V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.05/2025.27.12.FM.11.01.M RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

SULAYMONOV ILYOSXO‘JA ABDIRASUL O‘G‘LI

O‘ZGARMAS KOEFFITSIYENTLI SUBDIFFUZIYA TENGLAMALARI UCHUN TO‘G‘RI VA TESKARI MASALALAR

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

TOSHKENT – 2026

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Fizika – matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi avtoreferati mundarijasi

Contents of dissertation abstract of doctor of philosophy (PhD) on physical – mathematical sciences

Оглавление автореферата диссертации доктора философии (PhD) по физико – математическим наукам

**V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI
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Fizika-matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi mavzusi O‘zbekiston Respublikasi Oliy ta’lim, Fan va Innovatsiyalar Vazirligi huzuridagi Oliy attestatsiya komissiyasida № B2025.4.FM/T1392 raqam bilan ro‘yxatga olingan.

Dissertatsiya V.I. Romanovskiy nomidagi Matematika institutida bajarilgan.

Dissertatsiya avtoreferati uch tilda (o‘zbek, ingliz, rus (резюме)) Ilmiy kengash veb-sahifasi (http://kengash.mathinst.uz) va «Ziyonet» ta’lim axborot tarmog‘ida (www.ziyonet.uz) joylashtirilgan.

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Dissertatsiya bilan V.I.Romanovskiy nomidagi Matematika Institutining Axborot-resurs markazida tanishish mumkin (220-raqami bilan ro‘yxatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40.

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KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zaruriyati. So‘nggi o‘n yilliklarda kasr tartibli differensial tenglamalar turli xil anomal jarayonlarni modellashtirishda muhim matematik vosita sifatida keng qo‘llanila boshladi. Shu bois, ko‘plab olimlar bu turdagi tenglamalarni yechishning yangi yondashuvlarini ishlab chiqish va ularning yechim xususiyatlarini o‘rganishga alohida e’tibor qaratmoqdalar. Boshqa tomondan, kasr tartibli differensial tenglamalar uchun nafaqat boshlang‘ich-chegaraviy masalalar, balki koeffitsiyentlarni, manba funksiyasini, chegaraviy shartlarni hamda kasr tartibli hosilaning tartibini aniqlash bilan bog‘liq teskari masalalarni o‘rganish ham katta amaliy ahamiyatga ega. Ushbu dissertatsiya kasr tartibli differensial tenglamalar va tenglamalar sistemasi uchun boshlang‘ich-chegaraviy masalalarni, shuningdek, subdiffuziya tenglamalari uchun manba funksiyasini hamda kasr tartibli hosila tartibini aniqlashga doir teskari masalalarni o‘rganishga bag‘ishlangan.

Xususiy hosilali kasr tartibli tenglamalarni yechishni o‘rganish bilan bir qatorda tenglamaning koeffitsiyentlarini, uning o‘ng tomonini, chegaraviy funksiyani va kasr tartibli hosilaning tartibini aniqlashga oid teskari masalalarni yechishni o‘rganish juda muhim ahamiyatga ega. Teskari masalalarni o‘rganish bizga kasr tartibli tenglama bilan berilgan jarayonlarni o‘rganish, tahlil qilish va boshqarish imkoniyatlarini beradi. Manba funksiyasini aniqlashga oid teskari masalalar mexanika, seysmologiya, tibbiy tomografiya va geofizika kabi sohalardagi amaliy ehtiyojlar bilan bog‘liq. Shuningdek, kasr tartibli hosila tartibini aniqlashga oid teskari masalalar ham turli sohalarda katta amaliy ahamiyatga ega. Bunday masalalar orqali tizim yoki muhitdagi tarqalish jarayonining tabiati, ya’ni u odatiy yoki anomal diffuziya ekanligi aniqlanadi. Shuning uchun hosila tartibini aniqlash masalalari fizika, biologiya, kimyo, gidrologiya, materialshunoslik, hamda moliya matematikasi kabi fanlarda keng qo‘llanilib, murakkab jarayonlarning ichki tuzilishini chuqurroq tushunishga xizmat qiladi. Bunday masalalarning fizik mohiyati va amaliy ahamiyati ularni turli sohalarda o‘rganishga sabab bo‘lmoqda. Shuning uchun, kasr tartibli tenglamalar uchun to‘g‘ri va teskari masalalarning yechimlarini qurish maqsadli ilmiy tadqiqotlardan hisoblanadi.

Mamlakatimizda aniq va tabiiy fanlar, xususan, matematika, fizika, biologiya, geologiya sohalarini rivojlantirishga katta e’tibor qaratilmoqda. Jumladan, kasr tartibli va xususiy hosilali differensial tenglamalar nazariyasini rivojlantirish muhim ustuvor yo‘nalishlardan biri sifatida belgilangan. Chunki bu nazariya mexanika, elektronika, boshqaruv tizimlari, fiziologiya va biologik jarayonlarni tushunishda asosiy rol o‘ynaydi. Yuqorida ta’kidlanganidek, muhim yo‘nalishlar bo‘yicha xalqaro standartlar darajasida ilmiy tadqiqot olib borish, matematika fanining asosiy vazifasi va faoliyat yo‘nalishi etib belgilandi[1].

[1] O‘zbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi «Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash shuningdek, O‘zbekiston Respublikasi Fanlar akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risidagi» gi № PQ-4387-son qarori.

Mamlakatimiz mutaxassislari tomonidan mazkur sohalarda salmoqli natijalar qo‘lga kiritilib, ular yuqoridagi farmon ijrosini ta’minlashda muhim ahamiyat kasb etmoqda. Mamlakatimizda differensial tenglamalar va matematik fizika kabi muhim yo‘nalishlari bo‘yicha xalqaro miqyosda olib borilayotgan tadqiqotlar, fundamental tadqiqotlarning asosiy yo‘nalishi sifatida qaralayotgani quvonarlidir.

Mazkur dissertatsiya ishining predmeti va tadqiqot ob’yekti O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-sonli “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi” haqidagi farmonlarida belgilangan vazifalariga mos keladi, 2017 yil 17-fevraldagi PQ-2789 “Fanlar Akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi Prezident qarori, 2018 yil 27 apreldagi “Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliy joriy qilish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi PQ-3682-sonli qarori va 2019 yil 9 iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab quvvatlash, shuningdek, O‘zbekiston Respublikasi Fanlar Akademiyasining V.I.Romanovskiy nomidagi matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida”gi PQ-4387 sonli Prezident qarori hamda 2020 yil 7 maydagi “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi PQ-4708 sonli Prezident qarori hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga bog‘liqligi. Mazkur dissertatsiya Respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Muammoning o‘rganilganlik darajasi. Kasr tartibli differensial tenglamalar va ular bilan bog‘liq bo‘lgan to‘g‘ri va teskari masalalar zamonaviy matematikaning faol tadqiqot sohasiga aylandi. Bu sohada Sh.O. Alimov, R.R.Ashurov, S.R. Umarov, M. Yamamoto, Z. Li, A.V. Psxu, M. Kirane, M. Ruzhansky, D.Q. Durdiyev, Yu.E. Fayziyev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, X.T. Nguyn, A.S. Malik va boshqa ko‘plab matematik olimlarning salmoqli hissasi bor. Kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarning yechimining mavjudligi va yagonaligi yuqoridagi mualliflarning ishlarida batafsil o‘rganilgan. Kasr tartibli hosila tartibini aniqlashga oid teskari masalalar subdiffuziya tenglamalari uchun ko‘plab tadqiqotchilar tomonidan o‘rganilgan. Kasr tartibni bevosita o‘lchaydigan asbob mavjud bo‘lmagani sababli, bu parametr odatda yechimga oid bilvosita kuzatuvlar asosida tiklanadi. 2019-yilgacha bo‘lgan natijalar keng sharhi Z. Li, Y. Liu va M. Yamamotolarning maqolasida berilgan. Bu maqolaning “Ochiq muammolar” (Open Problems) bo‘limida mualliflar mavjud tadqiqotlar yetarli emasligini ta’kidlab, barcha natijalar uzoq vaqt oralig‘idagi kuzatuvlarga tayanganini aytadilar va ma’lumot faqat bitta vaqt nuqtasida berilgan holatdagi teskari masalalarni o‘rganish muhimligini qayd etadilar.

Ko‘plab ishlarning aksariyati faqat teskari masalaning yagonaligini isbotlaydi.

Oʻxshash yondashuv X. Jing va M. Yamamoto tomonidan qoʻllanilgan, ular uchta parametrni: kasr tartib, potensial koeffitsient va boshlangʻich shartni bir vaqtda aniqlash masalasini koʻrib chiqishgan. Masalaning yechimining mavjudligini taʼminlash uchun koʻplab ishlarda turli qoʻshimcha shartlar taklif qilingan. Masalan, Sh.Alimov va R.Ashurov ishlarida teskari masalaning yagona yechimga ega boʻlishi uchun yetarli shartlardan foydalanishgan. Bundan tashqari, qoʻshimcha umumlashmalar aralash tipdagi tenglamalar, kasr toʻlqin tenglamalari va kasr psevdo-differensial sistemalar uchun ham oʻrganilgan. Shuningdek, R.Ashurov va Yu.Fayziyevlar subdiffuziya tenglamasida kasr tartib va manba funksiyasini bir vaqtda aniqlashga oid teskari masalani tahlil qilganlar. Ikki parametrli teskari masalalar dastlab, 2013-yilda S. Tatar va S. Ulusoy tomonidan bir oʻlchamli holatda oʻrganilgan boʻlib, bu yerda Laplas operatori xos qiymatlarining asimptotik xatti-harakati hal qiluvchi rol oʻynagan. Ularning asosiy natijasi bu parametrlarning yagonaligini aniqlashgan. Keyinchalik, 2020-yilda M.Yamamoto ushbu yagonalik natijalarini yuqori oʻlchamli chegaralangan sohalarga kengaytirdi. Yaqinda G. Li va boshqalar kasr tartibni bitta fazo–vaqt nuqtasidagi qoʻshimcha maʼlumotdan aniqlash masalasini oʻrgandilar.

Kasr tartibli differensial tenglamalar sistemalari boʻyicha tadqiqotlar ham sezilarli darajada rivojlandi. 2019-yilgacha boʻlgan natijalar batafsil koʻrib chiqilishi A. N. Kochubei ishlarida berilgan boʻlib, unda Green funksiyalar va chiziqli sistemalar uchun asosiy baholar olingan. Keyingi yillarda S. Umarov bir qator fundamental natijalarni eʼlon qildi, ular kasr tartibli differensial tenglamalar sistemalari uchun umumiy yechim formulalarini berdi. Shuningdek, S.Umarov ishlarida keltirilgan eng umumiy holatda vektor tartib (0,1) oraligʻidagi ixtiyoriy haqiqiy sonlardan iborat boʻlishi mumkin, bu esa klassik oddiy kasr differensial tenglamalar nazariyasini ham kengaytiradi.

Dissertatsiya tadqiqotining dissertatsiya bajarilgan Oliy taʼlim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bogʻliqligi. Dissertatsiya tadqiqoti V.I. Romanovskiy nomidagi Matematika institutida Oʻzbekiston Respublikasi Oliy taʼlim, fan va innovatsiyalar vazirligining F-FA-2021-424-sonli ilmiy tadqiqot grantining rejalashtirilgan mavzusiga muvofiq amalga oshirildi.

Tadqiqotning maqsadi subdiffuziya tenglamasi va tenglamalar sistemasi uchun boshlangʻich-chegaraviy, subdiffuziya tenglamalarining manba funksiyasini hamda kasr tartibli hosila tartibini topish boʻyicha teskari masalalarning klassik yechimlarini topishdan iborat.

Tadqiqotning vazifalari:

Adamar kasr hosilali subdiffuziya tenglamasi uchun boshlangʻich–chegaraviy masalaning yagona yechimga ega ekanligini isbotlash va manba funksiyasini aniqlash;

Kaputo va Riman–Liuvill kasr hosilali subdiffuziya tenglamalari uchun kasr tartibni aniqlashga oid teskari masalalarning klassik maʼnodagi yagona yechimga ega ekanligini isbotlash;

Kaputo va Riman–Liuvill kasr hosilalari uchun subdiffuziya tenglamalari sistemasining boshlangʻich–chegaraviy masalasining yagona yechimga ega ekanligini isbotlash.

Tadqiqotning obyekti Kaputo, Riman-Liuvill va Adamar kasr hosilali subdiffuziya tenglamalari.

Tadqiqotning predmeti boshlang‘ich-chegaraviy masala, tenglamalarning o‘ng tomonini aniqlashning teskari masalalari va kasr tartibli hosila tartibini aniqlash masalalari.

Tadqiqotning usullari. Tadqiqotda funksional analiz usullari, spektral nazariya va Furye usullari qo‘llaniladi.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

Adamar kasr hosilali subdiffuziya tenglamasi uchun boshlang‘ich–chegaraviy masala va manba funksiyasini aniqlashga oid teskari masala yagona yechimga ega ekanligini isbotlangan;

Kaputo va Riman–Liuvill kasr hosilali subdiffuziya tenglamalari uchun kasr tartibni aniqlashga oid teskari masalalarning klassik ma’nodagi yagona yechimga ega ekanligini isbotlangan;

Kaputo va Riman–Liuvill kasr hosilalari uchun subdiffuziya tenglamalari sistemasining boshlang‘ich–chegaraviy masalasining yagona yechimga ega ekanligini isbotlangan.

Tadqiqotning amaliy natijalari. Ushbu dissertatsiyada olingan natijalar va qo‘llanilgan usullar oliy o‘quv yurtlari magistrantlari va doktorantlari uchun bitiruv kursi sifatida o‘qitilishi mumkin.

Tadqiqot natijalarining ishonchliligi. Natijalar funksional analiz, spektral nazariya va Furye usuli yordamida olingan. Olingan barcha natijalar matematik jihatdan to‘g‘ri.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati shundan iboratki, olingan ilmiy natijalar subdiffuziya tenglamalari uchun boshlang‘ich-chegaraviy va teskari masalalarni yanada chuqurroq tadqiq qilishda qo‘llanilishi mumkin.

Dissertatsiyaning amaliy ahamiyati shundan iboratki, olingan natijalar texnik, fizik va biologik jarayonlarni matematik modellashtirishda qo‘llanilishi mumkin.

Tadqiqot ishlarning joriy qilinishi. Subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalar bo‘yicha dissertatsiya ishida olingan natijalar asosida:

Kaputo kasr hosilali subdiffuziya tenglamalar sistemasi uchun boshlang‘ich chegaraviy masalalaning yechimlaridan 22-11-00064 raqamli «Geosferadagi dinamik jarayonlarni irsiyatni hisobga olgan holda modellashtirish» mavzusidagi xorijiy loyihada subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalarni tasniflashda foydalanilgan (Kosmofizika tadqiqotlari va radio to‘lqinlarining tarqalishi institutining 2025 yil 30 oktyabrdagi № 453-sonli ma'lumotnomasi, Rossiya Federatsiyasi). Ushbu ilmiy natijalar yordamida radonning to‘planish kameralaridagi diffuziya–konveksiya transporti jarayonini tavsiflovchi subdiffuziya modellari qurildi, bu esa radon konsentratsiyasining vaqt bo‘yicha evolyutsiyasini aniqlash va transport parametrlarini baholash imkonini berdi;

elliptik qismi Laplas operatoridan iborat bo‘lgan Kaputo kasr hosilali subdiffuziya tenglamalariga oid teskari masalalar yechimlaridan 125031904191-2 raqamli “Kasr va taqsimlangan tartibli differensial operatorlarga ega tenglamalar va sistemalar uchun chegaraviy masalalar va ularning qo‘llanishlari” mavzusidagi xorijiy loyihada turli fizik va biologik jarayonlarni matematik modellashtirishida qo‘llanilgan (Kabardin-Balkar ilmiy markazining Amaliy matematika va avtomatlashtirish institutining 2025 yil 6 noyabrdagi № 01-13/98-sonli ma'lumotnomasi, Rossiya Federatsiyasi). Ilmiy natijalarning qo‘llanilishi subdiffuziya va diffuziya–to‘lqin tenglamalari uchun teskari masalalarni yechimini topish va ularning turli fizik va biologik jarayonlarni matematik modellashtirishda qo‘llash imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Tadqiqotning asosiy natijalari 3 ta xalqaro va 1 ta respublika ilmiy anjumanlarida muhokama qilindi.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha 11 ta ilmiy ishlar chop etilgan bo‘lib, shundan, 7 ta maqola O‘zbekiston Respublikasi Oliy attestatsiya komissiyasining falsafa doktorlik dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda chop etilgan, shulardan 2 tasi xorijiy va 5 tasi respublika jurnallarida chop etilgan bo‘lib, ulardan 5 tasi SCOPUS ma’lumotlar bazalarida indekslangan va 4 tasi tezisdir.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish, to‘rtta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan iborat. Dissertatsiya hajmi 104 bet.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertasiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, mavzu bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob’ekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertasiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

Dissertatsiyaning **“Dastlabki ma’lumotlar”** deb nomlangan birinchi bobi yordamchi xususiyatga ega bo‘lib, dissertatsiyani o‘qish qulay bo‘lishi uchun yaratilgan. Bu yerda yangi natijalar yo‘q, faqat kerakli ta’riflar va tasdiqlar to‘plangan.

Keling, birinchi bobdagi ba’zi kerakli ma’lumotlarni keltiraylik.

Hilbert fazosida abstrakt operator. $(\cdot,\cdot)$ skalyar ko’paytma va norma $\|\cdot\|$ aniqlangan H separabel Hilbert fazosi bo'lsin. Aytaylik, $A: H \to H$ operator H fazoda aniqlangan bo‘lib, o‘z-o‘ziga qo‘shma, quyidan chegaralangan, musbat aniqlangan bo‘lsin. Faraz qilaylik, A operator kompakt A^{-1} teskari operatorga ega. U holda u to‘la ortonormal $\{v_k\}$ xos vektorlar sistemasiga va ularga mos $\{\lambda_k\}$ musbat xos sonlar to‘plamiga ega bo‘ladi, ya’ni v_k vektorlar va λ_k sonlar $Av_k = \lambda_k v_k$ tenglikni qanoatlantiradi. Xos sonlarni qayta nomerlash yordamida ularni kamaymaydigan qilib nomerlab olamiz, ya’ni $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$.

Keling, H Hilbert fazosida A operatornig darajasi tushunchasini eslatib o‘taylik. τ ixtiyoriy haqiqiy son bo‘lsin. Biz H da A operatorning darajasini quyidagicha kiritamiz:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

bu yerda $h_k=(h,v_k)$ lar $h\in H$ elementning Furye koeffitsiyentlari. Shubhasiz, ushbu operatorning aniqlanish sohasi quyidagi ko‘rinishga ega bo‘ladi:

$$D(A^\tau)=\{h\in H:\sum_{k=1}^{\infty}\lambda_k^{2\tau}|h_k|^2<\infty\}.$$

Laplas operatorining o‘z-o‘ziga qo‘shma kengaytmasi. Biz yuqorida Hilbert fazosida abstrakt o‘z-o‘ziga qo‘shma operatorni ko‘rib chiqdik. Biz qo‘llagan usulda, A operator faqat to‘la ortonormal xos vektorlar sistemasiga ega bo‘lishi talab qilinganligi sababli, M. Ruzhansky va boshqalarning ishida berilgan har qanday elliptik operatorni A operator deb hisoblash mumkin. Masalan, $L_2(\Omega)$, $\Omega\subset\mathbb{R}^N$, ni H Hilbert fazosi. Aytaylik, A operator $L_2(\Omega)$ da $Ag(x)=-\Delta g(x)$ kabi aniqlangan, $D(A)=\{g\in C^2(\overline{\Omega}): g(x)=0,\ x\in\partial\Omega\}$ aniqlanish sohaga ega bo‘lsin. A operatorning $L_2(\Omega)$ dagi o‘z-o‘ziga qo‘shma kengaytmasini $\hat{A}$ bilan belgilaymiz.

Aytaylik, σ ixtiyoriy haqiqiy son bo‘lsin. $L_2(\Omega)$ fazoda $\hat{A}$ operatorning darajasini quyidagicha aniqlaymiz:

$$\hat{A}^\sigma g(x)=\sum_{k=1}^{\infty}\lambda_k^\sigma g_k v_k(x),\quad g_k=(g,v_k),$$

va aniqlanish sohasi quyidagi shaklga ega

$$D(\hat{A}^\sigma)=\{g\in L_2(\Omega):\sum_{k=1}^{\infty}\lambda_k^{2\sigma}|g_k|^2<\infty\}.$$

V.A. Il’inning teoremasi. Tenglamaning elliptik qismi Laplas operatori bo‘lganda, Furye usulida boshlang‘ich-chegaraviy masalalarning yechimlarining mavjudligini isbotlash uchun

$$\sum_{k=1}^{\infty}\lambda_k^\tau|h_k|^2,\quad \tau>\frac{N}{2},\tag{1}$$

ko‘rinishdagi qatorlarning yaqinlashuvchi ekanligini o‘rganish kerak bo‘ladi, bunda h_k lar $h(x)$ funksiyaning Furye koeffitsiyentlari. τ sonining butun qiymatlarida (1) qatorning yaqinlashishi uchun $h(x)$ funksiyaning qaysi $W_2^k(\Omega)$ klassik Sobolev fazolariga tegishligi bo‘lishlik shartlari V.A. Il’inning fundamental ishida ko‘rsatilgan. Ushbu shartlarni keltirish uchun biz $\overset{\circ}{W}{}_2^1(\Omega)$ sinfni kiritamiz. Ω sohada uzluksiz differensiallanuvchi va $\partial\Omega$ chegaraning atrofida nolga teng barcha funksiyalar to‘plamini $W_2^1(\Omega)$ ning normasi bo‘yicha yopilmasini belgilaymiz.

Demak, agar $h(x)$ funksiya quyidagi

$$h(x)\in W_2^{\left[\frac{N}{2}\right]+1}(\Omega)\ \text{ va } h(x),\ \Delta h(x),\ \cdots,\ \Delta^{\left[\frac{N}{4}\right]}h(x)\in\overset{\circ}{W}{}_2^1(\Omega),\tag{2}$$

shartlarni qanoatlantirsa, u holda (1) qator $\tau=\left[\dfrac{N}{2}\right]+1$ bo‘lganda yaqinlashuvchi bo‘ladi.

Dissertatsiyaning asosiy natijalari ikkinchi bobdan boshlanadi. Ushbu bob **“Adamar kasr tartibli hosilali subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalar”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida H separable Hilbert fazosida Adamar kasr hosilali subdiffuziya tenglamasi uchun Koshi masala o‘rganildi.

1-masala. Aytaylik $\alpha \in (0,1)$ bo‘lsin. Quyidagi Koshi masalasini

$$\begin{cases} D_t^{\alpha} u(t) + Au(t) = f(t), \quad 1 < t \le T; \\ \lim\limits_{t\to 1+0} J_t^{1-\alpha} u(t) = \varphi, \end{cases} \tag{3}$$

qanoatlantiruvchi, hamda $D_t^{\alpha} u(x,t) \in C((1,T],H)$, $Au(x,t) \in C((1,T],H)$ xossalarga ega bo‘lgan $(\ln t)^{1-\alpha} u(x,t) \in C([1,T],H)$ funksiyani toping. Biz bu qidirilayotgan $u(x,t)$ funksiyani masalaning yechimi deb ataymiz. Bu yerda $f(t) \in C([1,T],H),$ va φ esa H fazoda berilgan uzluksiz funksiya.

1-teorema. *Aytaylik, $\varphi \in H$ va $f(t) \in C([1,T];D(A^{\varepsilon}))$ bo‘lsin, bu yerda $\varepsilon \in (0,1)$. U holda (3) Koshi masalasining yagona yechimi mavjud va u quyidagi qator ko‘rinishga ega bo‘ladi:*

$$u(x,t) = \sum_{k=1}^{\infty} \left[\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^{\alpha}) + \int_0^t (\ln \eta)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln \eta)^{\alpha}) f_k\left(\frac{t}{\eta}\right) d\eta \right] v_k, \tag{4}$$

va bu qator H da va barcha $t \in (1,T]$ lar uchun absolyut va tekis yaqinlashadi. Bu yerda φ_k, f_k lar mos ravishda, φ, $f(t)$ funksiyalarining Furye koeffitsientlaridir, masalan $\varphi_k = (\varphi, v_k)$.

Ushbu bobning ikkinchi bo‘limida (3) Koshi masalasidagi subdiffuziya tenglamasining o‘ng tomonini topish bo‘yicha teskari masalasi o‘rganildi.

2-masala. Aytaylik $\rho \in (0,1)$ bo‘lsin. (3) Koshi masalasini va quyidagi qo‘shimcha shartni

$$u(\tau) = \Psi, \tag{5}$$

qanoatlantiruvchi, hamda $D_t^{\alpha} u(t), Au(t) \in C((1,T],H)$ xossalarga ega bo‘lgan $(\ln t)^{1-\alpha} u(x,t) \in C([1,T],H)$ va $f \in H$ funksiyalar juftligini toping. Bu yerda ψ funksiya H fazoning elementi, τ esa $(1,T]$ segmentda berilgan ixtiyoriy fiksirlangan nuqta.

Bu teskari masalani yechish uchun, biz (3) Koshi masalasining (4) yechimiga (5) qo‘shimcha shartni qo‘llaymiz va Ψ_k orqali $\Psi_k = (\Psi, v_k)$ funksiyaning Furye koeffetsientlarini belgilaymiz. U holda quyidagi tenglikka ega bo‘lamiz:

$$\varphi_k (ln\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\tau)^{\alpha}) + f_k (ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha}) = \Psi_k.$$

Biz bu tenglikdan f_k koeffitsientlarni quyidagiga teng bo‘ladi:

$$f_k = \frac{\Psi_k}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})} - \frac{\varphi_k (ln\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\tau)^{\alpha})}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})}.$$

2-teorema. *Aytaylik $\varphi, \Psi \in D(A)$ bo‘lsin. U holda (3), (5) teskari masala yagona $\{u(t), f\}$ yechimga ega va u quyidagi qator ko‘rinishiga ega bo‘ladi:*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k (lnt)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (lnt)^{\alpha}) + f_k (lnt)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (lnt)^{\alpha}) \right] v_k, \tag{6}$$

bu yerda

$$f_k = \frac{\Psi_k}{(ln\tau)^\alpha E_{\alpha,\alpha+1}(-\lambda_k(ln\tau)^\alpha)} - \frac{\varphi_k(ln\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(ln\tau)^\alpha}{(ln\tau)^\alpha E_{\alpha,\alpha+1}(-\lambda_k(ln\tau)^\alpha)}$$

va

$$f = \sum_{k=1}^{\infty} f_k v_k.$$

Ushbu bobning ikkinchi bo'limida subdiffuziya tenglamasi uchun orqaga qaytish masalasi o'rganildi.

3-masala. Aytaylik $\alpha \in (0,1)$ bo'lsin. Quyidagicha masalani qaraylik:

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t), \quad 1 < t \le T; \\ u(T) = \Phi, \quad T > 1, \end{cases} \tag{7}$$

bu yerda $f(t) \in C([1,T];H)$ funksiya va Φ element berilganlar. Bu masala *orqaga qaytish masalasi* deb nomlanadi.

Bu masalani yechishdan avval quyidagi muhim teoremalarni keltirib o'tamiz.

3-teorema. *Aytaylik* $\varphi \in H$ *va* $f(t) \in C([1,T];D(A^{1+\varepsilon}))$, *bu yerda* $\varepsilon \in (0,1)$. *U holda* ε *ga bog'liq shunday* C *o'zgarmas son mavjudki quyidagi baho o'rinli bo'ladi*

$$\| u(T) \|_2 \le C(lnT)^{-2(\alpha+1)} \| \varphi \| + C_\varepsilon \max_{t\in[1,T]} \| f(t) \|_{1+\varepsilon}.$$

Bu yerda $\varphi \in H$ element (3) masaladagi boshlang'ich shartda keltirilgan element.

4-teorema. *Aytaylik* $f(t) \equiv 0$ *bo'lsin. U holda ixtiyoriy* $\Phi \in D(A^2)$ *elementlar uchun shunday* $C_1, C_2 > 0$ *o'zgarmas sonlar mavjudki, quyidagi tengsizliklar o'rinli bo'ladi*

$$C_1 \lim_{t\to 1+0} \| J_t^{1-\alpha} u(t) \| \le \| u(T) \|_2 \le C_2 \lim_{t\to 1+0} \| J_t^{1-\alpha} u(t) \|.$$

Endi bu teoremalardan foydalanib orqaga qaytish masalasi uchun quyidagi asosiy natijani keltiramiz.

5-teorema. *Aytaylik* $f(t) \in C([1,T];D(A^{1+\varepsilon}))$ *bo'lsin, bu yerda* $\varepsilon > 0$. *U holda ixtiyoriy* $\Phi \in D(A^2)$ *uchun (7) masala yagona yechimga ega. Bundan tashqari shunday o'zgarmas* $C > 0$ *son topilib, quyidagi baho o'rinli bo'ladi*

$$\lim_{t\to 1+0} \| J_t^{1-\alpha} u(t) \| \le C(\| \Phi \|_2 + \max_{t\in[1,T]} \| f \|_{1+\varepsilon}).$$

Uchinchi bob **"Mittag-Leffler funksiyasining yangi xossalari va vaqt bo'yicha kasr tartibli hosila tartibini aniqlash bo'yicha teskari masalalar"** deb nomlanadi.

Ushbu bobning birinchi bo'limida Mittag-Leffler funksiyasining kichik argumantlarda parametr bo'yicha monotonligi o'rganildi va uning darajani aniqlash bo'yicha teskari masalalarga tadbiqlari keltirildi.

6-teorema. *Aytaylik* $\rho_0 \in (0,1)$ *bo'lsin. U holda ixtiyoriy* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{7}{2}}}\right)\right]$ *lar uchun barcha* $\rho \in [\rho_0, 1]$ *larda ushbu* $E_\rho(-t^\rho)$ *Mittag-*

Leffler funksiyasi monoton o'suvchi bo'ladi.

7-teorema. *Aytaylik* $\rho_0 \in (0,1)$ *bo'lsin. U holda ixtiyoriy* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{13}{6}}}\right)\right]$ *lar uchun barcha* $\rho \in [\rho_0, 1]$ *larda ushbu* $t^{\rho-1}E_{\rho,\rho}(-t^{\rho})$ *Mittag-Leffler funksiyasi monoton kamayuvchi bo'ladi.*

Shu o'rinda muhim eslatmani keltirib o'tamiz. Agar subdiffuziya tenglamasida kasr hosila Kaputo ma'nosida bo'lsa, u holda kasr hosila tartibini aniqlash bo'yicha teskari masalani hal etish uchun biz 6-teoremadan foydalanimiz. Agar subdiffuziya tenglamasida kasr tartibli hosila Riman-Liuvill ma'nosida bo'lsa, u holda kasr hosila tartibini aniqlash bo'yicha teskari masalani hal etish uchun 7-teoremadan foydalanamiz.

Ushbu bobning ikkinchi bo'limida elliptik qismi kasr tartibli Laplas operatori bo'lgan Kaputo kasr hosilali subdiffuziya tenglamasida bir vaqtda Kaputo kasr hosila va Laplas operatorini tartibini aniqlash bo'yicha teskari masala o'rganildi.

4-masala. Aytaylik $\rho, \sigma \in (0,1]$ bo'lsin. Quyidagi masalani

$$\begin{cases} D_t^{\rho}u(x,t) + (-\Delta)^{\sigma}u(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad 0 < t \le T, \\ u(x,0) = \varphi(x), \quad x \in \Omega, \end{cases} \tag{8}$$

va

$$|(u(x,t_0), v_k(x))| = d_0, \quad |(u(x,t_1), v_k(x))| = d_1, \tag{9}$$

qo'shimcha shartni qanoatlantiruvchi, hamda $D_t^{\rho}u(t),\ Au(t) \in C((0,T];H)$ xossalarga ega bo'lgan $u(t) \in AC([0,T];H)$ funksiya va $\rho,\ \sigma$ parametrlarni toping. Bu yerda $\varphi(x) \in L_2(\Omega)$ berilgan funksiya, $t_0 \ne t_1, t_0, t_1 \in (0,T]$ va d_0, d_1 berilgan musbat sonlar. Shuningdek $v_k(x)$ bu Laplas operatorining Dirixle chegaraviy sharti bilan olingandagi k - xos funksiyasi.

Biz (8)-(9) teskari masalani yechish uchun, (8) masalaning yechimidan foydalanamiz. (8) masala yechimni keltirishdan avval quyida muhim bo'lgan tuhusnchani keltirib o'tamiz:

Aytaylik M *fikserlangan son bo'lsin. Berilgan funksiya* $f \in I_M(\Omega)$ *sinfga tegishli deyiladi agar quyidagi ikkita shartni qanoatlantirsa*

1. (2) shartni qanoatlnatirsin,

2. $\|(-\Delta)^{\frac{1}{2}\left([\frac{N}{2}]+1\right)} f\|_{L_2(\Omega)} \le M.$

Endi (8) masala yechimini keltiramiz.

Aytaylik $\varphi(x) \in I_M(\Omega)$ *bo'lsin. U holda (8) masala yagona yechimga ega va u quyidagi qator ko'rinishida bo'ladi:*

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k E_{\rho}(-\lambda_k^{\sigma} t^{\rho}) v_k(x), \tag{10}$$

bu yerda φ_k *lar* $\varphi(x)$ *funksiyaning Furye koeffitsiyentlari.*

(10) yechimni (9) qo’shimcha shartga olib borib biz quyidagi tengliklarga ega bo‘lamiz:

$$\begin{cases} F_1(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_0^\rho) - \dfrac{d_0}{|\varphi_1|} = 0, \\ F_2(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_1^\rho) - \dfrac{d_1}{|\varphi_1|} = 0. \end{cases}$$

Quyidagicha begilashni kiritamiz:

$$F(\rho,\sigma) = \begin{bmatrix} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right) \\ E_\rho\left(-\lambda_k^\sigma t_1^\rho\right) \end{bmatrix},$$

bu yerda $t_1, t_2 > 0$, $t_1 \neq t_2$, $\sigma \in (0,1)$, va $\rho \in [\rho_0, 1]$, $\rho_0 > 0$. Endi quyidagi teorema keltiramiz:

8-teorema. *Aytaylik $\lambda_k > 1$ Laplas operatorining k-xos soni bo‘lsin. U holda shunday musbat $T_1 = T_1(\lambda_k, \rho_0) > 0$ topilib, barcha $t_1 > T_1$ va $t_2 > T_1$ larda (8),(9) ikki parametrli teskari masala yagona $\{u(x,t), \rho, \sigma\}$ yechimga ega bo‘lishi uchun $\left(\dfrac{d_0}{|\varphi_1|}, \dfrac{d_1}{|\varphi_1|}\right)$ lar $F(\rho,\sigma)$ funksiyaning qiymatlar sohasiga tegishli bo‘lishi zarur va yetarli.*

Ushbu bobning uchinchi bo‘limida elliptik qismi kasr tartibli Laplas operatori bo‘lgan Riman-Liuvill kasr hosilali subdiffuziya tenglamasida bir vaqtda Kaputo kasr hosila va Laplas operatorini tartibini aniqlash bo‘yicha teskari masala o‘rganildi.

5-masala. Aytaylik $\rho, \sigma \in (0,1]$ bo‘lsin. Quyidagi masalani

$$\begin{cases} \partial_t^\rho u(x,t) + (-\Delta)^\sigma u(x,t) = 0, \quad x \in \Omega, 0 < t \le T, \\ Bu(x,t) \equiv \dfrac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, 0 < t \le T, \\ \lim\limits_{t \to 0} J_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{11}$$

va

$$|(u(x,t_0), v_1(x))| = d_0, \quad |(u(x,t_1), v_2(x))| = d_1, \tag{12}$$

qo‘shimcha shartni qanoatlantiruvchi, hamda $\partial_t^\rho u(t), Au(t) \in C((0,T];H)$ xossalarga ega bo‘lgan $t^{1-\rho}u(t) \in C([0,T];H)$ funksiya va ρ, σ parametrlarni toping. Bu yerda $\varphi(x) \in L_2(\Omega)$ berilgan funksiya, $t_0 \neq t_1, t_0, t_1 \in (0,T]$ va d_0, d_1 berilgan musbat sonlar. Shuningdek $v_k(x)$ lar bu Laplas operatorining Neyman chegaraviy sharti bilan olingandagi k - xos funksiyalari.

Biz (11)-(12) teskari masalani yechish uchun, (11) masalaning yechimidan foydalanamiz.

Aytaylik $\varphi(x) \in I_M(\Omega)$ bo‘lsin. U holda (11) masala yagona yechimga ega:

$$u(x,t)=\sum_{k=1}^{\infty}\varphi_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^{\rho})v_k(x), \tag{13}$$

bu yerda φ_k lar $\varphi(x)$ funksiyaning Furye koeffitsiyentlari.

(13) yechimni (12) qo'shimcha shartga olib borib biz quyidagi tengliklarga ega bo'lamiz (e'tibor bering $\lambda_1=0$):

$$\begin{cases} |\varphi_1|\dfrac{t_0^{\rho-1}}{\Gamma(\rho)}-d_0=0, \\ |\varphi_1|t_1^{\rho-1}E_{\rho,\rho}(-\lambda_2^{\sigma}t_1^{\rho})-d_1=0. \end{cases}$$

Bu sistemadagi birinchi tenglikdan ρ parametrni aniqlab ikkinchi tenglikka olib borib qo'yamiz va undan σ parametrni topamiz. Bu teskari masala uchun quyidagi asosiy natijani keltiramiz.

9-teorema. *Aytaylik $\lambda_2>1$ Laplas operatorining ikkinchi xos soni bo'lsin. U holda ixtiyoriy $t_0>1$ va $t_1>1$ lar uchun, (11),(12) teskari masala yagona $\{u(x,t),\rho,\sigma\}$ yechimga ega bo'lishi uchun d_0 va d_1 quyidagi tengsizliklarni qanoatlantirishi zarur va yetarli:*

$$\frac{d_0}{|\varphi_1|}<1, \quad \frac{d_1}{|\varphi_1|}<t_1e^{-\lambda_2 t_1}.$$

Ushbu bobning to'rtinchi bo'limida elliptik qismi Laplas operatori bo'lgan Kaputo kasr hosilali subdiffuziya tenglamasida kasr hosila tartibini bitta fazo va vaqt nuqtasida qo'shimcha shart berish orqali aniqlash bo'yicha teskari masala o'rganildi.

6-masala. Aytaylik $\rho\in(0,1)$ bo'lsin. Quyidagi boshlang'ich chegaraviy masalani

$$\begin{cases} D_t^{\rho}u(x,t)-\Delta u(x,t)=0, \quad x\in\Omega, \quad 0<t\le T, \\ u(x,t)|_{\partial\Omega}=0, \\ u(x,0)=\varphi(x), \quad x\in\Omega, \end{cases} \tag{14}$$

va

$$u(x_0,t_0)=d_0, \tag{15}$$

qo'shimcha shartni qanoatlantiruvchi, hamda $D_t^{\rho}u(t),\ Au(t)\in C((0,T];H)$ xossalarga ega bo'lgan $u(t)\in AC([0,T];H)$ funksiya va ρ parametrlarni toping. Bu yerda $\varphi(x)\in L_2(\Omega)$ berilgan funksiya, $x_0\in\Omega$, $t_0\in(0,T]$ va d_0 berilgan sonlar.

Bu teskari masalani yechishda biz (14) masalaning yechimidan foydalanamiz.

Aytaylik $\varphi(x)\in I_M(\Omega)$ bo'lsin. U holda (14) masala yagona yechimga ega va u quyidagi qator ko'rinishida bo'ladi:

$$u(x,t)=\sum_{k=1}^{\infty}\varphi_k E_{\rho}(-\lambda_k t^{\rho})v_k(x), \tag{16}$$

bu yerda φ_k lar $\varphi(x)$ funksiyaning Furye koeffitsiyentlari.

(16) yechimni (15) tenglikka olib borib biz quyidagi tenglikka ega bo'lamiz:

$$G(\rho)=\sum_{k=1}^{\infty}\varphi_k E_\rho(-\lambda_k t_0^\rho)v_k(x_0),$$

Teskari masala yechimi haqidagi asosiy natijani keltirishdan avval quyidagi qatorni tekis yaqinlashuvchiligini ko‘rib o‘tamiz:

$$\sum_{k=1}^{\infty}\varphi_k v_k(x)\frac{d}{d\rho}E_\rho(-\lambda_k t^\rho), \tag{17}$$

Bunda quyidagi Lemma o‘rinli:

1-lemma. *Aytaylik* $\rho_0\in(0,1)$ *va* $t_1>0$ *bo‘lsin. Agar* $\varphi(x)\in I_M(\Omega)$ *bo‘lsa, u holda (17) qator barcha* $x\in\overline{\Omega}, t\in[t_1,T], \rho\in[\rho_0,1]$ *lar bo‘yicha tekis yaqinlashadi.*

Bu lemmadan quyidagi natija kelib chiqadi:

1-natija. *Aytaylik* $\rho_0\in(0,1)$ *va* $t_1>0$ *bo‘lsin. U holda ixtiyoriy* $\varepsilon>0$ *uchun, shunday* $n_0=n_0(\varepsilon,M,\rho_0,t_1)$ *mavjudki, barcha* $x\in\overline{\Omega}, t\in[t_1,T], \rho\in[\rho_0,1]$ *va* $\varphi(x)\in I_M(\Omega)$ *lar uchun quyidagi tengsizlik o‘rinli:*

$$\sum_{k=n_0+1}^{\infty}\left|\varphi_k v_k(x)\frac{d}{d\rho}E_\rho(-\lambda_k t^\rho)\right|<\varepsilon. \tag{18}$$

Bu yerda n_0 son (18) tengsizlik bajariladigan eng kichik son.

Endi asosiy natijani keltirib o‘tamiz.

10-teorema. *Aytaylik* $\varepsilon>0$ *fikserlangan son va* $t_0\in\left(0,\min\left(\frac{1}{2^{1/\rho_0}},\frac{1}{e^{7/2}}\right)\right]$ *bo‘lsin.* $n_0=n_0(\varepsilon,M,\rho_0,t_0)$ *ni 1-natijadagi kabi belgilaymiz. Faraz qilamiz* $x_0\in\Omega$ *uchun* $v_k(x)$ *va* $\varphi(x)\in I_M(\Omega)$ *ning Furye koeffitsiyentlari* φ_k *quyidagi shartlarni qanoatlantirsin:*

1. $\varphi_k\ge 0$ *va* $v_k(x_0)\ge 0$ *barcha* $k=1,2,\ldots,n_0$;
2. φ_{k_0} *va* $v_{k_0}(x_0)$ *biror* $k_0\in\{1,2,\ldots,n_0\}$ *uchun quyidagi tengsizlikni qanoatlantirsin*

$$\varphi_{k_0}v_{k_0}(x_0)>\frac{\varepsilon}{M_{k_0}},$$

bu yerda $M_{k_0}=\frac{d}{d\rho}E_\rho(-\lambda_{k_0}t_0^\rho)$.

U holda chiziqsiz $G(\rho)$ *funksiya barcha* $\rho\in[\rho_0,1]$ *bo‘yicha monoton o‘suvchi bo‘ladi. Shuningdek, (14), (15) teskari masala yagona yechimga ega bo‘lishi uchun quyidagi shart bajarilishi zarur va yetarli*

$$d_0\in[G(\rho_0),G(1)].$$

Dissertatsiyaning **to‘rtinchi bobi “Vektorli kasr tartibli hosilali parabolik tenglamalar sistemasining yechimi”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida Kaputo kasr hosilali subdiffuziya tenglamalar sistemasida kasr hosilalar tartibi ixtiyoriy $\beta_j\in(0,1]$ bo‘lgan hol uchun boshlang‘ich-chegaraviy masala o‘rganildi.

7-masala. Quyidagi Koshi masalasini

$$D_t^{\beta}U(t,x)+\mathbb{A}(D)U(t,x)=H(x,t),\quad t>0,\, x\in\mathbb{R}^n, \tag{19}$$

$$U(0,x)=\Phi(x),\quad x\in\mathbb{R}^n, \tag{20}$$

shuningdek,

$$U(t,x)\in\mathbf{C}([0,T]\times\mathbb{R}^n),\quad D_t^{\beta}U(t,x)\,,\mathbb{A}(D)U(t,x)\in\mathbf{C}((0,T]\times\mathbb{R}^n),$$

shartlarni qanoatlantiruvchi $u_j(t,x)\in L_2^{q_j}(\mathbb{R}^n)$, $t\in(0,T]$, $j=1,\dots,m$ funksiyalarni toping. Bu yerda $H(t,x)$ va $\Phi(x)$ berilgan uzluksiz funksiyalar. $\mathbb{A}(D)$ esa quyidagicha aniqlangan matritsa

$$\mathbb{A}(D)=\begin{bmatrix} A_{1,1}(D) & 0 & \dots & 0\\ A_{2,1}(D) & A_{2,2}(D) & \dots & 0\\ \dots & \dots & \dots & \dots\\ A_{m,1}(D) & A_{m,2}(D) & \dots & A_{m,m}(D)\end{bmatrix}.$$

p^* orqali ushbu $p^*=\max\limits_{1\le j\le m}\{l_{j,j}\}$, ifodani belgilaymiz. Ya'ni diagonalda joylashgan differensial operatorlarning eng katta darajasi. Bu masala uchun quyidagicha asosiy natijani keltiramiz.

11-teorema. *Aytaylik* $\tau>\dfrac{n}{2}$ *va* $\varphi_i(x)\in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$ *va* $h_i(t,x)\in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$ *bo'lsin. U holda (19),(20) Koshi masalasining yechimi mavjud va yagona.*

Ushbu bobning ikkinchi bo'limida Riman-Liuvill kasr hosilali subdiffuziya tenglamalar sistemasida kasr hosilalar tartibi ixtiyoriy $\beta_j\in(0,1]$ bo'lgan hol uchun boshlang'ich-chegaraviy masala o'rganildi.

8-masala. Quyidagi Koshi masalasini

$$\partial_t^{\beta}U(t,x)+\mathbb{A}(D)U(t,x)=H(x,t),\quad t>0,\, x\in\mathbb{R}^n, \tag{21}$$

$$\lim_{t\to0}\partial_t^{\beta-1}U(0,x)=\Phi(x),\quad x\in\mathbb{R}^n, \tag{22}$$

shuningdek,

$$U(t,x)\in\mathbf{C}([0,T]\times\mathbb{R}^n),\partial_t^{\beta}U(t,x)\,\text{and}\,\mathbb{A}(D)U(t,x)\in\mathbf{C}((0,T]\times\mathbb{R}^n),$$

shartlarni qanoatlantiruvchi $u_j(t,x)\in L_2^{q_j}(\mathbb{R}^n)$, $t\in(0,T]$, $j=1,\dots,m$ funksiyalarni toping. Bu yerda $H(t,x)$ va $\Phi(x)$ berilgan uzluksiz funksiyalar. $\mathbb{A}(D)$ esa quyidagicha aniqlangan matritsa

$$\mathbb{A}(D)=\begin{bmatrix} A_{1,1}(D) & 0 & \dots & 0\\ A_{2,1}(D) & A_{2,2}(D) & \dots & 0\\ \dots & \dots & \dots & \dots\\ A_{m,1}(D) & A_{m,2}(D) & \dots & A_{m,m}(D)\end{bmatrix}.$$

p^* orqali ushbu $p^*=\max\limits_{1\le j\le m}\{l_{j,j}\}$, ifodani belgilaymiz. Ya'ni diagonalda joylashgan differensial operatorlarning eng katta darajasi. Bu masala uchun quyidagicha asosiy natijani keltiramiz.

11-teorema. *Aytaylik* $\tau > \dfrac{n}{2}$ *va* $\varphi_i(x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\ldots,m$ *va* $h_i(t,x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\ldots,m$ *bo'lsin. U holda (21),(22) Koshi masalasining yechimi mavjud va yagona.*

XULOSA

Ushbu dissertatsiya kasr tartibli subdiffuziya tenglamalari uchun to'g'ri va teskari masalalarni o'rganishga bag'ishlangan. Ushbu masalalarni yechishning asosiy usuli klassik Furye usulidir.

Tadqiqotning asosiy natijalari quyidagilardan iborat:

1. H separable Gilbert fazosida Adamar kasr tartibli subdiffuziya tenglamasi uchun boshlang'ich chegaraviy masala hamda manba funksiyasini aniqlash bo'yicha teskari masala o'rganildi. Yechim Furye usuli orqali qurildi hamda yechimning mavjudligi va yagonaligi isbotlandi.

2. H separable Gilbert fazosida Adamar kasr tartibli subdiffuziya tenglamasi uchun orqaga qaytish masalasi o'rganildi. Bu masala yechimining mavjudligi va yagonaligi isbotlandi hamda koersiv baholar olindi.

3. Elliptik qismi kasr tartibli Laplas operatori bo'lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun N o'lchovli sohada kasr tartibli hosila tartibini hamda Laplas operatori tartibini aniqlashga doir teskari masala o'rganilgan. Teskari masala yechimining mavjudligi va yagonaligi isbotlangan.

4. Elliptik qismi kasr tartibli Laplas operatori bo'lgan Riman-Liuvill kasr hosilali subdiffuziya tenglamasi uchun N o'lchovli sohada kasr tartibli hosila tartibini hamda Laplas operatori tartibini aniqlashga doir teskari masala o'rganilgan. Teskari masala yechimining mavjudligi va yagonaligi isbotlangan.

5. Elliptik qismi Laplas operatori bo'lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun N o'lchovli sohada kasr tartibli hosila tartibini bitta nuqtada qo'shimcha shart berish orqali aniqlashga doir teskari masala o'rganilgan. Teskari masala yechimining mavjudligi va yagonaligi isbotlangan.

6. Kaputo kasr hosilali subdiffuziya tenglamalar sistemasida kasr hosilalar tartibi ixtiyoriy $\beta_j \in (0,1]$ bo'lgan hol uchun boshlang'ich-chegaraviy masala o'rganildi. Bunda yechim Furye va Laplas almashtirishlari orqali qurildi hamda yechimning mavjudlik va yagonaligi haqidagi teorema isbotlandi.

7. Riman-Liuvill kasr hosilali subdiffuziya tenglamalar sistemasida kasr hosilalar tartibi ixtiyoriy $\beta_j \in (0,1]$ bo'lgan hol uchun boshlang'ich-chegaraviy masala o'rganildi. Bunda yechim Furye va Laplas almashtirishlari orqali qurildi hamda yechimning mavjudlik va yagonaligi haqidagi teorema isbotlandi.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.05/2025.27.12.FM.11.01.M AT V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

INSTITUTE OF MATHEMATICS

SULAYMONOV ILYOSKHUJA ABDIRASUL OGLI

FORWARD AND INVERSE PROBLEMS FOR SUBDIFFUSION EQUATIONS WITH CONSTANT COEFFICIENTS

01.01.02 – Differential equations and mathematical physics

ABSTRACT OF DISSERTATION OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

TASHKENT– 2026

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INTRODUCTION

Actuality and demand of the theme of the dissertation. In recent decades, fractional-order differential equations have been increasingly used as an important mathematical tool for modeling various anomalous processes. For this reason, many researchers have focused on developing new approaches to solving such equations and studying the qualitative properties of their solutions. On the other hand, for fractional-order differential equations, not only initial–boundary value problems but also inverse problems related to the determination of coefficients, source functions, boundary conditions, and the order of the fractional derivative are of great practical importance. This dissertation is devoted to the study of initial–boundary value problems for fractional-order differential equations and systems of equations, as well as inverse problems concerning the determination of the source function and the order of the fractional derivative for subdiffusion equations.

Along with studying methods for solving fractional-order partial differential equations, it is of significant importance to investigate inverse problems aimed at identifying the coefficients of the equation, its right-hand side, boundary functions, and the order of the fractional derivative. The study of inverse problems enables us to analyze, interpret, and control processes described by fractional-order equations. Inverse problems for determining source functions are closely related to practical needs in fields such as mechanics, seismology, medical tomography, and geophysics.Furthermore, inverse problems associated with identifying the order of the fractional derivative also have considerable practical significance in various applications. Such problems make it possible to determine the nature of the transport process in a system or medium, namely whether it corresponds to normal or anomalous diffusion. Therefore, problems of identifying the derivative order are widely applied in physics, biology, chemistry, hydrology, materials science, and financial mathematics, and they contribute to a deeper understanding of the internal structure of complex processes. The physical essence and practical relevance of these problems motivate their investigation across different scientific domains. Consequently, constructing solutions to direct and inverse problems for fractional-order equations constitutes a focused and important area of scientific research.

In our country, great importance is placed on the advancement of disciplines within the exact and natural sciences, such as mathematics, physics, biology, and geology. Particularly, fractional and partial differential equation theory development has been prioritized due to its essential role in understanding various phenomena in mechanics, electronics, control systems, physiology, and biological processes. Conducting research at the international level in these critical areas has been identified as a key priority in fundamental research[1]. Currently, progress in the theory of fractional and partial differential equations is pivotal to the successful implementation of this directive. Researchers in our country have made significant contributions to

[1] Decree of President of the Republic of Uzbekistan, dated July 9, 2019, "On measures to further develop mathematics education and mathematical sciences with state support, as well as to fundamentally improve the activities of the V.I. Romanovskiy Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan" № PQ-4387.

these fields, contributing significantly to the advancement of the decree. It is highly encouraging that the study of differential equations and mathematical physics, particularly at an international level, is recognized as a primary focus of fundamental scientific research in our nation.

The subject and object of research of this thesis are in line with tasks identified in the Decrees of the President of the Republic of Uzbekistan UP-4947 of February 7, 2017 "On the strategy of action for the further development Of the Republic of Uzbekistan", PP-2789 dated February 17, 2017 "On measures to further improve of the activities of the Academy of Sciences, organization, management and financing of research activities", PP-3682 from April 27, 2018 "On measures to further improve the system of practical implementation of innovative ideas, technologies and projects" and PP-4387 from July 9, 2019 "On measures to further development of mathematical education and science, total improvement of the activity of the Uzbekistan Academy of Sciences V.I. Romanovskiy Institute of Mathematics" and also PP-4708 from May 7, 2020 "On measures to improve the quality of education and research in mathematics" as well as in other regulations related to basic science.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, «Mathematics, Mechanics and Computer Science».

The degree of scrutiny of the problem. Fractional partial differential equations and their associated inverse problems have become an area of active research in contemporary mathematics. Initial-boundary value problems for fractional differential equations have been extensively studied by various authors. Notable contributions in this field have been made by researchers such as Sh.O. Alimov, R.R. Ashurov, S.R. Umarov, M. Yamamoto, J. Janno, Z. Li, A.V. Pskhu, M. Kirane, M. Ruzhansky, G.Li., D.K. Durdiev, Yu.E. Fayziev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, H.T. Nguyn, A.S. Malik, and others. Inverse problems for determining the order of fractional derivatives in subdiffusion equations have been studied by many researchers. Since no direct measurement tool exists for identifying the fractional order, this parameter is usually recovered from indirect observations of the solution. A broad survey of results up to 2019 is provided in the review article by Z. Li, Y. Liu, and M. Yamamoto. In the section Open Problems, the authors emphasized that existing studies are still insufficient, as all results rely on observations over a long time interval, and they highlighted the importance of investigating inverse problems based on data at a fixed time.

Most works establish only the uniqueness of the inverse problem, with the exception of J. Janno, who also proved existence. A similar approach is used in the recent paper by X. Jing and M. Yamamoto, where the authors considered the simultaneous recovery of three parameters: the fractional order, the potential coefficient, and the initial data. To guarantee solvability, various additional conditions were proposed in many works. For example, it was shown that the inverse problem admits a unique solution under the condition, where is the first eigenfunction of the Laplace operator, is sufficiently large, and is a given constant. Analogous results were obtained by Sh.Alimov and R.Ashurov with different additional conditions. Further generalizations were studied works some for equations of mixed type, fractional wave equations, and systems of fractional pseudo-differential equations. The authors analyzed the inverse problem of determining both the fractional order and the source term of the subdiffusion equation. Two-parameter inverse problems were investigated by S. Tatar and S. Ulusoy in the one-dimensional case, where the asymptotic behavior of the Laplacian eigenvalues played a crucial role. Their main result is the uniqueness of the determination of the parameters. Later, M. Yamamoto extended uniqueness results to higher-dimensional bounded domains. Recently, G. Li et al. studied the determination of the fractional order from additional data at a single space–time point. In this dissertation, we study the existence and uniqueness of a two-parameter inverse problem and a similar inverse problem based on observations at one space–time point.

The study of systems of fractional partial differential equations has also seen substantial progress. A detailed exposition up to 2019 is given in the works of A. N. Kochubei, where Green's functions and key estimates for linear systems were established. More recently, S. Umarov published a series of fundamental results providing general solution representations for systems of fractional differential equations. In the most general setting presented the vector order can consist of arbitrary real numbers in the interval (0,1), extending even the classical theory of ordinary fractional differential equations. In this dissertation, we apply these ideas to systems of subdiffusion equations and inverse problems involving two unknown parameters and pointwise observational data.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out. The dissertation research is done in accordance with the planned theme of the scientific research grant no. F-FA-2021-424 Ministry of higher education, science and innovations of the Republic of Uzbekistan, at the Institute of Mathematics after named V.I. Romanovskiy.

The aim of research work is to prove unique solvability in the classical sense of initial-boundary value and inverse problems for subdiffusion equations.

Research problems:

to prove the unique solvability of the initial–boundary value problem and to determine the source function for the subdiffusion equation with the Hadamard fractional derivative;

to prove unique solvability in the classical sense of the inverse problems on identifying the order of fractional derivative for subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives;

to prove unique solvability of the initial–boundary value problem for the system of subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives.

The research object. The subdiffusion equations with the Caputo, Riemann-Liouville and Hadamard fractional derivatives.

The research subject. The initial-boundary value problem and the inverse problems on determining the right-hand side and the order of fractional derivative of the equations.

Research methods. In the research the methods of functional analysis, spectral theory and the Fourier methods are used.

Scientific novelty of the research work consists of the following:

it is proved the unique solvability of the initial–boundary value problem and determined the source function for the subdiffusion equation with the Hadamard fractional derivative;

it is proved unique solvability in the classical sense of the inverse problems on identifying the order of fractional derivative for subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives;

it is proved unique solvability of the initial–boundary value problem for the system of subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives.

Practical results of the research. The results and methodologies presented in this dissertation can be incorporated into graduate-level courses designed for master's and doctoral students at higher education institutions.

The reliability of the results of the study. The results were derived using the techniques from functional analysis, spectral theory, and the Fourier method. All obtained results are mathematically correct.

Scientific and practical significance of the research results. The scientific significance of the research results is that they can be used for further study of the initial-boundary value and inverse problems for subdiffusion equations. The practical significance of the results of the dissertation is that its results can be used in mathematical modeling of technical, physical, and biological processes.

Implementation of the research results. Based on the results obtained on the problem of forward and inverse problems for subdiffusion equations:

the solutions to the initial-boundary problems for the system of subdiffusion equations with Caputo fractional derivative were used in the international project 22-11-00064 on the topic "Modeling of dynamic processes in geospheres considering heredity", in the study of the forward and inverse problems for the subdiffusion equations (Reference from Institute of Cosmophysical Research and Radio Wave Propagation dated October 30, 2025, № 453, Russian Federation). Based on the obtained scientific results, subdiffusion models describing diffusion–convection transport processes of radon in accumulation chambers were constructed, which made it possible to determine the temporal evolution of radon concentration and to estimate transport parameters;

the solutions to the inverse problems for the subdiffusion equation with Caputo fractional derivative when the elliptic part of the equation is the Laplace operator were used in the mathematical modeling of various physical and biological processes in the international project 125031904191-2 on the topic "Boundary value problems for equations and systems with fractional and distributed-order differential operators and their applications" (Reference from the Institute of Applied Mathematics and Automation of the Kabardino-Balkarian Scientific Center of the Russian Academy of Sciences dated November 6, 2025, № 01-13/98). The application of these scientific results enabled the solution of inverse problems for subdiffusion and diffusion–wave equations and their use in the mathematical modeling of various physical and biological processes.

Approbation of the research results. The main results of the research have been discussed at 3 international and 1 national scientific conferences.

Publications of the research results. On the topic of the dissertation 11 research papers have been published in the scientific journals, 7 of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the PhD thesis and 2 and 5 of them were published in international and national mathematical journals, respectively. 5 of them are included the SCOPUS information database, and 4 theses.

The structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, conclusion, and bibliography. The total volume of the dissertation is 104 pages.

MAIN CONTENT OF THE DISSERTATION

The introduction substantiates the actuality and demand of the theme of the dissertation, determines the correspondence of the study to priority areas of development of science and technology, provides an overview of foreign scientific research on the dissertation topic and the degree of study of the problem, formulates goals and objectives, identifies the object and subject of the study, outlines the scientific novelty and practical results of the study, the theoretical and practical significance of the results obtained is disclosed, information is given on the implementation of the research results, on published works and information on the structure of the dissertation.

The first chapter of the dissertation, titled **"Preliminary information"**, is auxiliary in nature and it was created for the convenience of reading the dissertation. There are no new results here, and only the necessary definitions and assertions are collected.

Abstract operator in Hilbert space. Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Suppose that the operator $A: H \to H$ is defined in H, is self-adjoint, bounded from below, and positive definite. Assume that an operator A has a compact inverse operator A^{-1}. Then it has a complete system of orthonormal eigenvectors $\{v_k\}$ and a set of corresponding positive eigenvalues $\{\lambda_k\}$, i.e. the vectors $\{v_k\}$ and values $\{\lambda_k\}$ satisfy the following equality:

$$Av_k = \lambda_k v_k.$$

It is assumed that the eigenvalues are ordered such that $0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$.

In order to formulate the main results of this dissertation, we introduce the Hilbert space of "smooth" functions related to the degree of operator A.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where h_k is the Fourier coefficient of the element $h: h_k = (h, v_k)$. Obviously, the domain of definition of this operator has the form

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

Self-adjoint extension of the Laplace operator. Note that we considered an abstract self-adjoint operator A in a Hilbert space H. Since the operator A is only required to have a complete orthonormal system of eigenvectors, then as A one can consider any of the elliptic operators given in the work by Ruzhansky et al. For example, let us take $L_2(\Omega)$, $\Omega\subset\mathbb{R}^N$, as the Hilbert space H. If A stands for the operator acting in $L_2(\Omega)$ as $Ag(x)=-\Delta g(x)$ with the domain of definition $D(A)=\{g\in C^2(\Omega): g(x)=0,\ x\in\partial\Omega\}$, then $\hat{A}\equiv\hat{A}^1$ is a self-adjoint extension of A in $L_2(\Omega)$.

Let σ be an arbitrary real number. Consider an operator $\hat{A}^\sigma$ acting in $L_2(\Omega)$ as:

$$\hat{A}^\sigma g(x)=\sum_{k=1}^{\infty}\lambda_k^\sigma g_k v_k(x),\quad g_k=(g,v_k),$$

with the domain of definition

$$D(\hat{A}^\sigma)=\{g\in L_2(\Omega):\sum_{k=1}^{\infty}\lambda_k^{2\sigma}|g_k|^2<\infty\}.$$

Theorem of V.A. Ilyin. In order to prove the existence of solutions of initial-boundary value problems by the Fourier method when the elliptic part of the equation is Laplace operator Δ, it is necessary to study the convergence of the following series:

$$\sum_{k=1}^{\infty}\lambda_k^\tau|h_k|^2,\quad \tau>\frac{N}{2},\tag{1}$$

where h_k are the Fourier coefficients of function $h(x)$. In the case of integers τ, in the fundamental paper by V.A. Il'in, conditions are obtained for the convergence of such series in terms of the membership of function $h(x)$ in the classical Sobolev spaces $W_2^k(\Omega)$, $\Omega\subset R^n$. To formulate these conditions, we introduce the class $\overset{\circ}{W}{}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$ norm of the set of all functions that are continuously differentiable in Ω and vanish near the boundary of Ω.

The theorem of V.A. Ilyin states that, if function $h(x)$ satisfies the conditions

$$h(x)\in W_2^{\left[\frac{N}{2}\right]+1}(\Omega)\quad and\quad h(x),\ \Delta h(x),....,\ \Delta^{\left[\frac{N}{4}\right]}h(x)\in\overset{\circ}{W}{}_2^1(\Omega),\tag{2}$$

then the number series (1) with $\tau=\left[\frac{N}{2}\right]+1$ converges. Here $[a]$ denotes the integer part of the number a.

The main results of the dissertation begin with the second chapter, titled **"Forward and inverse problems for fractional differential equations with Hadamard fractional derivative".**

The first paragraph of this chapter studied the Cauchy problem for the subdiffusion equation with the Hadamard fractional derivative in H separable Hilbert space.

Problem 1. Let $\alpha\in(0,1)$. Find a function $(\ln t)^{1-\alpha}u(x,t)\in C([1,T],H)$ with properties $D_t^\alpha u(x,t)\in C((1,T],H)$, $Au(x,t)\in C((1,T],H)$, that satisfies the following Cauchy problem

$$\begin{cases} D_t^{\alpha} u(t) + Au(t) = f(t), \quad 1 < t \le T; \\ \lim\limits_{t\to 1+0} J_t^{1-\alpha} u(t) = \varphi, \end{cases} \tag{3}$$

where $f(t) \in C([1,T], H)$, $\varphi \in H$ are given continuous functions.

Theorem 1. *Let $f(t) \in C([1,T]; D(A^{\varepsilon}))$ with some $\varepsilon \in (0,1)$ and $\varphi \in H$. Then the problem (3) has a unique solution*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k (lnt)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (lnt)^{\alpha}) + \int_1^t (ln\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\xi)^{\alpha}) f_k(\frac{t}{\xi}) \frac{d\xi}{\xi} \right] v_k. \tag{4}$$

where f_k, φ_k are the Fourier coefficients of the functions $f(t)$ and φ respectively.

In the second section of this chapter studied the inverse problem of finding the right-hand side of the subdiffusion equation in the Cauchy problem (3).

Problem 2. Let $\alpha \in (0,1)$ be a fixed number. Find a pair of functions $(\ln t)^{1-\alpha} u(x,t) \in C([1,T], H)$ and $f \in H$ with properties $D_t^{\alpha} u(t), Au(t) \in C((1,T], H)$, that satisfies the Cauchy problem (3) and the following the over-determination condition:

$$u(\tau) = \Psi, \tag{5}$$

where Ψ is a given element in H and τ is a given fixed point of the segment $(1,T]$.

We apply the additional condition (5) to solution (4) of the Cauchy problem (3) and denote by Ψ_k the Fourier coefficients of function $\Psi(x): \Psi_k = (\Psi, v_k)$. Then

$$\varphi_k (ln\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\tau)^{\alpha}) + f_k (ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha}) = \Psi_k.$$

From here, to find f_k, we obtain the following equality

$$f_k = \frac{\Psi_k}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})} - \frac{\varphi_k (ln\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\tau)^{\alpha})}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})}.$$

Theorem 2. *Let $\varphi, \Psi \in D(A)$. Then the inverse problem (3), (5) has a unique solution $\{u(t), f\}$ and this solution has the following form*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k (lnt)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (lnt)^{\alpha}) + f_k (lnt)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (lnt)^{\alpha}) \right] v_k, \tag{6}$$

where

$$f_k = \frac{\Psi_k}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})} - \frac{\varphi_k (ln\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (ln\tau)^{\alpha}}{(ln\tau)^{\alpha} E_{\alpha,\alpha+1}(-\lambda_k (ln\tau)^{\alpha})}$$

and

$$f = \sum_{k=1}^{\infty} f_k v_k.$$

In the third section of this chapter is studied the backward problem for the subdiffusion equation.

Problem 3. Let $\alpha \in (0,1)$. Consider the following problem:

$$\begin{cases} D_t^{\alpha} u(t) + Au(t) = f(t), \quad 1 < t \le T; \\ u(T) = \Phi, \quad T > 1, \end{cases} \tag{7}$$

where function $f(t) \in C([1,T];H)$ element Φ are given. This is called the *backward problem*.

Before solving this problem, let us state the following important theorems.

Theorem 3. *Let* $\varphi \in H$ *and* $f(t) \in C([1,T];D(A^{1+\varepsilon}))$. *Then there exists a constant* C, *depending on* ε *such that*

$$\| u(T) \|_2 \leq C(lnT)^{-2(\alpha+1)} \| \varphi \| + C_\varepsilon \max_{t \in [1,T]} \| f(t) \|_{1+\varepsilon}.$$

Here, $\varphi \in H$ is the element given in the initial condition in problem (3).

Theorem 4. *Let* $f(t) \equiv 0$. *Then for any* $\Phi \in D(A^2)$ *there exist constants* $C_1, C_2 > 0$, *such that*

$$C_1 \lim_{t \to 1+0} \| J_t^{1-\alpha} u(t) \| \leq \| u(T) \|_2 \leq C_2 \lim_{t \to 1+0} \| J_t^{1-\alpha} u(t) \|.$$

Now, using these theorems, we derive the following main result for the backward problem.

Theorem 5. *Let* $f(t) \in C([1,T];D(A^{1+\varepsilon}))$ *with some* $\varepsilon > 0$. *Then for any* $\Phi \in D(A^2)$ *problem (7) has a unique solution. Moreover there exists a constant* $C > 0$, *such that*

$$\lim_{t \to 1+0} \| J_t^{1-\alpha} u(t) \| \leq C(\| \Phi \|_2 + \max_{t \in [1,T]} \| f \|_{1+\varepsilon}).$$

The third chapter of the dissertation is titled **"new properties of the Mittag-Leffler function and inverse problems for determining the order of the time-fractional derivative"**.

In the first section of this chapter, In this section is devoted to new properties of the Mittag-Leffler function, particularly focusing on its monotonicity for small arguments. These properties are then applied to the solution of inverse problems, specifically for determining the order of the fractional derivative in subdiffusion equations.

Theorem 6. *Let* $\rho_0 \in (0,1)$. *Then, for any* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{7}{2}}}\right)\right]$, *the Mittag-Leffler function* $E_\rho(-t^\rho)$ *is monotonically increasing in* $\rho \in [\rho_0, 1]$.

Theorem 7. *Let* $\rho_0 \in (0,1)$. *Then, for any* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{13}{6}}}\right)\right]$, *the function* $t^{\rho-1} E_{\rho,\rho}(-t^\rho)$ *is monotonically decreasing in* $\rho \in [\rho_0, 1]$.

Here we make an important remark. If the fractional derivative in the subdiffusion equation is in the Caputo sense, then we use Theorem 6 to solve the inverse problem of determining the order of the fractional derivative. If the fractional derivative in the subdiffusion equation is in the Riemann-Liouville sense, then we use Theorem 7 to solve the inverse problem of determining the order of the fractional derivative.

In the second section of this chapter, the inverse problem of simultaneously determining the order of the Caputo fractional derivative and the Laplace operator in the Caputo fractional derivative subdiffusion equation, whose elliptic part is a fractional-order Laplace operator, is studied.

Problem 4. Let $\rho, \sigma \in (0,1]$ be a fixed numbers. Find a function $u(t) \in AC([0,T];H)$ and parameters ρ, σ with properties $D_t^\rho u(t),\ Au(t) \in C((0,T];H)$, that satisfy the following initial-boundary value problem

$$\begin{cases} D_t^\rho u(x,t) + (-\Delta)^\sigma u(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad 0 < t \le T, \\ u(x,0) = \varphi(x), \quad x \in \Omega, \end{cases} \tag{8}$$

and the conditions

$$|(u(x,t_0), v_k(x))| = d_0, \quad |(u(x,t_1), v_k(x))| = d_1, \tag{9}$$

where $\varphi(x) \in L_2(\Omega)$ is a given function, $t_0 \ne t_1, t_0, t_1 \in (0,T]$ and d_0, d_1 are given positive numbers. Here, the symbol $v_k(x)$ denotes the k-th eigenfunction of the Laplace operator with the Dirichlet boundary condition.

We use the solution of problem (8) to solve the inverse problem (8)-(9). Before presenting the solution to problem (8), we introduce the following important concept:

Let M be a fixed number. We will say that $f \in I_M(\Omega)$ if

1. *satisfy conditions (2),*
2. $\| A^{\frac{1}{2}\left([\frac{N}{2}]+1\right)} f \|_{L_2(\Omega)} \le M$.

Now we present the solution to problem (8).

Let function $\varphi(x) \in I_M(\Omega)$. Then problem (8) has a unique solution:

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k E_\rho(-\lambda_k^\sigma t^\rho) v_k(x), \tag{10}$$

where φ_k are the Fourier coefficients of function $\varphi(x)$.

Substituting the solution (10) into the additional condition (9), we obtain the following equations:

$$\begin{cases} F_1(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_0^\rho) - \dfrac{d_0}{|\varphi_1|} = 0, \\ F_2(\rho,\sigma) = E_\rho(-\lambda_k^\sigma t_1^\rho) - \dfrac{d_1}{|\varphi_1|} = 0. \end{cases}$$

We introduce the following notation:

$$F(\rho,\sigma) = \begin{bmatrix} E_\rho\left(-\lambda_k^\sigma t_0^\rho\right) \\ E_\rho\left(-\lambda_k^\sigma t_1^\rho\right) \end{bmatrix},$$

where $t_1, t_2 > 0$, $t_1 \ne t_2$, $\sigma \in (0,1)$, and $\rho \in [\rho_0, 1]$ with $\rho_0 > 0$. Now, we present the following theorem:

Theorem 8. *Let $\lambda_k > 1$ be the k-th eigenvalue of the Laplace operator. Then, there exists a positive number $T_1 = T_1(\lambda_k, \rho_0) > 0$ such that for all $t_1 > T_1$ and $t_2 > T_1$, the two-parameter inverse problem (8),(9) has a unique solution $\{u(x,t), \rho, \sigma\}$, if and only if $\left(\frac{d_0}{|\varphi_1|}, \frac{d_1}{|\varphi_1|}\right)$ belong to the range of the function $F(\rho,\sigma)$.*

In the third section of this chapter, the inverse problem of simultaneously determining the order of the Riemann-Liouville fractional derivative and the Laplace operator in the Riemann-Liouville fractional derivative subdiffusion equation, whose elliptic part is a fractional-order Laplace operator, is studied.

Problem 5. Let $\rho, \sigma \in (0,1]$ be a fixed numbers. Find a function $t^{1-\rho}u(t) \in C([0,T];H)$ and parameters ρ, σ with properties $\partial_t^\rho u(t), Au(t) \in C((0,T];H)$, that satisfy the following initial-boundary value problem

$$\begin{cases} \partial_t^\rho u(x,t) + (-\Delta)^\sigma u(x,t) = 0, \quad x \in \Omega, 0 < t \le T, \\ Bu(x,t) \equiv \dfrac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, 0 < t \le T, \\ \lim\limits_{t \to 0} J_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{11}$$

and the conditions

$$|(u(x,t_0), v_1(x))| = d_0, \quad |(u(x,t_1), v_2(x))| = d_1, \tag{12}$$

where $\varphi(x) \in L_2(\Omega)$ is a given function, $t_0 \ne t_1, t_0, t_1 \in (0,T]$ and d_0, d_1 are given positive numbers. Here, the symbol $v_k(x)$ denotes the k-th eigenfunction of the Laplace operator with the Neumann boundary condition.

We use the solution of problem (11) to solve the inverse problem (11)-(12).

Let function $\varphi(x) \in I_M(\Omega)$. Then problem (11) has a unique solution:

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x), \tag{13}$$

where φ_k are the Fourier coefficients of function $\varphi(x)$.

Substituting the solution (13) into the additional condition (12), we obtain the following equations (note $\lambda_1 = 0$):

$$\begin{cases} |\varphi_1| \dfrac{t_0^{\rho-1}}{\Gamma(\rho)} - d_0 = 0, \\ |\varphi_1| t_1^{\rho-1} E_{\rho,\rho}(-\lambda_2^\sigma t_1^\rho) - d_1 = 0. \end{cases}$$

We determine the parameter ρ from the first equation in this system and take it to the second equation, and from it we find the parameter σ. We give the following main result for this inverse problem.

Theorem 9. *Let $\lambda_2 > 1$ be the second eigenvalue of the Laplacian operator. Then for any $t_0 > 1$ and $t_1 > 1$, there is a unique solution $\{u(x,t), \rho, \sigma\}$ of the inverse* *problem (11), (12), if and only if d_0 and d_1 satisfy the following inequalities:*

$$\frac{d_0}{|\varphi_1|}<1, \quad \frac{d_1}{|\varphi_1|}<t_1 e^{-\lambda_2 t_1}.$$

In the fourth section of this chapter, the inverse problem of determining the order of the fractional derivative in the Caputo fractional derivative subdiffusion equation, whose elliptic part is the Laplace operator, by giving an additional condition at a single space-time point, is studied.

Problem 6. Let $\rho\in(0,1)$ be a fixed number. Find a function $u(t)\in AC([0,T];H)$ and parameter ρ with properties $D_t^\rho u(t),\ Au(t)\in C((0,T];H)$, that satisfy the following initial-boundary value problem

$$\begin{cases} D_t^\rho u(x,t)-\Delta u(x,t)=0, \quad x\in\Omega, \quad 0<t\le T, \\ u(x,t)|_{\partial\Omega}=0, \\ u(x,0)=\varphi(x), \quad x\in\Omega, \end{cases} \tag{14}$$

and the conditions

$$u(x_0,t_0)=d_0, \tag{15}$$

where $\varphi(x)\in L_2(\Omega)$ is a given function, $x_0\in\Omega,\ t_0\in(0,T]$ and d_0 are given numbers.

To solve this inverse problem, we use the solution to problem (14).

Let function $\varphi(x)\in I_M(\Omega)$. Then problem (14) has a unique solution:

$$u(x,t)=\sum_{k=1}^\infty \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x), \tag{15}$$

where φ_k are the Fourier coefficients of function $\varphi(x)$.

Substituting the solution (16) into equation (15), we obtain the following equation:

$$G(\rho)=\sum_{k=1}^\infty \varphi_k E_\rho(-\lambda_k t_0^\rho) v_k(x_0),$$

Before moving on to the inverse problem, let us examine the uniform convergence of the following series:

$$\sum_{k=1}^\infty \varphi_k v_k(x)\frac{d}{d\rho}E_\rho(-\lambda_k t^\rho), \tag{17}$$

where $\varphi_k=(\varphi,v_k)$. One has

Lemma 1. *Let $\rho_0\in(0,1)$ and $t_1>0$. If the function $\varphi(x)\in I_M(\Omega)$. Then the series (17) is uniformly convergent with respect to $x\in\overline{\Omega}, t\in[t_1,T], \rho\in[\rho_0,1]$.*

From Lemma 1 we get the following corollary.

Corollary 1. *Let $\rho_0\in(0,1)$ and $t_1>0$. Then for any number $\varepsilon>0$, there exists $n_0=n_0(\varepsilon,M,\rho_0,t_1)$ such that the following inequality holds for all $x\in\overline{\Omega}, t\in[t_1,T], \rho\in[\rho_0,1]$ and $\varphi(x)\in I_M(\Omega)$:*

$$\sum_{k=n_0+1}^{\infty}\left|\varphi_k v_k(x)\frac{d}{d\rho}E_\rho(-\lambda_k t^\rho)\right|<\varepsilon. \qquad (18)$$

Where n_0 is the smallest number that satisfies inequality (18).

Now we present the theorem about the solution of Inverse Problem (14), (15).

Theorem 10. *Let* $\varepsilon>0$ *be a fixed number and* $t_0\in\left(0,\min\left(\frac{1}{2^{1/\rho_0}},\frac{1}{e^{7/2}}\right)\right]$. *Denote* $n_0=n_0(\varepsilon,M,\rho_0,t_0)$ *as in Corollary 1. Suppose that for some* $x_0\in\Omega$ *the quantities* $v_k(x)$ *and* φ_k *of* $\varphi(x)\in I_M(\Omega)$ *satisfy the following conditions:*

1. $\varphi_k\geq 0$ *and* $v_k(x_0)\geq 0$ *for* $k=1,2,\ldots,n_0$;
2. φ_{k_0} *and* $v_{k_0}(x_0)$ *satisfying following inequality for some* $k_0\in\{1,2,\ldots,n_0\}$

$$\varphi_{k_0}v_{k_0}(x_0)>\frac{\varepsilon}{M_{k_0}},$$

where $M_{k_0}=\frac{d}{d\rho}E_\rho(-\lambda_{k_0}t_0^\rho)$.

Then the nonlinear function $G(\rho)$, *is strictly monotonically increasing on* $\rho\in[\rho_0,1]$. *Moreover, Inverse Problem (14), (15) has a unique solution if and only if*

$$d_0\in[G(\rho_0),G(1)].$$

The fourth chapter of the dissertation is titled **“Solution of fractional parabolic systems of vector order”.**

In the first section of this chapter, the initial-boundary problem was studied for the case where the order of Caputo fractional derivatives is arbitrary $\beta_j\in(0,1]$ in the system of subdiffusion equations.

Problem 7. Find functions $u_j(t,x)\in L_2^{q_j}(\mathbb{R}^n)$, $t\in(0,T]$, $j=1,\ldots,m$ (note that this inclusion is considered as a boundary condition at infinity), such that

$$U(t,x)\in\mathbf{C}([0,T]\times\mathbb{R}^n), D_t^{\beta}U(t,x) \text{ and } \mathbb{A}(D)U(t,x)\in\mathbf{C}((0,T]\times\mathbb{R}^n),$$

and satisfying the Cauchy problem

$$D_t^{\beta}U(t,x)+\mathbb{A}(D)U(t,x)=H(x,t),\quad t>0,\, x\in\mathbb{R}^n, \qquad (19)$$

$$U(0,x)=\Phi(x),\quad x\in\mathbb{R}^n, \qquad (20)$$

where $H(t,x)$ and $\Phi(x)$ are given continuous functions. $\mathbb{A}(D)$ is an arbitrary differential expressions $A_{i,j}(D)=\sum_{|\alpha|\leq l_{i,j}}a_\alpha D^\alpha$ with the order $l_{i,j}$ and constant coefficients:

$$\mathbb{A}(D)=\begin{bmatrix} A_{1,1}(D) & 0 & \ldots & 0\\ A_{2,1}(D) & A_{2,2}(D) & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ A_{m,1}(D) & A_{m,2}(D) & \ldots & A_{m,m}(D)\end{bmatrix}.$$

By p^* we denote the following expressions: $p^* = \max_{1\le j\le m}\{l_{j,j}\}$. That is, the largest degree of the differential operators on the diagonal. We give the following main result for this problem.

Theorem 11. *Let* $\tau > \frac{n}{2}$ *and* $\varphi_i(x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$ *and* $h_i(t,x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$. *Then the solution of Initial-Boundary Value Problem exists and is unique.*

In the second section of this chapter, the initial-boundary problem was studied for the case where the order of Riemann-Liouville fractional derivatives is arbitrary $\beta_j \in (0,1]$ in the system of subdiffusion equations.

Problem 8. Find functions $u_j(t,x) \in L_2^{q_j}(\mathbb{R}^n)$, $t \in (0,T]$, $j=1,\dots,m$ (note that this inclusion is considered as a boundary condition at infinity), such that

$$U(t,x) \in \mathbf{C}([0,T]\times\mathbb{R}^n), \partial_t^{\mathcal{B}} U(t,x) \text{ and } \mathbb{A}(D)U(t,x) \in \mathbf{C}((0,T]\times\mathbb{R}^n),$$

and satisfying the Cauchy problem

$$\partial_t^{\mathcal{B}} U(t,x) + \mathbb{A}(D)U(t,x) = H(x,t), \quad t>0,\, x\in\mathbb{R}^n, \tag{21}$$

$$\lim_{t\to 0}\partial_t^{\mathcal{B}-1} U(0,x) = \Phi(x), \quad x\in\mathbb{R}^n, \tag{22}$$

where $H(t,x)$ and $\Phi(x)$ are given continuous functions. $\mathbb{A}(D)$ is an arbitrary differential expressions $A_{i,j}(D) = \sum_{|\alpha|\le l_{i,j}} a_\alpha D^\alpha$ with the order $l_{i,j}$ and constant coefficients:

$$\mathbb{A}(D) = \begin{bmatrix} A_{1,1}(D) & 0 & \dots & 0 \\ A_{2,1}(D) & A_{2,2}(D) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{m,1}(D) & A_{m,2}(D) & \dots & A_{m,m}(D) \end{bmatrix}.$$

By p^* we denote the following expressions: $p^* = \max_{1\le j\le m}\{l_{j,j}\}$. That is, the largest degree of the differential operators on the diagonal. We give the following main result for this problem.

Theorem 12. *Let* $\tau > \frac{n}{2}$ *and* $\varphi_i(x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$ *and* $h_i(t,x) \in L_2^{\tau+p^*-l_{i,i}}(\mathbb{R}^n)$, $i=1,\dots,m$. *Then the solution of Initial-Boundary Value Problem exists and is unique.*

CONCLUSION

This dissertation is devoted to the study of direct and inverse problems for fractional-order subdiffusion equations. The main method for solving these problems is the classical Fourier method.

The main results of the research are as follows:

1. The initial boundary value problem and the inverse problem for determining the source function for the fractional-order Hadamard subdiffusion equation in the separable Gilbert space H were studied. The solution was constructed using the Fourier method and the existence and uniqueness of the solution were proved.

2. The problem of recursion for the fractional-order Hadamard subdiffusion equation in the separable Gilbert space H was studied. The existence and uniqueness of the solution of this problem were proved and coercive estimates were obtained.

3. An inverse problem is studied to determine the order of the fractional derivative and the order of the Laplace operator in a domain of dimension N for the Caputo fractional derivative subdiffusion equation with an elliptic part of the fractional Laplace operator. The existence and uniqueness of the solution of the inverse problem is proved.

4. An inverse problem is studied to determine the order of the fractional derivative and the order of the Laplace operator in a domain of dimension N for the Riemann-Liouville fractional derivative subdiffusion equation with an elliptic part of the fractional Laplace operator. The existence and uniqueness of the solution of the inverse problem is proved.

5. For the Caputo fractional derivative subdiffusion equation, the elliptic part of which is the Laplace operator, an inverse problem is studied to determine the order of the fractional derivative in a dimension N by giving an additional condition at one point. The existence and uniqueness of the solution of the inverse problem is proved.

6. An initial-boundary problem is studied for the case where the order of the fractional derivatives in the Caputo fractional derivative subdiffusion equation system is arbitrary $\beta_j \in (0,1]$. In this case, the solution is constructed using Fourier and Laplace transformations, and a theorem on the existence and uniqueness of the solution is proved.

7. An initial-boundary problem was studied for the case where the order of fractional derivatives is arbitrary $\beta_j \in (0,1]$ in the Riemann-Liouville system of subdiffusion equations. The solution was constructed using Fourier and Laplace transformations, and the theorem on the existence and uniqueness of the solution was proved.

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ИНСТИТУТ МАТЕМАТИКИ

СУЛАЙМОНОВ ИЛЁСХОДЖА АБДИРАСУЛ УГЛИ

ПРЯМЫЕ И ОБРАТНЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ СУБДИФФУЗИИ С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

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ВВЕДЕНИЕ (аннотация диссертации доктора философии (PhD))

Цель исследования состоит в доказательстве единственной разрешимости в классическом смысле краевых и обратных задач для уравнений субдиффузии.

Объектом исследования Уравнения субдиффузии с дробными производными в смысле Капуто, Римана–Лиувилля и Адамарда.

Предмет исследования. Краевые и обратные задачи по определению правой части и порядка дробной производной в уравнениях.

Методы исследования. В работе используются методы функционального анализа, спектральной теории и метод Фурье.

Научная новизна исследования состоит в следующем:

доказана единственная разрешимость краевой задачи и определена функция источника для уравнения субдиффузии с дробной производной Адамарда;

доказана единственная разрешимость в классическом смысле обратных задач по определению порядка дробной производной для уравнений субдиффузии с производными Капуто и Римана–Лиувилля;

доказана единственная разрешимость краевой задачи для системы уравнений субдиффузии с производными Капуто и Римана–Лиувилля.

Внедрение результатов исследования.

На основе полученных результатов по прямым и обратным задачам для уравнений субдиффузии:

решения краевых задач для системы уравнений субдиффузии с производной Капуто были использованы в международном проекте № 22-11-00064 «Моделирование динамических процессов в геосферах с учетом наследственности» при исследовании прямых и обратных задач для уравнений субдиффузии (Справка Института космофизических исследований и распространения радиоволн от 30.10.2025, № 453, Российская Федерация). На основе полученных научных результатов были построены субдиффузионные модели, описывающие процессы диффузионно–конвективного переноса радона в накопительных камерах, что позволило определить временную эволюцию концентрации радона и оценить параметры переноса;

решения обратных задач для субдиффузионных уравнений с дробной производной Капуто, эллиптическая часть которых содержит оператор Лапласа, были применены в зарубежном проекте № 125031904191-2 на тему «Краевые задачи для уравнений и систем с дробными и распределёнными порядковыми дифференциальными операторами и их приложения» при математическом моделировании различных физических и биологических процессов (Справка Институт прикладной математики и автоматизации Кабардино-Балкарского научного центра РАН от 6 ноября 2025 года № 01-13/98, Российская Федерация). Применение полученных научных результатов позволило находить решения обратных задач для субдиффузионных и диффузионно–волновых уравнений и использовать их при математическом моделировании различных физических и биологических процессов.

Объем и структура диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Общий объем диссертации составляет 104 стр.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (part I; I часть)

1. Ashurov R.R., Sulaymonov I.A., Determining the order of time and spatial fractional derivative, Mathematical Methods in the Applied Scinece, 48, 2, 1503-1518, 2025. (3. Scopus. IF= 0,63).
2. Ashurov R.R., Sulaymonov I.A., Fractional parabolic systems of vector order, Journal of Mathematical Sciences, 284, 2, 179-195, 2024. (3. Scopus. IF=0,28).
3. Sulaymonov I.A., On the backward problems for subdiffusion equations with the time-fractional Hadamard derivative, Uzbek Mathematical Journal, 67, 2, 135-141, 2023. (01.00.00. № 6).
4. Sulaymonov I.A., Determining the order of time and spatial fractional derivative in a subdiffusion equation, Uzbek Mathematical Journal, 69, 2, 281-286, 2025. (01.00.00. № 6).
5. Ashurov R.R., Fayziev Yu.E., Sulaymonov I.A., The inverse problem for determining the source function in the equation with the Hadamard fractional derivative, Bulletin of the Institute of Mathematics, 5, 5, 1-8, 2022. (01.00.00. № 17).
6. Sulaymonov I.A., New properties of the Mittag-Leffler function and its applications, Acta NUUz, No. 2/2/1, 132-142, 2025. (01.00.00. № 8)
7. Sulaymonov I.A., Fractional parabolic systems of vector order with Riemann-Liouville fractional derivative, Scientific reports of Bukhara state university, 2025, No. 12, 139-145, 2025. (01.00.00. № 3)

II bo‘lim (Part II; II часть)

1. Ashurov R.R., Fayziev Yu.E., Sulaymonov I.A., Inverse problem for determining a source function in the higher order equation with the Gerasimov-Caputo fractional derivative, Traditional International April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan, pp. 125-126, Almaty, April 6-8, 2022.
2. Fayziev Yu.E., Sulaymonov I.A., The Cauchy problem for equations of Hadamard fractional derivatives, International scientific and practical conference “Modern problems of applied mathematics and information technologies”, pp. 153-154, Bukhara, May 11-12, 2022.
3. Ashurov R.R., Sulaymonov, Determining the order of time and spatial fractional derivatives, Traditional International April Mathematical Conference in Honor of the Day of Scientists of the Republic of Kazakhstan, pp. 201-202, Almaty, April 1-4, 2024.
4. Ashurov R.R., Sulaymonov I.A., Monotonicity in the parameter of the Mittag-Leffler function and determining the fractional exponent of the subdiffusion equation, “Modern methods of mathematical physics and their applications”

republican scientific conference dedicated to the 80th anniversary of the birth of academician Sh.Alimov, vol. 2, p. 25, Tashkent, April 22-24, 2025.

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