THE IMPORTANCE OF CONDITIONS IN PRIVALOV'S THEOREM ABOUT SUBHARMONIC FUNCTIONS

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Annotatsiya (Rezyume)

Ushbu maqolada subgarmonik funksiyalar haqidagi Blyashke-Privalov hamda Privalov teoremalari keltirib o‘tilgan. Yangi singulyar to‘plamlar kiritilgan bo‘lib, ularning ba’zi xossalari ham isbotlangan.

Аннотация (Резюме)

В данной работе приведена теорема Бляшке-Привалова и теорема Привалова о субгармонических функций. Введены новые сингулярные множества, доказаны некоторые их свойства.

Subharmonic functions play an important role in the complex analysis and in the classical potential theory. One of the notable theorems in the theory of subharmonic functions is Privalov's theorem. In the present paper we give a structure of a removable set to Privalov's theorem.

Definition 1. A function $u: D \to [-\infty, \infty)$, given in domain $D \subset \mathbb{R}^n$ is called subharmonic (*sh*) if it satisfies following two conditions:

1) $u(x)$ is upper semi-continuous, i.e. $\forall x^0 \in D$ holds following inequality

$$\overline{\lim_{x \to x^0}} u(x) \le u(x^0);$$

2) for every point $x^0 \in D$ there exists a $r(x^0) > 0$ such that, for all $r \le r(x^0)$ holds inequality

$$u(x^0) \le \frac{1}{r^{n-1}\sigma_n} \int_{S(x^0, r)} u(x) d\sigma$$

where $S(x^0, r)$ is a sphere and σ_n is the area of $S(0,1) \subset \mathbb{R}^n$.

Note that, subharmonic function is summable, i.e. $u(x) \in L^1_{loc}(D)$ and its Laplace operator $\Delta u \geq 0$ in the generalization sense [1].

Theorem 1. [2] (Blaschke-Privalov). If the function $u(x), u(x) \not\equiv -\infty$, is upper semi-continuous in the domain $D \in \mathbb{R}^n$ holds following inequality

$$\bar{\Delta} u(x) \geq 0 \quad \forall x^0 \in D \setminus u_{-\infty}$$

then $u(x)$ is subharmonic in D.

Here, $u_{-\infty} := \{x \in D : u(x) = -\infty\}$, $\bar{\Delta} u(x)$ - *generalized Laplace operator* of a function u at the point x. For more information about *generalized Laplace operator* and its properties see [2-4].

I.Privalov got more deeper result with exceptional set E.

Theorem 2. [5] (Privalov). If the function $u(x), u(x) \not\equiv -\infty$, is upper semi-continuous in the domain $D \in \mathbb{R}^n$ and holds the following conditions

$$\bar{\Delta} u(x) \geq 0 \quad \forall x^0 \in [D \setminus u_{-\infty}] \setminus E,\ mesE = 0;$$

$$\bar{\Delta} u(x) > -\infty \quad \forall x^0 \in E \text{ except a polar set } P \in E,$$

then the function $u(x)$ is subharmonic in D.

The conditions $\bar{\Delta} u(x) > -\infty$ on $E, mesE = 0$ and polarity of P are necessary.

Example 1. Let's given a compact $P \in D: C(P) > 0, mesP = 0$. Then a harmonic measure [1] $w^*(x, P, D)$ possesses following properties:

1) $-1 \leq w^*(x, P, D) < 0$;
2) $w^*(x, P, D)$ is harmonic in $D \setminus P$ and subharmonic in D.

In that case the function $u(x) = -w^*(x, P, D)$ will be harmonic in $D \setminus E, E = P$, $mesE = 0$ but it is not subharmonic in D. On E the condition $\overline{\Delta}u(x) > -\infty$ is not fulfilled, excluding a polar set.

Example 2. Let $E = \{|x| \le 1\}, D = \mathbb{R}^3$. Then the function

$$u(x) = \begin{cases} \frac{1}{|x|} - 1, & if \ |x| \ge 1; \\ 0, & if \ |x| < 1 \end{cases}$$

is not subharmonic in $\mathbb{R}^3$, although $\overline{\Delta}u(x) > -\infty$ in $\mathbb{R}^3$. (Here $mesE > 0$).

In order to develop Privalov's theorem we introduce the following, so called singular sets.

Definition 2. E is called $\underline{S}$ (*singular*) – set, if there exists

$$v(x) \in sh(\mathbb{R}^n) : \underline{\Delta}v(x)\big|_E = +\infty.$$

E is called $\overline{S}$ (*singular*) – set, if there exists

$$v(x) \in sh(\mathbb{R}^n) : \overline{\Delta}v(x)\big|_E = +\infty.$$

Some properties of *singular sets:*

1. $\underline{S}-$ set is a $\overline{S}-$ set, i.e. $\underline{S}-$ sets $\subset$ $\overline{S}-$ sets, (clear).
2. Countable union of singular $\underline{S}-$ sets is singular $\underline{S}-$ set.

In fact, let $E_1, E_2 \in \underline{S}$, i.e. $\exists v_1(x), v_2(x) \in sh(\mathbb{R}^n)$, $\underline{\Delta}v_1(x)\big|_{E_1} = +\infty, \underline{\Delta}v_2(x)\big|_{E_2} = +\infty.$ Put $v(x) = v_1(x) + v_2(x)$. Then $v(x) \in sh(\mathbb{R}^n)$ and $\forall x^0 \in E = E_1 \cup E_2$, we have

$$\underline{\Delta}v(x) = \underline{\lim}_{r \to +0} \frac{M_v(x^0, r) - v(x^0)}{r^2} =$$

$$= \underline{\lim}_{r\to+0}\left[\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}\right]\geq$$

$$\geq \underline{\lim}_{r\to+0}\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\underline{\lim}_{r\to+0}\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}=+\infty.$$

So $E\in \underline{S}$.

If $E=\bigcup_{j=1}^{\infty}E_j$, $E_j\in\underline{S}\Rightarrow \exists v_j\left(x\right)\in sh\left(\mathbb{R}^n\right), v_j\left(x\right)\not\equiv-\infty, \underline{\Delta} v_j\left(x\right)\Big|_{E_j}=+\infty.$ The set $F_j=\left\{x\in\mathbb{R}^n : v_j\left(x\right)=-\infty\right\}\supset E_j$ has zero measure. Consequently, $F=\bigcup_{j=1}^{\infty}F_j$ also has zero measure, $mesF=0$. Hence there is a point $x^0\in\mathbb{R}^n$ such that $v_j\left(x^0\right)\neq-\infty, j=1,2,...$ Take the exhaustion $\mathbb{R}^n=\bigcup_{j=1}^{\infty}D_j, D_j\subset\subset D_{j+1}\subset\subset\mathbb{R}^n$, $x^0\in D_1$ and we denote $M_j=\sup_{D_j}v_j\left(x\right)$. Put $v\left(x\right)=\sum_{j=1}^{\infty}\varepsilon_j\left(v_j\left(x\right)-M_j\right)$, where $\varepsilon_j=\frac{1}{2^j}[M_j-v_j\left(x^0\right)]>0$. Then in any fixed compact $K\subset\subset\mathbb{R}^n$ starting from some number $j\geq N\left(K\right)$ terms of this series are negative. Consequently, the sum of the series is subharmonic in $\mathbb{R}^n$. Furthermore, $v\left(x^0\right)=-1$, i.e. $v\not\equiv-\infty$ and $\underline{\Delta} v\left(x\right)\geq\sum_{j=1}^{\infty}\varepsilon_j\underline{\Delta} v_j\left(x\right)\geq\varepsilon_1\underline{\Delta} v_1\left(x\right)=+\infty$, i.e., $E\in\underline{S}$.

3. Finite union of $\overline{S}-$ sets is singular $\overline{S}-$ set.

It is enough to prove for the union of two sets. Let $E_1,E_2\in\overline{S}\Rightarrow\exists v_1\left(x\right),v_2\left(x\right)\in sh\left(\mathbb{R}^n\right), \overline{\Delta} v_1\left(x\right)\Big|_{E_1}=+\infty, \overline{\Delta} v_2\left(x\right)\Big|_{E_2}=+\infty.$ Put $v\left(x\right)=v_1\left(x\right)+v_2\left(x\right)$. Then $v\left(x\right)\in sh\left(\mathbb{R}^n\right)$ and $\forall x^0\in E=E_1\cup E_2$, we have

$$\overline{\Delta} v\left(x\right)=\overline{\lim}_{r\to+0}\frac{M_v\left(x^0,r\right)-v\left(x^0\right)}{r^2}=$$

$$= \overline{\lim_{r\to+0}}\left[\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}\right] \geq$$

$$\geq \overline{\lim_{r\to+0}}\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\underline{\lim_{r\to+0}}\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}=+\infty$$

and

$$\overline{\Delta}v\left(x\right)=\overline{\lim_{r\to+0}}\frac{M_{v}\left(x^0,r\right)-v\left(x^0\right)}{r^2}=$$

$$= \overline{\lim_{r\to+0}}\left[\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}\right] \geq$$

$$\geq \underline{\lim_{r\to+0}}\frac{M_{v_1}\left(x^0,r\right)-v_1\left(x^0\right)}{r^2}+\overline{\lim_{r\to+0}}\frac{M_{v_2}\left(x^0,r\right)-v_2\left(x^0\right)}{r^2}=+\infty.$$

So $E \in \overline{S}$.

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