O’ZBEKISTON RESPUBLIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

JIZZAX POLITEXNIKA INSTITUTI

SERVIS fakulteti

OLIY MATEMATIKA kafedrasi

OLIY MATEMATIKADAN MISOL VA MASALALAR YECHISH

(*Bir o‘zgaruvchi funksiyasining integral hisobi*)

Jizzax – 2020

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"Oliy matematikadan misol va masalalar yechish" (*Bir o'zgaruvchi funksiyasining integral hisobi*) amaliy mashg`ulotlar bo`yicha uslubiy qo'llanma.–Jizzax: JizPI nashri, 2020. -60 bet.

Ushbu qo'llama oliy texnika o'quv yurtlarining "Oliy matematika" fani o'quv dasturi asosida yozilgan bo'lib barcha yo'nalishlarida ta'lim olayotgan bakalavr talabalari uchun mo'ljallangan.

Undan oliy texnika o'quv yurtlarining bakalavr yo'nalishlarida ta'lim olayotgan barcha talabalar matematik fanlarni o'rganishda qo'shimcha adabiyot sifatida foydalanishlari mumkin.

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K I R I SH

Ushbu qo’llama oliy texnika o’quv yurtlarining barcha yo’nalishlarida ta’lim olayotgan bakalavr talabalari uchun mo’ljallangan. Qo’llanma “Oliy matematika” fani o’quv dasturi asosida yozilgan bo’lib, bir o’zgaruvchili funksiyasining integral hisobi bo’limiga bag’ishlanadi . Unda ushbu bo’limni chuqur o’rganish va bilimlarni amaliy masalalarga tatbiq etish maqsadida zarur bo’lgan nazariy tushunchalar va amaliy topshiriqlar majmui berilgan.

ANIQMAS INTEGRAL

Boshlang‘ich funksiya va aniqmas integral.

Berilgan funksiyaning hosilasini topish differensial hisobning asosiy masalalaridan biri hisoblanadi. Matematik analiz masalalarining turliligi, uning geometriya, mexanika, fizika va texnikadagi keng miqyosdagi tatbiqi berilgan $f(x)$ funksiya uchun hosilasi shu funksiyaga teng bo‘lgan $F(x)$ funksiyani topishga olib keladi.

Funksiyaning berilgan hosilasiga ko‘ra uning o‘zini topish masalasi integral hisobning asosiy masalalaridan biri hisoblanadi.

$y=f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo‘lsin.

1-ta’rif. Agar $(a;b)$ intervalda differensiallanuvchi $F(x)$ funksiyanig hosilasi berilgan $f(x)$ funksiyaga teng, ya’ni $F'(x)=f(x)$ (yoki $dF(x)=f(x)dx$), $x\in(a;b)$ bo‘lsa, $F(x)$ funksiyaga $(a;b)$ intervalda $f(x)$ funksiyaning *boshlang‘ich funksiyasi* deyiladi.

Masalan: $F(x)=x^3$ funksiya sonlar oqida $f(x)=3x^2$ funksiyaning boshlang‘ich funksiyasi bo‘ladi, chunki $x\in R$ da $(x^3)'=3x^2$;

Lemma. Agar $F(x)$ va $\Phi(x)$ funksiyalar $(a;b)$ intervalda $f(x)$ funksiyaning boshlang‘ich funksiyalari bo‘lsa, u holda $F(x)$ va $\Phi(x)$ bir- biridan o‘zgarmas songa farq qiladi..

2-ta’rif. $f(x)$ funksiyaning $(a;b)$ intervaldagi boshlang‘ich funksiyalari to‘plamiga $f(x)$ funksiyaning *aniqmas integrali* deyiladi va $\int f(x)dx$ kabi belgilanadi.

Shunday qilib, ta’rifga ko‘ra

$$\int f(x)dx=F(x)+C, \qquad (1.1)$$

bu yerda $f(x)-$*integral ostidagi funksiya,* $f(x)dx-$*integral ostidagi ifoda,*
$x-$*integrallash o‘zgaruvchisi,* $\int$ $-$*integrallash belgisi* deb ataladi.

Aniqmas integralni topish, ya’ni berilgan funksiyaning boshlang‘ich funksiyalari to‘plamini aniqlash masalasi *funksiyani integrallash* deyiladi.

2-teorema.Agar $f(x)$ funksiya $[a;b]$kesmada uzluksiz bo‘lsa, u holda u bu kesmada uzluksiz bo‘lgan boshlang‘ich funksiyaga ega bo‘ladi.

Aniqmas integral quyidagi **xossalar**ga ega.

1^o. Aniqmas integralning hosilasi (differensiali) integral ostidagi funksiyaga (ifodaga) teng: $(\int f(x)dx)' = f(x)$. $(d\int f(x)dx = f(x)dx)$.

2^o. Funksiya differentsialining aniqmas integrali shu funksiya bilan o‘zgarmas sonning yig‘indisiga teng: $\int dF(x) = F(x) + C$.

3^o. Ozgarmas ko‘paytuvchini aniqmas integral belgisidan tashqariga chiqarish mumkin: $\int kf(x)dx = k\int f(x)dx,\ k = const, k \neq 0.$

4^o. Chekli sondagi funksiyalar algebraik yig‘indisining aniqmas integrali shu funksiyalar aniqmas integrallarining algebraik yig‘indisiga teng:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

5^o. Agar $\int f(x)dx = F(x) + C$ bo‘lsa, u holda$x$ ning istalgan differensiallanuvchi funksiyasi $u = u(x)$ uchun $\int f(u)du = F(u) + C$ bo‘ladi.

Quyida keltiriladigan integrallar *asosiy integrallar jadvali* deyiladi.

Asosiy integrallar jadvali

1. $\int u^\alpha du = \dfrac{u^{\alpha+1}}{\alpha+1} + C,\ (\alpha \neq -1)$; 2. $\int \dfrac{du}{u} = \ln|u| = C;$

3. $\int a^u du = \dfrac{a^u}{\ln a} + C,\ (0 < a \neq 1);$ 4. $\int e^u du = e^u + C;$

5. $\int \sin u du = -\cos u + C;$ 6. $\int \cos u du = \sin u + C;$

7. $\int tgu du = -\ln|\cos u| = C;$ 8. $\int ctgu du = \ln|\sin u| = C;$

9. $\int \dfrac{du}{\cos^2 u} = tgu + C;$ 10. $\int \dfrac{du}{\sin^2 u} = -ctgu + C;$

11. $\int \dfrac{du}{\sin u} = \ln\left|tg\dfrac{u}{2}\right| + C;$ 12. $\int \dfrac{du}{\cos u} = \ln\left|tg\left(\dfrac{u}{2} + \dfrac{\pi}{4}\right)\right| + C;$

13. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C;$ 14. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C.$

15. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} arctg\dfrac{u}{a} + C;$ 16. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C;$

17. $\int shudu = chu + C;$ 18. $\int chudu = shu + C;$

19. $\int \dfrac{du}{ch^2 u} = thu + C;$ 20. $\int \dfrac{du}{sh^2 u} = -cthu + C.$

Asosiy integrallar jadvalida integrallash o‘zgaruvchisi u erkli o‘zgaruvchi yoki erkli o‘zgaruvchining funksiyasi bo‘lishi mumkin.

Bevosita integrallash usuli

Integral ostidafi funksiyada (yoki ifodada) almashtirishlar bajarish va aniqmas integralning xossalarini qo‘llash orqali berilgan integralni bir yoki bir nechta jadval integraliga kelnitib integrallash usuliga *bevosita integrallash usuli*deyiladi.

Misollar:

1) $\int\left(5\sin x-\frac{2}{x^2+1}+x^3\right)dx=5\int\sin x dx-2\int\frac{dx}{x^2+1}+\int x^3dx=$

$=-5\cos x-2arctgx+\frac{x^4}{4}+C$

2) $\int\frac{\cos 2x}{\cos^2 x\sin^2 x}dx=\int\frac{\cos^2 x-\sin^2 x}{\cos^2 x\sin^2 x}dx=\int\left(\frac{1}{\sin^2 x}-\frac{1}{\cos^2 x}\right)dx=$

$=\int\frac{dx}{\sin^2 x}-\int\frac{dx}{\cos^2 x}=-ctgx-tgx+C=-\frac{2}{\sin 2x}+C;$

3) $\int\frac{x^4}{1+x^2}dx=-\int\frac{1-x^4-1}{1+x^2}dx=-\int(1-x^2)dx+\int\frac{dx}{1+x^2}=$

$=-\int dx+\int x^2dx+\int\frac{dx}{1+x^2}=-x+\frac{x^3}{3}+arctgx+C.$

Berilgan integralni jadval integrallariga keltirishda differensialning quyidagi almashtirishlari («differensial amali ostiga kiritish» jarayoni) qo‘llaniladi:

$$du=d(u+a),\ a-\text{son};\ \ du=\frac{1}{a}d(au);\ udu=\frac{1}{2}d(u^2);\ \cos udu=d(\sin u);$$

$$\sin udu=-d(\cos u);\ \frac{1}{u}du=d(\ln u);\ \frac{1}{\cos^2 u}du=d(tgu).$$

Umuman olganda, $f'(u)du=d(f(u))$. Bu formuladan integrallarni topishda ko‘p foydalaniladi.

Misollar:

$$1)\ \int\frac{dx}{16+9x^2}=\frac{1}{3}\int\frac{d(3x)}{16+(3x)^2}==\frac{1}{3}\cdot\frac{1}{4}arctg\frac{3x}{4}+C=\frac{1}{12}arctg\frac{3x}{4}+C;$$

$$2)\ \int\frac{\cos x+\sin x}{\sin x-\cos x}dx=\int\frac{d(\sin x-\cos x)}{\sin x-\cos x}=\ln|\sin x-\cos x|+C.$$

O‘rniga qo‘yish (o‘zgaruvchini almashtirish) usuli

Ko‘p hollarda integraldagi o‘zgaruvchini almashtirish uni bevosita integrallashga olib keladi. Integrallashning bu usuli *o‘rniga qo‘yish* (*o‘zgaruvchini* *almashtirish*) *usuli* deb yuritiladi. Bu usul quyidagi teoremaga asoslanadi.

2-teorema. Biror T oraliqda aniqlangan va differensiallanuvchi $x=\varphi(t)$ funksiyaning qiymatlar sohasi X oraliqdan iborat bo‘lib, X da $f(x)$ funksiya aniqlangan va uzluksiz, ya’ni T oraliqda $f(\varphi(t))$ murakkab funksiya aniqlangan va uzluksiz bo’lsin. U holda

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt \qquad (1.2)$$ bo‘ladi.

1-misol. $\int x\sqrt{x-3}dx$ integralni toping.

Yechish. $\sqrt{x-3}=t$ almashtirish bajaramiz. U holda $x=t^2+3,\ dx=2tdt.$

Shu sababli

$$\int x\sqrt{x-3}dx=\int(t^2+3)\,t\cdot 2tdt=2\int(t^4+3t^2)dt=$$

$$=2\int t^4dt+6\int t^2dt=2\cdot\frac{t^5}{5}+6\cdot\frac{t^3}{3}+C=\frac{2}{5}\sqrt{(x-3)^5}+2\sqrt{(x-3)^3}+C.$$

2- misol. $\int\frac{\sqrt{1+\ln x}}{x\ln x}dx$ integralni toping.

Yechish. $1+\ln x=t^2$ bo’lsin. Bundan $\ln x=t^2-1,\quad \frac{dx}{x}=2tdt.$ U holda

(1.2) formulaga ko‘ra

$$\int\frac{\sqrt{1+\ln x}}{x\ln x}dx=\int\frac{t\cdot 2tdt}{t^2-1}=2\int\frac{t^2dt}{t^2-1}=2\int\left(1+\frac{1}{t^2-1}\right)dt=2\left(t+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right)+C=$$

$$=2t+\ln\left|\frac{(t-1)^2}{t^2-1}\right|+C=2\sqrt{1+\ln x}+\ln\left|\frac{(\sqrt{1+\ln x})^2}{1+\ln x-1}\right|+C=$$

$$=2\sqrt{1+\ln x}+2\ln\left|\sqrt{1+\ln x}-1\right|-\ln\left|\ln x\right|+C.$$

3- misol. $\int\sqrt{1+\cos^2 x}\sin 2xdx$ integralni toping.

Yechish. $1+\cos^2 x=t^2$ deymiz.

Bundan $-2\cos x\sin xdx=2tdt$ yoki $\sin 2xdx=-2tdt$.

U holda $\int\sqrt{1+\cos^2 x}\sin 2xdx=\int t(-2t)dt=-2\cdot\frac{t^3}{3}+C=-\frac{2}{3}\sqrt{(1+\cos^2 x)^3}+C.$

Bo‘laklab integrallash usuli

Bo‘laklab integrallash usuli ikki funktsiya ko‘paytmasining differentsiali formulasiga asoslanadi.

3-teorema. $u(x)$ va $v(x)$ funksiyalar qandaydir X oraliqda aniqlangan va differentsiallanuvchi bo‘lib, $u'(x)v(x)$ funksiya bu oraliqdaboshlang‘ich funksiyaga ega, y’ani $\int u'(x)v(x)dx$ integral mavjud bo‘lsin. U holda X oraliqda $u(x)v'(x)$ funksiyaboshlang‘ich funksiyaga ega va

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \qquad (1.3)$$

bo‘ladi.

(1.3) formulaga *aniqmas integralni bo‘laklab integrallash* formulasi deyiladi. Ma’lumki, $v'(x)dx = dv, u'(x)dx = du$. Demak, (1.3) formulani

$$\int udv = uv - \int vdu \qquad (1.4)$$

ko‘rinishda yozish mumkin.

Bo‘laklab integrallash usulining mohiyati berilgan integralda integral ostidagi $f(x)dx$ ifodani udv ko‘paytma shaklida tasvirlash va (1.4) formulani qo‘llagan holda berilgan $\int udv$ integralni oson integrallanadigan $\int vdu$ integral bilan almashtirib topishdan iborat.

Bo‘laklab integrallash orqali topiladigan integrallarning asosan uchta guruhini ajratish mumkin:

1) $\int P(x)arctgxdx$, $\int P(x)arcctgxdx$, $\int P(x)\ln xdx$, $\int P(x)\arcsin xdx$, $\int P(x)\arccos xdx$ (bu yerda $P(x)$ −ko‘phad) ko‘rinishdagi 1-guruh integrallar. Bunda $dv = P(x)dx$ deb, qolgan ko‘paytuvchilar esa u bilan belgilanadi;

2) $\int P(x)e^{kx}dx$, $\int P(x)\sin kxdx$, $\int P(x)\cos kxdx$ ko‘rinishdagi 2-guruh integrallar. Bunda $u = P(x)$ deb, qolgan ko‘paytuvchilar dv deb olinadi;

3) $\int e^{kx}\sin kxdx$, $\int e^{kx}\cos kxdx$ ko‘rinishdagi 3-guruh integrallar bo‘laklab integrallash formulasini takroran qo‘llash orqali topiladi.

4- misol. $\int arctgxdx$ integralni toping.

Yechish. $$\int arctgxdx = \left| \begin{array}{c} u = arctgx, \ \ du = \dfrac{dx}{1+x^2}, \\ dv = dx, \ \ v = x \end{array} \right| = xarctgx - \int \frac{x}{1+x^2}dx =$$

$$= xarctgx - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}dx = xarctgx - \frac{1}{2}\ln\left|1+x^2\right| + C.$$

5- misol. $I = \int \sin xe^{2x}dx$ integralni toping.

Yechish.

$$I = \int \sin xe^{2x}dx = \left| \begin{array}{c} u = e^{2x}, \ \ du = 2e^{2x}dx \\ dv = \sin xdx, \ \ v = -\cos x \end{array} \right| = -e^{2x}\cos x + 2\int e^{2x}\cos xdx =$$

$$= \left| \begin{array}{c} u = e^{2x}, \ \ du = 2e^{2x}dx \\ dv = \cos xdx, \ \ v = \sin x \end{array} \right| = -e^{2x}\cos x + 2(^{2x}\sin x - 2\int e^{2x}\sin xdx) =$$

$$= e^{2x}(2\sin x - \cos x) - 4I.$$

Bundan

$$I=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C.$$

Ko‘rsatilgan uchta guruh bo‘laklab integrallanadigan barcha integrallarni o‘z ichiga olmaydi. Masalan, $\int\frac{xdx}{\cos^2 x}$ integral yuqorida keltirilgan integrallar guruhlariga kirmaydi, lekin uni bo‘laklab integrallash usuli bilan topish mumkin:

$$\int\frac{xdx}{\cos^2 x}=\left|\begin{array}{c} u=x,\ du=dx \\ dv=\frac{dx}{\cos^2 x},\ v=tgx \end{array}\right|=xtgx-\int tgxdx=xtgx-\ln|\cos x|+C.$$

Amaliy mashg’ulot uchun mashqlar

1. Berilgan integrallarni aniqmas integralning xossalari va integrallar jadvalini qo‘llab toping:

1) $\int\left(5tgx-\frac{2}{x^2+1}+x^4\right)\cdot dx;$ 2) $\int\frac{x^2-7}{x+3}dx;$

3) $\int\frac{\sqrt[3]{x}-x^2e^x-x}{x^2}dx;$ 4) $\int\left(\frac{3}{1+x^2}-\frac{2}{\sqrt{1-x^2}}\right)\cdot dx;$

5) $\int\frac{2\cdot 3^x-3\cdot 2^x}{3^x}dx;$ 6) $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2dx;$

7) $\int e^x\left(1+\frac{e^{-x}}{\cos^2 x}\right)dx;$ 8) $\int\frac{1-\sin^3 x}{\sin^2 x}dx;$

9) $\int ctg^2xdx;$ 10) $\int\frac{dx}{\cos^2 x-\cos 2x};$

11) $\int\frac{dx}{25+4x^2};$ 12) $\int\frac{dx}{\sqrt{3+4x-2x^2}}.$

2. Berilgan integrallarni differensial ostiga kiritish usuli bilan toping:

1) $\int\frac{tgx}{\cos^2 x}dx;$ 2) $\int\cos^2 x\sin xdx;$

3) $\int\frac{\sqrt[3]{arctg^5 2x}}{1+4x^2}dx;$ 4) $\int\frac{\sqrt[7]{\ln^3(x+5)}}{x+5}dx;$

5) $\int e^{\sin x}\cos xdx;$ 6) $\int e^{-x^3}x^2dx;$

7) $\int\frac{\cos x}{\sin^5 x}dx;$ 8) $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx;$

9) $\int\frac{e^xdx}{\sqrt{4-e^{2x}}};$ 10) $\int\frac{dx}{\sin^2 4x\sqrt[3]{ctg^2 4x}}.$

3. Berilgan integrallarni o‘rniga qo‘yish usuli bilan toping:

1) $\int \frac{e^x-1}{e^x+1}dx;$ 2) $\int \frac{x^5dx}{x^6+2}$

3) $\int \sqrt{16-x^2}dx;$ 4) $\int \frac{x^3dx}{\sqrt[3]{x^4+4}};$

5) $\int x^2\sqrt{x^3+3}dx;$ 6) $\int \frac{\cos 2xdx}{1+\sin x\cos x};$

7) $\int \frac{dx}{(\arcsin x)^3\sqrt{1-x^2}};$ 8) $\int \frac{4x-5}{x^2+5}dx;$

9) $\int \frac{dx}{\sqrt{5-4x-x^2}};$ 10) $\int \frac{dx}{\sqrt{3x^2-2x-1}};$

11) $\int x(2x+7)^{10}dx;$ 12) $\int \frac{dx}{\sqrt{x(1-x)}};$

13) $\int \frac{e^{2x}dx}{e^{4x}-9};$ 14) $\int \frac{\ln 2x}{\ln 4x}\cdot\frac{dx}{x}.$

4. Integrallarni bo‘laklab integrallash usuli bilan toping:

1) $\int x\,arctgx\,dx;$ 2) $\int \arcsin x dx;$

3) $\int x\ln xdx;$ 4) $\int x^2e^xdx;$

5) $\int x3^xdx;$ 6) $\int x\sin 2xdx;$

7) $\int x\ln(x+1)dx;$ 8) $\int \frac{x\sin xdx}{\cos^3 x};$

9) $\int \sin\ln xdx;$ 10) $\int e^{4x}\sin 4xdx.$

11) $\int \frac{\ln tgxdx}{\sin^2 x};$ 12) $\int \frac{\ln arctgxdx}{1+x^2}.$

5. Integrallarni toping:

1) $\int x^3\sqrt[3]{1+x^2}dx;$ 2) $\int \sin 3x\sin 5xdx;$

3) $\int e^x\cos^2(e^x)dx;$ 4) $\int \frac{xdx}{e^{3x}};$

5) $\int \frac{1-tgx}{1+tgx}dx;$ 6) $\int \frac{\ln xdx}{x(1-\ln^2 x)};$

7) $\int \frac{dx}{(x+1)(2x-3)};$ 8) $\int \frac{dx}{x(4+\ln^2 x)};$

9) $\int \frac{xdx}{\cos^2 x};$ 10) $\int \frac{dx}{x\sqrt{2x-9}};$

11) $\int \frac{e^{arctgx}dx}{1+x^2};$ 12) $\int \frac{e^{2x}dx}{\sqrt{3+e^{2x}}};$

13) $\int \sin^2\frac{3x}{2}dx;$ 14) $\int xtg^2x^2dx;$

15) $\int x^2 \ln^2 x dx$; 16) $\int \frac{1-2\cos x}{\sin^2 x} dx.$

RATSIONAL KASR FUNKSIYALARNI INTEGRALLASH

Ikkita $Q_m(x)$ va $P_n(x)$ ko‘phadning nisbatiga

$$R(x)=\frac{Q_m(x)}{P_n(x)}=\frac{b_0x^m+b_1x^{m-1}+...+b_{m-1}x+b_m}{a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n}$$

ratsional (*ratsional kasr*)*funksiya*deyiladi.
$m<n$ bo‘lganda ratsional kasr *to‘g‘ri kasr,* $m\geq n$ bo‘lganda *noto‘g‘ri kasr* deyiladi. Noto‘g‘ri kasrda uning $Q_m(x)$ suratini $P_n(x)$ maxrajiga odatdagidek bo‘lish yo‘li bilan kasrdan butun qismi $q(x)$ ajratiladi, ya’ni

$$R(x)=\frac{Q_m(x)}{P_n(x)}=q(x)+\frac{r(x)}{P_n(x)}$$

tenglik hosil qilinadi, bu yerda $q(x)$ - butun qism deb ataluvchi ko‘phad, $\frac{r(x)}{P_n(x)}$ -to‘g‘ri kasr, chunki $r(x)$ qoldiqning darajasi $P_n(x)$ ning darajasidan kichik.

3- misol. $R(x)=\frac{3x^4-2x^3+1}{x^2+2x+2}$ ratsional kasrdan butun qismini ajrating.

*Yechish.*Ko‘phadlarni bo‘lish qoidasi bo‘yicha kasr suratni maxrajga bo‘lamiz:

$$\begin{array}{r|l} 3x^4-2x^3+1 & x^2+2x+2 \\ -\;(3x^4+6x^3+6x^2) & 3x^2-8x+10 \\ \hline -8x^3-6x^2+1 & \\ -\;(-8x^3-16x^2-16x) & \\ \hline 10x^2+16x+1 & \\ -\;(10x^2+20x+20) & \\ \hline -4x-19 & \end{array}$$

Demak, $R(x)=3x^2-8x+10+\frac{-4x-19}{x^2+2x+2}$.

Quyidagi to‘g‘ri kasrlarga *sodda* (*elementar*) *kasrlar*deyiladi:

$I.\ \frac{A}{x-\alpha}$; $II.\ \frac{A}{(x-\alpha)^k}$, $(k\geq 2,\ k\in N)$;

$III.\ \frac{Mx+N}{x^2+px+q}$, $(p^2-4q<0)$;

$IV.\ \dfrac{Mx+N}{(x^2+px+q)^s}$, $(s \geq 2,\ s \in N,\ p^2-4q<0)$,

bu yerda $A,\ M,\ N,\ \alpha,\ p,\ q$ - haqiqiy sonlar.

7-teorema. Maxraji ko‘paytuvchilarga ajratilgan har qanday $\dfrac{Q_m(x)}{P_n(x)}$ to‘g‘ri kasrni sodda kasrlar yig‘indisiga yagona tarzda yoyish mumkin.

Bunda:

1) ifodaning $(x-\alpha)$ ko‘rinishdagi ko‘paytuvchisiga I turdagi $\dfrac{A}{x-\alpha}$ kasr mos keladi;

2) ifodaning $(x-\alpha)^k$ ko‘rinshidagi ko‘paytuvchisiga I va II turdagi k ta kasrlar yig‘indisi $\dfrac{A_1}{x-\alpha}+\dfrac{A_2}{(x-\alpha)^2}+...+\dfrac{A_k}{(x-\alpha)^k}$ mos keladi;

3) ifodaning x^2+px+q korinishdagi ko‘paytuvchisiga III turdagi $\dfrac{Mx+N}{x^2+px+q}$ kasr mos keladi;

4) ifodaning $(x^2+px+q)^s$ ko‘rinishdagi ko‘paytuvchisiga III va IV turdagi s ta kasrlar yig‘indisi $\dfrac{M_1x+N_1}{x^2+px+q}+\dfrac{M_2x+N_2}{(x^2+px+q)^2}+...+\dfrac{M_sx+N_s}{(x^2+px+q)^s}$ mos keladi.

Shunday qilib, teoremaga ko‘ra

$$\frac{Q_m(x)}{P_n(x)}=\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+...+\frac{A_k}{(x-\alpha)^k}+...+$$
$$+\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+...+\frac{M_sx+N_s}{(x^2+px+q)^s}, \qquad (2.1)$$

bu yerda $A_1,\ A_2,...,A_k,\ M_1,\ N_1,\ M_2,\ N_2,...,M_s,\ N_s-$ koeffitsiyentlar.

(2.1) tenglikdagi noma’lum koeffitsiyentlarini topishning turli usullari mavjud. *Masalan*,noma’lum koeffitsiyentlarni topishda*koeffitsiyentlarni tenglashtirish* usulini qo‘llash mumkin.

4- misol. $R(x)=\dfrac{x^4-2x+1}{x^2(x^3+1)}$ to‘g‘ri kasrni oddiy kasrlar yig‘indisiga yoying.

Yechish. $R(x)$ ning maxrajini ko‘paytuvchilarga ajratamiz:

$$x^2(x^3+1)=x^2(x+1)(x^2-x+1)$$

$R(x)$ ni 1-teoremaga asosan sodda kasrlar yifg‘indisiga yoyamiz:

$$R(x)=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A}{x+1}+\frac{Mx+N}{x^2-x+1}.$$

Noma'lum koeffitsiyentlarini koeffitsiyentlarni tenglashtirish usuli bilan topamiz. Buning uchun tenglikning o'ng qismini umumiy maxrajga keltiramiz, hosil bo'lgan tenglikning har ikkala qismidagi maxrajlarni tashlab yuboramiz va quyidagi tenglikni hosil qilamiz:

$$x^4-2x+1=A_1x(x^3+1)+A_2(x^3+1)+Ax^2(x^2-x+1)+Mx^3(x+1)+Nx^2(x+1).$$

x ning bir xil darajalari oldidagi koeffitsiyentlarni tenglashtiramiz:

$$\begin{cases} x^4: \ A+A_1+M=1, \\ x^3: \ -A+A_2+M+N=0, \\ x^2: \ A+N=0, \\ x^1: \ A_1=-2, \\ x^0: \ A_2=1. \end{cases}$$

Bu tenglamalar sistemasini yechamiz: $A=\frac{4}{3},\ A_1=-2,\ A_2=1,\ M=\frac{5}{3},\ N=-\frac{4}{3}.$

Demak,

$$R(x)=-\frac{2}{x}+\frac{1}{x^2}+\frac{4}{3(x+1)}+\frac{5x-4}{3(x^2-x+1)}.$$

1^o. I va II turdagi sodda kasrlar jadval integrallari orqali topiladi:

$$\int\frac{Adx}{x-\alpha}=A\int\frac{d(x-\alpha)}{x-\alpha}=A\ln|x-\alpha|+C; \qquad (2.1)$$

$$\int\frac{Adx}{(x-\alpha)^k}=A\int(x-\alpha)^{-k}d(x-\alpha)=$$

$$=A\frac{(x-\alpha)^{-k+1}}{-k+1}+C=\frac{A}{(1-k)(x-\alpha)^{k-1}}+C. \qquad (2.2)$$

2^o. III turdagi sodda kasrni qaraymiz. $\int\frac{Mx+N}{x^2+px+q}dx$ integralining suratida kasrning maxrajidan olingan hosila $(x^2+px+q)'=2x+p$ ni ajratamiz va natijani integrallaymiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\int\frac{\frac{M}{2}(2x+p)+N-\frac{Mp}{2}}{x^2+px+q}dx=\frac{M}{2}\int\frac{2x+p}{x^2+px+q}dx+$$

$$+\left(N-\frac{Mp}{2}\right)\int\frac{dx}{x^2+px+q}=\frac{M}{2}J_1+\left(N-\frac{Mp}{2}\right)J_2.$$

Tenglikning oxirgi qismidagi integrallardan birinchisi $J_1=\ln|x^2+px+q|$.

Ikkinchi integral maxrajida to'liq kvadrat ajratamiz va integralni quyidagicha hisoblaymiz:

$$J_2=\int\frac{dx}{x^2+px+q}=\int\frac{d\left(x+\frac{p}{2}\right)}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}=\frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}},$$

bunda $4q-p^2>0$, chunki $D<0.$

Natijada quyidagiga ega bo‘lamiz:

$$\int\frac{Mx+N}{x^2+px+q}dx=\frac{M}{2}\ln|x^2+px+q|+\frac{2N-Mp}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C. \qquad (2.3)$$

1- misol. $I=\int\frac{5x+11}{x^2+6x+13}dx$ integralni toping.

Yechish. $I=\int\frac{\frac{5}{2}(2x+6)+11-\frac{5}{2}\cdot 6}{x^2+6x+13}dx=\frac{5}{2}\int\frac{(2x+6)dx}{x^2+6x+13}-4\int\frac{dx}{x^2+6x+13}=$

$$=\frac{5}{2}\ln|x^2+6x+13|-4J.$$

Bu yerda

$$J=\int\frac{dx}{(x+3)^2+4}=\int\frac{d(x+3)}{(x+3)^2+2^2}=\frac{1}{2}arctg\frac{x+3}{2}.$$

Bundan

$$\int\frac{5x+11}{x^2+6x+13}dx=\frac{5}{2}\ln|x^2+6x+13|-2arctg\frac{x+3}{2}+C.$$

3^o. IV turdagi sodda kasrning integralini topamiz:

$$\int\frac{Mx+N}{(x^2+px+q)^s}dx=\frac{M}{2}\int\frac{(2x+p)dx}{(x^2+px+q)^s}+$$

$$+\left(N-\frac{Mp}{2}\right)\int\frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right)^s}. \qquad (2.4)$$

Bu tenglikning o’ng qismidagi birinchi integral jadvaldagi integralga keltirib, topiladi:

$$I=\int\frac{(2x+p)dx}{(x^2+px+q)^s}=\int(x^2+px+q)^{-s}d(x^2+px+q)=\frac{1}{(1-s)(x^2+px+q)^{s-1}}.$$

Ikkinchi integralga (uni I_s bilan belgilaymiz) $\left(x+\frac{p}{2}\right)=t$ almashtirish bajaramiz va $0<q-\frac{p^2}{4}=a^2$ belgilashkiritamiz. U holda

$$I_s=\int\frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right)^s}=\int\frac{dt}{(t^2+a^2)^s}=\frac{1}{a^2}\int\frac{(t^2+a^2)-t^2}{(t^2+a^2)^s}dt=$$

$$=\frac{1}{a^2}\int\frac{dt}{(t^2+a^2)^{s-1}}-\frac{1}{a^2}\int\frac{t^2dt}{(t^2+a^2)^s}.$$

Bu tenglikning o'ng qismidagi birinchi integral I_s ga o'xshash bo'lib, unda maxrajning darajasi s dan bir birlikka kichik. Shu sababli, belgilashga ko'ra u I_{s-1} bo'ladi. Ikkinchi integralni bo'laklab integrallaymiz:

$$\int\frac{t^2dt}{(t^2+a^2)^s}=\frac{1}{2}\int\frac{t\cdot 2tdt}{(t^2+a^2)^s}=\frac{1}{2}\left(\frac{-t}{(s-1)(t^2+a^2)^{s-1}}+\frac{1}{s-1}\int\frac{dt}{(t^2+a^2)^{s-1}}\right)=$$

$$=-\frac{t}{2(s-1)(t^2+a^2)^{s-1}}+\frac{1}{2(s-1)}I_{s-1}.$$

Demak, I_s integralni hisoblash uchun s darajani pasaytirish formulasini hosil qilamiz:

$$I_s=\frac{1}{a^2}I_{s-1}+\frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}}-\frac{1}{2a^2(s-1)}=$$

$$=\frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}}+\frac{2s-3}{2a^2(s-1)}I_{s-1}. \qquad (2.5)$$

Shunday qilib, (2.5) formula bo'yicha I_s integralni topamiz, keyin I_s dagi barcha t ni $x+\frac{p}{2}$ bilan almashtirib va I, I_s integrallarni (2.4) tenglikka qo'yib,IV turdagi sodda kasr integralini topish uchun ifoda hosil qilamiz.

(2.5) formula bo'yicha I_s integralni topish indeksi bittaga kichik bolgan I_{s-1} integralni topishga, I_{s-1} integralni topish esa o'z navbatida I_{s-2} integralni topishga keltiriladi va bu jarayon quyidagi jadval integralni topishgacha davom ettiriladi:

$$I_1=\int\frac{dt}{t^2+a^2}=\frac{1}{a}arctg\frac{t}{a}+C.$$

Demak, (2.5) formula orqali $I_s$dan I_{s-1}ga , so'ngra I_{s-2} o'tiladi va hokazo. Shu sababli bunday formulalar*keltirish* yoki *rekurrent* (*qaytuvchan*) formulalar deyiladi.

2- misol. $\int\frac{2x+5}{(x^2+4x+8)^2}dx$ integralni toping.

Yechish. $\int\frac{2x+4+1}{(x^2+4x+8)^2}dx=\int\frac{2x+4}{(x^2+4x+8)^2}dx+\int\frac{dx}{(x^2+4x+8)^2}=$

$$= -\frac{1}{x^2+4x+8} + \int \frac{d(x+2)}{[(x+2)^2+4]^2} = -\frac{1}{x^2+4x+8} + \int \frac{dt}{t^2+a^2},$$

bu yerda $t = x+2,\ a = 2.$ (2.5) integraldan foydalanib ayrim hisoblashlardan so'ng

$$I_2 = \frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^3} arctg\frac{t}{a} = \frac{x+2}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2}$$

ekanligini topamiz.

Demak,

$$\int \frac{2x+5}{(x^2+4x+8)^2}dx = -\frac{1}{x^2+4x+8} + \frac{x+2}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2} + C =$$

$$= \frac{x-6}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2} + C.$$

Ratsional kasr funksiyalarni integrallash tartibi.

Yuqorida aytilganlardan kelib chiqadiki, $R(x) = \frac{Q_m(x)}{P_n(x)}$ ratsional kasr funksiyani integrallash quyidagi tartibda amalga oshiriladi:

1) berilgan ratsional kasrning to‘g‘ri yoki noto‘g‘ri kasr ekanini tekshirish; agar kasr noto‘g‘ri bo‘lsa, kasrdan butun qismini ajratish;

2) to‘g‘ri kasrning maxrajini ko‘paytuvchilarga ajratish;

3) to‘g‘ri kasrni sodda kasrlar yig‘indisiga yoyish;

4) hosil bo‘lgan ko‘phad va sodda kasrlar yig‘indisini integrallash.

3- misol. $I = \int \frac{x^4+6}{x^3-2x^2+2x}dx$ integralni toping.

Yechish. $R(x) = \frac{x^4+6}{x^3-2x^2+2x}$ no to‘g‘ri kasr, chunki $m=4,\ n=3\ (m>n).$

Bu kasrning suratni maxrajga bo‘lish orqali kasrdan butun qismini ajratamiz:

$$\begin{array}{r|l} -\ x^4+6 & x^3-2x^2+2x \\ x^4-2x^3+2x^2 & \overline{x+2} \\ \hline -\ 2x^3-2x^2+6 & \\ 2x^3-4x^2+4x & \\ \hline 2x^2-4x+6 & \end{array}$$

Bundan

$$R(x) = x+2+\frac{2x^2-4x+6}{x^3-2x^2+2x}.$$

To‘g‘ri kasrning maxrajini ko‘paytuvchilarga ajratamiz:

$$x^3-2x^2+2x = x(x^2-2x+2).$$

To‘g‘ri kasrni sodda kasrlarga yoyilmasi ko’rinishida yozamiz:

$$\frac{2x^2-4x+6}{x(x^2-2x+2)}=\frac{A}{x}+\frac{Mx+N}{x^2-2x+2}.$$

Yoyilmaning noma’lum koeffitsiyentlarini topamiz:

$$2x^2-4x+6=A(x^2-2x+2)+Mx^2+Nx,$$

$$\begin{cases} x^2: \ A+M=2, \\ x^1: \ -2A+N=-4, \\ x^0: \ 2A=6. \end{cases}$$

Bundan $A=3,\ M=-1,\ N=2.$

Shunday qilib,

$$R(x)=x+2+\frac{3}{x}+\frac{-x+2}{x^2-2x+2}.$$

Ko‘phad va sodda kasrlar yig‘indisini integrallaymiz:

$$I=\int(x+2)dx+\int\frac{3dx}{x}+\int\frac{-x+2}{x^2-2x+2}dx=\frac{x^2}{2}+2x+3\ln|x|-$$

$$-\int\frac{\frac{1}{2}(2x-2)+1-2}{x^2-2x+2}dx=\frac{x^2}{2}+2x+3\ln|x|-\frac{1}{2}\int\frac{2x-2}{x^2-2x+2}dx+$$

$$\frac{x^2}{2}+2x+3\ln|x|-\frac{1}{2}\ln|x^2-2x+2|+arctg(x-1)+C.$$

Amaliy mashg’ulot uchun mashqlar

1. Berilgan to‘g‘ri kasrlarni sodda kasrlar yig‘indisiga yoying:

1) $\frac{x^2+4x+1}{x^3+x^2}$; 2) $\frac{3x^3-5x^2+8x-4}{x^4+4x^2}$;

3) $\frac{3x-2}{x^3+x^2-2x}$; 4) $\frac{x^2+5x+1}{x^4+x^2+1}$.

2. Berilgan to‘g‘ri kasrlarni sodda kasrlar yig‘indisiga yoying va koeffitsiyentlarni ixtiyoriy qiymatlar usuli bilan toping:

1) $\frac{x^2+2x+3}{x^4+x^3}$; 2) $\frac{2x^2-11x-6}{x^3+x^2-6x}$;

3) $\frac{3x^3-2x^2-2x+7}{x^4-x^2}$; 4) $\frac{2x-1}{x^4+x}$.

3. Integrallarni toping:

1) $\int\frac{2x+3}{(x-2)(x+5)}dx$; 2) $\int\frac{xdx}{(x+1)(2x+1)}$;

3) $\int\frac{xdx}{(x+1)(x+2)(x+3)}$; 4) $\int\frac{8xdx}{(x+1)(x^2+6x+5)}$;

5) $\int \frac{3x^2+2x-3}{x(x-1)(x+1)}dx;$ 6) $\int \frac{x^3-1}{4x^3-x}dx;$

7) $\int \frac{2x^3+2x^2+4x+3}{x^3+x^2}dx;$ 8) $\int \frac{2+5x^3}{x(x^2-5x+4)}dx;$

9) $\int \frac{x^3-3}{x^3-2x^2-x+2}dx;$ 10) $\int \frac{dx}{x^2(x^2+1)};$

11) $\int \frac{dx}{x(1+x^2)};$ 12) $\int \frac{dx}{1+x^3};$

13) $\int \frac{x^4+3x^3+2x^2+x+1}{x^2+x+1}dx;$ 14) $\int \frac{x^9dx}{x^4-1};$

15) $\int \frac{dx}{x^4-1};$ 16) $\int \frac{dx}{(x^2+9)^3};$

17) $\int \frac{3x+5}{(x^2+2x+2)^2}dx;$ 18) $\int \frac{x^4+2x^2+x}{(x-1)(x^2+4)^2}dx;$

19) $\int \frac{dx}{(x^2+4x+5)(x^2+4x+13)};$ 20) $\int \frac{dx}{(x+1)^2(x^2+1)};$

21) $\int \frac{dx}{(x^2+1)^4};$ 22) $\int \frac{2x-1}{(x^2-2x+5)^2}dx;$

23) $\int \frac{2x+3}{(x^2-3x+3)^2}dx;$ 24) $\int \frac{3x^2-10x+12}{x^4+13x^2+36}dx.$

TRIGONOMETRIK FUNKSIYALARNI INTEGRALLASH

1. $\int R(\sin x, \cos x)dx$ ko‘rinishidagi integralni $tg\frac{x}{2}=t$ almashtirish orqali hamma vaqt t o‘zgaruvchili ratsional funksiyaning integraliga almashtirish, ya’ni *ratsionallashtirish*mumkin. Shu sababli bu almashtirish *universal trigonometrik almashtirish* deyiladi.

Haqiqatan ham $\int R(\sin x, \cos x)dx$ ifodadan

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}, \quad x = arctgt, \; dx = \frac{2dt}{1+t^2}$$

tarzdagi o‘rniga qo‘yishlar yordamida t o‘zgaruvchili

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2} = \int R_1(t)dt$$

ratsional funksiya kelib chiqadi.

1- misol. $I=\int\frac{dx}{3\sin x+2\cos x+3}$ integralni toping.

Yechish. $tg\frac{x}{2}=t$ deymiz. U holda

$$I=\int\frac{\frac{2dt}{1+t^2}}{3\cdot\frac{2t}{1+t^2}+2\cdot\frac{1-t^2}{1+t^2}+3}=2\int\frac{dt}{t^2+6t+5}=2\int\frac{dt}{(t+1)(t+5)}=$$

$$=\int\left(\frac{A}{t+1}+\frac{B}{t+5}\right)dt=A\ln|t+1|+B\ln|t+5|+C.$$

No'malum koeffitsiyentlarni aniqlaymiz: $A=\frac{1}{2},\ B=-\frac{1}{2}.$

Demak,

$$I=\frac{1}{2}(\ln|t+1|-\ln|t+5|)+C=\frac{1}{2}\ln\left|\frac{t+1}{t+5}\right|=\frac{1}{2}\ln\left|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}+5}\right|+C.$$

Universal trigonometrik o'rniga qo'yish natijasida amalda ko'pinchaancha murakkab ratsional funksiyalar hosil bo'lishi mumkin. Bunday hollarda yuqorida keltirilgan integralni topishda quyidagi sodda almashtirishlardan foydalangan ma'qul:

a) agar $R(\sin x,\cos x)$ ifoda $\sin x$ ga nisbatan toq, ya'ni $R(-\sin x,\cos x)=-R(\sin x,\cos x)$ bo'lsa, u holda $\cos x=t$ o'rniga qo'yish bu funksiyani ratsionallashtiradi;

b) agar $R(\sin x,\cos x)$ ifoda $\cos x$ ga nisbatan toq, ya'ni $R(\sin x,-\cos x)=-R(\sin x,\cos x)$ bo'lsa, u holda $\sin x=t$ o'rniga qo'yish orqali bu funksiya ratsionallashtiriladi;

c) agar $R(\sin x,\cos x)$ ifoda $\sin x$ va $\cos x$ larga nisbatan juft, ya'ni $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ bo'lsa, u holda $tgx=t$ o'rniga qo'yish bu funksiyani ratsionallashtiradi. Bunda quyidagi almashtirishlardan foydalaniladi:

$$\sin^2 x=\frac{tg^2x}{1+tg^2x}=\frac{t^2}{1+t^2},\quad \cos^2 x=\frac{1}{1+tg^2x}=\frac{1}{1+t^2},$$

$$x=arctgt,\ dx=\frac{dt}{1+t^2}.$$

2- misol. $I=\int\frac{\cos xdx}{\sin^2 x-4\sin x+5}$ integralni toping.

Yechish. Integral ostidagi funksiya $\cos x$ ga nisbatan toq funksiya. Shu sababli $\sin x=t$ deb olamiz.U holda $\cos xdx=dt$ va

$$I=\int\frac{dt}{t^2-4t+5}=\int\frac{d(t-2)}{(t-2)^2+1}=arctg(t-2)+C=arctg(\sin x-2)+C.$$

3- misol. $I=\int\frac{dx}{1-2\sin^2 x}$ integralni toping.

Yechish. Integral ostidagi funksiya $\sin x$ ga nisbatanjuft funksiya. Shu sababli $tgx=t$ o'rniga qo'yishdan foydalanamiz. U holda

$$I=\int\frac{\frac{dt}{1+t^2}}{1-\frac{2t^2}{1+t^2}}=\int\frac{dt}{1-t^2}=\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|=\frac{1}{2}\ln\left|\frac{tgx+1}{tgx-1}\right|+C.$$

2. $\int\sin^n x\cos^m xdx$ ko'rinishidagi integrallar m va n butun sonlarga bog'liq holda quyidagicha topiladi:

a) $n>0$ va toq bo'lganida $\cos x=t$ o'rniga qo'yish integralni ratsionallashtiradi;

a) $m>0$ va toq bo'lganida $\sin x=t$ o'rniga qo'yish orqali integral ratsionallashtiriladi;

c) m va n sonlar juft va nomanfiy bo'lganida

$$\sin^2 x=\frac{1-\cos 2x}{2},\quad \cos^2 x=\frac{1+\cos 2x}{2}$$

formulalaridan foydalanib, darajalar pasaytiriladi;

d) $m+n<0$ hamda m va n juft bo'lganida $tgx=t$ yoki $ctgx=t$ o'rniga qo'yishdan foydalaniladi. Bunda $m<0$ va $n<0$ bo'lsa, suratda $1=(\sin^2 x+\cos^2 x)^k$, bu yerda $k=\frac{|m+n|}{2}-1$, almashtirishdan iborat usul qo'llab, ratsional funksiyalarni integrallashga keltiriladi;

e) $m,n\le 0$ va ulardan biri toq bo'lganida $\sin x$ va $\cos x$ lardan qaysi birining darajasi toqligiga qarab, surat va maxrajni shu funksiyaga qo'shimcha ko'paytirishdan foydalaniladi.

4- misol. $\int\sin^5 x\cos^2 xdx$ integralni toping.

Yechish. $\int\sin^5 x\cos^2 xdx\ (n>0\ \textit{va toq},\ \cos x=t)=\int\sin^4 x\cos^2 x\sin xdx=$

$$=-\int(1-t^2)^2t^2dt=-\int t^2dt+2\int t^4dt-\int t^6dt=-\frac{t^3}{3}+\frac{2t^5}{5}-\frac{t^7}{7}+C=$$

$$=-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+C.$$

5- misol. $\int\sin^4 x\cos^2 xdx$ integralni toping.

Yechish. $\int\sin^4 x\cos^2 xdx\ (n,m\ge 0\ \textit{va}\ n,m-\textit{juft})=\int(\sin x\cos x)^2\sin^2 xdx=$

$$=\int\left(\frac{\sin^2 2x}{4}\right)\cdot\left(\frac{1-\cos 2x}{2}\right)dx=\frac{1}{8}\int(\sin^2 2x-\sin^2 2x\cos 2x)dx=$$

$$=\frac{1}{8}\int\frac{1-\cos 4x}{2}dx-\frac{1}{16}\int\sin^2 2xd(\sin 2x)=$$

$$=\frac{1}{16}\left(x-\frac{\sin 4x}{4}\right)-\frac{\sin^3 2x}{48}+C=\frac{1}{16}\left(x-\frac{\sin 4x}{4}-\frac{\sin^3 2x}{3}\right)+C.$$

6- misol. $I=\int\frac{dx}{\sin^4 x\cos^2 x}$ integralni toping.

Yechish. Bunda $n=-4, m=-2,\ n+m=-6<0,\ k=\frac{|m+n|}{2}-1=2$.

Demak, $I=\int\frac{(\sin^2 x+\cos^2 x)^2}{\sin^4 x\cos^2 x}dx=\int\frac{\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x}{\sin^4 x\cos^2 x}dx=$

$$=\int\frac{dx}{\cos^2 x}+2\int\frac{dx}{\sin^2 x}+\int\frac{\cos^2 x}{\sin^4 x}dx=tgx-2ctgx-\int ctg^2 xd(ctgx)=$$

$$=tgx-2ctgx-\frac{1}{3}ctg^3 x+C.$$

3. $\int tg^n xdx$ va $\int ctg^n xdx$ (bu yerda $n>0$ butun son) ko‘rinishidagi integrallar mos rasvishda $tgx=t$ va $ctgx=t$ o‘rniga qo‘yish orqali topiladi.

Bunday integrallarni orniga qo‘yishllardan foydalanmasdan, bevosita

$$tg^2 x=\frac{1}{\cos^2 x}-1,\ ctg^2 x=\frac{1}{\sin^2 x}-1$$

formulalar yordamida hisoblash ham mumkin.

7- misol. $\int tg^5 xdx$ integralni hisoblang.

Yechish.1-usul. $\int tg^5 xdx=\left|tgx=t,\ dx=\frac{dt}{1+t^2}\right|=\int\frac{t^5 dt}{1+t^2}=\int t^3 dt-$

$$-\int tdt+\int\frac{tdt}{1+t^2}=\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{2}\int\frac{d(1+t^2)}{1+t^2}=\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{2}\ln|1+t^2|+C=$$

$$=\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\frac{1}{2}\ln|\cos^2 x|+C=\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\ln|\cos x|+C.$$

2-usul. $\int tg^5 xdx=\int tg^3 x\cdot tg^2 xdx=\int tg^3 x\cdot\left(\frac{1}{\cos^2 x}-1\right)dx=$

$$=\int tg^3 x\cdot\frac{dx}{\cos^2 x}-\int tg^3 xdx=\int tg^3 xd(tgx)-\int tgx\cdot\left(\frac{1}{\cos^2 x}-1\right)dx=$$

$$=\frac{1}{4}tg^4 x-\int tgxd(tgx)-\int tgxdx=\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\ln|\cos x|+C.$$

4. Ba’zi ko‘rinishdagi integrallarni hisoblashda

$$\sin mx\cos nx = \frac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right),$$

$$\sin mx\sin nx = \frac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right),$$

$$\cos mx\cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

trigonometrik formulalardan foydalaniladi.

8- misol. $\int \cos 3x \cdot \cos 5x dx$ integralni toping.

Yechish. $\int \cos 3x \cdot \cos 5x dx = \frac{1}{2}\int (\cos 8x + \cos 2x)dx =$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x\right) + C = \frac{1}{16}(\sin 8x + 4\sin 2x) + C.$$

Amaliy mashg'ulot uchun mashqlar.

1. Berilgan integrallarni toping:

1) $\int \frac{dx}{5+4\sin x}$; 2) $\int \frac{dx}{2\sin x + \sin 2x}$;

3) $\int \frac{dx}{3+5\sin x + 3\cos x}$; 4) $\int \frac{dx}{4+2\sin x + 3\cos x}$;

5) $\int \frac{\sin x dx}{\sqrt{3-\cos^2 x}}$; 6) $\int \frac{3\cos^3 x dx}{\sin^4 x}$;

7) $\int \frac{\cos^3 x dx}{1+\sin^2 x}$; 8) $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x} dx$;

9) $\int \sin^2 x\cos^4 x dx$; 10) $\int \frac{dx}{\sin x\cos^3 x}$;

11) $\int \frac{dx}{2+3\sin^2 x - 7\cos^2 x}$; 12) $\int ctg^3 2x dx$;

13) $\int \frac{\sin^2 x dx}{1+\cos^2 x}$; 14) $\int \cos 2x\cos 5x dx$;

15) $\int \sin^2 x\cos 3x dx$; 16) $\int \cos x\cos 2x\cos 3x dx$.

IRRATSIONAL IFODALARNI INTEGRALLASH

1. $\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}},...\right)dx$ ($R-$ratsional funksiya, $m_1,\ n_1,\ m_2,\ n_2,...-$ butun sonlar) ko‘rinishdagi integrallar $\frac{ax+b}{cx+d}=t^s$ o‘rniga qo‘ish yordamida ratsional funksiyaning integraliga keltiriladi, bunda $s=EKUK(n_1,n_2,...)$.

Xususan, $\int R\left(x,(ax+b)^{\frac{m_1}{n_1}},(ax+b)^{\frac{m_2}{n_2}},...\right)dx$ integrallar $ax+b=t^s$ o‘rniga qo‘yish yordamida, $\int R\left(x,x^{\frac{m_1}{n_1}},x^{\frac{m_2}{n_2}},...\right)dx$ integrallar esa $x=t^s$ o‘rniga qo‘yish yordamida t ozgaruvchili ratsional funksiyaga keltiriladi.

1- misol. $I=\int\frac{x^2+\sqrt[3]{1+x}}{\sqrt{1+x}}dx$ integralni toping.

Yechish. Bu yerda $EKUK(2,3)=6$ bo’lgani uchun $1+x=t^6$ tarzda belgilash kiritamiz.U holda

$$\sqrt{1+x}=t^3,\ \ \sqrt[3]{1+x}=t^2,\ \ dx=6t^5dt.$$

Demak,

$$I=\int\frac{(t^6-1)^2+t^2}{t^3}\cdot 6t^5dt=6\int t^2(t^{12}-2t^6+t^2+1)dt=$$

$$=6\left(\frac{t^{15}}{15}-2\frac{t^9}{9}+\frac{t^5}{5}+\frac{t^3}{3}\right)+C=\frac{2t^3}{15}(3t^{12}-10t^6+9t^2+15)+C=$$

$$=\frac{2\sqrt{1+x}}{15}\left(3(1+x)^2-10(1+x)+9\sqrt[3]{1+x}+15\right)+C.$$

2. $\int R(x,\sqrt{ax^2+bx+c})dx$ ko‘rinishidagi integrallar *Eylerning uchta o‘rniga qo‘yichusuli* orqali ratsional funksiyalardan olingan integrallarga keltiriladi:

a) $a>0$ bo‘lganida $\sqrt{ax^2+bx+c}=t\pm\sqrt{a}x$ almashtirish orqali integral ostidagi funksiya ratsionallashtiriladi (*Eylerning birinchi o‘rniga qo‘yishi*);

b) $c>0$ bo‘lganida $\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}$ almashtirish yordamida integral ostidagi funksiya ratsionallashtiriladi (*Eylerning ikkinchi o‘rniga qo‘yishi)*;

c) ax^2+bx+c kvadrat uchhad $a(x-x_1)(x-x_2)$ ko‘rinishda ko‘paytuvchilarga ajralganida integral ostidagi funksiya $\sqrt{ax^2+bx+c}=t(x-x_1)$ almashtirish bilan ratsionallashtiriladi (*Eylerning uchinchi o‘rniga qo‘yishi*).

2- misol. $I=\int\frac{dx}{1+\sqrt{x^2+2x+2}}$ integralni toping.

Yechish. Bunda $a>0$. Shu sababli $\sqrt{x^2+2x+2}=t-x$ ko'rinishdagi o'rniga qo'yish bajaramiz.
U holda

$$x^2+2x+2=t^2-2tx+x^2,\ 2x+2tx=t^2-2.$$

Bundan

$$x=\frac{t^2-2}{2(1+t)},\ dx=\frac{t^2+2t+2}{2(1+t)^2},\ 1+\sqrt{x^2+2x+2}=1+t-\frac{t^2-2}{2(1+t)}=\frac{t^2+4t+4}{2(1+t)}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I=\int\frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)2(1+t)^2}dt=\int\frac{t^2+2t+2}{(1+t)(2+t)^2}dt.$$

Integral ostidagi to'g'ri kasrni sodda kasrlarga yoyamiz:

$$\frac{t^2+2t+2}{(1+t)(2+t)^2}=\frac{A}{1+t}+\frac{B}{2+t}+\frac{C}{(2+t)^2}.$$

Koeffitsiyentlarni tenglashtirish usulini qo'llaymiz: $A=1,\ B=0,\ C=-2$. Bundan

$$I=\int\frac{dt}{1+t}-2\int\frac{dt}{(2+t)^2}=\ln|1+t|+\frac{2}{2+t}+C.$$

x o'zgaruvchiga qaytamiz:

$$I=\ln\left|1+x+\sqrt{x^2+2x+2}\right|+\frac{2}{x+2+\sqrt{x^2+2x+2}}+C.$$

3- misol. $I=\int\frac{dx}{\sqrt{x^2-3x+2}}$ integralni toping.

Yechish. $x^2-3x+2=(x-1)(x-2)$ bo'lgani uchun $\sqrt{(x-1)(x-2)}=(x-1)t$ shakldao'rniga qo'yish bajaramiz. U holda

$$(x-1)(x-2)=(x-1)^2t^2,\ t=\sqrt{\frac{x-2}{x-1}}.$$

Bundan

$$x=\frac{t^2-2}{t^2-1},\ dx=\frac{2tdt}{(t^2-1)^2},\ \sqrt{x^2-3x+2}=\left(\frac{t^2-2}{t^2-1}-1\right)t=-\frac{t}{t^2-1}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$I=\int\frac{-(t^2-1)2tdt}{(t^2-1)^2t}=-2\int\frac{dt}{t^2-1}=-\ln\left|\frac{t-1}{t+1}\right|+C=-\ln\left|\frac{1-2t+t^2}{t^2-1}\right|+C.$$

Dastlabki o'zgaruvchiga qaytamiz:

$$I=-\ln\left|\frac{1-2\sqrt{\frac{x-2}{x-1}}+\frac{x-2}{x-1}}{\frac{x-2}{x-1}-1}\right|+C=-\ln|3-2x+2\sqrt{x^2-3x+2}|+C.$$

Eyler o‘rniga qo‘yishlari ayrim integrallarda murakkab hisoblashlarga olib kelishi mumkin. Bunday hollarda integrallashning quyidagi usullaridan foydalanilsa bo’ladi.

1^o. $\int R(x,\sqrt{ax^2+bx+c})dx$ ko‘rinishidagi integrallarni hisoblashning *kvadrat uchhaddan to‘la kvadrat ajratish* usulida berilgan integrallar ax^2+bx+c kvadrat uchhaddan to‘la kvadrat ajratish yo‘li bilan ushbu integrallardan biriga keltiriladi:

a) agar $a>0$ va $b^2-4ac<0$ bo‘lsa, u holda $\int R(t,\sqrt{m^2+n^2t^2})dt,$ bu yerda

$$n^2=a,\ m^2=-\frac{b^2-4ac}{4a},\ \ t=x+\frac{b}{2a};$$

b) agar $a>0$ va $b^2-4ac>0$ bo‘lsa, u holda $\int R(t,\sqrt{n^2t^2-m^2})dt,$ bu yerda

$$n^2=a,\ m^2=\frac{b^2-4ac}{4a},\ t=x+\frac{b}{2a}\ ;$$

c) agar $a<0$ va $b^2-4ac>0$ bo‘lsa, u holda $\int R(t,\sqrt{m^2-n^2t^2})dt,$ bu yerda

$$n^2=-a,\ m^2=-\frac{b^2-4ac}{4a},\ t=x+\frac{b}{2a}.$$

Hosil qilingan integrallar mos ravishda $t=\frac{m}{n}tgz, t=\frac{m}{n\sin z},\quad t=\frac{m}{n}\sin z$ o‘rniga qo‘yishlar orqali $\int R(\sin z,\cos z)dz$ ko‘rinishga keltiriladi.

4-misol. $\int\sqrt{5+4x-x^2}dx$ integralni toping.

*Yechish.*Kvadrat uchhaddanto‘la kvadrat ajratamiz, yangi t o‘zgaruvchi kiritamiz va trigonometrik o‘rniga qo‘yishdan foydalanib, topamiz:

$$\int\sqrt{5+4x-x^2}dx=\int\sqrt{9-(x-2)^2}dx=\left|\begin{array}{c}x-2=t,\\ dx=dt\end{array}\right|=\int\sqrt{9-t^2}dt=$$

$$=\left|\begin{array}{c}t=3\sin z,\\ dt=\cos zdz\end{array}\right|=\int\sqrt{9-9\sin^2 z}3\cos zdz=\int 9\cos^2 zdz=$$

$$=\frac{9}{2}\int(1+\cos 2z)dz=\frac{9}{2}\left(z+\frac{\sin 2z}{2}\right)+C=\frac{9}{2}(z+\sin z\sqrt{1-\sin^2 z})+C=$$

$$=\left|z=\arcsin\frac{t}{3}\right|=\frac{9}{2}\left(\arcsin\frac{t}{3}+\frac{t}{3}\sqrt{1-\frac{t^2}{9}}\right)+C=\frac{9}{2}\arcsin\frac{t}{3}+\frac{t}{2}\sqrt{9-t^2}+C=$$

$$= \frac{9}{2}\arcsin\frac{x-2}{3} + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C.$$

2^o. $\int R(x, \sqrt{ax^2+bx+c})dx$ ko'rinishidagi ayrim integrallarni hisoblashning boshqa usullarini keltiramiz.

a) $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ ko‘rinishidagi integrallar, bu yerda $P_n(x)$ n- darajali ko‘phad:

1) $n=0$ da $\int \frac{Adx}{\sqrt{ax^2+bx+c}}$ ko'rinishda bo‘ladi; bu integral $a>0$ bo‘lganda jadvaldagi 14- integralga, $a<0$ bo‘lganda jadvaldagi 13- integralga keltiriladi;

2) $n=1$ da $\int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}$ ko'rinishda bo‘ladi; bu integral suratda kvadrat uchhadning hosilasini ajratish natijasida ikkita, biri jadvaldagi 1- integralga va ikkinchisi 1) banddagi integralga keltiriladi;

3) $n \geq 2$ bo‘lganda berilgan integraldan keltirish formulalari yordamida

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + M\int \frac{dx}{\sqrt{ax^2+bx+c}},$$

ko‘rinishdagi ifoda hosil qilinadi,bu yerda $Q_{n-1}(x)-$ koeffitsiyentlari noma'lum bo‘lgan $n-1$- darajali ko‘phad, $M-$ qandaydir o‘zgarmas son. Bunda ko‘phadning noma'lum koeffitsientlari va M soni oxirgi tenglikni differensiallash hamda x ning chap va o‘ng tomondagi bir xil darajalari oldidagi koeffitsientlarni tenglashtirish orqali topiladi.

b) $\int \frac{dx}{(\alpha x+\beta)\sqrt{ax^2+bx+c}}$ ko‘rinishidagi integral $\alpha x+\beta = \frac{1}{t}$ almashtirish yordamida 1) banddagi integralga keltiriladi;

c) $\int \frac{dx}{(\alpha x+\beta)^n\sqrt{ax^2+bx+c}}$ $(n \in Z, n>1)$ ko'rinishidagi integral $\alpha x+\beta = \frac{1}{t}$ o‘rniga qo‘yish orqali 3) banddagi integralga keltiriladi.

5- misol. $\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}}$ integralni toping.

Yechish. $x-2=\frac{1}{t}$ deymiz. U holda $dx = -\frac{dt}{t^2}$, $x^2-4x+5 = \frac{1}{t^2}+1.$

Bundan

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}} = -\int \frac{\frac{dt}{t^2}}{\frac{1}{t^3}\sqrt{\frac{1}{t^2}+1}} = -\int \frac{t^2dt}{\sqrt{t^2+1}}.$$

Demak, b) banddagi integral hosil qilindi. Bunda $n=2$ bo‘lgani uchun

$$\int \frac{t^2 dt}{\sqrt{t^2+1}} = (At+B)\sqrt{t^2+1} + M\int \frac{dt}{\sqrt{t^2+1}}.$$

Tenglikning har ikkala qismini differensiallaymiz:

$$\frac{t^2}{\sqrt{t^2+1}} = A\sqrt{1+t^2} + \frac{(At+B)t}{\sqrt{t^2+1}} + \frac{M}{\sqrt{t^2+1}}$$

yoki

$$t^2 = A(1+t^2) + (At+B)t + M .$$

x ning bir xil darajalari oldidagi koeffitsientlarni tenglab, topamiz:

$$A = \frac{1}{2},\ b=0,\ M = -\frac{1}{2}.$$

U holda

$$\int \frac{t^2 dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right| + C .$$

Dastlabki o‘zgaruvchiga qaytamiz:

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}} = -\frac{\sqrt{x^2-4x+5}}{2(x-2)^2} + \frac{1}{2}\ln\left|\frac{1+\sqrt{x^2-4x+5}}{x-2}\right| + C.$$

2. $\int x^m (a+bx^n)^p dx$ ko’rinishidagi integral *binominal differensial integrali* deyiladi. Bunda integral ostidagi ifoda $x^m(a+bx^n)^p$ ga *binominal differensial* deyiladi, bu yerda $m,\ n,\ p-$ ratsional sonlar.

Binominal differensial integrali uchta holdagina ratsional funksiyalarni integrallashga keltiriladi:

a) p butun son bo‘lganida integral $x=t^s$ (bu yerda $s=EKUK(m,n)$) o‘rniga qo‘yish orqali ratsionallashtiriladi;

b) $\frac{m+1}{n}$ butun son bo‘lganida integral $a+bx^n=t^s$(bu yerda $s-p$ sonning maxraji) o‘rniga qo‘yish yordamida ratsionallashtiriladi;

c) $\frac{m+1}{n}+p$ butun son bo‘lganida integralda $a+bx^n=t^s x^n$ (bu yerda $s-p$ sonning maxraji) almashtirish bajariladi.

Agar yuqorida keltirilgan shartlar bajarilmasa binominal differensial elementar funksiyalar orqali ifodalanmaydi, ya’ni integrallanmaydi.

Masalan, $\int\sqrt{1+x^3}dx$ integralning integral osti funksiyasi binominal differensial: $m=0,\ n=3,\ p=\frac{1}{2}$. Bunda $p=\frac{1}{2},\ \frac{m+1}{n}=\frac{1}{3},\ \frac{m+1}{n}+p=\frac{5}{6}$ sonlardan birortasi butun son emas. Shu sababli bu integral elementar funksiyalar orqali ifodalanmaydi.

6-misol. $I=\int\frac{dx}{x^3\sqrt{1+x^4}}$ integralni toping.

Yechich. Shartga ko‘ra $m=-3,\ n=4,\ p=-\frac{1}{2},\ \frac{m+1}{n}+p=\frac{-3+1}{4}-\frac{1}{2}=-1.$

c) bandda aytilganidek almashtirish bajaramiz:

$$1+x^4=t^2x^4,\ x=(t^2-1)^{-\frac{1}{4}},\ dx=-\frac{1}{2}t(t^2-1)^{-\frac{5}{4}},\ t=\frac{\sqrt{1+x^4}}{x^2}.$$

U holda $I=\int x^{-3}(1+x^4)^{-\frac{1}{2}}dx=-\frac{1}{2}\int(t^2-1)^{-\frac{1}{4}\cdot(-3)}(t^2)^{-\frac{1}{2}}\left((t^2-1)^{-\frac{1}{4}\cdot 4}\right)^{-\frac{1}{2}}t(t^2-1)^{-\frac{5}{4}}dt=$

$$=-\frac{1}{2}\int(t^2-1)^{\frac{3}{4}+\frac{1}{2}-\frac{5}{4}}\cdot t^{-1+1}dt=-\frac{1}{2}\int dt=-\frac{1}{2}t+C=-\frac{\sqrt{1+x^4}}{2x^2}+C.$$

Ma’lumki, har qanday uzluksiz funksiya boshlang‘ich funksiyaga ega bo‘ladi. Agar biror $f(x)$ elementar funksiyaning boshlang‘ich funksiyasi ham elementar funksiya bo‘lsa, u holda $\int f(x)dx$ *integral elementar funksiyalarda ifodalanadi* deyiladi.

Adabiyotlarda keltirilishicha $\int\sqrt{x}\cdot\cos x dx$ integral elementar funksiyalarda ifodalanmaydi, chunki hosilasi $\sqrt{x}\cdot\cos x$ ga teng bo‘lgan elementar funksiya mavjud emas. Amaliy tatbiqda muhim ahamiyatga ega bo‘lgan elementar funksiyalarda ifodalanmaydigan integrallarga misollar keltiramiz:

$\int e^{-x^2}dx$ – Puasson integrali (ehtimollar nazariyasi);

$\int\frac{dx}{\ln x}$ – integralli logarifm (sonlar nazariyasi);

$\int\cos x^2dx,\ \int\sin x^2dx$ – Frenel integrallari (fizika);

$\int\frac{\sin x}{x}dx,\ \int\frac{\cos x}{x}dx$ – integralli sinus va kosinus;

$\int\frac{e^x}{x}dx$ – integralli ko‘rsatkichli funksiya.

Elementar funksiyalarda ifodalanmasada, $e^{-x^2},\ \frac{1}{\ln x},\ \cos x^2,\ \sin x^2,\ \frac{\sin x}{x},\ \frac{\cos x}{x},\ \frac{e^x}{x}$ funksiyalarning boshlang‘ich funksiyalari etarlicha o‘rganilgan, x argumentning turli qiymatlarida ularning qiymatlari uchun mufassal jadvallar tuzilgan.

Amaliy mashg'ulot uchun mashqlar.

1. Berilgan integrallarni toping:

1) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$; 2) $\int \frac{dx}{\sqrt{x(1+\sqrt[4]{x})^3}}$;

3) $\int \frac{x^2+\sqrt[3]{1+x}}{\sqrt{1+x}}dx$ 4) $\int \frac{x-\sqrt{x+1}}{\sqrt[3]{x+1}}dx$;

5) $\int \frac{dx}{\sqrt{2x-1}+\sqrt[3]{(2x-1)^2}}$; 6) $\int \left(\sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} - \sqrt[6]{\left(\frac{x+1}{x-1}\right)^5} \right)\frac{dx}{1-x^2}$;

7) $\int \frac{dx}{\sqrt{x^2-3x+2}}$; 8) $\int \frac{dx}{\sqrt{x^2+2x+5}}$;

9) $\int \frac{dx}{x\sqrt{x^2+x+1}}$; 10) $\int \frac{dx}{x\sqrt{4-2x-x^2}}$;

11) $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$; 12) $\int \frac{dx}{1+\sqrt{x^2+2x+2}}$;

13) $\int \sqrt{5+4x-x^2}\,dx$; 14) $\int \sqrt{x^2-4}\,dx$;

15) $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$; 16) $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$;

17) $\int \frac{xdx}{\sqrt{3-2x-x^2}}$; 18) $\int \frac{(2x+3)dx}{\sqrt{6x-x^2-8}}$;

19) $\int \frac{dx}{x(1+\sqrt[3]{x})^2}$; 20) $\int \frac{dx}{x^3\sqrt[3]{2-x^3}}$;

21) $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$; 22) $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$;

23) $\int \frac{dx}{x^3\sqrt{1+x^4}}$; 24) $\int \frac{\sqrt{1+\sqrt[3]{x}}}{x\sqrt{x}}dx.$

ANIQ INTEGRAL
ANIQ INTEGRAL TA'RIFI VA HISOBLASH USULLARI. NYUTON-LEYBNIS FORMULASI

Aniq integral tabiat va texnikaning bir qancha masalalarini yechishda, xususan har xil geometrik va fizik kattaliklarni hisoblashda keng qo'llaniladi.

$y=f(x)$ funksiya $[a;b]$ kesmada aniqlangan bo'lsin.

$[a;b]$ kesmani ixtiyoriy ravishda $a=x_0<x_1<\ldots<x_{i-1}<x_i<\ldots<x_{n-1}<x_n=b$ nuqtalar bilan n ta qismga bo'lamiz, bunda $\{x_i\}$ ga $[a;b]$ kesmaning *bo'linishi*, $d=\max\limits_{1\le i\le n}(x_i-x_{i-1}),\ (i=\overline{1,n})$ kattalikka *bo'linish diametri* deymiz.

Har bir $[x_{i-1};x_i]$ kesmada ixtiyoriy ξ_i nuqtani tanlaymiz. Bunday nuqtalarni *belgilangan nuqtalar* deb ataymiz. Funksiyaning $f(\xi_i)$ qiymatni mos $\Delta x_i = x_i - x_{i-1}$ uzunlikka ko‘paytirib, bu ko‘paytmalardan

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (1.1)$$

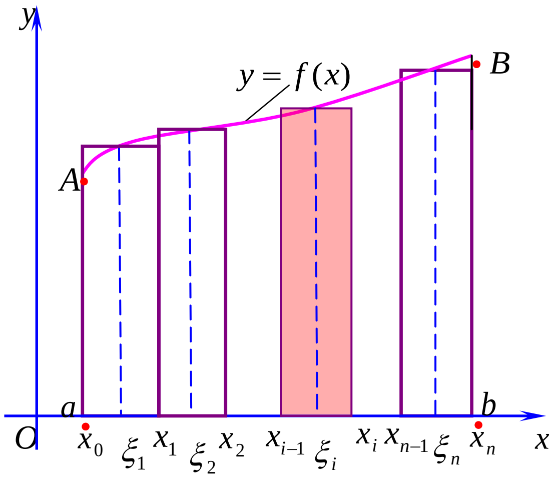
yig‘indini tuzamiz.(1.1) yig‘indiga $f(x)$ funksiya uchun $[a;b]$ kesmaning $\{x_i\}$ bo‘linishidagi *Riman integral yig‘indisi* deyiladi.

2-ta’rif. Agar (1.1) Riman integral yig‘indisi $d \to 0$ da chekli limitga ega bo‘lsa, u holda bu limitga $[a;b]$ kesmada $f(x)$ funksiyadan olingan *aniq integral* deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi.

Shunday qilib, ta’rifga ko‘ra

$$\int_a^b f(x)dx = \lim_{d\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i, \qquad (1.2)$$

bu yerda $f(x)$ – integral ostidagi funksiya, x – integrallash o‘zgaruvchisi, a, b – integralning quyi va yuqori chegarasi, $[a;b]$ – integrallash sohasi (kesmasi) deyiladi.



2-shakl

$[a;b]$ kesmada $\int_a^b f(x)dx$ anig integral mavjud bo‘lsa, $y = f(x)$ funksiya shu kesmada *integrallanuvchi* deyiladi.

Keltirilgan ta’riflarda $a < b$ bo‘lsin deb faraz qilindi. Aniq integral tushunchasini $a = b$ va $a > b$ bo‘lgan hollar uchun umumlashtiramiz.

$a > b$ bo‘lganida 2-ta’rifga ko‘ra

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \qquad (1.2)\text{bo‘ladi.}$$

2-ta’rifga ko‘ra $a = b$ bo‘lganida ((5.5) ga qarang).

$$\int_a^a f(x)dx = 0 \qquad (1.3)\text{bo‘ladi.}$$

1- misol. $\int_0^1 x^2 dx$ integralni aniq integralning ta’rifidan foydalanib hisoblang.

Yechish. $[0;1]$ kesmada $y = x^2$ funksiya uzluksiz. $[0;1]$ kesmani $0 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = 1$ nuqtalar bilan uzunliklari $\Delta x_i = \frac{1}{n}(i = \overline{1,n})$

bo‘lgan n ta bo‘lakka bo‘lamiz. Bunda $d=\max\limits_{1\le i\le n}\Delta x_i$. Denak, $d\to 0$ da $n\to\infty$. ξ_i nuqta sifatida qismiy kesmalarning oxirlarini olamiz:

$\xi_i = x_i = \frac{i}{n}$. Tegishli integral yig’indini tuzamiz:

$$w_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i=\sum_{i=1}^{n}\frac{i^2}{n^2}\cdot\frac{1}{n}=\frac{1}{n^3}(1^2+2^2+...+n^2)=$$

$$=\frac{n(n+1)(2n+1)}{6n^3}=\frac{(n+1)(2n+1)}{6n^2}.$$

Bundan

$$\lim_{d\to 0} w_n=\lim_{n\to\infty} w_n=\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}=\frac{1}{3}.$$

Demak, ta’rifga ko‘ra

$$\int_0^1 x^2dx=\frac{1}{3}.$$

Endi ξ_i nuqta sifatida qismiy kesmalarning boshlarini olamiz:

$$\xi_i=x_{i-1}=\frac{i-1}{n}.$$

Bundan

$$w_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i=\sum_{i=1}^{n}\frac{(i-1)^2}{n^2}\cdot\frac{1}{n}=\frac{(n-1)n(2n-1)}{6n^3}=\frac{(n-1)(2n-1)}{6n^2}$$

yoki

$$\int_0^1 x^2dx=\lim_{d\to 0\ (n\to\infty)} w_n=\lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2}=\frac{1}{3}.$$

Demak, berilgan integralning qiymati $[0;1]$ kesmani bo‘lish usuliga va bu kesmada ξ_i nuqtani tanlash usuliga bog‘liq emas va $\int_0^1 x^2dx=\frac{1}{3}$.

Aniq integral mavjud bo’lishi haqidagi teoremani isbotsiz keltiramiz.

1-teorema. (*Koshi teoremasi*). Agar $y=f(x)$ funksiya $[a;b]$ kesmada uzlukziz bo‘lsa, u holda $\int_a^b f(x)dx$ aniq integral mavjud bo’ladi.

Aniq integralning geometrik ma’nosi: agar $f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi va manfiy bo‘lmasa, u holda $[a;b]$ kesmada $f(x)$ funksiyadan olingan aniq integral $y=f(x)$, $y=0$, $x=a$ va $x=b$ chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasiga teng.

2- misol. $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ integralni uning geometrik ma’nosiga tayanib hisoblang.

Yechish. Bunda x ning -3 dan 3 gacha o‘zgarishida tenglamasi $y=\sqrt{9-x^2}$ bo‘lgan chiziq $x^2+y^2=9$ aylananing Ox o’qidan yuqorida joylashgan bo‘lagidan iborat bo‘ladi. Shu sababli $x=-3,\ x=3,\ y=0$ va $y=\sqrt{9-x^2}$ chiziqlar bilan chegaralangan egri chiziqli trapetsiya $x^2+y^2=9$ doiraning yarmidan tashkil topadi. Uning yuzi $S=\frac{9\pi}{2}$ ga teng. Demak,

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{9\pi}{2}.$$

Agar $v(t)$ funksiya $[a;b]$ kesmada integrallanuvchi va manfiy bo‘lmasa, u holda $v(t)$ tezlikdan $[a;b]$ vaqt oralig‘ida olingan aniq integral moddiy nuqtaning $t=a$ dan $t=b$ gacha vaqt oralig‘ida bosib o‘tgan yo‘liga teng. Bu jumla *aniq integralning mexanik ma’nosini* anglatadi.

Aniq integralning xossalari:

1^o. Agar integral ostidagi funksiya birga teng bo‘lsa, u holda

$$\int_a^b dx=b-a \text{ bo‘ladi.}$$

2^o. Ozgarmas ko‘paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya’ni

$$\int_a^b kf(x)dx=k\int_a^b f(x)dx,\ k=const.$$

3^o. Chekli sondagi funktsiyalar algebraik yig‘indisining aniq integrali qo‘shiluvchilar aniq integrallarining algebraik yig‘indisiga teng, ya’ni

$$\int_a^b (f(x)\pm\varphi(x))dx=\int_a^b f(x)dx\pm\int_a^b \varphi(x)dx.$$

4^o. Agar $[a;b]$ kesma bir necha qismga bo‘lingan bo‘lsa, u holda $[a;b]$ kesma bo‘yicha olingan aniq integral har bir qism bo‘yicha olingan aniq integrallar yig‘indisiga teng bo‘ladi.

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx,\quad c\in[a;b].$$

5^o. Agar $[a;b]$ kesmada funksiya o‘z ishorasini o‘zgartirmasa, u holda funksiya aniq integralining ishorasi funksiya ishorasi bilan bir xil bo‘ladi, ya’ni:

$[a;b]$ da $f(x)\ge 0$ bo‘lganda, $\int_a^b f(x)dx\ge 0$ bo’ladi;

$[a;b]$ da $f(x)\le 0$ bo‘lganda, $\int_a^b f(x)dx\le 0$ bo’ladi.

6^o. Agar $[a;b]$ kesmada $f(x)\ge\varphi(x)$ bo‘lsa, u holda

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx \text{ bo‘ladi.}$$

7^o. Agar m va M sonlar $f(x)$ funksiyaning $[a;b]$ kesmadagi eng kichik va eng katta qiymatlarii bo‘lsa, u holda

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \text{ bo‘ladi.}$$

Bu xossa aniq integralni *baholash haqidagi teorema* deb yuritiladi.

8^o. Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo‘lsa, u holda shunday $c \in [a;b]$ nuqta topiladiki, $\int_a^b f(x)dx = f(c)(b-a)$ (1.4) bo‘ladi.

Bu xossa *o‘rta qiymat haqidagi teorema* deb ataladi. (1.4) formulaga *o‘rta qiymat formulasi*, $f(c)$ ga $f(x)$ *funktsiyaning* $[a;b]$ *kesmadagi o‘rtacha qiymati* deyiladi.

O‘rta qiymat haqidagi teorema quyidagi geometrik talqinga ega: agar $f(x) > 0$ bo‘lsa, u holda $\int_a^b f(x)dx$ integral qiymati balandligi $f(c)$ ga va asosi $(b-a)$ ga teng bo‘lgan to‘g‘ri to’rtburchakning yuzasiga teng bo‘ladi.

Aniq integralning xossalaridan quyidagi natijalar kelub chiqadi.

1-natija. $[a;b]$ kesmada aniqlangan $f(x)$ funktsiya uchun

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx \text{ bo‘ladi.}$$

2-natija. Agar $[a;b]$ kesmada $|f(x)| \leq k$ bo‘lsa, u holda

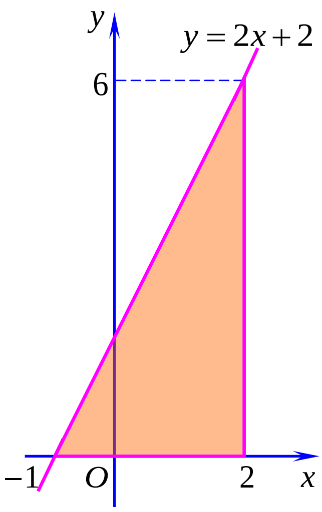
$$\left|\int_a^b f(x)dx\right| \leq k(b-a), \ (k = const.) \text{ bo‘ladi.}$$

3- misol. $y = 2x + 2$ funksiyaning $[-1;2]$ kesmadagi o‘rtacha qiymatini toping.

Yechish. O‘rta qiymat haqidagi teoremadan topamiz:

$$f_{o'rt} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Aniq integralning geometrik ma’nosiga ko‘ra $\int_{-1}^{2}(2x+2)dx$ integralning qiymati 3-shaklda keltirilgan uchburchakning yuzasiga teng, ya’ni $S = \frac{1}{2}\cdot(2+1)\cdot 6 = 9.$



3-shakl

Bundan

$$f_{o,rt} = \frac{1}{2-(-1)}\cdot 9 = 3.$$

1-teorema (*integral hisobning asosiy teoremasi*). Agar $F(x)$ funksiya $[a;b]$ kesmada uzluksiz bo‘lgan $f(x)$ funksiyaning boshlang‘ich funksiyasi bo‘lsa, u holda

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (1.5)$$

(1.5) formulaga ***Nyuton-Leybnis formulasi*** deyiladi.

$F(b) - F(a)$ ayirmani shartli ravishda $F(x)\Big|_a^b$ deb yozish kelishilgan. Bu kelishuv natijasida Nuyton-Leybnis formulasi

$$\int_a^b f(x)dx = F(x)\Big|_a^b \qquad (1.5)$$ ko‘rinishda ifodalanadi.

1- misol. $\int_0^3 \frac{dx}{\sqrt{1+x^2}}$ integralni hisoblang.

Yechish. $\int_0^3 \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{1+x^2}\right|\Big|_0^3 = \ln\left|3+\sqrt{10}\right| - \ln 1 = \ln\left|3+\sqrt{10}\right|.$

2- misol. $\int_1^4 \frac{dx}{x^2-2x+10}$ integralni hisoblang.

Yechish.

$$\int_1^4 \frac{dx}{x^2-2x+10} = \int_1^4 \frac{dx}{(x-1)^2+3^2} = \frac{1}{3}arctg\frac{x-1}{3}\Big|_1^4 = \frac{1}{3}(arctg1 - arctg0) = \frac{\pi}{12}.$$

Aniq integralda o‘zgaruvchini almashtirish

3-teorema. $y = f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo‘lsin. Agar: 1) $x = \varphi(t)$ funksiya $[\alpha;\beta]$ kesmada differensiallanuvchi va $\varphi'(t)$ funksiya $[\alpha;\beta]$ kesmada uzluksiz; 2) $x = \varphi(t)$ funksiyaning qiymatlar sohasi $[a;b]$ kesmadan iborat; 3) $\varphi(\alpha) = a$ va $\varphi(\beta) = b$ bo‘lsa, u holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt \qquad (1.6)$$ bo‘ladi.

(1.6) formula *aniq integralda o‘zgaruvchini almashtirish* formulasi deb yuritiladi. Aniq integralni hisoblashning bu usulida *aniq integralda o‘rniga qo‘yish* usuli deyiladi.

3- misol. $\int_0^{\sqrt{3}} \sqrt{4-x^2}dx$ integralni hisoblang.

Yechish. $x = 2\sin t,\ 0 \le t \le \frac{\pi}{3}$ belgilash kiritamiz. Bu o‘zgaruvchini almashtirish 3-teoremaning barcha shartlarini qanoatlantiradi: birinchidan $f(x) = \sqrt{4-x^2}$ funksiya $[0;\sqrt{3}]$ kesmada uzluksiz, ikkinchidan $x = 2\sin t$ funksiya $\left[0;\frac{\pi}{3}\right]$ kesmada differensiallanuvchi va $x' = 2\cos t$ bu kesmada uzluksiz, uchinchidan t o‘zgaruvchi 0 dan $\frac{\pi}{3}$ gacha o‘zgarganda $x=2\sin t$ funksiya 0 dan $\sqrt{3}$ gacha o‘sadi va bunda $\varphi(0)=0$ va $\varphi\left(\frac{\pi}{3}\right)=\sqrt{3}$. Bunda $dx=2\cos tdt$.

(1.6) formuladan topamiz:

$$\int_0^{\sqrt{3}}\sqrt{4-x^2}dx=4\int_0^{\frac{\pi}{3}}\cos^2 tdt=2\int_0^{\frac{\pi}{3}}(1+\cos 2t)dt=2\left(t+\frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{3}}=\frac{2\pi}{3}+\frac{\sqrt{3}}{2}.$$

4-misol. $\int_0^1 x\sqrt{1+x^2}dx$ integralni hisoblang.

Yechish. $t=\sqrt{1+x^2}$ o‘rniga qo‘yish bajaramiz. U holda

$$x=\sqrt{t^2-1},\ dx=\frac{tdt}{\sqrt{t^2-1}},\ x=0 \text{ da } t=1,\ x= \text{ da } t=\sqrt{2}.$$

$[1;\sqrt{2}]$ kesmada $\sqrt{t^2-1}$ funksiya monoton o‘sadi, demak o‘rniga qo‘yich to‘g‘ri bajarilgan. Bundan

$$\int_0^1 x\sqrt{1+x^2}dx=\int_1^{\sqrt{2}}\sqrt{t^2-1}\cdot t\cdot\frac{tdt}{\sqrt{t^2-1}}=\int_1^{\sqrt{2}}t^2dt=\frac{t^3}{3}\Bigg|_1^{\sqrt{2}}=\frac{2\sqrt{2}-1}{3}.$$

Izoh.(1.6) formulani qo‘llashda teoremada sanab o‘tilgan shartlarning bajarilishini tekshirish lozim. Agar bu shartlar buzilsa keltirilgan formula bo‘yicha o‘zgaruvchini almashtirish xato natijaga olib kelishi mumkin.

Aniq integralni bo‘laklab integrallash

4-teorema. Agar $u(x)$ va $v(x)$ funksiyalar $u'(x)$ va $v'(x)$ hosilalari bilan $[a;b]$ kesmada uzluksiz bo‘lsa, u holda

$$\int_a^b udv=uv\Big|_a^b-\int_a^b vdu \qquad (1.7)$$ bo‘ladi.

(1.7) formula *aniq integralni bo‘laklab integrallash formulasi* deb ataladi.

1) $\int_0^1 arctgxdx$ integral hisoblansin.

$$\int_0^1 arctgxdx=\begin{vmatrix} u=arctgx & du=\dfrac{dx}{1+x^2} \\ dv=dx & v=x\end{vmatrix}=xarctgx\Big|_0^1-\int_0^1\frac{xdx}{1+x^2}=arctg1-$$

$$\frac{1}{2}\ln(1+x^2)\Big|_0^1=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

2) $\int_0^1 xe^{-x}dx$ integral hisoblansin.

$$\int_0^1 xe^{-x}dx = \begin{vmatrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{vmatrix} = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx = -e^{-1} - e^{-x}\Big|_0^1 =$$

$$= -e^{-1} - e^{-1} + 1 = 1 - \frac{2}{e};$$

Amaliy mashg'ulot uchun mashhqlar.

Berilgan integrallarni hisoblang:

1) $\int_{-1}^{2}(x^2+2x+1)dx;$

2) $\int_0^{\frac{\pi}{4}}\sin 4xdx;$

3) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos xdx;$

4) $\int_1^e \frac{dx}{x};$

5) $\int_0^{\frac{\pi}{2}}\cos^2 xdx;$

6) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\sin^2 x};$

7) $\int_1^2 \frac{dx}{x+x^2};$

8) $\int_0^1 (2x^3+1)x^2dx;$

9) $\int_0^1 x\sqrt{1+x^2}dx;$

10) $\int_0^{\frac{\pi}{2}}\cos x\sin^3 xdx;$

11) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sin xdx}{1+\cos x};$

12) $\int_{\frac{\sqrt{2}}{3}}^{\frac{\sqrt{3}}{3}}\frac{dx}{\sqrt{4-9x^2}};$

13) $\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{dx}{3+4x^2};$

14) $\int_0^{\frac{\pi}{4}}\sin^3 xdx;$

15) $\int_0^{\frac{\pi}{2}}\frac{\cos xdx}{6-5\sin x+\sin^2 x};$

16) $\int_{\frac{\sqrt{2}}{2}}^{1}\frac{\sqrt{1-x^2}}{x^2}dx;$

17) $\int_0^1 \arcsin xdx;$

18) $\int_1^e \ln^2 xdx;$

19) $\int_0^{\pi} x\sin\frac{x}{2}dx;$

20) $\int_0^{\frac{\pi}{4}} e^x\sin 2xdx;$

21) $\int_0^1 x^2 e^{3x} dx$; 22) $\int_1^{\sqrt{e}} x \ln x dx$;

23) $\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x} dx$; 24) $\int_0^{e^{\frac{\pi}{2}}} \cos(\ln x) dx$.

XOSMAS INTEGRALLAR

Aniq integral qaralganida $\int_a^b f(x)dx$ integral mavjud bo‘lishi uchun ikkita shartning bajarilishi talab qilingan edi: 1) integrallash chegarasi chekli $[a;b]$ kesmadan iborat bo‘lishi; 2) integral ostidagi funksiya $[a;b]$ kesmada aniqlangan va chegaralangan bo‘lishi.

Aniq integral uchun keltirilgan shartlardan biri bajarilmaganda u *xosmas integral* deb ataladi: 1) faqat birinchi shart bajarilmasa, cheksiz chegarali xosmas integral (yoki I tur xosmas integral) deyiladi; 2) faqat ikkinchi shart bajarilmasa, chegaralanmagan funksiyaning xosmas integrali (yoki II tur xosmas integral) deyiladi.

1.

1-ta’rif. $f(x)$ funksiya $[a;+\infty)$ oraliqda uzluksiz bo‘lsin. Agar $\lim\limits_{b\to+\infty}\int_a^b f(x)dx$ limit mavjud va chekli bo‘lsa, bu limitga *yuqori chegarasi cheksiz xosmas integral* deyiladi va $\int_a^{+\infty} f(x)dx$ kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx. \qquad (1.1)$$

Bu holda $\int_a^{+\infty} f(x)dx$ integralga *yaqinlashuvchiintegral* deyiladi.

Agar $\lim\limits_{b\to+\infty}\int_a^b f(x)dx$ limit mavjud bo‘lmasa yoki cheksiz bo‘lsa, $\int_a^{+\infty} f(x)dx$ integralga *uzoqlashuvchiintegral* deyiladi.

Quyi chegarasi cheksiz va har ikkala chegarasi cheksiz xosmas integrallar shu kabi ta’riflanadi:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \qquad (1.2)$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to+\infty}\int_c^b f(x)dx, \qquad (1.3)$$

bu yerda $c-$ sonlar o‘qining biror fiksirlangan nuqtasi. Bunda (1.3) tenglikning chap tomonidagi xosmas integral, o‘ng tomondagi har ikkala xosmas integral yaqinlashgandagina yaqinlashadi.

1- misol. $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}(\alpha>0)$ integralni yaqinlashishga tekshiring.

Yechish. $\alpha\neq 1$ bo‘lsin. U holda

$$\int\limits_1^{+\infty}\frac{dx}{x^\alpha}=\lim_{b\to+\infty}\int\limits_1^{b}\frac{dx}{x^\alpha}=\lim_{b\to+\infty}\frac{x^{1-\alpha}}{1-\alpha}\Bigg|_1^b=\frac{1}{1-\alpha}(\lim_{b\to+\infty}b^{1-\alpha}-1).$$

Bunda: 1) $\alpha<1$ bo‘lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}=\frac{1}{1-\alpha}(\lim_{b\to+\infty}b^{1-\alpha}-1)=+\infty,$

2) $\alpha>1$ bo‘lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}=\frac{1}{1-\alpha}(\lim_{b\to+\infty}b^{1-\alpha}-1)=\frac{1}{1-\alpha},$

3) $\alpha=1$ bo‘lganda, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}=\lim_{b\to+\infty}\int\limits_1^{b}\frac{dx}{x}=\lim_{b\to+\infty}\ln x\Big|_1^b=\lim_{b\to+\infty}\ln b=+\infty.$

Demak, $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}$ xosmas integral $\alpha>1$ da yaqinlashadi va $0<\alpha\leq 1$ da uzoqlashadi.

2- misol. $\int\limits_{-\infty}^{0}x\cos x dx$ integralni yaqinlashishga tekshiring.

Yechish. $\int\limits_{-\infty}^{0}x\cos xdx=\lim_{a\to-\infty}\int\limits_a^0 x\cos xdx=\lim_{a\to-\infty}\left(x\sin x\Big|_a^0-\int\limits_a^0\sin xdx\right)=$

$=\lim_{a\to-\infty}(-a\sin a+\cos x\Big|_a^0=\lim_{a\to-\infty}(-a\sin a-\cos a+1).$

Bu limit mavjud emas. Shu sababli $\int\limits_{-\infty}^{0}x\cos xdx$ integral uzoqlashadi.

3- misol. $\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}$ integralni yaqinlashishga tekshiring.

Yechish. Oraliq nuqtani $c=0$ deymiz. U holda(1.3) tenglikga ko‘ra

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}=\int\limits_{-\infty}^{0}\frac{dx}{x^2+6x+10}+\int\limits_{0}^{+\infty}\frac{dx}{x^2+6x+10}.$$

Bundan

$$\int\limits_{-\infty}^{0}\frac{dx}{x^2+6x+10}=\lim_{a\to-\infty}\int\limits_a^0\frac{dx}{(x+3)^2+1}=\lim_{a\to-\infty}arctg(x+3)\Big|_a^0=$$

$$=arctg3-\lim_{a\to-\infty}arctg(a+3)=arctg3+\frac{\pi}{2},$$

$$\int\limits_{0}^{+\infty}\frac{dx}{x^2+6x+10}=\lim_{b\to+\infty}\int\limits_0^b\frac{dx}{(x+3)^2+1}=\lim_{b\to+\infty}arctg(x+3)\Big|_0^b=$$

$$= \lim_{b\to+\infty} arctg(b+3) - arctg3 = \frac{\pi}{2} - arctg3.$$

U holda

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+6x+10} = \int_{-\infty}^{0} \frac{dx}{x^2+6x+10} + \int_{0}^{+\infty} \frac{dx}{x^2+6x+10} = arctg3 + \frac{\pi}{2} + \frac{\pi}{2} - arctg3 = \pi.$$

Demak, xosmas integral yaqinlashadi.

2.

2-ta'rif. $f(x)$ funksiya $[a;b)$ oraliqda aniqlangan va uzluksiz bo'lib, $x=b$ da aniqlamagan yoki uzilishga ega bo'lsin. Agar $\lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx$ limit mavjud va chekli bo'lsa, bu limitga *chegaralanmagan funksiyadan olingan xosmas integral* deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx. \qquad (1.4)$$

Shu kabi: 1) $f(x)$ funksiya x ning a ga o'ngdan yaqinlashishida uzilishga ega bo'lsa,

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b} f(x)dx; \qquad (1.5)$$ bo'ladi;

2) agar $f(x)$ funksiya $c\in[a;b]$ da uzilishga ega bo'lsa,

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon\to 0}\int_{c+\varepsilon}^{b} f(x)dx \qquad (1.6)$$ bo'ladi.

II tur xosmas integrallar uchun yaqinlashish (uzoqlashish) tushunchalari I tur integrallardagi kabi kiritiladi.

4- misol. $\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}$ integralni yaqinlashishga tekshiring.

Yechish. $x=0$ da integral ostidagi funksiya uzilishga ega. U holda (1.6) tenglikka ko'ra

$$\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}} = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x\sqrt[3]{x}} + \lim_{\varepsilon\to 0}\int_{+\varepsilon}^{1}\frac{dx}{x\sqrt[3]{x}} =$$

$$= -3\lim_{\varepsilon\to 0} x^{-\frac{1}{3}}\Big|_{-1}^{-\varepsilon} - 3\lim_{\varepsilon\to 0} x^{-\frac{1}{3}}\Big|_{+\varepsilon}^{1} =$$

$$= 3\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}} - 1 - 1 + 3\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}} = 6\left(\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{3}} - 1\right) = +\infty.$$

Demak, xosmas integral uzoqlashadi. Berilgan integralga Nyuton-Leybnits formulasi formal qo'llanilishi xato natijaga olib keladi:

$$\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}=-\frac{3}{\sqrt[3]{x}}\Bigg|_{-1}^{1}=-6\,.$$

Ko‘pincha xosmas integralni (1.1) - (1.6) formulalar orqali hisoblash shart bo‘lmasdan, faqat uning yaqinlashuvchi yoyi uzoqlashuvchi bo‘lishini bilish yetarli bo‘ladi. Bunday hollarda berilgan integral yaqinlashishga yaqinlashuvchi yoki uzoqlashuvchiligi oldindan ma’lum bo‘lgan boshqa xosmas integral bilan taqqoslash orqali tekshiriladi. Xosmas integrallarning taqqoslash alomatlarini ifodalovchi teoremalarni isbotsiz keltiramiz.

1-teorema (*I tur xosmas integralning yaqinlashish alomati*). $[a;+\infty)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz va $0\le f(x)\le\varphi(x)$ bo’lsin.

U holda:

1) agar $\int_a^{+\infty}\varphi(x)dx$ integral yaqinlashsa, $\int_a^{+\infty}f(x)dx$ integral ham yaqinlashadi;

2) agar $\int_a^{+\infty}f(x)dx$ integral uzoqlashsa, $\int_a^{+\infty}\varphi(x)dx$ integral ham uzoqlashadi.

5- misol. $\int_0^{+\infty}e^{-x^2}dx$ integralni yaqinlashishga tekshiring.

Yechish. Puasson integrali deb ataluvchi bu integral boshlang’ich funksiyaga ega emas. Uni ikkita integralning yig’indisi ko’rinishida ifodalaymiz:

$$\int_0^{+\infty}e^{-x^2}dx=\int_0^{1}e^{-x^2}dx+\int_1^{+\infty}e^{-x^2}dx.$$

Tenglikning chap qismidagi integrallardan birinchisi chekli qiymatga ega bo’lgan aniq integral. Ikkinchi xosmas $\int_1^{+\infty}e^{-x^2}dx$ integralni yaqinlashishga tekshiramiz. $[1;+\infty)$ oraliqda $0<e^{-x^2}\le e^{-x}$ bo‘ladi, e^{-x^2} va e^{-x} funksiyalar uzluksiz. Bunda

$$\int_1^{+\infty}e^{-x}dx=\lim_{b\to+\infty}\int_1^{b}e^{-x}dx=\lim_{b\to+\infty}(-e^{-x})\Big|_1^{b}=\frac{1}{e}-\lim_{b\to+\infty}\frac{1}{e^b}=\frac{1}{e}.$$

Demak, bu integral yaqinlashadi va 1-teoremaning 1) bandiga asosan Puasson integrali ham yaqinlashadi.

6- misol. $\int_0^{1}\frac{dx}{e^x-\cos x}$ integralni yaqinlashishga tekshiring.

Yechish. Integral ostidagi funksiya $x=0$ da uzilishga ega.

$x\in(0;1]$ da $\frac{1}{e^x-\cos x}\ge\frac{1}{xe}$. Bundan

$$\int_0^{1}\frac{dx}{xe}=\frac{1}{e}\lim_{\varepsilon\to0}\int_{+\varepsilon}^{1}\frac{dx}{x}=\frac{1}{e}\lim_{\varepsilon\to0}\ln x\Big|_{+\varepsilon}^{1}=\frac{1}{e}(0-\lim_{\varepsilon\to0}\ln|\varepsilon|)=+\infty\,.$$

Demak, $\int_0^1 \frac{dx}{xe}$ integral uzoqlashadii va 1-teoremaning 2) bandiga asosan berilgan integral ham uzoqlashadi.

2-teorema(*II tur xosmas integralning yaqinlashish alomati*).$[a;b)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz bo‘lsin va $0 \le f(x) \le \varphi(x)$ tengsizlikni qanoatlantirsin, $x = b$da $f(x)$ va $\varphi(x)$ funksiyalar aniqlanmagan yoki uzilishga ega bo‘lsin. U holda:

1) agar $\int_a^b \varphi(x)dx$ integral yaqinlashsa, $\int_a^b f(x)dx$ integral ham yaqinlashadi;

2) agar $\int_a^b f(x)dx$ integral uzoqlashsa, $\int_a^b \varphi(x)dx$ integral ham uzoqlashadi.

Taqqoslash teoremalari integral ostidagi funksiy bir xil ishorali bo’lganida o’rinli bo’ladi. Ishorasi almashinaadigan funksiyalarning xosmas integrallari uchun quyidagi alomat o‘rinli bo‘ladi.

3-teorema. Agar $\int_a^{+\infty} |f(x)|dx \left(\int_a^b |f(x)|dx \right)$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx \right)$ integral ham yaqinlashadi.

Agar $\int_a^{+\infty} |f(x)|dx \left(\int_a^b |f(x)|dx \right)$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx \right)$ integralga *absolut yaqinlashuvchi xosmas integral* deyiladi.

Agar $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx \right)$ integral yaqinlashsa va $\int_a^{+\infty} |f(x)|dx \left(\int_a^b |f(x)|dx \right)$ integral uzoqlashsa, $\int_a^{+\infty} f(x)dx \left(\int_a^b f(x)dx \right)$ integralga *shartli yaqinlashuvchi xosmas integral* deyiladi.

7- misol. $\int_1^{+\infty} \frac{\cos x}{x^2}dx$ integralni yaqinlashishga tekshiring.

Yechish. Integral ostidagi funksiya $[1;+\infty)$ oraliqda ishorasini almashtiradi. Ma’lumki $\left| \frac{\cos x}{x^2} \right| \le \frac{1}{x^2}$. 1-misolga ko‘ra $\int_1^{+\infty} \frac{dx}{x^2}$ integral yaqinlashuvchi.

U holda 1-teoremaga asosan $\int_1^{+\infty} \left| \frac{\cos x}{x^2} \right| dx$ integral yaqinlashuvchi va 3-teorema va 3-ta’rifga ko‘ra $\int_1^{+\infty} \frac{\cos x}{x^2}dx$ integral absalut yaqinlashuvchi bo‘ladi.

(1.2), (1.3) ko‘rinishdagi ((1.5),(1.6) ko‘rinishdagi) xosmas integrallar uchun taqqoslash alomatlari hamda absolut va shartli yaqinlashish tusunchalari yuqorida (1.1) ko‘rinishdagi ((1.4) ko‘rinishdagi) integrallar uchun keltirilgandagi kabi kiritiladi.

Amaliy mashg’ulot uchun mashqlar.

1. Berilgan integrallarni hisoblang yoki uzoqlashuvchi ekanini aniqlang:

1) $\int_{1}^{+\infty}\frac{dx}{1+x^2}$; 2) $\int_{0}^{+\infty} xe^{-\frac{x}{2}}dx$;

3) $\int_{-\infty}^{0} x\cos x dx$; 4) $\int_{2}^{+\infty}\frac{\ln x dx}{x}$;

5) $\int_{2}^{+\infty}\frac{dx}{x\sqrt{x^2-1}}$; 6) $\int_{1}^{+\infty}\frac{arctgx dx}{x^2}$;

7) $\int_{0}^{+\infty} e^{-x}\sin x dx$; 8) $\int_{1}^{e}\frac{dx}{x\sqrt{\ln x}}$;

9) $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$; 10) $\int_{1}^{3}\frac{xdx}{\sqrt{(x-1)}}$;

11) $\int_{-1}^{1}\frac{3x^2+2}{\sqrt[3]{x^2}}dx$; 12) $\int_{0}^{2}\frac{dx}{x^2-4x+3}$;

13) $\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}$; 14) $\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}$.

2. Integrallarni yaqinlashishga tekshiring:

1) $\int_{1}^{+\infty}\frac{dx}{x^{\alpha}}$; 2) $\int_{0}^{+\infty}\frac{dx}{\sqrt{1+x^3}}$;

3) $\int_{0}^{+\infty}\sqrt{x}e^{-x}dx$; 4) $\int_{1}^{+\infty}\frac{\sin x dx}{x^2}$;

5) $\int_{1}^{+\infty}\frac{x^3+1}{x^4}dx$; 6) $\int_{0}^{1}\frac{dx}{e^{\sqrt{x}}-1}$;

7) $\int_{0}^{1}\frac{e^x dx}{\sqrt{1-\cos x}}$; 8) $\int_{0}^{1}\frac{dx}{e^x-\cos x}$;

9) $\int_{1}^{2}\frac{3+\sin x}{(x-1)^3}dx$; 10) $\int_{0}^{1}\frac{\sqrt{x}dx}{\sqrt{1-x^4}}$;

11) $\int_{1}^{+\infty}\frac{\cos x}{x^2}dx$; 12) $\int_{0}^{+\infty} e^{-x}\sin x dx$.

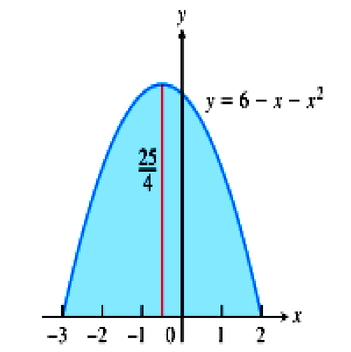
ANIQ INTEGRALNING TADBIQLARI

Yuzani dekart koordinatalarida hisoblash

Aniq integralning geometrik ma’nosiga asosan abssissalar o‘qidan yuqorida yotgan, ya’ni yuqoridan $y=f(x)$ ($f(x)\geq 0$) funksiya grafigi bilan, quyidan Ox o‘q bilan, yon tomonlaridan $x=a$ va $x=b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

$$S=\int_a^b f(x)dx \qquad (1.1)$$

integtegralga teng bo‘ladi.



5-shakl

Shu kabi, abssissalar o‘qidan pastda yotgan, ya’ni quyidan $y=f(x)$ ($f(x)\leq 0$) funksiya grafigi bilan, yuqoridan Ox o‘q bilan, yon tomonlaridan $x=a$ va $x=b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

$$S=-\int_a^b f(x)dx \qquad (1.2)$$

integtegralga teng bo‘ladi.

1-misol. Ox o’q va $y=6-x-x^2$ parabola bilan chegaralangan yassi shakl yuzasini toping[8].

Yechish. Parabolaning Ox o’q bilan kesishish nuqtasini topamiz (5-shakl):

$$y=0=6-x-x^2=(3+x)(2-x),\ x_1=-3,\ \ x_2=2.$$

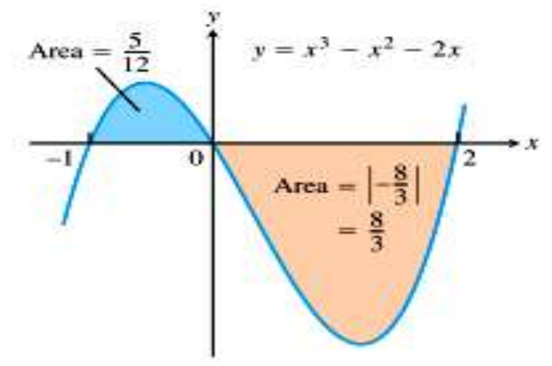
Yassi shakl yuzasini (1.1) formula bilan topamiz:

$$s=\int_{-3}^{2}(6-x-x^2)dx=\left(6x-\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_{-3}^{2}=$$

$$=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+27\right)=20\frac{5}{6}.$$

Yuzani hisoblashga oid murakkabroq masalalar yuzaning additivlik xossasiga asoslangan holda yechiladi. Bunda yassi shakl kesishmaydigan qismlarga ajratiladi va aniq integralning 4^o xossasiga ko‘ra yassi shaklning yuzasi qismlar yuzalarining yig‘indisiga teng bo‘ladi.

2- misol. $y=x^3-x^2-2x$ va $y=0$ chiziqlar bilan chegaralangan tekis shakl yuzasini hisoblang (6- shakl).



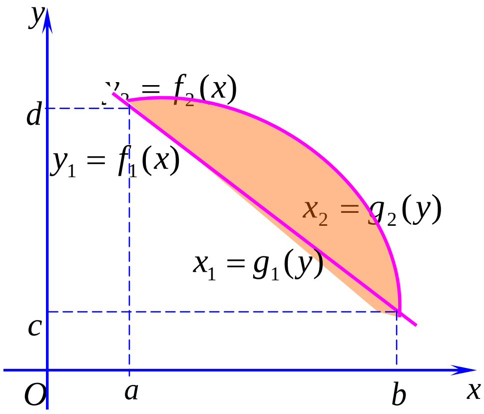
6-shakl

Yechish Berilgan tekis shaklni yuzalari S_1 va S_2 bo‘lgan kesishmaydigan qismlarga ajratamiz. U holda yuzaning additivlik xossasiga asosan berilgan tekis shaklning yuzasi qismlar yuzalarining yig‘indisiga teng bo‘ladi.

Demak, $S = S_1 + S_2 = \int\limits_{-1}^{0} (x^3 - x^2 - 2x)dx - \int\limits_{0}^{2} (x^3 - x^2 - 2x)dx =$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{0}^{2} = -\left(\frac{1}{4} + \frac{1}{3} - 1\right) + \left(4 - \frac{8}{3} - 4\right) = \frac{37}{12}.$$

$[a;b]$ kesmada ikkita $y_1 = f_1(x)$ va $y_2 = f_2(x)$ uzliksiz funksiyalar berilgan va $x \in [a;b]$ da $f_2(x) \ge f_1(x)$ bo‘lsin. Bu funksiyalarning grafiklari va $x = a$, $x = b$ to‘g‘ri chiziqlar bilan chegaralangan yassi shaklni qaraymiz.



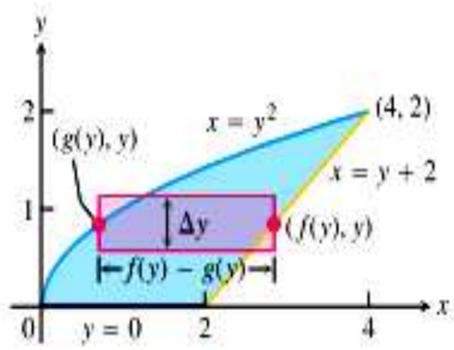
7-shakl

Bu yassi shaklning yuzasi yuqoridan $y_2 = f_2(x)$ va $y_1 = f_1(x)$ funksiyalar garfiklari bilan, quyidan Ox o‘q bilan, yon tomonlardan $x = a$ va $x = b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyalar yuzalarining ayirmasiga teng bo‘ladi:

$$S = \int\limits_a^b f_2(x)dx - \int\limits_a^b f_1(x)dx = \int\limits_a^b (f_2(x) - f_1(x))dx.$$

(1.3)

Ayrim hollarda yuzani hisoblashga oid masalalar yuzaning ko‘chishga nisbatan invariantlik xossasidan foydalangan holda soddalashtiriladi. Bunda yassi shakl yuzasi (1.3) formulada x va y o‘zgaruvchilar (Ox va Oy o‘qlar) ning o‘rnini almashtirish yo‘li bilan hisoblanadi, ya’ni (7-shakl)



7-shakl

$$S = \int\limits_a^b (f_2(x) - f_1(x))dx = \int\limits_c^d (g_2(y) - g_1(y))dy. \qquad (1.4)$$

3-misol. $x = y^2$ va $x = y + 2$ chiziqlar bilan chegaralangan yassi shaklning abssissalar o‘qidan yuqorida yotgan qismining yuzasini hisoblang (8-shakl).

Yechish. Berilgan chiziqlarning kesishish nuqtalarini topamiz:

$$y^2 = y + 2,\; y^2 - y - 2 = 0,\; y_1 = 0,\; y = 2.$$

Yassi shakl yuzasini (1.4) formula bilan topamiz:

$$s=\int_0^2 (y+2-y^2)dy=\left(\frac{y^2}{2}+2y-\frac{y^3}{3}\right)\Bigg|_0^2=2+4-\frac{8}{3}=\frac{10}{3}.$$

Agar egri chiziqli trapetsiya yuqoridan $x=\varphi(t),\ y=\psi(t),\ \alpha\le t\le\beta$ parametrik tenglamalar bilan berilgan funksiya grafigi bilan chegaralangan va $a=\varphi(\alpha),\ b=\varphi(b)$ bo‘lsa (1.1) formulada $x=\varphi(t),\ dx=\varphi'(t)dt$ ko’rinishdagi o‘rniga qo‘yish orqali o‘zgaruvchi almashtiriladi.

U holda

$$S=\int_\alpha^\beta \psi(t)\varphi'(t)dt \qquad (1.5)$$

bo‘ladi, bu yerda, $a=\varphi(\alpha)$ va $b=\varphi(\beta)$.

4- misol. Radiusi R ga teng doira yuzasini hisoblang.

Yechish. Koordinatalar boshini doiraning markaziga joylashtiramiz. Bu doiraning aylanasi $x=R\cos t,\ y=R\sin t$ parametrik tenglamalar bilan aniqlanadi va doira koordinata o‘qlariga nisbatan simmetrik bo‘ladi. Shu sababli uning birinchi chorakdagi yuzasini hisoblaymiz (bunda x o‘zgaruvchi 0 dan R gacha o‘zgarganda t parametr $\frac{\pi}{2}$ dan 0 gacha o‘zgaradi) va natijani to‘rtga ko‘paytiramiz. U holda (1.5) formulaga ko’ra:

$$S=4S_1=4\int_{\frac{\pi}{2}}^0 R\sin t(-R\sin t)dt=4R^2\int_0^{\frac{\pi}{2}}\sin^2 t dt=$$

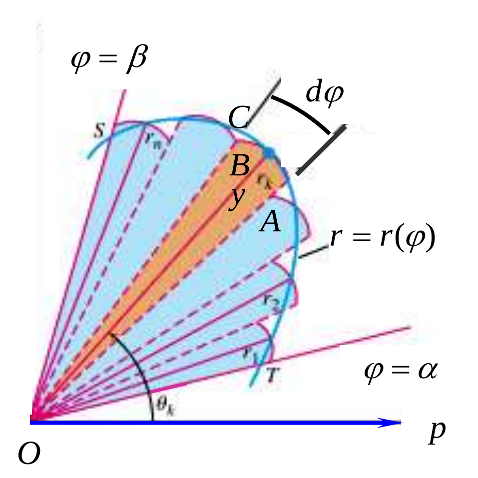
$$=2R^2\int_0^{\frac{\pi}{2}}(1-\cos 2t)dt=2R^2\left(t-\frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}}=\pi R^2.$$

Yuzani qutb koordinatalarida hisoblash

Qutb koordinatalar ($r-$qutb radiusi, $\varphi-$ qutb burchagi) sistemasida berilgan $r=r(\varphi)$ funksiya $\varphi\in[\alpha;\beta]$ kesmada uzluksiz bo‘lsin.

$r=r(\varphi)$ funksiya grafigi hamda O qutbdan chiqqan $\varphi=\alpha$ va $\varphi=\beta$ nurlar bilan chegaralangan yassi shaklga *egri chiziqli sektor* deyiladi.

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi. \qquad (1.6)$$



8-shakl

5- misol. $r=2\cos 3\varphi$ egri chiziq bilan chegaralangan figuraning yuzasini hisoblang.

Yechish. $r=2\cos 3\varphi$ egri chiziq bilan chegaralangan figuraga uch yaproqli gul deyiladi .

Uningning oltidan bir qismi yuzasini hisoblaymiz:

$$\frac{1}{6}S=\frac{1}{2}\int_0^{\frac{\pi}{6}}4\cos^2 3\varphi d\varphi=\int_0^{\frac{\pi}{6}}(1+\cos 6\varphi)d\varphi=\left(\varphi+\frac{\sin 6\varphi}{6}\right)\Bigg|_0^{\frac{\pi}{6}}=\frac{\pi}{6}.$$

Bundan $S=\pi.$

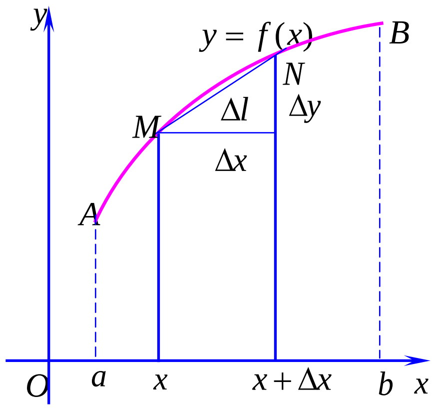
Agar egri chiziqli sektor $r_1=r_1(\varphi)$ va $r_2=r_2(\varphi)$ ($\varphi\in[\alpha;\beta]$ da $r_2(\varphi)>r_1(\varphi)$) funksiyalar grafiklari bilan chegaralangan bo'lsa,

$$S=\frac{1}{2}\int_\alpha^\beta (r_2^2(\varphi)-r_1^2(\varphi))d\varphi \qquad (1.7)\text{bo'ladi}$$

Egri chiziqli yoy uzunligi

Tekislikda AB egri chiziq $[a;b]$ kesmada uzluksiz $y=f(x)$ funksiya grafigi bo‘lsin. AB egri chiziq uzunligi l ni topamiz.

$$l=\int_a^b\sqrt{1+(y'_x)^2}\,dx. \qquad (1.8)$$



9-shakl

(1.7) tenglikka *yoy differensialining* to‘g‘ri burchakli koordinatalardagi formulasi deyiladi.

Agar egri chiziq $x=g(y),\ y\in[c;d]$, tenglama bilan berilgan bo‘lsa, yuqorida keltirilganlarni takrorlab, AB yoy uzunligini hisoblashning quyidagi formulasini hosil qilamiz:

$$l=\int_c^d\sqrt{1+(x'_y)^2}\,dy. \qquad (1.9)$$

6- nmisol. $y=\frac{3}{8}x\sqrt[3]{x}-\frac{3}{4}\sqrt[3]{x^2}$ egri chiziq yoyining Ox o‘q bilan kesishish nuqtalari orasidagi uzunligini hisoblang.

Yechish. $y=0$ deb egri chiziqning Ox oq bilan kesishish nuqtalarini aniqlaymiz: $x_1=0,\ x_2=2\sqrt{2}.$

Hosilani topamiz: $y'=\frac{3}{8}\cdot\frac{4}{3}x^{\frac{1}{3}}-\frac{3}{4}\cdot\frac{2}{3}x^{-\frac{1}{3}}=\frac{1}{2}\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right).$

Yoy uzunligini hisoblaymiz:

$$l=\int_0^{2\sqrt{2}}\sqrt{1+\frac{1}{4}\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right)^2}\,dx=\frac{1}{2}\int_0^{2\sqrt{2}}\sqrt{\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)^2}\,dx=$$

$$=\frac{1}{2}\int_0^{2\sqrt{2}}\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)dx=\frac{1}{2}\left(\frac{3}{4}x^{\frac{4}{3}}+\frac{3}{2}x^{\frac{2}{3}}\right)\Bigg|_0^{2\sqrt{2}}=3.$$

Agar egri chiziq $x=\varphi(t),\ y=\psi(t),\ \alpha\le t\le\beta$, parametrik tenglamalar bilan berilgan bo‘lsa, (1.8 formulada $x=\varphi(t),\ y=\psi(x),\ dx=\varphi'(t)dt$ o‘riniga qo‘yish orqali o‘zgaruvchi almashtiriladi. Bunda

$$l=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (1.10)$$

kelib chiqadi, bu yerda $a=\varphi(\alpha)$ va $b=\varphi(\beta)$.

Egri chiziq qutb koordinatalar sistemasida $r=r(\varphi),\ \alpha\le\varphi\le\beta$, tenglama bilan berilgan bo‘lsin, bunda $r(\varphi),\ r'(\varphi)$ funksiyalar $[\alpha;\beta]$ kesmada uzluksiz va A,B nuqtalarga qutb koordinatalarida α,β burchaklar mos keladi.

$x=r\cos\varphi,\ y=r\sin\varphi$ ekanligidan

$$x'(\varphi)=r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi,\ y'(\varphi)=r'(\varphi)\sin\varphi-r(\varphi)\cos\varphi.$$

(1.9) formulaga $x'(\varphi)$ va $y'(x)$ hosilalarni qo‘yamiz va almashtirishlar bajarib,

topamiz: $$l=\int_\alpha^\beta\sqrt{r^2(\varphi)+r'^2(\varphi)}d\varphi\ . \qquad (1.11)$$

7- misol. $r=a(1+\cos\varphi),\ a>0$, kardioida uzunligini toping.

Yechish. Egri chiziqning simmetrikligini (1-ilovaga qarng) hisobga olib, (1.11) formula bilan topamiz:

$$l=2\int_0^\pi\sqrt{a^2(1+\cos\varphi)^2+a^2(-\sin\varphi)^2}d\varphi=4a\int_0^\pi\sqrt{\frac{1+\cos\varphi}{2}}d\varphi=$$

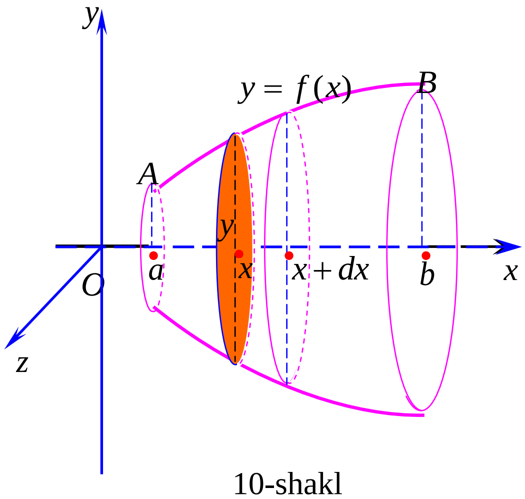
$$=4a\int_0^\pi\cos\frac{\varphi}{2}d\varphi=8a\sin\frac{\varphi}{2}\Big|_0^\pi=8a.$$

Aylanma jism sirti yuzi.

AB egri chiziq $y=f(x)\ge0$ funksiyaning grafigi bo‘lsin. Bunda $x\in[a;b],\ y=f(x)$ funksiya va uning $y'=f'(x)$ hosilasi bu kesmada uluksiz bo‘lsin.

AB egri chiziqning *Ox* o‘q atrofida aylanishidan hosil bo‘lgan jism sirti yuzini hisoblaymiz.

$$S=2\pi\int_a^b y\sqrt{1+(y'_x)^2}dx \qquad (1.12)$$



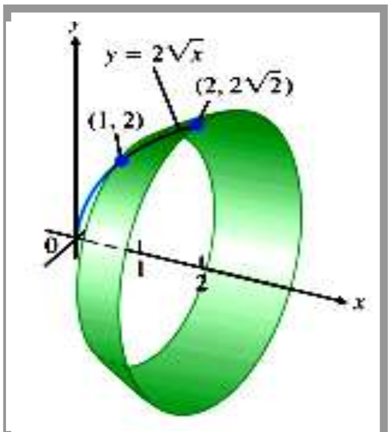
10-shakl

Shu kabi $x=g(y),\ \ y\in[c;d]$ funksiya grafigining *Oy* o‘q atrofida aylantirshdan hosil bo‘lgan jism sirtining yuzi ushbu

$$S = 2\pi\int_c^d x\sqrt{1+(x'_y)^2}\,dy \qquad (1.13)$$

formula bilan hisoblanadi.

5- misol. $y = 2\sqrt{x},\ 1 \le x \le 2$ egri chiziqning Ox o'qi atrofida aylanishidan hosil bo'lgan sirt (11-shakl) yuzini toping[2].



11-shakl

Yechish. Misol shartidan topamiz: $y' = \dfrac{1}{\sqrt{x}}$.

(1.12) formula bilan topamiz:

$$S = 2\pi\int_1^2 2\sqrt{x}\cdot\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx = 4\pi\int_1^2\sqrt{x+1}\,dx = 4\pi\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_1^2 = \frac{8\pi}{3}(3\sqrt{3}-2\sqrt{2}).$$

Agar AB egri chiziq $x=\varphi(t),\ y=\psi(t),\ \alpha\le t\le\beta$, parametrik tenglamalar bilan berilgan bo'lsa, u holda AB egri chiziqning $Ox(Oy)$ o'q atrofida aylanishidan aylanishidan hosil bo'lgan jism sirti yuzi quyidagicha hisoblanadi:

$$S = 2\pi\int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt\ \left(S = 2\pi\int_{\alpha_1}^{\beta_1}\varphi(t)\sqrt{\psi'^2(t)+\varphi'^2(t)}\,dt\right), \qquad (1.14)$$

bu yerda $a=\varphi(\alpha)$ va $b=\varphi(\beta)$ ($c=\psi(\alpha_1)$ va $d=\psi(\beta_1)$).

AB egri chiziq qutb koordinatalar sistemasida $r=r(\varphi),\ \alpha\le\varphi\le\beta$ tenglama bilan berilgan bo'lganida quyidagi formulalar o'rinli bo'ladi:

$$Ox:\ S = 2\pi\int_\alpha^\beta r\sin\varphi\sqrt{r^2+r'^2}\,d\varphi,\ Oy:\ S = 2\pi\int_\alpha^\beta r\cos\varphi\sqrt{r^2+r'^2}\,d\varphi. \qquad (1.15)$$

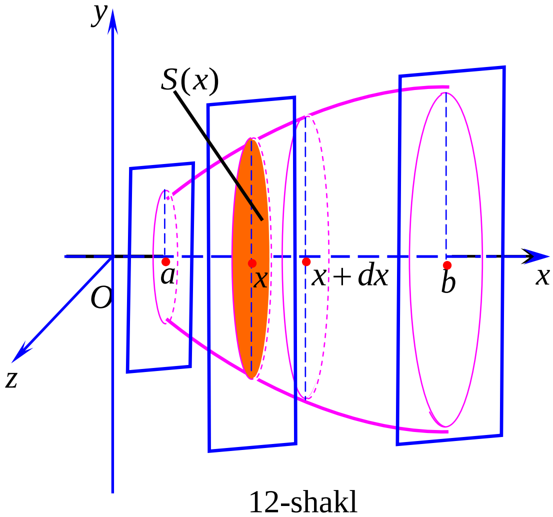
6- misol. Radiusi R ga teng bo'lgan shar sirti yuzini hisoblang.

Yechish. Shar parametrik tenglamasi $x=R\cos t,\ y=R\sin t$ bo'lgan yarim aylananing Ox o'q atrofida aylanishidan hosil bo'ladi. Sharning koordinata o'qlariga simmetrik bo'lishini inobatga olib, hisoblaymiz:

$$S = 2\cdot 2\pi\int_0^{\frac{\pi}{2}} R\sin t\sqrt{(-R\sin t)^2+(R\cos t)^2}\,dt = 4\pi R^2\int_0^{\frac{\pi}{2}}\sin t\,dt = -4\pi R^2\cos t\Big|_0^{\frac{\pi}{2}} = 4\pi R^2.$$

Hajmni ko'ndalang kesim yuzasi bo'yicha hisoblash

Hajmi hisoblanishi lozim bo'lgan qandaydir jism (12-shakl) uchun uning istalgan ko'ndalang kesim yuzasi S ma'lum bo'lsin. Bu yuza ko'ndalang kesim joylashishiga bog'liq bo'ladi: $S = S(x),\ x \in [a;b]$, bu yerda $S(x)-$ $[a;b]$ kesmada uzluksiz funksiya.



12-shakl

$$V = \int_a^b S(x)dx. \qquad (1.16)$$

7- misol. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ellipsoidning hajmini hisoblang.

Yechish. Ellipsoidning Ox o'qqa perpendikulyar va koordinatalar boshidan x $(-a \le x \le a)$ masofada o'tuvchi tekislik bilan kesamiz. Kesimda yarim o'qlari $b(x) = b\sqrt{1 - \dfrac{x^2}{a^2}}$ va $c(x) = c\sqrt{1 - \dfrac{x^2}{a^2}}$ bo'lgan ellips hosil bo'ladi. Uning yuzasi

$$S(x) = \pi b(x)c(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

U holda

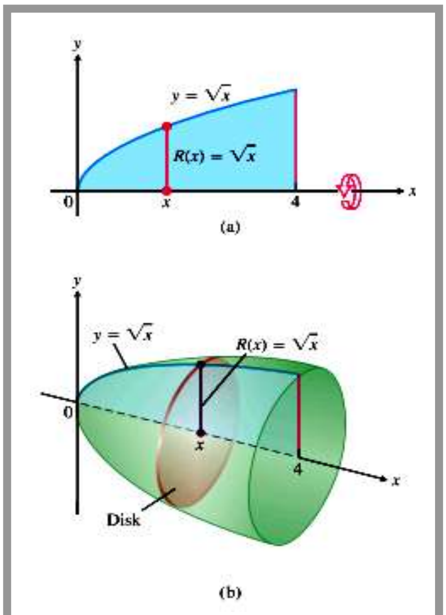
$$V = \int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \pi bc\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi abc.$$

Aylanish jismlarining hajmini hisoblash

Yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Ox o'q atrofida aylantirishdan hosil bo'lgan jism hajmini hisoblaymiz. Bu jismning ixtiyoriy ko'ndalang kesimi doiradan iborat. Shu sababli jismning $X = x$ tekislik bilan kesimining yuzasi $S(x) = \pi y^2$ bo'ladi.

U holda (1.16) formulaga ko'ra

$$V = \pi\int_a^b y^2 dx. \qquad (1.17)$$



13-shakl

Shu kabi yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Oy o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi quyidagi formula bilan hisoblanadi:

$$V = 2\pi\int_a^b yxdx. \quad (1.18)$$

8- misol[2]. $y=\sqrt{x},\ \ y=0,\ \ x=0,\ \ x=4$ chiziqlar bilan chegaralangan yassi shaklning Ox o’qi atrofida aylanishidan hosil bo’lgan jism (13-shakl) hajmini toping.

Yechish. Hajmni (1.17) formula bilan topamiz:

$$S = \pi\int_0^4 xdx = \pi\cdot\left.\frac{x^2}{2}\right|_0^4 = 8\pi.$$

Agar egri chiziqli trapetsiya $x=\varphi(y)$ uzluksiz funksiya grafigi, Oy o‘qi, $y=c$ va $y=d$ to‘g‘ri chiziqlar bilan chegaralangan bo‘lsa,

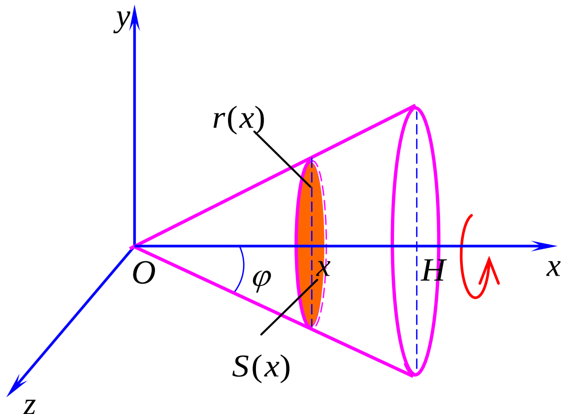
$$V = \pi\int_c^d x^2dy\ (Oy)\left(V = 2\pi\int_c^d xydy\ (Ox)\right) \qquad (1.19)$$

bo‘ladi.

9- misol. Radiusi R ga va balandligi H ga teng bo‘lgan konusning hajmini hisoblang.

Yechish. Konusnni katetlari R va H bo‘lgan to‘g‘ri burchakli uchburchakning balandlik bo‘ylab yo‘nalgan Ox o‘q atrofida aylanishidan hosil bo‘lgan jism deyish mumkin (14-shakl). Gipotenuza tenglamasi $y=kx$ bo‘lsin. U holda

$$y=kx,\ k=tg\varphi=\frac{R}{H},\ y=\frac{R}{H}x.$$



14-shakl

Bundan

$$V=\pi\int_0^H y^2dx=\pi\int_0^H\frac{R^2}{H^2}x^2dx=\frac{\pi R^2}{H^2}\cdot\left.\frac{x^3}{3}\right|_0^H=\frac{1}{3}\pi R^3H.$$

ANIQ INTEGRALNING MEXANIKAGA TADBIQLARI

Oxy tekislikda massalari mos ravishda $m_1, m_2, \ldots, m_n$ bo‘lgan $A_1(x_1;y_1), A_2(x_2;y_2), \ldots, A_n(x_n;y_n)$ nuqtalar sistemasi berilgan bo‘lsin.

Sistemaning Ox (Oy) o‘qqa nisbatan *statik momenti* M_x (M_y) deb nuqtalar massalarini ularning ordinatalariga (absissalariga) ko‘paytmalari yig‘indisiga aytiladi, ya’ni

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \left(M_y = \sum_{i=1}^{n} m_i x_i \right).$$

Sistemaning Ox (Oy) oqqa nisbatan *inersiya momenti* J_x (J_y) deb nuqtalar massalarini ularning ordinatalari (absissalari) kvadratiga ko‘paytmalari yig‘indisiga aytiladi, ya’ni

$$J_x = \sum_{i=1}^{n} m_i y_i^2 \quad \left(J_y = \sum_{i=1}^{n} m_i x_i^2 \right).$$

Sistemaning *og‘irlik markazi* deb koordinatalari $\left(\frac{M_y}{m}; \frac{M_x}{m} \right)$ bo‘lgan nuqtalarga aytiladi, bu yerda $m = \sum_{i=1}^{n} m_i$.

Yassi egri chiziqning momentlari va og‘irlik markazi

Oxy tekislikda AB egri chiziq $y = f(x)$ $(a \le x \le b)$ tenglama bilan berilgan bo’lib, egri chiziqning har bir $(x, f(x))$ nuqtasidagi zichlik $\gamma = \gamma(x)$ ga teng va $f(x)$ funksiya $f'(x)$ hosilasi bilan birga uzluksiz bo‘lsin.

U holda AB egri chiziqning momentlari va og’irlik markazining koordinatalari quyidagi formulalar bilan aniqlanadi:

$$M_x = \int_a^b \gamma y dl, \quad M_y = \int_a^b \gamma x dl; \tag{1.20}$$

$$J_x = \int_a^b \gamma y^2 dl, \quad J_y = \int_a^b \gamma x^2 dl; \tag{1.21}$$

$$x_c = \frac{\int_a^b \gamma x dl}{\int_a^b \gamma dl}, \quad y_c = \frac{\int_a^b \gamma y dl}{\int_a^b \gamma dl}, \tag{1.22}$$

bu yerda $y = f(x)$, $\gamma = \gamma(x)$, $dl = \sqrt{1 + y'^2} dx$, $a \le x \le b$.

10- misol. Zichligi $\gamma = 1$ ga teng bo‘lgan $y = \sqrt{R^2 - x^2}$, $|x| \le R$ yarim aylananing momentlari va og‘irlik markazini toping.

Yechish. $y' = -\frac{x}{\sqrt{R^2 - x^2}}$ bo‘lgani uchun $dl = \frac{Rdx}{\sqrt{R^2 - x^2}}$.

U holda (1.20) - (1.22) formulalardan topamiz:

$$M_x = \int_{-R}^{R} \sqrt{R^2 - x^2} \frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} dx = Rx\Big|_{-R}^{R} = 2R^2,$$

$$M_y = \int_{-R}^{R} \frac{xRdx}{\sqrt{R^2 - x^2}} = -R\sqrt{R^2 - x^2}\Big|_{-R}^{R} = 0.$$

$$J_x = \int_{-R}^{R} (R^2 - x^2) \frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2 \cos^2 t dt =$$

$$= R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = \frac{R^3}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi R^3}{2},$$

$$J_y = \int_{-R}^{R} \frac{x^2 Rdx}{\sqrt{R^2 - x^2}} = R\left(-x\sqrt{R^2 - x^2}\Big|_{-R}^{R} + \int_{-R}^{R} \sqrt{R^2 - x^2}\, dx\right) = R\left(0 + \frac{\pi R^2}{2}\right) = \frac{\pi R^3}{2}.$$

$$\int_{-R}^{R} dl = \int_{-R}^{R} \frac{Rdx}{\sqrt{R^2 - x^2}} = R \arcsin\frac{x}{R}\Big|_{-R}^{R} = \pi R,$$

$$x_c = 0,\ y_c = \frac{2R^2}{\pi R} = \frac{2R}{\pi}.$$

Yassi shaklning momentlari va og‘irlik markazi

Oxy yassilikda $[a;b]$ kesmada uzluksiz bo‘lgan $y = f(x)$ funksiya grafigi, *Ox* o‘q, $x = a$ va $x = b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya (yassi shakl) berilgan va yassi shaklning har bir nuqtasida $\gamma = \gamma(x)$ zichlik uzluksiz bo‘lsin. U holda yassi shaklning momentlari va og‘irlik markazining koordinatalari quyidagi formulalar orqali topiladi:

$$M_x = \frac{1}{2}\int_a^b \gamma y^2 dx, \quad M_y = \int_a^b \gamma xydx; \tag{1.23}$$

$$J_x = \frac{1}{2}\int_a^b \gamma y^3 dx, \quad J_y = \int_a^b \gamma x^2 ydx; \tag{1.24}$$

$$x_c = \frac{\int_a^b \gamma xydx}{\int_a^b \gamma ydx}, \quad y_c = \frac{\frac{1}{2}\int_a^b \gamma y^2 dx}{\int_a^b \gamma ydx}, \tag{1.25}$$

bu yerda $\gamma = \gamma(x)$, $y = y(x)$, , $a \le x \le b$.

11- misol. $y = \sin x$ sinusoida yoyi va *Ox* o‘qining $0 \le x \le \pi$ bo‘lagi bilan chegaralangan, zichligi $\gamma = 1$ ga teng figuraning og‘irlik markazini toping.

Yechish. Sinusoidaning simmetrikligidan $x_c = \frac{\pi}{2}$ bo‘ladi.

U holda

$$M_x = \frac{1}{2}\int_0^{\pi} y^2 dx = \frac{1}{2}\int_0^{\pi} \sin^2 x dx = \frac{1}{2}\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{4}\left(x - \frac{\sin 2x}{2} \right)\Bigg|_0^{\pi} = \frac{\pi}{4}.$$

$$\int_0^{\pi} y dx = \int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = 2,\ y_c = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}.$$

Demak, $x_c = \frac{\pi}{2},\ y_c = \frac{\pi}{8}.$

Aniq integralning ishni hisoblashga tatbiqi

Moddiy nuqta o‘zgaruvchan $\vec{F}$ kuch tasirida Ox o‘qi bo‘ylab harakatlanayotgan bo‘lsin va bunda kuchning yo‘nalishi harakat yo‘nalishi bilan bir xil bo‘lsin. U holda $\vec{F}$ kuchning moddiy nuqtani Ox o‘qi bo‘ylab $x = a$ nuqtadan $x = b$ $(a < b)$ nuqtaga ko‘chirishda bajargan ishi quyidagi formula bilan hisoblanadi:

$$A = \int_a^b F(x)dx, \qquad (1.26)$$

bu yerda $F(x)$ funksiya $[a;b]$ kesmada uzluksiz.

12- misol. m massali kosmik kemani yerdan h masofaga uchurish uchun qancha ish bajarish kerak?

Yechish. Butun olam tortishish qonuniga ko‘ra yerning jismni tortish kuchi $F = k\frac{mM}{x^2}$ ga teng bo’ladi, bu yerda $M-$yerning massasi, $x-$ yer markazidan kosmik kemagacha bo‘lgan masofa, $k-$gravitasiya doimiyligi. Yer sirtida, ya’ni $x = R$ da $F = mg$ ga teng, bu yerda $g-$ erkin tushish tezlanishi.

U holda

$$mg = k\frac{mM}{R^2}.$$

Bundan $kM = gR^2$ va $F = mg\frac{R^2}{x^2}.$

Izlanayotgan ishni (1.26) formula bilan topamiz:

$$A = \int_R^{R+h} mg\frac{R^2}{x^2}dx = -mgR^2\frac{1}{x}\Bigg|_R^{R+h} = -mgR^2\left(\frac{1}{R+h} - \frac{1}{R}\right) = mgR\frac{h}{R+h}.$$

Aniq integralning yo’lni hisoblashga tatbiqi

Moddiy nuqta (jism) to‘g‘ri chiziq bo‘ylab o‘zgaruvchan $v = v(t)$ tezlik bilan harakatlanayotgan bo‘lsin. Bu nuqtaning t_1 dan t_2 gacha vaqt oralig‘ida bosib o‘tgan yo‘lini topamiz.

$v(t)=\dfrac{dS}{dt}$. Bundan $dS=v(t)dt$.Bu tenglikni t_1 dan t_2 gacha integrallaymiz:

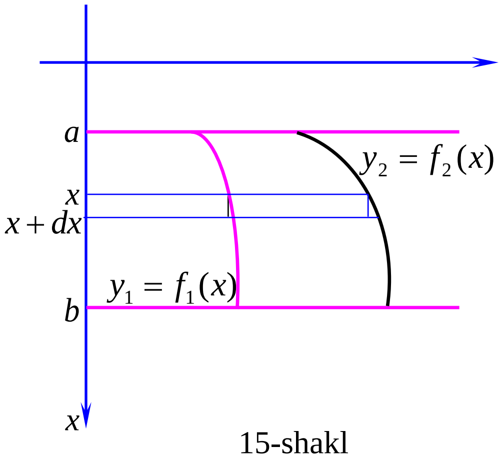
$$S=\int_{t_1}^{t_2}v(t)dt. \quad (1.27)$$

Aniq integralning bosimni hisoblashga tatbiqi

Paskal qonuniga ko‘ra suyuqlikning gorizontal plastinkaga bosimi

$$P=g\cdot\gamma\cdot S\cdot h$$

formula bilan topiladi, bu yerda $g-$ erkin tushish tezlanishi, $\gamma-$ suyuqlik zichligi, $S-$ plastinkaning yuzasi, $h-$ plastinkaning botish chuqurligi.



15-shakl

Plastinkaning vertikal botishida suyuqlikning plastinkaga bosimini bu formula bilan topib bo‘lmaydi, chunki plastinkaning har xil nuqtalari turli chuqurlikda yotadi.

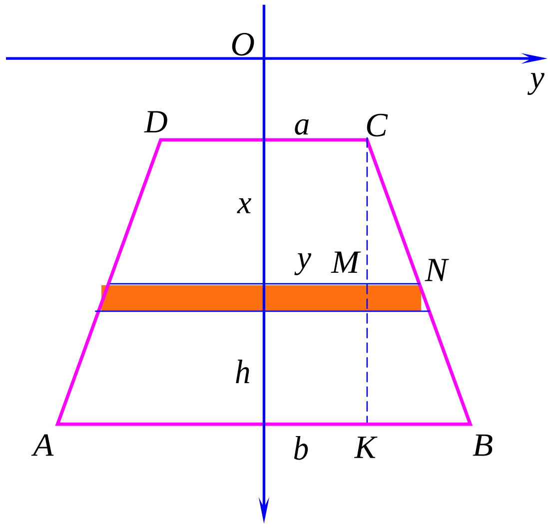
Suyuqlikka $x=a,\ x=b,\ y_1=f_1(x),$ $y_2=f_2(x)$ chiziqlar bilan chegaralangan plastinka vertikal botirilayotgan bo‘lsin. Bunda koordinatalar sistemasi 15-shaklda ko‘rsatilganidek tanlangan bo‘lsin. Shuyuqlikning plastinkaga P bosimi

$$P=g\cdot\gamma\cdot\int_a^b(y_2-y_1)xdx$$

yoki

$$P=g\cdot\gamma\cdot\int_a^b(f_2(x)-f_1(x))xdx. \quad (1.28)$$

24- misol. Asoslari a va b ga, balandligi h ga teng bo‘lgan teng yonli trapetsiya shaklidagi plastinka suyuqlikka c chuqurlikda botirilgan (13-shakl). Suyuqlikning plastinkaga bosimini toping.



16-shakl

Yechish. Izlanayotgan bosim (1.28) formulaga ko‘ra

$$P=g\gamma\int_c^{c+h}yxdx \text{ bo‘ladi.}$$

y o‘zgaruvchini x o‘zgaruvchi orqali ifodalash uchun CMN va CKB uchburchaklarning o‘xshashligidan foydalanamiz:

$$\frac{y-a}{b-a}=\frac{x-c}{h}.$$

Bundan $y=a+\frac{b-a}{h}(x-c)$. U holda

$$P=g\gamma\int_c^{c+h}\left(a+\frac{b-a}{h}(x-c)\right)xdx=g\gamma\left(\frac{ax^2}{2}+\frac{b-a}{h}\left(\frac{x^3}{3}-\frac{cx^2}{2}\right)\right)\Bigg|_c^{c+h}=$$

$$=g\gamma\left(\frac{a+b}{2}ch+\frac{h^2}{6}(a+2b)\right).$$

Amaliy mashg'ulot uchun mashqlar.

1 Berilgan chiziqlar bilan chegaralangan figuralar yuzalarini hisoblang:

1) $y=9-x^2,\ y=0;$ 2) $y=-x,\ y=2x-x^2;$

3) $y=\ln(x+6),\ y=3\ln x,\ y=0,\ x=0;$ 4) $y=\ln x,\ y=0,\ x=e^2;$

5) $x=y^2,\ x=|y+2|;$ 6) $xy=4,\ x=5-y;$

7) $y=x^2,\ y^2=-x;$ 8) $y=x^2,\ y=x^3,\ x=-1,\ x=1;$

9) $x=4\cos t,\ y=3\sin t,\ 0\le t\le 2\pi;$

10) $x=3(t-\sin t),\ y=3(1-\cos t)$, sikloida bitta arkasi;

11) $r=3\sqrt{\cos 2\varphi}$; 12) $r=3\sin 2\varphi.$

13) $r=2+3\cos\varphi;$ 14) $r=2\varphi$, bir o'rami.

2. Berilgan egri chiziqlar yoylari uzunliklarini toping:

1) $y=\frac{x^2}{2},\ x=0$ dan $x=\sqrt{3}$ gacha;

2) $y=chx,\ x=0$ dan $x=1$ gacha;

3) $y^2=x^3,\ x=0$ dan $x=5$ gacha;

4) $y=\arccos\sqrt{x}-\sqrt{x-x^2},\ x=0$ dan $x=1$ gacha;

5) $x=\frac{1}{4}y^2-\frac{1}{2}\ln y,\ y=1$ dan $y=2$ gacha;

6) $x=1-\ln(y^2-1),\ y=3$ dan $y=4$ gacha;

7) $x=t^2,\ y=\frac{t^3}{3}-t$, koordinata o'qlari bilan kesishish nuqtalari orasidagi;

8) $x=t^2,\ y=t^3,\ t=0$ dan $t=1$ gacha;

9) $x=2(t-\sin t),\ y=2(1-\cos t)$, sikloida bitta arkasi;

10) $x=3(2\cos t-\cos 2t),\ \ y=3(2\sin t-\sin 2t)$;

11) $r=a(1-\cos\varphi),\ \ r\le\frac{a}{2}$ kardioida bo‘lagining;

12) $r=8\cos^3\frac{\varphi}{3},\quad \varphi=0$ dan $\varphi=\frac{\pi}{2}$ gacha.

3. Chiziqlarning berilgan o‘q atrofida aylanishidan hosil bo‘lgan sirt yuzasini hisoblang:

1) $y^2=4x,\ \ x=0$ dan $x=3$ gacha, Ox o‘q;

2) $x^2+y^2=9,\ \ Oy$ o‘q;

3) $x=2(t-\sin t),\ \ y=2(1-\cos t)$, bitta arkasi, Ox o‘q;

4) $x=\sqrt{2}\cos t,\ \ y=\sin t,\ Ox$ o‘q;

4. R radiusli shar hajmini hisoblang.

5. Asosi $\frac{x^2}{16}+\frac{y^2}{9}=1$ ellipsdan iborat bo‘lgan va balandligi $h=3$ga teng elliptik konusning hajmini hisoblang.

6. $x^2+y^2+z^2=16$ shar hamda $x=2$ va $x=3$tekisliklar bilan chegaralangan jism hajmini hisoblang.

7. $\frac{y^2}{4}+\frac{z^2}{9}-x^2=1$ bir pallali giperboloid hamda $x=-1$ va $x=2$tekisliklar bilan chegaralangan jism hajmini hisoblang.

8. Berilgan chiziqlar bilan chegaralangan figuraning berilgan o‘q atrofida aylanishidan hosil bo‘lgan jism hajmini hisoblang:

1) $x^2=4-y,\ \ y=0,\ Ox$ o‘qi;

2) $x^2+y^2=4$ yarim aylana $(x\ge 0)$ va $y^2=3x$ parabola, Ox o‘qi;

3) $y=\arcsin x,\ \ y=0,\ \ x=1,\ \ Oy$ o‘qi;

4) $y^2=x^3,\ \ x=1,\ \ y=0,\ \ Oy$ o‘qi;

5) $x^2=4y,\ \ x=0,\ \ y=1,\ Oy$ o‘qi;

6) $\frac{x^2}{25}+\frac{y^2}{9}=1,\ \ Oy$ o‘qi;

7) $x=2(t-\sin t),\ \ y=2(1-\cos t)$, bitta arkasi, Ox o‘qi;

8) $x=t^2,\ \ y=t^3,\ \ x=0,\ \ y=1,\ Oy$ o‘qi;

9) $r=3(1+\cos\varphi)$, qutb o‘qi;

10) $r=2R\cos\varphi$, yarim aylana, qutb o‘qi;

9. $r=2R\sin\varphi$ bir jinsli aylananing og‘irlik markazini toping.

10. $x = a\cos^3 t,\ \ y = a\sin^3 t$ bir jinsli astroidaning Ox o'qdan yuqorida yotgan yoyining og'irlik markazini toping.

11. $4x + 3y - 12 = 0$ bir jinsli to'g'ri chiziqning koordinata o'qlari orasida joylashgan kesmasining koordinata o'qlariga nisbatan statik momentlarini toping.

12. $x = 0,\ \ y = 0,\ \ x + y = 2$ ciziqlar bilan chegaralangan bir jinsli tekis shaklning koordinata o'qlariga nisbatan statik va inersiya momentlarini, og'irlik markazini toping.

13. $y = 4 - x^2$ va $y = 0$ bir jinsli chiziqlar bilan chegaralangan figuraning og'irlik markazini toping.

14. Yarim o'qlari $a = 5$ va $b = 4$ bo'lgan bir jinsli ellipsning koordinata o'qlariga nisbatan inersiya momentini toping.

15. $x^2 + y^2 = R^2$ aylananing birinchi chorakda joylashgan bo'lagining o'girlik markazini toping. Bunda aylananing har bir nuqtasidagi chiziqli zichligi shu nuqta koordinatalarining ko'paytmasiga proporsional.

16. $x = 8\cos^3 t,\ 8 = 4\sin^3 t$ astroida birinchi chorakda yotgan yoyining koordinata o'qlariga nisbatan statik momentlarini va massasini toping. Bunda astroidaning har bir nuqtasidagi chiziqli zichligi x ga teng.

17. Prujinani $4\ sm.$ ga cho'zish uchun $24\ J$ ish bajariladi. $150\ J$ ish bajarilsa, prujinana qanday uzunlikka cho'ziladi?

18. Agar prujinani $1\ sm.$ ga siqish uchun $1\ kG$ kuch sarf qilinsa, prujinaning $8\ sm.$ ga siqishda sarf bo'ladigan F kuch bajargan ishni toping.

19. Uzunligi $0{,}5\ m.$ va radiusi $4\ mm.$ bo'lgan mis simni $2\ mm.$ cho'zish uchun qancha ish bajarish kerak? Bunda $F = E\dfrac{Sx}{l},\ \ E = 12\cdot 10^4\ \ N/mm^2.$

20. Og'irligi $P = 1{,}5\ T$ bo'lgan kosmik kemani yer sirtidan $h = 2000\ km.$ masofaga uchirish uchun bajarilishi kerak bo'ladigan ishni toping.

21. Jismning to'g'ri chiziqli harakat tezligi $v = 2t + 3t^2\ \ (m/s)$ formula bilan ifodalanadi. Jismning harakat boshlanishidan $5\ s.$ davomida bosib o'tgan yo'lini toping.

22. Nuqtaning harakat tezligi $v = 0{,}1t^3\ \ (m/s)$ ga teng. Nuqtaning $10\ s.$ davomidagi o'rtacha tezligini toping.

23. Sportchining parashyutdan tushish tezligi $v = \dfrac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$ formula bilan ifodalanadi, bu yerda g-erkin tushish tezlanishi, m- sportchining massasi, k- proporsionallik koeffitsiyenti. Agar parashyutdan tushish $3\ \min.$ davom etgan bo‘lsa, sportchi qanday balandlikdan sakragan?

24. Nuqtaning harakat tezligi $v = 0,1e^{-0,01t}$ (m/s) ga teng. Nuqtaning harakat boshlanishidan harakat to‘xtaguncha bosib o‘tgan yo‘lini toping.

25. Suyuqlikka vertikal botirilgan asoslari a va b $(b > a)$ ga, balandligi h ga teng bo‘lgan teng yonli trapetsiya shaklidagi plastinkaga suyuqlikning bosimini toping.

26. Asosi 18 m. Va balandligi 6 $m.$ bo‘lgan to‘rt burchakli shluzga suv bosimini toping.

27. Diametri 6 $m.$ bo‘lgan va suv sathida joylashgan yarim doira shaklidagi vertikal devorga suv bosimini toping. Suv zichligi $\gamma = 1000$ kg/m^3.

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