ushbu

$$S_N = \left\{\bigcup_{k \geq N} Q_k\right\} \cup Z_N\,, \ \ A = \bigcap_{N=1}^{\infty} S_N$$

belgilashlarni kiritib olamiz, bunda $Z_N = \left\{ z \in \mathbb{C} : v(z) < \frac{1}{N} \right\}$. Sig'imning sanoqli subadditiv ekanligidan quyidagi munosabat

$$C\left(S_N \cap D\right) \le \sum_{k \ge N} C(Q_k \cap D) + C(Z_N \cap D) \to 0$$

kelib chiqadi.

Bundan, S ning plyuripolyar to'plam bo'lishi ko'rinadi. S ning tashqarisida (1) qator tez yaqinlashadi va uning yig'indisi $\mathbb{C} \setminus S$ dagi $f(z)$ funksiyani aniqlaydi. (8) ga asosan tayinlangan natural N son uchun shunday natural $k_0 > N$ mavjudki, bunda $z \in \overline{D} \setminus \bigcup_{j \ge N} Q_j$ kompaktda quyidagi tekis baholash o'rinli bo'ladi

$$\left| r_{m_k}(z) - f(z) \right| = \left| \sum_{t \ge k} \left(r_{m_{t+1}}(z) - r_{m_t}(z) \right) \right| \le$$

$$\le \sum_{t \ge k} \left| r_{m_{t+1}}(z) - r_{m_t}(z) \right| \le \sum_{t \ge k} \left(\frac{R}{t} \right)^{d_t} \approx \left(\frac{R}{k} \right)^{d_k}, \; k \ge k_0. \qquad (9)$$

Endi $f(z)$ funksiya $\mathbb{C}$ da nozik analitik funksiya bo'lishini ko'rsatamiz. Shu maqsadda $z^0 \in \mathbb{C} \setminus S$ nuqtani tayinlaymiz va $z^0 \in D \setminus S$ deb olamiz. Aytaylik $F_N = \overline{D} \setminus S_N$, $F = \bigcup_{N=1}^{\infty} F_N$ bo'lsin. U holda $D \setminus F \subset S$ ayirma $z^0 \in F$ nuqtada "uzilgan" to'plam bo'ladi. F_N to'plam kompakt bo'lib, (9) ga asosan

$$\lim_{k \to \infty} \left\| r_{m_k}(z) - f(z) \right\|_{F_N} = 0\,.$$

Bu esa, $f|_{F_N} \in R(F_N)$ va $f(z)$ funksiyalarning z^0 nuqtada nozik analitik bo'lishini ko'rsatadi. **Teorema isbotlandi.**

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