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«Aniqmas integrallar va integrallash usullari »
mavzusida

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MUNDARIJA

KIRISH

Ta'lim tizimida yakuniy natija, bevosita ta'lim–tarbiya jarayonini amalga oshiradigan o'qituvchi mehnatining qanday tashkil etilishiga bog'liq. Ta'lim zimmasiga qo'yilayotgan ulkan vazifalar esa ta'lim berishga munosabatni, yondashuvni o'zgartirishni taqozo etmoqda. Shu munosabat va yondashuvni o'zida mujassam etishi lozim bo'lgan yangi pedagogik texnologiya xususida bir qancha maqsadlar e'lon qilindi. Biroq hozirdagi islohotlar jadalligi mavjud nazariyani amaliyotga tatbiq etishni talab etadi. Shu sababli ham birinchi navbatda, ta'lim mazmuni va uning tarkibini kengaytirish va chuqurlashtirish, xususan, bu mazmunga nafaqat bilim, ko'nikma va malaka, balki umuminsoniy madaniyatni tashkil qiluvchi – ijodiy faoliyat tajribasi, tevarak–atrofga munosabatlarni ham kiritish g'oyasi kun tartibiga ko'ndalang qilib qo'yildi.

Pedagogik tizimning moddiy ositalari kitoblardan, texnik jihozlardan va o'qitish metodikalaridan ko'pdan buyon foydalanib kelinadi yoki oldindan rejalashtirilgan ta'lim–tarbiya jarayoni professor–o'qituvchi faoliyati orqali amaliyotga joriy etilgan pedagogik tizimdir. SHu bilan birgalikda pedagogik tizim – bu qotib qolgan, siquvga olingan loyiha emas, balki ta'lim va tarbiya samaradorligini belgilovchi qator omillarni baholash imkonini bera oladigan ijodkorlik, yaratuvchilik, fidoyilik faoliyati natijasidir.

Bitiruv ishing dolzarbligi: Yangi ta'lim tizimi mazmunida o'quv reja, darsliklar asosida o'quv jarayonini loyihalashtirishga ham yangicha yondashish va tashkil etish zarurati tug'ilmoqda. Pedagogik texnologiyani joriy etish tajribasini o'rganish va unga ijodiy yondashish, o'quv jarayonini insonparvarlashtirish, bunda talabani sust ob'ektdan faol sub'ektga aylantirish, bilish faoliyatining aniq, maqsadlarga yo'nalganligini hamda o'quv jarayonini ishlab chiqarish jarayoni kabi takrorlanuvchanligini ta'minlashda muhim ahamiyatga ega bo'ladi.

Mazkur ishda zamonaviy tushuncha bo'lgan ta'lim texnologiyasida modulli o'qitish masalasi yoritilib uni mazmuni va qo'llash usul, uslublari xaqida mulohazalar yuritiladi.

Bitiruv ishining ob'ekti va predmeti: Bitiruv malakaviy ishida matematik analizning asosiy tushunchalaridan bo'lgan aniq integrallar va integrallash usullari hisobi masalalari ko'rib chiqiladi. Matematik analizda, aniq integrallar va integrallash usullarining turli masalalarni yyechishga tatbiqlari beriladi.

Bitiruv ishida aniq integrallar va integrallash usullarining klassifikatsiyasi, aniq integrallar va integrallash usullariga doir ma'lumotlar berilgan. Aniq integrallar va integrallash usullariga ko'plab misollar yechib ko'rsatilgan.

Bitiruv malakaviy ishi kirish, ikki bob, xulosa va adabiyotlar ro'yxatidan iborat bo'lib, birinchi bobda aniq integrallar va integrallash usullarining matematik analiz fanida tutgan o'rni va tarixiy rivojlanishi obzori keltirilgan. Ikkinchi bobda, oliy o'quv yurtlarining bakalavr talabalari uchun o'qitiladigan matematik analiz fanidagi «Aniq integrallar va integrallash usullari» modulining ta'lim texnologiyasi keltirilgan.

Bitiruv ishining maqsadi va vazifalari. "Matematik analiz" fanining muhim jihati uning amaliyot zarurati tufayli yuzaga kelganligidir. "Matematik analiz" fani talabalarda matematik modellarni qo'llash va tahlil qilish sohasidagi bilimlarini to'ldiradi. SHuning uchun muammolarni hal qilishda mazkur fanning optimal usullaridan foydalanish hozirgi zamon talabidir.

"Matematik analiz" fanini o'qitishning asosiy maqsadi:

– talabalarni mantiqiy, algoritmik, abstrakt fikrlashga;

– matematik tafakkurini shakllantirish va rivojlantirishga;

– fikr mulohazalar va xulosalarni asosli va tarzda aniq bayon eta olishga o'rgatishdan iboratdir.

"Matematik analiz" fanini o'qitishning asosiy vazifalar:

– talabalarda nazariy va amaliy masalalarini yyechishga yetarli matematik apparatni egallashiga;

– matematik masalalarni tahlil qila olishga o'rgatishdan iboratdir.

Bitiruv ishi nazariy va amaliy ahamiyati. Mazkur ishda “Matematik analiz” fanining integral hisob elementlari bo’limlari kiritilgan. SHuningdek, matematik tushuncha va tasdiqlarning matematik talqini misollar yordamida yoritildi. “Matematik analiz” kursini o’qitishdan asosiy maqsad talabalarda mantiqiy, algoritmik, abstrakt fikrlash, matematik tafakkurini shakllantirish va rivojlantirish, o’zining fikr–mulohazalarini, xulosalarini asosli tarzda aniq bayon etishga o’rgatish, nazariy va amaliy masalalarini yecha olishga yetarli matematik apparatni egallash va uni qo’llash, matematik masalalarning matematik modelini tuzish va tahlil qilishga o’rgatishdan iborat.

Matematika ta’limda asosiy (tayanch) fan hisoblanib, u algebra, matematik analiz, ehtimollar nazariyasi va matematika statistika, differentsial tenglamalar va boshqa fanlarga asoslanadi.

Matematik islohotlarni hayotga samarali tatbiq etishning yana bir sharti matematik ta’lim tizimini modernizatsiyalashdir, matematika sohasi uchun mutaxassislarni shakllantirish mexanizmini yaratishdir.

Hozirgi kunda O’zbekistonda ta’lim tizimidagi islohotlarning asosini shakllantiruvchi qator me’yoriy xujjatlar qabul qilingan va amalga oshirilib kelinmoqda. Bular asosida “Ta’lim to’g’risida”gi va “Kadrlar tayyorlash milliy dasturi to’g’risida”gi qonunlar alohida o’rin tutadi. Bu qonunlardan kelib chiqadigan vazifa ta’lim dasturlari mazmunining yuqori sifatiga erishish va yangi pedagogik texnologiyalarni joriy qilishdir.

Ushbu usullar talabalarning ijodiy faolligini oshirishda, matematik masalalarni hal qilishda, muammoni hal qilishning eng maqbul yo’llarini topishda yordam beradi.

Bitiruv ishining tuzilishi va xajmi. Bitiruv ishi kirish, ikki bob, beshta paragraf, xulosa va foydalanilgan adabiyotlardan iborat holda yoritib berilgan. Bitiruv ishida 25 ga yaqin adabiyotlardan foydalanilgan.

I–БОБ. "AHIQ INTEGRALLAR VA INTEGRALLASH USULLARI" MAVZUSINI O'QITISHNING NAZARIY MASALALARI.

1.1-§. Matematik analiz fanidagi integral hisobning yaratilish tarixi

Katta amaliy ahamiyatga ega bo'lish uchun bizga ma'lum bo'lgan qoidaga asosan, berilgan funktsiyaning hosilasini topish kerak. SHuningdek, berilgan qonunga ko'ra jismning harakatini topish uchun yo'ldan vaqt bo'yicha hosila olib uning tezligini topamiz; qandaydir noma'lum egri chiziqning ixtiyoriy nuqtasiga o'tkazilgan urinmaning burchak koeffitsienti aniqlanadi va h.k.

Ayrim hollarda teskari masalani yyechishga to'g'ri keladi: berilgan tezlik bo'yicha o'tilgan yo'l topiladi, berilgan burchak koeffitsientga ko'ra egri chiziq tenglamasini topish talab etaladi va h.k., boshqacha aytganda, berilgan hosilaga ko'ra hosilasi $f(x)$ funktsiyaga teng bo'lgan $y = F(x)$ funktsiyani topish talab etiladi.

Aniq integralning hozirgi zamon tushunchasiga hammadan ko'ra Paskalь yaqin edi va uning kuch–qudratini bayon qilgan edi. Bunga Paskalning tsikloidaga bog'liq masalalarni, turli yuzlarni, hajmlarni, yoylar uzunligini hisoblash masalalarini hal etganligini misol qilib keltirish mumkin. Ular «A. Dettonvilning geometriyadagi turli kashfiyotlar» nomli kitobida keltirilgan.

Integral xisobga oid 1686 yilda nashr qilingan Leybnitsning «O glubokoy geometrii i analize nedelimых, a takje beskonechnых» asarida birinchi marta integral $\int$ belgisi uchraydi (kichik s harfi shaklida). U integralni egri chiziqli figuraning yuzini asosi dx bo'lgan cheksiz ingichka to'rtburchaklar yuzlari yig'indisi sifatida qaraydi. Har bir bu tik egri to'rtburchaklar yuzi $f(x)dx$ bo'lgani uchun butun yuz bunday ifodalarning cheksiztasi yig'indisiga teng. Bu cheksiz kichiklarning cheksiz yig'indisini Leybnits integral (lotincha – integer – butun so'zidan olingan) deb atadi va

$$\int f(x)dx$$

belgilash kiritdi ($\int$ – ishora lotincha summa so'zi birinchi harfining boshqacha yozilishidan iborat).

Aniqmas integralning ta'rifi va xossalari

Berilgan $F(x)$ funktsiyani differentsiallashda uning $f(x)=F'(x)$ hosilasini topish talab qilinadi. Masalan: $F(x)=x^3$, $f(x)=3x^2$. Teskari masalani qaraylik: berilgan $f(x)$ hosilasi bo'yicha shunday $F(x)$ funktsiyani topingki, uning hosilasi $f(x)$ ga teng, ya'ni $F'(x)=f(x)$ bo'lsin.

1–ta'rif. *Hosilasi $f(x)$ ga teng bo'lgan $F(x)$ funktsiya $f(x)$ funktsiyaning boshlang'ich funktsiyasi (boshlang'ichi) deyiladi.*

1– misol. Berilgan $f(x)=3x^2$. $F(x)$ boshlang'ich funktsiyani toping.

Yechish: $F(x)=x^3$, chunki $F'(x)=\left(x^3\right)'=3x^2$.

2–misol. Berilgan: $f(x)=\dfrac{1}{2\sqrt{x}}$. $F(x)$ni toping.

Yechish: $F(x)=\sqrt{x}$, chunki $F'(x)=\left(\sqrt{x}\right)'=\dfrac{1}{2\sqrt{x}}$.

Ravshanki, agar $F(x)$ funktsiya $f(x)$ funktsiyaning boshlang'ichi bo'lsa, u holda $F(x)+C$ ko'rinishdagi istalgan funktsiya ham (bu yerda $C-$ixtiyoriy o'zgarmas) $f(x)$ ning boshlang'ich funktsiyasi bo'ladi, chunki $\left[F(x)+C\right]'=f(x)$.

Masalan, agar $f(x)=x^2$ bo'lsa, u holda $F(x)=\dfrac{x^3}{3}+2;$ $F(x)=\dfrac{x^3}{3}-5;$

$F(x)=\dfrac{x^3}{3}+\ln 6,$ chunki $F(x)=\left(\dfrac{x^3}{3}+2\right)'=\left(\dfrac{x^3}{3}-5\right)'=\left(\dfrac{x^3}{3}+\ln 6\right)'=x^2.$

1–ta'rif. *Agar $F(x)$ funktsiya $f(x)$ funktsiyaning boshlang'ich funktsiyasi bo'lsa, u holda $F(x)+C$ ifoda $f(x)$ funktsiyaning aniq integral deyiladi.*

U quyidagicha belgilanadi: $\int f(x)dx$.

SHunday qilib, ta'rifga ko'ra agar $F'(x)=f(x)$ bo'lsa, $\int f(x)dx=F(x)+C.$

Bu yerda $f(x)$ – integral ostidagi funktsiya, $f(x)dx$ – integral ostidagi ifoda, $\int$ – integral belgisi, x – integrallash o’zgaruvchisi, $f(x)$ ning boshlang’ich funktsiyasini topish amali funktsiyani integrallash (integral olish) deyiladi.

Aniqmas integralning xossalari

Agar $F'(x) = f(x)$ bo’lsa, u holda

1. $\left(\int f(x)dx\right)' = f(x)$.
2. $d\left(\int f(x)dx\right) = f(x)dx.$
3. $\int dF(x) = F(x) + C.$
4. $\int Af(x)dx = A\int f(x)dx,$ bu yerda $A - const.$
5. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx.$

Bu xossalarning to’g’riligi differentsiallash orqali tekshiriladi.

Asosiy integrallar jadvali

I. $\int x^m dx = \dfrac{x^{m+1}}{m+1} + C,\ (m = -1).$

II. $\int \dfrac{dx}{x} = \ln|x| + C.$

III. $\int \sin x dx = -\cos x + C.$

IV. $\int \cos x dx = \sin x + C.$

V. $\int a^x dx = \dfrac{a^x}{\ln a} + C.$

VI. $\int e^x dx = e^x + C.$

VII. $\int \dfrac{dx}{\cos^2 dx} = tgx + C.$

VIII. $\int \dfrac{dx}{\sin^2 dx} = -ctgx + C.$

IX. $\int \frac{dx}{1+x^2} = arctgx + C.$

X. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$

XI. $\int tgxdx = -\ln|\cos x| + C.$

XII. $\int ctgxdx = \ln|\sin x| + C.$

XIII. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + C.$

XIV. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C.$

XV. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

XVI. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C.$

Dastlabki, o’nta formula differentsiallashning ma’lum formulalariga mos keladi, XI– XVI formulalarni esa differentsiallash orqali hosil qilish mumkin. Masalan, XII va XV formulalarning o’rinli ekanini tekshirib ko’raylik:

$$\left(\ln|\sin x|\right)' = \frac{1}{\sin x} \cdot \cos x = ctgx.$$

$$\left(\ln\left|x + \sqrt{x^2 \pm a^2}\right|\right)' = \frac{1}{x + \sqrt{x^2 \pm a^2}} \cdot \left(1 + \frac{x}{\sqrt{x^2 \pm a^2}}\right)' = \frac{1}{\sqrt{x^2 \pm a^2}}.$$

Integrallashning asosiy usullarini qarab chiqishdan avval asosiy integrallar jadvalini jiddiy kengaytiradigan bir muhim integrallash qoidasini ko’rib chiqamiz. Agar $\int f(x)dx = F(x) + C$ va $z = \varphi(x)$ bo’lsa, u holda

$$\int f(z)dx = F(z) + C. \qquad (1)$$

Bu qoida integrallash fomulasining ko’rinishi integrallash o’zgaruvchisining xarakteriga bog’liq emasligini bildiradi. Bu qoidaning to’g’riligi (1) tenglikning har ikki tomonini differentsiallash orqali oson tekshiriladi. Jumladan,

1. $\int f(ax)dx = \frac{1}{a}\int f(ax)d(ax) = \frac{1}{a}F(ax) + C.$

2. $\int f(ax \pm b)dx = \frac{1}{a}\int f(ax \pm b)d(ax \pm b) = \frac{1}{a}F(ax \pm b) + C.$

Masalan:

$$\int \sin 3x dx = \frac{1}{3}\int \sin 3x d(3x) = -\frac{1}{3}\cos 3x + C.$$

$$\int e^{\frac{x}{2}} dx = 2\int e^{\frac{x}{2}} d\left(\frac{x}{2}\right) = 2e^{\frac{x}{2}} + C.$$

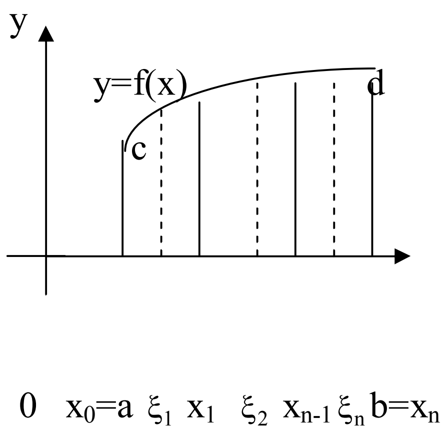
$$\int \frac{dx}{3x-5} = \frac{1}{3}\int \frac{d(3x-5)}{3x-5} = \frac{1}{3}\ln|3x-5| + C.$$

1.1-§. Aniq integral integral yig'indilarning limiti sifatida

Egri chiziqli trapesiyaning yuzini topish.

Yuqoridan tenglamasi y=f(x) egri chiziq bilan , pastdan OX o'qi bilan, yon tomonlaridan x=a, x=b to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapesiyaning yuzini toping.

Faraz qilaylik $[a,b]$ kesmada y=f(x) funksiya aniqlangan, uzluksiz va $f(x) \geq 0$ bo'lsin. $[a,b]$ kesmani $a = x_0 < x_1 < x_2 < .... < x_n = b$ nuqtalar bilan $n[x_{i-1}, x_i]$, $(i = \overline{1,n})$ ta bo'lakchalarga bo'lib va bo'linish nuqtalaridan OY o'qiga parallel to'g'ri chiziqlar o'tkazsak natijada acdb egri chiziqli trapesiyamiz n ta kichik egri chiziqli trapesiyalarga (trapesiyachalarga) ajraladi.



Endi har bir $[x_{i-1}, x_i]$ kesmachada ixtiyoriy ξ_i, $(i = \overline{1,n})$ nušta tanlab olib, f(x) funksiyaning bu ξ_i nuqtalardagi qiymatlarini $f(\xi_i)$ deb belgilaylik. Bu holda har bir kichik egri chiziqli trapesiyaning yuzi taxminan balandligi $f(\xi_i)$, asosi $\Delta x_i = x_i - x_{i-1}$ bo'lgan to'g'ri to'rtburchakning yuziga teng bo'ladi:

$$s_i \approx f(\xi_i)\Delta x_i \qquad (i=\overline{1,n})$$

Butun ya’ni acdb egri chiziqli trapesiyaning yuzi taxminan hamma kichik egri chiziqli trapesiyalar yuzalarining yig’indisiga teng bo’ladi:

$$s \approx f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (1)$$

Agar $[x_{i-1}, x_i]$ kesmalar uzunliklarining eng kattasini λ ya’ni $\lambda = \max\limits_{1\le i\le n}\{\Delta x_i\}$ desak, $\lambda \to 0$ da $[a,b]$ kesmaning mayda bo’lakchalarga bo’linish soni cheksiz o’sadi, natijada (1) yuza berilgan acdb egri chiziqli trapesiya yuziga cheksiz yaqinlashib boradi. Shuning uchun acdb egri chiziqli trapesiyaning yuzini

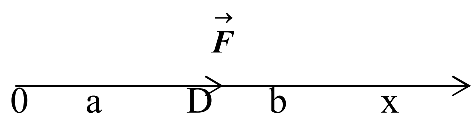
$$s \approx \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (2)$$

desak bo’ladi.

Kuch ta’sirida bajarilgan ishni hisoblash masalasi.

Faraz qilaylik biror D moddiy nuqtaga OX o’qi yo’nalishida biror o’zgaruvchan F=f(x) kuch ta’sir qilsin. Moddiy D nuqtaning F kuch ta’sirida biror a nuqtadan b nuqtagacha harakatlangandagi bajargan ishini hisoblaylik.

$[a,b]$ kesmani n ta $[x_{i-1}, x_i]$ $(i=\overline{1,n})$ bo’lakchalarga bo’lib, har bir $[x_{i-1}, x_i]$ bo’lakchada F kuchni deyarli



o’zgarmas deb qarasak, u holda har bir bo’lakchada bajarilgan ish taxminan $A_i \approx f(\xi_i)\Delta x_i$ bo’ladi. Bu yerda $\xi_i \in [x_{i-1}, x_i]$, $f(\xi_i)$ esa kesmadagi ta’sir etayotgan kuch.

U holda $[a,b]$ da F=f(x) kuch ta’sirida bajarilgan ish taxminan

$$A \approx f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

Agar $\max\limits_{1\le i\le n}\{\Delta x_i\}=\lambda$ desak va $\lambda\to 0$ bo'lsa, u holda bajarilgan ish quyidagicha bo'ladi:

$$A=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (3)$$

Juda ko'p texnika, mexanika va fizika masalalarini yyechishda (2),(3) ko'rinishdagi yig'indilarning limitini hisoblashga to'g'ri keladi.

Aniq integral va uning ta'rifi.

y=f(x) funksiya $[a,b]$ kesmada aniqlangan bo'lsin. $[a,b]$ ni $a=x_0<x_1<....<x_n=b$ nuqtalar bilan n ta bo'lakchalarga ajratib va har bir [x_{i-1} ,x_i] $(i=\overline{1,n})$ kesmada ixtiyoriy $\xi_i(i=\overline{1,n})$ nuqta olib, bu nuqtalardagi f(x) funksiyaning qiymatlarini $f(\xi_i)$ deylik. [x_{i-1} ,x_i] kesmalarning uzunliklarini $\Delta x_i=x_i-x_{i-1}$ deb belgilab quyidagi ko'paytmalar yig'indisini tuzaylik:

$$s=f(\xi_1)\Delta x_1+f(\xi_2)\Delta x_2+...+f(\xi_n)\Delta x_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad (1)$$

(1) ga f(x) funksiyaning $[a,b]$ kesmadagi integral yig'indisi deyiladi.

$\max\limits_{1\le i\le n}\{\Delta x_i\}=\lambda$ deylik

Ta'rif. Agar $[a,b]$ da aniqlangan f(x) funksiya uchun tuzilgan (1) integral yig'indi , $\lambda\to 0$ da $[a,b]$ ni ixtiyoriy n ta bo'lakchalarga bo'lish usuliga va har bir [x_{i-1} ,x_i] bo'lakchada ixtiyoriy ξ_i nuqtani tanlab olish usuliga bog'liq bo'lmagan limitga ega bo'lsa, bu limitga $[a,b]$ kesmada f(x) funksiyadan olingan aniq integral deyiladi va

$$\int_a^b f(x)dx$$

ko'rinishda yoziladi.

Shunday qilib ta’rifga ko’ra $\int_a^b f(x)dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ (2)

a,b-larga mos ravishda integralning quyi va yuqori chegarasi, $[a,b]$ ga integrallash sohasi deyiladi.

Agar f(x) funksiya uchun (2) limit mavjud bo’lsa, f(x) funksiyani $[a,b]$ kesmada integrallanuvchi funksiya deyiladi.

Aniq integralning geometrik ma’nosi yuqoridagi 1-masaladagi egri chiziqli trapesiyaning yuzini beradi:

$$s = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx$$

2-masalada esa bajarilgan ish F=f(x) kuchdan olingan integralga teng:

$$A = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx$$

1-teorema. $[a,b]$ kesmada uzluksiz bo’lgan har qanday f(x) funksiya shu kesmada integrallanuvchi bo’ladi.

2-teorema. f(x) funksiya $[a,b]$ da integrallanuvchi bo’lishi uchun shu kesmada chegaralangan bo’lishi zarur.

1-eslatma. $[a,b]$ da chegaralanmagan funksiya shu kesmada integrallanuvchi bo’lmaydi.

2-eslatma. Agar f(x) funksiya $[a,b]$ da chekli, ya’ni sanoqli uzilish nuqtalariga ega bo’lib, chegaralangan bo’lsa, bu funksiya shu kesmada integrallanuvchi bo’ladi.

Agar a>b bo’lsa $\int_a^b f(x)dx = -\int_b^a f(x)dx$ bo’ladi, chunki $a = x_0 > x_1 > .... > x_n = b$ bo’lib $\Delta x_i = x_i - x_{i-1} < 0$ bo’ladi. Agar a=b bo’lsa $\int_a^b f(x)dx = 0$ bo’ladi.

Integral o’zgaruvchini har qanday harf bilan belgilash mumkin:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz = \int_a^b f(y)dy$$

1.2-§. Aniqmas va aniq integralning asosiy xossalari .

1-xossa. O’zgarmas ko’paytuvchini aniq integral belgisidan tashqari chiqarish mumkin:

$$\int_a^b Af(x)dx = \mathrm{A}\int_a^b f(x)dx \quad \text{(A-o’zgarmas son)}$$

Isboti. $\int_a^b Af(x)dx = \begin{pmatrix} \text{aniq integral} \\ \text{ta'rifiga ko'ra} \end{pmatrix}$

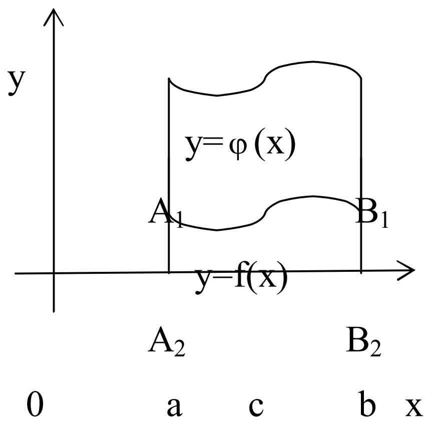
$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} Af(\xi_i)\Delta x_i = \begin{pmatrix} \text{limitning} \\ \text{hossasiga ko'ra} \end{pmatrix} = A \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \mathrm{A}\int_a^b f(x)dx$$

Keyingi xossalari ham aniq integralning ta’rifidan va limitning xossalaridan foydalanib osongina isbotlanadi. Shuning uchun biz ularning isbotini o’quvchilarga havola qilib, ba’zilarining geometrik ma’nosini ko’rsatib ketamiz.

2-xossa. $\int_a^b [f_1(x) \pm f_2(x)]dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$

3-xossa. Agar $[a,b]$ da f(x) va $\varphi(x)$ funksiyalar f(x)$\le$ φ(x) tengsizlikni qanoatlantirsalar, u holda $\int_a^b f(x)dx \le \int_a^b \varphi(x)dx$ munosabat o’rinli bo’ladi.

Isboti. Geometrik nuqtai nazardan ko’raylik. Agar f(x)>0 , φ(x)>0 bo’lib, f(x)$\le$ φ(x) bo’lsa u holda aA_1B_1b egri chiziqli trapesiyaning yuzi aA_2B_2b egri chiziqli trapesiyaning yuzidan kichik bo’lmaydi.



4-xossa. Agar f(x) funksiya [a,c],[c,b] (a<c<b) kesmalarning har birida integrallanuvchi bo'lsa, u holda bu funksiya shu [a,b] da ham integrallanuvchi bo'lib

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

bo'ladi.

Bu xossaning geometrik ma'nosi: asosi [a,b] bo'lgan egri chiziqli trapesiyaning yuzi asoslari [a,c] va [c,b] bo'lgan egri chiziqli trapesiyalar yuzalarining yig'indisiga teng bo'ladi.

5-xossa. Agar [a,b] da differensiallanuvchi bo'lgan f(x) funksiyaning shu kesmadagi eng kichik va eng katta qiymatlari mos ravishda m va M bo'lib, $a \le b$ bo'lsa, u holda

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

munosabat o'rinli bo'ladi.

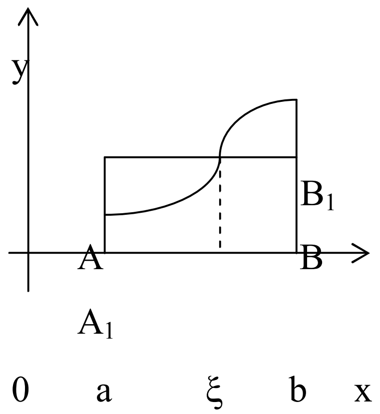
Isboti. $m \le f(x) \le M \Rightarrow \int_a^b \mathrm{m}dx \le \int_a^b \mathrm{f(x)}dx \le \int_a^b \mathrm{M}dx \Rightarrow \mathrm{m(b - a)} \le \int_a^b \mathrm{f(x)}dx \le M(b-a)$

6-xossa. (O'rta qiymat haqidagi teorema). Agar f(x) funksiya [a,b] kesmada uzluksiz bo'lsa, u holda bu kesmada shunday ξ nuqta topiladiki, bu nuqtada

$$\int_a^b f(x)dx = (b-a)f(\xi)$$

munosabat o'rinli bo'ladi.

Isboti. Geometrik nuqtai nazardan aA_1B_1b egri chiziqli trapesiyaning yuzi taxminan aABb to'g'ri to'rtburchakning yuziga teng.



Nyuton - Leybnits formulasi. Aniq integral hisobining asosiy formulasi

Faraz qilaylik [a,b] kesmada integrallanuvchi bo’lgan f(x) funksiyadan olingan integral

$$\int_a^b f(x)\,dx$$

bo’lsin

Biz bilamizki integral o’zgaruvchini istalgan harf bilan belgilash mumkin, shuning uchun

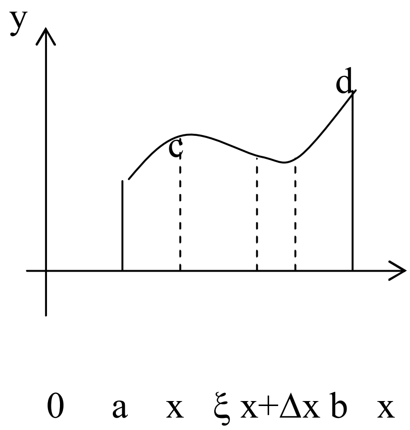
$$\int_a^b f(x)dx = \int_a^b f(u)du$$

deylik.

Biz bilamizki agar f(x) funksiya [a,b] da integrallanuvchi bo’lsa, [a,x] da ($a \le x \le b$) ya’ni $\forall x \in [a,b]$) ham integrallanuvchi bo’lib, integral qiymati x ning funksiyasi bo’ladi.

$$\Phi(x) = \int_a^x f(u)du \qquad (1)$$

(1) integral acdx egri chiziqli trapesiyaning yuzini ifodalaydi. Agar [a,b] da x o’zgarsa $\Phi(x)$ ham o’zgarishi ravshan.



Teorema. Agar f(u) funksiya u=$x \in [a,b]$ nuqtada uzluksiz bo’lsa, u holda $\int_a^x f(u)du$ integraldan yuqori chegarasi bo’yicha olingan hosila integral ostidagi funksiyaga teng bo'lib, unda integrallash o'zgaruvchisi o'rniga integralning yuqori chegarasi qo'yiladi, ya'ni

$$\Phi©(x) = f(x) \quad \text{yoki} \quad \left(\int_a^x f(u)du\right)' = \text{f(x)} \quad \text{bo'ladi.}$$

Isboti. x ga Δx orttirma bersak Φ(x) funksiya ham $\Delta\Phi$(x) orttirma oladi:

$$\Delta\Phi(x) = \Phi(x+\Delta x) - \Phi(x) = \int_a^{x+\Delta x} f(u)du - \int_a^x f(u)du = \int_a^x f(u)du + \int_x^{x+\Delta x} f(u)du - \int_a^x f(u)du = \int_x^{x+\Delta x} f(u)du$$

$\Delta\Phi(x) = \int_x^{x+\Delta x} f(u)du$. Agar $\int_x^{x+\Delta x} f(u)du$ integralga o'rta qiymat xaqidagi teoremani

tatbiq qilsak $\int_x^{x+\Delta x} f(u)du = f(\xi)\Delta x$ $(x < \xi < x+\Delta x)$ bo'lib, $\Delta\Phi(x) = f(\xi)\Delta x$

bo'ladi. Hosilaning ta'rifiga ko'ra

$$\Phi'(x) = \lim_{\Delta x\to 0}\frac{\Delta\Phi(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(\xi)\Delta x}{\Delta x} = \lim_{\Delta x\to 0} f(\xi) = f(x)$$

Chunki $\Delta x \to 0$ da $x+\Delta x \to x$ demak $x < \xi < x+\Delta x$ bo'lgani uchun $\Delta x \to 0$ da $\xi \to x$ ravshan. Shunday qilib $\Phi'(x) = f(x)$

Bu isbot qilingan, matematik analiz kursining asosiy teoremalarining biri bo'lgan teoremadan ko'rinadiki [a,b] da uzluksiz bo'lgan har qanday f(x) funksiya uchun $\int_a^x f(u)du$ aniq integral boshlang'ich funksiya bo'lar ekan. Uzluksiz bo'lgan f(x) funksiyaning boshlang'ich funksiyasi cheksiz ko'p bo'lgani uchun ularning ixtiyoriy bittasini F(x) boshqasini esa $\int_a^x f(u)du$ desak, ular bir-biridan o'zgarmas songa farq qiladi:

$$\text{F(x)} = \int_a^x f(u)du + \text{C}$$

O'zgarmas C ni topish uchun, x=a desak F(a)= $\int_a^a f(u)du$ +C $\Rightarrow$C=F(a), x=b desak

F(b)=$\int_a^b f(u)du + F(a) \Rightarrow \int_a^b f(u)du = F(b) - F(a)$ yoki $\int_a^b f(x)dx = F(b) - F(a)$ -

Nyuton-Leybnis formulasi kelib chiqadi.

F(b)-F(a)=$F(x)\Big|_a^b$ deb ham belgilanadi.

Misol. 1. $\int\limits_1^2 x^3 dx = \frac{x^4}{4}\Big|_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{1}{4}(16-1) = \frac{1}{4}15 = \frac{15}{4};$

2. $\int\limits_0^\pi \text{sinxdx} = -\cos x\Big|_0^\pi = 1 + 1 = 2$

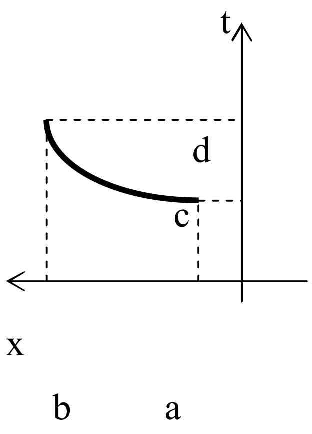
II BOB. ANIQMAS VA ANIQ INTEGRALLARDA INTEGRAL-LASHNING ASOSIY USULLARI

Aniq integralda o'zgaruvchini almashtirish.

Faraz qilaylik bizga $[a,b]$ kesmada integrallanuvchi bo'lgan funksiya f(x) berilgan bo'lib, undan shu kesmada olingan $\int\limits_a^b f(x)dx$ (1) aniq integral mavjud bo'lsin.

Bizning maqsad shu (1) aniq integralni hisoblash uchun o'zgaruvchini shunday almashtiraylikki natijada hosil bo'ladigan aniq integral berilgan aniq integralga nisbatan ancha sodda bo'lsin.

Teorema. Agar (1) da $x = \varphi(t)$ (2) almashtirish bajarganimizda $\varphi(t)$ funksiya quyidagi shartlarni qanoatlantirsa :



1. $\varphi(t)$ funksiya [c,d] kesmada aniqlangan va uzluksiz $\varphi'(t)$ hosilaga ega bo'lsin.
2. Yangi o'zgaruvchi t [c,d] kesmada o'zgarganda $\varphi(t)$ funksiyaning qiymati $[a,b]$ dan chiqmasin.
3. $\varphi(c) = a, \ \varphi(d) = b$ bo'lsin, bu holda $\int\limits_a^b f(x)dx = \int\limits_c^d f(\varphi(t))\varphi'(t)dt$

(3) tenglik o'rinli bo'ladi.

Isboti. f(x) funksiya $[a,b]$ da uzluksiz bo'lgani uchun uning shu kesmadagi boshlang'ich funksiyasini F(x) desak u holda Nyuton-Leybnis formulasiga ko'ra

$\int\limits_a^b f(x)dx = F(b) - F(a)$ (4) tenglik o'rinli bo'ladi. Agar $x = \varphi(t)$ desak F(x)=F($\varphi(t)$) funksiyaning $f(\varphi(t))\varphi'(t)$ funksiya uchun boshlang'ich funksiya ekanligini ko'rish qiyin emas.

haqiqatan $[F[\varphi(t)]]' = \begin{pmatrix} murakkab \\ funksiyaning \\ hosilasigako'ra \end{pmatrix} = F'[\varphi(t)]\varphi'(t) = f[\varphi(t)]\varphi'(t)$ chunki

$F'(x) = f(x)$

$x = \varphi(t)$ desak F' [$\varphi(t)$]=f[$\varphi(t)$].

Demak Nyuton-Leybnis formulasiga ko'ra

$\int\limits_c^d f[\varphi(t)]\varphi'(t)dt = F[\varphi(t)]\Big|_c^d = F[\varphi(d)] - F[\varphi(c)] =$

=F(b)-F(a)=((4) ga ko'ra)= $\int\limits_a^b f(x)dx \Rightarrow \int\limits_a^b f(x)dx = \int\limits_c^d f[\varphi(t)]\varphi'(t)dt$

Misol yechganda integral ostidagi funksiya yuqoridagi shartlarni qanoatlantirishi shart.

Misol. 1) $\int\limits_0^1 \sqrt{1 - x^2}dx = \begin{vmatrix} x = \sin t \\ 0 \le t \le \dfrac{\pi}{2} \\ dx = \cos tdt \end{vmatrix} = \int\limits_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t} \cdot \cos tdt = \int\limits_0^{\frac{\pi}{2}} \cos^2 tdt = \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} (1 + \cos 2t)dt = \frac{\pi}{4}$

2. $\int\limits_0^1 \frac{xdx}{x^2 + 1} = \begin{vmatrix} x^2 + 1 = t \\ 2xdx = dt \\ xdx = \dfrac{dt}{2} \\ x \to 0\partial a, t \to 1 \\ x \to 1\partial a, t \to 2 \end{vmatrix} = \int\limits_1^2 \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int\limits_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big|_1^2 = \frac{1}{2}[\ln 2 - \ln 1] = \frac{1}{2}[\ln 2] = \ln\sqrt{2}$;

Aniq integralni bo'laklab integrallash

Faraz qilaylik, *u(x)* va *v(x)* funksiyalar *[a,b]* kesmada differensiallanuvchi funksiyalar bo'lsin.

Ularning ko'paytmasini tuzib hosila olamiz:

$$(uv)' = u'v + uv'$$

Tenglikni ikkala tomonini *a* dan *b* gacha integrallaymiz:

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \qquad (2)$$

Bulardan

$$\int_a^b (uv)'dx = uv\Big|_a^b; \quad \begin{matrix} u'dx = du \\ v'dx = dv \end{matrix}$$

bo'lgani uchun (2) ni quyidagicha yozamiz .

$$uv\Big|_a^b = \int_a^b vdu + \int_a^b udv \quad \text{bundan}$$

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \quad (3)$$

kelib chiqadi.

Bu (3) formula aniq integralni bo'laklab integrallash formulasi deyiladi.

1-Misol .

$$\int_0^1 x \cdot e^x dx = \begin{vmatrix} u = x; & du = dx \\ dv = e^x dx & v = e^x \end{vmatrix} = x \cdot e^x\Big|_0^1 - \int_0^1 e^x \cdot dx = x \cdot e^x\Big|_0^1 - e^x\Big|_0^1 = 1 \cdot e^1 - 0 \cdot e^0 -$$

$$-e^1 + e^0 = e - e + 1 = 1 \;;$$

2-Misol .

$$\int_0^1 arctgxdx = \begin{vmatrix} u = arctgx; & dv = dx \\ du = \dfrac{dx}{1+x^2}; & v = x \end{vmatrix} = x \cdot arctgx\Big|_0^1 - \int_0^1 x \cdot \frac{dx}{1+x^2} = x \cdot arctgx\Big|_0^1 - \frac{1}{2}\ln\left|1+x^2\right|\Big|_0^1 =$$

$$1 \cdot arctgx1 - 0 \cdot arctgx0 - \frac{1}{2}\left[\ln(1+1^2) - \ln(1+0)\right] = \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2};$$

Integrallashning asosiy usullari

Quyidagilar integrallashning asosiy usullari hisoblanadi:

1. Yoyib integrallash usuli.

2. Bevosita inetgrallash usuli.
3. O'rniga qo'yish usuli.
4. Bo'laklab integrallash usuli.

Yoyish (integral ostidagi ifodani yoyib integrallash) usuli.

Agar $f(x) = f_1(x) \pm f_2(x)$ bo'lsa, u holda (5) xossaga ko'ra yozish mumkin:

$$\int f(x)dx = \int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx.$$

1–misol. $\int (3+2x)^2 dx$.

Yechish:
$$\int (3+2x)^2 dx = \int (9+12x+4x^2)dx = 9\int dx + 12\int xdx + 4\int x^2 dx =$$
$$= 9x + 12 \cdot \frac{x^2}{2} + 4 \cdot \frac{x^3}{3} + C = 9x + 6x^2 + 1\frac{1}{3}x^3 + C = 1\frac{1}{3}x^3 + 6x^2 + 9x + C.$$

2–misol. $\int \frac{(3+2x)^2}{\sqrt{x}}dx$ ni hisoblang.

$$\int \frac{(3+2x)^2}{\sqrt{x}}dx = \int \frac{9+12x+4x^2}{\sqrt{x}}dx = 9\int x^{\frac{1}{2}}dx + 12\int x^{\frac{1}{2}}dx + 4\int x^{\frac{3}{2}}dx =$$

$$= 9\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 12\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 4\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = 18\sqrt{x} + 12 \cdot \frac{2}{3}x^{\frac{3}{2}} + 4 \cdot \frac{2}{5}x^{\frac{5}{2}} + C =$$

$$= 18\sqrt{x} + 8\sqrt{x^3} + \frac{8}{5}\sqrt{x^5} + C$$

3–misol. $\int \left(7x^3 - 5\sqrt[4]{x^3} + \frac{3}{\sqrt[3]{x^2}}\right)dx$ ni hisoblang.

$$\int \left(7x^3 - 5\sqrt[4]{x^3} + \frac{3}{\sqrt[3]{x^2}}\right)dx = 7\int x^3 dx - 5\int x^{\frac{3}{4}}dx + 3\int x^{-\frac{2}{3}}dx =$$

$$= 7\frac{x^4}{4} - -5\frac{x^{\frac{3}{4}}+1}{\frac{3}{4}+1} + +3\frac{x^{-\frac{2}{3}}+1}{-\frac{2}{3}+1} + C = \frac{7}{4}x^4 - \frac{20}{7}\sqrt[4]{x^7} + 9\sqrt[3]{x} + C.$$

Bu misollardan ko'rinadiki, yoyish usuli bilan integrallaganimizda integral ostidagi funktsiyani elementar matematika vositalari yordamida shunday qo'shiluvchilarga yoyganimizda ulardan olingan integral jadvaldagi integraldan iborat bo'lar ekan.

Bevosita integrallash usuli.

Aniq integrallarni hisoblaganda integrallash o'zgaruvchisi x erkli o'zgaruvchi yoki $z=\varphi(x)$ funktsiyadan iborat bo'lishidan qat'iy nazar jadval integrallarini tatbiq qilish mumkin.

Bu usulni konkret misollarda ko'rsatamiz.

4–misol. $\int \frac{dx}{3-x} = -\int \frac{d(3-x)}{3-x} = -\ln|3-x| + C.$

5–misol. $\int \frac{dx}{\cos^2 7x} = \frac{1}{7}\int \frac{d(7x)}{\cos^2 7x} = \frac{1}{7} tg7x + C.$

6–misol. $\int (2-3x)^5 dx = -\frac{1}{3}\int (2-3x)^5 d(2-3x) = -\frac{1}{3}\cdot\frac{(2-3x)^6}{6} + C = -\frac{1}{18}(2-3x)^6 + C.$

7–misol. $\int \sqrt[3]{2x-1}dx = \frac{1}{2}\int (2x-1)^{\frac{1}{3}} d(2x-1) = \frac{1}{2}\cdot\frac{(2x-1)^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{8}(2x-1)^{\frac{4}{3}} + C.$

8–misol. $\int \frac{dx}{1+9x^2} = \frac{1}{3}\int \frac{d(3x)}{1+9x^2} = \frac{1}{3} arctg 3x + C.$

9-misol. $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{d(2x)}{\sqrt{3^2-(2x)^2}} = \frac{1}{2}\arcsin\frac{2x}{3} + C.$

10–misol. $\int 2^x e^x dx = \int (2e)^x dx = \frac{(2e)^x}{\ln 2e} + C = \frac{2^x e^x}{1+\ln 2} + C.$

11–misol. $\int e^{-x^3} x^2 dx = -\frac{1}{3}\int e^{-x^3} d(-x^3) = -\frac{1}{3}e^{-x^3} + C.$

Ko'pchilik hollarda dastlab integral ostidagi funktsiyani yoyib olib, so'ngra bevosita integrallashni tatbiq qilishga to'g'ri keladi.

12–misol. $\int tg^2(ax)dx = \int\left(\frac{1}{\cos^2(ax)} - 1\right)dx = \frac{1}{a} tgax - x + C.$

13–misol.

$$\int \frac{2x-1}{2x+3}dx = \int \frac{2x+3-4}{2x+3}dx = \int\left(1 - \frac{4}{2x+3}\right)dx = x - 4\int \frac{dx}{2x+3}x = x - 2\ln|2x+3| + C.$$

14–misol.

$$\int \frac{x-4}{\sqrt{x^2-9}}dx = \int \frac{x}{\sqrt{x^2-9}}dx - \int \frac{4}{\sqrt{x^2-9}}dx = \sqrt{x^2-9} - 4\ln|x+\sqrt{x^2-9}| + C.$$

$$\int \frac{(x+1)dx}{x^2-9} = \int \frac{xdx}{x^2-9} + \int \frac{dx}{x^2-9} = \frac{1}{2}\ln|x^2-9| + \frac{1}{2\cdot 3}\ln\left|\frac{x-3}{x+3}\right| + C =$$

15–misol.

$$= \frac{1}{2}\ln\left|\frac{(x^2-9)\sqrt[3]{x-3}}{\sqrt[3]{x-3}}\right| + C.$$

16–misol.

$$\int \frac{x^2+3}{x-2}dx = \int \frac{x^2-4+7}{x-2}dx = \int\left(x+2+\frac{7}{x-2}\right)dx = \frac{(x+2)^2}{2} + 7\ln|x-2| + C.$$

O'rniga qo'yish usuli.

O'rniga qo'yish usuli bilan integrallashning mohiyati shundan iboratki, integrallash o'zgaruvchisi $x$ni yangi t o'zgaruvchi bilan qulay almashtirib, berilgan integralni ancha soddaroq integralga yoki jadval integraliga keltiramiz.

Masalan, $\int f(x)dx$ integralda x o'zgaruvchini

$$x = \varphi(t), \quad dx = \varphi'(t)dt$$

formula bo'yicha t o'zgaruvchi bilan almashtiraylik. U holda

$$\int f(x)dx = \int f[\varphi(t)]\cdot\varphi'(t)dt.$$

Agar keyingi integralni topish mumkin bo'lsa (u $F(t)+C$ ga teng), u holda $x=\varphi(t)$ formuladan $t$ning qiymatini x orqali ifodalab, bu qiymatni $F(t)+C$ ga keltirib qo'yamiz va berilgan integralni topamiz.

17–misol. $\int \sqrt{4-x^2}dx.$

$x = 2\sin t$ deylik, u holda $dx = 2\cos t dt$. Demak,

$$\int \sqrt{4-x^2}dx = \int \sqrt{4(1-\sin^2 t)}\cdot 2\cos t dt = 4\int \frac{1+\cos 2t}{2}dt = 2\left(t+\frac{1}{2}\sin 2t\right) + C.$$

$x = 2\sin t \Rightarrow t = \arcsin\frac{x}{2}$, $\sin 2t = 2\sin t \cdot \cos t = 2 \cdot \frac{x}{2}\sqrt{1 - \frac{x^2}{2}} = \frac{1}{2}x\sqrt{1 - x^2}$.

$$\int \sqrt{4 - x^2}\,dx = 2\arcsin\frac{x}{2} + \frac{1}{2}x\sqrt{4 - x^2} + C.$$

Ko'pincha x o'zgaruvchini $x = \varphi(t)$ formula bo'yicha emas, balki t ning x orqali ifodasi bilan, ya'ni $t = \varphi(t)$ formula bilan almashtirish qulay bo'ladi. Masalan, agar integral ostidagi ifodani $f[\varphi(x)] \cdot \varphi'(x)dx = f(t)dt$ (bu yerda $t = \varphi(t)$ ko'rinishda yozishga erishgan bo'lsak, u holda $\int f(t)dt = F(t) + C$ integral quyidagiga teng bo'ladi:

$$\int f[\varphi(x)] \cdot \varphi'(x)dx = F[\varphi(x)] + C.$$

Buni misollar orqali ko'rsatamiz.

18-misol. $\int x^3\sqrt{3 - 2x^4}\,dx.$

$\sqrt{3 - 2x^4} = t$ deymiz, u holda $3 - 2x^4 = t^2$. Ikkala tomonini differentsiallaymiz:

$-8x^3dx = 2tdt,\ x^3dx = -\frac{1}{4}tdt.$ Demak,

$$\int x^3\sqrt{3 - 2x^4}\,dx = -\frac{1}{4}\int t^2dt = -\frac{1}{4} \cdot \frac{t^3}{3} + C = --\frac{1}{12}\sqrt{(3 - 2x^4)^3} + C.$$

O'zgaruvchini almashtirishning har ikkala usuliga doir misol keltiramiz.

19–misol. $\int \frac{x + 1}{x\sqrt{x - 2}}dx.$

Quyidagiga egamiz: $\int \frac{x + 1}{x\sqrt{x - 2}}dx = \int \frac{dx}{\sqrt{x - 2}} + \int \frac{dx}{x\sqrt{x - 2}}$.

O'ng tomondagi birinchi integral quyidagiga teng:

$$\int \frac{dx}{\sqrt{x - 2}} = 2\sqrt{x - 2} + C.$$

Ikkinchi integralda o'zgaruvchini almashtiramiz:

$x - 2 = t^2,\ \ x = t^2 + 2,\ \ dx = 2tdt.$

Natijada:

$$\int \frac{dx}{x\sqrt{x-2}} = 2\int \frac{tdt}{(t^2+2)t} = 2\int \frac{dt}{t^2+2} = \frac{2}{\sqrt{2}} arctg \frac{t}{\sqrt{2}} = \sqrt{2} arctg \sqrt{\frac{x-2}{2}} + C.$$

Binobarin, berilgan integral quyidagicha bo'ladi:

$$\int \frac{dx}{x\sqrt{x-2}} = 2\sqrt{x-2} + \sqrt{2} arctg \sqrt{\frac{x-2}{2}} + C.$$

Bo'laklab integrallash usuli

Ma'lumki, agar $u(x)$ va $v(x)$ lar x o'zgaruvchining birorta differentsiallanuvchi funktsiyalari bo'lsa, u holda

$$d(uv) = du \cdot v + dv \cdot u$$

va

$$udv = d(uv) - vdu.$$

Keyingi tenglikning har ikki tomonini integrallab

$$\int udv = \int d(uv) - \int vdu \text{ yoki } \int udv = uv - \int vdu \quad (*)$$

tenglikni hosil qilamiz. (*) ga *bo'laklab integrallash formulasi* deyiladi.

Bo'laklab integrallashning mohiyati shundan iboratki. Berilgan integralni hisoblashda integral ostidagi $f(x)dx$ ifodani $u \cdot dv$ ko'paytma shaklida tasvirlab va (*) formulani tatbiq qilib, berilgan $\int udv$ integralni $\int vdu$ jadval integrali yoki osngina olinadigan integral bilan almashtiramiz. Buni misollar yordamida tushuntiramiz.

20–misol. $\int x\cos xdx.$

$x = u,$ $\cos xdx = du$ deb olamiz. U holda $du = dx,$ $v = \int \cos xdx = \sin x.$

Bo'laklab integrallash formulasiga ko'ra:

$$\int x\cos xdx = x \cdot \sin x - \int \sin xdx = x\sin x + \cos x + C.$$

Bu yerda shuni aytish kerakki, agar $\cos x = u,\ xdx = dv$ desak, u holda $du = -\sin dx,\ v = \int xdx = \frac{x^2}{2} + C$ larga ega bo'lar edik va integral quyidagi ko'rinishga kelgan bo'lar edi:

$$\int x\cos xdx = \frac{1}{2}x^2\cos x - \int x^2\sin xdx.$$

O'ng tomondagi integral berilgan integraldan ham murakkabdir. Demak, keyingi holdagi o'rniga qo'yish maqsadga muvofiq emas.

21–misol. $\int x \cdot e^{-2x}dx.$

$x = u,\ du = dx$ deb olamiz, u holda $e^{-2x}dx = dv$, $v = -\frac{1}{2}e^{-2x}$. Bularni (*) formulaga qo'yib, topamiz:

$$\int x \cdot e^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C = -\frac{1}{4}e^{-2x}(2x+1) + C.$$

Ba'zan integralni hisoblash uchun bo'laklab integrallash formulasini bir necha marta tatbiq qilishga to'g'ri keladi.

22–misol. $\int x^3e^xdx.$

$x^3 = u,\ du = 3x^2dx$ deb olaylik, u holda $e^xdx = dv,\ v = e^x$. Natijada

$$\int x^3e^xdx = x^3e^x - 3\int x^2e^xdx.$$

Bir marta bo'laklab integrallash formulasini tatbiq qilib, x ning darajasini bir birlikka pasaytirdik. Yana bo'laklab integrallaymiz, buning uchun $x^2 = u$, $e^xdx = dv$ deb olamiz, $du = 2xdx,\ v = e^x$. Ntaijada:

$$\int x^3e^xdx = x^3e^x - 3\left(x^2e^x - 2\int x^2e^xdx\right)$$

Yana bir marta bo'laklab integrallash formulasini tatbiq qilib, uzil–kesil topamiz:

$$\int x^3e^xdx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C.$$

23–misol. $\int (x+1)\ln xdx.$

$$\int (x+1)\ln x dx = \left| \begin{matrix} u = \ln x & d\vartheta = (x+1)dx \\ du = \frac{dx}{x} & \vartheta = \frac{(x+1)^2}{2} \end{matrix} \right| = \frac{(x+1)^2}{2} \cdot \ln x - \frac{1}{2}\int (x+1)^2 \cdot \frac{dx}{x} =$$

$$= \frac{(x+1)^2}{2} \ln x - \frac{1}{2}\int \frac{x^2+2x+1}{x} dx =$$

$$= \frac{(x+1)^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2} + 2x + \ln|x| \right) + C = \frac{(x+1)^2}{2} \ln x - \frac{x^2}{4} - x - \frac{1}{2}\ln(x) + C.$$

24–misol.

$$\int xe^{2x}dx = \left| \begin{matrix} u = x & d\vartheta = e^{2x}dx \\ du = dx & \vartheta = \frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

25–misol. $$\int \arcsin x dx = \left| \begin{matrix} u = \arcsin x & d\vartheta = dx \\ du = \frac{dx}{\sqrt{1-x^2}} & \vartheta = x \end{matrix} \right| = x\arcsin x - \int \frac{xdx}{\sqrt{1-x^2}} =$$

$$= x\arcsin x + \frac{1}{2}\sqrt{1-x^2} + C.$$

2.1-§. Kvadrat uchhadni o'z ichiga olgan funktsiyalarni integrallari

Bu yerda quyidagi to'rtta integral nazarda tutiladi:

I. $\int \frac{dx}{ax^2+bx+c}$, III. $\int \frac{(Ax+B)dx}{ax^2+bx+c}$,

II. $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, IV. $\int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}$.

Agar kvadrat uchhadning bosh koefffitsienti $a > 0$ bo'lsa, I va II integrallar jadvalidagi $\int \frac{dx}{a^2+x^2} = \frac{1}{a}arctg\frac{x}{a} + C$ va $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$, hamda $a < 0$ bo'lganda mos ravishda $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$ va

$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$ integrallarga keltiriladi.

Buni konkret misollarda ko'rsataylik.

26–misol. $\int \frac{dx}{x^2+4x+6}$ integral hisoblansin.

Yechish: Kasr maxrajini to'la kvadratga ajratamiz va integralni hisoblaymiz:

$$\int \frac{dx}{x^2+4x+6} = \int \frac{dx}{x^2+4x+4+2} = \int \frac{dx}{(x+2)^2+2} = \int \frac{d(x+2)}{(x+2)^2+2} =$$

$$= \frac{1}{\sqrt{2}} arctg \frac{x+2}{\sqrt{2}} + C.$$

27–misol. $\int \frac{dx}{x^2+5x+7}$ integral hisoblansin.

Yechish: $\int \frac{dx}{x^2+5x+7} = \int \frac{dx}{x^2+2\cdot\frac{5}{2}x+\frac{25}{4}-\frac{25}{4}+7} = \int \frac{dx}{\left(x+\frac{5}{2}\right)^2+\frac{3}{4}} =$

$$= \frac{2}{\sqrt{3}} arctg \frac{x+\frac{5}{2}}{\frac{\sqrt{3}}{2}} + C = \frac{2}{\sqrt{3}} arctg \frac{x+5}{\sqrt{3}} + C.$$

28–misol. $\int \frac{dx}{2x^2-5x+11}$ integral hisoblansin.

Yechish:

$$\int \frac{dx}{2x^2-5x+11} = \frac{1}{2}\int \frac{dx}{(x^2-2x+1)+\left(\frac{11}{2}-1\right)} = \frac{1}{2}\int \frac{dx}{(x-1)^2+\frac{9}{2}} =$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{3} arctg \frac{x-1}{\frac{3}{\sqrt{2}}} + C = \frac{\sqrt{2}}{6} arctg \frac{\sqrt{2}(x-1)}{3} + C.$$

29–misol. $\int \frac{dx}{x^2-6x+5}$ integral hisoblansin.

Yechish:

$$\int \frac{dx}{x^2-6x+5} = \int \frac{dx}{x^2-6x+9-4} = \int \frac{dx}{(x-3)^2-4} = \frac{1}{2\cdot 2}\ln\left|\frac{x-3-2}{x-3+2}\right| + C =$$

$$= \frac{1}{4}\ln\left|\frac{x-5}{x-1}\right| + C.$$

30–misol. $\int \frac{dx}{3x^2-7x+4}$ integral hisoblansin.

$$\int\frac{dx}{3x^2-7x+4}=\frac{1}{3}\int\frac{dx}{x^2-\frac{7}{3}x+\frac{4}{3}}=\frac{1}{3}\int\frac{dx}{\left(x^2-2\cdot\frac{7}{6}x+\frac{49}{36}\right)+\left(\frac{4}{3}-\frac{49}{36}\right)}=$$

Yechish: $=\frac{1}{3}\int\frac{dx}{\left(x-\frac{7}{6}\right)^2-\frac{1}{36}}=\frac{1}{3}\cdot\frac{6}{2}\ln\left|\frac{x-\frac{7}{6}-\frac{1}{6}}{x-\frac{7}{6}+\frac{1}{6}}\right|+C=\ln\left|\frac{x-\frac{4}{3}}{x-1}\right|+C=$

$$=\ln\left|\frac{3x-4}{3(x-1)}\right|+C=\ln\left|\frac{3x-4}{x-1}\right|-\ln 3+C=\ln\left|\frac{3x-4}{x-1}\right|+C.$$

(S– $\ln 3$ o'zgarmasni yangi o'zgarmas S bilan almashtirdik.

31-misol. $\int\frac{dx}{4+2x-x^2}$ integral hisoblansin.

$$\int\frac{dx}{4+2x-x^2}=-\int\frac{dx}{x^2-2x-4}=-\int\frac{dx}{(x-1)^2-5}=\int\frac{dx}{5-(x-1)^2}=\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+x-1}{\sqrt{5}-x-1}\right|+C.$$

32–misol. $\int\frac{dx}{\sqrt{x^2-2x+5}}$ integral hisoblansin.

Yechish: $\int\frac{dx}{\sqrt{x^2-2x+5}}=\int\frac{dx}{\sqrt{(x-1)^2+4}}=\ln\left|x-1+\sqrt{x^2-2x+5}\right|+C.$

33–misol. $\int\frac{dx}{\sqrt{9x^2-6x+2}}$ integral hisoblansin.

$$\int\frac{dx}{\sqrt{9x^2-6x+2}}=\frac{1}{3}\int\frac{dx}{\sqrt{x^2-\frac{2}{3}x+\frac{2}{9}}}=\frac{1}{3}\int\frac{dx}{\sqrt{x^2-2\cdot\frac{1}{3}x+\frac{1}{9}+\frac{2}{9}}}=$$

$$=\int\frac{dx}{\sqrt{\left(x-\frac{1}{3}\right)^2+\frac{2}{9}}}=\frac{1}{3}\ln\left|x-\frac{1}{3}+\sqrt{x^2-\frac{2}{3}x+\frac{2}{9}}\right|+C=\frac{1}{3}\ln\left|\frac{3x-1+\sqrt{9x^2-6x+2}}{3}\right|+C=$$

$$=\frac{1}{3}\ln\left|3x-1+\sqrt{9x^2-6x+2}\right|+\left(C-\frac{1}{3}\ln 3\right)=\frac{1}{3}\ln\left|3x-1+\sqrt{9x^2-6x+2}\right|+C.$$

34-misol. $\int\frac{dx}{\sqrt{12x-9x^2-2}}$ integral hisoblansin.

$$\int \frac{dx}{\sqrt{12x-9x^2-2}} = \int \frac{dx}{\sqrt{-9\left(x^2-\frac{12}{9}x+\frac{2}{9}\right)}} = \int \frac{dx}{\sqrt{-9\left[\left(x^2-2\cdot\frac{2}{3}x+\frac{4}{9}\right)-\frac{2}{9}\right]}} =$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{\frac{2}{9}-\left(x-\frac{2}{3}\right)^2}} = \frac{1}{3}\arcsin\frac{x-\frac{2}{3}}{\frac{\sqrt{2}}{3}} + C = \frac{1}{3}\arcsin\frac{3x-2}{\sqrt{2}} + C.$$

35–misol. $\int \frac{dx}{\sqrt{6x-x^2}}$ integral hisoblansin.

$$\int \frac{dx}{\sqrt{6x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-6+9)+9}} = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\frac{x-3}{3} + C.$$

Endi (III) va (IV) ko’rinishdagi integrallarni ko’raylik. Bu integrallarning suratida ax^2+bx+c kvadrat uchhadning hosilasi “$2ax+b$”ni ajratish orqali hisoblanadi. Buni konkret misollarda ko’rsatamiz:

36–misol. $\int \frac{xdx}{x^2-7x+13}$ integral hisoblansin.

$$\int \frac{xdx}{x^2-7x+13} = \frac{1}{2}\int \frac{(2x-7)+7}{x^2-7x+13}dx = \frac{1}{2}\int \frac{2x-7}{x^2-7x+13}dx + \frac{7}{2}\int \frac{dx}{x^2-7x+13} =$$

$$= \frac{1}{2}\int \frac{d(x^2-7x+13)}{x^2-7x+13} + \frac{1}{2}\int \frac{dx}{\left(x-\frac{7}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}\ln\left(x^2-7x+13\right) + \frac{7}{\sqrt{13}}arctg\frac{2x-7}{\sqrt{3}} + C.$$

37–misol. $\int \frac{(x+2)dx}{x^2+2x+2}$ integral hisoblansin.

$$\int \frac{(x+2)}{x^2+2x+2}dx = \frac{1}{2}\int \frac{(2x+2)+2}{x^2+2x+2}dx = \frac{1}{2}\int \frac{(2x+2)dx}{x^2+2x+2} + \int \frac{dx}{x^2+2x+2} =$$

$$= \frac{1}{2}\ln\left(x^2+2x+2\right) + arctg(x+1) + C.$$

38–misol. $\int \frac{3x+2}{3x^2-5x+2}dx$ integral hisoblansin.

$$\int\frac{3x+2}{3x^2-5x+2}dx=\frac{1}{2}\int\frac{\left(6x-5\right)+9}{3x^2-5x+2}dx=\frac{1}{2}\int\frac{\left(6x-5\right)dx}{3x^2-5x+2}+\frac{9}{2}\int\frac{dx}{3x^2-5x+2}=$$

$$=\frac{1}{2}\ln\left|3x^2-5x+2\right|+\frac{9}{2}\ln\left|\frac{x-1}{3x-2}\right|+C=\frac{1}{2}\ln\left|(3x-2)(x-1)\right|+\frac{9}{2}\ln\left|\frac{x-1}{3x-2}\right|+C=$$

$$=\ln\left|\frac{\left(x-1\right)^5}{\left(3x-2\right)^4}\right|+C$$

39–misol. $\int\frac{(3x-1)dx}{\sqrt{x^2+2x+2}}$ integral hisoblansin.

$$\int\frac{(3x-1)dx}{\sqrt{x^2+2x+2}}=\int\frac{\frac{3}{2}\cdot(2x+2)-4}{\sqrt{x^2+2x+2}}dx=\frac{3}{2}\int\frac{(2x+2)dx}{\sqrt{x^2+2x+2}}-4\int\frac{dx}{\sqrt{(x+1)^2+1}}=$$

$$=3\sqrt{x^2+2x+2}-4\ln\left|x+1+\sqrt{x^2+2x+2}\right|+C.$$

40–misol. $\int\frac{(2-5x)dx}{\sqrt{4x^2+9x+1}}$ integral hisoblansin.

$$\int\frac{(2-5x)dx}{\sqrt{4x^2+9x+1}}=\int\frac{-\frac{5}{8}(8x+9)+\frac{45}{8}+2}{\sqrt{4x^2+9x+1}}dx=$$

$$=-\frac{5}{8}\int\frac{(8x+9)dx}{\sqrt{4x^2+9x+1}}+\frac{61}{8\cdot 2}\int\frac{dx}{\sqrt{x^2+\frac{9}{4}x+\frac{1}{4}}}=$$

$$=-\frac{5}{4}\int\frac{d(4x^2+9x+1)}{2\sqrt{4x^2+9x+1}}+\frac{61}{16}\int\frac{dx}{\sqrt{x^2+\frac{9}{4}x+\frac{1}{4}}}=$$

$$=-\frac{5}{4}\sqrt{4x^2+9x+1}+\frac{61}{16}\int\frac{dx}{\sqrt{\left(x+\frac{9}{8}\right)^2-\frac{65}{64}}}=$$

$$=-\frac{5}{4}\sqrt{4x^2+9x+1}+\frac{61}{16}\ln\left|8x+9+\sqrt{4x^2+9x+1}\right|+C.$$

41–misol. $\int \frac{(8x-11)dx}{\sqrt{5+2x-x^2}}$ integral hisoblansin.

$$\int \frac{(8x-11)dx}{\sqrt{5+2x-x^2}} = \int \frac{-4(-2x+2)-3}{\sqrt{5+2x-x^2}}dx = -4\int \frac{(2-2x)dx}{\sqrt{5+2x-x^2}} - 3\int \frac{dx}{\sqrt{5+2x-x^2}} =$$

$$= -8\cdot\sqrt{5+2x-x^2} - 3\cdot \arcsin\frac{x-1}{\sqrt{6}} + C.$$

42–misol. $\int \frac{xdx}{\sqrt{3x^2-11x+2}}$ integral hisoblansin.

$$\int \frac{xdx}{\sqrt{3x^2-11x+2}} = \frac{1}{6}\int \frac{6x-11+11}{\sqrt{3x^2-11x+2}}dx =$$

$$= \frac{1}{3}\int \frac{(3x^2-11x+2)}{2\sqrt{3x^2-11x+2}}dx + \frac{11}{6\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{11}{6}\right)^2 - \frac{95}{36}}} =$$

$$= \frac{1}{3}\sqrt{3x^2-11x+2} + \frac{11}{6\sqrt{3}}\ln\left|x-\frac{11}{6}+\sqrt{3x^2-11x+2}\right| + C.$$

Ko'rib chiqilgan misollardan quyidagi xulosalarni chiqarish mumkin:

Agar kvadrat uchhadning ildizlari kompleks bo'lsa, (I) ko'rinishdagi integral "*arctg*"ni, agar bu uchhadning ildizlari haqiqiy va har xil bo'lsa, kasrning natural logarifmini beradi.

Agar bosh koeffitsient $a>0$ bo'lsa, (II) ko'rinishdagi integral katta logarifmni agar $a<0$ bo'lsa "arcsin" ni beradi.

Agar maxrajning ildizlari kompleks bo'lsa, (III) ko'rinishdagi integral har doim natural logarifm va "*arctg*" ni, agar maxrajning ildizlari haqiqiy va turlicha bo'lsa, kasrning natural logarifmini beradi.

Agar $a>0$ bo'lsa, (IV) ko'rinishdagi integral maxrajning ildizini va "katta" logarifmni, agar $a<0$ bo'lsa "arcsin" ni beradi.

2.2-§. Ratsional va ba'zi irratsional funktsiyalarni integrallash

3-Ta`rif.** *Agar* $R(x)$ *funksiyani ikkita ko`phadning nisbati ko`rinishida yozish mumkin bo`lsa, u holda* $R(x)$ ***ratsional funksiyalar (yoki ratsional kasr)* *deyiladi, ya`ni*

$$R(x)=\frac{P_n(x)}{Q_m(x)}. \qquad \textbf{(6)}$$

$P_n(x)-n$-tartibli, $Q_m(x)-m$-tartibli ko`phad.

Agar $n\geq m$ bo`lsa kasr **noto`g`ri kasr**; $n<m$ bo`lsa **to`g`ri kasr** deyiladi.

Ixtiyoriy noto`g`ri kasr berilgan bo`lsa, ko`phadni ko`phadga bo`lish yordamida har doim uni ko`phad va to`g`ri kasrning yig`indisi shaklida ifodalash mumkin. Ixtiyoriy to`g`ri kasrni quyidagi 4ta ko`rinishdagi sodda kasrlarning yig`indisi kabi ifodalash mumkin.

$$\text{I. } \frac{A}{x-a}, \qquad \text{II. } \frac{A}{(x-a)^k}\,(k=2,3,4,...),$$

$$\text{III. } \frac{Mx+N}{x^2+px+q}, \qquad \text{IV. } \frac{Mx+N}{(x^2+px+q)^m}\,(m=2,3,..).$$

III va IV da $x^2+px+q\neq 0$, ya`ni $q-\frac{p^2}{4}>0$.

I va II ko`rinishidagi sodda kasrlar to`g`ridan to`g`ri integrallanadi. III va IV ko`rinishidagi sodda kasrlarni integrallash uchun esa $x+\frac{p}{2}=t$ almashtirish bajarish lozim.

43-masala. $\int\frac{2x^5-8x^3+2}{x^2-2x}dx$ **aniqmas integral hisoblansin.**

◁ Biz bu integralni ratsional funksiyani integrallash usulidan foydalanib hisoblaymiz. Avval noto`g`ri kasrni to`g`ri kasrga keltiramiz, so`ngra uni sodda kasrlarga yoyamiz:

$$R(x)=\frac{2x^5-8x^3+2}{x^2-2x}=2x^3+4x^2+\frac{2}{x(x-2)}=2x^3+4x^2+\frac{1}{x-2}-\frac{1}{x}\Rightarrow$$

$$\int\frac{2x^5-8x^3+3}{x^2-2x}=\int\left[2x^3+4x^2+\frac{1}{x-2}-\frac{1}{x}\right]dx=\frac{x^4}{2}+\frac{4x^3}{3}+\ln|x-2|-\ln|x|+C=$$

$$=\frac{x^4}{2}+\frac{4x^3}{3}+\ln\left|\frac{x-2}{x}\right|+C.$$

44-masala. $\int \frac{x^3+4x^2+4x+2}{(x+1)^2\cdot(x^2+x+1)}dx$ **aniqmas integral hisoblansin.**

◁ Bu integral ostida ham ratsional funksiya turibdi. Bu funksiyani sodda kasrlarga yoyaamiz.

$$R(x)=\frac{x^3+4x^2+4x+2}{(x+1)^2\cdot(x^2+x+1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}$$

Bu tenglikning o`ng tomonidagi noma`lum *A*, *B*, *C* va *D* larni noma`lum koeffitsientlar usulidan foydalanib topamiz. Buning uchun tenglikning o`ng tomonini umumiy maxrajga keltiramiz va berilgan kasr hamda hosil bo`lgan kasrlarning suratlarini bir-biriga tenglaymiz:

$$A(x+1)+B(x^2+x+1)+(Cx+D)\cdot(x+1)^2=x^3+4x^2+4x+2$$

$$(A+C)x^3+(2A+B+2C+D)x^2+(2A+B+C+2D)x+(A+B+D)=$$
$$=x^3+4x^2+4x+2$$

$$\begin{cases} A+C=1 \\ 2A+B+2C+D=4 \\ 2A+B+C+2D=4 \\ A+B+D=2 \end{cases}$$

Bu sistemani yechib $A=0$ va $B=C=D=1$ ekanligini topamiz. Demak, $R(x)=\frac{1}{(x+1)^2}+\frac{x+1}{x^2+x+1}$ ekan.

$$\Rightarrow \int R(x)dx=\int\frac{dx}{(x+1)^2}+\int\frac{x+1}{x^2+x+1}dx=I_1+I_2 \qquad (*)$$

$$I_1=\int\frac{dx}{(x+1)^2}=\int(x+1)^{-2}d(x+1)=-\frac{1}{x+1}+c_1$$

$$I_2=\int\frac{x+1}{x^2+x+1}dx\int\frac{x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\left(\left(\begin{array}{l}x+\frac{1}{2}=t\\ x=t-\frac{1}{2}\Rightarrow dx=dt\end{array}\right)\right)=\int\frac{t+\frac{1}{2}}{t^2+\frac{3}{4}}dt=$$

$$=\int\frac{tdt}{t^2+\frac{3}{4}}+\frac{1}{2}\int\frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}=$$

$$=\frac{1}{2}\int\frac{d\left(t^2+\frac{3}{4}\right)}{t^2+\frac{3}{4}}+\frac{1}{2}\int\frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}\ln\left|t^2+\frac{3}{4}\right|+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}+c_2=$$

$$=\frac{1}{2}\ln(x^2+x+1)+\frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+c_2.$$

I_1 va I_2 ning qiymatlarini (*) tenglikka olib borib qo`yib, ushbu natijaga kelamiz.

$$\int\frac{x^3+4x^2+4x+2}{(x+1)^2\cdot\left(x^2+x+1\right)}dx=-\frac{1}{x+1}+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+c \;\triangleright$$

Ba`zi irratsional ko`rinishidagi funksiyalarni integrallash. Eyler almashtirishlari.

$R(x,y)$ deganda x va y o`zgaruvchiga nisbatan ratsional bo`lgan funksiyani tushunamiz.

a) $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ integralni hisoblashda $t=\sqrt[n]{\frac{ax+b}{cx+d}}$ almashtirish bajarilsa, ratsional funksiyani integrallashga kelinadi.

b) $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ integralni hisoblashda quyidagi 3ta hol qaraladi.

1-hol. ax^2+bx+c kvadrad uchhad har xil x_1 va x_2 haqiqiy ildizlarga ega bo`lsin. $\Rightarrow ax^2+bx+c=a(x-x_1)(x-x_2)$. Bunda

$$\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1) \qquad \textbf{(7)}$$

almashtirish bajaramiz.

2-hol. $a>0$ bo`lsin. Unda

$$\sqrt{ax^2+bx+c}=t-\sqrt{a}x\left(ёки\sqrt{ax^2+bx+c}=t+\sqrt{a}x\right) \qquad \textbf{(8)}$$

almashtirish bajaramiz.

3-hol. $c>0$ bo\`lsin. U holda

$$\sqrt{ax^2+bx+c}=tx+\sqrt{c}\left(ёки\sqrt{ax^2+bx+c}=tx-\sqrt{c}\right) \qquad \textbf{(9)}$$

almashtirishni bajarish yordamida hisoblanishi kerak bo\`lgan integral ratsional funksiyani integrallashga keltiriladi.

(7)-(9) almashtirishlarga **Eyler almashtirishlari** deb ataladi.

45-masala. $\int\limits_0^1 \frac{xdx}{\sqrt{x^4+x^2+1}}$ **aniq integral hisoblansin.**

◁

$$\int\limits_0^1 \frac{xdx}{\sqrt{x^4+x^2+1}}=\int\limits_0^1 \frac{xdx}{\sqrt{\left(x^2+\frac{1}{2}\right)^2+\frac{3}{4}}}=\left(\left(\begin{array}{l} x^2+\frac{1}{2}=t, x=0\Rightarrow t=\frac{1}{2} \\ xdx=\frac{1}{2}dt, x=1\Rightarrow t=\frac{3}{2}\end{array}\right)\right)=\frac{1}{2}\int\limits_{\frac{1}{2}}^{3/2}\frac{dt}{\sqrt{t^2+\frac{3}{4}}}=$$

Binomial differensiallarni va trigonometrik funksiyalarni integrallash.

a) 4-ta\`rif.** *Ushbu* $x^m\left(a+bx^n\right)^p dx$ *ko\`rinishidagi ifodaga **binomial differensial deb ataladi.* Bu yerda m, n, p, -lar ratsional sonlar.

$$I=\int x^m\left(a+bx^n\right)^p dx \qquad \textbf{(10)}$$

integral quyidagi 3 ta holda ratsional funksiyaning integraliga kelar ekan**.**

1-hol. p-butun son. $x=t^N$ almashtirish bajariladi. Bu yerda N soni m va n ratsional sonlar (ya\`ni kasrlar) maxrajlarning eng kichik umumiy karralisi.

2-hol. $\frac{m+1}{n}-$**butun son.** Bu holda $a+bx^n=Z^N, N-p$ ratsional sonning maxaraji, almashtirish bajarish kerak.

3-hol. $\frac{m+1}{n}+p-$**butun son.** Bunda $\frac{a}{x^n}+b=Z^N, N-p$ ning maxraji, almashtirish bajarish yetarli.

b) $I=\int R(\sin x,\cos x)dx$

integral berilgan bo\`lsin. Bu integralni ushbu

$$tg\frac{x}{2}=t \quad -\pi<x<\pi$$

universal almashtirish yordamida har doim ratsional funksiyani integrallashga keltirish mumkin:

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}\ \ dx = \frac{2dt}{1+t^2}.$$

v) Aytaylik,

$$I = \int \sin^n x \cdot \cos^m x dx, (n, m \in Z)$$

integral berilgan bo`lsin. Bu integralni hisoblash uchun quyidagi hollar qaraladi.

1-hol. n-toq, m-juft $\Rightarrow \cos x = t$ almashtirish bajariladi.

2-hol. n-juft, m-toq $\Rightarrow \sin x = t$ almashtirish bajariladi

3-hol. n va m –toq. Bunda $\cos x = t,\ \sin x = t$ yoki $tgx = t$ almashtirishlardan biri bajariladi.

4-hol. n va m –juft. Bu holda

$$\sin 2x = 2\sin x \cdot \cos x \text{ va } \cos 2x = \cos^2 x - \sin^2 x$$

formulalardan foydalanib tartib pasaytiriladi va yuqoridagi hollardan biriga keltiriladi.

$$= \frac{1}{2}\ln\left|t + \sqrt{t^2 + \frac{3}{4}}\right|_{\frac{1}{2}}^{3/2} = \frac{1}{2}\left[\ln\left(\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{3}{4}}\right) - \ln\left(\frac{1}{2} + 1\right)\right] = \frac{1}{2}\ln\frac{3+\sqrt{12}}{3} = \frac{1}{2}\ln\frac{3+2\sqrt{3}}{3}. \triangleright$$

46-**masala.** $\int \frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^3}}{x^2 \cdot \sqrt[5]{x^2}}dx$ **aniqmas integral hisoblansin.**

$\triangleleft$ Integralni differentsial binomni integrallashdan foydalanib hisoblaymiz:

$$I = \int \frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^3}}{x^2 \cdot \sqrt[5]{x^2}}dx = \int x^{-\frac{12}{5}} \cdot \left(1 + x^{\frac{4}{5}}\right)^{\frac{3}{4}} dx \Rightarrow a = b = 1, m = -\frac{12}{5}, n = \frac{4}{5}, p = \frac{3}{4} \Rightarrow$$

$\Rightarrow \frac{m+1}{n} + p = \frac{-\frac{12}{5}+1}{\frac{4}{5}} + \frac{3}{4} = -\frac{7}{4} + \frac{3}{4} = -1 \Rightarrow \frac{1}{x^{4/5}} + 1 = z^4 -$ almashtirish bajarish lozim $\Rightarrow x = \left(z^4 - 1\right)^{-5/4} \Rightarrow dx = -5z^3 \cdot \left(z^4 - 1\right)^{-\frac{9}{4}} dz$ Bularni I ga olib borib qo`yib topamiz:

$$I = -5\int z^6 dz = -\frac{5z^7}{7} + c = -\frac{5}{7} \cdot \frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^7}}{x \cdot \sqrt[5]{x^2}} + c \triangleright$$

47-masala. $\int\limits_2^3 (x-1)^3 \cdot \ln^2(x-1)dx$ **aniq integral hisoblansin.**

$\triangleleft$ Bu integralni bo`laklab integrallash usulidan foydalanib hisoblaymiz:

$$\int\limits_2^3 (x-1)^3 \cdot \ln^2(x-1)dx = \left(\begin{pmatrix} u = \ln^2(x-1) \\ dv = (x-1)^3 dx \end{pmatrix} \Rightarrow \begin{pmatrix} du = \dfrac{2\ln(x-1)}{x-1}dx \\ v = \dfrac{(x-1)^4}{4} \end{pmatrix} \right) \overset{Б.И.}{=}$$

$$= \frac{(x-1)^4}{4}\ln^2(x-1)\Big|_2^3 - \frac{1}{2}\int\limits_2^3 (x-1)^3 \cdot \ln(x-1)dx =$$

$$= 4\ln^2 2 - \frac{1}{2}\int\limits_2^3 (x-1)^3 \cdot \ln^2(x-1)dx = \left(\begin{pmatrix} u = \ln^2(x-1) \\ dv = (x-1)^3 dx \end{pmatrix} \Rightarrow \begin{pmatrix} du = \dfrac{1}{x-1}dx \\ v = \dfrac{(x-1)^4}{4} \end{pmatrix} \right) \overset{Б.И.}{=}$$

$$= 4\ln^2 2 - \frac{1}{2}\left[\frac{(x-1)^4}{4} \cdot \ln(x-1)\ \Big|_2^3 - \frac{1}{4}\int\limits_2^3 (x-1)^3 dx \right] = 4\ln^2 2 - \frac{1}{2}\left[4\ln 2 - \frac{1}{16}(x-1)^4\ \Big|_2^3 \right] =$$

$$= 4\ln^2 2 - 2\ln 2 + \frac{1}{32}\cdot(16-1) = 4\ln^2 2 - 2\ln 2 + \frac{15}{32}. \triangleright$$

48-masala. $\int\limits_0^{\pi/2} \frac{\sin x dx}{(1+\sin x)^2}$ **aniq integral hisoblansin.**

$\triangleleft$ Bu integralni hisoblash uchun $tg\frac{x}{2} = t$ universal almashtirish bajaramiz. Unda $x = 0 \Rightarrow t = 0; x = \frac{\pi}{2} \Rightarrow t = 1.$va $dx = \frac{2dt}{1+t^2}; \sin t = \frac{2t}{1+t^2}$

Almashtirishdan so`ng berilgan integral quyidagi ko`rinishga keladi:

$$\int\limits_0^{\pi/2} \frac{\sin x dx}{(1+\sin x)^2} = \int\limits_0^1 \frac{2tdt}{(t+1)^2} = 2\int\limits_0^1 \frac{t+1-1}{(t+1)^2}dt = 2\left[\int\limits_0^1 \frac{dt}{t+1} - \int\limits_0^1 \frac{dt}{(t+1)^2} \right] = 2\left[\ln|t+1|\Big|_0^1 + \frac{1}{t+1}\Big|_0^1 \right]_0^1 =$$

$$= 2\left[\ln 2 + \frac{1}{2} - 1 \right] = 2\ln 2 - 1 \triangleright$$

49-masala. $\int\limits_{\frac{\pi}{2}}^{\pi} 2^8 \cdot \sin^2 x \cdot \cos^6 x dx$ **aniq integral hisoblansin.**

◁ Integralni $\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$, $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$ va $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ formulalaridan foydalanib hisoblaymiz:

$$I = \int_{\frac{\pi}{2}}^{\pi} 2^8 \sin^2 x\cdot\cos^6 x dx = 2^6\int_{\frac{\pi}{2}}^{\pi}(2\sin x\cdot\cos x)^2\cdot\cos^4 x dx = 2^4\int_{\frac{\pi}{2}}^{\pi}\sin^2 2x\cdot(1+\cos 2x)^2 dx =$$

$$= 2^4\int_{\frac{\pi}{2}}^{\pi}\left[\sin^2 2x + 2\sin^2 2x\cdot\cos 2x + \sin^2 2x\cdot\cos^2 2x\right]dx =$$

$$= 2^3\cdot\int_{\frac{\pi}{2}}^{\pi}(1-\cos 4x)dx + 2^4\int_{\frac{\pi}{2}}^{\pi}\sin 2x d(\sin 2x) +$$

$$+ 2^2\int_{\frac{\pi}{2}}^{\pi}\sin^2 4x dx = 2^3\left(x - \frac{1}{4}\sin 4x\right)\Big|_{\frac{\pi}{2}}^{\pi} + 2^4\cdot\frac{\sin^3 2x}{3}\Big|_{\frac{\pi}{2}}^{\pi} + 2\int_{\frac{\pi}{2}}^{\pi}(1-\cos 8x)dx =$$

$$= 4\pi + 2\left(x - \frac{1}{8}\sin 8x\right)\Big|_{\frac{\pi}{2}}^{\pi} = 5\pi. \triangleright$$

50-masala. $\int_0^2 \frac{dx}{(4+x^2)\sqrt{4+x^2}}$ **aniq integral hisoblansin.**

◁ Bu integralni hisoblashda uchun 2-hol uchun Eyler almashtirishini bajaramiz, ya`ni $\sqrt{x^2+4} = t + x$ almashtirish bajaramiz. $\Rightarrow x = \frac{4-t^2}{2t} \Rightarrow dx = -\frac{t^2+4}{2t^2}dt \Rightarrow \int_0^2 \frac{dx}{(4+x^2)\sqrt{4+x^2}} =$

$$\int_{2(\sqrt{2}-1)}^{2}\frac{4tdt}{(t^2+4)^2} = 2\int_{2(\sqrt{2}-1)}^{2}\frac{d(t^2+4)}{(t^2+4)^2} = -\frac{2}{t^2+4}\Big|_{2(\sqrt{2}-1)}^{2} = \frac{1}{4\sqrt{2}}. \triangleright$$

2.3-§. Trigonometrik va giperbolik funktsiyalarni integrallash

$R(\sin x,\ \cos x)dx$ ko'rinishdagi integrallar

$R(\sin x,\ \cos x)dx$ ko'rinishdagi integralar ($R - \sin x$ va $\cos x$ larga nisbatan ratsional funksiya) $tg\frac{x}{2} = t$ almashtirish yordamida ratsional funksiyalarning integrallariga keltiriladi.

$$\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2};$$

$$\sin x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2};$$

$$x=2\mathrm{arctg}t;\quad dx=\frac{2}{1+t^2}dt.$$

$R(\sin x,\ \cos x)dx=\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2}$ ko'rinishga keladi. Bunday almashtirish *universal almashtirish* deyiladi.

51–masala. Aniq integralni hisoblang.

$$\int_0^{\pi/2}\frac{\cos x-\sin x}{(1+\sin x)^2}dx=\left|\begin{matrix} tg\frac{x}{2}=t & \cos x=\frac{1-t^2}{1+t^2} \\ dx=\frac{2}{1+t^2}dt & \sin x=\frac{2t}{1+t^2}\end{matrix}\right|=\int_0^1\frac{\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}{\left(1+\frac{2t}{1+t^2}\right)^2}\cdot\frac{2dt}{1+t^2}=$$

$$=\int_0^1\frac{2(1-2t-t^2)}{(1+t)^4}dt.$$

$\frac{2(1-2t-t^2)}{(1+t^4)}$ to'g'ri kasrni sodda ratsional kasrlar yig'indisi ko'rinishida yozamiz:

$$\frac{2-4t-2t^2)}{(1+t)^4}=\frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{C}{(1+t^3)}+\frac{D}{(1+t)^4}=$$

$$=\frac{A(1+t)^3+B(1+t)^2+C(1+t)+D}{(1+t)^4}.$$

$$A(1+t)^3+B(1+t)^2+C(1+t)+D=2-4t-2t^2.$$

$t=-1$ da, $D=4;$

x ning bir xil darajalari oldidagi koeffitsiyentlarni tenglaymiz:

$t^3:\qquad A=0;$

$t^2:\qquad 3A+B=-2\Rightarrow B=-2;$

$t: \qquad 3A + 2B + C = -4 \Rightarrow C = 0;$

Demak,

$$\int_0^1 \left(\frac{4}{(1+t)^4} - \frac{2}{(1+t)^2} \right) dt = \left(-\frac{4}{3(1+t)^3} + \frac{2}{1+t} \right) \Bigg|_0^1 = -\frac{4}{3 \cdot 8} + 1 + \frac{4}{3} - 2 = \frac{1}{6}.$$

$\int tg^m x dx$ va $\int tg^m x dx$ (bu yerda $m-$butun musbat son) ko'rinishdagi integrallarda mos ravishda

$$tgt = t, \qquad dx = \frac{dt}{1+t^2}$$

$$ctgt = t, \qquad dx = -\frac{dt}{1+t^2}$$

$$\sin 2x = \frac{2tgx}{1+tg^2 x} = \frac{2t}{1+t^2}$$

o'rniga qo'yish orqali hisoblanadi.

52–masala. Aniq integralni hisoblang.

$$\int_{\pi/4}^{\operatorname{arctg} 3} \frac{dx}{(3\operatorname{tg}x + 5)\sin 2x} = \left| \begin{matrix} \operatorname{tg}x = t \\ dx = \dfrac{dt}{1+t^2} \end{matrix} \quad \sin 2x = \frac{2t}{1+t^2} \right| =$$

$$= \int_1^3 \frac{dt}{(3t+5)\dfrac{2t}{1+t^2}(1+t^2)} = \frac{1}{2}\int_1^3 \frac{dt}{t(3t+5)}.$$

$$\frac{1}{t(3t+5)} = \frac{A}{t} + \frac{B}{3t+5} = \frac{A(3t+5)+Bt}{t(3t+5)},$$

$$A(3t+5) + Bt = 1.$$

$t = 0$ da , $A = \dfrac{1}{5};$

$t=-\frac{5}{3}$ da , $B=-\frac{3}{5}$;

Shunday qilib,

$$\frac{1}{10}\int_1^3\left(\frac{1}{t}-\frac{3}{3t+5}\right)dt=\frac{1}{10}\left(\ln|t|-\ln|3t+5|\right)\Big|_1^3=\frac{1}{10}(\ln 3-\ln 14-0+\ln 8)=$$

$$=\frac{1}{10}\ln\frac{24}{14}=\frac{1}{10}\ln\frac{12}{7}.$$

$\int \sin^m x\cdot\cos^n xdx$ **ko'rinishdagi integrallar,**

bu yerda m va n –butun sonlar.

1. Agar m va n sonlarning hech bo'lmaganda bittasi toq musbat son, masalan, $m=2k+1$ bo'lsa bo'lsa, u holda quyidagicha yo'l tutamiz:

$$\int \sin^{2k+1} x\cdot\cos^n xdx=\int \sin^{2k} x\cdot\cos^n x\cdot\sin xdx=$$
$$=-\int (1-\cos^2 x)^k\cdot\cos^n xd(\cos x).$$

Masalan,

$$\int \sin^5 x\cdot\cos^2 xdx=\int \sin^4 x\cdot\cos^2 x\cdot\sin xdx=$$
$$=-\int (1-\cos^2 x)^2\cdot\cos^2 xd(\cos x)=$$
$$=-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C.$$

Agar m va n sonlardan biri toq musbat son, boshqasi istalgan haqiqiy son bo'lsa ham xuddi yuqoridek yo'l tutamiz.

2. Agar m va n juft musbat sonlar bo'lsa, integralni

$$\sin x\cdot\cos x=\frac{1}{2}\sin 2x,\quad \sin^2 x=\frac{1-\cos 2x}{2},\qquad \cos^2 x=\frac{1+\cos 2x}{2}$$

Trigonometrik formulalar yordamida hisoblaymiz.

Masalan,

$$\int \sin^2 x\cos^4 x dx = \int (\sin x\cdot\cos x)^2\cos^2 x dx =$$
$$= \int \frac{\sin^2 2x}{4}\cdot\frac{1+\cos 2x}{2}dx = \frac{1}{8}\int (\sin^2 2x + \sin^2 2x\cdot\cos 2x)dx =$$
$$= \frac{1}{8}\int \frac{1-\cos 4x}{2}dx + \frac{1}{8\cdot 2}\int \sin^2 2x d(\sin 2x) =$$
$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{48}\sin^3 2x + C =$$
$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4} + \frac{\sin^3 2x}{3}\right) + C.$$

3. Agar m va n juft-toqligi bir xil bo'lgan butun manfiy sonlar bo'lsa, integral

$$1+tg^2 x = \frac{1}{\cos^2 x},\quad 1-ctg^2 x = \frac{1}{\sin^2 x} \text{ yoki } \frac{1}{\sin^2 x} = \frac{1+tg^2 x}{tg^2 x}$$

formulalar yordamila hisoblanadi.

$$\int \frac{dx}{\cos^4 x} = \int \frac{1}{\cos^2 x}\cdot\frac{dx}{\cos^2 x} = \int (1+tg^2 x)^2 d(tgx) =$$
$$= tgx + \frac{tg^3 x}{3} + C.$$

53–masala. Aniq integralni hisoblang.

$$\int_0^{\pi} 2^4\cos^8\left(\frac{x}{2}\right)dx = \left|\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)\right| =$$
$$= \int_0^{\pi}(1+\cos x)^4 dx = \int_0^{\pi}(1+2\cos x+\cos^2 x)^2 dx =$$
$$= \int_0^{\pi}(1+3\cos x+6\cos^2 x+4\cos^3 x+\cos^4 x)dx =$$
$$= \int_0^{\pi}\left(\frac{35}{8}+3\cos x+\frac{7}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx + 4\int_0^{\pi}(1-\sin^2 x)\cos x dx =$$
$$= \left(\frac{35}{8}x+3\sin x+\frac{7}{4}\sin 2x+\frac{1}{32}\sin 4x\right)\Bigg|_0^{\pi} + 4\int_0^{\pi}(1-\sin^2 x)d(\sin x) =$$
$$= \frac{35}{8}\pi + 4(\sin x - \frac{1}{3}\sin^3 x)\Big|_0^{\pi} = \frac{35}{8}\pi.$$

$\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}},...\right]dx$ **ko'rinishdagi integrallar**, bu yerda $p_1, q_1, p_2, q_2, ... -$ butun sonlar. Agar barcha $q_1, q_2, ...$ maxrajlarning eng kichik karralisi k bo'lsa, u holda ushbu integral $\frac{ax+b}{cx+d} = t^k$ o'rniga qo'yish yordamida ratsional funksiyadan olingan integrallarga keltiriladi.

54–masala. Aniq integralni hisoblang.

$$\int_6^9 \sqrt{\frac{9-2x}{2x-21}}dx = \left| \begin{array}{l} \frac{9-2x}{2x-21} = t^2 \\ dx = \frac{12t}{(t^2+1)}dt \end{array} \right| = 12\int t\frac{t}{(t^2+1)^2}dt =$$

$$= 12\int \frac{t^2}{(t^2+1)^2} = \left| \begin{array}{l} t = tga \\ dt = \frac{da}{\cos^2 a} \end{array} \right| = 12\int tg^2 \cos^2 ada =$$

$$= 12\int \sin^2 ada = 6\int (1-\cos 2a)da = 6arctgT - 3\sin(2arctgt) =$$

$$= \left(6artctg\sqrt{\frac{9-2x}{2x-21}} - 3\sin\left(2arctg\sqrt{\frac{9-2x}{2x-21}}\right)\right)\Bigg|_6^9 =$$

$$= 6arctg\sqrt{3} - 3\sin(2arctg\sqrt{3}) - 6arctg\frac{1}{3} + 3\sin(2arctg\frac{1}{\sqrt{3}}) =$$

$$= 2\pi - 3\sin\frac{2\pi}{3} - -\pi + 3\sin\frac{\pi}{3} = \pi - 3\frac{\sqrt{3}}{2} + 3\frac{\sqrt{3}}{2} = \pi.$$

$R\left(x, \sqrt{a^2 \pm x^2}\right)$ **va** $R\left(x, \sqrt{x^2 - a^2}\right)$ **ko'rinishdagi integrallar.**

1. $\int_\alpha^\beta R\left(x, \sqrt{a^2 - x^2}\right)dx;$

2. $\int_\alpha^\beta R\left(x, \sqrt{a^2 + x^2}\right)dx;$

3. $\int_\alpha^\beta R\left(x, \sqrt{x^2 - a^2}\right)dx;$

bu yerda $R-$ratsional funksiya.

Agar a) $x=a\sin t$ yoki $x=a\cos t$

b) $x=atgt$ yoki $x=actgt$

c) $x=a\sec t$ yoki $x=a\cos ect$

trigonometrik o'rniga qo'yishlardan foydanilsa, bu integrallar $\int R(\sin t,\cos t)dt$ ko'rinishdagi integrallarga keltiriladi.

55–masala. Aniq integralni hisoblang.

$$\int_0^3 \frac{dx}{(9+x^2)^{3/2}}=\left|\begin{matrix} x=3tgt \\ dx=\dfrac{3dt}{\cos^2 t}\end{matrix}\right|=\int_0^{\pi/4}\frac{3dt}{(9+9tg^2t)^{3/2}\cos^2 t}=$$

$$=\frac{3}{27}\int_0^{\pi/4}\frac{\cos^3 t}{\cos^2 t}dt=\frac{3}{27}\int_0^{\pi/4}\cos tdt=\frac{3}{27}\sin t\Big|_0^{\pi/4}=\frac{\sqrt{2}}{18}.$$

Giperbolik funktsiyala quyidagicha aniqlangan:

$$shx=\frac{e^x-e^{-x}}{2},\quad chx=\frac{e^x+e^{-x}}{2},\quad thx=\frac{e^x-e^{-x}}{e^x+e^{-x}},\quad cthx=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

va ular mos ravishda giprebolik sinus, kosinus, tangens va kotangens deyiladi. Ular isbot qilinishi oson bo'lgan ushbu formulalar bilan bog'langan:

$$ch^2x-sh^2x=1,$$
$$ch2x=ch^2x+sh^2x,$$
$$sh2x=2shx\cdot chx,$$
$$sh^2x=\frac{1}{2}(ch2x-1),$$
$$ch^2x=\frac{1}{2}(ch2x+1).$$

Giperbolik funktsilarning hosilalari

$$(shx)'=chx,\qquad (chx)'=shx,\qquad (thx)'=\frac{1}{ch^2x},\qquad (cthx)'=-\frac{1}{sh^2x}$$

formulalar bilan ifodalanadi.

Giperbolik funktsiyalarni integrallash trigonometrik funktsiyalarni integrallash kabidir, ular quyidagi formulalarga asoslanadi:

$\int chxdx = shx + C, \quad \int shxdx = chx + C, \quad \int \frac{dx}{sh^2 x} = -cthx + C, \int \frac{dx}{sh^2 x} = -cthx + C.$

Ba'zan $\int R(x, \sqrt{x^2 - a^2})dx$ va $\int R(x, \sqrt{x^2 + a^2})dx$ ko'rinishdagi integrallarni mos ravishda $x = acht$ va $x = asht$ o'rniga qo'yishlardan foydalaniladi. Bunda:

agar $x = acht$ bo'lsa, $t = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right|$,

agar $x = asht$ bo'lsa, $t = \ln\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right|$.

Undan tashqari quyidagi o'rniga qo'yish ham mavjud:

agar $x = thx$ bo'lsa, $t = \frac{1}{2}\ln\frac{1+x}{1-x}$.

56-misol. $\int sh^2 xdx.$

Yechish. Quyidagiga egamiz:

$$\int sh^2 xdx = \frac{1}{2}\int (ch2x - 1)dx = \frac{1}{2}sh2x - \frac{1}{2}x + C.$$

57-misol. $\int sh^3 xdx.$

Yechish. Quyidagiga egamiz:

$$\int sh^3 xdx = \int sh^2 xd(chx) = \int (ch^2 x - 1)d(chx) = \frac{1}{3}ch^3 x - chx + C.$$

Integralashning rekurrent formulalari

Integrallarni hisoblashda tez-tez ishlatib turiladigan ba'zi rekurrent formulalarni keltiramiz (bu yerda $n -$ butun musbat son).

I.
$$\int \frac{dx}{(ax^2 + bx + x)^n} = \frac{2ax + b}{(n-1)(4ac - b^2)(ax^2 + bx + c)^{n-1}} + $$
$$+ \frac{2(2n-3)a}{(n-1)(4ac - b^2)}\int \frac{dx}{(ax^2 + bx + x)^{n-1}} \qquad (n \neq 1).$$

II. $$\int \frac{dx}{(x^2 + a^2)^n} = \frac{1}{2a^2(n-1)}\left[\frac{x}{(x^2 + a^2)^{n-1}} + (2n-3)\int \frac{dx}{(x^2 + a^2)^{n-1}}\right], \qquad (n \neq 1).$$

III. $$\frac{Mx + N}{(ax^2 + bx + c)^n} = -\frac{M}{2(n-1)(ax^2 + bx + c)^{n-1}} + \frac{2aN - bM}{2a}\int \frac{dx}{(ax^2 + bx + c)^n},$$

$(n \neq 1)$.

IV. $\int \sin^n axdx = -\frac{\sin^{n-1} ax \cdot \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} axdx.$

V. $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)} \cdot \frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}, \qquad (n > 1).$

VI. $\int \cos^n axdx = \frac{\cos^{n-1} ax \cdot \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} axdx.$

VII. $\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \cdot \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}, \quad (n > 1).$

VIII. $\int ch^n xdx = \frac{1}{n} shx \cdot ch^{n-1} x + \frac{n-1}{k}\int ch^{n-2} xdx.$

IX. $\int sh^n xdx = \frac{1}{n} chx \cdot sh^{n-1} x + \frac{n-1}{k}\int sh^{n-2} xdx.$

X. $\int tg^n axdx = \frac{tg^{n-1} ax}{a(n-1)} - \int tg^{n-2} axdx, \quad (n > 1).$

XI. $\int ctg^n axdx = -\frac{ctg^{n-1} ax}{a(n-1)} - \int ctg^{n-2} axdx, \ (n > 1).$

XII. $\int x^n \sin axdx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1} \cos axdx.$

XIII. $\int x^n \cos axdx = -\frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1} \sin axdx.$

XIV. $\int \ln^n xdx = x\ln^n x - n\int \ln^{n-1} xdx.$

XV. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx.$

Rekurrent formulalardan foydalanib, ba'zi integrallarni hisoblaylik.

58-misol. $\int \frac{dx}{(x^2 + a^2)^3}.$

Yechish: Quyidagiga egamiz: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg \frac{x}{a}$. (II) formulani $n = 2$ deb qo'llasak:

$$\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\left(\frac{x}{x^2+a^2}+\int\frac{dx}{x^2+a^2}\right)=\frac{1}{2a^2}\left(\frac{x}{x^2+a^2}+\frac{1}{a}arctg\frac{x}{a}\right).$$

(II) formulani yana bir marta, endi $n=3$ deb qo'llaymiz:

$$\int\frac{dx}{(x^2+a^2)^3}=\frac{1}{4a^2}\left(\frac{x}{(x^2+a^2)^2}+3\int\frac{dx}{(x^2+a^2)^2}\right).$$

Keyingi tenglikka $\int\frac{dx}{(x^2+a^2)^2}$ integralning qiymatini qo'yamiz:

$$\int\frac{dx}{(x^2+a^2)^3}=\frac{1}{4a^2}\left(\frac{x}{(x^2+a^2)^2}+\frac{3}{2a^2}\left(\frac{x}{x^2+a^2}+\frac{1}{a}arctg\frac{x}{a}\right)\right)+C=$$

$$=\frac{1}{4a^2}\left(\frac{x}{(x^2+a^2)^2}+\frac{3x}{2a^2(x^2+a^2)}+\frac{3}{2a^3}arctg\frac{x}{a}\right)+C=$$

$$=\frac{1}{4a^2}\left(\frac{x(3x^2+5a^2)}{2a^2(x^2+a^2)^2}+\frac{3}{2a^3}arctg\frac{x}{a}\right)+C.$$

59-misol. $\int\frac{dx}{\sin^3 x}$.

Yechish: Quyidagiga egamiz: $\int\frac{dx}{\sin x}=\ln\left|tg\frac{x}{2}\right|$. (IV) formulada $n=-1,\ a=1$ deb topamiz:

$$\int\frac{dx}{\sin x}=-\frac{\sin^{-2}x\cos x}{-1}+\frac{-2}{-1}\int\frac{dx}{\sin^3 x}=\frac{\cos x}{\sin^2 x}+2\int\frac{dx}{\sin^3 x}.$$

Oxirgi tenglikni $\int\frac{dx}{\sin^3 x}$ integralga nisbatan yechib, topamiz:

$$\int\frac{dx}{\sin^3 x}=\frac{1}{2}\left(-\frac{\cos x}{\sin^2 x}+\int\frac{dx}{\sin x}\right)=\frac{1}{2}\left(-\frac{\cos x}{\sin^2 x}+\ln\left|tg\frac{x}{2}\right|\right)+C.$$

60-misol. $\int x^4e^{2x}dx$.

Yechish: (XV) formulani $n=1$ va $a=2$ deb qo'llaymiz:

$$\int xe^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}.$$

$n=2$ va $a=2$ da shu formulaning o'zi quyidagini beradi:

$$\int x^2e^{2x}dx=\frac{1}{2}x^2e^{2x}-\frac{1}{2}\int xe^{2x}dx=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}.$$

$n=3$, $a=2$ deb topamiz:

$$\int x^3e^{2x}dx=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\int x^2e^{2x}dx=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\left(\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}\right)+C.$$

$n=4$ va $a=2$ da yana shu formulani tatbiq qilamiz:

$$\int x^4e^{2x}dx=\frac{1}{2}x^4e^{2x}-\frac{3}{2}\int x^3e^{2x}dx.$$

Oxirgi integralning qiymatini o'z o'rniga qo'yib, uzil-kesil topamiz:

XULOSA VA TAVSIYALAR

Yurtboshimiz Islom Abdug’anievich Karimovning “Bizga bitiruvchilar emas, maktab ta’lim va tarbiyasini ko’rgan shaxslar kerak”[1], – degan so’zlarini inobatga oladigan bo’lsak, oliy pedagogik ta’lim jarayoni oldiga pedagog kadrlarni, shu jumladan peadagogika ta’lim sohasi matematika ta’lim yo’nalishi mutaxassislarini tayyorlashni yanada yaxshilash vazifasini qo’yadi.

Ilmiy manbalarni tahlil qilish, hamda amaliy mashg’ulotlarni o’tkazish jarayonida kuzatish natijalari shuni ko’rsatadiki, “Matematika” ta’lim yo’nalishi talaba yoshlariga ta’lim berish va ularni bo’lajak pedagogik faoliyatga tayyorlash ko’pgina nazariy hamda amaliy muammolarni hal etishni taqoza etadi. Bo’lajak matematika o’qituvchilarini tayyorlashda “Aniq integral va integrallash usullari” moduli bo’yicha olib borilgan kuzatish natijasida biz qo’yidagi xulosalarga keldik.

- Uzluksiz ta’lim jarayonining eng muhim vazifalaridan biri bo’lajak kadrlarni tayyorlashda, avvalambor, uning kasbiy tayyorgarligini hisobga olish va maktab o’quvchilarini pedagoglik kasbiga tayyorlash sohasida maktablar, litseylar, kasb–hunar kollejlari bilan mustahkam aloqa bog’lashlari zarur. Chunki kasbiy mahoratning shakllanishida uzviylik, tizimlilik va izchillikning o’rni katta.
- Bo’lajak mutaxassislarni “Matematik analiz” fani mashg’ulotlari jarayonini ilg’or pedagogik–innovatsion texnologiyalar asosida tashkil etilishiga e’tibor qaratish zarur.
- Fanning mazmunini qayta ko’rib chiqish va uni zamonaviylashtirish hozirgi kun zarurati, shu jihatdan qaraganda biz tavsiya etgan pedagogika fanlarini o’qitish tizimi yaxshi natija beradi, deb hisoblaymiz.
- Bo’lajak mutaxassislarning nazariy jihatdan tayyorlashni amaliy tayyorgarlik bilan mustahkam aloqada yo’lga qo’yish zarur. CHunki amaliyot ularni pedagogik faoliyatga qanchalik tayyorligini ko’rsatuvchi mezondir. SHuning uchun ham bo’lajak kadrlarni uzluksiz ta’lim tizimiga qabul qilingandan

[1] Каримов И.А. Баркамол авлод – Ўзбекистон тараққиётининг пойдевори. –Т.: Шарқ. 1997 й.

boshlab amaliy tayyorlashga e'tibor qaratish, uzluksiz amaliyotni tashkil etish, amaliy va mustaqil tayyorgarlikka ko'proq vaqt ajratish yaxshi samara beradi deb o'ylaymiz.

Matematika ta'lim yo'nalishi talabalariga "Matematik analiz" fanini o'rgatishning samarali yo'llarini, o'quv jarayonida zamonaviy pedagogik–innovatsion texnologiyalarni tatbiq etishga o'rgatishda biz yuqorida tavsiya etgan ish tizimidan foydalansa kutilgan kafolatli samaraga erishish mumkin, deb hisoblaymiz.

Yuqorida keltirilgan muammolarni ijobiy tomonga o'zgartirishga qaratilgan fikr va tavsiyalarimiz quyidagilardan iborat:

- Oliy ta'lim muassasalari talabalari uchun "Matematik analiz" fanini o'qitish metodikasi bo'yicha ilg'or pedagogik–innovatsion ta'lim texnologiyalari asosida mukammal bo'lgan o'zbek tilidagi darslik, o'quv qo'llanma, metodik ishlanma, elektron darslik, elektron qo'llanmalarni, pedagogik dasturiy vositalarni va elektron ta'lim axborot resurslarini yetarli darajada yaratish va o'quv jarayoniga tatbiq qilish.
- "Matematik analiz" fanidan ilg'or pedagogik–innovatsion ta'lim texnologiyalari asosida yaratilgan mukammal darslik, o'quv qo'llanma, metodik ishlanma, elektron darslik, elektron qo'llanma, pedagogik dasturiy vositalar, elektron ta'lim axborot resurslari tanlovini o'tkazish va g'oliblarni rag'batlantirish mexanizmini yanada rivojlantirish.
- Oliy ta'lim muassasalarida bo'lajak "Matematik analiz" fani o'qituvchilariga ta'lim–tarbiya berayotgan professor–o'qituvchilarni ilg'or xorijiy mamlakatlarning tajribasini o'rganish uchun malaka oshirib kelish imkoniyatini yaratish.

Yuqoridagi tadbirlar to'g'ri va o'z o'rnida tashkil etilsa, o'quv mashg'ulotlarida, xususan "Matematik analiz" fanlarida turli xil ilg'or pedagogik–innovatsion metodlar talabalarga bilim berishda qo'llanilsa, bo'lajak pedagoglarni har tomonlama mukammal qilib tayyorlash, ularning kreativlik qobiliyatlarini shakllantirish va pedagogik–kasbiy kompetentligini rivojlantirish imkoniyatiga erishiladi degan umiddamiz.

Matematika fani o'qituvchisi doimo o'z pedagogik mahoratini oshirib, uni san'at darajasiga yetkazib borishi kerak. O'z fikrini tushunarli, ko'rgazmali ifodalay bilish, jahon va mamlakatimiz ijtimoiy–matematik hayotidagi voqealarga o'z munosabatini bildirishi, ta'lim berishga ijodiy yondashuvni rivojlantirib borishi kerak.

O'qituvchining ijodiy yondashuvi talaba tomonidan o'rganishga ijodiy yondashuvining bevosita shartidir.

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