V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.03/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

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ABDULLAYEV ABDUG‘ANI XOLMAMAT O‘G‘LI

DINAMIK SISTEMALAR TRAYEKTORIYALARINI TAQRIBIY HISOBLASH ANIQLIGINI BAHOLOVCHI TENGSIZLIKLAR

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

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KIRISH (falsafa doktori (PhD) dissertatsiaysi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zarurati. Jahon miqyosida matematika sohasida olib borilayotgan ilmiy-amaliy tadqiqotlarning salmoqli qismi dinamik sistemalar nazariyasini tadqiq qilishga keltiriladi. Dinamik sistemalar nazariyasi masalalarining o‘ta rang-barangligi, deyarli har qadamda murakkab va chuqur muammolarga duch kelinishi, ularni hal qilish differensial tenglamalar nazariyasidan tashqari geometriya, topologiya, funksional analiz, kompyuter texnologiyalari fanlarining usullarini qo‘llashni talab etishi kabi jihatlar bilan izohlanadi. Shuningdek, matematikaning boshqa fanlar va texnikaga tatbiqlari odatda dinamik sistema bilan modellashtirish orqali amalga oshiriladi. Shu sababli dunyoning yuzlab ilmiy markazlarida dinamik sistemalar va ularning tatbiqlari o‘rganiladi, shu bois ko‘plab aynan shu sohaga oid ilmiy jurnallar chop etiladi.

Hozirgi kunda kompyuter texnologiyalari sohasida dinamik sistemalar nazariyasi va ularning amaliy masalalarga tatbiqlari keng tadqiq qilinmoqda. Dinamik sistemalar nazariyasining asosiy o‘rganish ob’yekti bu – vaqt o‘tishi bilan holati o‘zgaradigan sistemalar va ulardagi jarayonlardir. Bugungi ilmiy-texnik taraqqiyot asrida dinamik sistemalarga oid masalalarni yechishda kompyuter texnologiyalari muhim ro‘l o‘ynamoqda, hatto bu yo‘nalishda hisoblash dinamikasi (Computational Dynamics) sohasi shakllanib, jadal rivojlanmoqda. Shuning uchun ushbu sohani tadqiq qilish maqsadli ilmiy tadqiqotlardan hisoblanadi.

Mamlakatimizda oxirgi yillarda fan va texnika, ayniqsa tabiiy va aniq fanlarni rivojlantirishga alohida e’tibor qaratila boshlandi. Bu yo‘nalishda matematika ta’limini takomillashtirish, fan sohasida fundamental tadqiqotlarni yanada yuqori bosqichga ko‘tarish, tadqiqotlar natijalarini amaliyot bilan bog‘lash, turdosh fan yo‘nalishlari va iqtisodiyot sohalarida tatbiqlarini yo‘lga qo‘yish bo‘yicha bir qancha natijalarga erishildi. Xususan, differensial tenglamalar va matematik fizika, dinamik sistemalar nazariyasi, algebra va funksional analiz, ehtimollar nazariyasi va matematik statistika, amaliy matematika va matematik modellashtirish kabi ustuvor yo‘nalishlarda ilmiy tadqiqotlar olib borish – O‘zbekiston Respublikasi Fanlar akademiyasi V. I. Romanovskiy nomidagi Matematika instituti faoliyatining asosiy yo‘nalishlari va vazifasi qilib belgilandi[1].

Mazkur dissertatsiya mavzusi va o‘rganish obyekti O‘zbekiston Respublikasi Prezidentining bir necha hujjatlarida belgilangan ustuvor vazifalarga to‘la muvofiqdir, jumladan O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-sonli “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi to‘g‘risida”gi, 2022-yil 28-yanvar PF-60 sonli “2022-2026- yillarga mo‘ljallangan Yangi O‘zbekistonning taraqqiyot strategiyasi to‘g‘risida”gi farmonlari, 2017-yil 17-fevral PQ-2789-sonli “Fanlar akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi qarori, 2017-yil 20-aprel

[1] O‘zbekiston Respublikasi Prezidentining 2019 yil 9-iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash shuningdek, O‘zbekiston Respublikasi Fanlar akademiyasining V.I. Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida” gi PQ-4387-sonli qarori.

PQ-2909-sonli “Oliy ta’lim tizimini yanada rivojlantirish chora-tadbirlari to‘g‘risida”gi qarori, 2018-yil 27-aprel PQ-3682-sonli “Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliyotga joriy qilish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi qarori, 2020-yil 7-may PQ-4708-sonli “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi qarori.

Shuningdek, mazkur qarorlarni hayotga joriy etishga qaratilgan tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga bog‘liqligi. Dissertatsiya doirasida amalga oshirilgan tadqiqot respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishiga muvofiqdir.

Muammoning o‘rganilganlik darajasi. Hisoblash dinamikasi 1845 yilda J. Adams tomonidan Neptun planetasi orbitasini taqribiy hisoblashdan boshlangan deyish mumkin. 20-asr boshlariga kelib differensial tenglamalar uchun Koshi masalasini taqribiy yechish usullari ishlab chiqish boshlandi (K.D.T.Runge, M.V.Kutta va b.). Bu davrda tadqiqotlar asosan nazariy nuqtai nazardan o‘rganilgan bo‘lsa, 20-asr o‘rtalaridan ilmiy-texnik taraqqiyot jadallashishi sababli differensial tenglamalarni taqribiy yechish bevosita amaliy ehtiyojga aylandi. Dastlabki Runge-Kutta usuli ikkinchi tartibli aniqlikka ega bo‘lsa, M.V.Kutta tomonidan uchinchi va to‘rtinchi tartibli aniqlikdagi usullar yaratildi. Bu usulning aniqlilik ko‘rsatgichi 10 gacha yetkazildi. Dormandning tadqiqotlarida ayrim sinflarga mansub tenglamalar uchun Runge Kutta usulining samaraliroq variantlari ishlab chiqildi. Bu boradagi tadqiqotlar natijalari J. Butcherning *“The numerical analysis of ordinary differential equations: Runge–Kutta and general linear methods”* monografiyasida batafsil jamlangan.

Tabiiy ravishda taqribiy yechim qay darajada aniq yechimga yaqin bo‘ladi, degan savol tug‘iladi. Chunki amaliy hisoblashlarda differensial tenglamalarning aniq yechimlari ko‘pincha noma’lum bo‘ladi yoki murakkab funksiyalar orqali ifodalanadi. Shunday vaziyatda taqribiy usullar, jumladan, Runge–Kutta kabi sonli usullar qo‘llaniladi. Ammo bu usullar orqali olingan natijalar aniq yechimni qanday aniqlikda aks ettiradi – bu masala har doim dolzarb bo‘lib kelgan. Taqribiy yechimning aniq yechimga yaqinligi odatda usulning tartibi bilan baholanadi. Masalan, to‘rtinchi tartibli Runge–Kutta usuli yechim xatoligini $O(h^4)$ tartibida baholaydi, bu esa qadam uzunligi h kichik bo‘lsa, juda yuqori aniqlikni kafolatlashini bildiradi.

Hisoblash dinamikasida yangi davr determinstik xaos kashf etilishidan boshlandi. E. N. Lorenz va O. E. Rossler kabi olimlarning ishlari – mos ravishda Lorenz sistemasi va Rossler sistemasi – deterministik, lekin oldindan bashorat qilib bo‘lmaydigan (ya’ni xaotik) sistemalarning mavjudligini isbotladi. Bu kashfiyotlar 1970–1980-yillarda hisoblash dinamikasi va xaos nazariyasining jadal rivojlanishiga turtki bo‘ldi.

1970 yillarda AQSHlik matematik J. Gukkenheimer “hisoblash dinamikasi” atamasini kiritdi. Ayni paytda u taqribiy hisoblash usullarini dinamik sistemalarning sifat nazariyasiga qo‘llash masalasini qo‘ydi. Avvalroq bu g‘oya A. A. Andronov, W. Gordon, M. A. Leontovich va M. G. Mayerning “Tekislikdagi dinamik sistemalarning sifat nazariyasi” monografiyasida eslatib o‘tilgan edi. Bu g‘oya L. P. Cherkas, L. A. Pchelintsev va boshqa matematiklar tomonidan ayrim masalalarga qo‘llandi. Biroq shuni ta’kidlash lozimki, barcha tadqiqotlarda asosan taqribiy yechim bilan vaqtni diskretlashtirish natijasida hosil bo‘ladigan diskret tenglama yechimi o‘rtasidagi farqni baholash masalasi qaralgan. Amalda esa diskret tenglama yechimi ham taqribiy hisoblanadi. Bunda xatolik hisoblashlarni yaxlitlash natijasida vujudga keladi. Bu ikki tur xatolik ma’lum ma’noda bir-biriga zid – taqribiy yechim qadami qancha kichik bo’lsa, birinchi tur xatolik shuncha kichiklashadi, ikkinchi tur xatolik esa aksincha o‘sadi. Bunda har ikki xatolik yig‘indisining eng kichik qiymati musbat son bo‘lib (“noaniqlik prinsipi”), taqribiy yechish usuli va qo‘llanadigan compyuter quvvatiga bog‘liq. Hisoblash dinamikasining samaradorligi ana shu kattalik bilan belgilanadi. Shuning uchun taqribiy yechim xatoligini aniq topish muhim ro‘l o‘ynaydi. A. Azamov va O. Ahmedov ishlarida quyi tartibli Runge-Kutta usullari uchun aniq tegishli ifodalar topilgan.

Dissertatsiyada mazkur masala Teylor formulasiga asoslangan taqribiy yechish usullari uchun amalga oshirilgan.

Dissertatsiya mavzusining dissertatsiya bajarilayotgan ilmiy tadqiqot muassasasining ilmiy-tadqiqot ishlari bilan bog‘liqligi. Dissertatsiya tadqiqoti V. I. Romanovskiy nomidagi Matematika instituti ilmiy tadqiqot rejasidagi “Dinamik jarayonlarning matematik modellari va ularning tatbiqlari” (2020-2025 yillar) mavzusidagi tadqiqot dasturlari bo‘yicha va FL-9524115148 raqamli “Dinamik sistemalarda topologik, optimal va raqamli usullar hamda ularning matematik modellashtirishga tatbiqlari” mavzusidagi fundamental loyihasida (2025 yil) olib borilayotgan ilmiy tadqiqot ishlari bilan bevosita bog‘liq.

Tadqiqotning maqsadi oddiy differensial tenglamalar bilan berilgan Koshi masalasi uchun Teylor formulasiga asoslangan past tartibli taqribiy yechish usullarining xatoliklarini baholovchi aniq tengsizliklarni keltirib chiqish hamda ushbu baholovchi tengsizliklar koeffitsiyentlari uchun oshkor formulalarni topishdan iborat.

Tadqiqot vazifalari.

Liuvill tenglamasini taqribiy yechimi bilan bog‘liq polinomial uchburchak qurish;

Avtonom differensial tenglamalar sistemasi uchun Teylor formulasi orqali hosil qilingan taqribiy yechimning xatoligini baholash;

Noavtonom differensial tenglamalar sistemalari uchun Teylor formulasi orqali hosil qilingan taqribiy yechimning xatoligini baholash;

Tadqiqotning ob’ekti: oddiy differensial tenglamalar sistemasi uchun Koshi masalasining yechimini taqribiy hisoblash usullarining aniqligini baholovchi tengsizliklar va dinamik sistemalar trayektoriyalarini taqribiy hisoblashning Teylor formulasiga asoslangan usullarining aniqligini ko‘rsatuvchi tengsizliklar.

Tadqiqotning predmeti. Chiziqsiz differensial tenglamalar, integrallanmaydigan dinamik sistemalar, Koshi masalasini taqribiy yechish usullari, Teylor formulasiga asoslangan taqribiy yechish sxemalari, taqribiy yechim xatoligini baholovchi tengsizliklar, baholash tengsizliklaridagi koeffitsiyentlar uchun aniq formulalar.

Tadqiqotning usullari: Tadqiqot ishida funksional analiz, oddiy differensial tenglamalar, matematik fizika tenglamalari, hisoblash matematikasi, differensial geometriya, dinamik sistemalar nazariyasi usullaridan foydalanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

Liuvill tenglamasi yechimining hosilasi uchun polinomial uchburchak hosil qilingan va bu yordamida rekurrent ifoda qurilgan;

avtonom differensial tenglamalar sistemasining aniq yechimi bilan Teylor formulasi orqali hosil qilingan taqribiy yechimlari orasidagi xatolikni baholovchi tengsizlik isbotlangan va bu tengsizliklardagi koeffitsiyentlarning qiymatlari topilgan;

noavtonom differensial tenglamalar sistemalarining aniq yechimi bilan Teylor formulasi orqali hosil qilingan taqribiy yechimlari orasidagi xatolikni baholovchi tengsizlik isbotlangan hamda ushbu tengsizliklardagi koeffitsiyentlarning qiymatlari topilgan.

Tadqiqotning amaliy natijalari. Olingan yangi natijalardan va dissertatsiyada qo‘llanilgan usullardan dinamik sistemalar nazariyasida va ayniqsa dinamik sistemalar nazariyasining texnik masalalarga tatbiqlarida samarali foydalanish mumkin.

Tadqiqot natijalarining ishonchliligi. Barcha natijalar matematikada qabul qilingan qat’iy deduktiv mushohadalar bilan asoslangan, ular qat’iy tengsizliklar tarzida bayon qilingan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati.

Tadqiqot natijalarining ilmiy ahamiyati dinamik sistemalarning taqribiy yechish usullarini rivojlantirishi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati dinamik sistemalarni tadqiq etishning sifat nazariyasi masalalarini yechishdagi muhim ro‘li bilan izohlanadi.

Tadqiqot natijalarining joriy qilinishi. Dinamik sistemalar trayektoriyalarini taqribiy hisoblash aniqligini baholovchi tengsizliklar bo‘yicha olingan natijalar asosida:

avtonom sistema uchun Koshi masalasini Teylor formulasiga asoslangan uchinchi tartibli taqribiy yechish usulining aniqligini bahovlovchi tengsizliklardan AUA-UAEU-12S141 raqamli “Pantograf tipidagi kechikishli differensial tenglamalarining sonli yechimlari” mavzusidagi xorijiy loyihada stoxastik pantograf kechikish differensial tenglamalarini yechimini o'rtacha kvadrat va kuchli yaqinlash tartiblarini aniqlashda foydalanilgan (Birlashgan Arab amirliklari Universitetining 2025 yil 24-avgustdagi ma’lumotnomasi). Ilmiy natijani qo‘llanilishi Teylor formulasi asosidagi yondashuvning an’anaviy Runge–Kutta usuli bilan taqqoslanishi yechimlarning aniqligini oshirish, xato bahosini qat’iy belgilash va sonli yaqinlashish usullarini yanada takomillashtirish imkonini bergan;

noavtonom differensial tenglamalar sistemalari uchun Teylor formulasi orqali hosil qilingan taqribiy yechimning xatoligini baholash bo‘yicha olingan natijalar UZB-Ind-2021-87 raqamli “Li simmetriyasi tahlili, giperbolik sistemalarning Lyapunov bo‘yicha turg‘unligini modellashtirish va tahlil qilish” mavzusidagi fundamental loyihada sistemalarning turg‘unligini tahlil qilishda foydalanilgan (O‘zbekiston Milliy Universitetining 2025 yil 16-oktyabrdagi №04/11-22965-sonli ma’lumotnomasi). Ilmiy natijaning qo‘llanilishi dinamik sistemalarni turg‘unligini o‘rganish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Tadqiqotning asosiy natijalari 6 ta ilmiy-amaliy anjumanlarda, jumladan 4 ta xalqaro va 2 ta respublika miqyosida anjumanlarida muhokamadan o‘tkazilgan.

Tadqiqot natijalarining e’lon qilinishi. Dissertatsiya mavzusi bo‘yicha jami 11 ta ilmiy ish chop etilgan bo‘lib, shundan O‘zbekiston Respublikasi Oliy Attestatsiya Komissiyasining falsafa doktorlik dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda 5 ta maqola, jumladan 1 tasi xorijiy va 4 tasi respublika jurnallarida nashr etilgan.

Dissertatsiyaning hajmi va tuzilishi. Dissertatsiya kirish qismi, uchta bob, xulosa va foydalanilgan adabiyotlar ro‘yhatidan iborat. Dissertatsiyaning umumiy hajmi 83 betni tashkil etadi.

DISSERTATSIYANING ASOSIY MAVZUSI

Kirish qismida dissertatsiya mavzusining dolzarbligi va zaruratini asoslash maqsadida dalillar bayon qilingan bo‘lib, ravishda tahlil qilingan bo‘lib tadqiqotning respublikada fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, muammoning o‘rganilganlik darajasi yoritilgan, tadqiqotning maqsadi, vazifalari, obyekti va predmeti bayon etilgan, tadqiqotda olingan natijalarning nazariy va amaliy ahamiyati ko‘rsatilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi haqida ham batafsil ma’lumotlar taqdim etilgan.

Dissertatsiyaning **“Oddiy differensial tenglamalar sistemasi uchun Koshi masalasini yechishning taqribiy usullari”** deb nomlanuvchi birinchi bobining birinchi bo‘limida dissertatsiya tadqiqotida qo‘llangan asosiy tushunchalar, keng tarqalgan taqribiy yechish usullari haqidagi ixcham ma’lumotlarni o‘z ichiga oladi. Sistema holatining vaqt o‘tishi bilan o‘zgarishining matematik modeli odatda dinamik sistema orqali ifodalandi. Dinamik sistema tushunchasi matematik ob’ekt sifatida birinchi marta A. Puankare asarlarida shakllangan, keyinchalik esa J. Birkgof, Danjua va boshqalar tomonidan rivojlantirilgan. Dinamik sistemalar nazariyasini avtonom differensial tenglamalar sistemasi sifatida o‘rganilishida A.A. Andronov rahbarligidagi ilmiy maktab katta hissa qo‘shgan bo‘lib, bu maktabga oid ko‘plab natijalar monografiyalarda jamlangan. Bu ishlarda, asosan, tekkislikdagi (ikki o‘lchamli) dinamik sistemalar ko‘rib chiqilgan.

Bizga

$$\frac{dy}{dx} = f(x,y) \tag{1.1}$$

sistema berilgan bo‘lsin. Bunda f_k funksiyalar $\mathbb{R}^{d+1}$ o‘lchovli fazoning biror D sohasida berilgan va yetarlicha tartibgacha xususiy hosilalari mavjud va uzluksiz degan shartni qanoatlantiradi. (1) sistema tayin $(x_0, y_0) \in \mathbb{R}^{d+1}$ nuqtadan o‘tuvchi yagona yechimga ega. Uni $y_\alpha(x)$ deb belgilaylik. Amalda $y_\alpha(x)$ kamdan-kam holda aniq topiladi. Shuning uchun taqribiy yechish usullarini qo‘llab taqribiy $y_t(x)$ topiladi. Hoh nazariy masalalar bo‘lsin, xoh amaliy masalalar, $y_t(x)$ qay darajada $y_\alpha(x)$ yechimga yaqin degan savol muhim. Agar har ikki funksiya tayin $k \in [x_0; x_*]$ kesmada mavjud bo‘lsa, u holda

$$\max_{x_0 \le x \le x_*} |y_\alpha(x) - y_t(x)| \le C \frac{e^{M_1(x_* - x_0)} - 1}{M_1} \tag{1.2}$$

ko‘rinishidagi baho o‘rinli bo‘ladi. Bunda M_1 Lipshits konstantasi, $C-$ taqribiy yechish usuliga bog‘liq konstanta. Taqribiy yechish usullari juda xilma-xil bo‘lib, amalda bir qadamli usullar keng qo‘llaniladi. Bunda $[x_0; x_*]$ kesmada h qadam bilan $[x_0, x_1, x_2, x_N = x_*]$ ajratiladi. Agar $h = \max\limits_{K}[x_k - x_{k-1}]$ bo‘lsa, biror natural s uchun $C = ch^s$ ifoda keltirib chiqariladi. Unda s taribiy yechish usulining aniqlik tartibi(darajasi) deyiladi, c esa funksiya va D sohaga hamda yechish usuliga bog‘liq konstanta.

Taqribiy yechish usullariga bag‘ishlangan adabiyotda (1.2) formula va $C = ch^s$ ifoda keltiriladi, ammo c koeffitsient uchun formula berilmaydi. Bunda odatda c ning qiymati $\dfrac{e^{M_1[x_* - x_0]} - 1}{M_1}$ ga nisbatan juda kichik degan evristik mulohaza bilan cheklaniladi. A.Azamov va shogirdlarining tadqiqotlari ko‘rsatadiki, bu mulohaza asosga ega emas. Qolaversa, nazariy masalalarda

$$\max_{x} |y_\alpha(x) - y_t(x)|$$

kattalik uchun aniq baho chiqarish muhim.

Bir qadamli taqribiy yechish usullaridan eng ko‘p qo‘llanadigani – bu Runge-Kutta usullaridir. Bu usulda $f(x,y)$ funksiyaning qiymatlari hisoblanadi, ammo hosilalarining qiymatlaridan foydalanilmaydi. Uzoq vaqt davomida bu qulay sanalgan. Ulardan farqli Teylor formulasiga asoslangan usullarda $f(x,y)$ ning hosilalari ham hisoblanadi. Bugun formulalar ustida amallar bajara oladigan dasturlar mavjud, ayniqsa, sun’iy intellect yaratilgan zamonda $f(x)$ ning hosilalarini hisoblashni avtomatlashtirish mumkin. Dissertatsiyada Teylor formulasiga asoslangan usullar qaralishi shu bilan izohlanadi. Bunda Teylor formulasiga asoslangan usullar boshqa afzalliklarga ega ekanligini ta’kidlash lozim. Chunonchi bu usullarda C konstantani topish sezilarli darajada o‘ng‘ay.

Shunday qilib, Runge-Kutta sxemalarida faqat f funksiyaning qiymatlari hisoblanadi, Teylor formulasida esa uning hosilalari ham hisoblanadi.

Teylor formulasiga asoslangan sxemalarni ikkita afzalligi bor:

birinchidan, istalgan aniqlikdagi sxemalar qurish mumkin va bu ishni kompyuterda bajarish mumkin;

ikkinchidan, C koeffitsient qiymatlarini topish sezilarli darajada yengil. Dissertatsiya asosan ikkinchi, uchunchi va to‘rtinchi tartibli Teylor sxemalari uchun C koeffitsient uchun oshkor qiymat topishga bag‘ishlangan.

Taqribiy yechim aniqligini baholash masalasi sonli usullarga bag‘ishlangan adabiyotda u yoki bu darajadagi tafsilotlar bilan yoritiladi. Bu yerda keng tarqalgan bir necha monografiya ustida to‘xtalamiz. Kamke ma’lumotnomasida va Сансоне-II kitobida xatoni baholash masalasi qaralmagan. Lambert, Kendal Atkinson, Бабенко, Hairer.E., Norsett.S va Wanner.G monografiyalarida faqat Runge-Kutta usullari qaralgan. Baxvalov fundamental monografiyasining III qismi oddiy differensial tenglamalarni taqribiy yechish usullariga bag‘ishlangan. Jumladan, bu qismning 1- paragrifida Teylor usuli metodologik nuqtai nazardan muhokama qilingan. Крилов. В.И., Бобков. В.В., Монастырный. П.И. Вычислительные методы monografiyasi II qismining 2-paragrifida Teylor usuliga to‘xtalib o‘tilgan, ammo xatoni baholash masalasi qaralmagan. Unda taqribiy yechim sxemasi hosil qilishda Teylor formulasini dastlabki n ta hadi olinsa, hatolik

$$\max_{0\le n\le N}\left|x_n - y(x_n)\right| = O(h^n)$$

bo‘lishini ta’kidlash bilan cheklanilgan.

Oddiy differensial tenglamalarni taqribiy yechish usullari bo‘yicha eng batafsil monografiya bo‘lgan Butcher kitobida aniq va taqribiy yechim orasidagi farq aniq yechim Teylor formulasining qoldiq hadi bilan baholanishi aytilib, bu had bir qadamda $O(h^{p+1})$ bo‘lishi takidlangan, xolos. Milne kitobida xatoni baholash masalasi

$$x(t+h) = x(t) + \int_t^{t+h} f[x(t)]dt$$

formulada integralni taqribiy hisoblashga asoslangan usullar uchun qaralgan va oshkor tengsizlik isbotlangan. Bu Teylor formulasining birinchi hadini qarashga teng kuchli, yuqori hadlarga asoslangan usullar esa qaralmagan. Bu kitobning keyingi bobida esa ko‘p qadamli taqribiy yechish usullari uchun hatolikni baholash o‘rganilgan, ammo

$$\sup_{0\le t\le T}\left|x_*(t) - \tilde{x}(t)\right| \le Ce^{LT}h^s$$

ko‘rinishdagi tengsizlik keltirib chiqarilmagan.

Ushbu bobning ikkinchi bo‘limida palinomial ko‘phad qurish masalasi ko‘rib o‘tilgan.

Differensial tenglamani taqribiy yechish tushunchasi bir necha xil talqin etilishi mumkin. Bulardan eng keng tarqalgani – Koshi masalasini sonli yechishdan iborat. Amaliyotda asosan shu talqinga mos ish tutiladi. Nazariy masalalarda esa yechimning taqribiy analitik ko‘rinishini izlash ham katta o‘rin tutadi. Matematikaning bus-butun sohasi bo‘lmish maxsus funksiyalar(Bessel, gipergeometrik va hakazo) aynan shu talqinga muvofiq vujudga kelgan.

Mutlaq ko‘pchilik holda analitik taqribiy yechim ko‘phad ko‘rinishida izlanadi. Dissertatsiyaning asosiy natijalarini tayin misolda - “Liuvill tenglamasi” ko‘rilar ekan analitik taqribiy yechim bilan bog‘liq polinomial uchburchak hosil qilindi. Bu yerda shu natija bayon qilinadi.

Liuvill tenglamasi $y''+xy=0$ ko‘rinishida bo‘lib, y chekli qadamlarda yechilmaydi - bu J.Liuvill tomonidan isbotlangan va integrallab bo‘lmasligi isbotlangan birinchi namunadir.

Shu maqsadda

$$y'_n(x)=a_n(x)y_n(x)+b_n(x)y_{n-1}(x)+c_n(x),\ \ n=0,1,2,... \tag{1.3}$$

ko‘rinishidagi rekurrent-differensial tenglamalar cheksiz sistemasiga keltirish mumkinligi yaxshi ma’lum, bunda $y_0(x)$ berilgan deb qaraladi. Bu sistemada no’malumlar birin-ketin integrallash yo‘li orqali hisoblanadi. Amaliyotda (1.3) sistemadan farqli

$$P_{n+1}^k(x)=a_n{}^k(x)P_n^{k+1}(x)+b_n{}^k(x)(P_n^k)'+c_n{}^k(x)P_n^{k-1}(x) \tag{1.4}$$

ko‘rinishdagi tenglamalar sistemasiga keltiriladigan masalalar uchraydi. Xususan, $a_n{}^k(x)=(k+1)x,\ \ b_n{}^k(x)=1,\ \ c_n{}^k(x)=k-1$ bo‘lgan hol Liuvill tenglamasi uchun kombinatorik masalaga tengkuchli. Bu yerda $P_n{}^k(x)$ ko‘phadlardan iborat bo‘lib, ular binomial koeffitsientlar kabi uchburchak hosil qiladi.

(1.4) formuladan

$$P_{n+1}^n(x)=(n+1)!x+(n-1)P_n{}^{n-1}(x),\ \ P_{n+1}^{n-1}(x)=\left(P_n{}^{n-1}(x)\right)'+(n-2)P_n{}^{n-2}(x),$$

$$P_{n+1}^k(x)=(k+1)xP_n^{k+1}(x)+(P_n^k)'+(k-1)P_n^{k-1}(x),\ \ P_{n+1}^1=\left(P_n{}^1(x)\right)'+2xP_n{}^2(x),$$

$$P_{n+1}^0(x)=\left(P_n{}^0(x)\right)'+xP_n^1(x)$$

tengliklarga ega bo‘lamiz, agarda

$$P_n^k(x)=\begin{cases}0, & k<0\\ 0, & k=n\\ n!, & k=n+1\end{cases}$$

bo‘lsa $(n=0,1,2,...,k=0,1,2,...)$.

Endi $y''+xy=0$ tenglamaga qaytaylik. Uning yechimi elementar funksiyalar orqali, ular ustida algebraik amallar va integrallash amali bilan ifodalanmaydi (J.Liuvill teoremasi). Bu tenglama

$$z'(x)=x+z^2(x) \tag{1.5}$$

Rikkati tenglamasiga tengkuchli.

1-teorema. (1.5) tenglamaning yechimi $z(x)$ uchun

$$z^{(n)} = n!z^{n+1} + \sum_{k=0}^{n-1} P_n^k(x) z^k$$

formula o‘rinli.

Dissertatsiyaning “**Teylor formulasi orqali hosil qilingan taqribiy yechimining xatoligini avtonom sistemalar uchun baholash**” nomli ikkinchi bobida dinamik sistema uchun Koshi masalasini Teylor formulasiga asoslangan ikkinchi, uchinchi va to‘rtinchi tartibli taqribiy yechish usulining aniqligini baholash masalasi ko‘rib o‘tilgan.

Ushbu

$$\dot{x}(t) = f(x(t)), \quad x(0) = \xi \tag{2.1}$$

ko‘rinishidagi Koshi masalasining sonli usullarda yechilishi nihoyatda muhim ahamiyatga ega. (2.1) –bu dinamik sistema bo‘lib, $x = (x_1, x_2, ..., x_n)$ - vektor-funksiya, $t \in \mathbb{R}$, ξ - berilgan boshlang‘ich nuqta. $f(x) = (f_1(x), f_2(x), ..., f_n(x))$ funksiya $\mathbb{R}^d$ fazoning biror D sohasida aniqlangan va yetarlicha marta differensiallanuvchi deb hisoblanadi.

Agar $x_*(t)$ — (2.1) Koshi masalasining aniq yechimi, x_n esa $t = t_n$ nuqtadagi taqribiy yechimi bo‘lsa, u holda sonli usulning aniqlik darajasi quyidagi tengsizlik orqali baholanadi:

$$\sup_n \left| x_*(t_n) - x_n \right| \le C(\sup_n \left| t_n - t_{n-1} \right|)^k (n = 1, 2, ...), \tag{2.2}$$

bu yerda $k-$ usulning aniqlik darajasi (ya’ni tartibi), $C-$o‘zgarmas son. Hisoblash matematikasida (2.1)-masalani sonli yechish uchun turli xil usullar ishlab chiqilgan va hozirgacha ishlab chiqilishi davom etmoqda. Bular qatoriga Runge–Kutta usullarining turli modifikatsiyalari, Adams usullari , chekli ayirma (ya’ni, farqli) usullari, neyron tarmoqlar yordamida yechish usullari va boshqalar kiradi.

Har doim usulning aniqlik darajasini baholovchi tengsizlikni chiqarish muhim ahamiyatga ega. Bu esa (2.1.2) tengsizlikdagi k va C kabi konstantalarni aniqlashga tengdir.

Mazkur bobda Teylor formulasi asosidagi sonli integrallash usuli ko‘rib chiqiladi. Unda usulning xatosi faqatgina Teylor formulasining qoldiq hadi bilan emas, boshqa omillar bilan ham bog‘liq ekani ko‘rsatiladi. So‘ngra, aniq va taqribiy yechimlar orasidagi farqni yuqoridan baholovchi tengsizlik shakllantiriladi va isbotlanadi. $x_*(t)$— (2.1) - masalaning aniq yechimi bo‘lsin (kelgusida $\tilde{x}(t)$ ifodasi (2.1)-masalaning taqribiy yechimini anglatadi), J esa $x_*(t)$ yechim mavjud bo‘ladigan o‘ng yarim interval bo‘lsin.

Faraz A. Izlanayotgan $x_*(t)$ yechim $[0;T]$ oraliqda mavjud bo‘lsin.

Odatda, sonli integrallash deganda $x_*(t)$ yechimning qiymatlarini to‘r nuqtalarida ya’ni $\{t_0 = 0 < t_1 < t_2 < ... < t_N = T\}$ to‘plamida topish tushuniladi. Biz faqat bir xil oraliqli tarmoqqa e’tibor qaratamiz, ya’ni $t_n = nh,\ h = \frac{T}{N}$. Mazkur ishda Teylor formulasi asosida qurilgan bir qadamli sonli usul ko‘rib chiqiladi:

$$x_*(t+h) = x_*(t) + h\dot{x}_*(t) + \frac{h^2}{2}\ddot{x}_*(t) + ... + \frac{h^m}{m!}x_*^{(m)}(t) + R_m,$$

bu yerda qoldiq had quyidagi ko‘rinishda ifodalanishi mumkin:

$$R_m = \frac{1}{m!}\int_t^{t+h}(t-s)^m x_*^{(m+1)}(s)ds$$

Qoldiq hadni e’tiborga olmasdan, taqribiy yechimning qiymatlarini hisoblash uchun quyidagi bir qadamli sonli usulni hosil qilamiz:

$$x_*\left[(n+1)h\right] \approx x_*(nh) + h\dot{x}_*(nh) + \frac{h^2}{2}\ddot{x}_*(nh) + \cdots + \frac{h^m}{m!}x_*^{(m)}(nh), \qquad (2.3)$$

bu yerda yechim hosilalari f funksiya va uning hosilalari orqali ifodalanadi:

$$\dot{x}_*(nh) = f\left[x_*(nh)\right],$$

$$\ddot{x}_*(nh) = f'\left[x_*(nh)\right]f\left[x_*(nh)\right],$$

$$\dddot{x}_*(nh) = f''\left[x_*(nh)\right]f\left[x_*(nh)\right]f\left[x_*(nh)\right] + f'\left[x_*(nh)\right]f'\left[x_*(nh)\right]f\left[x_*(nh)\right].$$

(2.4)

Eslatib o‘tamiz, agar (2.1)-masalada biz sistema bilan ish olib borayotgan bo‘lsak, ya’ni $x \in D \subset \mathbb{R}^d,\ f: D \to \mathbb{R}^d$ va $d \geq 2$, u holda $f'(x)$ — bu d-o‘lchamli chiziqli akslantirish (ya’ni, (1, 1)-rangli tenzor) bo‘lib, $\mathbb{R}^d$ ning standart bazisida $\left(\frac{\partial f_i}{\partial x_j}\right)$ Yakobi matritsasi yordamida ifodalanadi. Shuningdek, $f''(x)$ — bu d-o‘lchamli bichiziqli akslantirish yoki $(1, 2)$-rangli tenzor bo‘lib, uning komponentlari quyidagicha ifodalanadi: $\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right); i, j, k = 1, 2, ..., d.$ Shunga o‘xshash talqinlar f funksiyaning undan yuqori tartibli hosilalari uchun ham qo‘llaniladi.

Keyingi o‘rinlarda masalaning mohiyatini murakkab formulalar orqali ifodalamaslik uchun har bir qadamda uchinchi tartibli aniqlikka ega usulni ko‘rib chiqish bilan cheklanamiz:

$$x_*\left[(n+1)h\right] \approx x_*(nh) + h\dot{x}_*(nh) + \frac{h^2}{2}\ddot{x}_*(nh).$$

Agar $x_{*n} = x_*(nh)$ deb belgilasak, u holda:

$$x_{*(n+1)} \approx x_{*n} + hf(x_{*n}) + \frac{h^2}{2} f'(x_{*n}) f(x_{*n}). \qquad (2.5)$$

Endi yaqinlashtiruvchi (2.5)-formula o‘rniga quyidagi rekurrent formulani ko‘rib chiqamiz:

$$x_0 = \xi,\; x_{n+1} = x_n + hf(x_n) + \frac{h^2}{2} f'(x_n) f(x_n). \qquad (2.6)$$

Hisoblash matematikasida aynan shu (2.6)-formula bilan ishlanadi va ko‘pincha x_n va x_{*n} bir xil miqdor deb qaraladi. Biroq, aslida bu to‘g‘ri emas. Shu sababli hisoblash dinamikasi nuqtai nazaridan “muqaddas” deyish mumkin bo‘lgan savol yuzaga keladi: x_n ketma-ketligining $x_*(nh)$ aniq yechim qiymatiga qanday aloqasi bor?

Bizningcha, hisoblash dinamikasida ba’zan yetarli asoslarsiz holda, x_n qiymatlari asosida aniq yechim $x_*(t)$ ning xatti-harakatlari haqida uzoqni ko‘zlovchi xulosalar qilinadi — ayniqsa, Lorens, Ryossler, “bryusselator”, ko‘p jismli harakat masalalari kabi murakkab sistemalarda. Aslida esa vaziyat ancha murakkab. Eng avvalo, yuqorida keltirilgan Faraz A, shuningdek quyida bayon qilinadigan farazlar bajarilishini tekshirish zarur. Yana bir bor ta’kidlash joizki, amaliy qo‘llanmalar uchun evristik xulosa yoki umumiy mulohazalar yetarli bo‘lishi mumkin, ammo dinamik sistemalar nazariyasidagi teoremalarni isbotlash uchun esa quyidagi kabi aniq, matematik isbotlangan baholash talab qilinadi:

$$\left|x_*(nh) - x_n\right| \le C(f, T, K, \xi) h^2,$$

bu yerda C — funksiyaning xossalari, vaqt oralig‘i T, K kompakt to‘plam va boshlang‘ich shart ξ ga bog‘liq bo‘lgan doimiydir.

Asosiy maqsadi — ko‘rib chiqilayotgan usul uchun yuqoridagi kabi baholashning matematik isbotida mavjud bo‘lgan bo‘shliqni to‘ldirishdir. Shu bilan birga $m > 1$ bo‘lgan holatda (ya’ni Teylor formulasi orqali yuqori tartibli yaqinlashishda), (2.1) differensial tenglama uchun $\left|x_*(nh) - x_n\right|$ farqini baholovchi ifoda, Teylor formulasi qoldiq hadi asosida olingan baholashdan farq qilishini ko‘rsatib o‘tamiz**.**

Demak, endi bizni $x_*(nh) - x_n$ ayirmani baholash masalasi qiziqtiradi.

Bunda t vaqtga bog‘liq bo‘lmagan baholash olish uchun, yechimga yana bir qo‘shimcha shart qo‘yishga to‘g‘ri keladi. Buning uchun funksiyaning aniqlanish sohasi $D \subset \mathbb{R}^d$ ichida joylashgan, aniq yechimning $\left\{x_*(t) \middle| 0 \le t \le T\right\}$ yoyini va uning ma’lum bir sohasini o‘z ichiga oluvchi aniq kompakt K to‘plamni tanlash zarur bo‘ladi. Bu oson ish emas, chunki aniq yechim $x_*(t)$ bizga oldindan ma’lum emas. Shu sababdan biz quyidagi farazni qabul qilamiz:

Faraz B. $\left[0, T\right]$ oraliqda berilgan $x_*(t)$ aniq yechim K kompakt to‘plamdan chiqib ketmaydi (ya’ni, shu to‘plam ichida qoladi).

Bunday kompakt to‘plamni tanlash quyidagi qiymatlarni belgilash imkonini beradi:

$$M_0=\max_{x\in K}\left|f(x)\right|,\quad M_1=\max_{x\in K}\left\|f'(x)\right\|,\quad M_2=\max_{x\in K}\left\|f''(x)\right\|,...$$

va so‘ngra A va B farazlardan foydalanib, yaqinlashgan yechimning xatosi uchun aniq baho chiqariladi. Ushbu masalaga bag‘ishlangan nashrlarda quyidagi baholash ko‘rinishidagi tengsizlik keltiriladi:

$$\left|x_*(nh)-x_n\right|\le C\left(\frac{e^{M_1T}-1}{M_1}\right)\frac{h^2}{6}, \tag{2.7}$$

bu esa Teylor formulasi qoldiq hadining bahosiga tayanadi. (2.7)-tengsizlikning mutlaqo asosli va nuqsonsiz isboti aniqlanmadi. Quyida bu holatning sababi bayon etiladi hamda (2.7)-tengsizlikdan boshqa bir tengsizlikning qat’iy isboti keltiriladi.

1-lemma. $\tilde{x}(nh)=x_n$ tenglik barcha $n>0$ uchun o‘rinli.

1-ta’rif. Ushbu

$$\tilde{x}(t)=x_0+\int_0^t\left\{f\left[x(\sigma_s)\right]+(s-\sigma_s)f'\left[x(\sigma_s)\right]f\left[x(\sigma_s)\right]\right\}ds$$

funksiya (2.1) masalaning taqribiy yechimi deyiladi.

Bu yerda $\sigma(t)$ pog‘onali funksiya bo‘lib, quyida uning xossalarini keltiramiz:

1. $nh\le\sigma(t)<(n+1)h,$ bu yerda $n=\left\lfloor\frac{t}{h}\right\rfloor$ (ushbu xossa ta’rifga ekvivalentdir);
2. $\sigma(\sigma(t))=\sigma(t);$
3. agar $t\le s<t+h,$ bo‘lsa, unda $\sigma(s)=\sigma(t).$

Faraz C. $\tilde{x}(t)\in K,$ bunda $t\in\left[0,T\right].$

2-teorema. Agar A, B va C farazlar bajarilgan bo‘lsa, u holda (2.1)-masalaning aniq yechimi bilan taqribiy yechimi orasida quyidagi baho o‘rinli:

$$\left|x_*(t)-\tilde{x}(t)\right|\le\frac{e^{M_1T}-1}{M_1}\left(L_0+L_1h\right)\frac{h^2}{2}.$$

2-ta’rif. Ushbu

$$\tilde{x}(t)=x_0+$$
$$+\int_0^t\left\{f\left[\tilde{x}(\sigma_s)\right]+(s-\sigma_s)(f'f)\left[\tilde{x}(\sigma_s)\right]+\frac{(s-\sigma_s)^2}{2}(f''ff+f'f'f)\left[\tilde{x}(\sigma_s)\right]\right\}ds$$

funksiya (2.1) masalaning taqribiy yechimi deyiladi.

3-teorema. Aniq va taqribiy yechim orasidagi farq uchun

$$\left|x_*(t)-\tilde{x}(t)\right|\le\frac{e^{M_1T}-1}{6M_1}(L_0+L_1h+L_2h^2)h^3$$

tengsizlik o‘rinli.

3-ta’rif. Ushbu

$$\tilde{x}(t) = x_0 + \int_0^t \sum_{k=0}^{3} \frac{(s-\sigma_s)^k}{k!} F^k[\tilde{x}(\sigma_s)]ds$$

funksiya (2.1) masalaning taqribiy yechimi deyiladi.(bu yerda $x^{(k)}(t) = (F^k)[x_*(t)]$)

4-teorema. Ushbu $|x_*(t) - \tilde{x}(t)| \le Ch^4 \dfrac{e^{M_1 T} - 1}{M_1}$ tengsizlik o‘rinli, bunda

$$C = L_3 + L_{2,3} + L_{1,2,3} + H^1{}_{1,2,3} + H^2{}_{0,1,2,3} + H^3{}_{0,1,2,3} + H^4{}_{0,1,2,3} + H^5{}_{0,1,2,3}.$$

Dissertatsiyaning “**Teylor formulasi orqali hosil qilingan taqribiy yechimning xatoligini avtonom bo‘lmagan sistemalar uchun baholash”** deb nomlangan uchinchi bobida noavtonom sistema uchun Koshi masalasini Teylor formulasiga asoslangan ikkinchi va uchinchi tartibli taqribiy yechish usulining aniqligini baholash masalasi ko‘rib o‘tilgan.

Differensial tenglamalar nazariyasining bir qator boblarida, ayniqsa, tatbiqlarida

$$y' = f(x, y), \; y(x_0) = y_0, \tag{3.1}$$

Koshi masalasining taqribiy yechimini hisoblashga to‘g‘ri keladi. (3.1) masalada $y \in D \subset \mathbb{R}^d$, D – qavariq soha, $f: D \to \mathbb{R}^d$ uzluksiz va yetarlicha differensiallanuvchi funksiya. Agar $y_*(x)$ (3.1) masalaning aniq yechimi, $y(x)$ taqribiy yechimi bo‘lsa $(x_0 \le x \le x_*)$, u holda taqribiy yechim aniqligi

$$\sup_{x_0 \le x \le x_*} |y_*(x) - y(x)| \le Ce^{L(x-x_0)} h^s \tag{3.2}$$

ko‘rinishdagi tengsizlik tarzida ifodalanadi (L – Lipshits konstantasi, h – taqribiy yechim qiymatlari hisoblanadigan oraliqning eng katta qadami, s – taqribiy yechish usulining aniqlik darajasi,)

4-ta’rif. Ushbu

$$y(x) = y_0 + \int_{x_0}^{x} \{ f[\delta_s, y(\delta_s)] + (s-\delta_s)\Big(f'_y[\delta_s, y(\delta_s)] f[\delta_s, y(\delta_s)] + f'_x[\delta_s, y(\delta_s)]\Big)\} ds$$

funksiya (3.1) masalaning taqribiy yechimi deyiladi.

5-teorema. Aniq va taqribiy yechim orasidagi farq uchun

$$|y_*(x) - y(x)| \le \frac{e^{L(x-x_0)} - 1}{L}(L_1 + L_2 h)h^2$$

baho o‘rinli.

6-teorema. Aniq va taqribiy yechim orasidagi farq uchun

$$|y_*(x) - y(x)| \le \frac{e^{M_{01}(x-x_0)} - 1}{M_{01}}(L_0 + L_1 h + L_2 h^2)h^3$$

baho o‘rinli.

XULOSA

Mazkur dissertatsiya ishi dinamik sistemalar trayektoriyalarini taqribiy hisoblash aniqligini baholovchi tengsizliklarga bag‘ishlangan. Dissertatsiyaning asosiy natijalari quyidagilardan iborat:

Liuvill tenglamasi yechimining hosilasi uchun rekurrent ifoda qurilgan. Bu ifodani keltirib chiqarishda polinomial uchburch qo‘llangan;

avtonom differensial tenglamalar sistemalari uchun Teylor formulasiga asoslangan ikkinchi, uchinchi va to‘rtinchi tartibli taqribiy yechish usullarida xatolikni baholovchi tengsizliklarning qat’iy isboti keltirilgan. Ushbu tengsizliklardagi koeffitsientning qiymatlari oshkor holda aniqlangan;

avtonom bo‘lmagan differensial tenglamalar sistemalari uchun Teylor formulasiga asoslangan ikkinchi va uchinchi tartibli taqribiy yechish usullarida xatolikni baholovchi tengsizliklarning qat’iy isboti keltirilgan va tengsizliklardagi koeffitsientning qiymatlari oshkor holda aniqlangan.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 INSTITUTE OF MATHEMATICS NAMED AFTER V.I. ROMANOVSKIY

INSTITUTE OF MATHEMATICS

ABDULLAYEV ABDUGANI KHOLMAMAT O‘G‘LI

INEQUALITIES FOR ESTIMATING THE ACCURACY OF APPROXIMATE COMPUTATION OF TRAJECTORIES OF DYNAMICAL SYSTEMS

01.01.02-Differential equations and mathematical physics

ABSTRACT OF THESIS OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

TASHKENT-2025

The theme of dissertation of doctor of philosophy (PhD) on physical and mathematical sciences was registered at the Supreme Attestation Commission at the Ministry of Higher education, Science and Innovations of the Republic of Uzbekistan under number B2025.2.PhD/FM1285

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INTRODUCTION (Annotation to the doctor of philosophy (PhD) dissertation)

Actuality and demand of the theme of the dissertation. A number of scientific and practical tasks conducted worldwide in the field of mathematics is dedicated to the study of dynamical systems theory. The diversity of problems in dynamical systems, the frequent encounter with highly complex and profound challenges, and the need to employ methods not only from the theory of differential equations but also from geometry, topology, functional analysis, and computer technologies all contribute to this focus. Moreover, applications of mathematics to other sciences and engineering are typically carried out through modeling by means of dynamical systems. For this reason, dynamical systems and their applications are studied in hundreds of research centers around the world, and numerous scientific journals dedicated specifically to this field are published.

Currently, in the field of computer technologies is witnessing extensive research on dynamical systems theory and its applications to practical problems. The primary object of study in dynamical systems theory is systems and processes whose states evolve over time. In today's era of scientific and technological advancement, computer technologies play a crucial role in solving problems related to dynamical systems; indeed, a separate discipline—computational dynamics-has emerged and is rapidly developing. Therefore, research in this area is regarded as a purposeful and significant scientific endeavor.

In recent years, special attention has been paid in our country to the development of science and technology, particularly natural and exact sciences. Considerable progress has been made in improving mathematics education, elevating fundamental research in the sciences to a higher level, integrating research outcomes with practical applications, and establishing their relevance in related scientific fields and various sectors of the economy. In particular, conducting scientific research in such priority areas as differential equations and mathematical physics, dynamical systems theory, algebra and functional analysis, probability theory and mathematical statistics, applied mathematics, and mathematical modeling has been defined as one of the main directions and tasks of the V. I. Romanovskiy Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan[1].

The topic and object of study of this dissertation fully correspond to the priority tasks outlined in several official documents of the President of the Republic of Uzbekistan, including Presidential Decree No. PF-4947 of February 7, 2017, "On the Strategy of Actions for the Further Development of the Republic of Uzbekistan," Presidential Decree No. PF-60 of January 28, 2022, "On the Development Strategy of New Uzbekistan for 2022–2026," Resolution No. PQ-2789 of February 17, 2017, "On Measures to Further Improve the Activities of the

[1]Decree of the President of the Republic of Uzbekistan dated July 9, 2019 № PQ-4387 "On state support for the further development of mathematics education and subjects, as well as measures to fundamentally improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan".

Academy of Sciences, the Organization, Management, and Financing of Scientific Research," Resolution No. PQ-2909 of April 20, 2017, "On Measures to Further Develop the System of Higher Education," Resolution No. PQ-3682 of April 27, 2018, "On Measures to Further Improve the System for Implementing Innovative Ideas, Technologies, and Projects into Practice," and Resolution No. PQ-4708 of May 7, 2020, "On Measures to Improve the Quality of Education and Develop Scientific Research in the Field of Mathematics."

Additionally, this dissertation research contributes, to a certain extent, to the implementation of the tasks defined in other relevant normative and legal documents aimed at putting these resolutions into practice.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV,"Mathematics, Mechanics and Computer Science."

The degree of scrutiny of the problem. It can be said that computational dynamics began in 1845 with J. Adams's approximate calculations of the orbit of the planet Neptune. By the early 20th century, methods for obtaining approximate solutions to the Cauchy problem for differential equations started to be developed (notably by K. D. T. Runge, M. W. Kutta, and others). While early research focused mainly on theoretical aspects, from the mid-20th century onward the rapid advancement of science and technology made the approximate solution of differential equations an essential practical requirement. The original Runge–Kutta method provided second-order accuracy, whereas M. W. Kutta later developed third- and fourth-order accurate methods. The accuracy order of this class of methods was eventually extended to as high as ten. In Dormand's research, more efficient variants of Runge-Kutta methods were constructed for certain classes of differential equations. The results of these studies are comprehensively presented in J. Butcher's monograph "The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods."

Naturally, the question arises as to how closely an approximate solution approaches the exact one. In practical computations, the exact solutions of differential equations are often unknown or expressed through highly complicated functions. In such situations, approximate methods-particularly numerical techniques such as the Runge-Kutta methods are employed. However, how accurately these methods reproduce the exact solution has always been an important issue. The closeness of an approximate solution to the exact one is typically evaluated by the order of the method. For example, the classical fourth-order Runge-Kutta method estimates the solution error as $O(h^4)$, which means that when the step size h is sufficiently small, the method guarantees a very high level of accuracy.

A new era in computational dynamics began with the discovery of deterministic chaos. The works of scientists such as E. N. Lorenz and O. E. Rössler-namely the Lorenz system and the Rössler system-demonstrated the existence of systems that are deterministic yet unpredictable (that is, chaotic).

These discoveries stimulated the rapid development of computational dynamics and chaos theory during the 1970s and 1980s.

In the 1970s, the American mathematician J. Guckenheimer introduced the term computational dynamics. At the same time, he proposed the idea of applying approximate computational methods to the qualitative theory of dynamical systems. This idea had earlier been mentioned in the monograph “Qualitative Theory of Dynamical Systems on the Plane” by A. A. Andronov, W. Gordon, M. A. Leontovich, and M. G. Mayer. It was later applied to certain problems by mathematicians such as L. P. Cherkas, L. A. Pchelintsev, and others. However, it should be noted that in all previous studies the primary focus was on estimating the difference between the approximate solution and the solution of the discrete equation obtained by discretizing time. In practice, the solution of the discrete equation is itself only approximate, as numerical rounding inevitably introduces computational error. These two types of error are, in a sense, mutually opposing: the smaller the step size of the approximate method, the smaller the first type of error becomes, while the second type of error increases. The minimal possible sum of these two errors is a positive quantity often referred to as an “uncertainty principle”and depends on both the numerical method used and the computational power of the computer. The effectiveness of computational dynamics is determined precisely by this quantity. Therefore, accurately determining the error of an approximate solution plays a crucial role. In the works of A. Azamov and O. Ahmedov, exact corresponding expressions were obtained for lower-order Runge-Kutta methods.

In this dissertation, the same problem is addressed for approximate solution methods based on Taylor’s formula.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out. The research conducted in this dissertation is directly related to the scientific work carried out under the research programs of the V. I. Romanovskiy Institute of Mathematics, specifically within the topic “Mathematical Models of Dynamic Processes and Their Applications” (2020–2025), as well as the fundamental project FL-9524115148, “Topological, Optimal, and Numerical Methods in Dynamical Systems and Their Applications to Mathematical Modeling” *(2025).*

The aim of the research is to derive exact inequalities for estimating the errors of low-order Taylor formula-based approximate solution methods for the Cauchy problem defined by ordinary differential equations, and to obtain explicit expressions for the coefficients of these error-estimating inequalities.

Research problems:

To construct a polynomial triangle associated with the approximate solution of the Liouville equation;

To estimate the error of approximate solutions obtained via the Taylor formula for autonomous systems of ordinary differential equations;

To estimate the error of approximate solutions obtained via the Taylor formula for non-autonomous systems of ordinary differential equations.

The object of research: inequalities estimating the accuracy of approximate solutions to the Cauchy problem for systems of ordinary differential equations, and inequalities showing the accuracy of Taylor-based approximate computations of trajectories of dynamical systems.

The subject of research: nonlinear differential equations, non-integrable dynamical systems, approximate methods for solving the Cauchy problem, Taylor-based approximation schemes, error-estimating inequalities, and explicit formulas for coefficients in these inequalities.

Research methods: The research applies methods from functional analysis, ordinary differential equations, mathematical physics, computational mathematics, differential geometry, and dynamical systems theory.

Scientific novelty of the research consists of the following:

A polynomial triangle for the derivative of the solution of the Liouville equation has been constructed, and a recurrent formula has been derived on its basis;

For autonomous systems of differential equations, an inequality estimating the error between the exact and approximate solutions obtained via the Taylor formula has been proven, and the values of the coefficients appearing in these inequalities have been determined;

For non-autonomous systems of differential equations, an inequality estimating the error between the exact and approximate solutions obtained via the Taylor formula has been proven, and the values of the coefficients in these inequalities have been found.

Practical results of the research. The new results and methods applied in the dissertation can be effectively used in the theory of dynamical systems, particularly in the application of dynamical systems theory to technical problems.

Reliability of results. All results are grounded in strict deductive reasoning accepted in mathematics and are presented in the form of rigorous inequalities.

Scientific and practical significance of the results. The scientific significance of the results lies in the development of approximation methods for dynamical systems. The practical significance lies in their important role in solving problems of the qualitative theory of dynamical systems.

Implementation of research results. Based on the results obtained from inequalities estimating the accuracy of approximate solution of dynamical system trajectories:

For an autonomous system, the inequalities assessing the accuracy of the third-order Taylor formula-based approximate solution method for the Cauchy problem were utilized in the foreign project titled “Numerical Solutions of Pantograph-Type Delay Differential Equations”(Project No. AUA-UAEU-12S141) to determine the mean-square and strong convergence orders of stochastic pantograph delay differential equations (reference from United Arab Emirates University, August 24, 2025). The application of this scientific result, by comparing the Taylor formula-based approach with the conventional Runge–Kutta method, allowed for increasing the accuracy of solutions, strictly defining the error estimate, and further improving numerical approximation methods.

The results obtained from the error estimation of the approximate solution, constructed via the Taylor formula for non-autonomous systems of differential equations, were applied in the fundamental project "Li symmetry analysis, modeling and stability analysis of hyperbolic systems according to Lyapunov" (Project No. UZB-Ind-2021-87) for analyzing the stability of the systems (Reference: National University of Uzbekistan, october 16, 2025, No. 04/11-22965). The application of this scientific result has enabled the study of the stability of dynamical systems.

Aprobation of research results. The main results of the research have been discussed in 4 international and 2 national scientific conferences.

Publications of the research results. A total of 11 scientific papers have been published on the topic of the dissertation, including 5 articles in journals recommended by the Higher Attestation Commission of the Republic of Uzbekistan for publishing the main results of PhD dissertations, of which 1 was published abroad and 4 in national journals.

The structure and of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion, and a list of references. The total volume of the dissertation is 83 pages.

MAIN SUBJECT OF THE DISSERTATION

In the introduction, the relevance and necessity of the dissertation topic are substantiated through arguments and analytical discussions. It is demonstrated that the research corresponds to the priority directions of the development of science and technologies in the Republic. The degree of study of the problem is highlighted, the aim, objectives, object, and subject of the research are defined, the theoretical and practical significance of the obtained results is described, and detailed information is provided regarding the implementation of the research outcomes, the published works, and the structure of the dissertation.

The first section of the first chapter of the dissertation, titled **"Approximate methods for solving the Cauchy problem for systems of ordinary differential equations"**, contains a concise overview of the main concepts used in the research and commonly applied numerical solution methods. The mathematical model describing the evolution of the system's state over time is typically expressed through a dynamical system. The concept of a dynamical system, as a mathematical object, was first formulated in the works of A. Poincaré and later developed by J. Birkhoff, Denjoy, and others. The study of dynamical systems theory in the form of autonomous systems of differential equations has been significantly advanced by the scientific school led by A.A. Andronov, with many of its results compiled in monographs. These works primarily focused on dynamical systems in a plane (two-dimensional systems).

Let us consider the system

$$\frac{dy}{dx} = f(x,y) \tag{1.1}$$

where, the functions f_k are defined in some domain D of the space $\mathbb{R}^{d+1}$, and satisfy the conditions of being continuous with sufficiently many partial derivatives. System (1.1) has a unique solution passing through the point $(x_0, y_0) \in \mathbb{R}^{d+1}$. Let us denote this solution by $y_\alpha(x)$. In practice, the exact solution $y_\alpha(x)$ is rarely found explicitly. Therefore, approximate methods are employed to obtain an approximate solution $y_t(x)$. Whether for theoretical or applied problems, the question of how close $y_t(x)$ is to the exact solution $y_\alpha(x)$ is of crucial importance. If both functions exist on the interval $k \in [x_0; x_*]$, then the following estimate holds:

$$\max_{x_0 \le x \le x_*} |y_\alpha(x) - y_t(x)| \le C \frac{e^{M_1(x_* - x_0)} - 1}{M_1}, \tag{1.2}$$

where M_1 is the Lipschitz constant, and $C -$ is a constant depending on the chosen approximate solution method. Approximate solution methods are highly diverse, but in practice, one-step methods are most widely used. On the interval $[x_0; x_*]$, the partition $[x_0, x_1, x_2, x_N = x_*]$ is introduced with step size h. If $h = \max_K [x_k - x_{k-1}]$, then for some natural number s, the relation $C = ch^s$ is obtained. Here, s is called the order (degree) of accuracy of the approximate method, and c is a constant depending on the function, the domain D, and the chosen solution method.

In the literature devoted to approximate solution methods, formula (1.2) and the relation $C = ch^s$ are given, but no explicit formula is provided for the coefficient c. Usually, only the heuristic assumption is made that the value of c is very small compared to $\frac{e^{M_1[x_* - x_0]} - 1}{M_1}$. However, the studies of A. Azamov and his students show that this assumption is unfounded. Moreover, in theoretical problems, it is important to derive an explicit estimate for

$$\max_x |y_\alpha(x) - y_t(x)|.$$

Among one-step approximate methods, the most widely applied are the Runge–Kutta methods. In these methods, the values of the function $f(x,y)$ are computed, but its derivatives are not used, which for a long time was considered convenient. In contrast, methods based on the Taylor formula also take into account the derivatives of $f(x,y)$. Today, with the availability of programs capable of symbolic computations and especially in the era of artificial intelligence it is possible to automate the calculation of derivatives of $f(x)$. This explains the choice of Taylor-based methods as the focus of the dissertation.

It should be emphasized that Taylor-based methods possess other advantages as well. In particular, for these methods, determining the constant C becomes significantly more convenient.

Thus, while Runge–Kutta schemes compute only the values of the function f, Taylor schemes compute both its values and derivatives.

Taylor-based schemes have two main advantages:

1. Schemes of arbitrary accuracy can be constructed, and this can be done using a computer;
2. Determining the values of the coefficient C is considerably easier.

The dissertation is therefore mainly devoted to deriving explicit values of the coefficient C for Taylor schemes of the second, third, and fourth orders.

The problem of assessing the accuracy of approximate solutions is covered in the literature on numerical methods with varying levels of detail. Here, we shall focus on several well-known monographs. In Kamke and Sansone-II, the issue of error estimation is not addressed. In the monographs by Lambert, Kendall Atkinson, Babenko, and Hairer E., Norsett S., and Wanner G, only the Runge–Kutta methods are considered Bieberbach L. Part III of the fundamental monograph by Bakhvalov is devoted to methods of approximate solution of ordinary differential equations. In particular, Paragraph 1 of this part discusses the Taylor method from a methodological perspective. In the monograph Computational Methods by Krylov V.I., Bobkov V.V., Monastyrny P.I., Paragraph 2 of Part II touches upon the Taylor method; however, the problem of error estimation is not examined. In this case, when constructing an approximate solution scheme, if the first n terms of the Taylor formula are taken, it is only emphasized that the error satisfies

$$\max_{0\le n\le N}\left|x_n - y(x_n)\right| = O(h^n).$$

The most detailed monograph on methods for approximate solutions of ordinary differential equations is Butcher. In this book, it is stated that the difference between the exact and approximate solutions can be estimated by the remainder term of the Taylor expansion of the exact solution, and it is only emphasized that this term is of order $O(h^{p+1})$ in a single step. In Milne's book, the problem of error estimation is considered for methods based on the approximate evaluation of the integral in the formula

$$x(t+h) = x(t) + \int_t^{t+h} f[x(t)]dt,$$

and an explicit inequality is proved. This is essentially equivalent to considering the first term of the Taylor formula, while methods based on higher-order terms are not addressed. In a later chapter of this book, the estimation of errors for multi-step approximate solution methods is studied; however, an inequality of the form

$$\sup_{0\le t\le T}\left|x_*(t) - \tilde{x}(t)\right| \le Ce^{LT}h^s$$

is not derived. In the second section of this chapter, the problem of constructing polynomial approximations is considered. The concept of an approximate solution of a differential equation can be interpreted in several ways. The most widespread interpretation is the numerical solution of the Cauchy problem, which is primarily applied in practice. In theoretical problems, however, the search for approximate analytic representations of solutions also plays an important role. Entire branches of mathematics, such as the theory of special functions (Bessel, hypergeometric, etc.), have emerged precisely in accordance with this interpretation.

In the vast majority of cases, an approximate analytic solution is sought in the form of a polynomial. In the main results of this dissertation, using a concrete example the “Liouville equation” a polynomial triangle associated with the analytic approximate solution is constructed. This result is presented here.

The Liouville equation has the form $y''+xy=0$, and it cannot be solved in closed form in terms of elementary functions; this was proved by J. Liouville, making it the first known example of a differential equation that is non-integrable.

For this purpose, it is well known that the problem can be reduced to an infinite system of recurrent differential equations of the form

$$y'_n(x)=a_n(x)y_n(x)+b_n(x)y_{n-1}(x)+c_n(x),\;\; n=0,1,2,... \tag{1.3}$$

where $y_0(x)$ is considered given. In this system, the unknowns are computed sequentially through integration. In practice, however, problems often arise that are reducible to a system of equations of the form

$$P_{n+1}^k(x)=a_n{}^k(x)P_n^{k+1}(x)+b_n{}^k(x)(P_n^k)'+c_n{}^k(x)P_n^{k-1}(x) \tag{1.4}$$

where, in particular, the case $a_n{}^k(x)=(k+1)x,\;\; b_n{}^k(x)=1,\;\; c_n{}^k(x)=k-1$

is combinatorially equivalent to the Liouville equation. Here, the functions $P_n{}^k(x)$ are polynomials that form a triangular structure analogous to the binomial coefficients.

From formula (1.4), we obtain the following relations:

$$P_{n+1}^n(x)=(n+1)!x+(n-1)P_n{}^{n-1}(x),\;\; P_{n+1}^{n-1}(x)=\left(P_n{}^{n-1}(x)\right)'+(n-2)P_n{}^{n-2}(x),$$

$$P_{n+1}^k(x)=(k+1)xP_n^{k+1}(x)+(P_n^k)'+(k-1)P_n^{k-1}(x),\;\; P_{n+1}^1=\left(P_n{}^1(x)\right)'+2xP_n{}^2(x),$$

$$P_{n+1}^0(x)=\left(P_n{}^0(x)\right)'+xP_n^1(x)$$

with the initial conditions $P_n^k(x)=\begin{cases}0, & k<0\\ 0, & k=n\\ n!, & k=n+1\end{cases}\qquad (n=0,1,2,...,k=0,1,2,...)$.

Now, returning to the equation $y''+xy=0$, its solution cannot be expressed in terms of elementary functions through algebraic operations and integration (Liouville’s theorem). This equation is equivalent to the Riccati equation

$$z'(x)=x+z^2(x), \tag{1.5}$$

Theorem 1. For the solution $z(x)$ of equation (1.5), the following formula holds:

$$z^{(n)} = n! z^{n+1} + \sum_{k=0}^{n-1} P_n^k(x) z^k .$$

In the second chapter of the dissertation, entitled **"Error Estimation of Approximate Solutions Constructed via the Taylor Formula for Autonomous Systems",** the problem of assessing the accuracy of second-, third-, and fourth-order Taylor-based approximate solution methods for the Cauchy problem of a dynamical system is considered.

The numerical solution of the Cauchy problem of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = \xi \tag{2.1}$$

is of fundamental importance. Here, (2.1) represents a dynamical system where $x = (x_1, x_2, ..., x_n)$ is a vector-valued function, $t \in \mathbb{R},$ and ξ is a given initial point. The function $f(x) = (f_1(x), f_2(x), ..., f_n(x))$ is defined in some domain $D \subset \mathbb{R}^d$ and is assumed to be sufficiently differentiable.

Let $x_*(t)$ be the exact solution of the Cauchy problem (2.1), and let x_n denote its approximate solution at the point $t = t_n$. Then, the accuracy of the numerical method can be estimated by the inequality

$$\sup_n |\hat{x}(t_n) - x_n| \le C(\sup_n |t_n - t_{n-1}|)^k (n = 1, 2, ...), \ldots \tag{2.2}$$

where $k -$ is the order of accuracy of the method and $C -$ is a constant. In computational mathematics, various methods have been developed to numerically solve problem (2.1), and such developments continue to this day. These include different modifications of Runge–Kutta methods, Adams methods, finite difference methods, methods based on neural networks, and others.

It is always important to derive the inequality that assesses the accuracy of the method. This is equivalent to determining the constants such as k and C in inequality (2.1).

In this chapter, the numerical integration method based on the Taylor formula is considered. It is shown that the error of the method is not only related to the remainder term of the Taylor formula but also to other factors. Then, an inequality that provides an upper bound for the difference between the exact and approximate solutions is formulated and proved. Let $x_*(t)$ be the exact solution of problem (2.1) (hereafter, the expression $x(t)$ will denote the approximate solution of problem (2.1)), and let J be the right-hand semi-interval on which the solution $x_*(t)$ exists.

Assumption A. Let the desired solution $x_*(t)$ exist on the interval $[0;T]$. Usually, by numerical integration one understands the computation of the values of the solution $x_*(t)$ at the mesh points, that is, in the set $\{t_0 = 0 < t_1 < t_2 < ... < t_N = T\}$.

We restrict our attention to a uniform mesh, i.e., $t_n = nh,\ h = \frac{T}{N}$. In this work, we consider a one-step numerical method constructed on the basis of the Taylor formula:

$$x_*(t+h) = x_*(t) + h\dot{x}_*(t) + \frac{h^2}{2}\ddot{x}_*(t) + \ldots + \frac{h^m}{m!}x_*^{(m)}(t) + R_m,$$

where the remainder term can be expressed in the following form:

$$R_m = \frac{1}{m!}\int_t^{t+h}(t-s)^m x_*^{(m+1)}(s)ds.$$

Neglecting the remainder term, we obtain the following one-step numerical method for computing the approximate values of the solution:

$$x_*\left[(n+1)h\right] \approx x_*(nh) + h\dot{x}_*(nh) + \frac{h^2}{2}\ddot{x}_*(nh) + \cdots + \frac{h^m}{m!}x_*^{(m)}(nh), \tag{2.3}$$

where the derivatives of the solution are expressed in terms of the function f and its derivatives:

$$\dot{x}_*(nh) = f\left[x_*(nh)\right],$$

$$\ddot{x}_*(nh) = f'\left[x_*(nh)\right]f\left[x_*(nh)\right],$$

$$\dddot{x}_*(nh) = f''\left[x_*(nh)\right]f\left[x_*(nh)\right]f\left[x_*(nh)\right] + f'\left[x_*(nh)\right]f'\left[x_*(nh)\right]f\left[x_*(nh)\right].$$

(2.4)

Recall that if in problem (2.1) we consider a system, i.e., $x \in D \subset \mathbb{R}^d, f: D \to \mathbb{R}^d, d \geq 2,$ then $f'(x)$ is a d-dimensional linear map (i.e., a (1,1)-tensor) represented in the standard basis of $\mathbb{R}^d$ by the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$. Similarly, $f''(x)$ is a d-dimensional bilinear map, or a (1,2)-tensor, with components $\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right); i, j, k = 1, 2, \ldots, d.$ The same notation and interpretation extend naturally to higher-order derivatives of f.

To avoid expressing the essence of the problem in subsequent steps through complicated formulas, we restrict ourselves to a method of third-order accuracy at each step:

$$x_*\left[(n+1)h\right] \approx x_*(nh) + h\dot{x}_*(nh) + \frac{h^2}{2}\ddot{x}_*(nh).$$

If we denote $x_{*n} = x_*(nh)$, then

$$x_{*(n+1)} \approx x_{*n} + hf(x_{*n}) + \frac{h^2}{2}f'(x_{*n})f(x_{*n}). \tag{2.5}$$

Now, instead of the approximating formula (2.5), we consider the following recurrent formula:

$$x_0 = \xi,\ x_{n+1} = x_n + hf(x_n) + \frac{h^2}{2} f'(x_n) f(x_n). \tag{2.6}$$

In numerical mathematics, computations are often carried out using precisely the formula (2.6), and the x_n and x_{*n} are frequently regarded as identical. However, this is not actually the case. This gives rise to a question that can be considered “fundamental” from the viewpoint of computational dynamics: how is the sequence x_n related to the exact solution values $x_*(nh)$?

In our view, within computational dynamics, conclusions about the behavior of the exact solution $x_*(t)$ are sometimes drawn based on the values x_n, even when there is insufficient justification — especially in complex systems such as the Lorenz system, Rössler attractor, the Brusselator, or multi-body motion problems. In reality, however, the situation is significantly more complicated. First and foremost, it is necessary to verify Assumption A mentioned above, as well as the assumptions outlined below. It is worth emphasizing again that while heuristic conclusions or general reasoning might be sufficient for practical applications, proving theorems in the theory of dynamical systems requires precise, mathematically rigorous estimates such as the following:

$$\left| x_*(nh) - x_n \right| \le C(f, T, K, \xi) h^2,$$

where C is a constant depending on the properties of the function f, the time interval T, the compact set K, and the initial condition ξ.

The primary goal is to bridge the existing gap in the mathematical proof of an estimate like the one above for the considered numerical method. In addition, we will demonstrate that in the case $m > 1$ (i.e., when higher-order approximation via the Taylor expansion is applied), the expression estimating the difference $\left| x_*(nh) - x_n \right|$ for the differential equation (2.1) differs from the estimate obtained based solely on the remainder term in the Taylor expansion.

Thus, our interest now lies in estimating the difference $x_*(nh) - x_n$.

To obtain an estimate that is independent of the time variable t, it becomes necessary to impose an additional condition on the solution. Specifically, one needs to select a compact set K such that it lies within the domain of definition $D \subset \mathbb{R}^d$ of the function and contains the trajectory $\left\{ x_*(t) \middle| 0 \le t \le T \right\}$ along with a certain neighborhood of it. This is not a trivial task, as the exact solution $x_*(t)$ is not known in advance. For this reason, we adopt the following assumption:

Assumption B. We assume that the exact solution $x_*(t)$ defined on the interval $[0, T]$ remains within a compact set K, i.e., it does not leave this set during the interval.

Choosing such a compact set K makes it possible to define the following quantities:

$$M_0 = \max_{x\in K}|f(x)|, \quad M_1 = \max_{x\in K}\|f'(x)\|, \quad M_2 = \max_{x\in K}\|f''(x)\|, \dots$$

Then, using assumptions A and B, one can derive an explicit error bound for the approximate solution. In publications dedicated to this problem, the following inequality is often presented:

$$|x_*(nh) - x_n| \le C\left(\frac{e^{M_1T}-1}{M_1}\right)\frac{h^2}{6}, \tag{2.7}$$

which is based on an estimate derived from the remainder term in the Taylor expansion. However, a fully rigorous and flawless proof of inequality (2.7) has not yet been established. Below, we explain the reason for this and provide a strict proof of an alternative inequality that does not rely on (2.7).

Lemma 1. The equality $\tilde{x}(nh) = x_n$ holds for all $n > 0$.

Definition 1. We call the following function

$$\tilde{x}(t) = x_0 + \int_0^t \left\{ f\left[x(\sigma_s)\right] + (s-\sigma_s) f'\left[x(\sigma_s)\right] f\left[x(\sigma_s)\right]\right\} ds$$

the approximate solution of problem (2.1).

Assumption C. $x(t) \in K,$ for $t \in [0,T]$.

Theorem 2. If the assumptions A, B and C are satisfied, then the following estimate holds between the exact solution and the approximate solution of problem (2.1):

$$|x_*(t) - x(t)| \le \frac{e^{M_1T}-1}{M_1}\left(L_0 + L_1h\right)\frac{h^2}{2}.$$

Definition 2. The function

$$\tilde{x}(t) = x_0 +$$

$$+\int_0^t \left\{ f\left[\tilde{x}(\sigma_s)\right] + (s-\sigma_s)(f'f)\left[\tilde{x}(\sigma_s)\right] + \frac{(s-\sigma_s)^2}{2}(f''ff + f'f'f)\left[\tilde{x}(\sigma_s)\right]\right\} ds$$

is called the approximate solution of problem (2.1).

Theorem 3. For the difference between the exact and approximate solutions, the following inequality holds:

$$|x_*(t) - \tilde{x}(t)| \le \frac{e^{M_1T}-1}{6M_1}(L_0 + L_1h + L_2h^2)h^3.$$

Definition 3. The function

$$\tilde{x}(t) = x_0 + \int_0^t \sum_{k=0}^{3} \frac{(s-\sigma_s)^k}{k!} F^k[\tilde{x}(\sigma_s)] ds$$

is called the approximate solution of problem (2.1). (Here, $x^{(k)}(t) = (F^k)[x_*(t)]$.)

Theorem 4. The inequality

$$\left|x_*(t)-\tilde{x}(t)\right| \le Ch^4 \frac{e^{M_1T}-1}{M_1}$$

holds, where

$$C = L_3 + L_{2,3} + L_{1,2,3} + H^1{}_{1,2,3} + H^2{}_{0,1,2,3} + H^3{}_{0,1,2,3} + H^4{}_{0,1,2,3} + H^5{}_{0,1,2,3}.$$

In the third chapter of the dissertation, titled **"Estimating the error of the approximate solution derived via the Taylor formula for non-autonomous systems",** the problem of assessing the accuracy of second- and third-order approximate solution methods based on the Taylor formula for the Cauchy problem of a non-autonomous system is studied.

In several chapters of the theory of differential equations, especially in its applications, the Cauchy problem

$$y' = f(x,y),\ y(x_0) = y_0, \tag{3.1}$$

is considered for computing the approximate solution. In problem (3.1), $y \in D \subset \mathbb{R}^d$, where D is a convex domain, and $f : D \to \mathbb{R}^d$ is a continuous and sufficiently differentiable function. If $y_*(x)$ is the exact solution of problem (3.1) and $y(x)$ is the approximate solution on the interval $(x_0 \le x \le x_*)$, then the accuracy of the approximate solution is expressed by the inequality

$$\sup_{x_0 \le x \le x_*} \left|y_*(x) - y(x)\right| \le Ce^{L(x-x_0)}h^s, \tag{3.2}$$

where L is the Lipschitz constant, h is the maximum step size on the interval used for computing the approximate solution, and s is the order of accuracy of the approximation method.

Definition 4. The function

$$y(x) = y_0 + \int_{x_0}^{x} \{ f[\delta_s, y(\delta_s)] + (s-\delta_s)\Big(f'_y[\delta_s, y(\delta_s)] f[\delta_s, y(\delta_s)] + f'_x[\delta_s, y(\delta_s)] \Big) \} ds$$

is called the approximate solution of the problem (3.1).

Theorem 5. For the difference between the exact and approximate solutions, the following estimate holds:

$$\left|y_*(x) - y(x)\right| \le \frac{e^{L(x-x_0)}-1}{L}(L_1 + L_2h)h^2.$$

Theorem 6. For the difference between the exact and approximate solutions, the following estimate holds:

$$\left|y_*(x) - y(x)\right| \le \frac{e^{M_{01}(x-x_0)}-1}{M_{01}}(L_0 + L_1h + L_2h^2)h^3.$$

Conclusion

This dissertation is devoted to inequalities for estimating the accuracy of approximate computation of trajectories of dynamical systems. The main results of the dissertation are as follows:

A recurrent formula has been constructed for the derivative of the solution of the Liouville equation. A polynomial triangle was used in deriving this formula;

For autonomous systems of differential equations, rigorous proofs of inequalities estimating the errors in second-, third-, and fourth-order approximate methods based on the Taylor formula are presented. The values of the coefficients in these inequalities are explicitly determined;

For non-autonomous systems of differential equations, rigorous proofs of inequalities estimating the errors in second- and third-order approximate methods based on the Taylor formula are provided, with the coefficients in these inequalities explicitly determined.

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ИНСТИТУТЕ МАТЕМАТИКИ ИМЕНИ В.И.РОМАНОВСКОГО

ИНСТИТУТ МАТЕМАТИКИ

АБДУЛЛАЕВ АБДУҒАНИ ХОЛМАМАТ ЎҒЛИ

НЕРАВЕНСТВА ДЛЯ ОЦЕНКИ ТОЧНОСТИ ПРИБЛИЖЁННОГО ВЫЧИСЛЕНИЯ ТРАЕКТОРИЙ ДИНАМИЧЕСКИХ СИСТЕМ

01.01.02 – Дифференциальные уравнения и математическая физика

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ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ

Ташкент – 2025

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ВВЕДЕНИЕ (аннотация диссертации доктора философии(PhD))

Цель исследования состоит описания точные неравенства, оценивающие погрешность методов приближённого решения малых порядков для динамических систем и дифференциальных уравнений, основанных на формуле Тейлора, а также вывести явные формулы для коэффициентов этих оценочных неравенств.

Объект исследования: Неравенства, оценивающие точность приближённого вычисления решения задачи Коши для систем обыкновенных дифференциальных уравнений, а также неравенства, характеризующие точность приближённого вычисления траекторий динамических систем с использованием методов, основанных на формуле Тейлора.

Научная новизна исследования состоит в следующем:

Построена полиномиальный треугольник для производного решений уравнения Лиувиля и с помощью этого треугольника описана реккурентная выражения;

Доказано неравенство, оценивающее погрешность между точным решением автономной системы дифференциальных уравнений и приближёнными решениями, полученными с помощью формулы Тейлора, а также найдены значения коэффициентов в этих неравенствах;

Доказано неравенство, оценивающее погрешность между точным решением неавтономных систем дифференциальных уравнений и приближёнными решениями, полученными с помощью формулы Тейлора, а также найдены значения коэффициентов в этих неравенствах.

Внедрение результатов исследования.

Полученные результаты по неравенствам, оценивающим точность приближённого вычисления траекторий динамических систем внедрена следующих научных проектах:

неравенства, оценивающие точность третьего порядка приближённого метода решения задачи Коши для автономных системы, основанного на формуле Тейлора, были использованы в зарубежном проекте № AUA-UAEU-12S141 на тему «Численные решения дифференциальных уравнений пантографического типа с запаздыванием» для определения порядков среднеквадратичного и сильного сближения решений стохастических пантографических дифференциальных уравнений с запаздыванием (справка Университета Объединённых Арабских Эмиратов от 24 августа 2025 года**)**. Применение научного результата дает возможность сравнению подхода, основанного на формуле Тейлора, с традиционным методом Рунге-Кутты, следовательно позволило повысить точность решений, строго определить оценку ошибки и усовершенствовать численные методы аппроксимации;

результаты по оценке погрешности приближённого решения неавтономных систем дифференциальных уравнений:**,** полученного с помощью формулы Тейлора, были использованы в фундаментальном проекте

№ UZB-Ind-2021-87 на тему «Анализ симметрий Ли, моделирование и анализ устойчивости гиперболических систем по Ляпунову» при исследовании устойчивости систем (справка Национального университета Узбекистана № 04/11-22965 от 16 октября 2025 года**).** Применение данного научного результата позволило исследовать устойчивость динамических систем.

Структура и обьем диссертации. Диссертация состоит из введения, трёх глав, заключения и списка использованной литературы. Объем диссертации составляет 83 страниц.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (Част I; Part I)

1. Abdullayev A.Kh., Azamov A.A., Ruziboev M.B. An explicit estimate for approximate solutions of ODEs based on the Taylor formula. // Ural mathematical journal. Vol. 10, No. 1, 2024, pp. 18-27. (https://doi.org/10.15826/umj.2024.1.002) ***(Scopus, IF=1.3).***
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II bo‘lim (Част II; Part II)

6. Abdullayev. A.X**.** Error analysis for an approxin1ate solution of Cauchy's problem in the accuracy of second-order. // Topical issues and applications of modern mathematics (Scientific conference of young scientists), March 14–15, 2024, Tashkent, Uzbekistan. pp. 12–13.
7. Abdullayev. A.X**.** An explicit estimate for approximate solutions of ODEs based on the Taylor formula. // Actual problem of applied mathamatics and information technologies AlKhwarizmi *2024*, 22-23 October, 2024, Tashkent, Uzbekistan. pp. 27.
8. Abdullayev. A.X**.,** Ruzimuradova D.X. Error estimation for the third-order accuracy approximate solution of the Cauchy problem by the Taylor formula. // Nonclassical equations of mathematical physics and their applications, October 24–26, 2024, Tashkent, Uzbekistan. p. 51.
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11.Abdullayev A. Kh., Abdullayev Sh. A., Ruzimuradova D. Kh. About the Recurrent-differential Equation Related to the Liouville Equation. // Mathematical physics and related problems of the modern analysis. Bukhara city, October 10-11, 2025, Uzbekistan. pp. 141-142.

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