

CONSTRAINED OPTIMIZATION : THE METHOD OF LAGRANGE MULTIPLIERS

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In mathematical optimization, constrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables. The objective function is either a cost function or energy function which is to be minimized, or a reward function or utility function, which is to be maximized.

Constraints can be either hard constraints which set conditions for the variables that are required to be satisfied, or soft constraints which have some variable values that are penalized in the objective function if, and based on the extent that, the conditions on the variables are not satisfied.

In many applied problems, a function of two variables is to be optimized subject to a restriction or **constraint** on the variables. For example, an editor, constrained to stay within a fixed budget of \$60,000, may wish to decide how to divide this money between development and promotion in order to maximize the future sales of a new book. If x denotes the amount of money allocated to development, y the amount allocated to promotion, and $f(x, y)$ the corresponding number of books that will be sold, the editor would like to maximize the sales function $f(x, y)$ subject to the budgetary constraint that $x + y = 60000$

The method of Lagrange multipliers uses the fact that any relative extremum of the function $f(x, y)$ subject to the constraint $g(x, y) = k$ must occur at a critical point (a, b) of the function

$$F(x, y) = f(x, y) - \lambda(g(x, y) - k)$$

where λ is a new variable (the **Lagrange multiplier**). To find the critical points of F , compute its partial derivatives

$$F_x = f_x - \lambda g_x, \quad F_y = f_y - \lambda g_y, \quad F_\lambda = -(g - k)$$

and solve the equations $F_x = 0$, $F_y = 0$ and $F_\lambda = 0$ simultaneously, as follows:

$$F_x = f_x - \lambda g_x = 0 \quad \text{or} \quad f_x = \lambda g_x$$

$$F_y = f_y - \lambda g_y = 0 \quad \text{or} \quad f_y = \lambda g_y$$

$$F_\lambda = -(g - k) = 0 \quad \text{or} \quad g = k$$

Finally, evaluate $f(a, b)$ at each critical point (a, b) of F .

There is a version of the second partials test that can be used to determine what kind of constrained relative extremum corresponds to each critical point (a, b) of F .

For a geometric interpretation of the process of optimizing a function of two variables subject to a constraint, think of the function itself as a surface in three-dimensional space and of the constraint (which is an equation involving x and y) as a curve in the xy plane.

When you find the maximum or minimum of the function subject to the given constraint, you are restricting your attention to the portion of the surface that lies directly above the constraint curve. The highest point on this portion of the surface is the constrained maximum, and the lowest point is the constrained minimum.

Here we would solve one problem in economics by the method of Lagrange multipliers.

Problem. A jewel box is to be constructed of material that costs \$1 per square meter for the bottom, \$2 per square meter for the sides, and \$5 per square meter for the top. If the total volume is to be 96 m^3 what dimensions will minimize the total cost of construction?

Solution. Let the box be x meters deep, y meters long, and z meters wide. Then the volume of the box is $V = xyz$ and the total cost of construction is given by

$$C = 1yz + 2(2xy + 2xz) + 5yz = 6yz + 4xy + 4xz$$

You wish to minimize $C = 6yz + 4xy + 4xz$ subject to $V = xyz = 96$. The Lagrange equations are

$$C_x = \lambda V_x \quad \text{or} \quad 4y + 4z = \lambda yz$$

$$C_y = \lambda V_y \quad \text{or} \quad 6z + 4x = \lambda xz$$

$$C_z = \lambda V_z \quad \text{or} \quad 6y + 4x = \lambda xy$$

and $xyz = 96$. Simultaneously solving the equations gives us the result $x = 6$, $y = z = 4$. Thus, the minimal cost occurs when the jewel box is 6 meters deep with a square base, 4 meters on a side. So its minimal cost is \$288.

Another example to a constrained optimization is utility function, which is to be maximized. A **utility function** $U(x, y)$ measures the total satisfaction or *utility* a consumer receives from having x units of one particular commodity and y units of another. The problem is to determine how many units of each commodity the consumer should buy to maximize utility while staying within a fixed budget.

Although a rigorous explanation of why the method of Lagrange multipliers works involves advanced ideas beyond the scope of this text, there is a rather simple geometric argument one should find convincing. This argument depends on the fact that for the level curve $F(x, y) = C$, the slope is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

This result is true for any level curve of a function whose partial derivatives exist (provided $F_y \neq 0$).

Литература

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