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APPLICATION OF ASYNCHRONOUS ITERATION METHOD TO POLYNOMIAL NONLINEAR EQUATION

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Iterations methods for the solution linear and nonlinear equations are widely used because of their simplicity, fault tolerance, ease of parallelization. Historically, iterative algorithms were created and studied for use single processor computers.

In multiprocessor computers the parallel application of iterative methods often shows poor scaling and less than optimal parallel efficiency. As opposed to ordinary iterative asynchronous method often have much better parallel efficiency as they almost never need to wait to communicate between possessors.

Our investigation concerns to study of asynchronous algorithms with probabilistic approach and presenting a mathematical description of this computational process to the multiprocessor environment.

Convergence of the asynchronous method

$$\begin{aligned} \Psi_1 &= f_1(\Psi_1, ..., \Psi_N) \\ \Psi_2 &= f_2(\Psi_1, ..., \Psi_N) \\ \Psi_N &= f_N(\Psi_1, ..., \Psi_N) \end{aligned} \tag{1}$$

We can write the system (1) in operator form as

$$\psi = F(\psi)$$

.

Below we define [1,3] an asynchronous iterative method for solving the system (1). Let $\{J_n\}_{n=1}^{\infty}$ be a sequence of non-empty subsets of set {1,2,...,N} which is a called chaotic sequence of sets.

Let the given initial vector be $\psi(0)$. We will construct the sequence $\{\Psi(n)\}_{n=1}^{\infty}$ of iterations by the following way [4,5]

$$\psi(n+1) = F_{J_{n+1}}(\psi(n)), \quad n = 0, 1, 2. \tag{2}$$

Here

$$\Psi_i(n+1) = \begin{cases} \Psi_i(n), & \text{if } i \notin J_n \\ f_i(\Psi(n)), & \text{if } i \in J_n \end{cases} \tag{3}$$

The method of chaotic iterative (3) is a generalization of sequential iterative methods. If $J_n = J = \{1, 2, .., N\}$ then all components of the iteration vector are updated at the same time and as a result we get the simple Jacobi iterative method $\psi(n+1) = F(\psi(n))$ n=0,1,2,...

The components of the iteration vector are refreshed the cyclically, if $J_n = \{n(modN)\}$. In computing the following component, we use previous one, which already been computed using a Gauss-Seidel iteration. The main property of chaotic iterations is the random updating of the component iterative vector, which allows us to implement the efficiency on multiprocessor systems.

The generalization of method of chaotic iterative is the method of asynchronous iterations. The method of asynchronous iterations (1) is constructed with the following rule. Let (0) be given, then

$$\Psi_i(n) = \begin{cases} \Psi_i(n-1), & \text{if } i \notin J_n \\ f_i(Psi_1(S_1(n)), ..., \Psi_N(S_N(n))), & \text{if } i \in J_n \end{cases} \tag{4}$$

Here ψ_i is a component of the vector ψ, and $\{S_i(n)\}_{n=1}^{\infty}$, i=1,N is a sequence of sets of non-negative integers, satisfying the following conditions, for any i=1,N

$$S_i(n) \leq n-1, n = 1, 2, ..., i = 1, ..., N \tag{5}$$

$$S_i(n) \to \infty, n \to \infty; i = 1, ..., N \tag{6}$$

and every element i occurs infinitely many often in the sets J_n n=1,2... . The $S_i(n)$, i=1,N are called delays or lag. The condition (5) says that only components of previous iterates can be used in the evaluation of a new iterate. The condition (6) eventually say that values of an early iterate cannot be used any more in further evaluations, and more and more recent values of the components have to be used instead. The last condition "every element i occurs infinitely many often in the sets J_n n=1,2... "guarantees that no component is abandoned forever.

Let $0 \leq S_i(n) < S_1$, where S_1 is the maximum number of iterations saved, i.e. when computing iterations we use components of vectors of previous iterative with no more than S_1 in the past.

Let all elements in the sequence of subsets J_n have at least one element from {1,2,...,N}. Then the following statement is valid

Lemma. For convergence of the asynchronous iterative method in R^n to the solution of (1) it is necessary that the spectral radius p(F)<1.

The proof of this lemma is given in [6, p. 100)]. This condition also provides convergence of a simple iterative process for the system (1).

Let X be a Banach space. In X we consider the following polynomial nonlinear equation:

$$\phi = \sum_{m=1}^{\infty} K_m(\phi, ..., \phi) + f = P(\phi) \tag{7}$$

where $f \in X, K_m : X^m \to X$ is a polynomial map, m=1,2,...,n, which is continuous and which is equivalent to their bounded-ness. Then there are exist constants $C_m \geq 0$, so that for any $x_1, x_2, ..., x_m \in X$

$$\| K_m(x_1, ...x_m) \| \leq C_m \prod_{i=1}^{m} \| x_i \|, m = 1, n.$$

For the equation (7) we create the following iterative method. Suppose that $\phi^{-n+2} = \phi^{-n+3} = ... = \phi^0 = 0$ and at $i \geq 1$ we define $\phi^{(i)}$ as the solution of the following equation

$\phi^{(i)} = f + K_1(\phi^{(i)}) + \sum_{m=2}^{n} K_m(\phi^{i-m+1}, \phi^{i-m+2}, ..., \phi^i)$ It is known that [2], the system (1) is equivalent to equation (7) in the following sense. If the iterative method converges for (7) with initial value $\phi^{(0)} = f$, then the method for (1) will converge, starting from the initial vector (f,0,0,...,0), and vice versa. Using the results of lemma for the convergence of asynchronous iterations (4) the following assertion will be true [1].

Theorem. If the conditions of lemma are satisfied, then the asynchronous iterative process (4) will converge to the solution of the system (1).

The main difference between asynchronous iterative and the other iterative methods in parallel is the chaotic behavior of the vector components, which is expressed by the set of chaotic sequences J_n The chaotic iterative process has the following two main advantages:

a) it is possible to calculate each coordinate of the iteration vector independently from the others (like the Monte Carlo method),

b) the convergence rate is higher, because this method sometimes essentially becomes an implicit iterative method like Gauss-Seidel method.

To estimate the speed of convergence of the iterative method we usually use the value [4]

$$R = \liminf_{n\to\infty} \frac{-\log \parallel \Psi(n) - \xi \parallel_0}{n},$$

where ξ is a fixed point of the operator F, and

$$\parallel \Psi \parallel_0 = \max_{i=1,N} \parallel \Psi_i \parallel_i .$$

An efficiency of an iterative method we usually use the value:

$$E = \liminf_{n\to\infty} \frac{-\log \parallel \Psi(n) - \xi \parallel_0}{C_n},$$

where, c_n is the cost associated with the evaluation of the first n iterates. Usually we choose c_n, proportional to the number of arithmetic operations and are necessary to compute the first n iterations, or the computer time used for computing of n iterations. We note that if c_n/n tends to some finite τ (which corresponds to the average cost per step), then the efficiency is simple given by $E = R/\tau$.

In the asynchronous iteration case, we can determine the value of c_n in the following way: $c_n = \frac{|J_1|+\ldots+|J_n|}{n}$, where $|J_i|$ is the cardinality of the set $|J_n|$ i.e. number of components evaluated at the n-th step of the iterative. In this case, the cost is better suited for a parallel implementation, and can be evaluated through the classical tools of queuing theory. Let's denote the number of macro iterations (accumulated iterative) in the asynchronous process with as $|q_n|$. Then we obtain

$$R \geq -\lim_{n\to\infty} (\frac{q_n}{n}) \log p(L),$$

$$E \geq -\lim_{n\to\infty} (\frac{q_n}{c_n}) \log p(L),$$

where L is a contraction matrix of the operator F.

Numerical experiment

To illustrate our points we solved the following nonlinear boundary value problem in the unit square:

$$\triangle u(x,y) = u^2(x,y) + 4$$

$$u(x,y)|_{x=0} = y^2, u(x,y)|_{y=0} = x^2, u(x,y)|_{x=1} = 1 + y^2,$$

$$u(x,y)|_{y=1} = 1 + x^2.$$

The exact solution of this problem is $u(x,y) = x^2 + y^2$. Using the finite difference method, we get the following system of nonlinear algebraic equations:

$$u_{ij}^{(k)} = \frac{1}{4}(u_{i+1,j}^{(k-l)} + u_{i-1,j}^{(k-l)} + u_{i,j-1}^{(k-l)} + u_{i,j+1}^{(k-l)}) - \frac{h^2}{4} \times ((u_{ij}^{(k-l)})^2 + 4), i, j, = 1, n, k = 1, 2...$$

Here k is the iteration number. Suppose we use a five processor systems, due to the five point stencil we use. With probability p each processor is able to compute the next iteration in time. The results of the numerical experiments are given in the table below, where p is the probability of success. Here n is the number of iterations, T is the relative unit CPU time, û is iterative and u is the exact solution.

n-number of iterations	T	P	ũ (7,3)- iterative solution	u (7,3)- exact solution
100	1	1	1.26	1.30
200	2	0,5	1.27	1.30
300	1	0,6	1.16	1.30

The numerical experiments were done, with $\{J_n\}_{n=1}^{\infty}$ and $\{S_i(n)\}_{n\to\infty}^{\infty}$ having different probability distributions.

Conclusion

The asynchronous iterative method converges to the fixed point slower than the simple Jacobi iterative methods; however, the efficiency of asynchronous iterative process is better. The results of our simple computational experiments show the convergence and efficiency of asynchronous iterative processes for considered problems.

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