V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI
HUZURIDAGI ILMIY DARAJALAR BERUVCHI
DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

HAYDAROV FARHOD HALIMJONOVICH

PANJARALI SISTEMALARDA LIMIT VA GRADIYENT GIBBS O‘LCHOVLARI

01.01.01-Matematik analiz

FIZIKA-MATEMATIKA FANLARI DOKTORI (DSc) DISSERTATSIYASI AVTOREFERATI

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KIRISH (doktorlik dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zarurati. Jahon miqyosida olib borilayotgan ko‘plab ilmiy-amaliy tadqiqotlar statistik fizika va mexanikada uchraydigan spin sistemalarga bag‘ishlangan bo‘lib, ular ichida Gibbs o‘lchovlar nazariyasiga oid ilmiy ishlar salmoqli ulushga ega. Panjarali sistemalarda berilgan Gibbs o‘lchovlari statistik mexanikada alohida ahamiyatga ega bo‘lishi bilan bir qatorda boshqa turli ilmiy yo‘nalishlarda ham keng tatbiqlari mavjud. Gibbs o‘lchovlari ko‘p ta’sirli zarrachalarga bog‘liq sistemalarning muvozanat xossalarini tushuntirishda asosiy tayanch hisoblanadi. Sistemadagi holatlarning taqsimotlarini tavsiflashda Gibbs o‘lchovlari fazaviy o‘tishlarni aniqlash, termodinamik xossalarini tekshirish kabi muhim masalalarni tahlil qilish imkonini beradi. Shuningdek, Gibbs o‘lchovi potensial energiya va entropiya birgalikda qaralganda, tashqi ta’sir ostida kimyoviy reaksiyalarning sodir bo‘lish yoki bo‘lmasligini ifodalashga xizmat qiladi.

Hozirgi kunda jahonda panjarali sistemalarda aniqlangan Gamiltonian uchun gradiyent Gibbs o‘lchovlarini tavsiflash masalasi keng miqyosda o‘rganilmoqda. Gradiyent konfiguratsiyalar daraxtning uchlarida emas balki qirralarida aniqlanganligi sababli limit Gibbs o‘lchovlari mavjud bo‘lmay qolgan hollarda ham gradiyent Gibbs o‘lchovlari mavjud bo‘ladigan spin sistemalarga ko‘plab misollar keltirish mumkin. Bu esa statistik mexanikada gradiyent Gibbs o‘lchovlarining o‘rni qanchalik muhim ekanligini ko‘rsatadi. Shuningdek, ushbu o‘lchovlar turli energiya manbalariga asoslangan sistemalarda gradiyentlarni hisoblashga ham xizmat qiladi. Bundan tashqari, Gibbs o‘lchovlari chegaraviy qonunlar yordamida aniqlanishi hisobga olinib, dunyo olimlari tomonidan bir bog‘lamli komponentadan iborat graflarda berilgan chegaraviy qonunga oid ko‘plab ilmiy ishlar olib borilgan. Boshqa tomondan, agar Gibbs o‘lchovlarini daraxtda qaralsa, u holda chegaraviy qonunni bir nechta bog‘lamli kompotentalarga ega graflarda qarashga to‘g‘ri keladi. Shu sababli, hozirda daraxtlarda Gibbs o‘lchovlarini tadqiq etish masalasi jahon olimlari tomonidan katta qiziqishlarga sabab bo‘lmoqda.

Mamlakatimizda fundamental fanlarning ilmiy va amaliy tatbiqiga ega bo‘lgan statistik fizika va mexanikaning dolzarb yo‘nalishlariga e’tibor kuchaytirildi. Jumladan, panjarali sistemalarda berilgan Gamiltonian uchun limit va gradiyent Gibbs o‘lchovlari nazariyasini rivojlantirishda, o‘lchovlar nazariya va nochiziqli analizning amaliy masalalari yordamida sezilarli yutuqlarga erishildi. Keli daraxtida berilgan modellar uchun Gibbs o‘lchovlari to‘plamini tadqiq etishda salmoqli natijalarga erishildi. Matematik fizika, nochiziqli analiz, o‘lchovlar nazariyasi va stoxastik jarayonlar kabi ilmiy sohalarda xalqaro standartlar darajasida ilmiy tadqiqotlar olib borish matematika fanining asosiy vazifalari va faoliyat yo‘nalishlari etib belgilandi[1]. Qaror ijrosini ta’minlashda panjarali sistemalarda Gibbs o‘lchovlari

[1] O‘zbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash shuningdek, O‘zbekiston Respublikasi Fanlar akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida” gi № PQ-4387-son qarori

nazariyasini rivojlantirish muhim ahamiyatga ega.

O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi to‘g‘risida”gi PF-4947-son Farmoni, 2019-yil 9-iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash, shuningdek, O‘zbekiston Respublikasi Fanlar Akademiyasining V.I. Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida”gi PQ-4387-son Qarori va 2020-yil 7-maydagi “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi PQ-4708-sonli Qarori hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishi ustuvor yo‘nalishlariga bog‘liqligi. Mazkur tadqiqot respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Dissertatsiya mavzusi bo‘yicha xorijiy-ilmiy tadqiqotlar sharxi. Gibbs o‘lchovlari va ulardan amaliy foydalanish bo‘yicha butun dunyo bo‘ylab yirik ilmiy markazlar va universitetlarda ilmiy tadqiqotlar olib borilmoqda, xususan Bonn va Boxum-Rur universitetlarii (Germaniya), Britaniya Kolumbiya Texnologiya instituti (Kanada), Bari va Rim universitetlarida (Italiya) olib borilmoqda. Fizika instituti (Serbiya), Rossiya fanlar akademiyasi V.A. Steklov nomidagi matematika instituti (Rossiya), Fields instituti (Kanada), Xarran universiteti (Turkiya), Indiana universiteti (AQSh), Aix-Marsel universiteti va Parij universiteti (Fransiya), Lids universiteti va London kolleji (Buyuk Britaniya), Kyusyu universiteti (Yaponiya), Malayziya Xalqaro Islom universiteti (Malayziya), Berlin texnika universiteti (Germaniya) va boshqalar.

Panjarali sistemalarda berilgan Gibbs o‘lchovlari bo‘yicha olib borilgan ilmiy tadqiqotlar natijalari bir qancha dolzarb masalalarni muvaffaqiyatli hal qildi. Ulardan ayrimlarini keltirib o‘tamiz: daraxtlarda lokallashgan va lokal bo‘lmagan Gibbs o‘lchovlarini birgalikda mavjudligi (Boxum-Rur universiteti, Germaniya), uchburchak panjarada yuqori zichlikdagi qattiq yadroli tasodifiy konfiguratsiyalarning Gibbs statistikasini rivojlantirish (Kembrij universiteti, Buyuk Britaniya) , umumlashgan Gibbs o‘lchovlarining asosiy xossalarini o‘rganish (Parij universiteti, Fransiya), Keli daraxtlarida berilgan Gamiltonian uchun tartibsiz fazalarning chetkiligi (Harran universiteti, Turkiya), konfiguratsiyalar fazosida Gibbs klaster holatlarining ehtimollik taqsimotini rivojlantirish (Lids universiteti, Buyuk Britaniya), cheksiz spinli sanoqsiz zarrachalar sistemalarining Gibbs o‘lchovlarining mavjudligi va yagonaligi masalalari (York universiteti, Buyuk Britaniya), Gibbs o‘lchovlarining chegaraviy shartlariga mos erkin energiyalarni (va entropiyalarni) hisoblash (Aix-Marsel universiteti, Fransiya).

Bugungi kunda Gibbs o‘lchovlari nazariyasi bo‘yicha ko‘plab tadqiqotlar olib borilmoqda, ular quyidagilarni o‘z ichiga oladi: Berilgan panjarali modellar uchun Gibbs o‘lchovlari mavjudligi va yagonaligini ifodalovchi shartlarni tadqiq etish, Gibbs o‘lchovlarining termodinamik chegaradagi xususiyatlarini tahlil qilish, panjarali sistemalarda fazaviy o‘tishlarning paydo bo‘lishi va xossalarini o‘rganish, Gibbs o‘lchovlarining xususiyatlarini o‘rganishda gruppaviy tasniflarning usullarini qo‘llash, Monte-Karlo simulyatsiyalari, stoxastik evolyutsiya tenglamalari kabi turli jarayonlarda spin sistemalarning dinamikasini tadqiq etish.

Muammoning o‘rganilganlik darajasi. Gibbs o‘lchovi statistik mexanikadagi asosiy tushunchalardan biri bo‘lib, amerikalik fizik va matematik J. V. Gibbs tomonidan Gibbs holati tushunchasini kiritgan bo‘lib, bu tushuncha fizik sistemalarni band qilishi mumkin bo‘lgan holatlar to‘plamidir. Nemis fizigi Ernst Ising dastlab 1925 yilda doktorlik dissertatsiyasida panjaradagi magnit spinlarni tasvirlash uchun bir o‘lchamli matematik modelni fanga kiritadi. 1944 yilda Lars Onsager esa Ernst Ising modelining ikki o‘lchamli bo‘lgan holini kiritadi va uning aniq yechimlarini ko‘rsatadi. L. Dobrushin, D. Lanford, D. Ryuel kabi olimlar tomonidan ehtimollik o‘lchovining Gibbs o‘lchovi bo‘lishi uchun zarur va yetarlilik sharti olingan bo‘lib, keyinchalik ushbu shart DLR tenglamasi deya atala boshlandi. Hozirda, matematiklar tomonidan Gibbs o‘lchovining ta’rifini DLR tenglamasi yordamida berish keng tarqalgan. Yuriy Pirogov va Yakov Sinay nomi bilan atalgan Pirogov-Sinay nazariyasi panjarali sistemalarda Gibbs o‘lchovlari yordamida fazalar almashinuvlarini o‘rganish uchun qat’iy matematik asosni taqdim etadi. Ushbu nazariyada fazaviy o‘tishlar sodir bo‘ladigan shartlar aniqlanadi va kritik nuqtalar yaqinida Gibbs o‘lchovlarining xarakterini tavsiflaydi. Bundan tashqari, V. Minlos, H. Georgii, A. Bleher, B. Simon, S. Zaxari, R. Kotecky, S. Shlosman, K. Preston, S. Friedli, Y. Velenik va boshqa olimlar panjarali sistemalarda fazaviy o‘tish hodisalarini tadqiq etishga katta hissa qo‘shdilar va Gibbs o‘lchovlarining matematik xossalari va panjarali tizimlarda Gibbs o‘lchovlarining matematik xossalari va xarakterini ifodalovchi qimmatli ma’lumotlarni chop etishdi.

Ma’lumki, Gibbs o‘lchovlar nazariyasida Keli daraxtlarida aniqlangan Gibbs o‘lchovlarini tadqiq etish alohida ahamiyat kasb etadi. K. Kulske, K. Preston, R. Baxter, V. Malishev, D. Gandolfo, J. Ruiz, P. Bleher, Y. Suhov, L. Bogachev, H. Akin, S. Temur va boshqalar alohida hamda birgalikda hamkorlik qilib, statistik mexanikaning modellariga mos Gibbs o‘lchovlari to‘plamini tadqiq etishda katta hissa qo‘shdilar. Ularning ishlari asosan limit Gibbs o‘lchovlarining tahliliga, davriy Gibbs o‘lchovlariga va Gibbs o‘lchovlarini turli sohalarda, jumladan, biologiya, tibbiyot va iqtisodiyotda qo‘llash tahliliga qaratilgan. Ular turli sohalarda modellar uchun Gibbs o‘lchovlar to‘plamini bo‘sh bo‘lmasligi, fazaviy o‘tishlarni mavjudligiga oid nazariya va usullarni ishlab chiqdilar va rivojlantirdilar.

Hozirda tadqiqotchilar fizik sistemalarning Gamiltonianiga mos Gibbs holatlariga ko‘p e’tibor qaratmoqdalar. N. G‘anixo‘jaev, U. Rozikov va ularning ilmiy maktablarida Ising, SOS, HC va Potts kabi modellar uchun davriy Gibbs o‘lchovlari to‘plami faol tadqiq etib kelmoqdalar. Ayni paytda, F. Muhammedov, M. Rahmatullaev, O. Hakimov, I. Sattarov va A. To‘xtabaevlar yuqorida ko‘rsatilgan modellar bo‘yicha p-adik maydonlarda p-adik Gibbs o‘lchovlariga oid salmoqli natijalarga erishmoqdalar. Bundan tashqari, G‘. Botirov, M. Rahmatullaev, M. Rasulovalar ushbu modellarning asosiy holatlarini tavsiflashda daraxtlardagi kontur usulidan foydalanib, ma’lum haroratlarda davriy Gibbs o‘lchovlari sonini aniqladilar. Bir necha yil oldin nemis olimi K. Kulske daraxtlarda gradiyent Gibbs o‘lchovlarida oid maqola chop etdi. Hozirda A. Le Ny, L. Kokil, F. Henning, B. Jahnel, P. Shriever va boshqalar olimlar panjaralarda gradiyent Gibbs o‘lchovlari uchun ma’lum bo‘lgan natijalarga asoslanib, daraxtlarda gradiyent Gibbs o‘lchovlari oid ilmiy izlanishlar olib bormoqdalar. Aytish joizki, ushbu sohaning rivojlanish muddati nisbatan qisqa bo‘lishiga qaramasdan O‘zbekiston olimlari tomonidan ko‘plab ilmiy maqolalar chop etilmoqda.

Dissertasiya tadqiqotining dissertasiya bajarilgan oliy ta’lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog‘liqligi. Dissertatsiya tadqiqoti Matematika institutining YoF-4-3 “Sanoqli graflarda spin sistemalar extimollik o‘lchovlari” (2016-2017 yillar), O‘zbekiston Milliy universiteti YoF-4-8 “Matematik va statistik fizikaning noklassik masalalari» (2016-2017 yillar) va YoOT-Ftex-2018-154 raqamli “$\mathbb{Z}^d$ panjaralarida va Keli daraxtlarida aniqlangan Gamiltonianlar spektrlari va Gibbs o‘lchovlari” (2018-2019 y), OT-EA-4-02: “Kasr tartibli integro-differensial tenglamalar va bunday tenglamalarning yangi turlarini o‘rganish masalalari (2019-2020)” mavzusidagi ilmiy tadqiqot loyihalari doirasida bajarilgan.

Tadqiqot ishining maqsadi panjarali sistemalarda spin qiymatlari to‘plami chekli, sanoqli va kontinuum bo‘lgan modellar uchun spetsifikatsiyalarning yangi sinflarini qurish hamda ularga mos limit va gradiyent Gibbs o‘lchovlari to‘plamini tavsiflashdan iborat.

Tadqiqotning vazifalari:

panjarali sistemalarda ehtimollik bo‘lmagan yadrolar oilasi uchun yangi spetsifikatsiyani qurish;

spin qiymatlari sanoqli bo‘lgan HC va SOS modellari uchun davriy chegaraviy qonunga mos gradiyent va limit Gibbs o‘lchovlari to‘plamini tavsiflash;

uzluksiz spin modellarning limit Gibbs o‘lchovlari va nochiziqli integral operatorlarning qo‘zg‘almas nuqtalari o‘rtasida bog‘lanishni aniqlash va ushbu modellar uchun Gibbs o‘lchovlarining mavjudligi, yagonaligi va yagona bo‘lmasligini ta’minlovchi shartlar topish;

Keli daraxtidagi Izing modeli uchun translyatsion-invariant va davriy chegaraviy shartlar uchun erkin energiyalarni hisoblash, shuningdek, Keli daraxti gruppaviy tasvirining qism gruppalari uchun o‘zgarmaslik xossasini ta’minlash va ushbu xossani Izing modeliga qo‘llash.

Tadqiqotning ob’yekti: fazoviy va panjarali sistemalar, Gamiltonian, Gibbs o‘lchovi, gradiyent Gibbs o‘lchovi.

Tadqiqotning predmeti: Panjarali sistemalarni qurish, limit va gradiyent Gibbs o‘lchovlari haqida asosiy tushunchalar, Gibbs o‘lchovlari nazariyasi va nochiziqli operatorlar nazariyasi o‘rtasidagi bog‘lanishlar, tashqi maydonli Gamiltonian uchun erkin energiya, Keli daraxti gruppaviy tasvirining qism gruppalari.

Tadqiqotning usullari: Tadqiqot ishida o‘lchovlar nazariyasi usullari, funksional analiz, gruppalar nazariyasi, ehtimollar nazariyasi va nochiziqli integral operatorlar nazariyasi usullaridan foydalanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

panjarali sistemalarda ehtimollik bo‘lmagan yadrolar oilasi uchun yangi spesifikatsiya qurilgan;

Keli daraxtida, sanoqli spin qiymatli SOS modeli uchun 4-davriy chegaraviy qonun yordamida aniqlangan gradiyent Gibbs o‘lchovlari to‘plami tavsiflangan va HC modeli uchun cheksiz ko‘p gradiyent Gibbs o‘lchovlari qurilgan;

Keli daraxtida uzluksiz spin modellar uchun Gibbs o‘lchovlarining mavjud va yagonaligi isbotlangan hamda limit Gibbs o‘lchovlarining fazaviy o‘tishlarini mavjudligi nochiziqli integral operatorlarning qo‘zg‘almas nuqtalari orqali isbotlangan;

tashqi maydonga ega Ising modeli uchun translyatsion-invariant va davriy chegaraviy shartlarining erkin energiyalari hisoblangan va Keli daraxti gruppaviy tasvirining qism gruppalari uchun invariantlik xossasini saqlaydigan yetarlilik sharti olingan.

Tadqiqotning amaliy natijasi quyidagilardan iborat:

statistik mexanika modellari uchun berilgan Gibbs o‘lchovlari erkin energiyasining matematik formulasi va hisoblash usullari keltirilgan;

biologik modellar uchun zarrachalar va molekulalar spin sistemalarini tasniflashning matematik usullari bayon qilingan.

Tadqiqot natijalarining ishonchliligi. Nochiziqli operatorlar nazariyasi, funksional analiz, o‘lchovlar va ehtimollar nazariyasi, gruppa va graflar nazariyasi, Gibbs o‘lchovlar nazariyasining texnika va usullaridan foydalanilgan. Olingan natijalarning isboti qat’iy matematik mulohazalarga asoslangan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati panjarali sistemalarda berilgan statistik mexanikaning sanoqli spin qiymatga ega modellari uchun fazaviy o‘tishlar mavjudligi gradiyent Gibbs o‘lchovlari to‘plamini tavsiflashda qo‘llanilganligi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati Gammershteyn tipidagi nochiziqli integral tenglamaning qo‘zg‘almas nuqtalaridan foydalanib uzluksiz spin modellar uchun Gibbs o‘lchovlarini qurish va fazaviy o‘tishlar mavjudligini tekshirish imkonini berganligi bilan izohlanadi.

Tadqiqot natijalarining joriy qilinishi. Panjarali sistemalarda limit va gradiyent Gibbs o‘lchovlari bo‘yicha olingan natijalar asosida:

daraxtlarda berilgan tashqi maydonga ega Ising modeli uchun translatsion-invariant va davriy chegaraviy shartlarining erkin energiyalari hisoblash usullaridan G0003247 raqamli “Panjarali modellarning qayta normallashgan gruppalariga mos aralash p-adik dinamik sistemalari va xaotikligi” mavzusidagi xorijiy grant loyihasida Keli daraxtida raqobatlashuvchi ta’sirga ega bo‘lgan potensiallar energiyasi uchun yangi Gibbs o‘lchovlarini topishda foydalanilgan (Birlashgan Arab

Amirliklari universitetining 2024 yil 20-maydagi ma’lumotnoma, BAA). Ilmiy natijani qo‘llanishi sanoqli spin qiymatli fizik va biologik sistemalarning muvozanat holatlarini yoritish imkonini bergan;

Keli daraxtida berilgan uzluksiz spin qiymatli modellar uchun davriy Gibbs o‘lchovlarni ifodalaydigan nochiziqli operatorlarning qo‘zg‘almas nuqtalarining mavjud va yagonalik teoremalaridan FRGS21-230-0839 raqamli “Chekli o‘lchamli, ortogonallikni saqlovchi kubik stoxastik operatorlarning dinamikasi” mavzusidagi xorijiy grant loyihasida spin modellarga mos Gibbs o‘lchovlarning mavjudligi va yagonaligini tekshirishda qo‘llanilgan (Malayziya xalqaro islom universitetining 2024 yil 30-maydagi ma’lumotnoma, Malayziya). Ilmiy natijani qo‘llanishi spin sistemalarning faza almashishlarini mavjudligini tekshirish imkonini bergan;

panjarali sistemalarda ehtimollik bo‘lmagan yadrolar oilasi uchun yangi spesifikatsiya qurish usullaridan xorijiy ilmiy jurnallarda (Physical Review E, 2015, 92, 022106; Europhysics Letters, 2021, 133(2); Journal of Statistical Physics, 2014, 157(2); Chinese Journal of Physics, 2022, 77; Mathematical Physics, Analysis, and Geometry, 2016, 19(4)) statistik mexanikadagi Gauss, Potts-SOS va Vannimenus modellarining Gibbs o‘lchovlari va ularning erkin energiyalarini hisoblashda foydalanilgan. Ilmiy natijaning qo‘llanishi fizik sistemalardagi Gamiltonianga mos Gibbs o‘lchovlarini tahlil qilish imkonini bergan.

Tadqiqot natijalarining aprobasiyasi. Mazkur tadqiqot natijalari 4 ta xalqaro va 4 ta respublika ilmiy-amaliy anjumanlarida muhokamadan o‘tkazilgan.

Tadqiqot natijalarining e’lon qilinganligi. Tadqiqot mavzusi bo‘yicha jami 19 ta ish ilmiy jurnallarda chop etilgan bo‘lib, ularning barchasi O‘zbekiston Respublikasi Oliy Attestasiya komissiyasining fan doktori dissertasiyalari asosiy ilmiy natijalarini chop etish tavsiya etilgan ilmiy nashrlarda chop etilgan bo‘lib, jumladan, 16 tasi xorijiy va 3 tasi respublika jurnallarida nashr etilgan.

Dissertasiyaning tuzilishi va hajmi. Dissertatsiya kirish, to‘rt bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan tashkil topgan. Dissertatsiyaning hajmi 191 betni tashkil etgan.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismda dissertatsiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, mavzu bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, obyekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

Dissertatsiyaning birinchi bobi **“Sistemalarda $\mathbb{F}$-spesifikatsiyaning Gibbs o‘lchovlari”** deb nomlangan bo‘lib, keyingi boblar uchun asos bo‘lib xizmat qiladigan ma’lumot va natijalardan iborat. Shuningdek ayrim ma’lum natijalarni o‘lchovlar nazariyasi tilida, soddaroq usullarda ham isbotlari keltirilgan. Birinchi bobning asosiy yangi natijasi sifatida kvazi ehtimollik yadrosi bo‘lmagan yadrolar spesifikatsiyasi qurilganligini keltirish mumkin.

Bo‘sh bo‘lmagan D to‘plamda $\preceq$ qisman tartib munosabati berilgan bo‘lsin. Agar D to‘plamning ixtiyoriy bo‘sh bo‘lmagan qism to‘plami aniq yuqori chegaraga ega bo‘lsa, u holda qisman tartiblangan to‘plam *to‘la* deyiladi. Agar $\{z_n\}_{n\geq 1}$ ketma-ketlik, to‘la D to‘plamdan olingan o‘suvchi ketma-ketlik, ya’ni $z_n \preceq z_{n+1}$, $n \in \mathbb{N}$ bo‘lsa, u holda $\lim\limits_{n} z_n := \sup(\{z_n : n \geq 1\})$.

D to‘plamning qisman tartiblangan to‘la N va E qism to‘plamlari bo‘lib, $\psi : N \to E$ akslantirish bo‘lsin. Agar ψ akslantirish monoton (ya’ni, barcha $w \preceq z$ elementlar uchun $\psi(w) \preceq \psi(z)$) bo‘lib, N dan olingan barcha o‘suvchi $\{z_n\}_{n\geq 1}$ ketma-ketliklar uchun $\psi\left(\lim\limits_{n\to\infty} z_n\right) = \lim\limits_{n\to\infty} \psi(z_n)$ bo‘lsa, u holda ψ *uzluksiz* akslantirish deyiladi.

Agar ixtiyoriy $z_1, z_2 \in D$ elementlar uchun $z_1 \preceq a$ va $z_2 \preceq a$ shartlarni qanoatlantiruvchi $a \in D$ element (z_1 va z_2 ga bog‘liq ravishda) topilsa, u holda $\preceq$ qisman tartib *yo‘naltirilgan* deyiladi. Shuningdek, agar D ning sanoqli D_0 qism to‘plami mavjud bo‘lib, har bir $z \in D$ element uchun $z \preceq z_0$ shartni qanoatlantiradigan $z_0 \in D_0$ mavjud bo‘lsa, u holda $\preceq$ tartib munosabati *sanoqli tashkil qiluvchiga ega* deyiladi.

S bo‘sh bo‘lmagan to‘plam uchun $N \subset \mathcal{P}(S) := \{A \mid A \subset S\}$ shartni qanoatlantiradigan yo‘naltirilgan va sanoqli tashkil qiluvchiga ega bo‘lgan $(N, \subseteq)$ tartiblangan to‘plamni qaraymiz. Har bir $\Lambda \in N$ va bo‘sh bo‘lmagan X to‘plamga mos $\mathcal{F}^{\Lambda} \subset \mathcal{P}(X)$ sigma algebralar berilgan bo‘lsin. Barcha $\Lambda, \Delta \in N$ to‘plamlar uchun σ-algebralar oilasi $\mathcal{F}^{\Lambda\cup\Delta} = \mathcal{F}^{\Lambda} \vee \mathcal{F}^{\Delta}$ shartni qanoatlantirsin, bu yerda $\mathcal{F}^{\Lambda} \vee \mathcal{F}^{\Delta} := \mathcal{S}(\mathcal{F}^{\Lambda} \cup \mathcal{F}^{\Delta})$, ya’ni $\mathcal{F}^{\Lambda} \cup \mathcal{F}^{\Delta}$ to‘plamlar oilasini o‘z ichiga oluvchi minimal σ-algebra. Quyidagi belgilashni aniqlaylik: $\mathcal{A} := \bigcup\limits_{\in\in N} \mathcal{F}^{\Lambda}$, $\mathcal{F} = \mathcal{S}(\mathcal{A})$. U holda $(X, \mathcal{F})$ o‘lchovli fazo bo‘ladi. Endi, $\mathcal{F}$ ning qism σ-algebralaridan iborat kamayuvchi $\mathbb{F} = \{\mathcal{F}_A\}_{A\in N}$ oilani qaraymiz, bu yerda $\mathcal{F}_{\Lambda} := \mathcal{S}(\cup_{\Delta\in N, \Delta\in S\setminus\Lambda} F^{\Delta})$.

Yuqoridagi shartlarni qanoatlantiruvchi S, N, X va $\{\mathcal{F}_{\Lambda}\}_{\Lambda\in N}$ to‘plamlar to‘rtligi $\Sigma := \left(S, N, X, \{\mathcal{F}_{\Lambda}\}_{\Lambda\in N}\right)$, N da $\subset$ tartib munosabati bilan *fazoviy sistema* deyiladi.

$\mathcal{F}^{\Lambda}$ algebrada aniqlangan barcha ehtimollik o‘lchovlari oilasini $\mathrm{P}(\mathcal{F}^{\Lambda})$ bilan belgilaymiz. Barcha $\Lambda \in N$ to‘plamlar uchun $\mu_{\Lambda} \in \mathrm{P}(\mathcal{F}^{\Lambda})$ o‘lchovlarni $\{\mu_{\Lambda}\}_{\Lambda\in N}$ oilasini qaraylik. Agar ixtiyoriy $F \in \mathcal{F}^{\Lambda}$, $\Lambda \subset \Delta$ uchun $\mu_{\Lambda}(F) = \mu_{\Delta}(F)$ tenglik o‘rinli bo‘lsa, u holda $\{\mu_{\Lambda}\}_{\Lambda\in N}$ oila *muvofiqlashgan* deyiladi. Ixtiyoriy $\{\mu_{\Lambda}\}_{\Lambda\in N}$

muvofiqlashgan oila uchun, barcha $F \in \mathcal{F}^{\Lambda}, \Lambda \in N$ uchun $\mu(F) = \mu_{\Lambda}(F)$ tenglikni qanoatlantiradigan $\mu \in \mathrm{P}(\mathcal{F})$ o‘lchov topilsa, $\left\{\mathcal{F}^{\Lambda}\right\}_{\Lambda \in N}$ σ-algebralar oilasi Kolmogorov kengaytmasi xossasini qanoatlantiradi deyiladi va ushbu oilaga mos keladigan Σ fazoviy sistema ham Kolmogorov kengaytmasi xossasini qanoatlantiradi deyiladi.

X to‘plamni $\mathbb{R}_{\infty}^{+} = [0, \infty]$ ga akslantiruvchi barcha akslantirishlar oilasini $M(X)$ bilan belgilaymiz. $\mathbb{R}$ da Borel algebrasi $\mathcal{B}(\mathbb{R})$ va $(X, \mathcal{F})$ ni $\left(\mathbb{R}_{\infty}^{+}, \mathcal{B}_{\infty}^{+}\right)$ ga o‘tkazuvchi barcha o‘lchovli akslantirishlar oilasini $M(\mathcal{F})$ bilan belgilaymiz, bu yerda $\mathcal{B}_{\infty}^{+} = \mathcal{B}(\mathbb{R}) \cap \mathbb{R}_{\infty}^{+}$. Agar $M(X)$ ni standart qisman tartiblangan (ya’ni $f \preceq g$ faqat va faqat $f \leq g$ bo‘lsa) to‘plam sifatida qarasak, $M(\mathcal{F})$ fazo $M(X)$ ning to‘la qism fazosi bo‘ladi.

$(Z, \mathcal{E})$ va $(Z_1, \mathcal{E}_1)$ o‘lchovli fazolar uchun, $\kappa : Z \times \mathcal{E}_1 \to \mathbb{R}_{\infty}^{+}$ akslantirish, barcha $x \in Z$ elementlar uchun $\kappa(x, \cdot)$ o‘lchov va barcha $A \in \mathcal{E}_1$ to‘plamlar uchun $\kappa(\cdot, A)$ o‘lchovli akslantirish bo‘lsa, u holda ushbu akslantirish $(Z, \mathcal{E}) | (Z_1, \mathcal{E}_1)$*-yadro* deyiladi. Barcha $A \in \mathcal{E}_1$ to‘plamlar uchun $\Phi_{\kappa}\left(I_A\right) = \kappa(\cdot, A)$ tenglikni qanoatlantiradigan yagona chiziqli uzluksiz $\Phi_{\kappa} : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ akslantirish mavjud, bu yerda I_A xarakteristik funksiya. Demak, Φ_{κ} akslantirishni shunchaki κ ko‘rinishida yozishimiz mumkin ekan. Agar ixtiyoriy $x \in Z$ uchun $\kappa(I_{Z_1})(x) < \infty$ munosabat o‘rinli bo‘lsa, u holda $\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ yadro *chekli* deyiladi. Agar chekli yadro $\kappa(I_{Z_1}) = I_Z \cdot$tenglikni bajarsa, u holda *ehtimollik yadrosi* deyiladi. Shuningdek, $\kappa(I_{Z_1}) \in \{\mathbf{0}, I_Z\}$ bo‘lsa, u holda$\cdot\kappa$ yadro *kvazi-ehtimollik yadrosi* deyiladi, bu yerda $\mathbf{0}$ aynan nol funksiya.

Yadro $\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ va $\mu : \mathrm{M}(\mathcal{E}) \to \mathbb{R}_{\infty}^{+}$ o‘lchovning kompozitsiyasi deb ixtiyoriy $f \in \mathrm{M}(\mathcal{E}_1)$ uchun $(\mu\kappa)(f) = \mu(\kappa(f))$ tenglikni qanoatlantiruvchi $\mu\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathbb{R}_{\infty}^{+}$ akslantirishga aytiladi. Ya’ni, barcha $A \in \mathcal{E}$ to‘plamlar uchun $(\mu\kappa)(A) = \mu\left(\kappa\left(I_A\right)\right) = \mu(\kappa(\cdot, A))$ tenglikni qanoatlantiradi.

1-ta’rif. *i) Fazoviy sistema* $\left(S, N, X, \left\{\mathcal{F}_{\Lambda}\right\}_{\Lambda \in N}\right)$ *uchun,* κ_{Δ} *xos* $(Z, \mathcal{F}_{\Delta}) | (Z, \mathcal{F}_{\Delta})$ *-o‘lchovli kvazi-ehtimollik yadrosi* $\kappa_{\Delta} \in \mathrm{K}(\mathcal{F})$, $\Delta \in N$ *bo‘lsin. Ixtiyoriy* $A, B \in N$, $A \preceq B$ *to‘plamlar uchun* $\kappa_B = \kappa_B \kappa_A$ *tenglikni qanoatlantiruvchi kvazi-ehtimollik yadrolar oilasi* $\mathcal{V} = \left\{\kappa_A\right\}_{A \in N}$, $\mathbb{F}$*-**spesifikatsiya** deyiladi. Bu yerda va bundan keyin,* $\kappa_B \kappa_A$ *sifatida yadrolar kompozitsiyasini belgilangan.*

*ii) Ixtiyoriy $A \in N$ to'plam uchun $\mu = \mu\kappa_A$ tenglikni qanoatlantiruvchi $\mu \in \mathrm{P}(\mathcal{F})$ extimollik o'lchovi $\mathcal{V}$ spesifikatsiyaga mos **Gibbs o'lchovi** deyiladi. Oxirgi tenglik esa **DLR** tenglamasi deb ataladi.*

$\mathcal{V}$ spesifikatsiyaga mos Gibbs o'lchovlari to'plamini $\mathcal{G}(\mathcal{V})$ ko'rinishda belgilaymiz, ya'ni

$$\mathcal{G}(\mathcal{V}) = \left\{ \mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\kappa_A \text{ barcha } A \in N \right\}.$$

Agar fazoviy sistemadagi indekslar to'plamini sanoqli to'plam sifatida olinsa, u holda ushbu sistema *panjarali sistema* deyiladi.

Tartibi $k \geq 1$ bo'lgan *Keli daraxtini* (ya'ni, $k+1-$regulyar daraxt) $\Im^k = (V, L)$ ko'rinishda belgilaylik, V va L mos ravishda qirralar va uchlar to'plamini belgilaymiz. Odatda, modelning spin qiymatlari $\Phi \subset \mathbb{R}_+^\infty$ dan olinadi, va spin qiymatlar daraxtning uchlariga mos qo'yiladi. Ixtiyoriy $A \subset V$ to'plamda aniqlangan $\sigma_A : A \to \Phi$ akslantirish A to'plamda aniqlangan σ_A konfiguratsiya deyiladi. A da aniqlangan barcha konfiguratsiyalar oilasini $\Omega_A = \Phi^A$ ko'rinishida belgilaymiz; xususan $\Omega := \Phi^V$. Daraxni uchlari to'plamini nomerlab chiqaylik ya'ni $V = \{x_0, x_1, x_2, ...\}$.

Agar $\mathcal{I}_A$ xarakteristik funksiya bo'lsa, u holda ixtiyoriy ikkita $\sigma, \sigma' \in \Omega$ konfiguratsiyalar uchun $C(\sigma, \sigma') = \{x \in \{x_0, x_1, x_2, ...\} \mid \sigma(x) \neq \sigma'(x)\}$ to'plamni aniqlaymiz. Natijada, Ω ni quyidagi

$$\rho\left(\left\{ \sigma(x_n) \right\}_{x_n \in V}, \left\{ \sigma'(x_n) \right\}_{x_n \in V} \right) = \sum_{n \geq 0} 2^{-n} \mathcal{I}_{x_n \in C(\sigma, \sigma')} \qquad (1)$$

$\rho : \Omega \times \Omega \to \mathbb{R}^+$ metrika bo'yicha metrik fazo sifatida qarashimiz mumkin. $\mathcal{B}$ bilan Ω ning Borel qism to'plamlari σ-algebrasini belgilaylik.

Har bir $m \geq 0$ uchun $\pi_m : \Omega \to \Phi^{m+1}$ proyeksiyani quyidagi korinishda yozib olamiz $\pi_m(\sigma_0, \sigma_1, \sigma_2, ...) = (\sigma_0, \ldots, \sigma_m)$ va $\mathcal{C}_m = \pi_m^{-1}\left(\mathcal{P}\left(\Phi^{m+1}\right)\right)$, bu yerda $\sigma_i := \sigma(x_i)$ va $\mathcal{P}\left(\Phi^{m+1}\right)$ esa Φ^{m+1} ning barcha qism to'plamlari oilasi (Φ ning dekart ko'paymasi). Agar $\mathcal{C} = \bigcup_{m \geq 0} \mathcal{C}_m$ bo'lsa, u holda $\mathcal{C}$ algebra bo'ladi (the algebra of *silindr to'plamlar* algebrasi) va $\mathcal{C}$ dagi har bir element bir vaqtda ochiq ham yopiq bo'ladi. $\mathcal{S}(\mathcal{C})$ sifatida $\mathcal{C}$ ni o'z ichiga olgan minimal σ-algebrani belgilaymiz. Asosi birga teng bo'lgan silindrlarni quyidagicha yozib olaylik:

$$\sigma^{(m)}(q) = \left\{ \sigma \in \Omega : \sigma|_{\{x_m\}} = q \in \Phi \right\}.$$

$(X, \mathcal{E})$ o'lchovli fazo uchun $\mathcal{E}$ ning sanoqli $\mathcal{I}$ qism to'plami mavjud bo'lib, $\mathcal{E} = \mathcal{S}(\mathcal{I})$ tenglik o'rinli bo'lsa, u holda ushbu o'lchovli fazo sanoqli *tashkil qiluvchiga ega* deyiladi.

1-tasdiq. *Ushbu* $\mathcal{B}=\mathcal{S}(\mathcal{C})=\mathcal{S}\left(\left\{\sigma^{(m)}(q): m\geq 0,\, q\in\Phi\right\}\right)$ *tenglik o'rinli va agar* $|\Phi|<\infty$ *bo'lsa, u holda* $(\Omega,\mathcal{B})$ *sanoqli tashkil qiluvchiga ega bo'ladi.*

Natija sifatida, $\mathcal{B}=\mathcal{S}\left(\left\{\bar{\sigma}_{V_n}: n\in\mathbb{N}_0\right\}\right)$ tenglik o'rinli bo'ladi. O'lchovli fazo $(\Omega,\mathcal{F})$, indekslar to'plami V qism to'plam $\subseteq$ munosabatiga ko'ra va $\mathcal{F}$ ning kamayuvchi qism σ-algebralar oilasi $\mathbb{F}=\left\{F_\Lambda\right\}_{\Lambda\in\mathcal{N}}$, (bu yerda $F_\Lambda:=\mathcal{S}(\mathcal{C}_\Lambda)$) bo'lsin. Ravshanki, $(\mathcal{N},\subseteq)$ qisman tartiblangan to'plam yo'naltirilgan (ya'ni barcha $A_1,A_2\in\mathcal{N}$ uchun $A_1\subseteq A$, $A_2\subseteq A$ ni bajaradigan $A\in\mathcal{N}$ mavjud) va sanoqli tashkil qiluvchiga ega (ya'ni, $\mathcal{N}$ ning sanoqli qism to'plami $\{V_n\}_{n\in\mathbb{N}}$ mavjud, bunda har bir $A\in\mathcal{N}$ uchun $V_{n_0}\in\{V_n\,|\,n\in\mathbb{N}\}$, $A\subseteq V_{n_0}$ mavjud).

Ushbu $\Sigma:=\left(V,\mathcal{N},\Omega,\left\{F_\Lambda\right\}_{\Lambda\in\mathcal{N}}\right)$ to'rtlik *panjarali sistema bo'ladi,* bu yerda $(V,\mathcal{N})$, Ω va $\left\{F_\Lambda\right\}_{\Lambda\in\mathcal{N}}$ to'plamlar yuqorida aniqlangan va $\mathcal{N}$ dagi tartib munosabati $\subseteq$. Qo'shimcha sifatida shuni takidlash lozimki, (Ω,ρ) separabel va to'la metrik fazo.

1-teorema. $\mathcal{N}_1\subset\mathcal{N}$ *yo'nalishga ega va barcha* $\Lambda\in\mathcal{N}_1$ *uchun* $\mu_\Lambda\in\mathrm{P}\left(\mathcal{F}^\Lambda\right)$ *bo'lsin. Agar* $\left\{\mu_\Lambda\right\}_{\Lambda\in\mathcal{N}_1}$ *muvofiqlik shartini qanoatlantirsa, u holda barcha* $F\in F^\Lambda,\Lambda\in\mathcal{N}_1$ *uchun* $\mu(F)=\mu_\Lambda(F)$ *tenglikni qanoatlantiradigan yagona* $\mu\in\mathrm{P}(\mathcal{F})$ *extimollik o'lchovi mavjud.*

$\mathcal{B}_n$ sifatida $\Omega_n=\Omega_{V_n}$ ning barcha Borel to'plamlarining σ-halqasini belgilaylik. Shuningdek, $(\Omega_n,\mathcal{B}_n)$, $n\in\mathbb{N}$ da aniqlangan μ_n o'lchov berilgan bo'lsin.

2-teorema. *Agar* $\left\{\mu_n\right\}_{n=1}^{\infty}$ *o'lchovlar oilasidan biri* μ_{n_0} σ*-chekli bo'lsa, u holda* $\left\{\mu_n\right\}_{n=1}^{\infty}$ *oilani* $\mathcal{B}$ *da aniqlangan* σ*-additiv o'lchovgacha yagona tarzda davom etadi.*

2-ta'rif. *Barcha* $\Lambda\in\mathcal{N}$ *lar uchun* $P_\Lambda:\Omega\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,\infty\}$ *bilan* $\mathcal{F}_\Lambda$*-o'lchovli akslatirish belgilaylik, u holda* $P=\left\{P_\Lambda\right\}_{\Lambda\in\mathcal{N}}$ *oila potensial deyiladi. Shunindek, ushbu ifoda*

$$H_{\Delta,P}(\sigma)\overset{\mathrm{def}}{=}\sum_{\Delta\cap\Lambda\neq\varnothing,\Lambda\in\mathcal{N}}P_\Lambda(\sigma),\quad\forall\sigma\in\Omega.\tag{2}$$

P *potensialga mos* H *Gamiltonian deyiladi.*

$r(P)\overset{\mathrm{def}}{=}\inf\left\{R>0: P_\Lambda\equiv 0 \text{ barcha } \Lambda,\ \mathrm{diam}(\Lambda)>R\right\}$ bo'lsin. $r(P)<\infty, P$ chekli bo'lsa, u holda $H_{\Delta;P}$ to'g'ri aniqlangan bo'ladi. Agar $r(P)=\infty$ bo'lsa, u holda P ning qiymatlari cheksiz bo'ladi, Gamiltonian to'g'ri aniqlangan bo'lishi uchun, P norma bo'yicha absalyut yaqinlashuvchiligini talab qilamiz:

$$\sum_{\Lambda\in\mathcal{N},x\in\Lambda}\left\|P_\Lambda\right\|_\infty<\infty,\quad\forall x\in V,$$

(bu yerda $\| f \|_\infty \stackrel{\text{def}}{=} \sup_\omega | f(\omega) |\big)$, spinlarning ta'siri doim chegaralangan bo'lishi kerak, va $\left\| H_{\Delta;P} \right\|_\infty < \infty$.

$\mathcal{N}_1 = \{V_n : n \in \mathbb{N}\}$ bo'lsin, u holda $V_n, n \in \mathbb{N}$ da Gamiltonianni quyidagicha aniqlaymiz:

$$H_n(\sigma) = \sum_n P_{V_n}(\sigma), \quad \forall \sigma \in \Omega. \tag{3}$$

Endi spesifikatsiyani $\zeta^H = \left\{\zeta^H_{V_n}\right\}_{n\in\mathbb{N}}$ (qisqacha $\zeta^H_{V_n} := \zeta^H_n$) aniqlaymiz. Har bir $\tau_{V_n}\omega^c_{V_n}$ konfiguratsiya uchun, statistik mexanikadagi Boltzmann massasining muvozanatiga proporsional bo'lgan $\zeta^H_{V_n}(\cdot \mid \omega)$ ifodani beradi, ya'ni:

$$\zeta^P_n\left(\omega, \sigma_n\right) \stackrel{\text{def}}{=} \frac{1}{\mathbf{Z}^\omega_n} e^{-H_n\left(\sigma_n \omega_{\bar{V}_n}\right)}, \tag{4}$$

bu yerda, yozilgan ifoda $\omega_{\bar{V}_n}$ va taqsimot $\mathbf{Z}^\omega_n$ funksiyalarga bog'liq, ya'ni,

$$\mathbf{Z}^\omega_n \stackrel{\text{def}}{=} \sum_{\sigma_n \in \Omega_{V_n}} \exp\left(-H_n\left(\sigma_n \omega_{\bar{V}_n}\right)\right).$$

Ushbu $\mathbb{F}_1 := \left\{\zeta^P_n\right\}_{n\in\mathbb{N}}$ oilani aniqlab olaylik, u holda quyidagi teorema o'rinli:

3-teorema. $\mathbb{F}_1$ *spesifikatsiya bo'ladi.*

Dissertatsiyaning **"Panjarali sistemalarda gradiyent Gibbs o'lchovlari"** deb nomlangan ikkinchi bobida, sanoqli spin qiymatli HC modeli uchun barcha davriy Gibbs o'lchovlari to'liq tahlil qilingan va Bleher-G'anixo'jaev usuli yordamida kontinuum quvvatli Gibbs o'lchovlari to'plami qurilgan. Shuningdek, sanoqli spin qiymatli SOS modeli uchun 4 davriy chegaraviy qonun yordamida aniqlangan gradiyent Gibbs o'lchovlari o'rganiladi.

Uchlari $\mathbb{Z}$ butun sonlar to'plamida bo'lgan cheksiz G grafni qaraylik. Agar σ konfiguratsiya va x, y qo'shni uchlar uchun $\{\sigma(x), \sigma(y)\}$ juftlik G grafning qirrasi bo'lsa, u holda konfiguratsiya G-joiz konfiguratsiya deyiladi. Barcha joiz konfiguratsiyalar oilasi Ω^G ko'rinishda belgilanadi.

$\lambda : G \mapsto \mathbb{R}_+$ chegaralangan funksiya berilgan bo'lsin va λ_i lar λ funksiyaning $i \in \mathbb{Z}$ uchdagi "faolligi" deyiladi.

Berilgan G graf va λ ga mos HC-modeli quyidagi aniqlanadi:

$$H^\lambda_G(\sigma) = \begin{cases} J \sum_{x\in V} \ln \lambda_{\sigma(x)}, & \text{agar } \sigma \in \Omega^G, \\ +\infty, & \text{agar } \sigma \notin \Omega^G, \end{cases} \tag{5}$$

bu yerda $J \in \mathbb{R}$.

3-ta'rif *1) Ushbu* $l = \left\{l_{xy}\right\}_{\langle x,y\rangle \in L}$, $l_{xy} = \left\{l_{xy}(i) : i \in \mathbb{Z}\right\} \in (0,\infty)^{\mathbb{Z}}$ *vektorlar oilasi ixtiyoriy* $\langle x,y\rangle \in L$ *qirra va barcha* $i \in \mathbb{Z}$ *uchun*

$$l_{xy}(i) = c_{xy}\lambda_i \prod_{z \in \partial x \setminus \{y\}} \sum_{j \in \mathbb{Z}} a_{ij} l_{zx}(j) \tag{6}$$

tenglikni qanoatlantiradigan shunday $c_{xy} > 0$ *mavjud bo'lsa, u holda (5) model uchun chegaraviy qonun deyiladi.*

2) Agar l *chegaraviy qonun va barcha* $x \in V$ *uchlar uchun*

$$\sum_{i \in \mathbb{Z}} \left(\lambda_i \prod_{z \in \partial x} \sum_{j \in \mathbb{Z}} a_{ij}\, l_{zx}(j) \right) < \infty$$

munosabat o'rinli bo'lsa, u holda ushbu chegaraviy qonun normallashgan deyiladi.

Berilgan $\omega = \{\omega_x \mid x \in V\}$ konfiguratsiya uchun Q_b simmetrik o'tish matritsasi

$$Q_b(\omega_b) := \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y},$$

ko'rinishda bo'ladi, bu yerda $\omega_b = \{\omega_x, \omega_y\}$.

Chekli qism to'plam $\Lambda \subset V$ uchun (Markov) Gibbs spesifikatsiyasi

$$\gamma_\Lambda^\lambda(\omega : \omega|_\Lambda = \sigma_\Lambda) = (Z_\Lambda^\lambda)(\omega)^{-1} \prod_{\langle x,y\rangle, \{x,y\} \cap \Lambda \neq \varnothing} \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y},$$

ko'rinishda bo'ladi, bu yerda $Z_\Lambda^\lambda(\omega)$ taqsimot funksiya.

Berilgan $(Q_b)_{b \in L}$ o'tish matritsasiga mos keluvchi Gibbs spesifikatsiyasi uchun quyidagi mulohazalar o'rinli:

1. Har qanday normallashgan $(l_{xy})_{x,y}$ chegaraviy qonun uchun, $(Q_b)_{b \in L}$ operator ushbu

$$\mu(\sigma_{\Lambda \cup \partial\Lambda} = \omega_{\Lambda \cup \partial\Lambda}) = (Z_\Lambda^\lambda)^{-1} \prod_{y \in \partial\Lambda} l_{yy_\Lambda}(\omega_y) \prod_{b \cap \Lambda \neq \varnothing} Q_b(\omega_b) =$$

$$= (Z_\Lambda^\lambda)^{-1} \prod_{\langle x,y\rangle, \{x,y\} \cap \Lambda \neq \varnothing} \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y}, \tag{7}$$

tenglikni qanoatlantiradigan yagona μ Gibbs o'lchovini aniqlaydi, bu yerda $y \in \partial\Lambda$ va y_Λ esa Λ dagi y ning qo'shnisi.

2. Aksincha, har qanday μ Gibbs o'lchoviga (7) ko'rinishda ifodalanadigan normallashgan chegaraviy qonun (musbat o'zgarmas songa faktorlashga nisbatan yagona) mos keladi.

G^* sifatida quyidagicha grafni aniqlaymiz $a_{i0}=1$ barcha $i\in\mathbb{Z}$ va $a_{im}=0$ barcha $i,m\in\mathbb{Z}_0$. Berilgan $l_{xy}(i)$ chegaraviy qonun uchun, $z_{i,x}=l_{xy}(i)$ belgilashni kiritamiz, bu yerda x element y ning keyingi avlodi, barcha keyingi avlodlar to‘plami $S(y)$, Ushbu graf uchun (6) quyidagicha yozib olamiz:

$$z_{i,x}=\lambda_i\prod_{y\in S(x)}\frac{1}{1+\sum\limits_{j\in\mathbb{Z}_0} z_{j,y}},\quad i\in\mathbb{Z}_0. \tag{8}$$

Agar

$$\mathcal{U}^+:=\left\{z_x=\left(\ldots z_{-2,x},z_{-1,x},z_{0,x},z_{1,x},z_{2,x},\ldots\right)\mid z_{i,x}\in(0,+\infty),\ z_{0,x}=1,\ \forall x\in V\right\},$$

$$\mathcal{T}^+:=\left\{z_x\in\mathcal{U}^+\mid\sum_{i\in\mathbb{Z}} z_{i,x}<\infty\right\}.$$

belgilashlarni kiritsak, u holda (8) ni quyidagicha ko‘rinishda yozib olish mumkin

$$z_x=\lambda\prod_{y\in S(x)}\frac{1}{1+\|z_y\|},\quad z_x\in\mathcal{U}^+,\ \forall x\in V, \tag{9}$$

bu yerda $\lambda=(\ldots,\lambda_{-2},\lambda_{-1},1,\lambda_1,\lambda_2,\ldots)\in\mathcal{U}^+$ va $\|z_y\|:=\sum_{i\in\mathbb{Z}}\|z_{i,y}\|$.

Qisqaroq belgilash uchun (9) ni yangi $W:\mathcal{T}^+\to\mathcal{T}^+$ operatorning qo‘zg‘almas nuqtasi ko‘rinishda yozish mumkin, ya’ni $z_x=Wz_x$.

4-teorema. Agar $k\geq 2$ *va* $\lambda\in\mathcal{T}^+$ *bo‘lsa, u holda quyidagilar o‘rinli*

1. Agar $\|\lambda\|\leq\Lambda_{cr}$ *bo‘lsa, u holda* $\lim\limits_{n\to\infty}W^{(n)}(z_0)=\xi\lambda$, $z_0\in\mathcal{T}^+$.

2. Agar $\|\lambda\|>\Lambda_{cr}$ *va* $z_0=\alpha_0\lambda$, $\alpha_0\in(0,\xi)$ *bo‘lsa, u holda*

$$\lim_{n\to\infty}W^{(2n)}(z_0)=\alpha^*\lambda,\ \lim_{n\to\infty}W^{(2n-1)}(z_0)=\beta^*\lambda.$$

3. Agar $\|\lambda\|>\Lambda_{cr}$ *va* $z_0=\alpha_0\lambda$, $\alpha_0\in(\xi,1)$ *bo‘lsa, u holda*

$$\lim_{n\to\infty}W^{(2n)}(z_0)=\beta^*\lambda,\ \lim_{n\to\infty}W^{(2n-1)}(z_0)=\alpha^*\lambda.$$

4. Agar $\alpha_0=\xi$ *bo‘lsa, u holda* $\lim\limits_{n\to\infty}W^{(n)}(z_0)=\xi\lambda$.

Agar $\|\lambda\|>\Lambda_{cr}\approx 2.7$ bo‘lsa, u holda (8) ikkita davriy $(\alpha^*\lambda,\beta^*\lambda)$ yechimga ega. Agar $u=(\ldots,u_{-2},u_{-1},0,u_1,u_2,\ldots),v=(\ldots,v_{-2},v_{-1},0,v_1,v_2,\ldots)\in\mathcal{T}^+$ bo‘lsa, u holda $\mathcal{T}^+$ to‘plamda qisman tartib munosabatini quyidagicha aniqlaymiz: barcha $i\in\mathbb{Z}$ uchun $u_i\leq v_i$ tengsizlik o‘rinli bo‘lsa, u holda $u\preceq v$.

2-tasdiq. *Agar* z_x *(8) tenglamaning yechimi bo'lsa, u holda barcha* $x \in V$ *uchun* $\alpha^* \lambda \preceq z_x \preceq \beta^* \lambda$ *munosabat o'rinli.*

Yarim Keli daraxti $\Gamma_0^k = (V_0, L_0)$ ni qaraylik va x^0 sifatida daraxtning ildizini belgilaylik. Har bir $x \in V_0$ uchun x^0 va x uchlarni tutashtiruvchi yagona yo'lni π_x ko'rinishda belgilaymiz. Shuningdek, agar qaralayotgan yo'l cheksiz bo'lsa, u holda $\pi = \left\{ x^0 = x_0 < x_1 < \ldots \right\}$ ko'rinishda yozamiz.

Agar $z^\pi = \left\{ z_x^\pi, x \in V^0 \right\}$, (8) ning π yo'lga mos yechimi bo'lsa, u holda z^π ni quyidagicha aniqlaymiz

$$z_x^\pi = \begin{cases} \alpha^* \lambda, \; agar \; x \prec x_{2n}, \; x \in W_{2n}, \\ \beta^* \lambda, \; agar \; x \prec x_{2n+1}, \; x \in W_{2n+1}, \\ \beta^* \lambda, \; agar \; x_{2n} \prec x, \; x \in W_{2n}, \\ \alpha^* \lambda, \; agar \; x_{2n+1} \prec x, \; x \in W_{2n+1}. \end{cases} \tag{10}$$

5-teorema. *Agar* $\Lambda_{cr} < \| \lambda \| < \dfrac{1}{\beta^* - \alpha^*}, \; \lambda \in \mathcal{T}^+$ *tengsizlik o'rinli bo'lsa, u holda ixtiyoriy* π *yo'l uchun, (9) va (10) ni qanoatlantiradigan yagona* $z^\pi = \left\{ z_x^\pi, x \in V^0 \right\}$ *yechim mavjud. Shuningdek, turli yo'llarga mos keluvchi (9) tenglamaning yechimlariga mos Gibbs o'lchovlari ham turli bo'ladi.*

Endi SOS modeli uchun gradiyent Gibbs o'lchovlariga oid asosiy natijani qisqa ko'rinishda keltirib o'tamiz.

SOS modelining Gamiltoniani quyidagi ko'rinishda bo'ladi

$$H(\omega) = -J \sum_{\langle x, y \rangle \in L} \left| \omega_x - \omega_y \right|, \quad \omega \in \Omega, \tag{11}$$

bu yerda $J \in \mathbb{R}_+$ o'zgarmas son.

HC modeli singari SOS modeli uchun chegaraviy qonun tenglamasi (tranlatsion-invariant yechim uchun, ya'ni barcha $b \in L$ uchun $l_b \equiv l$)

$$z_i = \left(\frac{\theta^{|i|} + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} z_j}{1 + \sum_{j \in \mathbb{Z}_0} \theta^{|j|} z_j} \right)^k, \quad i \in \mathbb{Z}_0$$

ko'rinishda bo'ladi, bu yerda $\theta = \exp\{-\beta H\}$, $\beta = \dfrac{1}{T}$ teskari temperatura va $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$.

6-teorema. *Tartibi 3 ga teng bo‘lgan Keli daraxtida aniqlangan SOS modeli uchun* $\tau_{cr}^{(1)} \approx 3.13039, \tau_{cr}^{(2)} \approx 4.01009$ $(\tau = \theta^{-1} + \theta)$ *qiymatlarga mos ravishda quyidagi munosabatlar o‘rinli:*

(1) Agar $\tau \le \tau_{cr}^{(1)}$ *bo‘lsa, u holda berilgan chegaraviy qonunga mos keluvchi aynan bitta GGO‘ (gradiyent Gibbs o‘lchovi) mavjud..*

(2) Agar $\tau \in \left(\tau_{cr}^{(1)}, 4\right]$ *bo‘lsa, u holda aynan ikkita shunday GGO‘lari mavjud.*

(3) Agar $\tau \in \left(4, \tau_{cr}^{(2)}\right] \cup \{3\sqrt{2}\}$ *bo‘lsa, u holda ko‘pi bilan uchta shunday GGO‘lari mavjud.*

(4) Agar $\tau \in \left(\tau_{cr}^{(2)}, +\infty\right) \setminus \{3\sqrt{2}\}$ *bo‘lsa, u holda ko‘pi bilan to‘rtta shunday GGO‘lari mavjud.*

Dissertatsiyaning **“Keli daraxtlarida uzluksiz spin modellarning Gibbs o‘lchovlari”** deb nomlanuvchi uchinchi bobida, uzluksiz spin modellarning limit Gibbs o‘lchovlari va nochiziqli integral operatorlarning qo‘zg‘almas nuqtalari o‘rtasida bog‘lanishni aniqlash va ushbu modellar uchun Gibbs o‘lchovlarining mavjudligi, yagonaligi ifodalovchi shartlar topilgan. Bundan tashqari, kamida translyatsion-invariant Gibbs o‘lchovlariga ega uzluksiz modellar qurilgan.

Keli daraxtida spin qiymatlari [0,1] bo‘lgan $A \subset V$ to‘plamda aniqlangan barcha konfiguratsiyalar oilasini $\Omega_A = [0,1]^A$ bilan belgilaylik va $\Omega := [0,1]^V$. Quyidagi ko‘rinishda aniqlangan Gamiltonianni qaraylik:

$$H(\sigma) = -J \sum_{\langle x,y \rangle \in L} \xi_{\sigma(x),\sigma(y)}, \tag{12}$$

bu yerda $J \in R \setminus \{0\}$ va $\xi : (u,v) \in [0,1]^2 \to \xi_{u,v} \in R$ chegaralangan, Lebeg ma’nosida o‘lchovli funksiya. Odatda, $\langle x, y \rangle$ bilan eng yaqin qo‘shni uchlarni belgilaymiz, ya’ni $d(x,y) = 1$.

Ildizdan farqli $x \in V \setminus \{x^0\}$ uchlar uchun $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ funksiyani aniqlaylik. Ixtiyoriy n natural son uchun, Ω_{V_n} da aniqlangan $\mu^{(n)}$ extimollik o‘lchovini quyidagicha kiritamiz:

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right), \tag{13}$$

bu yerda $\sigma_n : x \in V_n \mapsto \sigma(x)$ va Z_n esa taqsimot funksiyasi. $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, extimollik o‘lchovlari oilasi muvofiqlashgan bo‘lishi uchun ixtiyoriy $x \in V \setminus \{x^0\}$ uchun

$$f(t,x)=\prod_{y\in S(x)}\frac{\int_0^1 \exp(J\beta\xi_{t,u})f(u,y)du}{\int_0^1 \exp(J\beta\xi_{0,u})f(u,y)du} \tag{14}$$

tenglik o‘rinli bo‘lishi zarur va yetarli. Bu yerda, $f(t,x)=\exp(h_{t,x}-h_{0,x}),\, t\in[0,1]$ va $du=\lambda(du)$-Lebeg o‘lchovi.

Agar A operatorni

$$Av(x)=\int_{[a,b]} K(x,y)v(y)dy,$$

ko‘rinishda aniqlaylik va f funksiyaga mos N_f Nemistki operatorini

$$\tilde{N}_f u(x)=f(u(x))$$

ko‘rinishda yozib olamiz. Dastlab, $A\tilde{N}_f$ operatorning musbat qo‘zg‘almas nuqtalarini o‘rganamiz, ya’ni

$$\int_a^b K(x,y)f(u(y))dy=u(x),$$

bu yerda, $K:[a,b]^2\to(0,+\infty)$ va $f:[0,+\infty)\to[0,+\infty)$ uzluksiz akslantirishlar, hamda f nochiziqli.

$\mathcal{C}_1$***-shart***: Barcha $x,y\in[a,b]$ elementlar uchun $K(x,y)\le\Re(y)$ tengsizlikni va $x\in[\alpha,\beta]$, $y\in[a,b]$ elementlar uchun $c\cdot\Re(y)\le K(x,y)$ tengsizlikni qanoatlantiradigan $\Re:[a,b]\to[0,+\infty)$ uzluksiz funksiya va $c\in(0,1)$, $\alpha,\beta\in[a,b]\,(\alpha<\beta)$ o‘zgarmas sonlar mavjud bo‘lsin.

Quyidagi belgilashlarni kiritib olaylik

$$C^+[a,b]=\{\omega\in C[a,b]:\omega(x)\ge 0,\text{ barcha } x\in[a,b]\},$$

$$\hat{\mathcal{K}}:=\left\{\omega\in C^+[a,b]:\min_{x\in[\alpha,\beta]}\omega(x)\ge c\,\|\omega\|\right\},$$

Bu yerd α,β va c o‘zgarmaslar $\mathcal{C}_1$-shartni qanoatlantiradigan sonlar.

$\mathcal{C}_2$***-shart***: Ixtiyoriy $\varepsilon>0$ musbat son uchun $\inf\left\{f(\delta t):t\in[1,c^{-1}]\right\}>\varepsilon\cdot\delta$ tengsizlikni qanoatlantiradigan $\delta(\varepsilon)>0$ son mavjud bo‘lsin.

3-tasdiq. *$K(x,y)$ va f funksiyalar uchun $\mathcal{C}_1$ va $\mathcal{C}_2$ shartlar o‘rinli bo‘lsin hamda:*

1. There exists $\tau>0$ such that $f(\cdot)$ is non-decreasing on $[0,\tau]$.

2. Ushbu tengsizlik o‘rinli bo‘lsin:

$$\sup_{s\in(0,\tau)}\frac{(1-c)s}{f(s)\cdot\|\int_a^b K(x,y)dy\|}>1.$$

U holda $A\tilde{N}_f$ *operator* $\hat{\mathcal{K}}$ *da kamida bitta musbat qo'zg'almas nuqtaga ega.*

7-teorema. *Faraz qilaylik 3-tasdiqning shartlariga qo'shimcha ravishda quyidagi shartlar ham o'rinli bo'lsin:*

(i) f *funksiya ixtiyoriy* $\nu \in [1,+\infty)$ *va* $\omega_i(x) \in C^+[a,b], i=\overline{1,2}$ *uchun ushbu munosabatni qanoatlantirsin:*

$$\omega_1(x) \geq \nu\omega_2(x) \Rightarrow f(\omega_1(x)) \geq \nu f(\omega_2(x)).$$

(ii) f *funksiya uchun ushbu shart ham o'rinli bo'lsin:*

$$\gamma_1 x^{\sigma_1} \leq f(x) \leq \gamma_2 x^{\sigma_2}, \ \gamma_i > 0, \sigma_i > 1, i=\overline{1,2}$$

va $f(x) - \gamma_1 x^{\sigma_1}$, $\gamma_2 x^{\sigma_2} - f(x)$ *kamaymaydigan monoton funksiyalar.*

(iii) $A\tilde{N}_f$ *ning yadrosining eng katta (*M *) va eng kichik (m) qiymatlari uchun ushbu tengsizlik o'rinli bo'lsin:*

$$\Im_2(M,m) - \Im_1(m,M) < \frac{1}{b-a},$$

bu yerda

$$\Im_i(x,y) = x\sigma_i\gamma_i\left(\frac{x}{y}\cdot(m\gamma_{3-i}(b-a))^{\frac{1}{1-\sigma_{3-i}}}\right)^{\sigma_i-1}.$$

U holda $A\tilde{N}_f$, $f \in C_+^{'}[a,b]$ *operator* $\hat{\mathcal{K}}$ *da aynan bitta musbat qo'zg'almas nuqtaga ega.*

1-izoh. *Kamida ikkita musbat qo'zg'almas nuqtalarga ega bo'lgan* $A\tilde{N}_f$ *operatorning yadrosiga ko'plab misollar qurilgan.*

Ma'lumki, (3)-tenglamaning translyatsion-invariant yechimlari topish masalasi ushbu

$$(H_k f)(t) := \int_0^1 K(t,u) f^k(u)du = f(t).$$

tenglamaning musbat yechimlarini topish masalasiga ekvivalent, bu yerda $K(t,u) = \exp\{-J\beta\xi_{tu}\}$. O'z navbatida, H_k operator $A\tilde{N}_f$ operatorning hususiy ko'rinishi va ushbu operator uchun 7-teoremani quyidagicha yozish mumkin:

8-teorema. *Agar* $\tau_k = \sqrt[k]{\frac{1+\sqrt{k^2+1}}{k}}$, $k \geq 2$ *va* H_k *ning yadrosi*

$$M \leq m\tau_k \tag{15}$$

tengsizlikni qanoatlantirsa, u holda H_k *operator yagona musbat qo'zg'almas nuqtaga ega.*

Gibbs o‘lchovlar nazariyasi tilida, yozadigan bo‘lsak quyidagi teoremani olamiz:

9-teorema. *$k \geq 2$ bo‘lsin. Agar (12) Gamiltonianning $K(t,u)=\exp\{-J\beta\xi_{tu}\}$ funksiyasi (15) shartni qanoatlantirsa, u holda k-tartibli Keli daraxtida ushbu Gamiltonianning yagona translyatsion-invariant Gibbs o‘lchovi mavjud.*

Dissertatsiyaning **“Izing modelining erkin energiyasi va invariantlik xossasi”** deb nomlanuvchi to‘rtinchi bobida, Keli daraxtidagi Ising modeli uchun translyatsion-invariant va davriy chegaraviy shartlar uchun erkin energiyalar hisoblangan. Shuningdek, Keli daraxti gruppaviy tasvirining qism gruppalari uchun o‘zgarmaslik xossasini ma’lum shartlar asosida isbotlandi va ushbu xossani Ising modeliga qo‘llash orqali kuchsiz davriy Gibbs o‘lchovlar sinfi topildi.

Ikkinchi tartibli $k+1$ ta siklik gruppalarning erkin ko‘paytmasini G_k bilan belgilaylik va $a_1, a_2, ..., a_{k+1}$ elementlar siklik gruppani yasovchilari bo‘lsin. Ma’lumki, k-tartibli Keli daraxtining uchlari to‘plami V hamda G_k gruppaning elementlari o‘rtasida o‘zaro bir qiymatlik moslik mavjud.

$N_k=\{1,2,...,k+1\}$ va $A_0 \subset N_k$, $0 \leq |A_0| \leq k-2$ bo‘lsin. Shuningdek, $\{A_1, A_2, A_3\}$ to‘plamlar oilasi $N_k \setminus A_0$ to‘plamning bo‘linishlari va m_j sonlar esa mos ravishda A_j, $j \in \{1,2,3\}$ to‘plamlardagi eng kichik elementlar bo‘lsin. U holda $u_{A_1A_2A_3}: G_k \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ gomomorfizmni quyidagi ko‘rinishda aniqlaymiz

$$u_{A_1A_2A_3}(x)=\begin{cases} e, \text{ agar } x=a_i, i \in N_k \setminus (A_1 \cup A_2 \cup A_3) \\ a_{m_j}, \text{ agar } x=a_i, i \in A_j, j=1,2,3 \end{cases},$$

bu yerda e gruppaning birlik elementi.

Har bir $i \in \{1,2,3,...,7,8\}$ soni uchun, $\gamma_i : \langle a_{m_1}, a_{m_2}, a_{m_3}\rangle \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ akslantirishlarni quyidagicha aniqlab olamiz:

$$\gamma_1(x)=\begin{cases} e, \text{ agar } x=e, \\ a_{m_1}, \text{ agar } x \in \{a_{m_3}a_{m_1}, a_{m_2}a_{m_3}\}, \\ a_{m_2}, \text{ agar } x \in \{a_{m_1}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \text{ agar } x \in \{a_{m_1}a_{m_2}, a_{m_2}a_{m_1}\}, \\ \gamma_1(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases} \quad \gamma_2(x)=\begin{cases} e, \text{ agar } x=e, \\ a_{m_1}, \text{ agar } x \in \{a_{m_3}a_{m_2}, a_{m_2}a_{m_1}\}, \\ a_{m_2}, \text{ agar } x \in \{a_{m_1}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \text{ agar } x \in \{a_{m_1}a_{m_2}, a_{m_2}a_{m_3}\}, \\ \gamma_2(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases}$$

$$\gamma_3(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_3}a_{m_1}, a_{m_2}a_{m_1}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_2}a_{m_3}\}, \\ \gamma_3(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases} \quad \gamma_4(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_1}\}, \\ \gamma_4(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases}$$

$$\gamma_5(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_2}a_{m_1}, a_{m_3}a_{m_1}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_3}\}, \\ \gamma_5(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases} \quad \gamma_6(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_1}\}, \\ \gamma_6(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases}$$

$$\gamma_7(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_2}a_{m_1}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_3}\}, \\ \gamma_7(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases} \quad \gamma_8(x)=\begin{cases} e, \text{ agar } \ x=e, \\ a_{m_1}, \text{ agar } x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \text{ agar } x\in\{a_{m_1}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \text{ agar } x\in\{a_{m_1}a_{m_2}, a_{m_2}a_{m_1}\}, \\ \gamma_8(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), l(x)>2, \end{cases}$$

bu yerda $l(x)$ sifatida x so‘zning uzunligi belgilangan hamda barcha $s\in\{1,2,...,n\}$ uchun $i_s\in\{m_1,m_2,m_3\}$.

Quyidagi belgilashni kiritib olaylik:

$$\Im^j_{A_1A_2A_3}(G_k)=\{x\in G_k \mid \gamma_j(u_{A_1A_2A_3}(x))=e\}, \ \ j=\overline{1,8}.$$

10-teorema. *i) Agar* $A_0\subset N_k, 0\le |A_0|\le k-2$ *va* $\{A_1,A_2,A_3\}$ *oila* $N_k\setminus A_0$ *to‘plamning bo‘linishlari bo‘lsin. U holda barcha* $j=\overline{1,8}$ *uchun* $\Im^j_{A_1A_2A_3}(G_k)$ *berilgan* G_k *gruppaning qism gruppasi bo‘ladi.*

ii) G_k *gruppaning indeksi 4 ga teng bo‘lgan ixtiyoriy* K *qism gruppasi qaraylik. U holda,* $K=\Im^j_{A_1A_2A_3}(G_k)$ *tenglikni qanoatlantiradigan* $N_k\setminus A_0$ *ning* A_1,A_2,A_3 *bo‘linishlari va* $j\in\{1,2,...,8\}$ *soni topiladi va aksincha.*

Endi, qo‘shni ta’sirli Izing modelini aniqlaylik

$$H(\sigma)=-J\sum_{\langle x,y\rangle\subset V}\sigma(x)\sigma(y),$$

bu yerda $\sigma(x)$ ning spin qiymatlari ± 1, va J haqiqiy parametr.

Teskari $\beta = 1/T$ temperaturada berilgan konfiguratsiyalar orqali Gibbs (chekli o'lchamli) taqsimotlarini quyidagicha aniqlab olamiz:

$$\mu_n(\sigma_n) = Z_n^{-1}(h)\exp\{\beta J \sum_{\langle x,y\rangle\subset V_n} \sigma(x)\sigma(y) + \sum_{x\in W_n} h_x\sigma(x)\},$$

$Z_n(h)$ taqsimot funksiya va $h := \{h_x \in \mathbb{R} \,|\, x \in V\}$.

Ravshanki, ushbu Gibbs taqsimotlari oilasi muvofiqlik shartini qanoatlantirishi uchun barcha $x \in V$ uchun

$$h_x = \sum_{y\in S(x)} f_\theta(h_y), \tag{16}$$

tenglik o'rinli bo'lishi zarur va yetarli, bu yerda $f_\theta(h) = \operatorname{arctanh}(\theta \tanh h)$, $\theta = \tanh(J)$.

Endi, yarim daraxtda (16) funksional tenglamaning yangi yechimlari oilasini aniqlaymiz. Ya'ni, q va r nomanfiy butun sonlar uchun $1 \le q \le k-1$, $0 \le r \le k$ tengsizliklar o'rinli bo'lsin va r bilan k ning juft toqligi bir hil bo'lsin.

Qiymatlari $0, \pm h_1, \pm h_2$ dan olingan $h = \{h_x, x \in V\}$ chegaraviy shartni quyidagi qadamlar orqali qurib olamiz:

• agar x uchda $h_x = 0$ bo'lsa, u holda keyingi avlod uchlaridagi qiymatlarini quyidagicha olamiz

$$\begin{cases} 0 & S(x) \text{ ning } r \text{ ta uchida} \\ h_1 & \text{qolgan uchlarining yarmida} \\ -h_1 & \text{qolgan uchlarda;} \end{cases}$$

• agar x uchda $h_x = h_1$ (mos ravishda $-h_1$) bo'lsa, u holda h ning $S(x)$ dagi q uchlaridagi qiymatlarini h_2 (mos ravishda $-h_2$) va qolgan uchlardagi qiymatlarini 0 qilib olamiz;

• agar x uchda $h_x = h_2$ (mos ravishda $-h_2$) bo'lsa, u holda h ning $S(x)$ dagi barcha uchlaridagi qiymatlarini h_1 (mos ravishda $-h_1$) qilib olamiz.

Agar chegaraviy qonunni yuqorida ko'rinishda aniqlasak, u holda chegaraviy qonun muvofiqlik shartini qanoatlantirishi uchun h_1 va h_2 qiymatlar

$$\begin{cases} h_1 = qf_\theta(h_2) \\ h_2 = kf_\theta(h_1) \end{cases} \tag{17}$$

tenglikni bajarishi zarur va yetarli ekanligini tekshirish qiyin emas, bu yerda $1 \le q \le k-1$. $\theta_c = 1/\sqrt{qk}$ bo'lsin.

11-teorema. *Agar* $-\theta_c \le \theta \le \theta_c$ *bo'lsa, u holda (17) tenglamalar sistemasi yagona* $(0,0)$ *yechimga ega,* $|\theta| > \theta_c$ *bo'lganda esa uchta* $(0,0)$, (h_1^*, h_2^*), *va* $(-h_1^*, -h_2^*)$ *(*$h_1^*, h_2^* > 0$*), turli yechimlarga ega.*

Muvofiqlik shartini qanoatlantiruvchi chegaraviy qonuning (ch.q.) erkin energiyasi mavjud bo‘lsa, u holda erkin energiya quyidagi limit ko‘rinishida aniqlanadi:

$$\lim_{n\to\infty}\frac{1}{\beta|V_n|}\ln Z_n(h)=F(h), \qquad .$$

bu yerda $|\cdot|$ belgilash to‘plam quvvatini bildiradi.

4-tasdiq. *Agar* $r\neq 0$ *bo‘lsa, u holda (17) chegaraviy qonunning erkin energiyasi* F_{ALT}

$$\frac{k(k-q)}{(2k^2-rk-rq)}\left(a(0)+\frac{k-r}{k-q}a(h_1)+\frac{q(k-r)}{k(k-q)}a(h_2)\right),$$

ga teng, bu yerda

$$a(t)=-\frac{1}{2\beta}\ln[4\cosh(t+J)\cosh(t-J)].$$

• *Agar daraxt ildizidagi qiymatlari* $0,\ \pm h_2$ *va* $r=0$ *bo‘lsa, u holda erkin energiya quyidagi limit nuqtalariga ega*

$$\begin{cases}\dfrac{k-q}{k+1}a(0)+\dfrac{1}{k+1}a(h_1)+\dfrac{q}{k+1}a(h_2), & \text{agar} \quad n=2m,\ m\to\infty\\[2mm] \dfrac{k-q}{k(k+1)}a(0)+\dfrac{k}{k+1}a(h_1)+\dfrac{q}{k(k+1)}a(h_2), & \text{agar} \quad n=2m+1\end{cases}$$

• *Agar daraxt ildizidagi qiymatlari* $\pm h_1$ *va* $r=0$ *bo‘lsa, u holda*

$$\begin{cases}\dfrac{k-q}{k(k+1)}a(0)+\dfrac{k}{k+1}a(h_1)+\dfrac{q}{k(k+1)}a(h_2), & \text{agar} \quad n=2m,\\[2mm] \dfrac{k-q}{k+1}a(0)+\dfrac{1}{k+1}a(h_1)+\dfrac{q}{k+1}a(h_2), & \text{agar} \quad n=2m+1\end{cases}.$$

XULOSA

Dissertatsiya panjarali sistemalarda berilgan spesifikatsiyalar uchun limit va gradiyent Gibbs oʻlchovlari toʻplamini tavsiflashga, bunday oʻlchovlar toʻplamining strukturasini tadqiq etishga va statistik mexanikadagi ayrim modellar uchun tadbiq qilishga bagʻishlangan.

Muxtasar qilib aytganda, ilmiy ishning asosiy xulosalari quyidagilardan iborat:

1. Panjarali sistemalarda ehtimollik boʻlmagan yadrolar oilasi uchun spesifikatsiya qurish.

2. Panjarali sistemalarda spin qiymatlari sanoqli HC va SOS modellari uchun limit va gradiyent Gibbs oʻlchovlari tadqiq etilgan. HC modeli uchun kontinuum quvvatli Gibbs oʻlchovlari qurildi hamda SOS modeli uchun 4-davriy chegaraviy qonunga mos keluvchi gradiyent Gibbs oʻlchovlari toʻplamining tasviri berildi.

3. Izing, Potts, SOS va HC kabi bir qancha klassik modellarning umumlashmasi boʻlgan sanoqsiz spin qiymatli Gamiltonian qaraladi va bu Gamiltonian uchun Gammershteyn tipidagi nochiziqli integral operatorning musbat qoʻzgʻalmas nuqtalarini tadqiq etish orqali quyidagi natijalar olinadi: Gibbs oʻlchovlari toʻplami boʻsh emasligi, Gibbs oʻlchovining yagona boʻlishligi uchun yetarlilik sharti, yetarlilik shartini qanoatlantirmaydigan hollarda fazaviy oʻtishlarning mavjudligi kabi natijalar olindi.

4. Keli daraxti gruppaviy tasvirining normal boʻlmagan qism gruppalari orqali davriy va kuchsiz davriy Gibbs oʻlchovlarini sinfi kengaytirildi. Shuningdek, Izing modeli uchun yangi kuchsiz davriy Gibbs oʻlchovlari topildi.

5. Keli daraxtida Izing modeli uchun erkin energiyasi mavjud boʻlmaydigan Gibbs oʻlchovlari oilasi qurildi va unga mos fazaviy oʻtishlarning sodir boʻlishi tekshirildi.

SCIENTIFIC COUNCIL AWARDING SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 INSTITUTE OF MATHEMATICS NAMED AFTER V.I.ROMANOVSKY

INSTITUTE OF MATHEMATICS

HAYDAROV FARHOD HALIMJONOVICH

LIMITING AND GRADIENT GIBBS MEASURES ON LATTICE SYSTEMS

01.01.01-Mathematical analysis

ABSTRACT OF DISSERTATION OF THE DOCTOR OF SCIENCE (DSc) ON PHYSICAL AND MATHEMATICAL SCIENCES

TASHKENT-2024

The theme of dissertation of doctor of philosophy (PhD) on physical and mathematical sciences was registered at the Supreme Attestation Commission at the of Ministers of Higher education, Science and Innovations of the Republic of Uzbekistan under number B2024.1.DSc/FM254

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INTRODUCTION (abstract of DSc thesis)

Actuality and demand of the theme of the dissertation. In the world of scientific and applied research, there is a significant focus on Gibbs measures for modeling systems in statistical physics and mechanics. Gibbs measure on lattice systems lies in its foundational importance in statistical mechanics and its wide-ranging applications in various scientific disciplines. Gibbs measures provide a probabilistic framework for understanding the equilibrium properties of systems with many interacting particles. By characterizing the distribution of states in a system at thermal equilibrium, Gibbs measures enable researchers to analyze phase transitions, calculate thermodynamic properties, and predict the behavior of diverse systems.

Nowadays in the world, the study of Gibbs measures has expanded beyond its traditional applications in statistical mechanics to encompass a broader range of disciplines, including mathematics, biology, computer science, and machine learning. Mathematicians are currently focusing on the study of phase transitions in spin systems. Phase transitions occur when a system undergoes a sudden change in its macroscopic properties, such as the transition from a solid to a liquid or from a liquid to a gas.

In our country, much attention has been paid to developing important directions of statistical physics that have applications to the applied and fundamental sciences. Particularly, notable accomplishments have been made in gradient and limiting Gibbs measures for Hamiltonians delineated in lattice systems, as well as in solving practical problems via nonlinear analysis and measure theory. Explorations at the global level in critical fields like stochastic processes, mathematical physics, measure theory, and nonlinear analysis are deemed fundamental research[1] priorities. Currently, the progress in investigating Gibbs measures on lattice systems takes a central role in the implementation this decree.

The subject and object of research of this dissertation are in line with tasks identified in the Decrees and Resolutions of the President of the Republic of Uzbekistan of February 7, 2017, PF-4947, “On the strategy of action for the further development of the Republic of Uzbekistan”, PQ-4387 dated July 9, 2019 “On state support for the further development of mathematics education and science, as well as measures to radically improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan”, PQ-4708 of May 7, 2020 “On measures to improve the quality of education and research in the field of mathematics” as well as in other regulations related to basic sciences.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the

[1] Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017 year 18 May "On measures on the organization of activities of the first created scientific research institutions of the Academy of Sciences of the Republic of Uzbekistan" No 292.

priority areas of science and technology of the Republic of Uzbekistan IV, "Mathematics, Mechanics, and Computer Science".

Review of foreign research on the topic of the dissertation. Scientific research on Gibbs measures and their practical uses is happening at major scientific centers and universities globally, notably at the Universities of Bonn and Bochum Ruhr University (Germany), University of British Columbia (Canada), Universities of Bari and Rome (Italy), Institute of Physics (Serbia), V.A.Steklov Mathematical Institute of Russian Academy of Science (Russia), The Fields Institute (Canada), University of Harran (Turkey), Indiana University (USA), University of Aix-Marseille and University of Paris-Est (France), Universities of Leeds and College London (UK), University of Kyushu (Japan), International Islamic University of Malaysia (Malaysia), Technische Universitaet Muenchen (Germany), etc.

The outcomes of scientific investigations into Gibbs measures on lattice systems have successfully addressed several pressing issues. This includes the following breakthroughs: construction, coexistence of localized and delocalized Gibbs measures on trees (Bochum Ruhr University, Germany), development of the Gibbs statistics of high-density hard-core random configurations on a triangular lattice (University of Cambridge, UK), study of basic properties of generalized Gibbsian measures University of Paris-Est (France), The extremality of disordered phases for Hamiltonians on Cayley trees (University of Harran, Turkey), development of probability distribution of a Gibbs cluster states on configuration spaces (University of Leeds, UK), problems of existence and uniqueness of Gibbs measures of uncountable particle systems with unbounded spins (University of York, UK), computing free energies (and entropies) according to boundary conditions Gibbs measures (University of Aix-Marseille, France).

Today there are numerous investigations into the theory of Gibbs measures, encompassing: investigating conditions under which Gibbs measures exist and are unique for a given lattice model, analyzing the behavior of Gibbs measures in the thermodynamic limit, studying the occurrence and properties of phase transitions in lattice systems, applying renormalization group techniques to study the behavior of Gibbs measures, investigating the dynamics of lattice systems under various processes, such as Monte Carlo simulations or stochastic evolution equations.

The degree of scrutiny of the problem. The Gibbs measure is a fundamental concept in statistical mechanics, developed by Josiah Willard Gibbs, an American physicist and mathematician. Gibbs introduced the concept of an ensemble, which is a collection of possible states that a physical system can occupy. Ernst Ising, a German physicist, initially proposed to describe magnetic spins of one dimensional model on a lattice in his 1925 doctoral thesis. Lars Onsager's improvement upon Ernst Ising's model involved providing an exact solution for the two-dimensional Ising model in 1944. R. Dobrushin, D. Lanford, and D. Ruelle discovered the necessary and sufficient conditions for the probability measure to be the Gibbs measure, which they later dubbed the DLR equation. Currently, it is common for mathematicians to define the Gibbs measure using the DLR equation. The Pirogov-Sinai theory, named after Yuri Pirogov and Yakov Sinai, provides a rigorous mathematical framework for studying phase transitions in lattice systems governed by Gibbs measures. It establishes the conditions under which phase transitions occur and characterizes the behavior of Gibbs measures near critical points. In addition, V. Minlos, H. Georgii, A. Bleher, B. Simon, S. Zachary, R. Kotecky, S. Shlosman, Ch. Preston, S. Friedli, Y. Velenik made significant contributions to understanding the phase transition phenomena on lattice systems and provided valuable insights into the mathematical properties and behavior of Gibbs measures in the context of lattice systems.

It is known that the theory of Gibbs measures is of particular importance in the study of Gibbs measures defined on Cayley trees. Ch. Kulske, Ch. Preston, R. Bexter, V. Malyshev, D. Gandolfo, J. Ruiz, K. Kulske, P. Bleher, Y. Suhov, L. Bogachev, H. Akin, S. Temur, and others have collaborated and individually contributed significantly to the study of Gibbs measures for models in statistical mechanics. Their works often focus on the analysis of limiting Gibbs measures, periodic Gibbs measures, and the application of Gibbs measures in various fields, including biology, medicine, and economics. They have found and developed theories and methods of existence of phase transitions and non-emptiness of Gibbs measures in various fields.

Right now, researchers are focusing a lot on Gibbs states connected to the Hamiltonian of physical systems. N. Ganikhodzhaev, U. Rozikov and in their scientific school the set of periodic Gibbs measures is always filled for models like Ising, SOS, HC, and Potts has been investigating. Meanwhile, F. Muhammedov, M. Rahmatullaev, O. Khakimov, I. Sattarov, and A. Tukhtabaev are investigating p-adic Gibbs measures for the aforementioned models on p-adic fields. Additionally, G. Botirov, R. Rahmatullaev, M. Rasulova have determined the count of periodic Gibbs measures at certain temperatures by using the contour method on trees when describing the ground states of these models. Just a few years ago, the German scientist K. Kulske blazed a trail in gradient Gibbs measures on trees. Right now, A. Le Ny, L. Coquille, F. Henning, B. Jahnel, P. Schriever, and others are looking into analogous results for gradient Gibbs measures, based on what's already known about gradient Gibbs measures on the lattice $\mathbb{Z}^d$. In this direction, it's worth mentioning that despite this field's relatively short development time, numerous scientific articles are being published by our native scientists: N. Ganikhodzhaev, U. Rozikov, R. Khakimov, M. Rahmatullaev, N. Khatamov, G. Botirov, O. Khakimov, M. Makhammadaliev, and R. Ilyasova.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out.

The dissertation work is done in accordance with the planned theme of scientific research "YoF-4-3: Spin systems probability measures on a countable graph" Institute of Mathematics (2016-2017 y), "YoF-4-8 Non-classic problems on mathematical and statistical physics" National university of Uzbekistan (2016-2017 y) and "YoOT-Ftex-2018-154: Gibbs measures and spectrum of Hamiltonian on a lattice $\mathbb{Z}^d$ and on a Cayley tree" National University of Uzbekistan (2018-2019 y), OT-EA-4-02: “Integro-differential equations of fractional order and the study of problems of a new type for such equations (2019-2020)”.

The aim of the research work is to construct new classes of specifications for models with a finite, countable, and continuum set of spin values on lattice systems and to describe a set of corresponding limit and gradient Gibbs measures.

Research problems:

to construct new specification for a family of non-probability kernels on lattice systems;

to give the set of gradient and limiting Gibbs measures with periodic boundary conditions for HC and SOS models with a countable set of spin values;

to establish connections between the limiting Gibbs measures of continuous spin models and the fixed points of nonlinear integral operators, and to determine the conditions of existence, uniqueness, and non-uniqueness of Gibbs measures for the model;

to find the free energies for translation-invariant and periodic boundary conditions for the Ising model on the Cayley tree, as well as to provide an invariance property for subgroups of the group representation of a Cayley tree and applying this property to the Ising model.

The research object: spatial and lattice systems, Hamiltonian, Gibbs measure, gradient Gibbs measure.

The research subject: Construction of lattice systems, main conceptions of limiting Gibbs and gradient Gibbs measures, connections between Gibbs measure theory and nonlinear operator theory, free energy for Hamiltonian with an external field, subgroups of group representation of Cayley tree.

Research methods: In the dissertation the methods of measure theory, functional analysis, group theory, probability theory, and nonlinear integral operators are applied.

Scientific novelty of the research work consists of the following:

a new specification with a family of non-probability kernels on lattice systems is constructed;

on lattice systems, a set of gradient Gibbs measures with periodic boundary laws for the SOS model with a countable set of spin values is described and infinitely many gradient Gibbs measures for the HC model with a countable set of spin values are constructed;

the existence and uniqueness of Gibbs measures on the Cayley tree are proved and by using fixed points of nonlinear integral operators, the existence of phase transitions of the limiting Gibbs measures is proved;

the free energies of translation-invariant and periodic boundary conditions for the Ising model with an external field are computed and a sufficiency condition on the invariance property of subgroups of the group representation of Cayley trees is obtained.

Practical results of the research consists of the following:

providing a mathematical model for calculating the free energy of Gibbs measures for models in statistical mechanics;

describing mathematical methods of the collection of spin systems of particles and molecules for biological models.

The reliability of the results of the study. The results have been obtained by using the techniques and methods of nonlinear operator theory, functional analysis, measure and probability theory, group and graph theory, Gibbs measure theory. The obtained results are strongly proved.

Scientific and practical significance of the research results. The scientific importance of the results of the research work lies in the fact that by checking the occurrence the phase transitions for various models in physics and statistical mechanics. The results of the research will help to better understand phase transitions.

The practical significance is the use of limit and gradient Gibbs measures and a complete description of the set of such measures for the analysis of changes in the state of a physical system.

Implementation of the research results. The scientific outcomes acquired during the research of the dissertation are implemented in the following research projects:

from methods of computing the free energies of translation-invariant and periodic boundary conditions for the Ising model with an external field on trees were used in the research of the foreign project number G0003247 on the topic "Chaotic and mixing p-adic dynamical systems associated with renormalized groups of lattice models" (United Arab Emirates University, reference dated May 20, 2024, UAE) to find new Gibbs measures for energies of potentials with competing interactions on the Cayley tree. The application of the scientific result made it possible to illuminate the equilibrium states of physical and biological systems with a countable set of spin values;

from theorems regarding the existence and uniqueness of fixed points of nonlinear operators representing periodic Gibbs measures for continuous spin models on the Cayley tree and methods for constructing kernels such that the operator has at least two fixed points were used in the research of the foreign project number FRGS21-230-0839 on the topic "Dynamics of Finite Dimensional Orthogonality Preserving Cubic Stochastic Operators" (International Islamic University of Malaysia, reference dated May 30, 2024, Malaysia) to check the existence and uniqueness of Gibbs measures for spin models. The application of the scientific result enabled the verification of the existence of phase transitions;

from constructing a new specification with a family of non-probability kernels on lattice systems is used to compute Gibbs measures and its free energies in papers of foreign scientific journals (Physical Review E, 2015, 92, 022106; Europhysics Letters, 2021, 133(2); Journal of Statistical Physics, 2014, 157(2); Chinese Journal of Physics, 2022, 77; Mathematical Physics, Analysis, and Geometry, 2016, 19(4)). The application of the scientific results made it possible to analyze Gibbs measures corresponding to Hamiltonians in physics systems.

Approbation of the research results. The main results of the research have been discussed in 4 international and 4 national scientific conferences.

Publications of the research results. On the topic of the dissertation 19 research papers have been published in scientific journals; all of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the DSc thesis. In addition, 16 of them were published in international journals of mathematics and physics, and 3 papers published in national mathematical journals.

The structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, a conclusion, and a bibliography. The volume of the thesis is 191 pages.

THE MAIN CONTENT OF THE DISSERTATION

In the introduction besides the motivation of research theme and correspondence to the priority research areas of science and technology of the Republic, we present a review of international research on the theme of the dissertation and the degree of scrutiny of the problem, formulate our goals and objectives, identify the object and subject of study, and state scientific novelty and practical results of the research. Moreover, we reduce the theoretical and practical importance of the obtained results, and give information on the implementation of the research results, the published works and the structure of dissertation.

In the first chapter of the dissertation, titled **“Gibbs measures of $\mathbb{F}$-specification on systems”** we give main definitions and concepts of Gibbs measure on spatial systems and construction of non-probability kernels.

Let $(D,\preceq)$ be a partially ordered set (poset) with $D\neq\varnothing$. The poset D is said to be *complete* if each non-empty subset of D possesses a least upper bound. If $\{z_n\}_{n\geq 1}$ is an increasing sequence (i.e., $z_n\preceq z_{n+1}$) of elements from D then $\lim_n z_n := \sup\left(\{z_n : n\geq 1\}\right)$.

Let $N\subset D$ and N, E be complete posets then a monotone mapping $\psi : N\to E$ (i.e. $\psi(w)\preceq\psi(z)$ for all $w\preceq z$) is said to be *continuous* if $\psi\left(\lim_{n\to\infty} z_n\right)=\lim_{n\to\infty}\psi\left(z_n\right)$ holds for each increasing sequence $\{z_n\}_{n\geq 1}$ from N.

The partial order $\preceq$ is called *directed* if for all $z_1, z_2\in D$ there exists $a\in D$ with $z_1\preceq a$ and $z_2\preceq a$. Also, the partial order $\preceq$ is called *countably generated* if there is a countable subset D_0 of D such that for each $z\in D$ there is an element $z_0\in D_0$ with $z\preceq z_0$.

Let S be a non-empty set and $(N,\subseteq)$ be a directed and countably generated poset with $N\subset\mathcal{P}(S):=\{A\,|\,A\subset S\}$. For each $\Lambda\in N$ and a non-empty set X we define a σ-algebra $\mathcal{F}^{\Lambda}\subset\mathcal{P}(X)$. The family of such σ-algebras satisfies the following condition $\mathcal{F}^{\Lambda\cup\Delta}=\mathcal{F}^{\Lambda}\vee\mathcal{F}^{\Delta}$ for all $\Lambda,\Delta\in N$, where $\mathcal{F}^{\Lambda}\vee\mathcal{F}^{\Delta}:=\mathcal{S}(\mathcal{F}^{\Lambda}\cup\mathcal{F}^{\Delta})$ denotes the minimal σ-algebra containing $\mathcal{F}^{\Lambda}\cup\mathcal{F}^{\Delta}$. Put

$\mathcal{A} := \bigcup_{\Lambda \in N} \mathcal{F}^{\Lambda}$, $F = \mathcal{S}(\mathcal{A})$. Then $(X, \mathcal{F})$ is a measurable space. Now, we define a decreasing family $\mathbb{F} = \{\mathcal{F}_A\}_{A \in N}$ of sub σ-algebras of $\mathcal{F}$, where $\mathcal{F}_{\Lambda} := \mathcal{S}(\cup_{\Delta \in N, \Delta \in S \setminus \Lambda} F^{\Delta})$.

A quadruple $\Sigma := \left(S, N, X, \{\mathcal{F}_{\Lambda}\}_{\Lambda \in N}\right)$ with (S, N), X and $\{\mathcal{F}_{\Lambda}\}_{\Lambda \in N}$ as above and with the order $\subset$ on N countably generated will be called a *spatial system*.

The set of all probability measures on $\mathcal{F}^{\Lambda}$ is denoted by $\mathrm{P}\left(\mathcal{F}^{\Lambda}\right)$. Let $\mu_{\Lambda} \in \mathrm{P}\left(\mathcal{F}^{\Lambda}\right)$ for each $\Lambda \in N$. The collection of distributions $\{\mu_{\Lambda}\}_{\Lambda \in N}$ is called *compatible (consistent)* if $\mu_{\Lambda}(F) = \mu_{\Delta}(F)$ for $F \in \mathcal{F}^{\Lambda}$ whenever $\Lambda \subset \Delta$. For any compatible family $\{\mu_{\Lambda}\}_{\Lambda \in N}$ there is a measure $\mu \in \mathrm{P}(\mathcal{F})$ with $\mu(F) = \mu_{\Lambda}(F)$ for all $F \in \mathcal{F}^{\Lambda}, \Lambda \in N$ then we say that the family of σ-algebras $\{\mathcal{F}^{\Lambda}\}_{\Lambda \in N}$ has the Kolmogorov extension property. Ultimately, we say that the spatial system Σ has the Kolmogorov extension property if the collection of σ-algebras $\{\mathcal{F}^{\Lambda}\}_{\Lambda \in N}$ does.

Let $M(X)$ be the set of all mappings from X to $\mathbb{R}_{\infty}^{+} = [0, \infty]$. Put $\mathcal{B}(\mathbb{R})$ is Borel algebra on $\mathbb{R}$ and $M(\mathcal{F})$ is the set of all measurable mappings from $(X, \mathcal{F})$ to $\left(\mathbb{R}_{\infty}^{+}, \mathcal{B}_{\infty}^{+}\right)$ whenever $\mathcal{B}_{\infty}^{+} = \mathcal{B}(\mathbb{R}) \cap \mathbb{R}_{\infty}^{+}$. It is easy to check that $M(\mathcal{F})$ is a complete subspace of $\mathrm{M}(X)$ if we consider $\mathrm{M}(X)$ as the standard poset. For $(Z, \mathcal{E})$ and $(Z_1, \mathcal{E}_1)$ measurable spaces, we say that a mapping $\kappa : Z \times \mathcal{E}_1 \to \mathbb{R}_{\infty}^{+}$ is an $(Z, \mathcal{E}) | (Z_1, \mathcal{E}_1)$*-kernel* if $\kappa(x, \cdot)$ is a measure for all $x \in Z$ and $\kappa(\cdot, A)$ is a measurable mapping for each $A \in \mathcal{E}_1$. There exists a unique continuous linear mapping $\Phi_{\kappa} : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ with $\Phi_{\kappa}\left(I_A\right) = \kappa(\cdot, A)$ for all $A \in \mathcal{E}_1$, where I_A is the indicator function. Hence, we can just write κ instead of Φ_{κ}. We say that the kernel $\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ is *finite* if and only if $\kappa(1)(x) < \infty$ for each $x \in Z$. If finite kernel satisfies the condition $\kappa(I_{Z_1}) = I_Z \cdot$ then it is called *a probability kernel*. Also, we say that $\kappa(1)$ is *a quasi-probability kernel* if $\kappa(I_{Z_1}) \in \{\mathbf{0}, I_Z\}$ where $\mathbf{0}$ is zero function.

The mappings $\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathrm{M}(\mathcal{E})$ and measure $\mu : \mathrm{M}(\mathcal{E}) \to \mathbb{R}_{\infty}^{+}$ can be composed, it means that the mapping $\mu\kappa : \mathrm{M}(\mathcal{E}_1) \to \mathbb{R}_{\infty}^{+}$ with $(\mu\kappa)(f) = \mu(\kappa(f))$ for each $f \in \mathrm{M}(\mathcal{E}_1)$. In other words, it is given by $(\mu\kappa)(A) = \mu\left(\kappa\left(I_A\right)\right) = \mu(\kappa(\cdot, A))$ for each $A \in \mathcal{E}$.

Definition 1. *i) For the spatial system* $\left(S, N, X, \{\mathcal{F}_{\Lambda}\}_{\Lambda \in N}\right)$ *let* κ_{Δ} *be a proper* $(Z, \mathcal{F}_{\Delta}) | (Z, \mathcal{F}_{\Delta})$*-measurable quasi-probability kernel* $\kappa_{\Delta} \in \mathrm{K}(\mathcal{F})$ *for each* $\Delta \in N$*. Then the family* $\mathcal{V} = \{\kappa_A\}_{A \in N}$ *will be called an* $\mathbb{F}$***-specification*** *if* $\kappa_B = \kappa_B \kappa_A$ *whenever* $A, B \in N$ *with* $A \preceq B$*. Here and below,* $\kappa_B \kappa_A$ *is the composition of kernels.*

ii) A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is called ***a Gibbs measure with specification*** *$\mathcal{V}$ if $\mu = \mu\kappa_A$ for each $A \in N$. The last equation is well-known as* ***DLR*** *equation.*

The set of Gibbs measures with specification $\mathcal{V}$ will be denoted by $\mathcal{G}(\mathcal{V})$, i.e.

$$\mathcal{G}(\mathcal{V}) = \left\{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\kappa_A \text{ for all } A \in N\right\}.$$

If we take the index set defined in spatial systems as a countable set, then this system is called a lattice system. Let $\Im^k = (V, L)$ be a *Cayley tree* of order $k \geq 1$ (i.e. $k+1-$regular tree) with V and L is the set of vertices and edges respectively.

As usual, we consider models with spin values in $\Phi \subset \mathbb{R}_+^\infty$, and is assigned to the nodes of the tree and Φ is referred to as the collection of spin values. For $A \subset V$ a configuration σ_A on A is a mapping $\sigma_A : A \to \Phi$. Let $\Omega_A = \Phi^A$ be the collection of all configurations on A; the collection of all configurations is $\Omega := \Phi^V$. We consider all elements of V are numerated by the numbers: $0,1,2,3,...$. In other words, we can write $V = \{x_0, x_1, x_2, ...\}$.

Let $\mathcal{I}_A$ be the indicator function. For any two configurations $\sigma, \sigma' \in \Omega$, we define $C(\sigma, \sigma') = \{x \in \{x_0, x_1, x_2, ...\} \mid \sigma(x) \neq \sigma'(x)\}$. Ω can be viewed as a metric space with respect to the metric $\rho : \Omega \times \Omega \to \mathbb{R}^+$ given by

$$\rho\left(\left\{\sigma(x_n)\right\}_{x_n \in V}, \left\{\sigma'(x_n)\right\}_{x_n \in V}\right) = \sum_{n \geq 0} 2^{-n} \mathcal{I}_{x_n \in C(\sigma, \sigma')} \tag{1}$$

and let $\mathcal{B}$ be the σ-algebra of Borel subsets of Ω.

For every $m \geq 0$ let $\pi_m : \Omega \to \Phi^{m+1}$ be given by $\pi_m(\sigma_0, \sigma_1, \sigma_2, ...) = (\sigma_0, \ldots, \sigma_m)$ and let $\mathcal{C}_m = \pi_m^{-1}\left(\mathcal{P}\left(\Phi^{m+1}\right)\right)$, where $\sigma_i := \sigma(x_i)$ and $\mathcal{P}\left(\Phi^{m+1}\right)$ is the collection of all subsets of Φ^{m+1} (Cartesian product of Φ). Let $\mathcal{C} = \bigcup_{m \geq 0} \mathcal{C}_m$; then $\mathcal{C}$ is an algebra (the algebra of *cylinder sets*) and each of the sets in $\mathcal{C}$ is both open and closed. Let $\mathcal{S}(\mathcal{C})$ be the minimal σ-algebra containing $\mathcal{C}$. Put

$$\sigma^{(m)}(q) = \left\{\sigma \in \Omega : \sigma|_{\{x_m\}} = q \in \Phi\right\}.$$

A measurable space $(X, \mathcal{E})$ is called *countably generated* if $\mathcal{E} = \mathcal{S}(\mathcal{I})$ for some denumerable subset $\mathcal{I}$ of $\mathcal{E}$.

Proposition 1. *$\mathcal{B} = \mathcal{S}(\mathcal{C}) = \mathcal{S}\left(\left\{\sigma^{(m)}(q) : m \geq 0, q \in \Phi\right\}\right)$ and in particular if $|\Phi| < \infty$ then $(\Omega, \mathcal{B})$ is countably generated.*

As a corollary, we can say $\mathcal{B} = \mathcal{S}\left(\left\{\bar{\sigma}_{V_n} : n \in \mathbb{N}_0\right\}\right)$. A measurable space $(\Omega, \mathcal{F})$ a non-empty index set V equipped with a partial order $\subseteq$ and a decreasing family $\mathbb{F} = \left\{F_\Lambda\right\}_{\Lambda \in \mathcal{N}}$, (where $F_\Lambda := \mathcal{S}(\mathcal{C}_\Lambda)$) of sub σ-algebras of $\mathcal{F}$. It is easy to check that the poset $(\mathcal{N}, \subseteq)$ is directed (i.e., for all $A_1, A_2 \in \mathcal{N}$ there exists $A \in \mathcal{N}$ with $A_1 \subseteq A$

and $A_2 \subseteq A$) and countably generated (i.e., there exists a countable subset $\{V_n\}_{n\in\mathbb{N}}$ of $\mathcal{N}$ such that for each $A \in \mathcal{N}$ there is an element $V_{n_0} \in \{V_n \mid n \in \mathbb{N}\}$ with $A \subseteq V_{n_0}$).

A collection $\Sigma := \left(V, \mathcal{N}, \Omega, \{F_\Lambda\}_{\Lambda\in\mathcal{N}}\right)$ with $(V, \mathcal{N})$, Ω and $\{F_\Lambda\}_{\Lambda\in\mathcal{N}}$ as above and with the inclusion order $\subseteq$ on $\mathcal{N}$ countably generated will be called a *lattice system*. In addition, (Ω, ρ) is a separable and complete metric space.

Theorem 1. *Let* $\mathcal{N}_1 \subset \mathcal{N}$ *be a directed and for each* $\Lambda \in \mathcal{N}_1$ *let* $\mu_\Lambda \in \mathrm{P}\left(\mathcal{F}^\Lambda\right)$. *If* $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_1}$ *is consistent then there exists a unique probability measure* $\mu \in \mathrm{P}(\mathcal{F})$ *such that* $\mu(F) = \mu_\Lambda(F)$ *for all* $F \in F^\Lambda, \Lambda \in \mathcal{N}_1$.

Note that $\Omega_n = \Omega_{V_n}$ and $\mathcal{B}_n$ is the σ-ring of all Borel sets of Ω_n. Also, μ_n is a measure on $(\Omega_n, \mathcal{B}_n)$, $n \in \mathbb{N}$.

Theorem 2. *If one of measures* $\{\mu_n\}_{n=1}^{\infty}$, *say* μ_{n_0}, *is* σ*-finite, then* $\{\mu_n\}_{n=1}^{\infty}$ *can be extended uniquely to a* σ*-additive measure on* $\mathcal{B}$.

Definition 2. *Let* $P_\Lambda : \Omega \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ *be* $\mathcal{F}_\Lambda$*-measurable mapping for all* $\Lambda \in \mathcal{N}$ *, then the collection* $P = \{P_\Lambda\}_{\Lambda\in\mathcal{N}}$ *is called a potential. Also, the following expression*

$$H_{\Delta,P}(\sigma) \overset{\text{def}}{=} \sum_{\Delta\cap\Lambda\neq\varnothing,\Lambda\in\mathcal{N}} P_\Lambda(\sigma), \quad \forall\sigma \in \Omega. \tag{2}$$

is called Hamiltonian H *associated to the potential* P.

Put $r(P) \overset{\text{def}}{=} \inf\{R > 0 : P_\Lambda \equiv 0 \text{ for all } \Lambda \text{ with } diam(\Lambda) > R\}$. If $r(P) < \infty, P$ has finite range and $H_{\Delta;P}$ is well defined. If $r(P) = \infty$, P has infinite range and, for the Hamiltonian to be well defined, we will assume that P is absolutely summable in the sense that

$$\sum_{\Lambda\in\mathcal{N},x\in\Lambda} \|P_\Lambda\|_\infty < \infty, \quad \forall x \in V,$$

(remember that $\| f \|_\infty \overset{\text{def}}{=} \sup_\omega | f(\omega)|$) which ensures that the interaction of a spin with the rest of the system is always bounded, and therefore that $\|H_{\Delta;P}\|_\infty < \infty$.

Let $\mathcal{N}_1 = \{V_n : n \in \mathbb{N}\}$ then we define the following Hamiltonian in the box $V_n, n \in \mathbb{N}$:

$$H_n(\sigma) = \sum_n P_{V_n}(\sigma), \quad \forall\sigma \in \Omega. \tag{3}$$

Let us define a specification $\zeta^H = \left\{\zeta^H_{V_n}\right\}_{n\in\mathbb{N}}$ (in short $\zeta^H_{V_n} := \zeta^H_n$) such that $\zeta^H_{V_n}(\cdot \mid \omega)$ gives to each configuration $\tau_{V_n}\omega^c_{V_n}$ a probability proportional to the Boltzmann weight prescribed by equilibrium statistical mechanics:

$$\zeta_n^P\left(\omega,\sigma_n\right)\overset{\text{def}}{=}\frac{1}{\mathbf{Z}_n^{\omega}}e^{-H_n\left(\sigma_n\omega_{\bar{V}_n}\right)}, \tag{4}$$

where we have written explicitly the dependence on $\omega_{\bar{V}_n}$, and $\mathbf{Z}_n^{\omega}$ is a partition function, i.e.,

$$\mathbf{Z}_n^{\omega}\overset{\text{def}}{=}\sum_{\sigma_n\in\Omega_{V_n}}\exp\left(-H_n\left(\sigma_n\omega_{\bar{V}_n}\right)\right).$$

We denote $\mathbb{F}_1:=\left\{\zeta_n^P\right\}_{n\in\mathbb{N}}$, then the following theorem holds:

Theorem 3. $\mathbb{F}_1$ *is a specification.*

In the second chapter of the dissertation, entitled **"Gradient Gibbs measures on lattice systems"**, all periodic Gibbs measures for the countable spin value HC model are fully analyzed and the existence of an infinite number of Gibbs measures for the HC model with countable spin values using the Bleher-Ganikhodjaev construction is proved.

Consider the set $\mathbb{Z}$ as the set of vertices of some infinite graph G. Using the graph G, we define a G-admissible configuration as follows: a configuration σ is called a G-admissible configuration on a Cayley tree, if $\{\sigma(x),\sigma(y)\}$ is an edge of the graph G for any nearest neighbors x,y from V. Denote the set of G-admissible configurations by Ω^G.

For the graph G, $\lambda:G\mapsto\mathbb{R}_+$ is a bounded function. The value λ_i of the function λ at the vertex $i\in\mathbb{Z}$ is called its "activity".

For given G and λ, we define the $G-$ HC-model Hamiltonian as

$$H_G^{\lambda}(\sigma)=\begin{cases}J\sum_{x\in V}\ln\lambda_{\sigma(x)}, & \text{if}\,\sigma\in\Omega^G,\\ +\infty, & \text{if}\,\sigma\notin\Omega^G,\end{cases} \tag{5}$$

where $J\in\mathbb{R}$.

Definition 3. *A family of vectors* $l=\left\{l_{xy}\right\}_{\langle x,y\rangle\in L}$ *with* $l_{xy}=\left\{l_{xy}(i):i\in\mathbb{Z}\right\}\in(0,\infty)^{\mathbb{Z}}$ *is called the boundary law for the Hamiltonian (5) if*

1) for each $\langle x,y\rangle\in L$ *there exists a constant* $c_{xy}>0$ *such that the consistency equation*

$$l_{xy}(i)=c_{xy}\lambda_i\prod_{z\in\partial x\setminus\{y\}}\sum_{j\in\mathbb{Z}}a_{ij}l_{zx}(j) \tag{6}$$

holds for any $i\in\mathbb{Z}$, *where* λ_i *is an activity of* λ *at the vertex* $i\in\mathbb{Z}$.

2) the boundary law l *is said to be normalisable if and only if*

$$\sum_{i\in\mathbb{Z}}\left(\lambda_i\prod_{z\in\partial x}\sum_{j\in\mathbb{Z}}a_{ij}\,l_{zx}(j)\right)<\infty$$

for all $x \in V$.

Now, for a configuration $\omega = \{\omega_x \mid x \in V\}$ we introduce symmetric transfer matrices Q_b.

$$Q_b(\omega_b) := \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y},$$

where $\omega_b = \{\omega_x, \omega_y\}$.

For a finite subset $\Lambda \subset V$ define the (Markov) Gibbsian specification as

$$\gamma_\Lambda^\lambda(\omega : \omega|_\Lambda = \sigma_\Lambda) = (Z_\Lambda^\lambda)(\omega)^{-1} \prod_{\langle x,y\rangle, \{x,y\}\cap\Lambda\neq\varnothing} \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y},$$

where $Z_\Lambda^\lambda(\omega)$ is a partition function.

For any Gibbsian specification γ with associated family of transfer matrices $(Q_b)_{b\in L}$ we have

1. Each normalisable boundary law $(l_{xy})_{x,y}$ for $(Q_b)_{b\in L}$ defines a unique Gibbs measure μ (corresponding to γ) via the equation given for any connected set $\Lambda \subset V$

$$\mu(\sigma_{\Lambda\cup\partial\Lambda} = \omega_{\Lambda\cup\partial\Lambda}) = (Z_\Lambda^\lambda)^{-1} \prod_{y\in\partial\Lambda} l_{yy_\Lambda}(\omega_y) \prod_{b\cap\Lambda\neq\varnothing} Q_b(\omega_b) =$$

$$= (Z_\Lambda^\lambda)^{-1} \prod_{\langle x,y\rangle, \{x,y\}\cap\Lambda\neq\varnothing} \lambda_{\omega_x} a_{\omega_x,\omega_y} \lambda_{\omega_y}, \tag{7}$$

where for any $y \in \partial\Lambda$, y_Λ denotes the unique nearest-neighbor of y in Λ.

2. Conversely, every Gibbs measure μ admits a representation of the form (7) in terms of a normalisable boundary law (unique up to a constant positive factor).

We consider a **concrete graph** G^* with $a_{i0} = 1$ for all $i \in \mathbb{Z}$ and $a_{im} = 0$ for all $i, m \in \mathbb{Z}_0$.Given a boundary law $l_{xy}(i)$, we define $z_{i,x} = l_{xy}(i)$ when x is direct successor of y, the set of all direct successors is $S(y)$, then (6) can be written as

$$z_{i,x} = \lambda_i \prod_{y\in S(x)} \frac{1}{1 + \sum_{j\in\mathbb{Z}_0} z_{j,y}}, \quad i \in \mathbb{Z}_0. \tag{8}$$

If we introduce the following notations

$$\mathcal{U}^+ := \left\{ z_x = \left(\ldots z_{-2,x}, z_{-1,x}, z_{0,x}, z_{1,x}, z_{2,x}, \ldots\right) \mid z_{i,x} \in (0,+\infty),\ z_{0,x} = 1,\ \forall x \in V \right\},$$

$$\mathcal{T}^+ := \left\{ z_x \in \mathcal{U}^+ \mid \sum_{i\in\mathbb{Z}} z_{i,x} < \infty \right\}.$$

then, (8) can be written as

$$z_x = \lambda \prod_{y\in S(x)} \frac{1}{1+\| z_y \|}, \quad z_x \in \mathcal{U}^+, \forall x \in V, \tag{9}$$

where $\lambda = (...,\lambda_{-2},\lambda_{-1},1,\lambda_1,\lambda_2,...) \in \mathcal{U}^+$ and $\| z_y \| := \sum_{i\in\mathbb{Z}} \| z_{i,y} \|$.

For a short notation, (9) can be considered as fixed point of new operator $W : \mathcal{T}^+ \to \mathcal{T}^+$ i.e. $z_x = Wz_x$.

Theorem 4. *If $k \geq 2$ and $\lambda \in \mathcal{T}^+$ then the following statements hold:*

1. If $\| \lambda \| \leq \Lambda_{cr}$ then $\lim_{n\to\infty} W^{(n)}(z_0) = \xi\lambda$, for all $z_0 \in \mathcal{T}^+$.

2. If $\| \lambda \| > \Lambda_{cr}$ and $z_0 = \alpha_0\lambda$, with $\alpha_0 \in (0,\xi)$ then

$$\lim_{n\to\infty} W^{(2n)}(z_0) = \alpha^*\lambda, \quad \lim_{n\to\infty} W^{(2n-1)}(z_0) = \beta^*\lambda.$$

3. If $\| \lambda \| > \Lambda_{cr}$ and $z_0 = \alpha_0\lambda$, with $\alpha_0 \in (\xi,1)$ then

$$\lim_{n\to\infty} W^{(2n)}(z_0) = \beta^*\lambda, \quad \lim_{n\to\infty} W^{(2n-1)}(z_0) = \alpha^*\lambda.$$

4. If $\alpha_0 = \xi$ then $\lim_{n\to\infty} W^{(n)}(z_0) = \xi\lambda$.

If $\| \lambda \| > \Lambda_{cr} \approx 2.7$ then there is only one periodic solution with period two $(\alpha^*\lambda, \beta^*\lambda)$ such that $\alpha^* < \beta^*$, i.e.

$$W(\alpha^*\lambda) = \beta^*\lambda, \quad W(\beta^*\lambda) = \alpha^*\lambda.$$

Let $u = (...,u_{-2},u_{-1},0,u_1,u_2,...), v = (...,v_{-2},v_{-1},0,v_1,v_2,...) \in \mathcal{T}^+$. We introduce partial ordered relation $\preceq$ on the set $\mathcal{T}^+$ such that $u \preceq v$ if $u_i \leq v_i$ for all $i \in \mathbb{Z}$.

Proposition 2. *If z_x is a solution to (9) then $\alpha^*\lambda \preceq z_x \preceq \beta^*\lambda$, for any $x \in V$.*

Let $\Gamma_0^k = (V^0, L^0)$ be a semi-infinite rooted tree and x^0 is the root. For $x \in V_0$ there is the unique path from x^0 to x and we denote the path by π_x. Also, let $\pi = \{x^0 = x_0 < x_1 < \ldots\}$ be an infinite path.

For $x \in W_n$, $n = 1,2,\ldots$ the set z^π is unambiguously defined by

$$z_x^\pi = \begin{cases} \alpha^*\lambda, \text{ if } x \prec x_{2n}, & x \in W_{2n}, \\ \beta^*\lambda, \text{ if } x \prec x_{2n+1}, & x \in W_{2n+1}, \\ \beta^*\lambda, \text{ if } x_{2n} \prec x, & x \in W_{2n}, \\ \alpha^*\lambda, \text{ if } x_{2n+1} \prec x, & x \in W_{2n+1}. \end{cases} \tag{10}$$

Theorem 5. *Let $\Lambda_{cr} < \| \lambda \| < \dfrac{1}{\beta^* - \alpha^*}$, $\lambda \in \mathcal{T}^+$ then for any infinite path π, there exists a unique set of numbers $z^\pi = \{z_x^\pi, x \in V^0\}$ satisfying equations (9) and (10). Also, Gibbs measures corresponding to the distinct solutions to (9) are distinct*

Now we give a brief overview of the main result of the gradient Gibbs measures for the SOS model.

The (formal) Hamiltonian of the SOS model is

$$H(\omega) = -J \sum_{\langle x,y\rangle \in L} \left|\omega_x - \omega_y\right|, \quad \omega \in \Omega, \tag{11}$$

where $J \in \mathbb{R}_+$ is a constant.

Then the boundary law equation (for translation-invariant case, i.e. $l_b \equiv l$, for all $b \in L$) reads

$$z_i = \left(\frac{\theta^{|i|} + \sum_{j\in\mathbb{Z}_0} \theta^{|i-j|} z_j}{1 + \sum_{j\in\mathbb{Z}_0} \theta^{|j|} z_j} \right)^k, \quad i \in \mathbb{Z}_0,$$

where $\theta = \exp\{-\beta H\}$ with inverse temperature $\beta = \dfrac{1}{T}$, and $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$

Theorem 6. *For the SOS model (11) on the Cayley tree of order $k = 3$ there are critical values $\tau_{cr}^{(1)} \approx 3.13039, \tau_{cr}^{(2)} \approx 4.01009$ ($\tau = \theta^{-1} + \theta$) such that the following assertions hold:*

(1) If $\tau \le \tau_{cr}^{(1)}$ then there is precisely one GGM associated to a boundary law.

(2) If $\tau \in \left(\tau_{cr}^{(1)}, 4\right]$ then there are precisely two such GGMs.

(3) If $\tau \in \left(4, \tau_{cr}^{(2)}\right] \cup \{3\sqrt{2}\}$ then there are at most three such GGMs.

(4) If $\tau \in \left(\tau_{cr}^{(2)}, +\infty\right) \setminus \{3\sqrt{2}\}$ then there are at most four such measures.

In the third chapter of the dissertation, entitled **"Gibbs measures of continuous spin models on Cayley trees"**, connections between the limiting Gibbs measures of continuous spin models and the fixed points of nonlinear integral operators are sought to be established, and the conditions of existence, uniqueness, and non-uniqueness of Gibbs measures for the model are aimed to be determined.

On the Cayley tree, the set of all configurations σ_A on $A \subset V$ is denoted by $\Omega_A = [0,1]^A$ and $\Omega := [0,1]^V$. The (formal) hamiltonian is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \tag{12}$$

where $J \in R \setminus \{0\}$ and $\xi : (u,v) \in [0,1]^2 \to \xi_{u,v} \in R$ is a given bounded, Lebesgue measurable function. As usual, $\langle x, y\rangle$ stands for nearest neighbor vertices, i.e., $d(x, y) = 1$.

Let $h: x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right), \tag{13}$$

where $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function. The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (13) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$f(t,x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta \xi_{t,u}) f(u,y) du}{\int_0^1 \exp(J\beta \xi_{0,u}) f(u,y) du}. \tag{14}$$

Here $f(t,x) = \exp(h_{t,x} - h_{0,x}), t \in [0,1]$ and $du = \lambda(du)$ is the Lebesgue measure.

If we define the operator A by

$$Av(x) = \int_{[a,b]} K(x,y) v(y) dy,$$

and N_f to be the Nemystkii operator associated with f :

$$\tilde{N}_f u(x) = f(u(x)).$$

Firstly, we study positive fixed points of the operator $A\tilde{N}_f$, i.e.

$$\int_a^b K(x,y) f(u(y)) dy = u(x),$$

where $K : [a,b]^2 \to (0,+\infty)$ and $f : [0,+\infty) \to [0,+\infty)$ are continuous functions, moreover f is nonlinear.

Condition $\mathcal{C}_1$: There exists continuous function $\Re : [a,b] \to [0,+\infty)$ and constants $c \in (0,1)$ and $\alpha, \beta \in [a,b]$ $(\alpha < \beta)$ such that $K(x,y) \le \Re(y)$ for $x, y \in [a,b]$ and $c \cdot \Re(y) \le K(x,y)$ for $x \in [\alpha, \beta]$ and $y \in [a,b]$.

Denote

$$C^+[a,b] = \{\omega \in C[a,b] : \omega(x) \ge 0, \text{ for all } x \in [a,b]\},$$

$$\hat{\mathcal{K}} := \left\{ \omega \in C^+[a,b] : \min_{x \in [\alpha,\beta]} \omega(x) \ge c \|\omega\| \right\},$$

where α, β and c are defined in the condition $\mathcal{C}_1$.

Condition $\mathcal{C}_2$: For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\inf\{ f(\delta t) : t \in [1, c^{-1}]\} > \varepsilon \cdot \delta$.

Proposition 3. *Assume that the conditions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *hold and moreover:*

1\. *There exists* $\tau > 0$ *such that* $f(\cdot)$ *is non-decreasing on* $[0,\tau]$.

2. The following inequality holds:

$$\sup_{s\in(0,\tau)} \frac{(1-c)s}{f(s)\cdot \| \int_a^b K(x,y)dy \|} > 1.$$

Then the operator $A\tilde{N}_f$ *has at least one positive fixed point in* $\hat{\mathcal{K}}$.

Theorem 7. *Assume that the assumptions of Proposition 3 hold and moreover:*

(i) For all $\nu \in [1,+\infty)$ *and* $\omega_i(x) \in C^+[a,b], i=\overline{1,2}$, f *satisfies the following relation:*

$$\omega_1(x) \geq \nu\omega_2(x) \Rightarrow f(\omega_1(x)) \geq \nu f(\omega_2(x)).$$

(ii) f *satisfies the following condition:*

$$\gamma_1 x^{\sigma_1} \leq f(x) \leq \gamma_2 x^{\sigma_2},\ \gamma_i > 0, \sigma_i > 1, i=\overline{1,2}$$

and $f(x)-\gamma_1 x^{\sigma_1}$, $\gamma_2 x^{\sigma_2} - f(x)$ *are monotone non-decreasing functions.*

(iii) For the maximum (M*) and minimum (*m*) values of the kernel of* $A\tilde{N}_f$*, the following inequality holds:*

$$\Im_2(M,m) - \Im_1(m,M) < \frac{1}{b-a},$$

where

$$\Im_i(x,y) = x\sigma_i\gamma_i \left(\frac{x}{y} \cdot (m\gamma_{3-i}(b-a))^{\frac{1}{1-\sigma_{3-i}}} \right)^{\sigma_i - 1}.$$

Then the operator $A\tilde{N}_f$, $f \in C_+^{'}[a,b]$ *has exactly one positive fixed point in* $\hat{\mathcal{K}}$.

Remark 1. *Note that there are some examples of the kernel of* $A\tilde{N}_f$ *such that the operator has at least two positivy fixed points.*

The equation (3) has a translation-invariant solutions iff the equation

$$(H_k f)(t) := \int_0^1 K(t,u) f^k(u)du = f(t).$$

has a strictly positive solution, where $K(t,u) = \exp\{-J\beta\xi_{tu}\}$. In turn, the operator H_k is a special type of the operator $A\tilde{N}_f$ and for this operator Theorem 7 can be written as the following form:

Theorem 8. *Let* $\tau_k = \sqrt[k]{\frac{1+\sqrt{k^2+1}}{k}}, k \geq 2$. *If the kernel of the operator* H_k *satisfies the condition*

$$M \leq m\tau_k \tag{15}$$

then H_k *has exactly one positive fixed point.*

In term of Gibbs measures, we obtain the following theorem:

Theorem 9. *Let* $k \geq 2$. *If the function* $K(t,u) = \exp\{-J\beta\xi_{tu}\}$ *of the Hamiltonian (12) satisfies the condition (15), then the Hamiltonian has a unique translation-invariant Gibbs measure on the Cayley tree of order* k.

In the fourth chapter of the dissertation, entitled **"The free energy of the Ising model and invariance property"**, the free energies for translation-invariant and periodic boundary conditions for the Ising model on the Cayley tree are to be found, along with an invariance property for subgroups of the group representation of a Cayley tree, which will be applied to the Ising model.

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$ respectively. It is known that there exists a one to one correspondence between the set of vertices V of the Cayley tree Γ^k and elements of the group G_k.

Put $N_k = \{1,2,\dots,k+1\}$ and $A_0 \subset N_k$, $0 \leq |A_0| \leq k-2$. Let (A_1, A_2, A_3) be a partition of the set $N_k \setminus A_0$ and m_j be the minimal element of A_j, $j \in \{1,2,3\}$. Then we consider the homomorphism $u_{A_1A_2A_3} : G_k \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ given by

$$u_{A_1A_2A_3}(x) = \begin{cases} e, \ if \quad x = a_i, i \in N_k \setminus (A_1 \cup A_2 \cup A_3), \\ a_{m_j}, \ if \quad x = a_i, i \in A_j, \ j = 1,2,3. \end{cases}$$

where e is the identity element.

For $i \in \{1,2,3,\dots,7,8\}$ we define the following mappings: $\gamma_i : \langle a_{m_1}, a_{m_2}, a_{m_3} \rangle \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ by the formula:

$$\gamma_1(x) = \begin{cases} e, \ if \quad x = e, \\ a_{m_1}, \ if \ x \in \{a_{m_3}a_{m_1}, a_{m_2}a_{m_3}\}, \\ a_{m_2}, \ if \ x \in \{a_{m_1}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \ if \ x \in \{a_{m_1}a_{m_2}, a_{m_2}a_{m_1}\}, \\ \gamma_1(a_{i_1} \dots a_{i_{n-2}} \gamma_1(a_{i_{n-1}} a_{i_n})), \ if \ l(x) > 2, \end{cases} \qquad \gamma_2(x) = \begin{cases} e, \ if \quad x = e, \\ a_{m_1}, \ if \ x \in \{a_{m_3}a_{m_2}, a_{m_2}a_{m_1}\}, \\ a_{m_2}, \ if \ x \in \{a_{m_1}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \ if \ x \in \{a_{m_1}a_{m_2}, a_{m_2}a_{m_3}\}, \\ \gamma_2(a_{i_1} \dots a_{i_{n-2}} \gamma_1(a_{i_{n-1}} a_{i_n})), \ if \ l(x) > 2, \end{cases}$$

$$\gamma_3(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_3}a_{m_1}, a_{m_2}a_{m_1}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_2}a_{m_3}\}, \\ \gamma_3(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases} \qquad \gamma_4(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_1}\}, \\ \gamma_4(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases}$$

$$\gamma_5(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_2}a_{m_1}, a_{m_3}a_{m_1}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_3}\}, \\ \gamma_5(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases} \qquad \gamma_6(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_2}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_1}\}, \\ \gamma_6(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases}$$

$$\gamma_7(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_2}a_{m_1}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_2}a_{m_3}\}, \\ \gamma_7(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases} \qquad \gamma_8(x)=\begin{cases} e, \ if \quad x=e, \\ a_{m_1}, \ if \ \ x\in\{a_{m_2}a_{m_3}, a_{m_3}a_{m_2}\}, \\ a_{m_2}, \ if \ \ x\in\{a_{m_1}a_{m_3}, a_{m_3}a_{m_1}\}, \\ a_{m_3}, \ if \ \ x\in\{a_{m_1}a_{m_2}, a_{m_2}a_{m_1}\}, \\ \gamma_8(a_{i_1}...a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), \ if \ \ l(x)>2, \end{cases}$$

where $l(x)$ is the length of x and $i_s\in\{m_1,m_2,m_3\}$ for $s\in\{1,2,...,n\}$.

Denote

$$\Im^j_{A_1A_2A_3}(G_k)=\{x\in G_k \mid \gamma_j(u_{A_1A_2A_3}(x))=e\}, \ \ j=\overline{1,8}.$$

Theorem 10. *i) Let $A_0\subset N_k, 0\le |A_0|\le k-2$ and (A_1,A_2,A_3) be a partition of the set $N_k\setminus A_0$. Then $\Im^j_{A_1A_2A_3}(G_k)$ $\ j=\overline{1,8}$ is a subgroup of the group G_k.*

ii) For any subgroup K of G_k of index 4, there exists a partion A_1,A_2,A_3 of $N_k\setminus A_0$ and $j\in\{1,2,...,8\}$ such that $K=\Im^j_{A_1A_2A_3}(G_k)$ and conversely.

The n.n. Ising model is then defined by the formal Hamiltonian

$$H(\sigma)=-J\sum_{\langle x,y\rangle\subset V}\sigma(x)\sigma(y).$$

The spins $\sigma(x)$ take values ±1, and the real parameters J.

The (finite-dimensional) Gibbs distributions over configurations at inverse temperature $\beta=1/T$ are defined by

$$\mu_n(\sigma_n)=Z_n^{-1}(h)\exp\{\beta J\sum_{\langle x,y\rangle\subset V_n}\sigma(x)\sigma(y)+\sum_{x\in W_n}h_x\sigma(x)\}$$

with partition function $Z_n(h)$ and $h:=\{h_x\in\mathbb{R}\,|\,x\in V\}$.

It is well known that the family of such Gibbs distributions satisfies the compatibility condition if and only if for any $x\in V$ the following equation holds

$$h_x=\sum_{y\in S(x)} f_\theta(h_y), \tag{16}$$

where $\theta=\tanh(J)$, $\quad f_\theta(h)=\operatorname{arctanh}(\theta\tanh h)$.

Now, we consider the half tree and construct below new solutions of the functional equation (16). Namely, we let q and r be non-negative integers such that $1\le q\le k-1$, $0\le r\le k$ and r has the same parity as k.

Consider the boundary condition $h=\{h_x, x\in V\}$ with fields taking values $0,\pm h_1,\pm h_2$ defined by the following steps:

• if at vertex x we have $h_x=0$, then the function has values

$$\begin{cases} 0, & \text{on } r \text{ vertices of } S(x),\\ h_1, & \text{on half of remaining vertices},\\ -h_1, & \text{on the remaining vertices}.\end{cases}$$

• if at vertex x we have $h_x=h_1$ (resp. $-h_1$) then on q vertices in $S(x)$ the function has value h_2 (resp. $-h_2$) and on other vertices, it takes value 0;

• if at vertex x we have $h_x=h_2$ (resp. $-h_2$) then on each vertices in $S(x)$ the function has value h_1 (resp. $-h_1$).

It is easy to see that the boundary conditions in the above construction are compatible iff h_1 and h_2 satisfy the following system of equations:

$$\begin{cases} h_1=qf_\theta(h_2),\\ h_2=kf_\theta(h_1),\end{cases} \tag{17}$$

where $1\le q\le k-1$. Put $\theta_c=1/\sqrt{qk}$.

Theorem 11. *The system of equations (17) has a unique solution* $(0,0)$*, if* $-\theta_c\le\theta\le\theta_c$ *and three distinct solutions* $(0,0)$, (h_1^*,h_2^*)*, and* $(-h_1^*,-h_2^*)$ *(*$h_1^*,h_2^*>0$*), when* $|\theta|>\theta_c$.

The free energy of a compatible boundary condition is defined as the limit:

$$\lim_{n\to\infty}\frac{1}{\beta|V_n|}\ln Z_n(h)=F(h)$$

if it exists. Hereafter $|\cdot|$ denotes the cardinality of a set.

Proposition 4. *If* $r\neq 0$ *then free energies* F_{ALT} *of b.c. (17) equal to*

$$\frac{k(k-q)}{(2k^2-rk-rq)}\left(a(0)+\frac{k-r}{k-q}a(h_1)+\frac{q(k-r)}{k(k-q)}a(h_2)\right),$$

where

$$a(t) = -\frac{1}{2\beta}\ln[4\cosh(t+J)\cosh(t-J)].$$

• *If* $r = 0$ *then free energies has the following accumulating points*

$$\begin{cases} \frac{k-q}{k+1}a(0)+\frac{1}{k+1}a(h_1)+\frac{q}{k+1}a(h_2), & \text{if} \quad n=2m, m\to\infty \\ \frac{k-q}{k(k+1)}a(0)+\frac{k}{k+1}a(h_1)+\frac{q}{k(k+1)}a(h_2), & \text{if} \quad n=2m+1 \end{cases}$$

when the root takes value 0, $\pm h_2$ *and*

$$\begin{cases} \frac{k-q}{k(k+1)}a(0)+\frac{k}{k+1}a(h_1)+\frac{q}{k(k+1)}a(h_2), & \text{if} \quad n=2m, \\ \frac{k-q}{k+1}a(0)+\frac{1}{k+1}a(h_1)+\frac{q}{k+1}a(h_2), & \text{if} \quad n=2m+1 \end{cases}$$

when the root takes value $\pm h_1$.

CONCLUSION

The dissertation is devoted to describing the set of limiting and gradient Gibbs measures for given specifications on lattice systems, studying the structure of the set of such measures, and its applications to models in statistical mechanics.

In short, the scientific work's main conclusions are as follows:

1. The specification for a family of non-probability kernels on lattice systems is constructed.

2. Limiting and gradient Gibbs measures for HC and SOS models with a countable set of spin values on lattice systems were studied. An uncountable number of Gibbs measures for the HC model was constructed, and the set of gradient Gibbs measures for the SOS model is described.

3. A Hamiltonian with an uncountable set of spin values, which is a generalization of several classical models such as Ising, Potts, SOS, and HC, is considered and by using positive fixed points of the nonlinear integral operator of the Hammerstein type for the model, the following results are obtained: The set of Gibbs measures is non-empty; a sufficient condition for the uniqueness of Gibbs measure is obtained; except for the sufficient condition, the existence of phase transitions is proven.

4. It was proved that the class of periodic and weakly periodic Gibbs measures can be enlarged by non-normal subgroups of the group representation of the Cayley tree. Also, new weakly periodic Gibbs measures for the Ising model were presented.

5. A family of Gibbs measures was constructed in which free energy does not exist for the Ising model on the Cayley tree, and the occurrence of corresponding phase transitions was examined.

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ХАЙДАРОВ ФАРХОД ХАЛИМЖОНОВИЧ

ПРЕДЕЛЬНЫЕ И ГРАДИЕНТНЫЕ МЕРЫ ГИББСА НА РЕШЕТЧАТЫХ СИСТЕМАХ

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ВВЕДЕНИЕ (аннотация докторской диссертации)

Целью исследования является построение нового класса спецификаций на решетчатых системах, описание множества предельных и градиентных мер Гиббса заданных спецификаций на решетчатых системах, расширение класса периодических и слабо периодических мер Гиббса на деревьях Кэли.

Объект исследования: пространственные и решетчатые системы, гамильтониан, мера Гиббса, градиентная мера Гиббса.

Научная новизна исследования состоит в следующем:

построена новая спецификация с семейством невероятностных ядер на решетчатых системах;

на решеточных системах описан набор градиентных мер Гиббса с периодическими граничными законами для модели SOS со счетным набором значений спина и построено бесконечное число градиентных мер Гиббса для модели НС со счетным набором значений спина;

доказано существование и единственность мер Гиббса на дереве Кэли, а с помощью неподвижных точек нелинейных интегральных операторов доказано существование фазовых переходов предельных мер Гиббса;

рассчитаны свободные энергии трансляционно-инвариантных и периодических граничных условий для модели Изинга с внешним полем и получено достаточное условие свойства инвариантности подгрупп группового представления деревьев Кэли.

Внедрение результатов исследования. Научные результаты, полученные в ходе исследования диссертации, реализованы в следующих научно-исследовательских проектах:

из методов вычисления свободных энергий трансляционно-инвариантных и периодических граничных условий для модели Изинга с внешним полем на деревьях были использованы в исследовании зарубежного проекта номер G0003247 на тему «Хаотические и перемешивающиеся р-адические динамические системы, связанные с перенормированные группы решеточных моделей» (Университет Объединенных Арабских Эмиратов, справка от 20 мая 2024 г., ОАЭ) для поиска новых мер Гиббса для энергий потенциалов с конкурирующими взаимодействиями на дереве Кэли. Применение научного результата позволило осветить состояния равновесия физических и биологических систем со счетным набором значений спина;

из теорем о существовании и единственности неподвижных точек нелинейных операторов, представляющих периодические меры Гиббса для моделей с непрерывным спином на дереве Кэли, и методов построения ядер таких, что оператор имеет не менее двух неподвижных точек, были использованы при исследовании номера зарубежного проекта ФРГС21-230-0839 на тему «Динамика конечномерных ортогональностей, сохраняющих кубические стохастические операторы» (Международный исламский университет Малайзии, справка от 30 мая 2024 года, Малайзия) для проверки существования и единственности мер Гиббса для спиновых моделей.

Применение научного результата позволило проверить существование фазовых переходов;

вычисление свободных энергий трансляционно-инвариантных граничных условий мер Гиббса для модели Изинга с внешним полем используется для вычисления свободных энергий некоторых моделей статистической механики в статьях зарубежных научных журналов (Physical Review E, 2015, 92, 022106; Europhysical Letters, 2021, 133(2); Journal of Statistical Physics, 2014, 157(2); Chinese Journal of Physics, 2022, 77; Mathematical Physics, Analysis, and Geometry, 2016, 19(4)). Применение научных результатов позволило проанализировать свободные энергии, соответствующие гамильтонианам в физических системах.

Объем и структура диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Объем диссертации составляет 191 страницы.

E’LON QILINGAN ISHLAR RO‘YHATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

1-bo‘lim (1-часть; part 1)

1. Haydarov F. H., Kolmogorov extension theorem for non-probability measures on Cayley trees // *Reviews in Mathematical Physics*, 2450010, (2023), 12 pages. (3. Scopus IF=0.67)

2. Rozikov U., Haydarov F., Invariance Property on Group Representations of the Cayley Tree and Its Applications. // *Results in Mathematics,* 77(6), (2022), 15 pages. (3. Scopus IF=0.62)

3. Rozikov U. A., Haydarov F. H., A HC model with countable set of spin values: Uncountable set of Gibbs measures. // *Reviews in Mathematical Physics*, 35(1), (2023), 17 pages. (3. Scopus IF=0.67)

4. Haydarov F. H., On normal subgroups of the group representation of the Cayley tree. // *Vladikavkaz Mathematical Journal*, 25(4), (2023), 135-142. (3. Scopus IF=0.21)

5. Haydarov F. H., Rozikov U. A., Gradient Gibbs measures of an SOS model on Cayley trees: 4-periodic boundary laws. // *Reports on Mathematical Physics*, 90(1), (2022), 81-101. (3. Scopus IF=0.41)

6. Botirov G., Haydarov F., On the set of Gibbs measures for model with a countable set of spin values on Cayley trees. // *Positivity*, 26(50), (2022), 43-51. (3. Scopus IF=0.6)

7. Haydarov F.H., Ilyasova R.A., On periodic Gibbs measures of the Ising model corresponding to new subgroups of the group representation of a Cayley tree. // *Theoretical and Mathematical Physics,* 210, (2022), 261-274. (3. Scopus IF=0.33)

8. Haydarov F. H., New condition on uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree. // *Mathematical Physics, Analysis and Geometry*, 24(31), (2021), 23-30. (3. Scopus IF=0.62)

9. Haydarov F. H., Existence and uniqueness of fixed points of integral operator of Hammerstein type. // *Theoretical and Mathematical Physics*, 208(3), (2021), 1228-1238. (3. Scopus IF=0.33)

10. Haydarov F., Akhtamaliyev Sh., Nazirov M., Qarshiyev B., Uniqueness of Gibbs measures for an Ising model with continuous spin values on a Cayley tree. // *Reports on Mathematical Physics*, 86(3), (2020), 293-302. (3. Scopus IF=0.41)

11. Haydarov F.H., Fixed points of Lyapunov integral operators and Gibbs measures. // *Positivity*, 22(4), (2018), 1165-1172. (3. Scopus IF=0.6)

12. Eshkabilov Yu. Kh., Haydarov F. H., Lyapunov operator $\mathcal{L}$ with degenerate kernel and Gibbs measures. // *Nanosystems: Physics, Chemistry, Mathematics*, 8(5), (2017), 553-558. (3. Scopus IF=0.22)

13. Rozikov U. A., Haydarov F. H., Four competing interactions for models with an uncountable set of spin values on a Cayley tree. // *Theoretical and Mathematical Physics*, 191(2), (2017), 748-761. (3. Scopus IF=0.33)

14. Haydarov F.H., Characterization of the normal subgroups of finite index for the group representation of a Cayley tree. // *Nanosystems: Physics, Chemistry,*

Mathematics, 7(5), (2016), 1-6. (3. Scopus IF=0.22)

15. Eshkabilov Yu.Kh., Haydarov F.H., On positive solutions of the homogeneous Hammerstein integral equation. // *Nanosystems: Physics, Chemistry, Mathematics*, 6(5), (2015), 618-627. (3. Scopus IF=0.22)

16. Gandolfo D., Haydarov F.H., Rozikov U.A., Ruiz J., New phase transitions of the Ising model on Cayley trees. // *Journal of Statistical Physics* 153(3), (2013), 400-411. (3. Scopus IF=0.8)

17. Haydarov F.H., Positive fixed points of Lyapunov integral operators and Gibbs measures. // *Bulletin of the National University of Uzbekistan: Mathematics and Natural Sciences*, (2), (2017), 45-49.

18. Haydarov F.H., Gibbs measures of models with uncountable set of spin values on lattice systems. // *Bulletin of the National University of Uzbekistan: Mathematics and Natural Sciences*, 6(3), (2023), 166-178.

19. Haydarov F.H., Gradient Gibbs measures of a SOS model with countable set of spin values on a Cayley tree. // *Acta NUUz*, 1(1), (2023), 129-134.

2-bo‘lim (2-часть; part 2)

20. Haydarov F.H., Periodic Gibbs measures for continuous spin models on Cayley trees // International conference: *Gibbs measures and the theory of dynamical systems*, Tashkent, May 20-21, (2024), 62-63.

21. Haydarov F.H., σ-algebra generated by cylinder sets on Cayley trees // Republican conference: *Current problems and applications of modern mathematics*, Tashkent, March 14-15, (2024) 71-72.

22. Haydarov F.H., Mavlonov I.M., On positive fixed points of operator of Hammerstein type with degenerate kernel and Gibbs measures // International conference: *Gibbs measures and the theory of dynamical systems*, Tashkent, May 20-21, (2024), 64-65.

23. Ilyasova R.A., Haydarov F.H., The existence of Gibbs measures on Cayley trees. // Republican conference: *Current problems and applications of modern mathematics*, Tashkent, March 14-15, (2024), 75-76.

24. Haydarov F.H., Ilyasova R.A., Gradient Gibbs measures of SOS model with alternating magnetism on Cayley tree: 3-Periodic, mirror symmetric boundary law. // International conference: *Actual Problems of Physics, Mathematics and Mechanics*, Bukhara, May 24-25, (2023), 28-30.

25. Haydarov F.H., Ilyasova R.A., Gradient Gibbs measures of SOS model with alternating magnetism on Cayley tree: 2-periodic boundary law. // Republican conference: *Modern problems of analysys*, Karshi, June 2-3, (2023), 41-43.

26. Kucharov R.R., Haydarov F.H., On fixed points of nonlinear integral operator.// Republican conference: *Problems of Modern Topology and Its Applications*, Tashkent, September 11-12, (2018), 71-72.

27. Eshkabilov Yu.Kh, Haydarov F.H., Non-uniqueness of a fixed point of Hammerstein’s integral operator. // International conference: *Operator Algebras and Related Topics*, Tashkent, September 12-14, (2012), 26-28.

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