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# A Cauchy-Goursat problem for the generalized Euler-Poisson-Darboux equation

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**Abstract.** In the present work, a Cauchy-Goursat problem in the characteristic triangle for the generalized Euler-Poisson-Darboux equation is investigated. Solution of the problem is defined by using Riemann's method. Theorems on uniqueness and existence of the problem are proved.

**Key words.** Euler-Poisson-Darboux equation, equation of hyperbolic type, Cauchy-Goursat problem, Riemann-Hadamard function, existence of solution, uniqueness of solution.

## 1 Introduction

Mathematical simulations of many problems such as mathematical biology [1], gas and fluid dynamics [2-7], shell theory [8, 9] and mechanics [10, 11] are degenerated by partial differential equations. On the other hand, degenerated differential equations can be expressed by equations with singular coefficients for small terms of equation.

One of the representation of hyperbolic type differential equation with singular coefficients is called Euler-Poisson-Darboux equation, that is,

$$E_{\alpha,\beta}\left(u\right) \equiv u_{\xi\eta} - \frac{\beta}{\xi-\eta}u_{\xi} + \frac{\alpha}{\xi-\eta}u_{\eta} = 0.$$

This equation and more generalized equation $E_{\alpha,\beta}\left(u\right) + \gamma(\xi-\eta)^{-2}u = 0$ were investigated for the first time by Euler for $\alpha = \beta = m,\ \gamma = n\ (m, n \in \mathbb{N})$ in [12]. The famous mathematicians Darboux [9], Poisson [13], Riemann [14] and also many other scientists of the XX century have been engaged in with these equations.

The next hyperbolic type equation with singular coefficients is

$$L^{\gamma}_{\alpha,\beta}(u) \equiv u_{\xi\eta} + \left(\frac{\alpha}{\eta+\xi} + \frac{\beta}{\eta-\xi}\right) u_{\xi} + \left(\frac{\alpha}{\eta+\xi} - \frac{\beta}{\eta-\xi}\right) u_{\eta} + \gamma u = 0. \tag{1.1}$$

In the case $\alpha = 0$, we have $E_{\beta,\beta}\left(u\right) = 0$ and also in the case $\alpha = \beta \neq 0$, by changing variables as $t = \sqrt{\xi}$, $z = \sqrt{\eta}$, the equation $L^{\gamma}_{\alpha,\beta}\left(u\right) = 0$ is shifted to the equation $E_{\beta,\beta}\left(v\right) = 0$, where $v\left(t,z\right) = u\left(\sqrt{\xi}, \sqrt{\eta}\right)$. For this reason, equation $L^{\gamma}_{\alpha,\beta}\left(u\right) = 0$ is called as generalized Euler-Poisson-Darboux equation. In the present work equation (1.1) is considered in the domain $\Delta = \left\{(\xi,\eta) : 0 < \xi < \eta < 1\right\}$.

**Cauchy-Goursat problem.** *Find a function* $u(\xi,\eta) \in C\left(\bar{\Delta}\right)$ *satisfying equation (1.1) and the following conditions in domain* $\Delta$

$$\lim_{\eta-\xi\to+0} (\eta-\xi)^{2\beta}(u_{\xi} - u_{\eta}) = \nu(\xi),\ 0 < \xi < 1, \tag{1.2}$$

$$u(0,\eta) = \psi_1(\eta),\ 0 \leq \eta \leq 1, \tag{1.3}$$

*where* $\nu(\xi)$ *and* $\psi_1(\eta)$ *are given continuous functions.*

The Cauchy-Goursat problem for equation $E_{\alpha,\beta}\left(u\right) = 0$, in the case $0 < \alpha = \beta < 1/2$ in domain $\Delta$, was studied by Gellersted [15]. Riemann-Hadamard function of the Cauchy-Goursat problem was constructed and using it the solution of considered problem was written explicitly. From the results which derived from books [1, pages 235-236] and [16, pages 100-102] for equation $y^m u_{xx} - u_{yy} + ay^{(m/2)-1}u_x = 0$, where $m$ and $a$ are constants with $m > 0$, $|a| < (m/2)$, it follows that Cauchy-Goursat's problem for equation $E_{\alpha,\beta}\left(u\right) = 0$ in the case $\alpha = \left(m-2a\right)/\left(2m+4\right)$, $\beta = \left(m+2a\right)/\left(2m+4\right)$ is uniquely solvable. However, explicit solution of the problem was not given. In [17], Riemann-Hadamard function of Cauchy-Goursat problem was constructed for equation $E_{\alpha,\beta}\left(u\right) = 0$ and solution of the problem was found by Riemann's method. Moreover, theorems on the existence of the solution of considered problem were proved for $0 \leq \alpha, \beta, \alpha+\beta < 1$. Cauchy-Goursat problem for equation $\left(-y\right)^m u_{xx} - x^m u_{yy} = 0$, where $x > 0$, $y < 0$, $m > 0$, in the characteristic triangle was investigated in [18]. Riemann-Hadamard function was constructed, a formula of solution was found and the uniqueness and the existence of the problem was proved.

In [19, 20], for equation

$$u_{xx} - u_{yy} + \frac{2p}{x}u_x - \frac{2q}{y}u_y + \lambda^2 u = 0 \tag{1.4}$$

a formula for the solution of Cauchy-Goursat problem has been found in the case $\psi(x) = 0$. But this formula was not investigated, i. e , theorems on existence were not proved.

In the present article solution of Cauchy-Goursat problem for equation $L^{\gamma}_{\alpha,\beta}(\eta) = 0$, in the case $\alpha > 0$, $0 < \beta < (1/2)$ $\forall\gamma \in \mathbb{R}$ is formulated and the unique solvability of this problem is established.

## 2 Investigation of Cauchy-Goursat problem

We shall solve this problem by Riemann's method. For the solution, we use a function which is called as Riemann-Hadamard function $V(\xi, \eta; \xi_0, \eta_0; \gamma)$, satisfying the following conditions:

***(i)*** The function $V(\xi, \eta; \xi_0, \eta_0; \gamma)$ is a solution of equation (1.1) with respect to $\xi_0, \eta_0$, and also with respect to $\xi$, $\eta$ it satisfies adjoint equation of (1.1), that is,

$$M^{\gamma}_{\alpha,\beta}(V) \equiv V_{\xi\eta} - \frac{\partial}{\partial\xi}\left[\left(\frac{\alpha}{\eta+\xi} + \frac{\beta}{\eta-\xi}\right)V\right] - \frac{\partial}{\partial\eta}\left[\left(\frac{\alpha}{\eta+\xi} - \frac{\beta}{\eta-\xi}\right)V\right] + \gamma V = 0;$$

***(ii)***

$$V_{\eta} - \left[\alpha(\eta+\xi)^{-1} + \beta(\eta-\xi)^{-1}\right]V = 0 \, for \, \xi = \xi_0,$$

and

$$V_{\xi} - \left[\alpha(\eta+\xi)^{-1} - \beta(\eta-\xi)^{-1}\right]V = 0 for \eta = \eta_0;$$

***(iii)*** $V(\xi_0, \eta_0; \xi_0, \eta_0; \gamma) = 1$;

***(iv)*** $\lim\limits_{\eta-\xi\to+0} V(\xi, \eta; \xi_0, \eta_0; \gamma) = 0$;

***(v)*** $\lim\limits_{\eta-\xi\to+0}\left[V_{\eta} - V_{\xi} - 4\beta(\xi-\eta)^{-1}V\right] = 0$;

***(vi)***

$$\lim_{\varepsilon\to0}\Big(\left\{V_{\xi} - \left[\alpha(\eta+\xi)^{-1} - \beta(\eta-\xi)^{-1}\right]V\right\}\big|_{\eta=\xi_0+\varepsilon} - \left\{V_{\xi} - \left[\alpha(\eta+\xi)^{-1} - \beta(\eta-\xi)^{-1}\right]V\right\}\big|_{\eta=\xi_0-\varepsilon}\Big) = 0, \ \varepsilon > 0.$$

Using Riemann and Green-Hadamard functions which were constructed in the work of Kapilevich [20, pages 1480-1481] for equation (1.4) it is not difficult to show that the function $V(\xi,\eta;\xi_0,\eta_0;\gamma)$ has above-mentioned properties *(i)-(vi)* and is defined by the following equality:

$$V(\xi,\eta;\xi_0,\eta_0;\gamma)=\begin{cases} R_1\left(\xi,\eta;\xi_0,\eta_0;\gamma\right) & \text{for} \quad \eta>\xi_0, \\ R_2(\xi,\eta;\xi_0,\eta_0;\gamma) & \text{for} \quad \eta<\xi_0, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} R_1\left(\xi,\eta;\xi_0,\eta_0;\gamma\right) &= \sigma_0\sum_{k=0}^{+\infty}\frac{\sigma_3^k}{(k!)^2}F_3\left(\beta,\alpha,1-\beta,1-\alpha;1+k;\sigma_2,\sigma_1\right), \\ R_2\left(\xi,\eta;\xi_0,\eta_0;\gamma\right) &= \chi\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\frac{(\eta-\xi)^{2\beta}}{\left[(\xi_0-\xi)\left(\eta_0-\eta\right)\right]^{\beta}} \\ &\quad\times\sum_{k=0}^{+\infty}\frac{\sigma_3^k}{k!(1-\beta)_k}H_2\left(\beta-k,\beta,\alpha,1-\alpha;2\beta;\frac{1}{\sigma_2},-\sigma_1\right). \end{aligned}$$

Note that here $\sigma_0=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}$, $\sigma_1=-\frac{(\xi-\xi_0)(\eta-\eta_0)}{(\eta+\xi)(\eta_0+\xi_0)}$, $\sigma_2=\frac{(\xi-\xi_0)(\eta-\eta_0)}{(\eta-\xi)(\eta_0-\xi_0)}$, $\sigma_3=-\gamma\left(\xi-\xi_0\right)\left(\eta-\eta_0\right)$ and $\chi=\Gamma(\beta)/\Gamma(1-\beta)\Gamma(2\beta)$, $\Gamma(z)$ is the Euler's gamma function [21]. Note also that here

$$\begin{aligned} F_3(a,a',b,b',c;x,y) &= \sum_{m,n=0}^{\infty}\frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n}\,m!n!}x^m y^n, \\ H_2(a,b,c,d;e;x,y) &= \sum_{m,n=0}^{+\infty}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!n!}x^m y^n \end{aligned}$$

are Appell's and Horn's function [21], respectively. Moreover, $(z)_n=z(z+1)...(z+n-1)=\Gamma\left(z+n\right)/\Gamma\left(z\right)$ is Pochgammer's symbol [21].

Based on the equalities [20, 22]

$$\begin{aligned} F_3(a,a',b,b';c;x,y) &= \sum_{m=0}^{+\infty}\frac{(a')_m(b')_m}{(c)_m m!}y^m F(a,b;c+m;x), \\ H_2(a,b,c,d;e;x,y) &= \sum_{m=0}^{+\infty}\frac{(-1)^m(c)_m(d)_m}{(1-a)_m m!}y^m F(a-m,b;e;x), \end{aligned}$$

where $F(a,b;c;x)=\sum\limits_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$ is Gauss' hypergeometric function [21], the functions $R_1$ and $R_2$ can be written as

$$R_1\left(\xi,\eta;\xi_0,\eta_0;\gamma\right)=\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}$$

$$\times\sum_{k=0}^{+\infty}\frac{\sigma_3^k}{(k!)^2}\sum_{m=0}^{+\infty}\frac{(\alpha)_m(1-\alpha)_m}{m!(1+k)_m}\sigma_1^m F\left(\beta,1-\beta;1+k+m;\sigma_2\right), \tag{2.1}$$

$$R_2\left(\xi,\eta;\xi_0,\eta_0;\gamma\right)=\chi\left(\frac{\eta+\xi}{\eta_0+\xi_0}\right)^{\alpha}\frac{(\eta-\xi)^{2\beta}}{\left[\left(\xi_0-\xi\right)\left(\eta_0-\eta\right)\right]^{\beta}}$$
$$\times\sum_{k=0}^{+\infty}\frac{\sigma_3^k}{k!(1-\beta)_k}\sum_{m=0}^{+\infty}\frac{(\alpha)_m(1-\alpha)_m}{m!(1-\beta+k)_m}\sigma_1^m F\left(\beta-k-m;\beta;2\beta;\frac{1}{\sigma_2}\right). \quad (2.2)$$

Using (2.2) and (2.3) expansions of functions $R_1$ and $R_2$, one can show that function (2.1) satisfies conditions *(i)-(vi)*.

Now, let the function $u(\xi,\eta)$ be solution of Cauchy-Goursat problem for equation (1.1), and $P(\xi_0,\eta_0)$ be an arbitrary point in domain $\Delta$. We shall find $u(\xi_0,\eta_0)$. In the triangle $O'A'B'$, which is bounded by segments $O'A'$, $A'B'$, $O'B'$ of straight lines $\eta=\xi+\varepsilon$, $\eta=\xi_0-\varepsilon$, $\xi=0$, respectively, and rectangle $B''A''P''P'$, which is bounded by segments $B''A''$, $A''P''$, $P''P'$, $P'B''$ of straight lines $\eta=\xi_0+\varepsilon$, $\xi=\xi_0-2\varepsilon$, $\eta=\eta_0$, $\xi=0$, the following identity is valid:

$$\begin{aligned}2\left[VL^{\gamma}_{\alpha,\beta}(u)-uM^{\gamma}_{\alpha,\beta}(V)\right]&\equiv\frac{\partial}{\partial\eta}\left[Vu_\xi-uV_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uV\right]\\&\quad+\frac{\partial}{\partial\xi}\left[Vu_\eta-uV_\eta+\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uV\right]\\&\equiv 0.\end{aligned} \quad (2.3)$$

Integrating this identity along the triangle $O'A'B'$ and the rectangle $B''A''P''P'$, and then changing the order of the integration of the resultant, we get

$$\left(\int\limits_{\partial O'A'B'}+\int\limits_{\partial B''A''P''P'}\right)\left[Vu_\eta-uV_\eta+\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uV\right]d\eta$$

$$-\left[Vu_\xi-uV_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uV\right]d\xi=0.$$

After computing the integrals in this equality, we obtain

$$\int\limits_0^{\xi_0-2\varepsilon}\left[V\left(u_\eta-u_\xi\right)+u\left(V_\xi-V_\eta+\frac{4\beta}{\eta-\xi}V\right)\right]_{\eta=\xi+\varepsilon}d\xi$$
$$+\int\limits_0^{\xi_0-2\varepsilon}\left[Vu_\xi-uV_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uV\right]_{\eta=\xi_0-\varepsilon}d\xi$$
$$+\int\limits_0^{\xi_0-\varepsilon}\left[uV_\eta-Vu_\eta-\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uV\right]_{\xi=0}d\eta$$

$$
\begin{aligned}
&-\int_{0}^{\xi_0-2\varepsilon}\left[Vu_\xi-uV_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uV\right]_{\eta=\xi_0+\varepsilon}d\xi\\
&+\int_{\xi_0+\varepsilon}^{\eta_0}\left[Vu_\eta-uV_\eta+\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uV\right]_{\xi=\xi_0-2\varepsilon}d\eta\\
&+\int_{0}^{\xi_0-2\varepsilon}\left[Vu_\xi-uV_\xi+\left(\frac{2\alpha}{\eta+\xi}-\frac{2\beta}{\eta-\xi}\right)uV\right]_{\eta=\eta_0}d\xi\\
&+\int_{\xi_0+\varepsilon}^{\eta_0}\left[uV_\eta-Vu_\eta-\left(\frac{2\alpha}{\eta+\xi}+\frac{2\beta}{\eta-\xi}\right)uV\right]_{\xi=0}d\eta\\
=\;&0.
\end{aligned}
$$

Integrating by parts the first terms of all integrals from the second to the sixth, after some evaluations, we obtain

$$
\begin{aligned}
&u\left(\xi_0-2\varepsilon,\eta_0\right)V\left(\xi_0-2\varepsilon,\eta_0;\xi_0,\eta_0;\gamma\right)\\
=\;&u\left(\xi_0-2\varepsilon,\xi_0+\varepsilon\right)V\left(\xi_0-2\varepsilon,\xi_0+\varepsilon;\xi_0,\eta_0;\gamma\right)+\frac{1}{2}u\left(0,\varepsilon\right)V\left(0,\varepsilon;\xi_0,\eta_0;\gamma\right)\\
&-\frac{1}{2}u\left(\xi_0-2\varepsilon,\xi_0-\varepsilon\right)V\left(\xi_0-2\varepsilon,\xi_0-\varepsilon;\xi_0,\eta_0;\gamma\right)\\
&+\int_{\xi_0+\varepsilon}^{\eta_0}u\left[V_\eta-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)V\right]_{\xi=\xi_0-2\varepsilon}d\eta\\
&+\int_{0}^{\xi_0-2\varepsilon}u\left[V_\xi-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)V\right]_{\eta=\eta_0}d\xi\\
&-\int_{0}^{\xi_0-2\varepsilon}\left\{u\left[V_\xi-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)V\right]_{\eta=\xi_0+\varepsilon}\right.\\
&\left.-u\left[V_\xi-\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right)V\right]_{\eta=\xi_0-\varepsilon}\right\}d\xi\\
&+\frac{1}{2}\int_{0}^{\xi_0-2\varepsilon}\left[V\left(u_\xi-u_\eta\right)+u\left(V_\eta-V_\xi-\frac{4\beta}{\eta-\xi}V\right)\right]_{\eta=\xi+\varepsilon}d\xi\\
&+\left(\int_{0}^{\xi_0-\varepsilon}+\int_{\xi_0+\varepsilon}^{\eta_0}\right)V\left[u_\eta+\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)u\right]_{\xi=0}d\eta.
\end{aligned}
\tag{2.4}
$$

We shall search expression (2.5) in the case when $\varepsilon$ converges to zero. By *(iii)* property of $V\left(\xi,\eta;\xi_0,\eta_0;\gamma\right)$ function left-hand side of (2.5) converges to $u\left(\xi_0,\eta_0\right)$. The first, second and third terms in the right-hand side of (2.5), by *(iv)* property of $V\left(\xi,\eta;\xi_0,\eta_0;\gamma\right)$

function, the fourth and fifth terms by *(ii)* property of $V(\xi,\eta;\xi_0,\eta_0;\gamma)$ function, by *(vi)* property of the function sixth term converge to zero. Moreover, by *(v)* property of $V(\xi,\eta;\xi_0,\eta_0;\gamma)$, we have

$$\lim_{\varepsilon\to 0} u\Big[V_\eta - V_\xi - 4\beta(\eta-\xi)^{-1}V\Big]_{\eta=\xi+\varepsilon} = 0.$$

By condition (1.2) and expansion function (2.3) of $R_2(\xi,\eta;\xi_0,\eta_0;\gamma)$, the equality

$$\lim_{\varepsilon\to 0} V(u_\xi - u_\eta)|_{\eta=\xi+\varepsilon} = \chi\left(\frac{2\xi}{\xi_0+\eta_0}\right)^\alpha \frac{\nu(\xi)\,\Xi_2(\alpha,1-\alpha;1-\beta;\sigma_1,\sigma_3)|_{\eta=\xi}}{[(\xi_0-\xi)(\eta_0-\xi)]^\beta}$$

holds, where $\Xi_2(a,b;c;x,y)$ is Hymbert's hypergeometric function [21], that is,

$$\Xi_2(a,b,c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_m}{(c)_{m+n}\,m!n!}x^m y^n, \qquad |x|<1.$$

Finally, by condition (1.3), eigth term converges to

$$\int_0^{\eta_0}\left[\psi'_1(\eta) + \frac{\alpha+\beta}{\eta}\psi_1(\eta)\right]V(0,\eta;\xi_0,\eta_{0,};\gamma)\,d\eta.$$

Now, passing to the limit when $\varepsilon$ converges to zero in (2.5), taking into account the results proved above, we get the representation of solution of Cauchy-Goursat problem (1.1) – (1.3) as

$$\begin{aligned} u(\xi_0,\eta_0) &= \frac{1}{2}\chi\int_0^{\xi_0}\left(\frac{2\xi}{\xi_0+\eta_0}\right)^\alpha \frac{\nu(\xi)\,\Xi_2(\alpha,1-\alpha;1-\beta;\sigma_1,\sigma_3)|_{\eta=\xi}}{[(\xi_0-\xi)(\eta_0-\xi)]^\beta}d\xi \\ &+\int_0^{\eta_0}\left[\psi'_1(\eta)+\frac{\alpha+\beta}{\eta}\psi_1(\eta)\right]V(0,\eta;\xi_0,\eta_0;\gamma)\,d\eta. \end{aligned} \quad (2.5)$$

Therefore, by the induction method on formula (2.6) one can state that if Cauchy-Goursat problem (1.1) – (1.3) has a solution, then it is unique.

**Lemma 2.1** *If $\nu(\xi)$ satisfies Hölder's condition on $[0,1)$ with degree $\delta>\beta$, then the function*

$$I(\xi,\eta) = \frac{1}{2}\chi\int_0^{\xi}\left(\frac{2t}{\xi+\eta}\right)^\alpha \frac{\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)}{[(\xi-t)(\eta-t)]^\beta}dt,$$

*where $y_1 = -(\xi-t)(\eta-t)/[2t(\xi+\eta)]$, $y_2 = -\gamma(\xi-t)(\eta-t)$ possesses the following properties:*

**1.** *The partial derivatives $I_\xi$ and $I_\eta$ can be represented as*

$$\begin{aligned}\frac{\partial I(\xi,\eta)}{\partial \xi} &= \frac{-\alpha I(\xi,\eta)}{\eta+\xi}+\frac{1}{2}\beta\chi\int\limits_0^\xi\left(\frac{2t}{\xi+\eta}\right)^\alpha\frac{\nu(\xi)-\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)}{(\xi-t)^{1+\beta}(\eta-t)^\beta}dt\\ &+\frac{1}{2}\chi\int\limits_0^\xi\left(\frac{2t}{\xi+\eta}\right)^\alpha\frac{\nu(t)}{[(\xi-t)(\eta-t)]^\beta}\frac{\partial}{\partial\xi}\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\,dt\\ &+\frac{1}{2}\chi\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\nu(\xi)\left(\frac{2\xi}{\xi+\eta}\right)^\alpha\\ &\times\left(\frac{\xi}{\eta}\right)^{-\beta}(\eta-\xi)^{-2\beta}F\left(-\beta,1+\alpha-2\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right); \qquad (2.6)\end{aligned}$$

$$\begin{aligned}\frac{\partial I(\xi,\eta)}{\partial \eta} &= \frac{-\alpha I(\xi,\eta)}{\eta+\xi}+\frac{1}{2}\beta\chi\int\limits_0^\xi\left(\frac{2t}{\xi+\eta}\right)^\alpha\frac{\nu(\xi)-\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)}{(\xi-t)^{\beta}(\eta-t)^{1+\beta}}dt\\ &+\frac{1}{2}\chi\int\limits_0^\xi\left(\frac{2t}{\xi+\eta}\right)^\alpha\frac{\nu(t)}{[(\xi-t)(\eta-t)]^\beta}\frac{\partial}{\partial\eta}\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\,dt\\ &-\frac{1}{2}\beta\chi\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(2+\alpha-\beta)}\nu(\xi)\left(\frac{2\xi}{\xi+\eta}\right)^\alpha\\ &\times\left(\frac{\xi}{\eta}\right)^{1-\beta}(\eta-\xi)^{-2\beta}F\left(1-\beta,1+\alpha-2\beta;2+\alpha-\beta;\frac{\xi}{\eta}\right). \qquad (2.7)\end{aligned}$$

**2.** *The function $I(\xi,\eta)$ satisfies equation (1.1) and the following boundary conditions*

$$\lim_{\eta-\xi\to0}(\eta-\xi)^{2\beta}(I_\xi-I_\eta) = \nu(\xi),\ \ 0<\xi<1, \qquad (2.8)$$

$$I(0,\eta) = 0,\ \ 0\le\eta\le1. \qquad (2.9)$$

**Proof.** Firstly, we prove Assertion 1. Let us consider the function

$$I^\varepsilon(\xi,\eta)=\frac{1}{2}\chi\int\limits_0^{\xi-\varepsilon}\frac{\nu(t)}{[(\xi-t)(\eta-t)]^\beta}\left(\frac{2t}{\xi+\eta}\right)^\alpha\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\,dt,$$

where $\varepsilon>0$ is a small enough real number. It is obvious that $\lim\limits_{\varepsilon\to0}I^\varepsilon(\xi,\eta)=I(\xi,\eta)$.

By direct differentiating, we obtain

$$\begin{aligned}\frac{\partial}{\partial\xi}I^\varepsilon(\xi,\eta) &= -\frac{\alpha}{\xi+\eta}I^\varepsilon(\xi,\eta)+\frac{1}{2}\chi\nu(\xi-\varepsilon)[\varepsilon(\eta-\xi+\varepsilon)]^{-\beta}\\ &\times\left(\frac{2\xi-2\varepsilon}{\xi+\eta}\right)^\alpha\Xi_2(\alpha,1-\alpha,1-\beta;y_1,y_2)\big|_{t=\xi-\varepsilon}\end{aligned}$$

$$+\frac{1}{2}\beta\chi\int\limits_0^{\xi-\varepsilon}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu(\xi)-\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)}{(\xi-t)^{1+\beta}(\eta-t)^{\beta}}dt$$

$$+\frac{1}{2}\chi\int\limits_0^{\xi-\varepsilon}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu(t)}{[(\xi-t)(\eta-t)]^{\beta}}\frac{\partial}{\partial\xi}\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\,dt$$

$$-\frac{1}{2}\beta\chi\nu(\xi)\int\limits_0^{\xi-\varepsilon}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}(\xi-t)^{-1-\beta}(\eta-t)^{-\beta}dt. \tag{2.10}$$

We also have

$$\begin{aligned}-\beta\int\limits_0^{\xi-\varepsilon}t^{\alpha}(\xi-t)^{-1-\beta}(\eta-t)^{-\beta}dt &= -\int\limits_0^{\xi-\varepsilon}t^{\alpha}(\eta-t)^{-\beta}\frac{\partial}{\partial t}(\xi-t)^{-\beta}dt\\ &= -(\xi-\varepsilon)^{\alpha}(\eta-\xi+\varepsilon)^{-\beta}\varepsilon^{-\beta}\\ &\quad+\alpha\int\limits_0^{\xi-\varepsilon}t^{\alpha-1}(\xi-t)^{-\beta}(\eta-t)^{-\beta}dt\\ &\quad+\beta\int\limits_0^{\xi-\varepsilon}t^{\alpha}(\xi-t)^{-\beta}(\eta-t)^{-1-\beta}dt.\end{aligned}$$

Taking this into account, equality (2.10) can be written as

$$\begin{aligned}\frac{\partial}{\partial\xi}I^{\varepsilon}(\xi,\eta) &= -\frac{\alpha}{\xi+\eta}I^{\varepsilon}(\xi,\eta)+\frac{1}{2}\chi\left(\frac{2\xi-2\varepsilon}{\xi+\eta}\right)^{\alpha}(\eta-\xi+\varepsilon)^{-\beta}\\ &\quad\times\varepsilon^{-\beta}\Big[\nu(\xi-\varepsilon)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)|_{t=\xi-\varepsilon}-\nu(\xi)\Big]\\ &\quad+\frac{1}{2}\beta\chi\int\limits_0^{\xi-\varepsilon}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu(\xi)-\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)}{(\xi-t)^{1+\beta}(\eta-t)^{\beta}}dt\\ &\quad+\frac{1}{2}\chi\int\limits_0^{\xi-\varepsilon}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu(t)}{[(\xi-t)(\eta-t)]^{\beta}}\frac{\partial}{\partial\xi}\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\,dt\\ &\quad+\frac{1}{2}\chi\nu(\xi)\left(\frac{2}{\xi+\eta}\right)^{\alpha}\left[\alpha\int\limits_0^{\xi-\varepsilon}t^{\alpha-1}(\xi-t)^{-\beta}(\eta-t)^{-\beta}dt\right.\\ &\quad\left.+\beta\int\limits_0^{\xi-\varepsilon}t^{\alpha}(\xi-t)^{-\beta}(\eta-t)^{-1-\beta}dt\right].\end{aligned} \tag{2.11}$$

Now, we investigate the expression

$$l=\alpha\int\limits_0^{\xi}t^{\alpha-1}(\xi-t)^{-\beta}(\eta-t)^{-\beta}dt+\beta\int\limits_0^{\xi}t^{\alpha}(\xi-t)^{-\beta}(\eta-t)^{-1-\beta}dt.$$

Substituting $t = \xi s$ into the integral and taking integral representation of Gauss' hypergeometric function [21]

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int\limits_0^1 t^{a-1} (1-t)^{c-a-1} (1-xt)^{-b} \, dt$$

into account, one can show that

$$\begin{aligned} l &= \xi^{\alpha-\beta}\eta^{-\beta}\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}F\left(\alpha,\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right) \\ &\quad + \frac{\beta\xi^{1+\alpha-\beta}}{\eta^{1+\beta}}\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(2+\alpha-\beta)}F\left(1+\alpha,1+\beta;2+\alpha-\beta;\frac{\xi}{\eta}\right) \\ &= \frac{\xi^{\alpha-\beta}}{\eta^{\beta}}\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(2+\alpha-\beta)}\left[(1+\alpha-\beta)F\left(\alpha,\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right)\right. \\ &\quad \left. + \beta\frac{\xi}{\eta}F\left(1+\alpha,1+\beta;2+\alpha-\beta;\frac{\xi}{\eta}\right)\right]. \end{aligned}$$

Then, using the equalities [21, 23]

$$cF(a,b;c;z) - cF(a+1,b;c;z) + bzF(a+1,b+1;c+1;z) = 0,$$

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z),$$

we get

$$l = \frac{\xi^{\alpha-\beta}\eta^{\beta}}{(\eta-\xi)^{2\beta}}\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}F\left(-\beta,1+\alpha-2\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right). \tag{2.12}$$

Now, passing to limit when $\varepsilon \to 0$ in (2.11), considering equalities (2.12) and

$$\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)|_{t=\xi-\varepsilon} = 1 + \varepsilon O(1),$$

and also properties of the function $\nu(\xi)$, we get equality (2.6).

Similarly, equality (2.7) can be proved, so it is omitted. Hence, Assertion 1 is proved.

Secondly, we prove Assertion 2. Taking into account (2.6), (2.7) and the equality [23]

$$cF(a,b;c;z) + a\,z\,F(a+1,b+1;c+1;z) = cF(a,b+1;c;z), \tag{2.13}$$

we derive

$$\begin{aligned} I_\xi - I_\eta &= \frac{1}{2}\beta\chi\int\limits_0^\xi \left(\frac{2t}{\xi+\eta}\right)^\alpha \frac{(\eta-\xi)\left[\nu(\xi)-\nu(t)\,\Xi_2(\alpha,1-\alpha;1-\beta;y_1,y_2)\right]}{\left[(\xi-t)(\eta-t)\right]^{1+\beta}}dt \\ &\quad + \frac{1}{2}\chi\int\limits_0^\xi \left(\frac{2t}{\xi+\eta}\right)^\alpha \frac{\nu(t)}{\left[(\xi-t)(\eta-t)\right]^{\beta}} \end{aligned}$$

$$\times\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)dt+\frac{1}{2}\chi\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1-\beta\right)}{\Gamma\left(1+\alpha-\beta\right)}$$
$$\times\nu\left(\xi\right)\left(\eta-\xi\right)^{-2\beta}\left(\frac{2\xi}{\xi+\eta}\right)^{\alpha}\left(\frac{\xi}{\eta}\right)^{-\beta}F\left(-\beta,\alpha-2\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right). \quad (2.14)$$

Then, multiplying both sides by $(\eta-\xi)^{2\beta}$ and passing to limit as $\eta-\xi\to+0$, we get

$$\begin{aligned}\lim_{\eta-\xi\to+0}(\eta-\xi)^{2\beta}\left(I_\xi-I_\eta\right) &= \frac{\chi}{2}\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1-\beta\right)}{\Gamma\left(1+\alpha-\beta\right)}\nu\left(\xi\right)F\left(-\beta,\alpha-2\beta;1+\alpha-\beta;1\right)\\ &= \frac{\chi}{2}\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1-\beta\right)}{\Gamma\left(1+\alpha-\beta\right)}\frac{\Gamma\left(1+\alpha-\beta\right)\Gamma\left(1+2\beta\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(1+\beta\right)}\nu\left(\xi\right)\\ &= \nu\left(\xi\right),\ 0<\xi<1,\end{aligned}$$

Note that, here, the following equalities from [21]

$$\Gamma\left(1+z\right)=z\Gamma\left(z\right),\quad F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c-a-b>0$$

are used.

Now, we show that equality (2.9) holds. To do this we prove $L^{\gamma}_{\alpha,\beta}\left(I\right)=0$. From (2.6) it follows that

$$\begin{aligned}\frac{\partial^2 I}{\partial\xi\,\partial\eta} &= \frac{\alpha\left(1-\alpha\right)}{\left(\eta+\xi\right)^2}I-\frac{\alpha}{\eta+\xi}\left(I_\xi+I_\eta\right)\\ &-\frac{1}{2}\beta^2\chi\int_0^{\xi}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu\left(\xi\right)-\nu\left(t\right)\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)}{\left[\left(\xi-t\right)\left(\eta-t\right)\right]^{1+\beta}}dt\\ &-\frac{1}{2}\chi\int_0^{\xi}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu\left(t\right)}{\left[\left(\xi-t\right)\left(\eta-t\right)\right]^{\beta}}\\ &\times\left[\frac{\beta}{\xi-t}\frac{\partial}{\partial\eta}+\frac{\beta}{\eta-t}\frac{\partial}{\partial\xi}-\frac{\partial^2}{\partial\xi\,\partial\eta}\right]\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)dt\\ &+\frac{1}{2}\chi\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1-\beta\right)}{\Gamma\left(1+\alpha-\beta\right)}\nu\left(\xi\right)\left(\frac{2\xi}{\xi+\eta}\right)^{\alpha}\\ &\times\frac{\partial}{\partial\eta}\left[\left(\frac{\xi}{\eta}\right)^{-\beta}\left(\eta-\xi\right)^{-2\beta}F\left(-\beta,1+\alpha-2\beta;2+\alpha-\beta;\frac{\xi}{\eta}\right)\right]. \quad (2.15)\end{aligned}$$

Considering (2.6), (2.7), (2.14) and (2.15), we find

$$L^{\gamma}_{\alpha,\beta}(I)=\frac{1}{2}\chi\int_0^{\xi}\left(\frac{2t}{\xi+\eta}\right)^{\alpha}\frac{\nu\left(t\right)}{\left[\left(\xi-t\right)\left(\eta-t\right)\right]^{\beta}}$$

$$\begin{aligned}&\times\left[\frac{\partial^2}{\partial\xi\,\partial\eta}-\frac{\beta}{\xi-t}\frac{\partial}{\partial\eta}-\frac{\beta}{\eta-t}\frac{\partial}{\partial\xi}+\frac{\beta}{\eta-\xi}\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right.\\&\left.+\frac{\alpha\left(1-\alpha\right)}{\left(\eta+\xi\right)^2}+\gamma\right]\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)dt\\&+\frac{1}{2}\chi\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1-\beta\right)}{\Gamma\left(1+\alpha-\beta\right)}\nu\left(\xi\right)\left(\frac{2\xi}{\xi+\eta}\right)^{\alpha}\\&\times\left\{\beta\left(\frac{\xi}{\eta}\right)^{-\beta}\left(\eta-\xi\right)^{-1-2\beta}F\left(-\beta,\alpha-2\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right)\right.\\&\left.+\frac{\partial}{\partial\eta}\left[\left(\frac{\xi}{\eta}\right)^{-\beta}\left(\eta-\xi\right)^{-2\beta}F\left(-\beta,1+\alpha-2\beta;1+\alpha-\beta;\frac{\xi}{\eta}\right)\right]\right\}.\qquad(2.16)\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}\frac{\beta}{\eta-\xi}\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)&=-\frac{\beta}{2t\left(\xi+\eta\right)}\frac{\partial}{\partial y_1}-\beta\gamma\frac{\partial}{\partial y_2},\\\frac{\beta}{\xi-t}\frac{\partial}{\partial\eta}+\frac{\beta}{\eta-t}\frac{\partial}{\partial\xi}&=-\left[\frac{\beta}{2t\left(\xi+\eta\right)}+\frac{\beta}{\left(\eta+\xi\right)^2}\right]\frac{\partial}{\partial y_1}-2\beta\gamma\frac{\partial}{\partial y_2},\\\frac{\partial^2}{\partial\xi\,\partial\eta}&=\frac{1}{\left(\xi+\eta\right)^2}\left[y_1\left(y_1-1\right)\frac{\partial^2}{\partial y_1^2}-y_2\frac{\partial^2}{\partial y_1\partial y_2}+\left(2y_1-1\right)\frac{\partial}{\partial y_1}\right]\\&\quad-\gamma\left[y_2\frac{\partial^2}{\partial y_2^2}+y_1\frac{\partial^2}{\partial y_1\partial y_2}+\frac{\partial}{\partial y_2}\right].\end{aligned}$$

Then,

$$\begin{aligned}&\left[\frac{\partial^2}{\partial\xi\,\partial\eta}-\frac{\beta}{\xi-t}\frac{\partial}{\partial\eta}-\frac{\beta}{\eta-t}\frac{\partial}{\partial\xi}+\frac{\beta}{\eta-\xi}\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right.\\&\left.+\frac{\alpha\left(1-\alpha\right)}{\left(\eta+\xi\right)^2}+\gamma\right]\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)\\=\;&-\frac{1}{\left(\eta+\xi\right)^2}\left[y_1\left(1-y_1\right)\frac{\partial^2}{\partial y_1^2}+y_2\frac{\partial^2}{\partial y_1\partial y_2}+\left(1-\beta-2y_1\right)\frac{\partial}{\partial y_1}\right.\\&\left.-\alpha\left(1-\alpha\right)\right]\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)\\&-\gamma\left[y_2\frac{\partial^2}{\partial y_2^2}+y_1\frac{\partial^2}{\partial y_1\partial y_2}+\left(1-\beta\right)\frac{\partial}{\partial y_2}-1\right]\Xi_2\left(\alpha,1-\alpha;1-\beta;y_1,y_2\right)\\=\;&0,\qquad(2.17)\end{aligned}$$

in which the function $\Xi_2\left(a,b;c;x,y\right)$ satisfies the system [21]

$$\begin{cases}\left[x(1-x)\dfrac{\partial^2}{\partial x^2}+y\dfrac{\partial^2}{\partial x\partial y}+\left[c-(a+b+1)x\right]\dfrac{\partial}{\partial x}-ab\right]\Xi_2=0,\\\left[y\dfrac{\partial^2}{\partial y^2}+x\dfrac{\partial^2}{\partial x\partial y}+c\dfrac{\partial}{\partial y}-1\right]\Xi_2=0.\end{cases}$$

Now, we consider the expression

$$\begin{aligned} l_1 &= \beta(\eta-\xi)^{-1-2\beta}\left(\frac{\xi}{\eta}\right)^{-\beta} F\left(-\beta, \alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right) \\ &+\frac{\partial}{\partial\eta}\left[(\eta-\xi)^{-2\beta}\left(\frac{\xi}{\eta}\right)^{-\beta} F\left(-\beta, 1+\alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right)\right]. \end{aligned}$$

Applying the formula [21]

$$\frac{d}{dz}z^a F(a,b;c;z) = az^{a-1}F(a+1,b;c;z),$$

and in pursuance of some transformations, we get

$$\begin{aligned} l_1 &= \beta(\eta-\xi)^{-1-2\beta}(\xi/\eta)^{-\beta}\left\{F\left(-\beta, \alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right)\right. \\ &-F\left(-\beta, 1+\alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right) \\ &+\left[\left(1-\frac{\xi}{\eta}\right)F\left(1-\beta, 1+\alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right)\right. \\ &\left.\left.-F\left(-\beta, 1+\alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right)\right]\right\}. \end{aligned}$$

By virtue of the equality [21]

$$c(1-z)F(a,b;c;z) - cF(a-1,b;c;z) = (b-c)zF(a,b;c+1;z),$$

the sum in the quadratic brackets equals to

$$-\left[\beta/(1+\alpha-\beta)\right]\frac{\xi}{\eta}F\left(1-\beta, 1+\alpha-2\beta; 2+\alpha-\beta; \frac{\xi}{\eta}\right).$$

Taking this into account, we get

$$\begin{aligned} l_1 &= \beta(\eta-\xi)^{-1-2\beta}(\xi/\eta)^{-\beta}\left[F\left(-\beta, \alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right)\right. \\ &-F\left(-\beta, 1+\alpha-2\beta; 1+\alpha-\beta; \frac{\xi}{\eta}\right) \\ &\left.-\left[\beta/(1+\alpha-\beta)\right]\frac{\xi}{\eta}F\left(1-\beta, 1+\alpha-2\beta; 2+\alpha-\beta; \frac{\xi}{\eta}\right)\right]. \end{aligned}$$

According to equality (2.13), the expression in the quadratic brackets equals to zero. Thus, $l_1 = 0$. Considering this and (2.17), from (2.16) it follows that $L_{\alpha,\beta}^{\gamma}(I) = 0$. The proof of Lemma 2.1 is completed. ■

**Lemma 2.2** *If $\psi_1(\eta) \in C^2[0,1]$, then the function*

$$\Phi(\xi,\eta) = \int_0^{\eta}\left[\psi_1{}'(t) + (\alpha+\beta)t^{-1}\psi_1(t)\right]V(0,t;\xi,\eta;\gamma)\,dt$$

*possesses the following properties:*

**1.** *The function $\Phi(\xi,\eta)$ and its derivatives are represented as*

$$\begin{aligned}\Phi(\xi,\eta) &= \varphi(\xi)\int_0^\eta s^{-1}V(0,s;\xi,\eta;\gamma)\,ds-\int_0^\xi \varphi'(t)\,dt\int_0^t s^{-1}R_2(0,s;\xi,\eta;\gamma)ds\\ &+\int_\xi^\eta \varphi'(t)\,dt\int_t^\eta s^{-1}R_1(0,s;\xi,\eta;\gamma)ds, \qquad (2.18)\\ \frac{\partial}{\partial\xi}\Phi(\xi,\eta) &= \varphi(\xi)\cos(\beta\pi)\,\xi^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}\\ &+\varphi(\xi)\int_0^\eta s^{-1}\frac{\partial}{\partial\xi}V(0,s;\xi,\eta;\gamma)\,ds\\ &-\int_0^\xi \varphi'(t)\,dt\int_0^t s^{-1}\frac{\partial}{\partial\xi}R_2(0,s;\xi,\eta;\gamma)\,ds\\ &+\int_\xi^\eta \varphi'(t)\,dt\int_t^\eta s^{-1}\frac{\partial}{\partial\xi}R_1(0,s;\xi,\eta;\gamma)\,ds, \qquad (2.19)\\ \frac{\partial}{\partial\eta}\Phi(\xi,\eta) &= \varphi(\eta)\,\eta^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}\\ &+\varphi(\xi)\int_0^\eta s^{-1}\frac{\partial}{\partial\eta}V(0,s;\xi,\eta;\gamma)\,ds\\ &-\int_0^\xi \varphi'(t)\,dt\int_0^t s^{-1}\frac{\partial}{\partial\eta}R_2(0,s;\xi,\eta;\gamma)\,ds\\ &+\int_\xi^\eta \varphi'(t)\,dt\int_t^\eta s^{-1}\frac{\partial}{\partial\eta}R_1(0,s;\xi,\eta;\gamma)\,ds, \qquad (2.20)\end{aligned}$$

*where $\varphi(t)=t\psi_1'(t)+(\alpha+\beta)\,\psi_1(t)$.*

**2.** *The function $\Phi(\xi,\eta)$ satisfies equation (1.1) and the conditions*

$$\lim_{\eta-\xi\to+0}(\eta-\xi)^{2\beta}(\Phi_\xi-\Phi_\eta) = 0,\ \ 0<\xi<1, \qquad (2.21)$$

$$\Phi(0,\eta) = \psi_1(\eta)\,,\ \ 0\le\eta\le 1. \qquad (2.22)$$

**Proof.** Taking into account the process of the induction of formula (2.5) and the designation $\varphi(t)$, the function $\Phi(\xi,\eta)$ can be rewritten as

$$\Phi(\xi,\eta)=\lim_{\varepsilon\to+0}\left(\int_0^{\xi-\varepsilon}+\int_{\xi+\varepsilon}^{\eta}\right)\varphi(t)\,t^{-1}V(0,t;\xi,\eta;\gamma)\,dt.$$

Applying integration by parts with $u = \varphi(t)$, $dv = t^{-1}V(0,t,\xi,\eta;\gamma)dt$, we get

$$\begin{aligned}\Phi(\xi,\eta) &= \lim_{\varepsilon\to+0}\left\{[\varphi(\xi-\varepsilon)\mu_2(\xi-\varepsilon;\xi,\eta)-\varphi(\xi+\varepsilon)\mu_1(\xi+\varepsilon;\xi,\eta)]\right.\\ &\left.-\int_0^{\xi-\varepsilon}\varphi'(t)\mu_2(t,\xi,\eta)\,dt-\int_{\xi+\varepsilon}^{\eta}\varphi'(t)\mu_1(t,\xi,\eta)\,dt\right\},\end{aligned}$$

where

$$\mu_1(t,\xi,\eta) = -\int_t^{\eta} s^{-1}R_1(0,s;\xi,\eta;\gamma)\,ds,\ \ \mu_2(t,\xi,\eta) = \int_0^{t} s^{-1}R_2(0,s;\xi,\eta;\gamma)\,ds.$$

Substituting the expressions for $\mu_1(t,\xi,\eta)$ and $\mu_2(t,\xi,\eta)$, and passing to limit, one can reach equality (2.18).

Now, we rewrite equality (2.18) as

$$\begin{aligned}\Phi(\xi,\eta) &= \lim_{\varepsilon\to+0}\left[\varphi(\xi)\left(\int_0^{\xi-\varepsilon}+\int_{\xi+\varepsilon}^{\eta}\right)s^{-1}V(0,s;\xi,\eta;\gamma)\,ds\right.\\ &-\int_0^{\xi-\varepsilon}\varphi'(t)\,dt\int_0^{t}s^{-1}R_2(0,s;\xi,\eta;\gamma)\,ds\\ &\left.+\int_{\xi+\varepsilon}^{\eta}\varphi'(\eta)\,dt\int_t^{\eta}s^{-1}R_1(0,s;\xi,\eta;\gamma)\,ds\right].\end{aligned}\tag{2.23}$$

Differentiating this equality with respect to $\xi$, we obtain

$$\begin{aligned}\frac{\partial\Phi}{\partial\xi} &= \lim_{\varepsilon\to+0}\left[\varphi(\xi)\left\{s^{-1}V(0,s;\xi,\eta;\gamma)\right\}\Big|_{s=\xi+\varepsilon}^{s=\xi-\varepsilon}\right.\\ &+\varphi'(\xi)\left(\int_0^{\xi-\varepsilon}+\int_{\xi+\varepsilon}^{\eta}\right)s^{-1}V(0,s,\xi,\eta;\gamma)\,ds\\ &+\varphi(\xi)\left(\int_0^{\xi-\varepsilon}+\int_{\xi+\varepsilon}^{\eta}\right)s^{-1}\frac{\partial}{\partial\xi}V(0,s,\xi,\eta;\gamma)\,ds\\ &-\varphi'(\xi-\varepsilon)\int_0^{\xi-\varepsilon}s^{-1}R_2(0,s;\xi,\eta;\gamma)ds\\ &-\varphi'(\xi+\varepsilon)\int_{\xi+\varepsilon}^{\eta}s^{-1}R_1(0,s;\xi,\eta;\gamma)ds\end{aligned}$$

$$-\int\limits_{0}^{\xi-\varepsilon} \varphi'(t)\,dt \int\limits_{0}^{t} s^{-1}\frac{\partial}{\partial\xi}R_2(0,s,\xi,\eta;\gamma)\,ds$$
$$\left.+\int\limits_{\xi+\varepsilon}^{\eta} \varphi'(t)\,dt \int\limits_{t}^{\eta} s^{-1}\frac{\partial}{\partial\xi}R_1(0,s,\xi,\eta;\gamma)\,ds\right].$$

Then, passing to limit, we get

$$\begin{aligned}\frac{\partial\Phi}{\partial\xi} &= \varphi(\xi)\lim_{\varepsilon\to+0}\left[s^{-1}V(0,s;\xi,\eta;\gamma)\right]\Big|_{s=\xi+\varepsilon}^{s=\xi-\varepsilon}\\ &+\varphi(\xi)\int\limits_{0}^{\eta} s^{-1}\frac{\partial}{\partial\xi}V(0,s;\xi,\eta;\gamma)\,ds\\ &-\int\limits_{0}^{\xi}\varphi'(t)\,dt\int\limits_{0}^{t} s^{-1}\frac{\partial}{\partial\xi}R_2(0,s;\xi,\eta;\gamma)\,ds\\ &+\int\limits_{\xi}^{\eta}\varphi'(t)\,dt\int\limits_{t}^{\eta} s^{-1}\frac{\partial}{\partial\xi}R_1(0,s;\xi,\eta;\gamma)\,ds. \end{aligned}\tag{2.24}$$

Using representations (2.2) and (2.3) of functions $R_1$ and $R_2$, and also the formulas [21]

$$\begin{aligned} F(a,b;c;1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c-a-b>0;\\ F(a,b;c;1-x) &= -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}F(a,b;1;x)\ln x\\ &+\frac{\Gamma(a+b)}{\Gamma^2(a)\Gamma^2(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{(k!)^2}\\ &\times\left[\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(a+k)}{\Gamma(a+k)}-\frac{\Gamma'(b+k)}{\Gamma(b+k)}\right]x^k, \end{aligned}$$

it is easy to prove that

$$\lim_{\varepsilon\to+0}\left[s^{-1}V\,(0,s,\xi,\eta;\gamma)\right]\Big|_{s=\xi+\varepsilon}^{s=\xi-\varepsilon}=\cos(\pi\beta)\,\xi^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}.\tag{2.25}$$

Then, substituting (2.23) into (2.22), we obtain (2.19).

Further, differentiating (2.23) with respect to $\eta$, we find

$$\begin{aligned}\frac{\partial\Phi}{\partial\eta} &= \lim_{\varepsilon\to+0}\Big[\varphi(\xi)\,\eta^{-1}\,R_1(0,\eta;\xi,\eta;\gamma) \\ &+\varphi(\xi)\left(\int\limits_0^{\xi-\varepsilon}+\int\limits_{\xi+\varepsilon}^{\eta}\right)s^{-1}\frac{\partial}{\partial\eta}V(0,s;\xi,\eta;\gamma)\,ds \\ &+\eta^{-1}R_1(0,\eta;\xi,\eta;\gamma)\left[\varphi(\eta)-\varphi(\xi+\varepsilon)\right] \\ &-\int\limits_0^{\xi-\varepsilon}\varphi'(t)\,dt\int\limits_0^{t}s^{-1}\frac{\partial}{\partial\eta}R_2(0,s;\xi,\eta;\gamma)\,ds \\ &+\int\limits_{\xi+\varepsilon}^{\eta}\varphi'(t)\,dt\int\limits_0^{t}s^{-1}\frac{\partial}{\partial\eta}R_1(0,s;\xi,\eta;\gamma)\,ds\Big].\end{aligned}$$

Hence, passing to limit and considering

$$\eta^{-1}R_1(0,\eta;\xi,\eta;\gamma)=\eta^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta},$$

one can get (2.20).

Similarly, differentiating equality (2.20) with respect to $\xi$, we obtain

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\xi\partial\eta} &= -\varphi(\eta)\,\eta^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right) \\ &+\lim_{\varepsilon\to+0}\Big[\varphi(\xi)\left\{s^{-1}\frac{\partial}{\partial\eta}V\;(0,s,\xi,\eta;\gamma)\right\}\Big|_{s=\xi+\varepsilon}^{s=\xi-\varepsilon} \\ &+\varphi'(\xi)\left(\int\limits_0^{\xi-\varepsilon}+\int\limits_{\xi+\varepsilon}^{\eta}\right)s^{-1}\frac{\partial}{\partial\eta}V(0,s;\xi,\eta;\gamma)\,ds \\ &+\varphi(\xi)\left(\int\limits_0^{\xi-\varepsilon}+\int\limits_{\xi+\varepsilon}^{\eta}\right)s^{-1}\frac{\partial^2}{\partial\xi\partial\eta}V(0,s;\xi,\eta;\gamma)\,ds \\ &-\varphi'(\xi-\varepsilon)\int\limits_0^{\xi-\varepsilon}s^{-1}\frac{\partial}{\partial\eta}R_2(0,s;\xi,\eta;\gamma)ds \\ &-\varphi'(\xi+\varepsilon)\int\limits_{\xi+\varepsilon}^{\eta}s^{-1}\frac{\partial}{\partial\eta}R_1(0,s;\xi,\eta;\gamma)ds\end{aligned}$$

$$
\left. -\int\limits_0^{\xi-\varepsilon} \varphi'(t)\,dt \int\limits_0^t s^{-1} \frac{\partial^2}{\partial\xi\partial\eta} R_2(0,s;\xi,\eta;\gamma)\,ds + \int\limits_{\xi+\varepsilon}^{\eta} \varphi'(t)\,dt \int\limits_t^{\eta} s^{-1} \frac{\partial}{\partial\xi\partial\eta} R_1(0,s;\xi,\eta;\gamma)\,ds \right] .
$$

Hence, passing to limit and considering equality (2.25), we get

$$
\begin{aligned}
\frac{\partial^2\Phi}{\partial\xi\partial\eta} &= -\varphi(\eta)\,\eta^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}\left(\frac{\alpha}{\eta+\xi}-\frac{\beta}{\eta-\xi}\right) \\
&\quad -\cos(\pi\beta)\varphi(\xi)\,\xi^{\alpha+\beta-1}(\eta+\xi)^{-\alpha}(\eta-\xi)^{-\beta}\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right) \\
&\quad +\varphi(\xi)\int\limits_0^{\eta} s^{-1}\frac{\partial^2}{\partial\xi\partial\eta}V(0,s;\xi,\eta;\gamma)\,ds \\
&\quad -\int\limits_0^{\xi}\varphi'(t)\,dt\int\limits_0^t s^{-1}\frac{\partial^2}{\partial\xi\partial\eta}R_2(0,s;\xi,\eta;\gamma)\,ds \\
&\quad +\int\limits_{\xi}^{\eta}\varphi'(t)\,dt\int\limits_t^{\eta} s^{-1}\frac{\partial}{\partial\xi\partial\eta}R_1(0,s;\xi,\eta;\gamma)\,ds. \qquad (2.26)
\end{aligned}
$$

From (2.18), (2.19), (2.20) and (2.26) it follows that

$$
\begin{aligned}
L^{\gamma}_{\alpha,\beta}(\Phi) &= \varphi(\xi)\int\limits_0^{\eta} s^{-1}L^{\gamma}_{\alpha,\beta}\left[V(0,s;\xi,\eta;\gamma)\right]ds \\
&\quad -\int\limits_0^{\xi}\varphi'(t)\,dt\int\limits_0^t s^{-1}L^{\gamma}_{\alpha,\beta}\left[R_2(0,s;\xi,\eta;\gamma)\right]ds \\
&\quad +\int\limits_{\xi}^{\eta}\varphi'(t)\,dt\int\limits_t^{\eta} s^{-1}L^{\gamma}_{\alpha,\beta}\left[R_1(0,s;\xi,\eta;\gamma)\right]ds,
\end{aligned}
$$

whence on the basis of the properties of the functions $R_1$ and $R_2$ it follows that $L^{\gamma}_{\alpha,\beta}(\Phi) \equiv 0$, i. e., the function $\Phi(\xi,\eta)$ satisfies equation (1.1).

Now, using equalities (2.19), (2.20) and easy-tested equality

$$
\lim_{\eta-\xi\to+0}(\eta-\xi)^{2\beta}\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)R_2(0,s;\xi,\eta;\gamma)\,ds = 0,
$$

it is easy to validate equation (2.21).

We now prove equality (2.22). By virtue of $\sigma_0 = (\eta/\eta_0)^{\alpha+\beta}$, $\sigma_1=\sigma_2=\sigma_3=0$ for $\xi=\xi_0=0$, from (2.1) and (2.2) it follows that $V(0,t;0,\eta;\gamma)=(t/\eta)^{\alpha+\beta}$. Considering this and the equality

$$\left[\psi'_1(t)+(\alpha+\beta)\,t^{-1}\psi_1(t)\right]t^{\alpha+\beta}=\left[t^{\alpha+\beta}\psi_1(t)\right]',$$

we get

$$\begin{aligned}\Phi(0,\eta) &= \int\limits_0^\eta \left[\psi_1'(t)+(\alpha+\beta)\,t^{-1}\psi_1(t)\right]V(0,t;0,\eta;\gamma)\,dt\\ &= \int\limits_0^\eta \left[\psi_1'(t)+(\alpha+\beta)\,t^{-1}\psi_1(t)\right](t/\eta)^{\alpha+\beta}dt\\ &= \psi_1(\eta),\ \ 0\le\eta\le 1,\end{aligned}$$

Thus, the proof of Lemma 2.2 is completed. ■

On the basis of proof of Lemma 2.2 we present the following theorem.

**Theorem 2.3** *If $\nu(\xi)$ satisfies Hölder's condition with degree $\delta>\beta$ on $[0,1)$ and if $\psi_1(\eta)\in C^2[0,1]$, then the function defined by formula (2.6)is the unique solution of Cauchy-Goursat problem (1.1)-(1.3).*

Now, we consider Cauchy-Goursat problem for equation (1.1) which is formulated as follows: find a function $u(\xi,\eta)\in C\left(\bar{\Delta}\right)$ in domain $\Delta$ satisfying equation (1.1), conditions (1.2) and

$$u(\xi,1)=\psi_2(\xi),\ \ 0\le\xi\le 1. \tag{2.27}$$

The Riemann-Hadamard function $\tilde{V}(\xi,\eta;\xi_0,\eta_0;\gamma)$ of this problem is defined by

$$\tilde{V}(\xi,\eta;\xi_0,\eta_0;\gamma)=\begin{cases}R_1(\xi,\eta;\xi_0,\eta_0;\gamma) & \text{for } \xi<\eta_0,\\ R_2(\xi,\eta;\xi_0,\eta_0;\gamma) & \text{for } \xi>\eta_0,\end{cases}$$

where $R_1$ and $R_2$ are the functions which are formulated by (2.2) and (2.3).

The function $\tilde{V}(\xi,\eta;\xi_0,\eta_0;\gamma)$ satisfies conditions *(i)-(v)* of the function $V$ and

$$\lim_{\varepsilon\to 0}\left\{\left[\tilde{V}_\eta-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)\tilde{V}\right]_{\xi=\eta_0+\varepsilon}-\left[\tilde{V}_\eta-\left(\frac{\alpha}{\eta+\xi}+\frac{\beta}{\eta-\xi}\right)\tilde{V}\right]_{\xi=\eta_0-\varepsilon}\right\}=0,\varepsilon>0.$$

Applying the same method, which is used above, one can prove the following theorem.

**Theorem 2.4** *If $\nu(\xi)$ satisfies Hölder's condition with degree $\delta>\beta$ on $(0,1]$ and if $\psi_2(\eta)\in C^2[0,1]$, then the solution of Cauchy-Goursat problem $\{(1.1),(1.2),(2.26)\}$ uniquely* *exists and it is defined by the formula*

$$\begin{aligned} u(\xi_0,\eta_0) &= \frac{1}{2}\chi\int\limits_{\eta_0}^{1}\left(\frac{2\xi}{\eta_0+\xi_0}\right)^{\alpha}\frac{\nu(\xi)}{[(\xi-\xi_0)(\xi-\eta_0)]^{\beta}} \\ &\quad \times\Xi_2(\alpha,1-\alpha;1-\beta;\sigma_1,\sigma_3)|_{\eta=\xi}d\xi \\ &\quad -\int\limits_{\xi_0}^{1}\left[\psi'_2(\xi)+\left(\frac{\alpha}{1+\xi}-\frac{\beta}{1-\xi}\right)\psi_2(\xi)\right]\tilde{V}(\xi,1;\xi_0,\eta_0;\gamma)\,d\xi. \quad (2.28)\end{aligned}$$

# 3 Conclusion

In this paper, the solutions of Cauchy-Goursat problems (1.1)−(1.3) and {(1.1), (1.2), (2.26)} are studied by above-mentioned method, in the case when condition (1.2) is replaced by

$$\lim_{\eta-\xi\to+0}u(\xi,\eta)=\tau(\xi)\,,\ 0\le\xi\le1.$$

Moreover, we note that formulas (2.6) and (2.28), which were taken for solution of Cauchy-Goursat problem, can be used for investigating boundary-value problems for mixed type equations.

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